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Topological methods and the existence/non-existence problem of Einstein metrics on homogeneous spaces

Métodos topológicos aplicados ao problema de existência/não-existência de métricas de Einstein em espaços homogêneos

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Resumo

Sejam G um grupo de Lie e H um subgrupo fechado de G. Nesta dissertação, associaremos ao espaço homogêneo compacto G/H um complexo simplicial abstrato $\Delta_{G/H}^{T}$. Foi provado que a não-contrabilidade de $\Delta_{G/H}^{T}$ implica a existência de uma métrica Einstein G-invariante em G/H. Será mostrada uma maneira de calcular uma classe de homologia reduzida não-nula de $\Delta_{G/H}^{T}$ para subgrupos intermediários $H < K < G$ sobre algumas hipóteses, o que implica a não-contrabilidade de $\Delta_{G/H}^{T}$. Esse método será aplicado ao caso em que G é simples clássico e H tem posto maximal.

Palavras-Chave: Espaços homogêneos, Teoria de Lie, Geometria Riemanniana, Métricas de Einstein

Abstract

Let G be a compact Lie group and H be closed subgroup of G. In this dissertation, we will associate to the compact homogeneous space G/H a abstract simplicial complex $\Delta^T_{G/H}$. It was proved that the non-contractility of $\Delta^T_{G/H}$ implies that the existence of a G-invariant Einstein metric in G/H. It will be shown a way to compute a non-zero reduced homology class of $\Delta^T_{G/H}$ for intermediate subgroups $H < K < G$ under some hypothesis, which implies the non-contractility of $\Delta^T_{G/H}$. This method will be applied to case where G is classical simple and H has maximal rank.

Keywords: Homogeneous spaces, Lie theory, Riemannian geometry, Einstein metrics

Contents

INTRODUCTION . . . **9**

1 Simplicial Complexes and Homology . . . **12**

1.1 Simplicial Complexes . . . 12

1.2 Simplicial Homology . . . 19

1.3 Singular homology . . . 21

1.3.1 Reduced homology . . . 24

1.3.2 Homology with coefficients . . . 25

1.3.3 The Mayers-Vietoris sequence . . . 27

2 The Simplicial Complex of a Homogeneous Space . . . **28**

2.1 Order complexes . . . 28

2.2 The Simplicial Complexes $\Delta_{G/H}$, $\Delta_{G/H}^{min}$ and $\Delta_{G/H}^{T}$ . . . 29

2.2.1 $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ . . . 30

2.2.2 $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \neq \mathfrak{h}$ . . . 31

2.3 The simplicial complex theorems for invariant Einstein metrics . . . 33

2.4 Algebraic topology of order complexes . . . 37

2.4.1 Rauße's theorem . . . 39

3 Lie Theory . . . **44**

3.1 Root Systems and Subalgebras of Maximal Rank . . . 44

3.1.1 Abstract Root Systems . . . 44

3.1.2 Root Systems of Semisimple Lie Algebras . . . 46

3.1.3 Compact Real Forms . . . 47

3.1.4 Subalgebras of Maximal Rank . . . 48

4 The Simplicial Complex for G simple classic and H connected of maximal rank **50**

4.1 $SU(n+1), n \geq 1$ . . . 50

4.2 $SO(n), n \geq 3$ . . . 54

4.2.1 n even, $n \geq 8$ . . . 55

4.2.2 n odd, $n \geq 3$ . . . 62

4.3 $Sp(n), n \geq 1$ . . . 67

5 An application in a real flag manifold . . . **73**

BIBLIOGRAPHY . . . **77**

Introduction

In the studies of Riemannian metrics, we are faced with the concept of Einstein metrics, a type of Riemannian metrics well studied and of great importance to Geometry and applications to Physics. Its definition is

Definition 0.0.1. *[Lee19, p. 210] Given a differentiable manifold M, a Riemannian metric g on M is said to be an Einstein metric if there exists $\lambda \in \mathbb{R}$ such that*

$$Ric(g) \; = \lambda \cdot g$$

where $Ric(g)$ is the Ricci tensor of the metric g.

When the manifold is a homogeneous space of a Lie group G, i.e., a manifold with a transitive action of G, many studies restrict themselves to G-homogeneous Einstein metrics, i.e., Einstein metrics preserved by the action of G.

It is well known that if H is a closed subgroup of G, then G/H, the set of left cosets of H in G, has a unique differentiable structure such that the natural projection is a submersion [War83, p. 120]. If H is a isotropy group for the action at a point of M, then M is diffeomorphic to G/H [War83, p. 123]. Since G acts in G/H, by $(g_1, g_2H) \mapsto g_1g_2H$ for all $g_1, g_2 \in G$ with isotropy group equal to H at $o := eH$ and the above diffeomorphism between M and G/H is equivariant between the two actions, then we can always understand a homogeneous space as a coset space G/H for a Lie group G and a closed Lie group G/H. We also fix $\mathfrak{g} := Lie(G)$ and $\mathfrak{h} := Lie(H)$.

Many studies have already been done about homogeneous Einstein metrics, but we do not have a general result about the existence/non-existence of these type of metrics.

The beginning of the methods described in this work about existence/non-existence of homogeneous space is this variational characterisation:

Theorem 0.0.2. *[Bes87, p. 121] Let $\mathcal{M}_1^G$ be the space of G-homogeneous metrics of volume 1 in G/H. Then, $\mathcal{M}_1^G$ is a finite dimensional manifold and a metric $g \in \mathcal{M}_1^G$ is an Einstein metric if, and only if, it is a critical point of the scalar curvature functional $sc : \mathcal{M}_1^G \to \mathbb{R},\; g \mapsto sc(g)$.*

In this work, sc always means scalar curvature and the scalar curvature functional above is well define, since scalar curvature is preserved by isometries and $G \subseteq Isom(G/H, g)$ acts transitively.

Using this characterisation, in [WZ86], W. Ziller and M. Wang proved the following theorem

Theorem 0.0.3. *[WZ86, p. 183] The scalar curvature functional sc is bounded from above and proper if, and only if, H is a maximal connected subgroups of G, or, equivalently, $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$. In this case, sc has a global maximum, which must be a G-invariant Einstein metric on G/H.*

(Here, proper means a continuous map such its inverse images of compacts are compacts.)

The studies of existence or non-existence of homogeneous Einstein metrics in G/H follows the idea of trying to find the hypothesis for the existence/non-existence in terms of the algebraic structure of $\mathfrak{g}$ and $\mathfrak{h}$, used in the theorem above.

In this dissertation, we study the following method:

In [Bö04], Böhm introduced for a compact homogeneous space a abstract simplicial complex $\Delta^T_{G/H}$, which can be thought as polyhedron in a Euclidean space, and proved that if $\Delta^T_{G/H}$ is a not contractible topological space, then G/H admits a G-invariant Einstein metric.

The concept of contractible space is given by

Definition 0.0.4. *[Mau96, p. 27, 30] Let X, Y be two topological spaces. Then, a continuous map $f : X \to Y$ is called a homotopy equivalence if there exist a continuous map $g : Y \to X$, called homotopy inverse of f, such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X. In this case, X and Y are called homotopy equivalent. A homeomorphism is clearly a homotopy equivalence.*

A topological space X is called contractible if it is homotopic equivalent to a point. The empty set $\emptyset$ is non-contractible by vacuous truth.

A topological space X is contractible if, and only if, the identity $Id_X : X \to X$ is homotopic to a constant map: Let $p \in X$, $f : X \to \{p\}$ and $g : \{p\} \to X$ be any continuous functions, then $f \circ g$ is a constant function and any constant function can be described this way. So f is a homotopy equivalence if, and only if, the constant function $f \circ g$ is homotopic to Id_X.

In this work, we write, for groups, $H < G$ if H is a subgroup of G different of G and, for Lie algebras, $\mathfrak{h} < \mathfrak{g}$ if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ different of $\mathfrak{g}$. Whenever G is a Lie group, G_0 denotes the connected component of the neutral element of G.

Now, to construct $\Delta^T_{G/H}$, let $H < G$ be compact Lie groups and $\mathfrak{g} := Lie(G), \mathfrak{h} := Lie(H)$. An intermediate subalgebra $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$ is called an H-subalgebra if it is $Ad(H)$-invariant. Let $\mathfrak{m}$ be a complement $Ad(H)$-invariant to $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{m}_0 := \{X \in \mathfrak{m} \mid [X, \mathfrak{h}] = 0\}$. In [Bö04, p. 109], it was proved that $\mathfrak{m}_0$ is a compact Lie subalgebra of $\mathfrak{g}$. So, let T be a torus of the compact Lie subgroup of G associated with $\mathfrak{m}_0$. By

[Bö04, p.154], there exists only finitely many H-subalgebras $\mathfrak{k}$ which are minimal among all H-subalgebras that are *non-toral*, i.e., for $\mathfrak{m}_{\mathfrak{k}} := \mathfrak{m} \cap \mathfrak{k}$, we have that $[\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}] \neq \{0\}$, and $\mathfrak{k}$ is T-adapted, i.e., $\mathfrak{k}$ is $Ad(T)$-invariant.

$\Delta^T_{G/H}$ is defined as the abstract simplicial complex with vertices being H-subalgebras which are generated by minimal non-toral T-adapted H-subalgebras and the n-simplices of $\Delta^T_{G/H}$ are given by all chains, i.e., totally ordered sets, $(\mathfrak{k}_0 < \ldots < \mathfrak{k}_n)$ with $\mathfrak{k}_i$ a vertex of $\Delta^T_{G/H}$.

For the case that $\mathfrak{m}_0 = \{0\}$, we have $T = \{e\}$, the condition of T-adapted is always satisfied and we define $\Delta^{min}_{G/H} := \Delta^{\{e\}}_{G/H}$. In addition, if, in this case, there exists only finitely many H-subalgebras, if we consider the simplicial complex $\Delta_{G/H}$ defined the same way as before, but with vertices being all H-subalgebras. It was proved in [Bö04, p. 153], that $\Delta_{G/H}$ and $\Delta^{min}_{G/H}$ are homotopy equivalent. So, $\Delta_{G/H}$ is not contractible if, and only if, $\Delta^{min}_{G/H}$ is not contractible and, in this case, G/H admits a G-invariant Einstein metric.

In chapter 1, we define what are simplicial complexes and introduce the concept of homology, a classical method to show that some simplicial complexes are not contractible.

Let $\mathfrak{g}$ be a simple classical real Lie algebra with rank n. Up to covering G is one of $SU(n+1), SO(2n), SO(2n+1), Sp(n)$. Let $T < G$ be a fixed maximal torus of G, we will prove that the T-subalgebras are finite for G simple classical and if $\Delta_{G/H}$ is contractible or not, where H is connected of maximal rank, i.e., contains a torus of G.

In chapter 4, we prove that $\Delta_{G/H}$ is not-contractible if G and H of maximal rank are given by

$$\begin{aligned}
&G = SU(n), && H \cong S(U(n_1) \times \ldots \times U(n_k)) \\
&G = SO(2n), && H \cong U(n_1) \times \ldots \times U(n_k) \text{ or} \\
& && H \cong SO(2n_1) \times \ldots \times SO(2n_k) \\
&G = SO(2n+1), && H \cong SO(2n_1) \times \ldots \times SO(2n_k) \text{ or} \\
& && H \cong SO(2n_1) \times \ldots \times SO(2n_{k-1}) \times SO(2n_k+1) \\
&G = Sp(n) && H \cong U(n_1) \times \ldots \times U(n_k) \text{ or} \\
& && H \cong Sp(n_1) \times \ldots \times Sp(n_k)
\end{aligned}$$

and $\Delta_{G/H}$ is contractible if H is of "mixed type", example $G = Sp(8)$ and $H = U(5) \times Sp(3)$.

In chapter 5, we describe the simplicial complex $\Delta^T_{G/H}$ of the real flag manifold $G/H = SO(4)/S(O(1) \times O(1) \times O(1) \times O(1))$. It is a space with just two points, hence not-contractible.

The notation of this introduction will be used in the rest of this work.

1 Simplicial Complexes and Homology

We begin with basic notions of simplicial complex and homology based on [Arm13], [Mau96] and [Hat02].

Simplicials complexes are the main tools of this work and homology is a possible way to discover if a simplicial complex is not contractible.

1.1 Simplicial Complexes

Definition 1.1.1. *Points $\{v_0, ..., v_k\} \subset \mathbb{R}^n$ are said affinely independent if they span an affine n-plane, i.e., if*

$$\sum_{i=0}^{k} \lambda_i v_i = 0 \quad \text{and} \quad \sum_{i=0}^{k} \lambda_i = 0 \implies \lambda_i = 0 \quad \forall i \in \{0, ..., k\}$$

Observe that, from definition, any $\{v_{i_0}, \ldots, v_{i_p}\}$ with $\{i_0, \ldots, i_p\} \subset \{0, ..., k\}$ subset of $\subseteq \{v_0, ..., v_k\} \subset \mathbb{R}^n$ affinely independent is also affinely independent if and only if its points are all different, since the sums in the definition applied to $\{v_{i_0}, \ldots, v_{i_p}\}$ can be extended to $\{v_0, ..., v_k\}$ with 0's in the missing constants.

Proposition 1.1.2. *Let $\{v_0, ..., v_k\} \subset \mathbb{R}^n$, then the following statements are equivalent:*

1. *$\{v_0, ..., v_k\}$ is affinely independent;*
2. *$\{v_1 - v_0, ..., v_k - v_0\}$ is linearly independent;*
3. *For every $\{\lambda_0, ..., \lambda_k, \mu_0, ..., \mu_k\} \subset \mathbb{R}$ such that $\sum_{i=0}^{k} \lambda_i = \sum_{i=0}^{k} \mu_i = 1$ and $\sum_{i=0}^{k} \lambda_i v_i = \sum_{i=0}^{k} \mu_i v_i$, we have $\lambda_0 = \mu_0, ..., \lambda_n = \mu_k$.*

Proof. (1) $\implies$ (2): Let $\{\lambda_1, ..., \lambda_k\} \subset \mathbb{R}$ be such that $\sum_{i=1}^{k} \lambda_i(v_i - v_0) = 0$. Define $\lambda_0 := -\sum_{i=1}^{k} \lambda_i$. Then $\sum_{i=0}^{k} \lambda_i v_i = 0$ and

$$\sum_{i=1}^{k} \lambda_i(v_i - v_0) = 0 \implies \sum_{i=1}^{k} \lambda_i v_i = \lambda_0 v_0 \implies \sum_{i=0}^{k} \lambda_i v_i = 0.$$

Since $\{v_0, ..., v_k\}$ is affinely independent, we have $\lambda_0 = ... = \lambda_k = 0$. So $\{v_1 - v_0, ..., v_k - v_0\}$ is linearly independent.

(2) $\implies$ (1): Let $\{\lambda_0, ..., \lambda_k\} \subset \mathbb{R}$ be such that $\sum_{i=0}^{k} \lambda_i v_i = 0$ and $\lambda_0 + \sum_{i=1}^{k} \lambda_i = 0$. So

$$\left(-\sum_{i=1}^{k} \lambda_i\right) v_0 + \sum_{i=1}^{k} \lambda_i v_i = 0 \implies \sum_{i=1}^{k} \lambda_i (v_i - v_0) = 0$$

Since $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent, we have that $\lambda_1 = \ldots = \lambda_k = 0$. And $\lambda_0 = -\sum_{i=1}^{k} \lambda_i = 0$. So $\{v_0, ..., v_k\}$ is affinely independent.

(1) $\implies$ (3): Since $\sum_{i=0}^{k} (\lambda_i - \mu_i) v_i = 0$, $\sum_{i=0}^{k} (\lambda_i - \mu_i) = 0$ and $\{v_0, ..., v_k\}$ is affinely independent, then $\lambda_0 = \mu_0, ..., \lambda_k = \mu_k$.

(3) $\implies$ (1): Suppose that $\{v_0, ..., v_k\}$ is not affinely independent, so we can suppose, without loss of generality, that there exists $\{\lambda_0, ..., \lambda_k\} \subset \mathbb{R}$ such that $\sum_{i=0}^{k} \lambda_i v_i = 0$ and $\lambda_0 = -\sum_{i=1}^{k} \lambda_i \neq 0$. Then

$$v_0 = \sum_{i=1}^{k} \left(-\frac{\lambda_i}{\lambda_0}\right) v_i \quad \textit{and} \quad \sum_{i=1}^{k} \left(-\frac{\lambda_i}{\lambda_0}\right) = 1$$

So, $1v_0 + 0v_1 + ... + 0v_n = 0v_0 + \left(\frac{-\lambda_1}{\lambda_0}\right) v_1 + ... + \left(\frac{-\lambda_k}{\lambda_0}\right) v_k$ and $1 + 0 + ... + 0 = 0 + \frac{-\lambda_1}{\lambda_0} + ... + \frac{-\lambda_k}{\lambda_0} = 1$, which contradicts the hypothesis, since $1 \neq 0$ in $\mathbb{R}$. We conclude that $\{v_0, ..., v_k\}$ is affinely independent. □

The last proposition, in particular, says that the maximum cardinality of an affinely independent set in $\mathbb{R}^n$ is $n + 1$.

Remark 1.1.3. In this work, we use $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ for the definition of the naturals numbers. Given $X \subseteq \mathbb{R}^n$, its convex hull will be denoted by $conv(X)$.

Definition 1.1.4. *If $v_0, ..., v_k$ are affinely independent in $\mathbb{R}^n$, then*

$$conv(v_0, \ldots, v_k) = \left\{ \sum_{i=0}^{k} \lambda_i v_i \;\middle|\; \sum_{i=0}^{k} \lambda_i = 1,\ 0 \leq \lambda_i \leq 1 \right\}$$

is said to be a geometric simplex of dimension k *with vertices* $\{v_0, \ldots, v_k\}$. *If σ is a geometric simplex, its dimension is denoted by $\dim \sigma$. A geometric simplex of dimension k can be denoted as a k-simplex.*

Given $\sigma := conv(v_0, \ldots, v_k)$ a geometric simplex and $w \in \sigma$, the real numbers $\lambda_0 \geq 0, \ldots, \lambda_k \geq 0$ such that $w = \sum_{i=0}^{n} \lambda_i v_i$ are called the barycentric coordinates of w, which are well defined by the last proposition.

For $p \leq k$, a k-face of the simplex $conv(v_0, ..., v_k)$ is a set $conv(v_{i_0}, ..., v_{i_p})$ with $\{i_0, ..., i_p\} \subseteq \{0, ..., k\}$ and the elements of $\{v_{i_0}, \ldots, v_{i_p}\}$ are all distinct, then affinely independent.

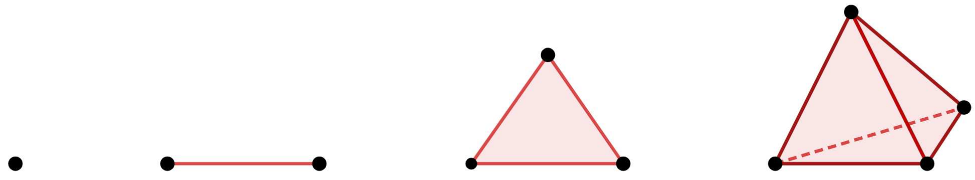

Figure 1 – From left to right: a 0-simplex, a 1-simplex, a 2-simplex and a 3-simplex

Definition 1.1.5. *A* geometric simplicial complex *K of $\mathbb{R}^n$ is a finite set of geometric simplices of $\mathbb{R}^n$ such that:*

(a) If $\sigma \in K$ and if τ is a face of σ, then $\tau \in K$.

(b) If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or a commom face of σ and τ.

The dimension *of K is $dim\ K := max\{dim\ \sigma \mid \sigma \in K\}$ for $K \neq \emptyset$ and $dim\ K = -1$ for $K = \emptyset$. A* subcomplex *L of K is a subset of K which satisfies (a) and (b).*

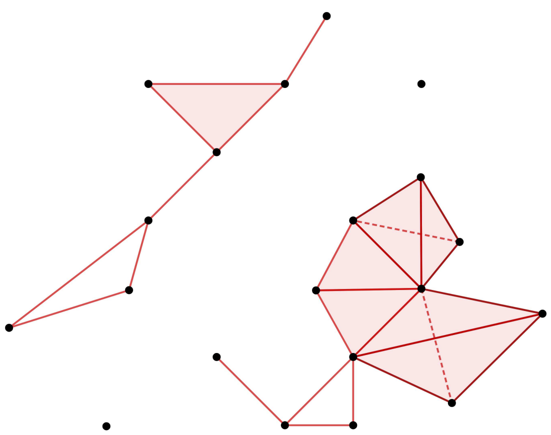

Figure 2 – A geometric simplicial complex

The polyhedron *of K is defined as*

$$||K|| := \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^n$$

given the subspace topology of $\mathbb{R}^n$. If $L \subseteq K$ is a subcomplex, then $||L||$ is called a subpolyhedron *of $||K||$.*

*For $m \in \mathbb{N}$, the m-*skeleton *of K is defined as $K^m := \{\sigma \in K \mid dim\ \sigma \leq m\}$, the subcomplex consisting of of simplices of dimension less or equal m. The 0-skeleton K^0 is also called the* vertex set *of K. If $\{v\} \in K^0$, then v is called a* vertex *of K and we write $v \in K$ instead of $\{v\} \in K$.*

Given a geometric simplicial complex K, we have the following basic results that can be found in [Mau96]:

- If L_1 and L_2 are subcomplexes of K, then $L_1 \cap L_2$ and $L_1 \cup L_2$ also are.
- For any subset $S \subseteq K$, there is a minimal subcomplex $\langle S \rangle$ of K containing S. $\langle S \rangle$ is called the *subcomplex of K generated by S*. For any $\sigma \in K$, the subcomplex $\langle \sigma \rangle = \{\tau \in K \mid \emptyset \subsetneq \tau \subseteq \sigma\}$ is also denoted by σ.
- $||K||$ is compact.
- $A \subseteq ||K||$ is closed in $||K||$ if and only if $A \cap ||\sigma||$ is closed in $||\sigma||$ for all $\sigma \in K$.
- If X is any topological space, then a map $f : ||K|| \to X$ is continuous if and only if $f|_{||\sigma||} : ||\sigma|| \to X$ is continuous for every $\sigma \in K$.

Definition 1.1.6. *Let K_1 be a geometric simplicial complex of $\mathbb{R}^{n_1}$ and K_2 be a geometric simplicial complex of $\mathbb{R}^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$. For any $\sigma = conv(v_0, \ldots, v_{k_1}) \in K_1$ and $\tau = conv(w_0, \ldots, w_{k_2}) \in K_2$, the set*

$$\{(v_0, 0, 0), \ldots, (v_{k_1}, 0, 0), (0, w_0, 1), \ldots, (0, w_{k_2}, 1)\}$$

is an independent subset of $\mathbb{R}^{n_1+n_2+1}$. So, we can define the joins

$$\sigma * \tau := conv((v_0, 0, 0), \ldots, (v_{k_1}, 0, 0), (0, w_0, 1), \ldots, (0, w_{k_2}, 1))$$
$$\sigma * \emptyset := conv((v_0, 0, 0), \ldots, (v_{k_1}, 0, 0))$$
$$\emptyset * \tau := conv((0, w_0, 1), \ldots, (0, w_{k_2}, 1))$$

Then,

$$K_1 * K_2 := \{\sigma * \tau \mid \sigma \in K_1,\ \tau \in K_2\} \cup \{\sigma * \emptyset \mid \sigma \in K_1\} \cup \{\emptyset * \tau \mid \tau \in K_2\}$$

is a geometric simplicial complex of $\mathbb{R}^{m_1+m_2+1}$ with dimension $dim\ K_1 + dim\ K_2 + 1$, called the join *of K_1 and K_2. Its polyhedron is*

$$||K_1 * K_2|| = \{(tp, (1-t)q, t) \mid p \in ||K_1||,\ q \in ||K_2||,\ t \in [0, 1]\}$$

Remark 1.1.7. Let $v \in \mathbb{R}^n$ and K a geometric simplicial complex of $\mathbb{R}^n$, then $||v * K||$ is homeomorphic to $conv(v, ||K||) = \{tv + (1-t)p \mid t \in [0,1]\ ,\ p \in ||K||\}$ in $\mathbb{R}^n$.

Definition 1.1.8. *Let K, L be geometric simplicial complexes and let $f^0 : K^0 \to L^0$ be a map such that whenever $v_0, \ldots, v_k$ are vertices of a simplex of K, then $f^0(v_0) \ldots, f^0(v_k)$ are vertices of a simplex of L. The induced map*

$$f : K \to L; \quad conv(v_0, \ldots, v_k) \mapsto conv(f^0(v_0), \ldots, f^0(v_k))$$

is called a simplicial map. Additionally, if $f^0 : K^0 \to L^0$ is a bijection whose inverse also induces a simplicial map, then $f : K \to L$ is called a simplicial isomorphism and the complexes K and L are called isomorphic, written as $K \cong L$.

A simplicial map f also induces a map of polyhedrons $||f|| : ||K|| \to ||L||$, which is also called a simplicial map [Mun18, p. 12], by:

$$||f|| : \ ||K|| \to ||L||; \quad \sum_{i=0}^{n} \lambda_i v_i \mapsto \sum_{i=0}^{n} \lambda_i f(v_i)$$

$||f||$ is well defined and continuous. Furthermore, $||f||$ is a homeomorphism if and only if f is a simplicial isomorphism [Mun18, p. 12,13].

For this work, we need a generalisation of the concept of geometric simplicial complex that allows every type of object to be vertices of simplices.

Remark 1.1.9. In this work, # denotes the cardinality of a set.

Definition 1.1.10. *An* abstract simplicial complex Δ *is a finite set of non-empty finite sets, which are called* abstract simplices*, such that if $\sigma \in \Delta$ and $\emptyset \neq \tau \subseteq \sigma$, then $\tau \in \Delta$. A subset $\Gamma \subseteq \Delta$ is called a* subcomplex *of Δ if Γ is an abstract simplicial complex.*

If $\sigma \in \Delta$ and $n := \#\sigma - 1$, then σ is called an n-simplex and $\dim\,\sigma := n$ is the dimension *of σ. The elements of σ are called the* vertices *of σ. As in the geometric case, the* dimension *of Δ is $\dim\,\Delta := max\{\dim\,\sigma \mid \sigma \in \Delta\}$ for $\Delta \neq \emptyset$ and $\dim\,\Delta := -1$ for $\Delta = \emptyset$.*

A simplex $\tau \subseteq \sigma$ is called a face *of σ. Moreover, if σ is not a face of any other simplex, then it is called a* maximal simplex *or a* facet*.*

The subcomplex $\Delta^m := \{\sigma \in \Delta \mid \dim\,\sigma \leq m\}$, $m \in \mathbb{N}$, is called the m-skeleton *of Δ and Δ^0 is called the* vertex set *of Δ. If $\{v\} \in \Delta^0$, then v is called a* vertex *of Δ and one writes $v \in \Delta$ instead of $\{v\} \in \Delta$.*

Given an abstract simplicial complex Δ, we have the following basic results that can be found in [Mau96]:

- If Γ_1, Γ_2 are subcomplexes of Δ, then so are $\Gamma_1 \cap \Gamma_2$ and $\Gamma_1 \cup \Gamma_2$.

- As in the geometric case, for any subset $S \subseteq \Delta$, there is a minimal subcomplex $\langle S \rangle$ of Δ containing S. For $\sigma \in \Delta$, the subcomplex $\langle \sigma \rangle = \{\tau \in \Delta \mid \emptyset \subsetneq \tau \subseteq \sigma\}$ is denoted by σ.

Definition 1.1.11. *Let Δ_1, Δ_2 be any two simplicial complexes. Then we can define*

$$\Delta_1 * \Delta_2 := \{\sigma \sqcup \tau \mid \sigma \in \Delta_1 \cup \{\emptyset\},\ \tau \in \Delta_2 \cup \{\emptyset\},\ \sigma \sqcup \tau \neq \emptyset\}$$

is an abstract simplicial complex of dimension dim $\Delta_1 + \dim\ \Delta_2 + 1$, called the join of Δ_1 and Δ_2. In particular, $\Delta * \emptyset = \Delta = \emptyset * \Delta$. The join $\Delta * S^0$ is called the suspension *over Δ, in which $S^0 := \{-1, 1\}$. Moreover, $*$ is commutative and associative, i.e., $\Delta_1 * \Delta_2 = \Delta_2 * \Delta_1$ and $(\Delta_1 * \Delta_2) * \Delta_3 = \Delta_1 * (\Delta_2 * \Delta_3)$.*

Definition 1.1.12. *Let Δ, Γ be abstracts simplicial complexes. Let $f^0 : \Delta^0 \to \Gamma^0$ be a map such that, if $\{v_0, ..., v_k\} \in \Delta$, then $\{f^0(v_0), \ldots, f^0(v_k)\} \in \Gamma$. The induced map*

$$f : \Delta \to \Gamma;\ \{v_0, \ldots, v_k\} \mapsto \{f^0(v_0), \ldots, f^0(v_k)\}$$

is called a simplicial map*. Furthermore, if $f^0 : \Delta^0 \to \Gamma^0$ is a bijection whose inverse induces a simplicial map, them $f : \Delta \to \Gamma$ is called a* simplicial isomorphism *and Δ and Γ are called isomorphic, written as $\Delta \cong \Gamma$.*

The correspondence between geometric and abstract simplicial complexes is given as follows:

Let K be a geometric simplicial complex of $\mathbb{R}^n$ and $\sigma = conv(v_0, ..., v_k) \in K$. Since all subsets $conv(v_{i_0}, ..., v_{i_m})$, $\emptyset \neq \{i_0, \ldots, i_k\} \subseteq \{0, \ldots, n\}$, are elements of K, we can define an abstract simplex

$$\mathcal{K} := \{\{v_0, ..., v_n\} \subseteq \mathbb{R}^n |\ conv(v_0, ..., v_n) \in K\}$$

called the *abstraction* of K.

Now let Δ be an abstract simplicial complex. A geometric simplicial complex $K(\Delta)$ is called a realisation of Δ if its abstraction is isomorphic to Δ. For the existence and uniqueness of a realisation we have the following Lemma:

Lemma 1.1.13 ([Mau96],p. 37-40)**.** *Let Δ be an abstract simplicial complex of dimension m. Then, Δ has a realisation $K(\Delta)$ in $\mathbb{R}^{2m+1}$. Furthermore, if $K_1(\Delta)$ in $\mathbb{R}^{n_1}$ and $K_2(\Delta)$ in $\mathbb{R}^{n_2}$ are two realisations of Δ, then $K_1(\Delta)$ is isomorphic to $K_2(\Delta)$. In particular, their polyhedrons $||K_1(\Delta)||$ and $||K_2(\Delta)||$ are homeomorphic. Moreover, for abstract simplicial complexes Δ_1, Δ_2, $K(\Delta_1) * K(\Delta_2)$ is a realisation of $\Delta_1 * \Delta_2$.*

A realisation $K(\Delta)$ of Δ can be constructed as a geometric simplicial complex with vertices in bijection with the vertices of Δ and vertices of a simplex in $K(\Delta)$ correspond to vertices of a simplex in Δ.

The above lemma guarantees that, given Δ an abstract simplicial complex, we gain a topological space $||\Delta|| := ||K(\Delta)||$ which is unique up to homeomorphism. This way we can assign to Δ topological and homotopical properties and topological and homotopical invariants from $||\Delta||$ without ambiguity.

So, Δ is called *connected or compact*, if and only if, $||\Delta||$ is contractible or connected or compact, respectively. Furthermore, if $v \in \Delta$ is a vertex and $k \in \mathbb{N}$, we define the k-th *homotopy group* of Δ as $\pi_k(\Delta, v) := \pi_k(||\Delta||, ||v||)$.

Definition 1.1.14. *An abstract simplicial complex Δ is said to be contractible if $||\Delta||$ is contractible.*

One of the simple contractible simplicial complexes are the cones.

Definition 1.1.15. *A geometric simplicial complex K is said to be a* cone *if there are a vertex $v \in K^0$ and a subsimplex K' such that $K = v * K'$. We also say that K is a cone over v. An abstract simplicial complex Δ is called a* cone *if there is a vertex $v \in \Delta^0$ such that, given $\sigma \in \Delta$, then $\sigma \cup \{v\} \in \Delta$. As before, Δ is also called a cone over v.*

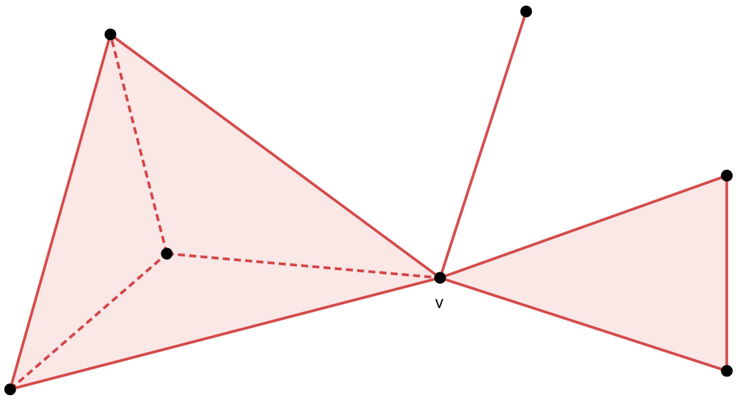

Figure 3 – A cone over v

Remark 1.1.16. An abstract simplicial complex Δ is a cone if and only if a realisation $K(\Delta)$ is a cone by Remark 1.1.7 and the construction of a realisation of an abstract simplicial complex.

Proposition 1.1.17. *Cones are contractible.*

Proof. We just need to prove for a geometric simplicial complex K that is a cone by the remark above. Let v like in the definition 1.1.17, we define the homotopy $H : ||K|| \times [0, 1] \to ||K||$ given by $H(p, t) := tv + (1 - t)p$, which is a homotopy between the identity of $||K||$ and a constant map. $\square$

1.2 Simplicial Homology

Definition 1.2.1. *Let K a non-empty geometric simplicial complex. For $n \in \mathbb{N}$, define $C_n(K) := F_n(K)/R_n(K)$ where F_n is the free abelian group generated by all n-tuples*

$$\{(v_0, \dots, v_n) \mid conv(v_0, \dots, v_n) \text{ is } n\text{-simplex of } K\}$$

and $R_n(K)$ is the free abelian subgroup of $F_n(K)$ generated by

$$\{(v_{\rho(0)}, \dots, v_{\rho(n)}) - sgn(\rho) \cdot (v_0, \dots, v_n) \mid \rho \in Sym(n+1)\}$$

where $Sym(n+1)$ is $(n+1)$-th symmetric group of permutations and sgn is the sign function of $Sym(n+1)$.

For $n \in \mathbb{Z},\ n < 0$, define $C_n(K) := \{0\}$ and let

$$C_*(K) := \bigoplus_{n \in \mathbb{Z}} C_n(K)$$

The coset of $(v_0, \dots, v_n)$ in $C_n(K)$ is denoted by $[v_0, \dots, v_n]$.

The boundary operator $\partial_n : C_n(K) \to C_{n-1}(K)$ is the group homomorphism defined by

$$\partial_n([v_0, \dots, v_n]) := \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$$

if $n > 0$ and the zero map for $n \leq 0$. The boundary operator is a well-defined group homomorphism since $\partial_n([v_{\rho(0)}, \dots, v_{\rho(n)}]) = sgn(\rho) \cdot \partial([v_0, \dots, v_n])$ for every $\rho \in S_n$.

Lemma 1.2.2. *[Hat02, p. 105] For every $n \in \mathbb{Z}$, we have that $\partial_{n-1} \circ \partial_n = 0$.*

Proof. For $n > 2$, we have that

$$\begin{aligned}\partial_{n-1}(\partial_n([v_0, \dots, v_n])) = &\sum_{j<i} (-1)^i (-1)^j [v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n] + \\ &\sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]\end{aligned}$$

for every $[v_0, \dots, v_n] \in C_n(K)$. The latter two summations cancel, since after switching i and j in the second sum, it becomes the negative of the first.

For $n \leq 1$, $\partial_{n-1} = 0$, so we have the wanted result. □

So, define

$Z_n(K) := Ker\ \partial_n$	the group of *n-cycles* of K
$B_n(K) := Im\ \partial_{n+1}$	the group of *n-boundaries* of K

The Lemma above guarantees that $B_n(K) \subseteq Z_n(K)\ \forall n \in \mathbb{Z}$, so we can define

$$H_n(K) := Z_n(K)/B_n(K) \qquad \text{the } \textit{n-th homology group} \text{ of K}$$

We observe that if $dim\ K = n$, then $H_m(K) = 0$ if $m > n$. The *simplicial homology group* of K is

$$H_*(K) := \bigoplus_{n\in\mathbb{Z}} H_n(K) \tag{1.2.1}$$

The zero-th group of homology $H_0(K)$ is given by

Proposition 1.2.3. *[Hat02, p. 109] Let K be a non-empty geometric simplicial complex. Then $H_0(K) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, m times, such that m is the quantity of path connected components of $||K||$. So $||K||$ is path connected if, and only if, $H_0(K) = \mathbb{Z}$.*

Remark 1.2.4. If $||K||$ is not connected, then let $p, q \in K^0$ in two different connected components in $||K||$, then $[p - q]$ is a non-zero homology class in $H_0(K)$.

Example 1.2.5. Let K be a a simplicial complex of dimension 2 of a full triangle, i.e., let $a, b, c \in \mathbb{R}^2$ affinely independent, then $K := \{\{a\},\ \{b\},\ \{c\},\ conv(a,b),\ conv(b,c),\ conv(a,c),\ conv(a,b,c)\}$. Observe that an abstraction of K is given by $\mathcal{K} := \{\{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{b,c\},\ \{a,c\},\ \{a,b,c\}\}$.

We have that $H_n(K) = 0$ if $n > 2$ and $n < 0$.

$C_0(K) = \mathbb{Z}a + \mathbb{Z}b + \mathbb{Z}c$ is the free abelian group generated by a, b, c, the vertices of the triangle, $C_1(K) = \mathbb{Z}[a,b] + \mathbb{Z}[b,c] + \mathbb{Z}[c,a]$ is the free abelian group generated by $[a,b],\ [b,c],\ [a,c]$, the sides of the triangle, and $C_2(K) = \mathbb{Z}[a,b,c]$ the free group generated by $[a,b,c]$, the triangle itself. $C_n(K) = 0$ if $n < 0$ or $n > 2$.

The boundary operators $\partial_2 : C_2(K) \to C_1(K)$, $\partial_1 : C_1(K) \to C_0(K)$ are the only ones that are not the zero operator and they are given by

$$\partial_2[a,b,c] = [b,c] - [a,c] + [a,b] = [a,b] + [b,c] + [c,a]$$
$$\partial_1[a,b] = b - a, \quad \partial_1[b,c] = c - b, \quad \partial_1[c,a] = a - c$$

Observe that $\partial_1(\partial_2[a,b,c]) = 0$.

We have that $B_2(K) = Im\ \partial_3 = 0$ and $Z_2(K) = Ker\ \partial_2 = 0$, so the second homology group of K is given by $H_2(K) = 0$.

Let $x \in C_1(K)$ and $p, q, r \in \mathbb{Z}$ such that $x = p \cdot [a,b] + q \cdot [b,c] + r \cdot [c,a]$, so

$$\partial_1 x = p \cdot b - p \cdot a + q \cdot c - q \cdot b + r \cdot a - r \cdot c = (r - p) \cdot a + (p - q) \cdot b + (q - r) \cdot c$$

So $\partial_1 x = 0 \iff p = q = r \iff x = p([a,b] + [b,c] + [c,a])$. Then $[a,b] + [b,c] + [c,a]$ is the generator of $Z_1(K)$. As $\partial_2[a,b,c] = [a,b] + [b,c] + [c,a]$, we have that $B_1(K) = Im\ \partial_2 = Z_1(K)$. So we conclude that $H_1(K) = 0$.

By the proposition 1.2.3, $H_0(K) \cong \mathbb{Z}$, since $|K|$ is connected, in fact, a full triangle in $\mathbb{R}^2$.

1.3 Singular homology

We defined homology groups for simplicial complexes. Now we will generalise it for topological spaces.

Definition 1.3.1. *For $n \in \mathbb{N}$, let*

$$\Delta^n_{std} := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 0,\ t_i \geq 0\ \forall i \in \{0, \dots, n\}\}$$

be the standard n-simplex *of $\mathbb{R}^{n+1}$, the convex hull of the standard basis of $\mathbb{R}^{n+1}$.*

Let X be a non-empty topological space. For $n \in \mathbb{N}$, a singular n-simplex *σ_n in X is a continuous map*

$$\sigma_n : \Delta^n_{std} \to X$$

and the points in X given by $\{\sigma_n(e_1), \dots, \sigma_n(e_{n+1})\}$ are called the vertices of σ_n.

The free abelian group generated by all singular n-simplices is denoted by $S_n(X)$ and its elements are called singular n-chains *of X. For $n \in \mathbb{Z},\ n < 0$, we define $S_n(X) := 0$ and*

$$S_*(X) := \bigoplus_{n \in \mathbb{Z}} S_n(X)$$

Observe that, for $n \geq 0$, $S_n(X)$ may not be finitely generated as the case for simplicial homology.

For the standard n-simplex, given $i \in \{0, \dots, n\}$, we define the i-th face map *by*

$$\begin{aligned} \delta^n_i :\ \Delta^{n-1}_{std} &\to \Delta^n_{std} \\ (x_0, \dots, x_n) &\mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_n) \end{aligned}$$

This map can be seen as the embedding of Δ^{n-1}_{std} in Δ^n_{std} as the $(n-1)$-subsimplex of Δ^n_{std} opposing the vertex e_i. Observe that if σ_n is a singular n-simplex, then $\sigma_n \circ \delta^n_i$ is a singular $(n-1)$-simplex for every $i \in \{1, \dots, n\}$.

The boundary operator *$\partial_n : S_n(X) \to S_{n-1}(X)$ is the group homomorphism defined by*

$$\partial_n(\sigma_n) := \sum_{i=1}^{n} (-1)^{i-1} \sigma_n \circ \delta^n_i \in S_{n-1}(X) \quad \textit{for all singular } n\textit{-simplices } \sigma_n$$

for $n \geq 0$ and the zero map for $n < 0$.

As in the case in simplicial homology, for $n \in \mathbb{Z}$ we have that $\partial_{n-1} \circ \partial_n = 0$ [Hat02, p. 108]. And, for $n \in \mathbb{Z}$, we define

$$Z_n(X) := Ker\ \partial_n, \quad \text{the group of } \textit{singular n-cycles} \text{ of } X,$$
$$B_n(X) := Im\ \partial_{n+1}, \quad \text{the group of } \textit{singular n-boundaries} \text{ of } X,$$

and

$$H_n(X) := Z_n(X)/B_n(X), \quad \text{the n-th } \textit{singular homology group} \text{ of } X$$

The *singular homology group* of X is

$$H_*(X) := \bigoplus_{n \in \mathbb{Z}} H_n(X)$$

From now on we write ∂ instead of ∂_n for the boundary maps for both simplicial and singular homology. In this notation, $\partial_{n-1} \circ \partial_n = 0$ is written as $\partial^2 = 0$ which is more concisely.

Proposition 1.3.2. *[Mun18, p. 164] Let X be a non-empty topological space. Then, $H_0(X)$ is free abelian. If $\{C_\alpha\}_{\alpha \in \Lambda}$ is the family of path components of X and if $\{p_\alpha\}_{\alpha \in \Lambda} \subseteq X$ is a family of points in X such that $p_\alpha \in C_\alpha$ for all $\alpha \in \Lambda$, then the homology classes of all p_α form a basis for $H_0(X)$. In particular, X is path connected if, and only if, $H_0(X) \cong \mathbb{Z}$.*

The fundamental property of the singular homology group of X is that it is a homotopy invariant, so it becomes a strong tool for any problem that deals with homotopy type of topological spaces. This property is given by:

Theorem 1.3.3. *[Hat02, p. 111] Let X, Y be non-empty topological spaces and let $f : X \to Y$ be a continuous map. the group homomorphism $f_* : S_*(X) \to S_*(Y)$ defined $f_*(\sigma_n) := f \circ \sigma_n$ for all $\sigma_n \in S_n(X)$ satisfies $\partial \circ f_* = f_* \circ \partial$. Hence, it induces a homomorphism*

$$f_* : H_n(X) \to H_n(Y)$$

for every $n \in \mathbb{Z}$. The assignment $f \mapsto f_*$ is functorial, i.e., $(g \circ f)_* = g_* \circ f_*$ and $id_* = id$. Moreover, if $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$. In particular, if f is a homotopy equivalence, then f_* is a group isomorphism.

Example 1.3.4. [Hat02, p. 110](Homology of a point)

Let $X = \{p\}$ be a unitary set with trivial topology, for every $n \in \mathbb{Z}$, there is a unique $\sigma_n \in S_n(X)$ the constant map $\sigma_n(x) = p\ \forall x \in \Delta^n_{std}$. So $\partial(\sigma_n)(x) = \sum_{i=0}^{n}(-1)^i \sigma_n(\delta^n_i(x)) = \sum_{i=0}^{n}(-1)^i\, p$. If n is odd, we have that $\partial(\sigma_n) = 0$. If n is even and $n \neq 0$, we have that $\partial(\sigma_n)$ is the unique element of S_{n-1}, so we have the singular homology chain complex for $n \geq 0$:

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{0} 0 \longrightarrow \cdots$$

with the boundary maps alternately being isomorphisms and zero maps, except at the last $\mathbb{Z}$. So, the singular homology groups are given by $H_n(X) = 0$ for $n \neq 0$ and $H_0(X) = \mathbb{Z}$.

Corollary 1.3.5. *Let X be a non-empty contractible topological space, its singular homology groups are given by:*

$$H_n(X) := \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.3.1}$$

The following result shows the relation between the simplicial homology of a simplicial complex and the singular homology of its polyhedron

Theorem 1.3.6. *[Mau96, p. 117,119] Let K be a non-empty geometric simplicial complex. Then $H_n(K)$ and $H_n(||K||)$ are isomorphic groups for all $n \in \mathbb{Z}$. More precisely, the homomorphism $\alpha : C_*(K) \to S_*(||K||)$ which maps the coset $[v_0, \dots, v_n]$ to the corresponding simplicial map, i.e., $\alpha([v_0, \dots, v_n])$ is the singular n-simplex in $||\Delta||$ given by*

$$\begin{aligned} \alpha([v_0, \dots, v_n]) : \ \Delta^n_{std} &\to ||K|| \\ (\lambda_0, \dots, \lambda_n) &\mapsto \sum_{i=0}^{n} \lambda_i v_i \end{aligned}$$

is a chain homotopy equivalence. Hence, α induces group isomorphisms

$$\alpha_K : H_n(K) \to H_n(||K||) \tag{1.3.2}$$

for each $n \in \mathbb{Z}$.

Corollary 1.3.7. *Let K and L be geometric simplicial simplexes and let $f : ||K|| \to ||L||$ be continuous.*

- *f induces a group homomorphism $f_* : H_n(K) \to H_n(L)$ for each $n \in \mathbb{Z}$;*
- *The correspondence $f \mapsto f_*$ is functorial;*
- *If $f, g : ||K|| \to ||L||$ are homotopic maps, then $f_* = g_*$;*
- *If f is a homotopy equivalence, then f_* is an isomorphism.*

In particular, since a simplicial isomorphism $h : K \to L$ induces a homeomorphism $||h|| : ||K|| \to ||L||$, we have that $||h||_* : H_n(K) \to H_n(L)$ is a isomorphism for every $n \in \mathbb{Z}$. So, with this topology invariance, we can define the homology groups of abstract simplicial complexes:

Definition 1.3.8. *If Δ is an abstract simplicial complex and $n \in \mathbb{Z}$, the n-homology group of Δ is given by*

$$H_n(\Delta) := H_n(K(\Delta)) \tag{1.3.3}$$

and the homology group is given by

$$H_*(\Delta) := H_*(K(\Delta)) \tag{1.3.4}$$

for a realisation $K(\Delta)$ of Δ.

These are well-defined, since the polyhedrons of two realisations of Δ are homeomorphic.

In sight of the last results and definition, we write just simplicial complex for an abstract simplicial complex and the result is valid for a geometric simplicial complex too.

1.3.1 Reduced homology

Definition 1.3.9. *Let X be a non-empty topological space. We define $\tilde{S}_{-1}(X) := \mathbb{Z}$, $\tilde{S}_n(X) := S_n(X)$ for $n \neq -1$ and the homomorphisms of groups $\tilde{\partial}_{-1} := 0$, $\tilde{\partial}_n := \partial_n$ for $n \notin \{0, 1\}$ and $\tilde{\partial}_0 : \tilde{S}_0(X) \to \tilde{S}_{-1}(X) = \mathbb{Z}$, given by $\tilde{\partial}_0(\sum n_i \sigma_i) := \sum n_i$ for all $n_i \in \mathbb{Z}$ and $\sigma_i \in S_0(X)$.*

As before, we have that $\tilde{\partial}_{n-1} \circ \tilde{\partial}_n = 0$. Hence, we define the n-th reduced singular homology group of X as

$$\tilde{H}_n(X) := \frac{Ker\ \tilde{\partial}_n}{Im\ \tilde{\partial}_{n+1}}$$

The reduced singular homology group of X is defined as

$$\tilde{H}_*(X) := \bigoplus_{n \in \mathbb{Z}} \tilde{H}_n(X)$$

From now on, we write $\tilde{\partial}$ instead of $\tilde{\partial}_n$ like before.

Of course, the reduced singular homology group is closely related to the singular homology group.

Lemma 1.3.10. *Given X and Y non-empty topological spaces and $n \in \mathbb{Z}$, we have the following properties*

1. *$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$;*
2. *For $n \neq 0$, $H_n(X) = \tilde{H}_n(X)$;*
3. *X is path connected if, and only if, $\tilde{H}_0(X) = 0$;*

4. *Given a continuous map $f : X \to Y$, we have the induced homomorphism $\tilde{f}_* : \tilde{H}_n(X) \to \tilde{H}_n(Y)$ like 1.3.3 and the association $f \mapsto \tilde{f}_*$ is functorial.*

5. *If the map f above is a homotopy equivalence, then $\tilde{f}_* : \tilde{H}_n(X) \to \tilde{H}_n(Y)$ is an isomorphism of groups for every $n \in \mathbb{Z}$.*

Now, let Δ be a simplicial complex. We define the n-th reduced homology group of Δ as

$$\tilde{H}_n(\Delta) = \tilde{H}_n(||\Delta||)$$

and the reduced homology group as

$$\tilde{H}_*(\Delta) = \tilde{H}_*(||\Delta||)$$

which are well-defined by the Lemma above.

We define $\tilde{H}_n(\emptyset) := \{0\}$ if $n \neq -1$ and $\tilde{H}_{-1}(\emptyset) = \mathbb{Z}$. So, $\tilde{H}_{-1}(\Delta) = 0$ if and only if $\Delta \neq \emptyset$.

Corollary 1.3.11. *Let X be a contractible topological space. Then, $\tilde{H}_*(X) = 0$.*

Proof. This is direct consequence of corollary 1.3.5 and parts 1 and 2 of 1.3.10 and the definition of reduced homology of the empty space. □

1.3.2 Homology with coefficients

This subsection is based on section 50 of [Mun18] and section 4.5 of [Mau96]. In these sections, we can found the notion of tensor products of abelian groups that will be used in the subsection. The notion of rank of a finitely generated abelian group will also be used and can be found in section 4 of [Mun18].

Definition 1.3.12. *Let X be a non-empty topological space. Given G abelian group, so also a $\mathbb{Z}$-module, for any $n \in \mathbb{Z}$, we have that $\partial_n \otimes id_G : S_n(X) \otimes G \to S_{n-1}(X) \otimes G$ and $\tilde{\partial}_n \otimes id_G : \tilde{S}_n(X) \otimes G \to \tilde{S}_{n-1}(X) \otimes G$ satisfy $(\partial_{n-1} \otimes id_g) \circ (\partial_n \otimes id_G) = 0$ and $(\tilde{\partial}_{n-1} \otimes id_G) \circ (\tilde{\partial}_n \otimes id_G) = 0$.*

Then, we define the n-th singular homology group with coefficients in G and reduced n-th singular homology group with coefficients in G as

$$H_n(X, G) := \frac{Ker\ (\partial_n \otimes id_G)}{Im\ (\partial_{n+1} \otimes id_G)}$$

$$\tilde{H}_n(X, G) := \frac{Ker\ (\tilde{\partial}_n \otimes id_G)}{Im\ (\tilde{\partial}_{n+1} \otimes id_G)}$$

and the correspondent homology groups

$$H_*(X,G) := \bigoplus_{n\in\mathbb{Z}} H_n(X,G) \qquad \tilde{H}_*(X,G) := \bigoplus_{n\in\mathbb{Z}} \tilde{H}_n(X,G)$$

Let Δ be a simplicial complex and let G be any abelian group. Given $n \in \mathbb{Z}$, we define $H_n(\Delta, G), \tilde{H}_n(\Delta, G), H_*(\Delta, G), \tilde{H}_(\Delta, G)$ as above by replacing $S_n(X)$ with $C_n(\Delta)$. Given $\sigma \in C_n(\Delta)$ and $g \in G$, we will write $\sigma \otimes g$ as $g \cdot \sigma$.

Observe that $H_n(\Delta, \mathbb{Z}) = H_n(X)$, $\tilde{H}_n(X, \mathbb{Z}) = \tilde{H}_n(X)$ for every $n \in \mathbb{Z}$. Like before, we have the correspondence $H_n(||\Delta||, G) \cong H_n(\Delta, G), \tilde{H}_n(||\Delta||, G) \cong \tilde{H}_n(\Delta, G)$ for every $n \in \mathbb{Z}$.

Lemma 1.3.13. *Given X and Y non-empty topological spaces, G abelian group and $n \in \mathbb{Z}$, we have the following properties*

- $H_0(X, G) \cong \tilde{H}_0(X, G) \oplus G$;
- *For $n \neq 0$, $H_n(X, G) = \tilde{H}_n(X, G)$;*
- *X is path connected if, and only if, $\tilde{H}_0(X, G) = 0$;*
- *Given a continuous map $f : X \to Y$, we have the induced homomorphisms $f_* : H_n(X, G) \to H_n(X, G)$ and $\tilde{f}_* : \tilde{H}_n(X, G) \to \tilde{H}_n(X, G)$ like 1.3.3 and the associations $f \mapsto f_*, f \mapsto \tilde{f}_*$ are functorial;*
- *If the map f above is a homotopy equivalence, then f_* and $\tilde{f}_*$ are isomorphism of groups;*
- *$H_n(X, G) \cong (H_n(X) \otimes G) \oplus Tor(H_{n-1}(X), G)$ and $\tilde{H}_n(X, G) \cong (\tilde{H}_n(X) \otimes G) \oplus Tor(\tilde{H}_{n-1}(X), G)$ where Tor is the Tor-functor defined in [Mun18, p. 317] and this result can be found in [Mun18, p. 332]*

If $G = \mathbb{F}$ is a field, then $H_n(\Delta, \mathbb{F})$ and $\tilde{H}_n(\Delta, \mathbb{F})$ also have the structure of $\mathbb{F}$-vectors spaces such that $f_*, \tilde{f}_*$ are $\mathbb{F}$-linear. From 1.3.13, we have that $H_n(X, \mathbb{Q}) = H_n(X) \otimes \mathbb{Q}$, since $\mathbb{Q}$ is torsion free [Hat02, p. 265]. Since $\mathbb{Z}_p \otimes \mathbb{Q} = \mathbb{Q}$ [Mun18, p. 305] for $p \in \mathbb{N} \backslash \{0, 1\}$, if $H_n(X)$ is finitely generated, we have that the dimension of $H_n(X, \mathbb{Q}) = H_n(X) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space is the rank of $H_n(X)$, also known as the n-th Betti number of X.

The result $H_n(X, \mathbb{Q}) = H_n(X) \otimes \mathbb{Q}$ implies that, if $H_n(X, \mathbb{Q}) \neq 0$, we have that $H_n(X)$ has non-zero rank (possibly infinite), then $H_n(X, \mathbb{F}) \neq 0$ for every field $\mathbb{F}$. This fact and the non-existence of a torsion part in $H_n(X, \mathbb{Q})$ are the reason we use homology over the rationals in chapter 4.

Corollary 1.3.14. *Let X be a contractible topological space and G any abelian group, then $\tilde{H}_*(X,G) = 0$.*

Proof. Follows from Lemma 1.3.10 and 1.3.13. □

The corollary above is extremely important for this work, since it says that if, for a G abelian group, a Δ simplicial complex and $n \in \mathbb{Z}$, $\tilde{H}_n(\Delta, G) \neq 0$, then we have that Δ is not contractible, which are the hypothesis in 2.3.2 and 2.3.1 theorems, the motivations of this work.

1.3.3 The Mayers-Vietoris sequence

In this subsection, we will present the Mayers-Vietoris sequence for reduced simplicial homology. We just have to remember that any notion of homology have a equivalent Mayers-Vietoris sequence, so the next result can be much more general.

Theorem 1.3.15. *[Mau96, p. 128] Let Δ be a simplicial complex and let Δ_1, Δ_2 be subcomplexes such that $\Delta_1 \cup \Delta_2 = \Delta$ and $\Delta_1 \cap \Delta_2 \neq \emptyset$. We have the inclusions*

$$i_1 : \Delta_1 \cap \Delta_2 \hookrightarrow \Delta_1, \quad i_2 : \Delta_1 \cap \Delta_2 \hookrightarrow \Delta_2, \quad i_3 : \Delta_1 \hookrightarrow \Delta, \quad i_4 : \Delta_2 \hookrightarrow \Delta$$

Let R be a commutative unitary ring and $n \in \mathbb{Z}$ Then, there exists an R-linear homomorphism $\partial_* : \tilde{H}_n(\Delta, R) \to \tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, R)$ such that

$$\begin{aligned} \ldots \longrightarrow \tilde{H}_n(\Delta_1 \cap \Delta_2, R) \overset{(\tilde{\mathrm{i}}_{1*}, -\tilde{\mathrm{i}}_{2*})}{\longrightarrow} \tilde{H}_n(\Delta_1, R) \oplus \tilde{H}_n(\Delta_2, R) \overset{\tilde{\mathrm{i}}_{3*}+\tilde{\mathrm{i}}_{4*}}{\longrightarrow} \ldots \\ \ldots \longrightarrow \tilde{H}_n(\Delta, R) \overset{\partial_*}{\longrightarrow} \tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, R) \longrightarrow \ldots \end{aligned}$$

is a long exact sequence. This sequence is called the Mayer-Vietoris sequence for reduced homology of the triple $(\Delta, \Delta_1, \Delta_2)$.

Observe that, if Δ_1 and Δ_2 are contractible, their reduced homology groups are trivial, so the exactness of the Mayer-Vietoris sequence gives that ∂_* is an isomorphism between the homology of Δ and the homology of $\Delta_1 \cap \Delta_2$.

2 The Simplicial Complex of a Homogeneous Space

2.1 Order complexes

The simplicial complexes associated to a homogeneous space are order complexes. In this section, we will introduce the concept of order complexes.

Definition 2.1.1. *[Bjö96, p. 1843],[BW96, p. 1312] Let $P = (P, \leq)$ be a* poset*, i.e., a partially ordered set. A finite totally ordered subset $C := \{p_0 < \ldots < p_k\}$ is called a chain of P. The number k is called the* length *of C and is denoted by $l(C)$.*

A maximal chain is a chain $C = \{p_0 < \ldots < p_k\}$ such that there exists no $p \in P \backslash C$ such that $C \cup \{p\}$ is a chain.

For a given poset P, let $\hat{0}$ and $\hat{1}$ be two distinct elements not contained in P. Then $\hat{P} := P \dot{\cup} \{\hat{0}, \hat{1}\}$ becomes a poset by $\hat{0} < x < \hat{1}$ for all $x \in P$. $\hat{0}$ and $\hat{1}$ are called the bottom element and the top element of $\hat{P}$, respectively. Moreover, $\hat{P}$ is called a *lattice*, if for all $x, y \in P$ there exists a least upper bound (join) in $\hat{P}$, denoted by $x \vee y$, and a greatest lower bound (meet) in $\hat{P}$, denoted by $x \wedge y$. Furthermore, for all $x \in P$ let

$$P_{\leq x} := \{z \in P \mid z \leq x\}$$

and similarly $P_{<x}, P_{\geq x}$ and $P_{>x}$.

Definition 2.1.2 ([Bjö96], p.1844)**.** *Let $(P, \leq)$ be a finite poset. We define the order complex $\Delta(P, \leq) =: \Delta(P)$ of P as the abstract simplicial complex whose k-simplices are the chains of length k of P for $k \geq 0$. A polyhedron of $\Delta(P)$ will be denoted by $\|P\|$ instead of $\|\Delta(P)\|$.*

To understand the topology of order complexes, we first need to understand when a map between posets can induce a map in the associated order complexes.

Let P, Q be finite posets and let $f : P \to Q$ be a monotonic map, i.e., f preserves order or reverses order. Then, for any chain C of $P, f(C)$ is a chain of Q. Thus, it induces a simplicial map from $\Delta(P)$ to $\Delta(Q)$, which is again denoted by f. Two monotonic maps $f, g : P \to Q$ are called homotopic, denoted by $f \sim g$, if the induced maps $\|f\|, \|g\| : \|P\| \to \|Q\|$ are homotopic.

Proposition 2.1.3. *Let $(P, \leq)$ be a finite poset and $p \in P$, then the subset $\mathcal{C} = \{\sigma \in \Delta(P) \mid min\ \sigma \geq p\}$ of $\Delta(P)$ is a cone over p. Hence, it is contractible by 1.1.17.*

Proof. $\mathcal{C}$ is a subcomplex of $\Delta(P)$: if $C_1 \in \mathcal{C}$ and $C_2 \subseteq C_1$, we have that $min\ \sigma \geq p$ for all $\sigma \in C_2$, then $C_2 \in \mathcal{C}$.

If $C = \{p_0 < \ldots < p_k\} \in \mathcal{C}$, then $C \cup \{p\} = \{p \leq p_0 < \ldots < p_k\} \in \mathcal{C}$, since $p \leq p$. □

2.2 The Simplicial Complexes $\Delta_{G/H}$, $\Delta_{G/H}^{min}$ and $\Delta_{G/H}^{T}$

In this section, the simplicial complexes $\Delta_{G/H}$, $\Delta_{G/H}^{min}$, $\Delta_{G/H}^{T}$ associated to homogeneous spaces will be introduced. Let $H < G$ be compact Lie groups such that G/H is connected with finite fundamental group and G acts almost effectively, i.e., $\mathfrak{h}$ does not contain an ideal of $\mathfrak{g}$. Moreover, let Q be a fixed $Ad(G)$-invariant inner product on $\mathfrak{g}$ which exists by [BD13, p. 68]. The orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ will be denoted by $\mathfrak{m}$, which is $Ad(H)$-invariant so it can be identified with $T_o(G/H)$ where $o := eH$: If $X \in \mathfrak{m}$, $h \in H$ and $Y \in \mathfrak{h}$, then

$$Q(Ad(h)X, Y) = Q(X, Ad(h^{-1})Y) = 0$$

since $Ad(h^{-1})Y \in \mathfrak{h} = \mathfrak{m}^{\perp_Q}$ and Q is $Ad(G)$-invariant. So, $Ad(h)X \in \mathfrak{m}$.

Furthermore, let

$$\mathfrak{m}_0 := \{X \in \mathfrak{m} \mid [X, \mathfrak{h}] = 0\}$$

The Lie algebra of $N_{G_0}(H_0)$ is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ that can be described by:

Lemma 2.2.1. *[BK23, p. 102] Let G/H be a compact homogeneous space. Then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h} \oplus \mathfrak{m}_0$ and this decomposition is Q-orthogonal. Moreover, $\mathfrak{m}_0$ is a compact subalgebra of $\mathfrak{g}$.*

Definition 2.2.2. *A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called an H-subalgebra, if the following hold:*

1. $\mathfrak{h} < \mathfrak{k} < g$,
2. *$\mathfrak{k}$ is $Ad(H)$-invariant.*

Furthermore, let $\mathfrak{m}_{\mathfrak{k}} := \mathfrak{m} \cap \mathfrak{k}$, then $\mathfrak{k}$ is called toral *if $[\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}] = 0$, otherwise $\mathfrak{k}$ is called non-toral.*

Observe that $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_{\mathfrak{k}}$ and it is a Q-orthogonal decomposition, since $\mathfrak{m}_{\mathfrak{k}}$ is the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$, and $\mathfrak{m}_{\mathfrak{k}}$ is $Ad(H)$-invariant.

Every intermediary Lie subalgebra $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$ is $ad(\mathfrak{h})$-invariant, so, for H connected, every intermediary Lie subalgebra is $Ad(H)$-invariant, thus an H-subalgebra. Hence, in the case of H connected, H-subalgebras are in one-to-one correspondence with connected Lie subgroups K of G with $H < K < G$ and we also use the term intermediary subalgebra for an H-subalgebra.

Remark 2.2.3. In this work, if V is a vector space over a field $\mathbb{F}$ and $A_1, \ldots, A_k \subseteq V$, then $span_{\mathbb{F}}\{A_1, \ldots, A_k\}$ means the subspace of V generated by $A_1 \cup \ldots \cup A_k$. If $\mathfrak{g}$ is a Lie algebra over a field $\mathbb{F}$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_k \subseteq \mathfrak{g}$, then $\langle \mathfrak{a}_1, \ldots, \mathfrak{a}_k \rangle$ means the Lie subalgebra generated by $\mathfrak{a}_1 \cup \ldots \cup \mathfrak{a}_k$. Also, if $\mathfrak{h} < \mathfrak{g}$, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) := \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$ is the normalizer of $\mathfrak{h}$.

2.2.1 $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$

Let $H < G$ be compact Lie groups as above such that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. In particular, $\mathfrak{m}_0 = \{0\}$ and every H-subalgebra is non-toral: if $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_{\mathfrak{k}}$ is a H-subalgebra, then $[\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}] = 0 \implies Q([\mathfrak{h}, \mathfrak{m}_{\mathfrak{k}}], \mathfrak{m}_{\mathfrak{k}}) = Q(\mathfrak{h}, [\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}]) = \{0\} \implies [\mathfrak{h}, \mathfrak{m}_{\mathfrak{k}}] \subseteq \mathfrak{h} \cap \mathfrak{m}_{\mathfrak{k}} = \{0\} \implies \mathfrak{m}_{\mathfrak{k}} \subseteq \mathfrak{m}_0 = \{0\}$ which contradicts $\mathfrak{m}_{\mathfrak{k}} \neq 0$.

Let $P_{G/H}$ be the set of all H-subalgebras of $\mathfrak{g}$, which might be infinite. However, by [Bö04, p. 144], there exists at most finitely many minimal, by inclusion, H-subalgebras. Hence, let $\{\mathfrak{k}_1, \ldots, \mathfrak{k}_n\}$ be the set of all minimal H-subalgebras and let $P_{G/H}^{min}$ be the set of all H-subalgebras generated by minimal ones, i.e.

$$P_{G/H}^{min} = \{\langle \mathfrak{k}_{i_1}, \ldots, \mathfrak{k}_{i_l} \rangle \mid 1 \leq l \leq n,\ 1 \leq i_1 < \ldots < i_l \leq n\}$$

$P_{G/H}$ and $P_{G/H}^{min}$ are partially ordered by inclusion $\subseteq$.

Definition 2.2.4. *With the notation as above, the order complex of $P_{G/H}^{min}$ is called the* simplicial complex *of G/H and is denoted by $\Delta_{G/H}^{min}$. If $P_{G/H}$ is finite, the order complex of $P_{G/H}$ is called the* extended simplicial complex *G/H and is denoted by $\Delta_{G/H}$.*

The following properties of $\Delta_{G/H}^{min}$ and $\Delta_{G/H}$ are very important for this work.

1. If H is connected, then $\Delta_{G/H}^{min}$ and $\Delta_{G/H}$ only depend on the Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ instead of the Lie groups H and G. In particular, if $\pi : \tilde{G} \to G$ is a covering map and $\tilde{H} := \pi^{-1}(H)_0$, it follows that $\Delta_{G/H}^{min} = \Delta_{\tilde{G}/\tilde{H}}^{min}$ ($\Delta_{G/H} = \Delta_{\tilde{G}/\tilde{H}}$).

2. If $rank\ H = rank\ G$, it folows that $\mathfrak{m}_0 = \{0\}$, i.e., $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$: Suppose that exists $X \in \mathfrak{m}_0 \backslash \{0\}$ and let $\mathfrak{t}$ be a maximal abelian Lie subalgebra (a Cartan subalgebra) of $\mathfrak{h}$. Then $\mathfrak{t} \oplus \langle X \rangle$ is also an abelian Lie subalgebra of $\mathfrak{g}$, since $[\mathfrak{t}, X] = 0$. It follows that $rank\ G > rank\ H$, which contradicts the hypothesis.

3. If $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{h} < \mathfrak{k}$, then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{k}$: Suppose that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{k}) \neq \mathfrak{k}$. By 2.2.1, there exists $X \notin \mathfrak{k}$ such that $[X, \mathfrak{k}] = 0$. In particular, $[X, \mathfrak{h}] = 0$ which implies $X \in \mathfrak{n}(\mathfrak{h}) = \mathfrak{h} < \mathfrak{k}$, a contradiction with the hypothesis.

If, in addition, H is connected and $P_{G/H}$ is finite, then $P_{G/K}$ is finite and $\Delta_{G/K}$ is well-defined for every compact Lie subgroup $H < K < G$. Moreover, every K-subalgebra is also an H-subalgebra. Hence, $P_{G/K}$ can be identified as a subposet of $P_{G/H}$ and $\Delta_{G/K}$ can be identified as a subcomplex of $\Delta_{G/H}$

2.2.2 $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \neq \mathfrak{h}$

Let $H < G$ be compact Lie groups as above such that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \neq \mathfrak{h}$,i.e., $\mathfrak{m}_0 \neq 0$. By 2.2.1, $\mathfrak{m}_0 \cong \mathfrak{n}(\mathfrak{h})/\mathfrak{h}$ is a compact Lie subalgebra of $\mathfrak{g}$. So, fix a maximal torus T of the compact connected Lie subgroup of G with Lie algebra $\mathfrak{m}_0$ and let $\mathfrak{t} := Lie(T)$.

Definition 2.2.5. *An H-subalgebra $\mathfrak{k}$ is called T-*adapted *if it is invariant under the adjoint action of T, i.e., $Ad(T)\mathfrak{k} \subseteq \mathfrak{k}$. A non-toral T-adapted H-subalgebra is called T-*minimal non-toral *if it is minimal by inclusion in the set of all non-toral T-adapted H-subalgebras.*

Remark 2.2.6. Since T is connected, if $\mathfrak{t} := Lie(T)$, the above condition of $Ad(T)$-invariance is equivalent to $ad(\mathfrak{t})$-invariance.

By [Bö04, p. 154] there exists at most finitely many T-minimal non-toral H-subalgebras. As above, let $P^T_{G/H}$ be the set of all non-toral T-adapted H-subalgebras which are generated by minimal ones. Again, $P^T_{G/H}$ is a finite, partially ordered set by inclusion $\subseteq$.

Definition 2.2.7. *The order complex of the chains of $P^T_{G/H}$ is called the* simplicial complex *of G/H and is denoted by $\Delta^T_{G/H}$.*

The following properties of $\Delta^T_{G/H}$ are of interest, see [Bö04, p. 154].

1. $\Delta^T_{G/H}$ is a generalisation of $\Delta^{min}_{G/H}$, since both definitions coincide for $\mathfrak{m}_0 = \{0\}$ defining $T := \{e\}$.

2. If H is connected, then $\Delta^T_{G/H}$ only depends on $\mathfrak{g}$ and $\mathfrak{h}$ and not on the choice of T up to isomorphism. In particular, again, $\Delta^T_{G/H} = \Delta^T_{\tilde{G}/\tilde{H}}$ for a covering map $\pi : \tilde{G} \to G$ and $\tilde{H} := \pi^{-1}(H)_0$.

3. If both G and H are connected and $\#\pi_1(G/H, eH) < \infty$, then there is a bijection between the minimal non-toral TH-subalgebras and the T-minimal non-toral H-subalgebras. Furthermore, $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{t} \oplus \mathfrak{h}) = \mathfrak{t} \oplus \mathfrak{h}$. By [Bö04, p. 154], it follows $\Delta^T_{G/H} \cong \Delta^{min}_{G/TH}$.

We can always may assume the following conditions

1. G/H is connected (but G and H might be disconnected).

2. The group action of G on G/H is almost effective, i.e., any normal subgroups of G which is contained in H is discrete or $\mathfrak{h}$ does not contain ideals of $\mathfrak{g}$,

3. $\pi_1(G/H, eH)$ is finite.

For the first condition, if $\hat{G}$ is the union of all connected components of G which intersects H, then $\hat{G}$ is a compact subgroup of G such that $\hat{G}/H$ is the connected component of G/H which contains o. Moreover, $\Delta^T_{\hat{G}/H} = \Delta^T_{G/H}$, since the complex depends only on $\mathfrak{h}, \mathfrak{g}, H$ and T.

For the second condition, if $N \trianglelefteq G, N \leq H$, one may consider the homogeneous space $M := (G/N)/(H/N) \cong G/H$. Then $\Delta^T_{G/H} \cong \Delta^T_M$, since $\mathfrak{k}$ is $\mathrm{Ad}(H)$-invariant if and only if $\mathfrak{k}/\mathfrak{n}$ is $\mathrm{Ad}(H/N)$-invariant.

For the third condition, $\#\pi(G/H, eH) = \infty$ implies that $\Delta^T_{G/H}$ is contractible or $\Delta^T_{G/H} = \emptyset$ and G/H is a torus, see [Bö04, p. 155].

When dealing with product spaces $G_1 \times G_2/H_1 \times H_2$, $\Delta^{T_1 \times T_2}_{G_1 \times G_2/H_1 \times H_2}$ is obtained from $\Delta^{T_1}_{G_1/H_1}$ and $\Delta^{T_2}_{G_2/H_2}$ in the following manner:

Lemma 2.2.8. *For $i \in \{1, 2\}$ let $H_i < G_i$ be compact Lie groups as above and let T_i be a maximal torus as above. Then:*

$$\Delta^{T_1 \times T_2}_{G_1 \times G_2/H_1 \times H_2} \simeq \Delta^{T_1}_{G_1/H_1} * \Delta^{T_2}_{G_2/H_2} * S^0$$

where $*$ is the join defined in 1.1.11.

where $\simeq$ means homotopy equivalence. A proof is given by [Bö04, p. 95].

Since $\underbrace{S^0 * \ldots * S^0}_{n \text{ times}} = S^{n-1}$, it follows

Lemma 2.2.9. *For $i \in \{0, \ldots, n\}$, let $H_i < G_i$ be compact Lie groups as above and T_i be a maximal torus as above. Then:*

$$\Delta^{\prod_{i=1}^n T_i}_{\prod_{i=1}^n G_i / \prod_{i=1}^n H_i} \simeq \Delta^{T_1}_{G_1/H_1} * \ldots * \Delta^{T_n}_{G_n/H_n} * S^{n-2}$$

by induction for compact Lie groups $H_i < G_i$ and maximal tori T_i as above.

Lemma 2.2.10. *Let G/H be a compact homogeneous space, P a generic notation for $P_{G/H}, P_{G/H}$ or $P^T_{G/H}$. Then, $\hat{P} := P \dot{\cup} \{\mathfrak{g}, \mathfrak{h}\}$ is a lattice.*

Proof. $\mathfrak{h}$ is the bottom element of $\hat{P}$ and $\mathfrak{g}$ is the top element pf $\hat{P}$. For $\mathfrak{k}_1, \mathfrak{k}_2 \in P$, there exists a least upper bound $\langle \mathfrak{k}_1, \mathfrak{k}_2 \rangle$ in $\hat{P}$ and a greatest lower bound $\mathfrak{k}_1 \cap \mathfrak{k}_2$ in $\hat{P}$. □

2.3 The simplicial complex theorems for invariant Einstein metrics

In [Bö04], studying the problem of existence or non-existence of Einstein invariant metrics on compact homogeneous spaces, it was proved the following theorem, that is the central result of this work:

Theorem 2.3.1. *[Bö04, p. 156] Let G/H be a compact homogeneous space. If a simplicial complex $\Delta^T_{G/H}$ is not contractible, then G/H admits a G-invariant Eintein metric.*

And

Theorem 2.3.2. *[Bö04, p. 87] Let G/H be a compact homogeneous space, such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$. If the simplicial complex $\Delta^{min}_{G/H}$ is not contractible, then G/H admits a G-invariant Einstein metric. And, if $\Delta_{G/H}$ is well-defined, then $\Delta_{G/H}$ being not contractible also implies that G/H admits a G-invariant Einstein metric.*

The second part of the corollary above comes from the fact that, if $\Delta_{G/H}$ is well-defined, then $\Delta^{min}_{G/H}$ and $\Delta_{G/H}$ are homotopic equivalent, which will be proved in the section [2.4.6].

We will give a brief outline of the mains steps for the proof of both theorems above in this section based on [Bö04] and [BK23].

Let $\mathcal{M}^G_1$ be the space of G-invariant, unit volume metrics on G/H. Fix Q a $Ad(G)$-invariant inner product in $\mathfrak{g}$ with volume 1 after rescaling. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the $Ad(H)$-invariant decomposition of $\mathfrak{g}$ with $Q(\mathfrak{h}, \mathfrak{m}) = 0$. The set of G-invariant metric on G/H, denoted by $\mathcal{M}^G$, can be identified with the set of $Ad(H)$-invariant inner products on $\mathfrak{m}$. Furthermore, for every $g \in \mathcal{M}^G$, there exists α_g an $Ad(H)$-equivariant, Q-self-adjoint and positive definite endomorphism of $\mathfrak{m}$. Then, we identify g with α_g and $\mathcal{M}_G$ will be thought as the space of $Ad(H)$-equivariant, Q-self-adjoint and positive definite endomorphism of $\mathfrak{m}$ and it is a finite dimensional manifold.

In $\mathcal{M}^G$, we can define L^2-metric, denoted by $\langle \cdot, \cdot \rangle$, given by

$$\langle \varphi, \psi \rangle_{\alpha_g} = tr(\alpha_g^{-1} \varphi \alpha_g^{-1} \psi)$$

for $\alpha_g \in \mathcal{M}^G$ and $\varphi, \psi \in T_{\alpha_g}\mathcal{M}^G$ which is the space of $Ad(H)$-equivariant, Q-self-adjoint endomorphism of $\mathfrak{m}$.

Lemma 2.3.3. *Let G/H be a compact homogeneous space. Then, $(\mathcal{M}^G, L^2)$ is a non-compact symmetric space.*

Corollary 2.3.4. *Let G/H be a compact homogeneous space with dim $\mathcal{M}^G \geq 2$. Then, $\mathcal{M}^G_1$ is the subspace of $Ad(H)$-equivariant, Q-self-adjoint endomorphism of $\mathfrak{m}$ with determinant 1 and $(\mathcal{M}^G, L^2)$ is a non-compact symmetric space.*

Corollary 2.3.5. *Let G/H be a compact homogeneous space with dim $\mathcal{M}^G \geq 2$. Then, for any $v \in S := \{v \in T_Q\mathcal{M}_1^G \mid ||v|| = 1\}$, the curve*

$$\gamma_v(t) = exp(t \cdot v), \ t \in \mathbb{R} \tag{2.3.1}$$

is a unit speed geodesic in $(\mathcal{M}_1^G, L^2)$, where denotes exp the exponential of linear operators.

Proof. See [Hel78, p. 226]. □

Lemma 2.3.6. *For any $v \in T_Q\mathcal{M}^G$, there exists a Q-orthogonal decomposition $\mathfrak{m} = \oplus_{i=1}^l \mathfrak{m}_i$ of into irreducible, $Ad(H)$-invariant summands $\mathfrak{m}_i$ and $v_i \in \mathbb{R}$ for $i = 1, \cdots, l$, such that*

$$v = v_1 \cdot Id_{\mathfrak{m}_1} + \cdots + v_l \cdot Id_{\mathfrak{m}_l} \tag{2.3.2}$$

Any such decomposition will be called a good decomposition with respect to v.

Proof. The eigenspace of v are $Ad(H)$-invariant and pairwise Q-orthogonal. Decomposing an eigenspace further into Q-orthogonal $Ad(H)$-irreducible summands shows the claim. □

By last Lemma, for each $\alpha_g \in \mathcal{M}_1^G$, there exists $v \in S$ unity sphere in $T_Q\mathcal{M}_1^G$ and $t_0 \geq 0$ such that $\alpha_g = \gamma_v(t_0)$. By above lemma,

$$\gamma_v(t) = e^{tv_1} \cdot Id_{\mathfrak{m}_1} + \ldots + e^{tv_l} \cdot Id_{\mathfrak{m}_l} \tag{2.3.3}$$

for the good decomposition $\mathfrak{m} = \oplus_{i=1}^l \mathfrak{m}_i$.

Let $v \in T_Q\mathcal{M}^G$ and the good decomposition for v from above. We denote by

$$\hat{v}_1 < \ldots < \hat{v}_{l_v} \tag{2.3.4}$$

the distinct eigenvalues of v ordered by size, $1 \leq l_v \leq l$. For each eigenvalue, $1 \leq m \leq l_v$, we define the index set (which depends on our choice of good decomposition)

$$I_m^v := \{i \in \{1, \ldots, l\} \mid v_i = \hat{v}_m\} \tag{2.3.5}$$

Let

$$\lambda(v) = \hat{v}_1 \quad \text{and} \quad \Lambda(v) = \hat{v}_{l_v} \tag{2.3.6}$$

denote the smallest and the largest eigenvalues of v, respectively.

So we have

Lemma 2.3.7. *Let G/H be a compact homogeneous space. Any $v \in S$ must have at least 2 distinct eingenvalues and there exists a constant $c_{G/H} < 0$ such that the following holds:*

$$\lambda(v) \leq c_{G/H} \quad \textit{and} \quad -c_{G/H} \leq \Lambda(v) \tag{2.3.7}$$

Let $\mathfrak{m} = \oplus_{i=1}^{l}\mathfrak{m}_i$ be a decomposition of $\mathfrak{m}$ into Q-orthogornal, $Ad(H)$-irreducible summands, that will be called a decomposition of $\mathfrak{m}$, for any non-empty subset I of $\{1, \ldots, l\}$, we define

$$\mathfrak{m}_I := \oplus_{i\in I}\mathfrak{m}_i \quad \text{and} \quad d_I := dim\ \mathfrak{m}_I \tag{2.3.8}$$

For $v \in T_Q\mathcal{M}_1^G$ and a good decompostion for v, the spaces $\mathfrak{m}_{I_m^v}$ are the eigenspaces of v, thus we have

$$\gamma_v(t) = e^{t\hat{v_1}} \cdot Id_{\mathfrak{m}_{I_1^v}} + \ldots_e^{t\hat{v_{l_v}}} \cdot Id_{m_{I_{l_v}^v}} \tag{2.3.9}$$

Let $\mathfrak{m} = \oplus_{i=1}^{l}\mathfrak{m}_i$ be a fixed decomposition of $\mathfrak{m}$, let $\{e_1, \ldots, e_n\}$ be a Q-orthonormal basis of $\mathfrak{m}$ adapted to the decomposition and I, J, K be non-empty subsets of $\{1, \ldots, l\}$. We define, following [WZ86],

$$[IJK] := \sum_{\alpha,\beta,\gamma} Q([e_\alpha, e_\beta], e_\gamma)^2$$

where we sum over all indices $\alpha, \beta, \gamma \in \{1, \ldots, n\}$ with $e_\alpha \in \mathfrak{m}_I, e_\beta \in \mathfrak{m}_J$ and $e_\gamma \in \mathfrak{m}_K$.

Since the adjoint maps $ad(X)$ are Q-skew-adjoint $\forall\ X \in \mathfrak{g}$, it follows that $[IJK]$ is symmetric in all three entries and is independent of the Q-orthonormal bases chosen for $\mathfrak{m}_I, \mathfrak{m}_J$ and $\mathfrak{m}_K$. In case, $I = \{i\}, J = \{j\}$ and $K = \{k\}$, we write $[ijk]$ instead of $[IJK]$.

We have that $[ijk] \geq 0$, with $[ijk] = 0$ if, and only if, $Q([\mathfrak{m}_i, \mathfrak{m}_j], \mathfrak{m}_k) = 0$.

Definition 2.3.8. *[Hel78, p. 131] The $Ad(G)$-invariant symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by $B(X,Y) = tr(ad(X) \circ ad(Y))$ for all $X, Y \in \mathfrak{g}$ is called the* Cartan-Killing form *of $\mathfrak{g}$. Observe that this definition generalises to Lie algebras over any field.*

Since both Q and the $-B$ are $Ad(G)$-invariant non-negative forms on $\mathfrak{g}$, there exists $b_i \geq 0$ for $1 \leq i \leq l$ such that

$$B|_{\mathfrak{m}_i} = -b_i \cdot Q|_{\mathfrak{m}_i} \tag{2.3.10}$$

Lemma 2.3.9. *Let $v \in T_Q\mathcal{M}^G$ and let $\mathfrak{m} = \oplus_{i=1}^{l}\mathfrak{m}_i$ be a good decomposition with respect to v. Then, the scalar curvature of $\gamma_v(t)$ is given by*

$$sc(\gamma_v(t)) = \frac{1}{2}\sum_{i=1}^{l} d_i b_i \cdot e^{t(-v_i)} - \frac{1}{4}\sum_{i,j,k=1}^{l} [ijk] \cdot e^{t(v_i - v_j - v_k)} \tag{2.3.11}$$

$$= \frac{1}{2}\sum_{i=1}^{l_v}\left(\sum_{j\in I_i^v} d_j b_j\right) \cdot e^{t(-\hat{v}_i)} - \frac{1}{4}\sum_{i,j,k=1}^{l_v} [I_i^v I_j^v I_k^v] \cdot e^{t(v_i - v_j - v_k)} \tag{2.3.12}$$

Definition 2.3.10. *For each H-subalgebra $\mathfrak{k}$, the* canonical direction $v^{\mathfrak{k}} \in S$ *associated to $\mathfrak{k}$ is defined by*

$$v^{\mathfrak{k}} := v_1^{\mathfrak{k}} \cdot Id_{\mathfrak{m}_{\mathfrak{k}}} + v_2^{\mathfrak{k}} \cdot Id_{\mathfrak{m}_{\mathfrak{k}}^{\perp}} \tag{2.3.13}$$

where $\perp$ means orthogonal complement with respect to Q, $(dim\ \mathfrak{m}_{\mathfrak{k}}) \cdot v_1^{\mathfrak{k}} + (dim\ \mathfrak{m}_{\mathfrak{k}}^{\perp}) \cdot v_2^{\mathfrak{k}} = 0$, $||v^{\mathfrak{k}}|| = 1$ and $v_1^{\mathfrak{k}} < v_2^{\mathfrak{k}}$.

With Lemma 2.3.9 we can have information on the asymptotic behaviour of the scalar curvature functional:

Lemma 2.3.11. *For any H-subalgebra $\mathfrak{k}$,*

$$\lim_{t\to+\infty} sc(\gamma_{v^{\mathfrak{k}}}(t)) = \begin{cases} +\infty \\ 0 \end{cases} \iff \begin{cases} \mathfrak{k} \text{ non-toral} \\ \mathfrak{k} \text{ toral} \end{cases} \tag{2.3.14}$$

If in addition G/H is not a torus, then $sc(\gamma_{v^{\mathfrak{k}}}(t)) > 0$ for all $t \geq 0$.

Now, the results about the boundness of the curvature scalar curvature.

Lemma 2.3.12. *If the scalar curvature functional is bounded from below along a geodesic γ_v for $v \in S$, that is, $sc(\gamma_v(t)) \geq C$ for all $t \geq 0$, then $\mathfrak{h} \oplus \mathfrak{m}_{I_1^v}$ is an H-subalgebra.*

Theorem 2.3.13. *Let G/H be a compact homogeneous space. Then, the scalar curvature functional $sc : \mathcal{M}_1^G \to \mathbb{R}$ is bounded from above if, and only if, there exist no non-toral H-subalgebras.*

Using theorems like the above one and variational methods, if $\mathfrak{m}_0 = \{0\}$, the non-contrability of $\Delta_{G/H}^{min}$ implies the existence of a Palais-Smale sequence of G-invariant metrics of volume one with scalar curvature bounded from below by a positive constant, see [Bö04, p. 156]. A sequence (g_i) in $\mathcal{M}_1^G$ is called a Palais-Smale sequence if $sc(g_i)$ is bounded and $||(grad\ sc)_{g_i}||$ converges to 0 in the norm induced by the L^2-metric.

Using this Palais-Smale sequence and the fact that a metric in $\mathcal{M}_1^G$ are Einstein if, and only if, it is a critical point of $sc : \mathcal{M}_1^G \to \mathbb{R}$ [0.0.2], we obtain theorem 2.3.2.

In the case $\mathfrak{m}_0 \neq \{0\}$, like before, T is the maximal torus of the compact Lie subgroup associated with $\mathfrak{m}_0$ and we define $(\mathcal{M}_1^G)^T := \mathcal{M}_1^G \cap \{g \in \mathcal{M}_1^{G_0} \mid g = (R_t)_* g$ for all $t \in T\}$ where R_g is the right translation by $g \in G$ and G_0 is the connected component of G.

Lemma 2.3.14. *Let G/H be a compact homogeneous space. Then, $(\mathcal{M}_1^G)^T$ is a totally geodesic subspace of $(\mathcal{M}_1^G, L^2)$ invariant under the Ricci flow.*

As a consequences we can also apply the variational methods described above to $sc : (\mathcal{M}_1^G)^T \to \mathbb{R}$ and obtain Theorem 2.3.1.

2.4 Algebraic topology of order complexes

In order to apply the Theorem 2.3.2 and 2.3.1, several tools will be introduced to determine whether the complex $\Delta_{G/H}$ is contractible or not. Of course, these tools will be used in examples of chapter 4.

By [Mau96, p. 274], the polyhedron of a simplicial complex is a CW-complex(for a definition see [Hat02, p. 6]). Thus, we can apply the theorem of Whitehead for homotopy groups to simplicial complexes.

We use that a CW-complex is connected if, and only if, is path connected. If X is path connected and $x_0 \in X$, then the k-homotopy group $\pi_k(X, x_0)$ does not depend on the basepoint x_0 up to isomorphism, so we write just $\pi_k(X)$.

Theorem 2.4.1. *(Theorem of Whitehead)[Hat02, p. 346] Let X, Y be non-empty connected CW-complexes and let $f : X \to Y$ be a weak homotopy equivalence, i.e. f is continuous and $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n \in \mathbb{N}$. Then f is a homotopy equivalence. If, in addition, X is a subcomplex of Y and $f : X \hookrightarrow Y$ is the inclusion map, then X is a strong deformation retract of Y.*

Corollary 2.4.2. *Let X be a CW-complex. If X is simply connected and $\tilde{H}_*(X) = 0$, then X is contractible.*

Proof. X is a non-empty connected CW-complex by assumption. By Theorem 2.4.1, it is enough to prove that for any vertex x_0 of X the inclusion $\iota : \{x_0\} \to X$ is a weak homotopy equivalence, which is equivalent to prove that $\pi_n(X) = 0$ for all $n \geq 1$.

We prove this by induction. For $n = 1$, we have by hypothesis that $\pi_1(X) = 0$. Now, suppose $n \geq 2$ and $\pi_k(X) = 0$ for all $1 \leq k \leq n - 1$, which are the hypothesis to the theorem of Hurewicz, see [Hat02, p. 369], that implies that $\pi_n(X) \cong \tilde{H}_n(X) = 0$. □

Now, using the corollary above and the Mayer-Vietoris, we prove that the union of contractible complexes is contractible, if the intersection of the complexes is contractible.

Lemma 2.4.3. *[Rau16, p. 16] Let Δ_1, Δ_2 be contractible subcomplexes of a simplicial complex Δ such that $\Delta_1 \cap \Delta_2$ is contractible. Then $\Delta_1 \cup \Delta_2$ is also contractible.*

Proof. $\Delta_1 \cap \Delta_2$ is non-empty by assumption. So, let $v \in \Delta_1 \cap \Delta_2$ be any vertex. Since Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$ are path-connected and since $\pi_1(\Delta_1, v) = \pi_1(\Delta_2, v) = 0$, it follows $\pi_1(\Delta_1 \cup \Delta_2, v) = 0$ by van Kampen's theorem, see [Hat02, Theo. 1.20]. By Corollary 2.4 , it remains to prove that $\tilde{H}_*(\Delta_1 \cup \Delta_2, \mathbb{Z}) = 0$. This follows from the Mayer-Vietoris sequence for reduced homology, see 1.3.15. More precisely, for all $n \in \mathbb{Z}$ there is an exact sequence:

$$\tilde{H}_n(\Delta_1, \mathbb{Z}) \oplus \tilde{H}_n(\Delta_2, \mathbb{Z}) \xrightarrow{j_*} \tilde{H}_n(\Delta_1 \cup \Delta_2, \mathbb{Z}) \xrightarrow{\partial_*} \tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, \mathbb{Z})$$

But $\tilde{H}_n(\Delta_1, \mathbb{Z}) \oplus \tilde{H}_n(\Delta_2, \mathbb{Z}) = 0$ and $\tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, \mathbb{Z}) = 0$. It follows $\tilde{H}_n(\Delta_1 \cup \Delta_2, \mathbb{Z}) = 0$ for all $n \in \mathbb{Z}$ and $\Delta_1 \cup \Delta_2$ is contractible. □

Corollary 2.4.4. *[Rau16, p. 16] Let $n \in \mathbb{N}, n \geq 2$ and let Δ be a simplicial complex with subcomplexes $\Delta_1, \dots, \Delta_n$ such that*

$$\Delta = \bigcup_{i=1}^{n} \Delta_i$$

If $\Delta_{i_1} \cap \dots \cap \Delta_{i_l}$ is contractible for all non-empty subsets $\{i_1, \dots, i_l\} \subseteq \{1, \dots, n\}$, then Δ is contractible.

Proof. If $n = 2$, then Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$ are contractible by assumption. Hence, $\Delta_1 \cup \Delta_2$ is contractible by Lemma 2.4.3. Now, let $n \geq 3$ and let the claim be true for all $n' \in \{2, \dots, n-1\}$. In particular, $\bigcup_{i=1}^{n-1} \Delta_i$ is contractible. By Lemma 2.4.3, it remains to prove that

$$\Gamma := \left(\bigcup_{i=1}^{n-1} \Delta_i \right) \cap \Delta_n = \bigcup_{i=1}^{n-1} \Delta_i \cap \Delta_n$$

is contractible. For $i \in \{1, \dots, n-1\}$ let $\Gamma_i := \Delta_i \cap \Delta_n$. Then $\Gamma = \bigcup_{i=1}^{n-1} \Gamma_i$ and by assumption $\Gamma_{i_1} \cap \dots \cap \Gamma_{i_l} = \Delta_{i_1} \cap \dots \cap \Delta_{i_l} \cap \Delta_n$ is contractible for all non-empty subsets $\{i_1, \dots, i_l\} \subseteq \{1, \dots, n-1\}$. Hence, by the induction hypothesis, Γ is contractible. This proves the claim. □

Our first result about the homology type of simplicial complexes associated to homogeneous spaces is that, whenever $P_{G/H}$ is finite, then $\Delta_{G/H}^{min}$ is homotopic equivalent to $\Delta_{G/H}$. Moreover, $\Delta_{G/H}$ is a strong deformation retract of $\Delta_{G/H}$. To prove this we need the following techinical lemma:

Lemma 2.4.5. *[Wal81, p. 375] Let P, Q be finite posets and let $f : P \to Q$ be a monotonic map. Suppose that either $\Delta(f^{-1}(Q_{\leq q}))$ is a contractible subcomplex of $\Delta(P)$ for all $q \in Q$ or $\Delta\left(f^{-1}(Q_{\geq q})\right)$ is a contractible subcomplex for all $q \in Q$. Then*

$$\|f\| : \|P\| \longrightarrow \|Q\|$$

is a homotopy equivalence.

Theorem 2.4.6. *[Bö04, p. 153] Let G/H be a compact homogeneous space such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $P_{G/H}$ is finite. Then $\Delta_{G/H}^{\min}$ is a strong deformation retract of $\Delta_{G/H}$.*

Proof. Let $P^{\min} := P^{\min}_{G/H}$ and $P := P_{G/H}$. Consider the inclusion map

$$\iota : P^{\min} \hookrightarrow P$$

ι is monotonic by defintion. We have that for each $\mathfrak{k} \in P$, the set $||\iota^{-1}(P_{\leq \mathfrak{k}})||$ is contractible: if $\mathfrak{l} \in P^{\min}$, then $\mathfrak{l} \leq \mathfrak{k}$ if and only if $\mathfrak{l}$ is generated by minimal H-subalgebras which are all contained in $\mathfrak{k}$. Then, for any $\mathfrak{k} \in P$, define $\mathfrak{a} := \langle \mathfrak{l}_1, \dots, \mathfrak{l}_s \rangle$, where $\mathfrak{l}_1, \dots, \mathfrak{l}_s$ are all minimal H-subalgebras contained in $\mathfrak{k}$, and we have that

$$\iota^{-1}(P_{\leq \mathfrak{k}}) = P_{\leq \mathfrak{a}}.$$

Hence, $\Delta\left(\iota^{-1}(P_{\leq \mathfrak{k}})\right) = \Delta\left(P^{\min}_{\leq \min(\mathfrak{k})}\right)$ is a cone over $\mathfrak{a}$ by 2.1.3. Thus, by Lemma 2.4.5, $\|\iota\| : ||\Delta^{\min}_{G/H}|| \to ||\Delta_{G/H}||$ is a homotopy equivalence. The theorem of Whitehead 2.4.1 implies that $\Delta^{\min}_{G/H}$ is a strong deformation retract of $\Delta_{G/H}$. □

Now, we need a result about a method to obtain that certain order complexes are contractible.

Definition 2.4.7. *Let P be a finite poset such that $\hat{P} := P\dot{\cup}\{\hat{0}, \hat{1}\}$ is a lattice. For $p \in P$, the complement of p is defined as*

$$c(p) := \{x \in P \mid x \wedge p = \hat{0} \text{ and } x \vee p = \hat{1}\}$$

Theorem 2.4.8. *[Bjö96, p. 1852] Let P be a finite poset such that $\hat{P} = P\dot{\cup}\{\hat{0}, \hat{1}\}$ is a lattice. For a fixed $p \in P$ let $Q := P \backslash c(p)$. Then $\Delta(Q)$ is contractible.*

When we apply this theorem to the simplicial complexes associated to a homogeneous space, we obtain the result used in this work to prove the some simplicial complexes are contractible.

Corollary 2.4.9. *Let P be a generic notation for $P^{min}_{G/H}, P_{G/H}$ or $P^{T}_{G/H}$ and Δ for the correspondents order complexes. Given $\mathfrak{k} \in P$, if there is no $\mathfrak{l} \in P$ such that $\langle \mathfrak{k}, \mathfrak{l} \rangle = \mathfrak{g}$ or $\mathfrak{k} \cap \mathfrak{l} = \mathfrak{h}$, then Δ is contractible.*

Proof. The hypothesis is equivalent to $c(\mathfrak{k}) = \emptyset$ for $\hat{P} = P \,\dot{\cup}\, \{\mathfrak{h}, \mathfrak{g}\}$, then apply the last theorem. □

2.4.1 Rauße's theorem

In this subsection, we prove the main theorem used in this work to show that some simplicial complexes associated to homogeneous spaces are non-contractible. In the case of $H < G$ compact connected Lie groups with $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and $P_{G/H}$ finite, item 3 implies that for every K compact connected intermediary Lie group $H < K < G$

with $\mathfrak{k} := Lie(K)$ we have $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{k}$ and $\Delta_{G/K}$ is a subcomplex of $\Delta_{G/H}$. In this case, the theorem describes how a non-zero homology class $[\theta_{\text{new}}] \in \tilde{H}_m\left(\Delta_{G/H}, R\right)\backslash\{0\}$ can be constructed if a non-zero homology class $[\theta] \in \tilde{H}_{m-1}\left(\Delta_{G/K}, R\right)\backslash\{0\}$ is given, where $m \in \mathbb{N},\ \ H < K < G$, with H maximal in K and R is a commutative unitary ring. Since this theorem is proved using induction, we need to describe when $\tilde{H}_0\left(\Delta_{G/H}, R\right)$ is non-zero:

Lemma 2.4.10. *Let $H < G$ and R as above. Assume that there exists a maximal H-subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is maximal in $\mathfrak{k}_0$. Then $\tilde{H}_0\left(\Delta_{G/H}, R\right) \neq 0$ if and only if there exists another H-subalgebra $\mathfrak{k}_1 \neq \mathfrak{k}_0$.*

Proof. The conditions of $\mathfrak{h}$ being maximal in $\mathfrak{k}_0$ and $\mathfrak{k}_0$ being maximal in $\mathfrak{g}$ imply that $\mathfrak{k}_0$ is a maximal simplex, i.e. a facet, of $\Delta_{G/H}$. Hence, the existence of another vertex $\mathfrak{k}_1$ is equivalent to $\Delta_{G/H}$ being disconnected, since $\mathfrak{k}_0 - \mathfrak{k}_1$ is a 0-cycle with $[\mathfrak{k}_0 - \mathfrak{k}_1] \in \tilde{H}_0(\Delta_{G/H}, R)\backslash\{0\}$. □

The following definitions are required for the theorems of this subsection:

Definition 2.4.11. *[Kah09, p. 12] Let Δ be a simplicial complex, R a commutative unitary ring and $\theta = \sum_{i=1}^{N} r_i \cdot [v_0^i, \ldots, v_n^i] \in C_n(\Delta) \otimes R$ an n-chain with coefficients in R with $r_i \in R\backslash\{0\}$ for $1 \leq i \leq N$. We define the* support *of θ by*

$$\operatorname{supp}(\theta) := \{\{v_0^1, \ldots, v_n^1\}, \ldots, \{v_0^N, \ldots, v_n^N\}\}$$

i.e. the set of all n-simplices whose coset is a non-zero summand of θ. If one of the elements of $\operatorname{supp}(\theta)$ *is a facet of Δ, then we say that θ is supported by a facet.*

The vertex set of θ, or 0-skeleton of θ,

$$\operatorname{vsupp}(\theta) := \bigcup_{i=1}^{N} \left\{v_0^i, \ldots, v_n^i\right\}$$

is also called the vertex support *of θ.*

Let P be a finite poset and $\Delta := \Delta(P)$ be its order complex. A subset $L \subseteq P$ is called a lower bound set *of θ, if there exists an $l \in L$ such that $l \leq v$ for all vertices $v \in \operatorname{vsupp}(\theta)$.*

Theorem 2.4.12. *[Rau16, p. 26] Let $H < G$ be compact connected Lie groups such that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and $P_{G/H}$ is finite and let R be a commutative unitary ring. Furthermore, let $m \in \mathbb{N}, \mathfrak{k}_0$ a minimal H-subalgebra with K_0 as the corresponding connected Lie subgroup of G and $[\theta] \in \tilde{H}_{m-1}\left(\Delta_{G/K_0}, R\right)\backslash\{0\}$ a non-zero homology class with a representative θ such that the following holds:*

Given a lower bound set $\{\mathfrak{l}_1, \ldots, \mathfrak{l}_N\}$ *of* θ *in* P_{G/K_0}, *there exist* H*-subalgebras* $\mathfrak{k}_1, \ldots, \mathfrak{k}_N$, *not necessarily distinct, with the following properties:*

$$\mathfrak{k}_0 \neq \mathfrak{k}_i \qquad \forall i \in \{1, \ldots, N\}; \tag{2.4.1}$$

$$\mathfrak{l}_i \geq \mathfrak{k}_i \qquad \forall i \in \{1, \ldots, N\}; \tag{2.4.2}$$

$$\langle \mathfrak{k}_1, \ldots, \mathfrak{k}_N \rangle < \mathfrak{g} \tag{2.4.3}$$

and θ *is supported by a facet of* Δ_{G/K_0}. *Then*

$$\tilde{H}_m(\Delta_{G/H}, R) \neq 0$$

More precisely, there exists an m*-cycle* θ_{new} *with* $[\theta_{new}] \in \tilde{H}_m(\Delta_{G/H}, R) \backslash \{0\}$, *in which* θ_{new} *is supported by a facet of* $\Delta_{G/H}$ *and a lower bound set of* θ_{new} *is given by* $\{\mathfrak{k}_0, \ldots, \mathfrak{k}_N\}$.

Proof. Let $\mathfrak{k}_1, \ldots, \mathfrak{k}_N$ be H-subalgebras satisfying properties 2.4.2, 2.4.3 and 2.4.4. For any simplex $\sigma = (\mathfrak{k}_0^\sigma < \ldots < \mathfrak{k}_{\dim \sigma}^\sigma) \in \Delta_{G/H}$, we denote its minimal element by $\mathfrak{k}_{\sigma_{\min}} := \mathfrak{k}_0^\sigma$. For $l \in \{0, \ldots, N\}$, define

$$C_l := \left\{\sigma \in \Delta_{G/H} \mid \mathfrak{k}_{\sigma_{\min}} \geq \mathfrak{k}_l\right\} \text{ and}$$
$$D := \bigcup_{l=1}^{N} C_l$$

The C_l's are contractible since they are cones over the $\mathfrak{k}_l$'s by 2.1.3.

Since $\langle \mathfrak{k}_1, \ldots, \mathfrak{k}_N \rangle < \mathfrak{g}$ by 2.4.4, we have that $\langle \mathfrak{k}_{i_1}, \ldots, \mathfrak{k}_{i_s} \rangle$ is an H-subalgebra for every non-empty subset $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, N\}$. Thus,

$$C_{i_1} \cap \ldots \cap C_{i_s} = \left\{\sigma \in \Delta_{G/H} \mid \mathfrak{k}_{\sigma_{\min}} \geq \langle \mathfrak{k}_{i_1}, \ldots, \mathfrak{k}_{i_s} \rangle\right\}$$

is also a cone, hence contractible.

By Corollary 2.4.4, we have that D is contractible. Thus, $\tilde{H}_k(C_0, R) \oplus \tilde{H}_k(D, R) = 0$ for all $k \in \mathbb{Z}$. Moreover, $C_0 \cap D \neq \emptyset$: $\mathfrak{l}_i > \mathfrak{k}_0$, then $\mathfrak{l}_i \in C_0 \cap D$ for all $i \in \{1, \ldots, N\}$. Therefore, the Mayer-Vietoris sequence for reduced homology of the triple $(C_0 \cup D; C_0, D)$, 1.3.15, yields the following exact sequence for $k \in \mathbb{Z}$:

$$0 \longrightarrow \tilde{H}_k(C_0 \cup D, R) \xrightarrow{\partial_*} \tilde{H}_{k-1}(C_0 \cap D, R) \longrightarrow 0$$

Hence, the exactness of the sequence gives that the homomorphism $\partial_* : \tilde{H}_m(C_0 \cup D, R) \to \tilde{H}_{m-1}(C_0 \cap D, R)$, whose existence is given by 1.3.15, is an isomorphism.

The central idea of this proof is to use the isomorphism ∂_* to lift the homology class $[\theta]$ using ∂_*. To do this, we first need to show that $[\theta] \in \tilde{H}_{m-1}(C_0 \cap D, R) \backslash \{0\}$.

Let $\sigma \in \operatorname{supp}(\theta)$, so $\mathfrak{k}_{\sigma_{\min}} \in \operatorname{vsupp}(\theta)$. By 2.4.3, $\mathfrak{k}_{\sigma_{\min}} \geq \mathfrak{l}_i \geq \mathfrak{k}_i$ for some $i \in \{1, \ldots, N\}$, it follows $\sigma \in C_i \subseteq D$. Furthermore, $\sigma \in \Delta_{G/K_0} \subseteq C_0$. Thus, $\operatorname{supp}(\theta) \subseteq C_0 \cap D$ and θ represents an element $[\theta] \in \tilde{H}_{m-1}(C_0 \cap D, R)$. It remains to show that $[\theta] \neq 0$.

Every vertex $\mathfrak{l} \in C_0 \cap D$ satisfies $\mathfrak{l} \geq \langle \mathfrak{k}_0, \mathfrak{k}_{\mathrm{i}} \rangle$ for some $i \in \{1, \ldots, N\}$. By 2.4.2, $\mathfrak{k}_0 \neq \mathfrak{k}_i \; \forall i \in \{1, \ldots, N\}$, so $\mathfrak{l} > \mathfrak{k}_0$. Hence, $C_0 \cap D \subseteq \Delta_{G/K_0}$. Now, θ is a facet of $\Delta_{G/H}$ which implies it is not a boundary of $\Delta_{G/H}$, so it is also not a boundary of $C_0 \cap D$. Therefore, $[\theta] \in \tilde{H}_{m-1}(C_0 \cap D, R) \setminus \{0\}$.

With this information, we can lift the homology class. Let

$$[\theta'] := \partial_*^{-1}([\theta]) \in \tilde{H}_m(C_0 \cup D, R) \setminus \{0\}$$

where θ' is a fixed representative. By the definition of ∂_*, see [Mun18, p. 137], θ' can be constructed the following way:

θ is a boundary of C_0 and D, since $\tilde{H}_{m-1}(C_0, R) = 0$ and $\tilde{H}_{m-1}(D, R) = 0$. So, there are m-chains τ_1 of C_0 and τ_2 of D, such that $\partial(\tau_1) = \theta, \partial(\tau_2) = -\theta$. Then

$$\theta' := \tau_1 + \tau_2$$

is an m-cycle of $C_0 \cup D$ such $\partial_*([\theta']) = [\theta]$.

We have that θ is supported by a facet, so let $s \in \operatorname{supp}(\theta)$ be a facet of Δ_{G/K_0}, i.e. $s = \{\tilde{\mathfrak{l}}_0 < \ldots < \tilde{\mathfrak{l}}_{m-1}\}$ is a maximal chain of K_0-subalgebras. Since $\mathfrak{k}_0$ is a minimal subalgebra over $\mathfrak{h}$, $t := \left\{\mathfrak{k}_0 < \tilde{\mathfrak{l}}_0 < \ldots < \tilde{\mathfrak{l}}_{m-1}\right\}$ is a simplex of C_0 and facet of $\Delta_{G/H}$, i.e. a maximal chain of H-subalgebras. t is the only simplex of C_0 such that s is a proper face. Since $s \in \mathit{supp}(\theta) = \mathit{supp}(\partial(\tau_1))$, we have that $t \in \operatorname{supp}(\tau_1)$. On the other hand, $\mathfrak{k}_0 \neq \mathfrak{k}_i$ given in 2.4.2 and the minimality of $\mathfrak{k}_0$ imply $\mathfrak{k}_0 \not> \mathfrak{k}_i$ for $i \in \{1, \ldots, N\}$. Hence, t is not a simplex of D, which guarantees that $t \notin \operatorname{supp}(\tau_2)$ and therefore, $t \in \operatorname{supp}(\theta')$ since it cannot be cancelled out by the sum $\theta' := \tau_1 + \tau_2$.

Now, the inclusion map $\iota : C_0 \cup D \hookrightarrow \Delta_{G/H}$ induces a homomorphism in the homology groups $i_* : \tilde{H}_m(C_0 \cup D, R) \to \tilde{H}_m(\Delta_{G/H}, R)$ and we can define

$$[\theta_{\text{new}}] := i_*([\theta'])$$

with θ_{new} being an m-cycle of $\Delta_{G/H}$ such that $\operatorname{supp}(\theta_{\text{new}}) = \operatorname{supp}(\theta')$, so $\operatorname{supp}(\theta_{\text{new}})$ contains the facet t. This implies θ_{new} cannot be a boundary of $\Delta_{G/H}$, so $[\theta_{\text{new}}] \in \tilde{H}_m(\Delta_{G/H}, R) \setminus \{0\}$.

Moreover, since for $\mathfrak{k} \in \operatorname{vsupp}(\theta_{new}) \subseteq C_0 \cap D$ there exists $i \in \{0, \ldots, N\}$ with $\mathfrak{k} \geq \mathfrak{k}_i$, we have that a lower bound set of θ_{new} is given by $\{\mathfrak{k}_0, \ldots, \mathfrak{k}_N\}$. □

Last theorem 2.4.12 yields a method to be applied iteratively along maximal chains of H-subalgebras, which is given by the next theorem.

Theorem 2.4.13. *[Rau16, p. 28] Let $H < G$ be compact connected Lie groups such that $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, $P_{G/H}$ is finite and let $\left(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_N^0\right)$ be a maximal chain of H-subalgebras for some $N \in \mathbb{N}$ with associated maximal chain of connected Lie subgroup $(K_0^0 > \ldots > K_N^0)$ of G. Furthermore, let $\mathfrak{k}_{N+1}^0 := \mathfrak{h}$ and assume that for each $m \in \{1, \ldots, N+1\}$ there exist K_m^0-subalgebras $\mathfrak{k}_{m-1}^1, \ldots, \mathfrak{k}_{m-1}^m$, not necessarily distinct, with the following properties:*

1. $\mathfrak{k}_0^1 \neq \mathfrak{k}_0^0$
2. *If $N \geq 1$, then for each $m \in \{1, \ldots, N\}$ the K_{m+1}^0-subalgebras $\mathfrak{k}_m^1, \ldots, \mathfrak{k}_m^{m+1}$ satisfy the following properties:*

- $\mathfrak{k}_m^0 \neq \mathfrak{k}_m^i \qquad \forall i \in \{1, \ldots, m+1\}$
- $\mathfrak{k}_{m-1}^{i-1} \neq \mathfrak{k}_m^i \qquad \forall i \in \{1, \ldots, m+1\}$
- $\langle \mathfrak{k}_m^1, \ldots, \mathfrak{k}_m^{m+1} \rangle < \mathfrak{g}$

Then $\tilde{H}_N(\Delta_{G/H}, R) \neq 0$ for any commutative unitary ring R.

Proof. Property 1 implies $\theta_0 := \mathfrak{k}_0^0 - \mathfrak{k}_0^1 \neq 0$ and $[\theta_0] \in \tilde{H}_0\Delta_{G/K_1^0}, R)\backslash\{0\}$ as in the proof of Lemma 2.4.10. It follows that θ_0 is supported by the facet $\mathfrak{k}_0^0$ of Δ_{G/K_1^0} and $\{\mathfrak{k}_0^0, \mathfrak{k}_0^1\}$ is a lower bound set for θ_0.

Now suppose that for $m \in \{1, \ldots, N\}$ there exists an $(m-1)$-cycle θ_{m-1} of Δ_{G/K_m^0} with $[\theta_{m-1}] \in \tilde{H}_{m-1}(\Delta_{G/K_m^0}, R)\backslash\{0\}$, that θ_{m-1} is supported by a facet of Δ_{G/K_m^0} and a lower bound set of θ_{m-1} is given by $\{\mathfrak{k}_{m-1}^0, \ldots, \mathfrak{k}_{m-1}^m\}$. By assumption, the K_{m+1}^0-subalgebras $\mathfrak{k}_m^1, \ldots, \mathfrak{k}_m^{m+1}$ satisfy the properties hypotheses of Theorem 2.4.12 with respect to θ_{m-1}. Hence, Theorem 2.4.12 yields an m-cycle θ_m of Δ_{G/K_{m+1}^0} such that $[\theta_m] \in \tilde{H}_m(\Delta_{G/K_{m+1}^0}, R)\backslash\{0\}$, θ_m is supported by a facet of Δ_{G/K_{m+1}^0} and a lower bound set of θ_m^{m+1} is given by $\{\mathfrak{k}_m^0, \ldots, \mathfrak{k}_m^{m+1}\}$. By iteration, the case $m = N$ gives a non-trivial homology class of $\tilde{H}_N(\Delta_{G/H}, R)$. $\square$

3 Lie Theory

3.1 Root Systems and Subalgebras of Maximal Rank

When we have $H < G$ compact connected, $\Delta^T_{G/H}$ only depends on the corresponding Lie algebras by Remark 2.2.2. So, if we want to understand the contractibility or not-contractibility of $\Delta^T_{G/H}$ for G real compact semisimple and H connected of maximal rank, all possibilities of real compact semisimple Lie algebras $\mathfrak{g}$ will be classified and we will show how to obtain all Lie subalgebras $\mathfrak{h}$ of maximal rank.

3.1.1 Abstract Root Systems

Definition 3.1.1. *A finite subset $R \subseteq V \backslash \{0\}$ of a finite-dimensional Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ is called a* root system *if the following conditions hold:*

1. *R spans V.*
2. *If $\alpha, \beta \in R$ and $s_\alpha : V \to V$ is the orthogonal reflection at α with respect to $\langle \cdot, \cdot \rangle$, then $s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\alpha \in R$, i.e., R is invariant by the orthogonal reflections at the roots.*
3. *$n_{\alpha\beta} := 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$ for all $\alpha, \beta \in R$.*
4. *For $\alpha \in R$, the only multiples of α in R are $-\alpha, \alpha$.*

The elements of R are called roots.

Two root systems $R \subseteq V$ and $R' \subseteq V'$ are called isomorphic, denoted by $R \cong R'$, if there exists a isomorphism of vector spaces $\phi : V \to V'$ such that $\phi(R) = R'$ and $n_{\phi(\alpha)\phi(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in R$.

Definition 3.1.2. *Let $R \subseteq V$ be a root system. A subset $\Lambda = \{\alpha_1, \ldots, \alpha_n\} \subseteq R$ is called a set of simple roots, if the following two conditions hold:*

1. *Λ is a basis of V.*
2. *For each $\alpha \in R$, the unique $\{k_1, \ldots, k_n\} \subseteq \mathbb{R}$ such that*
$$\alpha = \sum_{i=1}^{n} k_i \cdot \alpha_i$$
are either all non-negative integers or all non-positive integers.

A root α is called a positive root (negative root), with respect to Λ, if the coefficients in 2 satisfy $k_i \geq 0\,(k_i \leq 0)$.

Each root system contains a set of simple roots and the simple roots determine the root system up to isomorphism as the following lemma shows.

Lemma 3.1.3. *[Hum94, p. 55] Let $R \subseteq V$ and $R' \subseteq V'$ be root systems with simple roots $F = \{\alpha_1, \dots, \alpha_n\}$ and $F' = \{\alpha'_1, \dots, \alpha'_n\}$. If $n_{\alpha_i\alpha_j} = n_{\alpha'_i\alpha'_j}$ for all $1 \leq i, j \leq n$, then $R \cong R'$.*

We want to classify all root systems. For this purpose, let R be a root system and $\Lambda = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots. Then $n_{\alpha_i\alpha_j} \cdot n_{\alpha_j\alpha_i} \in \{0, 1, 2, 3\}$ for all $1 \leq i < j \leq n$, see [Hum94, p. 44, 45]. The Dynkin diagram of R, denoted by $D(R)$, is the graph with n vertices such that for $i \neq j$ the i-th and the j-th vertex are connected by $n_{\alpha_i\alpha_j} \cdot n_{\alpha_j\alpha_i}$ edges, if there are more than two edges, we use an arrow that points to the short root. We observe that $D(R)$ is independent of the choice of Λ.

Two isomorphic root systems have the same Dynkin diagram. Hence, root systems are classified by their Dynkin diagrams. By [Hel78, p. 470], the connected Dynkin diagrams are given by four infinite series $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4)$, the classical Dynkin diagrams, and five exceptional Dynkin diagrams E_6, E_7, E_8, F_4, G_2.

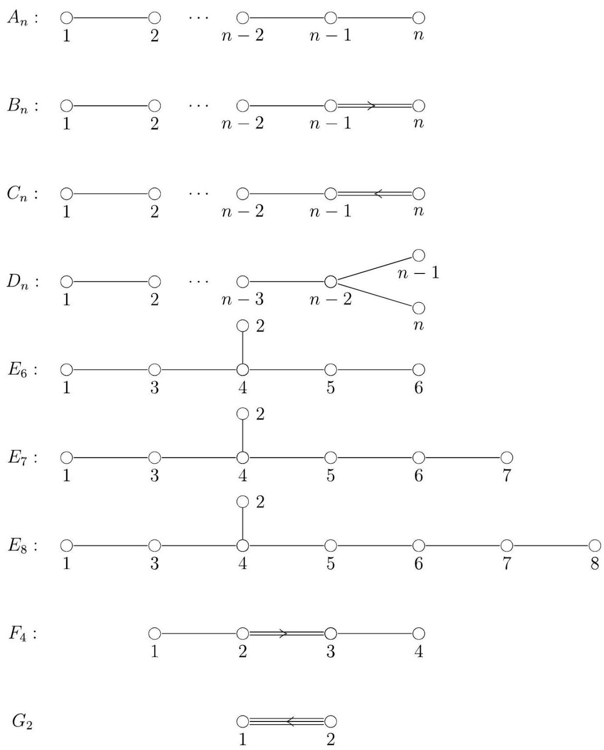

Figure 4 – Dynkin diagrams

3.1.2 Root Systems of Semisimple Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ with Cartan-Killing form $B(X,Y) = tr(\mathrm{ad}_X \circ \mathrm{ad}_Y)$, $X, Y \in \mathfrak{g}$ as defined in 2.3.8. $\mathfrak{g}$ is called semisimple, if its Killing form B is nondegenerate, that is, its ideal $rad\ B = \{X \in \mathfrak{g} \mid B(X,Y) = 0\ \forall Y \in \mathfrak{g}\}$ is equal to zero. Moreover, $\mathfrak{g}$ is called simple, if $\mathfrak{g}$ is non-abelian and if it has no non-trivial ideals. Since $rad\ B$ is an ideal of $\mathfrak{g}$, simple Lie algebras are semisimple. A Lie group G is called semisimple (simple), if its Lie algebra is semisimple (simple).

If $\mathfrak{g}$ is a semisimple Lie algebra, there exists a unique decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$ where each $\mathfrak{g}_i$ is an simple ideal of $\mathfrak{g}$ and these are the only ideals of $\mathfrak{g}$, see [Hum94, p. 23].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and $ad_X : \mathfrak{g} \to \mathfrak{g}$ is a semisimple endomorphism for all $X \in \mathfrak{h}$. Let $\alpha \in \mathfrak{h}^*$ and define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid\ [h, X] = \alpha(h) \cdot X \text{ for all } h \in \mathfrak{h}\}$$

α is called a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$. The set of all roots with respect to $\mathfrak{h}$ is denoted by $R(\mathfrak{g}, \mathfrak{h})$. Since $\mathfrak{h}$ is maximal abelian subalgebra of $\mathfrak{g}$, we have $\mathfrak{h} = \mathfrak{g}_0 = \{X \in \mathfrak{g} \mid [h, X] = 0\ \forall h \in \mathfrak{h}\}$. Furthermore, the endomorphisms $\mathrm{ad}_X, X \in \mathfrak{h}$, are simultaneously diagonalizable, since $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]} = 0$ for $X, Y \in \mathfrak{h}$. Then, we have the called decomposition of roots spaces of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha \tag{3.1.1}$$

Lemma 3.1.4. *[Hel78, p. 166, 170] The restriction of the Killing form to $\mathfrak{h}$, $B_{\mathfrak{h}\times\mathfrak{h}}$, is non-degenerate. Hence, given $\alpha \in \mathfrak{h}^*$ there exists a unique $h_\alpha \in \mathfrak{h}$ such that*

$$B(h_\alpha, h) = \alpha(h) \textit{ for all } h \in \mathfrak{h}$$

Using this lemma, we define a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$ by

$$\langle \alpha, \beta \rangle := B(h_\alpha, h_\beta) \text{ for all } \alpha, \beta \in \mathfrak{h}^*$$

Since B is real valued and positive definite in $\mathfrak{h}_\mathbb{R} := span_\mathbb{R}\{h_\alpha \mid \alpha \in R(\mathfrak{g}, \mathfrak{h})\}$, we have that $\langle \cdot, \cdot \rangle$ is real valued and positive definite in $\mathfrak{h}^*_\mathbb{R} = span_\mathbb{R}\{\alpha \mid \alpha \in R(\mathfrak{g}, \mathfrak{h})\}$.

As expected, $R(\mathfrak{g}, \mathfrak{h})$ is a root system as defined in the last subsection:

Theorem 3.1.5. *[Hum94, p. 73] Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathfrak{h}^*_\mathbb{R}$ as above. Then, $R(\mathfrak{g}, \mathfrak{h})$ is a root system of $\mathfrak{h}^*_\mathbb{R}$ as in Definition 3.1.1.*

Given two Cartan subalgebras $\mathfrak{h}, \tilde{\mathfrak{h}}$ of $\mathfrak{g}$, then $R(\mathfrak{g}, \mathfrak{h})$ and $R(\mathfrak{g}, \tilde{\mathfrak{h}})$ are isomorphic [Hum94, p. 75], then we can write just $R(\mathfrak{g})$ for a root system of $\mathfrak{g}$ since it is unique up to isomorphism.

The Dynkin diagram $D(\mathfrak{g})$ determines $\mathfrak{g}$ up to isomorphism:

Theorem 3.1.6. *[Hum94, p. 173][Hel78, p. 490] The assignment $\mathfrak{g} \mapsto D(\mathfrak{g})$ yields a one-to-one correspondence between the isomorphism classes of complex semisimple (simple) Lie algebras and (connected) Dynkin diagrams.*

3.1.3 Compact Real Forms

Let $\mathfrak{g}$ be a real compact semisimple Lie algebra, i.e. $\mathfrak{g} = Lie\ G$ for a compact semisimple Lie group G. The complexification of $\mathfrak{g}$ is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ that becomes a complex semisimple Lie algebra by defining a Lie bracket in the pure tensors by

$$[X_1 \otimes z_1, X_2 \otimes z_2] := [X_1, X_2] \otimes z_1 z_2 \quad \text{for all } X_1, X_2 \in \mathfrak{g}, z_1, z_2 \in \mathbb{C}$$

and extended by bilinearity. Observe that $\mathfrak{g}_{\mathbb{C}}$ is simple if and only if $\mathfrak{g}$ is simple, since complexification and realification take ideals to ideals. $\mathfrak{g}$ is called a compact real form of $\mathfrak{g}_{\mathbb{C}}$ which is unique up to isomorphism, see [Hel78, p. 184].

Given $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, then $\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and we define the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$ as

$$R(\mathfrak{g}) := R(\mathfrak{g}, \mathfrak{t}) := R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) = R(\mathfrak{g}_{\mathbb{C}})$$

Since the root system $R(\mathfrak{g}_{\mathbb{C}})$ does not depend on the choice of the Cartan subalgebra up to isomorphism, the root system $R(\mathfrak{g})$ does not depend on the choice of $\mathfrak{t}$ up to isomorphism. Moreover,

$$\operatorname{rank} \mathfrak{g} := \dim_{\mathbb{R}} \mathfrak{t} = \dim_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}^*$$

We define the Dynkin diagram of $R(\mathfrak{g})$ or $\mathfrak{g}$ by $D(\mathfrak{g}) := D(\mathfrak{g}_{\mathbb{C}})$. The Dynkin diagram $D(\mathfrak{g})$ defines $\mathfrak{g}$ up to isomorphism: If $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are compact semisimple Lie algebras such that $D(\mathfrak{g}_{\mathbb{C}}) = D(\tilde{\mathfrak{g}}_{\mathbb{C}})$, Theorem 3.1.6 implies $\mathfrak{g}_{\mathbb{C}} \cong \tilde{\mathfrak{g}}_{\mathbb{C}}$. The uniqueness of a compact form implies $\mathfrak{k} \cong \tilde{\mathfrak{k}}$. Furthermore, by [Hel78, p. 81], every complex semisimple Lie algebra has a compact real form. It follows from Theorem 3.1.6:

Theorem 3.1.7. *The assignment $\mathfrak{g} \mapsto D(\mathfrak{g})$ yields a one-to-one correspondence between the isomorphism classes of real compact semisimple (simple) Lie algebras and (connected) Dynkin diagrams.*

By the Theorem above, a compact real Lie algebra is said to classical if it is associated to the classical Dynkin diagrams. The ones with rank n are $\mathfrak{su}(n+1), n \geq 1$ for A_n, $\mathfrak{so}(2n+1), n \geq 2$ for B_n, $\mathfrak{sp}(n), n \geq 3$ for C_n and $\mathfrak{so}(2n), n \geq 4$.

A compact real Lie algebra is said to be exceptional if it is associated to the exceptional Dynkin diagrams. They are denoted by $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$, see [Hel78, p. 516].

3.1.4 Subalgebras of Maximal Rank

Now, let G be any semisimple compact Lie group with real compact semisimple Lie algebra $\mathfrak{g}$ and let T be a maximal torus of G with Lie algebra $\mathfrak{t} = Lie(T)$ that is a Cartan subalgebra of $\mathfrak{g}$.

To determine the simplicial complex of G/H for all connected compact Lie subgroups $T \leq H < G$, first we have to determine all the possibilities of H. By [Djo81, p. 2], every subgroup $T \leq H < G$ is closed in G, hence a compact Lie subgroup. Thus, there is a bijection between all connected compact Lie subgroups $T \leq H < G$ and all Lie subalgebras $\mathfrak{t} \leq \mathfrak{h} < \mathfrak{g}$. These Lie subalgebras are given by the following lemma.

Lemma 3.1.8. *Let $\mathfrak{g}$ and $\mathfrak{t}$ be as above. Let a set of simple roots of $R(\mathfrak{g})$ be given and $R(\mathfrak{g})^+$ be the associated subset of positive roots. For $\alpha \in R(\mathfrak{g}_{\mathbb{C}})$ and $\mathfrak{g}_\alpha$ the associated root space as in 3.1.1, define $\mathfrak{m}_\alpha := \mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. Observe that $\mathfrak{m}_{-\alpha} = \mathfrak{m}_\alpha$. Then*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(\mathfrak{g})^+} \mathfrak{m}_\alpha \tag{3.1.2}$$

We will call 3.1.2 the decomposition of roots spaces of $\mathfrak{g}$.

All intermediary subalgebras $\mathfrak{t} < \mathfrak{h} < \mathfrak{g}$ are given by

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in I} \mathfrak{m}_\alpha \tag{3.1.3}$$

where $I \subseteq R(\mathfrak{g})^+$is any non-empty subset with the following property:

$$\alpha, \beta \in I, \alpha \pm \beta \in R(\mathfrak{g}) \quad \Longrightarrow \quad \alpha \pm \beta \in I \ \dot{\cup} - I \tag{3.1.4}$$

Proof. Let $\alpha \neq \beta \in R(\mathfrak{g})^+$. From [Bö04, p. 89], we have the brackets between the root spaces $\mathfrak{m}_\alpha$ and $\mathfrak{m}_\beta$:

$$[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] = \begin{cases} \mathfrak{m}_{\alpha+\beta} \oplus \mathfrak{m}_{\alpha-\beta}, & \text{if } \alpha+\beta \in R(\mathfrak{g}) \text{ and } \ \alpha-\beta \in R(\mathfrak{g}) \\ \mathfrak{m}_{\alpha+\beta}, & \text{if } \alpha+\beta \in R(\mathfrak{g}) \text{ and } \alpha-\beta \notin R(\mathfrak{g}) \\ \mathfrak{m}_{\alpha-\beta}, & \text{if } \alpha-\beta \in R(\mathfrak{g}) \text{ and } \alpha+\beta \notin R(\mathfrak{g}) \\ \{0\}, & \text{if } \alpha+\beta \notin R(\mathfrak{g}) \text{ and } \alpha-\beta \notin R(\mathfrak{g}) \end{cases} \tag{3.1.5}$$

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h} = \mathfrak{t} \oplus \mathbb{C}$, we have that $[\mathfrak{m}_\alpha, \mathfrak{m}_\alpha] \subseteq \mathfrak{t}$.

A vector space $\mathfrak{h}$ as in 3.1.3 is closed under Lie brackets, hence a Lie subalgebra of $\mathfrak{g}$: if $\alpha, \beta \in I$, then $\alpha \pm \beta \in I \dot{\cup} - I \subseteq R(\mathfrak{g})$ implies $\mathfrak{m}_{\pm\alpha\pm\beta} \in \mathfrak{h}$ and $[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \in \mathfrak{h}$ by 3.1.5.

On the other hand, any subalgebra $\mathfrak{t} < \mathfrak{h} < \mathfrak{g}$ is also an $\mathrm{Ad}(T)$-module, so $\mathfrak{h}$ is the sum of $\mathfrak{t}$ and root spaces $\mathfrak{m}_\alpha, \alpha \in I$, for some $I \subseteq R(\mathfrak{g})^+$, see [Bö04, p. 89]. Now, 3.1.5 implies that I has to satisfy property 3.1.4: if $\alpha, \beta \in I$ and $\alpha + \beta \in R(\mathfrak{g})$, then $\mathfrak{m}_{\alpha+\beta} = \mathfrak{m}_{-(\alpha+\beta)} \subseteq [\mathfrak{m}_\alpha, \mathfrak{m}_\beta] \subseteq \mathfrak{h}$ which implies $\alpha + \beta \in I \dot{\cup} - I$ and equivalently to $\alpha - \beta \in R(\mathfrak{g})$. □

The last Lemma 3.1.8 implies that T-subalgebras of $\mathfrak{g}$ are indexed by subsets of $R(\mathfrak{g})^+$ that satisfy the condition 3.1.4. Hence, $\mathfrak{g}$ contains only finitely many T-subalgebras. Moreover, $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ for every $\mathfrak{t} \leq \mathfrak{h} < \mathfrak{g}$ by 2 and 3, see also [Djo81, p. 1]. Hence, $P_{G/H}$ is finite and the extended simplicial complex $\Delta_{G/H}$ is well-defined for all $T \leq H < G$.

Our purpose is to determine the contractility or non-contractility of $\Delta_{G/H}$ for G semisimple classical and H a subgroup that has maximal rank, i.e., H contains a maximal torus of G. However, it is just necessary to determine it to G simple as we will see next. By Remark 2.2.2, we may also assume that G is simply-connected, so $G = G_1 \times \ldots \times G_k$ with G_i simply connected and simple for all $1 \leq i \leq k$. Let T_i be a maximal torus of G_i for $1 \leq i \leq k$. Then $T := T_1 \times \ldots \times T_k$ is a maximal torus of G and $T \leq H < G$ is a subgroup of maximal rank if and only if $H = H_1 \times \ldots \times H_k$ with $T_i \leq H_i \leq G_i$.

We assumed that the canonical action of G in G/H is almost effectively, which implies $H_i \neq G_i$ for all $1 \leq i \leq k$. By 2.2.8, it follows

$$\Delta_{G/H} \cong \Delta_{G_1/H_1} * \ldots * \Delta_{G_k/H_k} * S^{k-2}$$

Thus, $\Delta_{G/H}$ is contractible, if Δ_{G_i/H_i} is contractible for at least one $1 \leq i \leq k$, see [Bö04, p. 96].

Moreover, it was proved in [Mil56, p. 431] that $\tilde{H}_*(X * Y, \mathbb{F}) \neq 0$, if $\tilde{H}_*(X, \mathbb{F}) \neq 0$ and $\tilde{H}_*(Y, \mathbb{F}) \neq 0$ for any spaces X, Y and any field $\mathbb{F}$. In particular, $\tilde{H}_*(\Delta_{G/H}, \mathbb{Q}) \neq 0$, if $\tilde{H}_*(\Delta_{G_i/H_i}, \mathbb{Q}) \neq 0$ for all $1 \leq i \leq k$.

From the discussion above follows that:

$$\Delta_{G/H} \text{ non-contractible } \iff \Delta_{G_i/H_i} \text{ non-contractible for all } i \in \{1, \ldots, k\}$$

4 The Simplicial Complex for G simple classic and H connected of maximal rank

From the last chapter, we obtained that, if G is a compact semisimple Lie group and H is a connected Lie subgroup of maximal rank, then $\Delta_{G/H}$ is well-defined and to determine the contractility of $\Delta_{G/H}$ it is just necessary to to do it for G simple.

Since $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, we are in the hypothesis of 2.4.13, which will be the main theorem to prove that some $\Delta_{G/H}$ are non-contractible. Theorem 2.4.9 will be the main tool to prove that some $\Delta_{G/H}$ are contractible.

This chapter follows [Rau16].

Remark 4.0.1. Here $\mathfrak{gl}(k, \mathbb{F})$ denotes the Lie algebra of $k \times k$ matrices with coefficients in $\mathbb{F} = \mathbb{R}, \mathbb{C}$, with Lie bracket being the canonical commutator of matrices, and $GL(n, \mathbb{F})$ the group of invertible $k \times k$ matrices with coefficients in $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

4.1 $SU(n+1), n \geq 1$

Let $G = SU(n+1) = \{A \in GL(n+1, \mathbb{C}) \mid A^{-1} = \overline{A^T} \text{ and } det(A) = 1\}$, where $\overline{A^T}$ is conjugate transpose of A. Its Lie algebra is given by

$$\mathfrak{su}(n+1) = \{A \in \mathfrak{gl}(n+1, \mathbb{C}) \mid A = -\overline{A^T} \text{ and } tr(A) = 0\}$$

and its complexification is given by

$$\mathfrak{su}(n+1)_{\mathbb{C}} = \{A \in \mathfrak{gl}(n+1, \mathbb{C}) \mid tr(A) = 0\} = \mathfrak{sl}(n+1, \mathbb{C})$$

A Cartan subalgebra of $su(n+1)$ is given by

$$\mathfrak{t} := \left\{ \text{diag}(i\alpha_1, \dots, i\alpha_{n+1}) \mid \alpha_k \in \mathbb{R}, 1 \leq k \leq n+1, \sum_{k=1}^{n+1} \alpha_k = 0 \right\}$$

and the corresponding Cartan subalgebra of $\mathfrak{su}(n+1)_{\mathbb{C}} = \mathfrak{sl}(n+1, \mathbb{C})$ is

$$\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C} = \left\{ \text{diag}\,(z_1, \dots, z_{n+1}) \mid z_k \in \mathbb{C}, 1 \leq k \leq n+1, \sum_{k=1}^{n+1} z_k = 0 \right\}$$

For $1 \leq k \leq n+1$ consider $\nu_k \in \mathfrak{h}^*$ defined by $\nu_k\,(\text{diag}\,(z_1, \dots, z_n)) := z_k$. Let $\{E_{kl}\}_{1 \leq k,l \leq n+1}$ be the canonical basis of $\mathfrak{gl}(n+1, \mathbb{C})$. If $k \neq l$, $E_{kl} \in \mathfrak{sl}(n+1, \mathbb{C}) = \mathfrak{su}(n+1)_{\mathbb{C}}$. We have that

$$[\text{diag}(z_1, \dots, z_n), E_{kl}] = (z_k - z_l) \cdot E_{kl}$$

Thus, $\nu_k - \nu_l$ is a root for $\mathfrak{h}$ with root space $\langle E_{kl}\rangle_{\mathbb{C}}$ and the fact $\mathfrak{su}(n+1)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{k\neq l} span_{\mathbb{C}}\{E_{kl}\}$ implies that all roots are given this way. Furthermore, for indices $i, j, k, l \in \{1, \ldots, n+1\}$ such that $i > j,\ k > l, \{i, j\} \neq \{k, l\}$, by 3.1.5 the following equivalence holds:

$$(\nu_i - \nu_j) + (\nu_k - \nu_l) \text{ is a root } \Leftrightarrow i = l \text{ or } j = k \Leftrightarrow \#(\{i, j\}\triangle\{k, l\}) = 2 \tag{4.1.1}$$

where $\triangle$ denotes the symmetric difference. From 4.1.1, it follows that a set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n\}$$

with Dynkin diagram A_n and a set of positive roots is given by

$$\{\nu_k - \nu_l \mid 1 \leq k < l \leq n+1\}$$

see [Hel78, p. 462].

We described the root space decomposition for $\mathfrak{su}(n+1)_{\mathbb{C}} = \mathfrak{sl}(n+1, \mathbb{C})$. Now, we will describe the root decomposition for $\mathfrak{su}(n+1)$ using 3.1.8. For $k < l$, let

$$\mathfrak{m}_{kl} := \mathfrak{m}_{lk} := \mathfrak{m}_{\{k,l\}} := \mathfrak{m}_{\nu_k - \nu_l} = \langle E_{kl}, E_{lk}\rangle_{\mathbb{C}} \cap \mathfrak{su}(n+1)$$

This subspace consists of all matrices of $\mathfrak{su}(n+1)$ whose entries are all zero except the $(kl)-$th and the $(lk)-$th one, then

$$\mathfrak{m}_{kl} \cong \left\{ \left(\begin{array}{cc} 0 & z \\ -\bar{z} & 0 \end{array} \right) \middle| \; z \in \mathbb{C} \right\}$$

and $\mathfrak{su}(n+1) = \mathfrak{t} \oplus \bigoplus_{k<l} \mathfrak{m}_{kl}$. It follows:

Proposition 4.1.1. *[Rau16, p. 38] Let $\mathfrak{t}$ be as in as above. Moreover, let $r \in \{2, \ldots, n\}$ and $I_1 \dot\cup \ldots \dot\cup I_r = \{1, \ldots, n+1\}$ be a partition of the index set $\{1, \ldots, n+1\}$. Let $n_i := |I_i|, 1 \leq i \leq r$. Then*

$$\mathfrak{s}\left(\oplus_{i=1}^r \mathfrak{u}\,(n_i)_{I_i}\right) := \mathfrak{t} \oplus \bigoplus_{i,j\in I_1, i<j} \mathfrak{m}_{ij} \oplus \ldots \oplus \bigoplus_{i,j\in I_r, i<j} \mathfrak{m}_{ij} \tag{4.1.2}$$

is a T-subalgebra of $\mathfrak{su}(n+1)$. Moreover, every T-subalgebra is of this type.

Proof. For $i < j, k < l, (i, j) \neq (k, l)$ it follows from 4.1.1 and 3.1.8 that

$$[\mathfrak{m}_{ij}, \mathfrak{m}_{kl}] = \begin{cases} \mathfrak{m}_{\{i,j\}\triangle\{k,l\}}, & \{i, j\} \cap \{k, l\} \neq \emptyset, \\ \{0\} & \{i, j\} \cap \{k, l\} = \emptyset \end{cases} \tag{4.1.3}$$

So, if $\mathfrak{m}_{I_s} := \bigoplus_{i,j\in I_s, i<j} \mathfrak{m}_{ij}$ for $1 \leq s \leq r$, then $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ and $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_{s'}}] = 0$ for $s' \neq s$. It follows that $\mathfrak{s}(\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i})$ is a T-subalgebra of $\mathfrak{su}(n+1)$.

Now, let $\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{k<l} \mathfrak{m}_{kl}$ be any T-subalgebra of $\mathfrak{su}(n+1)$. Consider a graph Γ with vertex set $\{1, \dots, n+1\}$ where two vertices i and j are connected by an edge if and only if $\mathfrak{m}_{ij} \subseteq \mathfrak{k}$. By 4.1.3, if i and j are connected, $\mathfrak{m}_{ij} \subseteq \mathfrak{k}$, and if j and k are connected, $\mathfrak{m}_{jk} \subseteq \mathfrak{k}$, then i and k are also connected, since $\mathfrak{m}_{ik} = [\mathfrak{m}_{ij}, \mathfrak{m}_{jk}] \subseteq \mathfrak{k}$. By induction, this implies that $\mathfrak{m}_{ij} \subseteq \mathfrak{k}$ for every vertices i and j in the same connected component of Γ. If $I_1, \dots, I_r$ are the vertices sets of the connected components of Γ, then $\mathfrak{m}_{ij} \subseteq \mathfrak{k}$ for $i < j \in I_m, 1 \leq m \leq r$. This implies that $\mathfrak{k}$ is of type 4.1.2. We have that $r \geq 2$, since $r = 1$ implies that Γ is connected, all root spaces are contained in $\mathfrak{k}$ and $\mathfrak{k} = \mathfrak{su}(n+1)$ which contradicts $\mathfrak{k} \neq \mathfrak{su}(n+1)$. Also, $r \leq n$: if $r = n+1$, we have that the connected components are just all the vertices, none of them are connected and $\mathfrak{k} = \mathfrak{t}$ which contradicts $\mathfrak{k} \neq \mathfrak{t}$. □

Observe in the last Proposition that, given a partition with more amount of subsets from $\{1, \dots, n+1\}$, we produce a T-subalgebra with less dimension. So, to obtain a T-subalgebra with more dimension, we need a partition with less amount subsets.

The subalgebras from last Proposition can be understand as diagonals block matrices. After conjugation the subalgebra $\mathfrak{k} = \mathfrak{s}\left(\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i}\right)$ just consists of all block matrices

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}, \quad A_i \in \mathfrak{u}(n_i), \ \sum_{i=1}^r \operatorname{trace}(A_i) = 0 \tag{4.1.4}$$

In fact, if $n_0 := 0$ and $\sigma \in S_n$ with $I_i = \sigma\left(\left\{\sum_{j=0}^{i-1} n_j + 1, \dots, \sum_{j=0}^{i} n_j\right\}\right)$ for $1 \leq i \leq r$, then $\mathrm{Ad}_P(\mathfrak{k})$ is of type 4.1.4, where $P = \begin{pmatrix} \operatorname{sgn}(\sigma) & 0 \\ 0 & I_n \end{pmatrix} P_\sigma \in SU(n+1)$ and P_σ is the permutation matrix.

Using the notation $\mathfrak{t} := \mathfrak{s}\left(\oplus_{i=1}^{n+1} \mathfrak{u}(1)_{\{i\}}\right)$, the non-contractibility of $\Delta_{G/H}$ for $G = SU(n+1)$ and $H < G$ connected and of maximal rank is given by the following theorem.

Theorem 4.1.2. *[Rau16, p. 39] Let $\mathfrak{h} = \mathfrak{s}(\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i})$ as in Proposition 4.1.1 or as above if $r = n+1$. Then*

$$\tilde{H}_{r-3}(\Delta_{G/H}, \mathbb{Q}) \neq 0$$

Proof. If $r = 2$, then $\mathfrak{h} = \mathfrak{s}(\mathfrak{u}(n_1)_{I_1} \oplus \mathfrak{u}(n_2)_{I_2})$ is a maximal subalgebra of $\mathfrak{su}(n+1)$ by Proposition 4.1.1, since partitioning I with in other two subsets just produce other subalgebras that are not contained in $\mathfrak{h}$ and the only way to partition I with less subsets is just I itself. Hence, $\Delta_{G/H} = \emptyset$ which implies $\tilde{H}_{-1}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.

So assume $r \geq 3$. To simplify notation, for $s \in \{1, \ldots, r\}$ and $1 \leq i_1 < \ldots < i_s \leq r$ let $I_{i_1,\ldots,i_s} := \cup_{j=1}^{s} I_{i_j}$ and $n_{i_1,\ldots,i_s} := \sum_{j=1}^{s} n_{i_j}$. Now, for $p \in \{0, \ldots, r-3\}$ and $q \in \{0, \ldots, p+1\}$ let

$$\mathfrak{k}_p^q := \mathfrak{s}\left(\mathfrak{u}\left(n_{1,\ldots,r-2-p,r-1-p+q}\right)_{I_{1,\ldots,r-2-p,r-1-p+q}} \oplus \bigoplus_{\substack{l=r-1-p \\ l\neq r-1-p+q}}^{r} \mathfrak{u}\left(n_l\right)_{I_l}\right)$$

It follows that $\left(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{r-3}^0\right)$ is a maximal chain of H-subalgebras. Furthermore, for $q \neq 0$ it holds

$$\begin{aligned} &\mathfrak{k}_p^q \neq \mathfrak{k}_p^0 \text{ and} \\ &\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0 \text{ with } \mathfrak{k}_{-1}^0 := \mathfrak{g}, \mathfrak{k}_{r-2}^0 := \mathfrak{h} \end{aligned}$$

Moreover, for all $p \geq 1$ it holds:

$$\langle \mathfrak{k}_p^1, \ldots, \mathfrak{k}_p^{p+1} \rangle = \mathfrak{s}(\mathfrak{u}(n_{1,\ldots,r-2-p,r-p,\ldots,r})_{I_{1,\ldots,r-2-p,r-p,\ldots,r}} \oplus \mathfrak{u}\left(n_{r-1-p}\right)_{I_{r-1-p}}) < \mathfrak{g}$$

Hence, $\tilde{H}_{r-3}(\Delta_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.4.13.

□

Corollary 4.1.3. *If H is a connected Lie subgroup of maximal rank of $SU(n+1)$, then $SU(n+1)/H$ admits a $SU(n+1)$-invariant Einstein metric.*

Proof. It follows from 4.1.2, 2.3.2 and 1.3.14. □

Example 4.1.4. We will illustrate the subalgebras and their relations in the proof of theorem 4.1.2 for the case $\mathfrak{g} = \mathfrak{su}(5)$ with $r = 4$ and $I_1 = \{1, 3\}, I_2 = \{2\}, I_3 = \{4\}, I_4 = \{5\}$, then

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}_{13} = \left\{ \begin{pmatrix} i\alpha_1 & & z_1 & & \\ & i\alpha_2 & & & \\ -\bar{z}_1 & & i\alpha_3 & & \\ & & & i\alpha_4 & \\ & & & & i\alpha_5 \end{pmatrix} \middle| \; z_1 \in \mathbb{C}, \alpha_i \in \mathbb{R} \right\}$$

$$\mathfrak{k}_0^0 = \mathfrak{s}(\mathfrak{u}(4)_{I_{1,2,3}} \oplus \mathfrak{u}(1)_{I_4}) = \left\{ \begin{pmatrix} i\alpha_1 & z_1 & z_2 & z_4 & \\ -\bar{z}_1 & i\alpha_2 & z_3 & z_5 & \\ -\bar{z}_2 & -\bar{z}_3 & i\alpha_3 & z_6 & \\ -\bar{z}_4 & -\bar{z}_5 & -\bar{z}_6 & i\alpha_4 & \\ & & & & i\alpha_5 \end{pmatrix} \middle| \; z_i \in \mathbb{C}, \alpha_i \in \mathbb{R} \right\}$$

$$\mathfrak{k}_0^1 = \mathfrak{s}(\mathfrak{u}(4)_{I_{1,2,4}} \oplus \mathfrak{u}(1)_{I_3}) = \left\{ \begin{pmatrix} i\alpha_1 & z_1 & z_2 & & z_4 \\ -\bar{z}_1 & i\alpha_2 & z_3 & & z_5 \\ -\bar{z}_2 & -\bar{z}_3 & i\alpha_3 & & z_6 \\ & & & i\alpha_4 & \\ -\bar{z}_4 & -\bar{z}_5 & -\bar{z}_6 & & i\alpha_5 \end{pmatrix} \middle| \; z_i \in \mathbb{C}, \alpha_i \in \mathbb{R} \right\}$$

$$\mathfrak{k}_1^0 = \mathfrak{s}(\mathfrak{u}(3)_{I_{1,2}} \oplus \mathfrak{u}(1)_{I_3} \oplus \mathfrak{u}(1)_{I_4}) = \left\{ \begin{pmatrix} i\alpha_1 & z_1 & z_2 & & \\ -\bar{z}_1 & i\alpha_2 & z_3 & & \\ -\bar{z}_2 & -\bar{z}_3 & i\alpha_3 & & \\ & & & i\alpha_4 & \\ & & & & i\alpha_5 \end{pmatrix} \middle| \; z_i \in \mathbb{C}, \alpha_i \in \mathbb{R} \right\}$$

$$\mathfrak{k}_1^1 = \mathfrak{s}(\mathfrak{u}(3)_{I_{1,3}} \oplus \mathfrak{u}(1)_{I_2} \oplus \mathfrak{u}(1)_{I_4}) = \left\{ \begin{pmatrix} i\alpha_1 & & z_1 & z_2 & \\ & i\alpha_2 & & & \\ -\bar{z}_1 & & i\alpha_3 & z_3 & \\ -\bar{z}_2 & & -\bar{z}_3 & i\alpha_4 & \\ & & & & i\alpha_5 \end{pmatrix} \middle| \; z_i \in \mathbb{C}, \alpha_i \in \mathbb{R} \right\}$$

$$\mathfrak{k}_1^2 = \mathfrak{s}(\mathfrak{u}(3)_{I_{1,4}} \oplus \mathfrak{u}(1)_{I_2} \oplus \mathfrak{u}(1)_{I_3}) = \left\{ \begin{pmatrix} i\alpha_1 & & z_1 & & z_2 \\ & i\alpha_2 & & & \\ -\bar{z}_1 & & i\alpha_3 & & z_3 \\ & & & i\alpha_4 & \\ -\bar{z}_2 & & -\bar{z}_3 & & i\alpha_5 \end{pmatrix} \middle| \; z_i \in \mathbb{C}, \alpha_i \in \mathbb{R} \right\}$$

Then, $(\mathfrak{k}_0^0 > \mathfrak{k}_1^0)$ is a maximal chain of H-subalgebras, $\mathfrak{k}_0^1 \neq \mathfrak{k}_0^0$, $\mathfrak{k}_1^0 \neq \mathfrak{k}_1^2, \mathfrak{k}_0^0 > \mathfrak{k}_1^1, \mathfrak{k}_0^1 > \mathfrak{k}_2^1$ and, since $\mathfrak{k}_1^1 = \mathfrak{t} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{14} \oplus \mathfrak{m}_{34}$ and $\mathfrak{k}_1^2 = \mathfrak{t} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{15} \oplus \mathfrak{m}_{35}$,

$$\langle \mathfrak{k}_1^1, \mathfrak{k}_1^2 \rangle = \mathfrak{t} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{14} \oplus \mathfrak{m}_{34} \oplus \mathfrak{m}_{15} \oplus \mathfrak{m}_{35} \oplus \mathfrak{m}_{45} < \mathfrak{so}(5)$$

and, for H the connected Lie subgroup of $SU(5)$ with Lie algebra $\mathfrak{h}$,

$$\tilde{H}_1(\Delta_{SU(5)/H}, \mathbb{Z}) \neq 0$$

4.2 $SO(n), n \geq 3$

Let $G = SO(n) = \{A \in GL(n, \mathbb{R}) \mid A^{-1} = A^T \text{ and } det\ A = 1\}$. Its Lie algebra is given by

$$\mathfrak{so}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^T = -A\}$$

and its complexification is given by

$$\mathfrak{so}(n)_{\mathbb{C}} = \{A \in \mathfrak{gl}(n,\mathbb{C}) \mid -A^T = A\}$$

The cases n even and n odd have to be considered separately. Moreover, for n even one may assume $n \geq 8$, since $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is not simple and $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ is covered by the the last case.

4.2.1 n even, $n \geq 8$

For any $z \in \mathbb{C}$ let $I(z) := \left(\begin{smallmatrix} 0 & z \\ -z & 0 \end{smallmatrix}\right) \in \mathfrak{so}(2)_{\mathbb{C}}$. A Cartan subalgebra of $\mathfrak{so}(2n)$ is given by

$$\mathfrak{t} := \{\mathrm{diag}(I(\alpha_1), \ldots, I(\alpha_n)) \mid \alpha_k \in \mathbb{R},\ 1 \leq k \leq n\}$$

and its complexification is given by

$$\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C} = \{\mathrm{diag}(I(z_1), \ldots, I(z_n)) \mid z_k \in \mathbb{C},\ 1 \leq k \leq n\}$$

By [Hel78, p. 186], the root system is given as follows: For $1 \leq k \leq n$ let $\nu_k \in \mathfrak{h}^*$ defined by $\nu_k \left(\mathrm{diag}\left(I\left(z_1\right), \ldots, I\left(z_n\right)\right)\right) := i \cdot z_k$. Now, consider the following matrices:

$$M_{++} := \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, M_{--} := \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, M_{+-} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, M_{-+} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{4.2.1}$$

For $1 \leq k < l \leq n$ let $E^{++}_{kl} \in \mathfrak{so}(2n)_{\mathbb{C}}$ be defined as the matrix with all entries are zero except its submatrix induced by the indices $2k-1, 2k, 2l-1$ and $2l$ is of type

$$\begin{pmatrix} 0 & M_{++} \\ -M_{++}^T & 0 \end{pmatrix}$$

Let $E^{--}_{kl}, E^{+-}_{kl}, E^{-+}_{kl} \in \mathfrak{so}(2n)_{\mathbb{C}}$ be defined similarly. For $z_1 \ldots, z_n \in \mathbb{C}$ and $H := \mathrm{diag}\left(I(z_1), \ldots, I\left(z_n\right)\right)$, we have that:

$$\begin{aligned}
[H, E^{++}_{kl}] &= i(z_k + z_l) \cdot E^{++}_{kl} = (\nu_k + \nu_l)(E^{++}_{kl}) \\
[H, E^{--}_{kl}] &= -i(z_k + z_l) \cdot E^{-}_{kl} = -(\nu_k + \nu_l)(E^{--}_{kl}) \\
[H, E^{+-}_{kl}] &= i(z_k - z_l) \cdot E^{+-}_{kl} = (\nu_k - \nu_l)(E^{+-}_{kl}) \\
H, E^{-+}_{kl}] &= -i(z_k - z_l) \cdot E^{-+}_{kl} = -(\nu_k - \nu_l)(E^{-+}_{kl})
\end{aligned}$$

Hence, for $1 \leq k < l \leq n$, $\pm\nu_k \pm \nu_l$, are roots and since $\mathfrak{so}(2n)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{k<l} span_{\mathbb{C}}\{E^{++}_{kl}, E^{--}_{kl}, E^{+-}_{kl}, E^{-+}_{kl}\}$, all roots are given this way and this is the root space decomposition for $\mathfrak{so}(2n)_{\mathbb{C}}$. Again, for indices $i, j, k, l \in \{1, \ldots, n\}$ with $i \neq j, k \neq l$, $\{i, j\} \neq \{k, l\}$, the following equivalences hold:

$$\begin{aligned} \alpha &:= (\nu_i - \nu_j) + (\nu_k - \nu_l) \text{ is a root } \Leftrightarrow i = l \text{ or } j = k \\ \beta &:= (\nu_i + \nu_j) - (\nu_k + \nu_l) \text{ is a root } \Leftrightarrow \{i,j\} \cap \{k,l\} \neq \emptyset \\ \gamma &:= (\nu_i - \nu_j) + (\nu_k + \nu_l) \text{ is a root } \Leftrightarrow j \in \{k,l\} \\ \delta &:= (\nu_i - \nu_j) - (\nu_k + \nu_l) \text{ is a root } \Leftrightarrow i \in \{k,l\} \end{aligned} \tag{4.2.2}$$

which can be written as: if the left-hand side of 4.2.2 is a root, then $\alpha, \beta \in \{\pm(\nu_p - \nu_q)\}$ and $\gamma, \delta \in \{\pm(\nu_p + \nu_q)\}$, where $p < q$, $\{p,q\} = \{i,j\} \triangle \{k,l\}$. Neither $(\nu_i + \nu_j) + (\nu_k + \nu_l)$ nor $(\nu_i + \nu_j) \pm (\nu_i + \nu_j)$ is a root. It follows from [Hel78, p. 464] that a set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_{n-1} + \nu_n\}$$

with Dynkin diagram D_n and a set of positive roots is given by

$$\{\nu_k \pm \nu_l \mid 1 \leq k < l \leq n\}$$

Then, for $k < l$, the root spaces for $\mathfrak{so}(2n)$ are given by

$$\mathfrak{m}_{kl}^+ := \mathfrak{m}_{lk}^+ := \mathfrak{m}_{\{k,l\}}^+ := \mathfrak{m}_{\nu_k+\nu_l} := span_{\mathbb{C}}\{E_{kl}^{++}, E_{kl}^{--}\} \cap \mathfrak{so}(2n)$$

and

$$\mathfrak{m}_{kl}^- := \mathfrak{m}_{lk}^- := \mathfrak{m}_{\{k,l\}}^- := \mathfrak{m}_{\nu_k-\nu_l} := span_{\mathbb{C}}\{E_{kl}^{+-}, E_{kl}^{-+}\} \cap \mathfrak{so}(2n)$$

So

$$\mathfrak{m}_{kl}^+ \cong \left\{ \left(\begin{smallmatrix} & & \alpha & \beta \\ & & \beta & -\alpha \\ -\alpha & -\beta & & \\ -\beta & \alpha & & \end{smallmatrix} \right) \middle| \ \alpha, \beta \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{m}_{kl}^- \cong \left\{ \left(\begin{smallmatrix} & & \alpha & -\beta \\ & & \beta & \alpha \\ -\alpha & -\beta & & \\ \beta & -\alpha & & \end{smallmatrix} \right) \middle| \ \alpha, \beta \in \mathbb{R} \right\}$$

and the root space decomposition of $\mathfrak{so}(2n)$ is given by

$$\mathfrak{so}(2n) = \mathfrak{t} \oplus \bigoplus_{k<l} \left(\mathfrak{m}_{kl}^+ \oplus \mathfrak{m}_{kl}^- \right)$$

Remark 4.2.1. The results above are also true for $SO(4)$ and $SO(6)$ too.

To determine the T-subalgebras of $\mathfrak{so}(2n)$, the following lemmas are needed.

Lemma 4.2.2. *[Rau16, p. 41] Let $i,j,k,l \in \{1,\dots,n\}, i < j, k < l, (i,j) \neq (k,l)$. Then, for any signs $\epsilon_{ij}, \epsilon_{kl} \in \{-,+\}$, it follows*

$$[\mathfrak{m}_{ij}^{\epsilon_{ij}}, \mathfrak{m}_{kl}^{\epsilon_{kl}}] = \begin{cases} \mathfrak{m}_{pq}^-, & \epsilon_{ij} = \epsilon_{kl},\ \{i,j\} \triangle \{k,l\} = \{p,q\} \\ \mathfrak{m}_{pq}^+, & \epsilon_{ij} \neq \epsilon_{kl},\ \{i,j\} \triangle \{k,l\} = \{p,q\} \\ \{0\}, & \{i,j\} \cap \{k,l\} = \emptyset \end{cases} \tag{4.2.3}$$

Proof. From 4.2.2 and 3.1.8, if $\{i,j\}\triangle\{k,l\}=\emptyset$, then $[\mathfrak{m}_{ij}^{\epsilon_{ij}},\mathfrak{m}_{kl}^{\epsilon_{kl}}]=\{0\}$ and, if $\{i,j\}\triangle\{k,l\}=\{p,q\}$, then $[\mathfrak{m}_{ij}^{+},\mathfrak{m}_{kl}^{+}]=\mathfrak{m}_{pq}^{-}$, $[\mathfrak{m}_{ij}^{-},\mathfrak{m}_{kl}^{-}]=\mathfrak{m}_{pq}^{-}$ and $[\mathfrak{m}_{ij}^{+},\mathfrak{m}_{kl}^{-}]=\mathfrak{m}_{pq}^{+}$. □

Definition 4.2.3. *Observing in the last Lemma that the bracket of roots spaces with the same signal results in* 0 *or a root space with signal* $-$ *and if the signals are different the bracket results in* 0 *or a root space with signal* $+$*, we can simplify 4.2.3 defining a multiplication on* $\{-,+\}$ *by* $-\circ- := +\circ+ := -$ *and* $-\circ+ := +\circ- := +$*, then* $(\{-,+\},\circ)\cong\mathbb{Z}_2$*. Now 4.2.3 can be rewritten as*

$$[\mathfrak{m}_{ij}^{\epsilon_{ij}},\mathfrak{m}_{kl}^{\epsilon_{kl}}]=\begin{cases}\mathfrak{m}_{pq}^{\epsilon_{ij}\circ\epsilon_{kl}}, & \{i,j\}\triangle\{k,l\}=\{p,q\}\\ \{0\}, & \{i,j\}\cap\{k,l\}=\emptyset\end{cases} \tag{4.2.4}$$

Lemma 4.2.4. *[Rau16, p. 42] Let* $r\in\{2,\dots,n\}$ *and* $I=\{i_1<\dots<i_r\}\subseteq\{1,\dots,n\}$*. For any* $(r-1)$*-tuple* $(\epsilon_1,\dots,\epsilon_{r-1})\in\{-,+\}^{r-1}$ *of signs, there exist unique signs* $\epsilon_{pq}\in\{-,+\}$ *for* $1\le p<q\le r$*, satisfying:*

1. $\epsilon_{p,p+1}=\epsilon_p$ *for all* $1\le p\le r-1$*.*
2. *The subspace*
$$\mathfrak{u}(r)_I^{\epsilon_1,\dots,\epsilon_{r-1}} := \mathfrak{t}\oplus\bigoplus_{1\le p<q\le r}\mathfrak{m}_{i_pi_q}^{\epsilon_{pq}} \tag{4.2.5}$$
is a T*-subalgebra.*

Proof. If $r=2$, we just have one sign to define, so let $\epsilon_{1,2}:=\epsilon_1$, then $\mathfrak{t}\oplus\mathfrak{m}_{i_1i_2}^{\epsilon_1}\cong\mathfrak{u}(2)\oplus\mathfrak{so}(2)^{n-1}$ is a subalgebra.

First, we will prove that the signs ϵ_{p1} Let $r\ge3$ and let $1\le k<l\le r$ such that $l\ge l+2$. Suppose that we are given signs ϵ_{pq} for $1\le p<q\le n$ that satisfy the conditions 1 and 2, then 4.2.4 yields

$$\mathfrak{m}_{i_k,i_l}^{\epsilon_{kl}}=[\mathfrak{m}_{i_k,i_{k+1}}^{\epsilon_{k,k+1}},\mathfrak{m}_{i_{k+1},i_l}^{\epsilon_{k+1}}]=\mathfrak{m}_{i_k,i_l}^{\epsilon_{k,k+1}\circ\epsilon_{k+1,l}}$$

which implies

$$\epsilon_{kl}=\epsilon_{k,k+1}\circ\epsilon_{k+1,l}=\epsilon_k\circ\epsilon_{k+1,l}.$$

Iterating the result above $l-k$ times, we obtain

$$\epsilon_{kl}=\epsilon_k\circ\dots\circ\epsilon_{l-1} \tag{4.2.6}$$

Hence, the sign ϵ_{kl} is unique.

Now, we will prove the existence satisfying the conditions 1 and 2. For $1\le p<q\le n$, define ϵ_{kl} as above, then $\epsilon_{k,k+1}=\epsilon_k$. The only thing we still need to prove is that $\mathfrak{u}(r)_I^{\epsilon_1,\dots,\epsilon_{r-1}}$ is a Lie subalgebra of $\mathfrak{so}(2n)$. For this purpose, let $i,j,k,l\in I, i<j, k<l$, $(i,j)\ne(k,l)$. By 4.2.4, it just remains to prove $\epsilon_{ij}\circ\epsilon_{kl}=\epsilon_{pq}$, if $\{i,j\}\triangle\{k,l\}=\{p<q\}$.

Since $\epsilon_{ij} \circ \epsilon_{kl} = \epsilon_{kl} \circ \epsilon_{ij}$, we assume $i < k$ or $i = k$ and $j < l$. We have to verify in three situations:

$$j = k : p = i, q = l \Rightarrow \epsilon_{ij} \circ \epsilon_{kl} = (\epsilon_p \circ \ldots \circ \epsilon_{j-1}) \circ (\epsilon_j \circ \ldots \circ \epsilon_{q-1}) = \epsilon_{pq}.$$

$$j = l : p = i, q = k \Rightarrow \epsilon_{ij} \circ \epsilon_{kl} = (\epsilon_p \circ \ldots \circ \epsilon_{q-1} \circ \epsilon_q \circ \ldots \circ \epsilon_{j-1}) \circ (\epsilon_q \circ \ldots \circ \epsilon_{j-1}) = \epsilon_{pq}$$

since $\epsilon_s^{-1} = \epsilon_s$ for $1 \leq s \leq n$, so the factors $\epsilon_q, \ldots, \epsilon_{j-1}$ cancel out. Similarly,

$$i = k : p = j, q = l \Rightarrow \quad \epsilon_{ij} \circ \epsilon_{kl} = \epsilon_i \circ \ldots \circ \epsilon_{p-1} \circ \epsilon_i \circ \ldots \circ \epsilon_{q-1} = \epsilon_{pq}$$

since the factors $\epsilon_i, \ldots, \epsilon_{p-1}$ cancel out. Hence, $\mathfrak{u}(r)_I^{\epsilon_1, \ldots, \epsilon_{r-1}}$ is a subalgebra. □

The T-subalgebras of $\mathfrak{so}(2n)$ are now given by the following proposition.

Proposition 4.2.5. *[Rau16, p. 43] Let $\mathfrak{t}$ as before. Moreover, let $r \in \{1, \ldots, n-1\}$, $I_1 \dot\cup \ldots \dot\cup I_r = \{1, \ldots, n\}$ be a partition of the index set $\{1, \ldots, n\}$ and $n_i := |I_i|$, $1 \leq i \leq r$. Then the T-subalgebras of $\mathfrak{so}(2n)$ are precisely given by*

$$\begin{aligned} &\mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1 \ldots \epsilon_{n_1-1}^1} \oplus \ldots \oplus \mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l \ldots \epsilon_{n_l-1}^l} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus \ldots \oplus \mathfrak{so}(2n_r)_{I_r} \\ := \mathfrak{t} \oplus &\bigoplus_{\substack{i,j \in I_1 \\ i<j}} \mathfrak{m}_{ij}^{\epsilon_{ij}^1} \oplus \ldots \oplus \bigoplus_{\substack{i,j \in I_l \\ i<j}} \mathfrak{m}_{ij}^{\epsilon_{ij}^l} \oplus \bigoplus_{\substack{i,j \in I_{l+1} \\ i<j}} \left(\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-\right) \oplus \ldots \oplus \bigoplus_{\substack{i,j \in I_r \\ i<j}} \left(\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-\right) \end{aligned} \tag{4.2.7}$$

for some given $l \in \{0, \ldots, r\}$ and $\epsilon_1^k, \ldots, \epsilon_{n_k-1}^k \in \{-,+\}$ for $1 \leq k \leq l$. The signs $\epsilon_{ij}^k \in \{-,+\}, i < j, i, j \in I_k$ are given as in 4.2.6. For $n_s = 1$, this notation means that $\mathfrak{u}(1)_{I_s} = \mathfrak{so}(2)_{I_s} = \mathbb{R}$ is contained in $\mathfrak{t}$. Moreover, $n_s \geq 2$ for at least one $1 \leq s \leq r$ and if $l = 0$, then $r \geq 2$.

Proof. For $s \in \{1, \ldots, r\}$, let

$$\mathfrak{m}_{I_s} := \bigoplus_{i,j \in I_s, i<j} \mathfrak{m}_{ij}^{\epsilon_{ij}^s} \text{ for } s \leq l \text{ and } \mathfrak{m}_{I_s} := \bigoplus_{i,j \in I_s, i<j} \left(\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-\right) \text{ for } s > l$$

By Lemma 4.2.4, $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ for $s \leq l$. Furthermore, from 4.2.3, it follows $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ for $s > l$ and $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_{s'}}] = 0$ for $s \neq s'$. Hence, 4.2.7 defines a T-subalgebra.

Now, let $\mathfrak{k}$ be any T-subalgebra of $\mathfrak{so}(2n)$. As in the proof of Proposition 4.1.1, let Γ be a graph with vertex set $\{1, \ldots, n\}$ where i and j are connected by an edge if and only $\mathfrak{m}_{ij}^+ \subseteq \mathfrak{k}$ or $\mathfrak{m}_{ij}^- \subseteq \mathfrak{k}$. By 4.2.3, if i and j are connected and if j and k are connected, then so are i and k. Thus, if $I_1, \ldots, I_r$ denote the connected components of Γ, then

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\substack{s=1 \\ n_s \geq 2}}^{r} \Big(\bigoplus_{\substack{i,j \in I_s \\ i<j}} \mathfrak{k}_{ij} \Big)$$

where $\mathfrak{k}_{ij} \in \left\{\mathfrak{m}_{ij}^+, \mathfrak{m}_{ij}^-, \mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-\right\}$. Now, fix some $s \in \{1, \ldots, r\}$ such that $n_s \geq 2$. If $\mathfrak{k}_{ij} \in \left\{\mathfrak{m}_{ij}^+, \mathfrak{m}_{ij}^-\right\}$ for all $i < j, i, j \in I_s$, then by the uniqueness statement of Lemma 4.2.4,

$\mathfrak{t}\oplus \bigoplus_{i,j\in I_s, i<j} \mathfrak{k}_{ij}$ must be of type $\mathfrak{u}(n_s)_{I_s}^{\epsilon_1^1,\ldots,\epsilon_{n_s-1}^s}$. If $\mathfrak{k}_{ij}=\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-$ for any $i<j, i,j\in I_s$, then $\mathfrak{k}_{pq}=\mathfrak{m}_{pq}^+\oplus\mathfrak{m}_{pq}^-$for all $p<q, p,q\in I_s$. In fact, if $i\neq p$, then by 4.2.3,

$$\mathfrak{m}_{ip}^+\oplus\mathfrak{m}_{ip}^-=[\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-,\mathfrak{k}_{jp}]\subseteq\mathfrak{k}$$

and

$$\mathfrak{m}_{pq}^+\oplus\mathfrak{m}_{pq}^-=[\mathfrak{m}_{ip}^+\oplus\mathfrak{m}_{ip}^-,\mathfrak{k}_{iq}]\subseteq\mathfrak{k}$$

and similarly $\mathfrak{m}_{pq}^+\oplus\mathfrak{m}_{pq}^-\subseteq\mathfrak{k}$ for the case $j\neq p$. This proves that $\mathfrak{k}$ is of type 4.2.7. Moreover, $n_s\geq 2$ for at least one s, since $\mathfrak{k}\neq\mathfrak{t}$ and if $l=0$, then $r\geq 2$ since $\mathfrak{k}\neq\mathfrak{so}(2n)$. □

Using the embedding

$$\mathfrak{gl}(m,\mathbb{C})\hookrightarrow\mathfrak{gl}(2m,\mathbb{R});$$

$$\begin{pmatrix} x_{1,1}+i\cdot y_{1,1} & \cdots & x_{1,m}+i\cdot y_{1,m} \\ \vdots & & \vdots \\ x_{m,1}+i\cdot y_{m,1} & \cdots & x_{m,m}+i\cdot y_{m,m}\end{pmatrix}\mapsto\begin{pmatrix} x_{1,1} & -y_{1,1} & \cdots & x_{1,m} & -y_{1,m} \\ y_{1,1} & x_{1,1} & \cdots & y_{1,m} & x_{1,m} \\ \vdots & & \vdots & & \\ x_{m,1} & -y_{m,1} & \cdots & x_{m,m} & -y_{m,m} \\ y_{m,1} & x_{m,1} & \cdots & y_{m,m} & x_{m,m}\end{pmatrix}$$

for $m\in\mathbb{N}^*$ with $x_{k,l},y_{k,l}\in\mathbb{R}, 1\leq k,l\leq m,\ \mathfrak{u}(m)$ can be considered as a subalgebra of $\mathfrak{so}(2m)$. Moreover, $\mathfrak{k}=\mathfrak{u}\left(n_1\right)_{I_1}^{\epsilon_1^1\ldots\epsilon_{n_1-1}^1}\oplus\ldots\oplus\mathfrak{u}\left(n_l\right)_{I_l}^{\epsilon_1^l\ldots\epsilon_{n_l-1}^l}\oplus\mathfrak{so}\left(2n_{l+1}\right)_{I_{l+1}}\oplus\ldots\oplus\mathfrak{so}\left(2n_r\right)_{I_r}$ is isomorphic to the subalgebra of block matrices of type

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r\end{pmatrix},\quad A_i\in\mathfrak{u}\left(n_i\right), i\leq l, A_i\in\mathfrak{so}\left(2n_i\right), i>l \tag{4.2.8}$$

More precisely, for $\sigma\in S_n$ let $P_\sigma\in SO(2n)$ be the permutation matrix acting on the 2×2-blocks in a canonical way. After conjugation with some P_σ, one may assume $I_i=\left\{\sum_{j=0}^{i-1}n_j+1,\ldots,\sum_{j=0}^{i}n_j\right\}$ for $n_0:=0, 1\leq i\leq r$. Moreover, if $A:=\operatorname{diag}\left(\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, I_2,\ldots,I_2\right)\in O(2n)$, then the automorphism $\operatorname{Ad}(A)$ maps $\mathfrak{m}_{1,2}^{\pm}$to $\mathfrak{m}_{1,2}^{\mp}$ and leaves $\mathfrak{m}_{p,p+1}^{\pm}$invariant for all $p\geq 2$. Combining A with appropriate cyclic permutations, it follows that $\mathfrak{u}\left(n_i\right)^{\epsilon_1^i\ldots\epsilon_{n_i-1}^i}$ is $\operatorname{Aut}(SO\left(2n_i\right))$-conjugate to $\mathfrak{u}\left(n_i\right)^{-\cdots-}$. Hence, $\mathfrak{k}$ is of type 4.2.8 up to automorphism.

With $\mathfrak{t}=:\oplus_{i=1}^n\mathfrak{so}(2)_{\{i\}}$ or $\mathfrak{t}=:\oplus_{i=1}^n\mathfrak{u}(1)_{\{i\}}$, the contractility or not-contractility of $\Delta_{G/H}$ for $G=SO(2n)$ and $H<G$ connected of maximal rank is given by the following theorem.

Theorem 4.2.6. *[Rau16, p. 44] Let* $\mathfrak{h} = \mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1\dots\epsilon_{n_1-1}^1} \oplus \dots \oplus \mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l\dots\epsilon_{n_l-1}^l} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus \dots \oplus \mathfrak{so}(2n_r)_{I_r}$ *as in Proposition 4.2.5. If* $n_s = 1$ *for some* $s \in \{1, \dots, r\}$*, the corresponding summand will be written as* $\mathfrak{so}(2)_{I_s}$*, whenever there exists a summand of type* $\mathfrak{so}(2n_{s'})_{I_{s'}}$*, for some* $n_{s'} \geq 2$*. Otherwise, it will be written as* $\mathfrak{u}(1)_{I_s}$*. Using this notation, the following statements hold:*

1. *If* $l = 0$*, i.e.* $\mathfrak{h} \cong \oplus_{i=1}^r \mathfrak{so}(2n_i)$*, then* $\tilde{H}_{r-3}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.
2. *If* $l = r$*, i.e.* $\mathfrak{h} \cong \oplus_{i=1}^r \mathfrak{u}(n_i)$*, then* $\tilde{H}_{r-2}(\Delta_{G/H}, \mathbb{Q}) \neq 0$.
3. *If* $l \notin \{0, r\}$*, then* $\Delta_{G/H}$ *is contractible.*

Proof. Let $l = 0$. If $r = 2$, then $\mathfrak{h}$ is maximal, like in the beginning of the proof of Therema 4.1.2. Hence, $\Delta_{G/H} = \emptyset$ which implies $\tilde{H}_{-1}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.

Let $r \geq 3$. Using the notation of the proof of Theorem 4.2.6, let

$$\mathfrak{k}_p^q := \mathfrak{so}(2n_{1,\dots,r-2-p,r-1-p+q})_{I_{1,\dots,r-2-p,r-1-p+q}} \oplus \bigoplus_{\substack{l=r-1-p \\ l\neq r-1-p+q}}^{r} \mathfrak{so}(2n_l)_{I_l}$$

for $p \in \{0, \dots, r-3\}, q \in \{0, \dots, p+1\}$. Without loss of generality, suppose that $n_1 \geq 2$, so $\left(\mathfrak{k}_0^0 > \dots > \mathfrak{k}_{r-3}\right)$ is again a maximal chain of H-subalgebras and for $q \neq 0$, it holds

$$\begin{aligned}
&\mathfrak{k}_p^q \neq \mathfrak{k}_p^0 \\
&\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0 \text{ with } \mathfrak{k}_{-1}^0 := \mathfrak{g}, \mathfrak{k}_{r-2}^0 := \mathfrak{h} \text{ and} \\
&\left\langle \mathfrak{k}_p^1, \dots, \mathfrak{k}_p^{p+1} \right\rangle = \mathfrak{so}\left(2n_{1,\dots,r-2-p,r-p,\dots,r}\right)_{I_{1,\dots,r-2-p,r-p,\dots,r}} \oplus \mathfrak{so}\left(2n_{r-1-p}\right)_{I_{r-1-p}} < \mathfrak{g}
\end{aligned}$$

Hence, $\tilde{H}_{r-3}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$ by Theorem 2.4.13.

Now, let $l = r$, i.e. $\mathfrak{h} = \mathfrak{u}\left(n_1\right)_{I_1}^{\epsilon_1^1\dots\epsilon_{n_1-1}^1} \oplus \dots \oplus \mathfrak{u}\left(n_r\right)_{I_r}^{\epsilon_1^r\dots n_{n_r-1}^r}$. As mentioned above, we may assume $I_i = \{\sum_{l=1}^{i-1} n_l + 1, \dots, \sum_{l=1}^{i} n_l\}$ with $n_0 := 0$ and $\epsilon_j^i = -$ for all $i \in \{1, \dots, r\}, j \in \{1, \dots, n_i - 1\}$. If $r = 1$, then $\mathfrak{h} = \mathfrak{u}(n)^{-\dots-}$ is maximal and $\tilde{H}_{-1}(\Delta_{G/H}, \mathbb{Q}) \neq 0$. So, let $r \geq 2$. For $p \in \{0, \dots, r-2\}$ let

$$\mathfrak{k}_p^q := \mathfrak{u}(n_{1,\dots,r-p-1,r-p+q})_{I_{1,\dots,r-p-1,r-p+q}}^{-\dots-} \oplus \bigoplus_{\substack{l=r-p \\ l\neq r-p+q}}^{r} \mathfrak{u}(n_l)_{I_l}^{-\dots-} \text{ for } q \in \{0, \dots, p\} \text{ and}$$

$$\mathfrak{k}_p^{p+1} := \mathfrak{u}(n_{1,\dots,r-p-1,r})_{I_{1,\dots,r-p-1,r}}^{-\dots-+} \oplus \bigoplus_{l=r-p}^{r-1} \mathfrak{u}(n_l)_{I_l}^{-\dots-}$$

i.e. $\mathfrak{k}_p^{p+1}$ is generated by $\mathfrak{u}\left(n_{1,\dots,r-p-1}\right)_{I_{1,\dots,r-p-1}}^{-\dots-}$ and $\mathfrak{u}\left(n_r\right)_{I_r}^{-\dots-}$, adding the root space $\mathfrak{m}_{1,n}^+$. Moreover, $\mathfrak{k}_p^0 = \mathfrak{u}\left(n_{1,\dots,r-p}\right)_{I_{1,\dots,r-p}}^{-\dots}$, so $\left(\mathfrak{k}_0^0 > \dots > \mathfrak{k}_{r-2}^0\right)$ is a maximal chain of H-subalgebras and with $\mathfrak{k}_{-1}^0 := \mathfrak{g}$ and $\mathfrak{k}_{r-1}^0 := \mathfrak{h}$ it follows for $q \neq 0$:

$$\mathfrak{k}_p^q \neq \mathfrak{k}_p^0$$
$$\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0$$

Moreover, for $p \geq 1$ it follows

$$\begin{aligned}\langle \mathfrak{k}_p^1, \ldots, \mathfrak{k}_p^{p+1} \rangle \quad &= \langle \mathfrak{u}(n_{1,\ldots,\widehat{r-p},\ldots,r})^{-}_{I^{-\cdots-p}_{1,\ldots,},\ldots,r} \oplus \mathfrak{u}(n_{r-p})^{-\cdots-}_{I_{r-p}}, \mathfrak{k}_p^{p+1} \rangle \\ &= \mathfrak{so}\left(2n_{1,\ldots,\widehat{r-p},\ldots,r}\right)_{I_{1,\ldots,\widehat{r-p},\ldots,r}} \oplus \mathfrak{u}\left(n_{r-p}\right)^{-\cdots-}_{I_{r-p}} < \mathfrak{g}\end{aligned}$$

Where a hat denotes the omission of an element.

Thus, $\tilde{H}_{r-2}(\Delta_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.4.13.

Now, let $l \notin \{0, r\}$ and assume $\mathfrak{h} = \mathfrak{u}(n_1)^{-\cdots-}_{I_1} \oplus \ldots \oplus \mathfrak{u}(n_l)^{-\cdots-}_{I_l} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus \ldots \oplus \mathfrak{so}(2n_r)_{I_r}$. Consider the H-subalgebra

$$\mathfrak{k} := \mathfrak{so}\left(2n_1\right)_{I_1} \oplus \ldots \oplus \mathfrak{so}\left(2n_l\right)_{I_l} \oplus \mathfrak{so}\left(2n_{l+1}\right)_{I_{l+1}} \oplus \ldots \oplus \mathfrak{so}\left(2n_r\right)_{I_r}$$

In fact, $\mathfrak{k} \neq \mathfrak{h}$, since $n_i \geq 2$ for at least one $i \leq l$. To show that $\Delta_{G/H}$ is contractible, it suffices to show, by Theorem 2.4.9, that that there exists no H-subalgebra $\mathfrak{l}$ with $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{h}$ and $\langle \mathfrak{l}, \mathfrak{k} \rangle = \mathfrak{g}$. So, let $\mathfrak{l}$ be any H-subalgebra. Since $n_i \geq 2$ for at least one $i > l, \mathfrak{l} \neq \mathfrak{u}(n)$. Hence,

$$\mathfrak{l} = \mathfrak{u}\left(m_1\right)^{\epsilon_1^1 \cdots \epsilon_{m_1-1}^1}_{J_1} \oplus \ldots \oplus \mathfrak{u}\left(m_{l'}\right)^{\epsilon_1^l \cdots \epsilon_{m_{l'}-1}^{l'}}_{J_{l'}} \oplus \mathfrak{so}\left(2m_{l'+1}\right)_{J_{l'+1}} \oplus \ldots \oplus \mathfrak{so}\left(2m_{r'}\right)_{J_{r'}}$$

for some $r' \geq 2$. Moreover, for all $i \in \{1, \ldots, r\}$ there exists some $j \in \{1, \ldots, r'\}$ with $I_i \subseteq J_j$. In particular,

$$\langle \mathfrak{k}, \mathfrak{l} \rangle \leq \mathfrak{so}\left(2m_1\right)_{J_1} \oplus \ldots \oplus \mathfrak{so}\left(2m_{r'}\right)_{J_{r'}} < \mathfrak{g}$$

Hence, $\Delta_{G/H}$ is contractible by Theorem 2.4.9.

□

Corollary 4.2.7. *If H is a connected Lie subgroup of maximal rank of $SO(n)$, n even, $n \geq 8$. Then, following notation from 4.2.5,*

1. *If $H \cong \prod_{i=1}^{r} SO(2n_i)$, then $SO(n)/H$ admits a $SO(n)$-invariant Einstein metric*

2. *If $H \cong \prod_{i=1}^{r} U(n_i)$, then $SO(n)/H$ admits a $SO(n)$-invariant Einstein metric.*

Proof. It follows from 4.1.2, 2.3.2 and 1.3.14.

Observe that the case for $l \notin \{0, r\}$ is inconclusive by 2.3.2. □

Example 4.2.8. As in the computations of example 4.1.4, we will ilustrate some subalgebras used in the proof of Theorem 4.2.6 for the case $n = 4$, that is, we are in $\mathfrak{so}(8)$, with $r = 3$, $I_1 = \{1, 2\}$, $I_2 = \{3\}$, $I_3 = \{4\}$ and

$$\mathfrak{h} = \mathfrak{u}(2)^{-}_{I_1} \oplus \mathfrak{u}(1)_{I_2} \oplus \mathfrak{u}(1)_{I_3} = \mathfrak{t} \oplus \mathfrak{m}^{-}_{12} =$$

$$= \left\{ \left(\begin{array}{cccccccc} 0 & \alpha_1 & a & -b & & & & \\ -\alpha_1 & 0 & b & a & & & & \\ -a & -b & 0 & \alpha_2 & & & & \\ b & -a & -\alpha & 0 & & & & \\ & & & & 0 & \alpha_3 & & \\ & & & & -\alpha_3 & 0 & & \\ & & & & & & 0 & \alpha_4 \\ & & & & & & -\alpha_4 & 0 \end{array} \right) \middle| \; \alpha_i, a, b \in \mathbb{R} \right\}$$

$$\mathfrak{k}^0_1 = \mathfrak{u}(3)^{---}_{I_{1,2}} \oplus \mathfrak{u}(1)_{I_3} = \mathfrak{t} \oplus \mathfrak{m}^{-}_{12} \oplus \mathfrak{m}^{-}_{13} \oplus \mathfrak{m}^{-}_{23} =$$

$$= \left\{ \left(\begin{array}{cccccccc} 0 & \alpha_1 & a & -b & c & -d & & \\ -\alpha_1 & 0 & b & a & d & c & & \\ -a & -b & 0 & \alpha_2 & e & -f & & \\ b & -a & -\alpha & 0 & f & e & & \\ -c & -d & -e & -f & 0 & \alpha_3 & & \\ d & -c & f & -e & -\alpha_3 & 0 & & \\ & & & & & & 0 & \alpha_4 \\ & & & & & & -\alpha_4 & 0 \end{array} \right) \middle| \; \alpha_i, a, \dots, f \in \mathbb{R} \right\}$$

$$\mathfrak{k}^2_1 = \mathfrak{u}(3)^{-+}_{I_{1,3}} \oplus \mathfrak{u}(1)_{I_2} = \mathfrak{t} \oplus \mathfrak{m}^{-}_{12} \oplus \mathfrak{m}^{+}_{14} \oplus \mathfrak{m}^{+}_{24} =$$

$$= \left\{ \left(\begin{array}{cccccccc} 0 & \alpha_1 & a & -b & & & c & d \\ -\alpha_1 & 0 & b & a & & & d & -c \\ -a & -b & 0 & \alpha_2 & & & e & f \\ b & -a & -\alpha & 0 & & & f & -e \\ & & & & 0 & \alpha_3 & & \\ & & & & -\alpha_3 & 0 & & \\ -c & -d & -e & -f & & & 0 & \alpha_4 \\ -d & c & -f & e & & & -\alpha_4 & 0 \end{array} \right) \middle| \; \alpha_i, a, \dots, f \in \mathbb{R} \right\}$$

4.2.2 n odd, $n \geq 3$

First, consider $\mathfrak{so}(3)$. A maximal abelian subalgebra is given by

$$\mathfrak{t} = \left\{ \left(\begin{array}{ccc} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| \; \alpha \in \mathbb{R} \right\}$$

Hence, $\mathfrak{so}(3)$ has rank 1, i.e., only one simple root, then postive root. It follows from Lemma 3.1.8, that there exist no T-subalgebras of $\mathfrak{so}(3)$, since adding the one root space generates $\mathfrak{so}(3)$. Follows that $\Delta_{SO(3)/T} = \emptyset$, so contractible and $SO(3)/H$ admits a $SO(3)$-invariant Einstein metric by 2.3.2.

So, assume $n \geq 5$, i.e. $SO(n) = SO(2m + 1)$ for some $m \geq 2$. Subalgebras of $\mathfrak{so}(2m), \mathfrak{so}(2m)_{\mathbb{C}}$ can be considered subalgebras of $\mathfrak{so}(2m + 1), \mathfrak{so}(2m + 1)_{\mathbb{C}}$ by the canonical embedding $X \in \mathfrak{so}(2m) \mapsto \left(\begin{smallmatrix} X & \\ & 0 \end{smallmatrix} \right) \in \mathfrak{so}(2m + 1)$. Let $\mathfrak{t}$ be the Cartan subalgebra of $\mathfrak{so}(2m)$ as in 4.2.6. By [Hel78, p. 187], $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{so}(2m + 1)$ and $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{so}(2m + 1)_{\mathbb{C}}$. For $1 \leq k < l \leq m$ let $\pm\nu_k \pm \nu_l$ be as before. These are roots with root space $span_{\mathbb{C}}\{E^{++}_{kl}\}, span_{\mathbb{C}}\{E^{--}_{kl}\}, span_{\mathbb{C}}\{E^{+-}_{kl}\}$ and

$span_{\mathbb{C}}\{E_{kl}^{-+}\} \subseteq \mathfrak{so}(2m)_{\mathbb{C}} \subseteq \mathfrak{so}(2m{+}1)_{\mathbb{C}}$. Furthermore, for $1 \leq k \leq m$, let $E_k^+ \in \mathfrak{so}(2m{+}1)_{\mathbb{C}}$ be the matrix whose entries are all zero but its submatrix induced by the indices $2k-1, 2k$ and $2m+1$ is of type

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}$$

Analogously, let $E_k^- \in \mathfrak{so}(2n+1)_{\mathbb{C}}$ be the matrix whose entries are all zero but its submatrix induced by the indices $2k-1, 2k$ and $2n+1$ is of type

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}$$

It follows for $z_1, \ldots, z_n \in \mathbb{C}, H = \operatorname{diag}\left(I\left(z_1\right), \ldots, I\left(z_n\right), 0\right) \in \mathfrak{h}$:

$$\begin{aligned} [H, E_k^+] &= iz_k \cdot E_k^+ \\ [H, E_k^-] &= -iz_k \cdot E_k^- \end{aligned}$$

Hence, ν_k and $-\nu_k$ are roots with root spaces $span_{\mathbb{C}}\{E_k^+\}, span_{\mathbb{C}}\{E_k^-\}$, respectively.

The root space decomposition is given by

$$\mathfrak{so}(2n+1)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{1\leq k<l\leq n} span_{\mathbb{C}}\{E_{kl}^{++}, E_{kl}^{--}, E_{kl}^{+-}, E_{kl}^{-+}\} \oplus \bigoplus_{1\leq k\leq n} span_{\mathbb{C}}\{E_k^+, E_k^-\}$$

and all roots are given by $\pm\nu_k$ and $\pm\nu_k \pm \nu_l, 1 \leq k < l \leq n$. In addition to 4.2.2, for indices $i, k, l \in \{1, \ldots, n\}, k \neq l$, the following equivalences hold:

$$\begin{aligned} \epsilon &:= \nu_i + (\nu_k - \nu_l) \text{ is a root } \Leftrightarrow i = l \\ \zeta &:= -\nu_i + (\nu_k - \nu_l) \text{ is a root } \Leftrightarrow i = k \\ \eta &:= \pm\nu_i \mp (\nu_k + \nu_l) \text{ is a root } \Leftrightarrow i \in \{k, l\} \end{aligned} \tag{4.2.9}$$

A set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_n\}$$

with Dynkin diagram B_n and the corresponding set of positive roots is given by

$$\{\nu_k \pm \nu_l \mid 1 \leq k < l \leq n\} \cup \{\nu_k \mid 1 \leq k \leq n\}$$

see [Hel78, p. 462]. For $1 \leq k < l \leq n$ let $\mathfrak{m}_{kl}^+ = \mathfrak{m}_{\nu_k+\nu_l}, \mathfrak{m}_{kl}^- = \mathfrak{m}_{\nu_k-\nu_l}$ as above and

$$\mathfrak{m}_k := \mathfrak{m}_{\nu_k} := span_{\mathbb{C}}\{E_k^+, E_k^-\} \cap \mathfrak{so}(2n+1) \cong \left\{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ -\alpha & -\beta & 0 \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{R} \right\}$$

And the root space decomposition of $\mathfrak{so}(2n+1)$ is given by

$$\mathfrak{so}(2n+1) = \mathfrak{h} \oplus \bigoplus_{1\le k<l\le n} (\mathfrak{m}_{kl}^{+} \oplus \mathfrak{m}_{kl}^{-}) \oplus \bigoplus_{1\le k\le n} \mathfrak{m}_k$$

Thus, the T-subalgebras of $\mathfrak{so}(2n+1)$ are given by the following proposition.

Proposition 4.2.9. *[Rau16, p. 47] With the notation as above, all T-subalgebras $\mathfrak{k}$ of $\mathfrak{so}(2n+1)$ are precisely given by the following three cases:*

1. $\mathfrak{k} = \mathfrak{so}(2n)$.
2. $\mathfrak{k} < \mathfrak{so}(2n)$ *and* $\mathfrak{k}$ *given as in 4.2.6.*
3. $\mathfrak{k} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \dots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r-1} \mathfrak{so}(2n_j)_{I_j} \oplus \mathfrak{so}(2n_r+1)_{I_r} := \mathfrak{k}' \oplus \bigoplus_{k\in I_r} \mathfrak{m}_k$, *where* $\mathfrak{k}' = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \dots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r} \mathfrak{so}(2n_j)_{I_j}$ *is equal to* $\mathfrak{t}$ *or a T-subalgebra of* $\mathfrak{so}(2n)$ *as in 4.2.6 with* $0 \le l < r \le n$.

Proof. Every subalgebra of $\mathfrak{so}(2n)$ is also a subalgebra of $\mathfrak{so}(2n+1)$. Hence, $\mathfrak{k}$ like in 1 and 2 are T-subalgebras in $\mathfrak{so}(2n+1)$. Now, let $\mathfrak{k} = \mathfrak{k}' \oplus \bigoplus_{k\in I_r} \mathfrak{m}_k$ as in 3 . For $i, j, k \in \{1, \dots, n\}, i < j$, 3.1.8 and 4.2.9 imply

$$[\mathfrak{m}_k, \mathfrak{m}_{ij}^{\pm}] = \begin{cases} \mathfrak{m}_i, & k = j \\ \mathfrak{m}_j, & k = i \\ \{0\}, & k \notin \{i, j\} \end{cases} \tag{4.2.10}$$

and furthermore,

$$[\mathfrak{m}_i, \mathfrak{m}_j] = \mathfrak{m}_{ij}^{+} \oplus \mathfrak{m}_{ij}^{-} \tag{4.2.11}$$

Let $\mathfrak{m}_{I_s}$ be defined as in the proof of Proposition 4.2.5 for $1 \le s \le r$. For all $k, l \in I_r$, $k < l$, it follows $[\mathfrak{m}_k, \mathfrak{m}_l] \subseteq \mathfrak{m}_{I_r}, [\mathfrak{m}_k, \mathfrak{m}_k] \subseteq \mathfrak{t}, [\mathfrak{m}_k, \mathfrak{m}_{I_s}] = 0$ for $s < r$ and $[\mathfrak{m}_k, \mathfrak{m}_{I_r}] \subseteq \bigoplus_{i\in I_r} \mathfrak{m}_i$. Thus, 3 defines a subalgebra of $\mathfrak{so}(2n+1)$.

Now, let $\mathfrak{k}$ be any T-subalgebra of $\mathfrak{so}(2n+1)$. We may assume that $\mathfrak{m}_k \subseteq \mathfrak{k}$ for at least one k, since otherwise $\mathfrak{k} = \mathfrak{so}(2n)$ or $\mathfrak{k}$ would be a T-subalgebra of $\mathfrak{so}(2n)$ and thus, it would be of type 4.2.6. By 4.2.10 and 4.2.11, $[\mathfrak{m}_k, \mathfrak{so}(2n)] = \mathfrak{so}(2n+1)$ and $[\mathfrak{m}_k, \mathfrak{u}(n)^{\epsilon_1,\dots,\epsilon_{n-1}}] = \mathfrak{so}(2n+1)$ for any $1 \le k \le n$ and $\epsilon_1, \dots, \epsilon_{n-1} \in \{-, +\}$. Thus, for $\mathfrak{k}' := \mathfrak{k} \cap \mathfrak{so}(2n)$, it follows $\mathfrak{k}' = \mathfrak{t}$ or $\mathfrak{k}' = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \dots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r} \mathfrak{so}(2n_j)_{I_j}$ as in 4.2.6 for some $0 \le l \le r, r \ge 2$. Now, let

$$I := \{k \in \{1, \dots, n\} \mid \mathfrak{m}_k \subseteq \mathfrak{k}\}$$

By 4.2.11, $\mathfrak{m}_{ij}^{+} \oplus \mathfrak{m}_{ij}^{-} \subseteq \mathfrak{k}'$ for all $i, j \in I, i < j$. Hence, if $\mathfrak{k}' = \mathfrak{t}$, then $I = \{k\}$ for some $k \in \{1, \ldots, n\}$ and $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{m}_k$ is of type 3 . Now, assume $\mathfrak{k}' \neq \mathfrak{t}$. By 4.2.11, there is an $s > l$ such that $I \subseteq I_s$, so assume $I \subseteq I_r$. But for any $i \in I, j \in I_r, i \neq j$, it follows from 4.2.10 that $\mathfrak{k} \supseteq \left[\mathfrak{m}_i, \mathfrak{m}_{ij}^{\pm}\right] = \mathfrak{m}_j$. Hence, $I = I_r$ and $\mathfrak{k}$ is of type 3.

□

Let $\mathfrak{k} = \bigoplus_{i=1}^{l} \mathfrak{u}\left(n_i\right)^{\epsilon_1^i \ldots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r-1} \oplus \mathfrak{so}\left(2n_j\right)_{I_j} \left(\oplus \mathfrak{so}\left(2n_r\right)_{I_r}\right) < \mathfrak{so}(2n{+}1)$ be any T-subalgebra of $\mathfrak{so}(2n{+}1)$. Similarly to the case $\mathfrak{g} = \mathfrak{so}(2n)$, after conjugation with appropriate matrices of type $P_\sigma \subseteq SO(2n) \subseteq SO(2n+1), \sigma \in S_n$, and $\operatorname{diag}\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), I_2, \ldots, I_2, -1\right) \in SO(2n+1)$, we may assume that $\mathfrak{k}$ consists of all block matrices of type

$$\begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_{r-1} & \\ & & & 0 \end{pmatrix} \text{ or } \begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_r & \\ & & & B \end{pmatrix}$$

with $A_i \in \mathfrak{u}\left(n_i\right), i \leq l, A_i \in \mathfrak{so}\left(2n_i\right), i > l$ and $B \in \mathfrak{so}\left(2n_r + 1\right)$.

The contractility or not-contractility of $\Delta_{G/H}$ for $G = SO(2n+1)$ and $H < G$ connected of maximal rank is now given by the following theorem.

Theorem 4.2.10. *[Rau16, p. 49] With the notation as above, let $\mathfrak{h} = \mathfrak{so}(2n)$ or $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}\left(n_i\right)_{I_i}^{\varepsilon_1^i \ldots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r-1} \mathfrak{so}\left(2n_j\right)_{I_j} \oplus \mathfrak{so}\left(2n_r+1\right)_{I_r}$ or $\mathfrak{h} = \mathfrak{t}$. Here, $n_r = 0, I_r = \emptyset$ means that $\mathfrak{h} \leq \mathfrak{so}(2n)$.*

As above, if $n_s = 1$ for some $s \in \{1, \ldots, r\}$, the corresponding summand will be written as $\mathfrak{so}(2)_{I_s}$, whenever there exists a summand of type $\mathfrak{so}\left(2n_{s'}\right)_{I_{s'}}$ for some $n_{s'} \geq 2$ or a summand of type $\mathfrak{so}\left(2n_r+1\right)_{I_r}$ with $n_r \geq 1$. Otherwise, it will be written as $\mathfrak{u}(1)_{I_s}$. Using this notation, the following statements hold:

1. *If $\mathfrak{h} = \mathfrak{so}(2n)$, then $\Delta_{G/H} = \emptyset$.*
2. *If $l = 0$, then $\tilde{H}_{r-3}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.*
3. *If $\mathfrak{h} = \mathfrak{t}$, then $\tilde{H}_{n-2}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.*
4. *If $l \neq 0$ and $\mathfrak{h} \neq \mathfrak{t}$, then $\Delta_{G/H}$ is contractible.*

Proof. $\mathfrak{h} = \mathfrak{so}(2n)$ is maximal in $\mathfrak{so}(2n+1)$, hence $\Delta_{SO(2n+1)/SO(2n)} = \emptyset$ as in the proof of 4.1.2.

Now, assume $l = 0$. Under this assumption, $r = 2$ implies that $\mathfrak{h}$ is of type $\mathfrak{so}\left(2n_1\right)_{I_1} \oplus \mathfrak{so}\left(2n_2+1\right)_{I_2}$. Hence, $\mathfrak{h}$ is maximal, i.e. $\Delta_{G/H} = \emptyset$ and $\tilde{H}_{-1}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$. So, assume $r \geq 3$. With the notation as in Theorem 4.2.5 let $p \in \{0, \dots, r-3\}$, $q \in \{0, \dots, p+1\}$ and

$$\mathfrak{k}_p^q := \mathfrak{so}\left(2n_{1,\dots,r-3-p,r-2-p+q,r}+1\right)_{I_{1,\dots,r-3-p,r-2-p+q,r}} \oplus \bigoplus_{\substack{l=r-2-p \\ l\neq r-2-p+q}}^{r-1} \mathfrak{so}\left(2n_l\right)_{I_l} \tag{4.2.12}$$

This yields a maximal chain of H-subalgebras $(\mathfrak{k}_0^0 > \dots > \mathfrak{k}_{r-3}^0)$. Furthermore, for $q \neq 0$, $\mathfrak{k}_{-1}^0 := \mathfrak{g}$ and $\mathfrak{k}_{r-2}^0 := \mathfrak{h}$ it follows

$$\mathfrak{k}_p^q \neq \mathfrak{k}_p^0$$
$$\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0$$

and for $p \geq 1$:

$$\langle \mathfrak{k}_p^1, \dots, \mathfrak{k}_p^{p+1} \rangle = \mathfrak{so}\left(2n_{1,\dots,r-3-p,r-1-p,\dots,r}+1\right)_{I_{1,\dots,r-3-p,r-1-p,\dots,r}} \oplus \mathfrak{so}\left(2n_{r-2-p}\right)_{I_{r-2-p}} < \mathfrak{g}$$

So, $\tilde{H}_{r-3}(\Delta_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.4.13.

Let $\mathfrak{h} = \mathfrak{t}$. Then $\mathfrak{h} = \bigoplus_{j=1}^{r-1} \mathfrak{so}\left(2n_j\right)_{I_j} \oplus \mathfrak{so}\left(2n_r+1\right)_{I_r}$ with $r = n+1, I_i = \{i\}, n_i = 1$ for $1 \leq i \leq n$ and $I_r = \emptyset, n_r = 0$. As above, 4.2.12 yields H-subalgebras $\mathfrak{k}_p^q \cong \mathfrak{so}(2(n-1-p)+1) \oplus \mathfrak{so}(2)^{p+1}$ and $\tilde{H}_{n-2}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$ by Theorem 2.4.13.

Now, let $l \neq 0$. If $\mathfrak{h} \cong \mathfrak{u}(n)$, then $\Delta_{G/H} = \{\mathfrak{so}(2n)\}$ is just a point. So, let $\mathfrak{h} \not\subseteq \mathfrak{u}(n)$. After conjugation one may assume $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)^{-\dots-} \oplus \bigoplus_{j=1}^{r-1} \mathfrak{so}\left(2n_j\right)_{I_j} \oplus \mathfrak{so}\left(2n_r+1\right)_{I_r}$, where $n_r = 0, I_r = \emptyset$ is possible. Since $\mathfrak{h} \neq \mathfrak{t}$, there exists at least one $i \leq l$ with $n_i \geq 2$. So,

$$\mathfrak{k} := \bigoplus_{i=1}^{r-1} \mathfrak{so}\left(2n_i\right)_{I_i} \oplus \mathfrak{so}\left(2n_r+1\right)_{I_r}$$

is an H-subalgebra. As in the proof of Theorem 4.2.5, any H-subalgebra is of type

$$\mathfrak{l} = \bigoplus_{i=1}^{l'} \mathfrak{u}\left(m_i\right)_{J_i}^{\epsilon_1^i \dots \epsilon_{m_i-1}^i} \oplus \bigoplus_{i=l'+1}^{r'-1} \mathfrak{so}\left(2m_i\right)_{J_i} \oplus \mathfrak{so}\left(2m_{r'}\right)_{J_{r'}}$$

for some $r' \geq 2$ and for all $i \in \{1, \dots, r\}$ there is a $j \in \{1, \dots, r'\}$ with $I_i \subseteq J_j$. Hence,

$$\langle \mathfrak{k}, \mathfrak{l} \rangle \leq \bigoplus_{i=1}^{r'-1} \mathfrak{so}\left(2m_i\right)_{J_i} \oplus \mathfrak{so}\left(2m_{r'}+1\right)_{J_{r'}} < \mathfrak{so}(2n+1)$$

Thus, there exits no H-subalgebra in $P_{G/H}$ such together with $\mathfrak{k}$ generate $\mathfrak{so}(2n+1)$ and $\Delta_{G/H}$ is contractible by Theorem 2.4.9.

□

Corollary 4.2.11. *Following the notation from last theorem, if H is a connected Lie subgroup of maximal rank of $SO(n)$, n odd, $n \geq 3$. Then*

1. *If* $H \cong SO(2n)$*, then* $SO(n)/H$ *admits a* $SO(n)$*-invariant Einstein metric*
2. *If* $H \cong \prod_{i=1}^{r-1} SO(2n_i) \times SO(2n_r+1)$*, then* $SO(n)/H$ *admits a* $SO(n)$*-invariant Einstein metric*
3. *If* $H \cong \prod_{i=1}^{r} SO(1)$*, a maximal torus of* $SO(n)$*, then* $SO(n)/H$ *admits a* $SO(n)$*-invariant Einstein metric.*

Proof. It follows from 4.1.2, 2.3.2 and 1.3.14.

Observe that, for the case $l \neq 0$ and $\mathfrak{h} \neq \mathfrak{t}$, we have that 2.3.2 inconclusive. □

The examples from 4.2.8 also apply in this case.

4.3 $Sp(n), n \geq 1$

Let $G = Sp(n) = \{A \in GL(2n, \mathbb{C}) \mid A^T J_n A = J_n \text{ and } \overline{A^T} = A^{-1}\}$, with $J_n := \left(\begin{smallmatrix} 0 & Id_{\mathbb{C}^n} \\ -Id_{\mathbb{C}^n} & 0 \end{smallmatrix}\right)$.

Its lie algebra is given by

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \middle| A, B \in \mathfrak{gl}(n, \mathbb{C}), \overline{A^T} = -A \text{ and } B = B^T \right\}$$

and its complexification is

$$\mathfrak{sp}(n)_{\mathbb{C}} = \left\{ \begin{pmatrix} U & W \\ V & -U^T \end{pmatrix} \in \mathfrak{gl}(2n, \mathbb{C}) \middle| U, V, W \in \mathfrak{gl}(n, \mathbb{C}),\ V = V^T \text{ and } W = W^T \right\}$$

A Cartan subalgebra of $\mathfrak{sp}(n)$ is given by

$$\mathfrak{t} := \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A = \operatorname{diag}(i\alpha_1, \ldots, i\alpha_n), \alpha_i \in \mathbb{R}, 1 \leq i \leq n \right\} \tag{4.3.1}$$

and a Cartan subalgebra of $\mathfrak{sp}(n)_{\mathbb{C}}$ is given by

$$\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \middle| A = \operatorname{diag}(z_1, \ldots, z_n), z_i \in \mathbb{C}, 1 \leq i \leq n \right\}$$

For $k \in \{1, \ldots, n\}$ and $H := \operatorname{diag}(z_1, \ldots, z_n, -z_1, \ldots, -z_n)) \in \mathfrak{h}$ let $\nu_k \in \mathfrak{h}^*$ be defined by $\nu_k(H) := z_k$. Moreover, for $1 \leq k, l \leq n$, let E_{kl} the matrices of the canonical basis of $\mathfrak{gl}(2n, \mathbb{C})$. It follows:

$$\begin{aligned}
[H, E_{k,n+l} + E_{l,n+k}] &= (z_k + z_l) \cdot E_{k,n+l} + E_{l,n+k}, && k \leq l \\
[H, E_{n+k,l} + E_{n+l,k}] &= -(z_k + z_l) \cdot E_{n+k,l} + E_{n+l,k}, && k \leq l \\
[H, E_{kl} - E_{n+l,n+k}] &= (z_k - z_l) \cdot E_{kl} - E_{n+l,n+k}, && k \neq l.
\end{aligned}$$

and

$$\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{k\leq l} span_{\mathbb{C}}\{E_{k,n+l} + E_{l,n+k}\} \oplus \bigoplus_{k\leq l} span_{\mathbb{C}}\{E_{n+k,l} + E_{n+l,k}\} \oplus$$
$$\oplus \bigoplus_{k\neq l} span_{\mathbb{C}}\{E_{kl} - E_{n+l,n+k}\}$$

is the root space decomposition and the roots are given by $\pm\nu_k \pm \nu_l, 1 \leq k < l \leq n$ and $\pm 2\nu_k, 1 \leq k \leq n$. For indices $i, j, k, l \in \{1, \dots, n\}, i \neq j, k \neq l, \{i, j\} \neq \{k, l\}$, the equivalences in 4.2.2 hold and, in addition, we have:

$$\begin{aligned} \epsilon &:= \pm\left((\nu_i + \nu_j) - 2\nu_k\right) \text{ is a root } \Leftrightarrow k \in \{i, j\} \\ \zeta &:= \quad (\nu_i - \nu_j) - 2\nu_k \text{ is a root } \Leftrightarrow k = i \\ \eta &:= \quad (\nu_i - \nu_j) + 2\nu_k \text{ is a root } \Leftrightarrow k = j \end{aligned} \tag{4.3.2}$$

Moreover, $(\nu_i - \nu_j) \pm (\nu_i + \nu_j)$ is always a root. It follows that a set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2\nu_n\}$$

with Dynkin diagram C_n and a set of positive roots is given by

$$\{\nu_k \pm \nu_l \mid 1 \leq k < l \leq n\} \cup \{2\nu_k \mid 1 \leq k \leq n\}$$

see [Hel78, p. 463].

Furthermore, for $k < l$, the root spaces for $\mathfrak{sp}(n)$ are given by

$$\mathfrak{m}_{kl}^{+} := \mathfrak{m}_{lk}^{+} := \mathfrak{m}_{\{k,l\}}^{+} := \mathfrak{m}_{\nu_k+\nu_l} = span_{\mathbb{C}}\{E_{k,n+l} + E_{l,n+k}, E_{n+k,l} + E_{n+l,k}\} \cap \mathfrak{sp}(n)$$
$$\mathfrak{m}_{kl}^{-} := \mathfrak{m}_{lk}^{-} := \mathfrak{m}_{\{k,l\}}^{-} := \mathfrak{m}_{\nu_k-\nu_l} = span_{\mathbb{C}}\{E_{kl} - E_{n+l,n+k}, E_{lk} - E_{n+k,n+l}\} \cap \mathfrak{sp}(n)$$

$$\mathfrak{m}_k := \mathfrak{m}_{2\nu_k} = span_{\mathbb{C}}\{E_{k,n+k}, E_{n+k,k}\} \cap \mathfrak{sp}(n)$$

In other words,

$$\mathfrak{m}_{kl}^{+} \cong \left\{ \begin{pmatrix} & & & -\bar{z} \\ & & -\bar{z} & \\ & z & & \\ z & & & \end{pmatrix} \;\middle|\; z \in \mathbb{C} \right\}, \mathfrak{m}_{kl}^{-} \cong \left\{ \begin{pmatrix} & -\bar{z} & & \\ z & & & \\ & & & -z \\ & & \bar{z} & \end{pmatrix} \;\middle|\; z \in \mathbb{C} \right\} \text{ and}$$

$$\mathfrak{m}_k \cong \left\{ \begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix} \;\middle|\; z \in \mathbb{C} \right\}$$

And the decomposition of root space for $\mathfrak{sp}(n)$ is

$$\mathfrak{sp}(n) = \mathfrak{t} \oplus \bigoplus_{k\leq l} (\mathfrak{m}_{kl}^{+} \oplus \mathfrak{m}_{kl}^{-}) \oplus \bigoplus_{k\leq l} \mathfrak{m}_k$$

Since 4.2.2 holds for roots of $\mathfrak{sp}(n)$, Lemma 4.2.2 and Lemma 4.2.4 also hold for subalgebras of $\mathfrak{sp}(n)$, i.e. $\mathfrak{u}(r)_I^{\epsilon_1,\dots,\epsilon_{r-1}}$ defined as in 4.2.5 is a T-subalgebra of $\mathfrak{sp}(n)$. It then follows:

Proposition 4.3.1. *[Rau16, p. 52] Let $\mathfrak{t}$ be as in 4.3.1. Furthermore, let $r \in \{1, \ldots, n\}$, $I_1 \dot\cup \ldots \dot\cup I_r = \{1, \ldots, n\}$ be a partition of the index set $\{1, \ldots, n\}$ and $n_i := |I_i|, 1 \leq i \leq r$. Then the T-subalgebras of $\mathfrak{sp}(n)$ are precisely given by*

$$\begin{aligned} \mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1 \ldots \epsilon_{n_1-1}^1} \oplus \ldots \oplus \mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l \ldots \epsilon_{n_1-1}^l} \oplus \mathfrak{sp}(n_{l+1})_{I_{l+1}} \oplus \ldots \oplus \mathfrak{sp}(n_r)_{I_r} :=& \\ := \mathfrak{t} \oplus \bigoplus_{\substack{i,j \in I_1 \\ i<j}} \mathfrak{m}_{ij}^{\epsilon_{ij}^1} \oplus \ldots \oplus \bigoplus_{\substack{i,j \in I_l \\ i<j}} \mathfrak{m}_{ij}^{\epsilon_{ij}^l} \oplus& \\ \oplus \left(\bigoplus_{\substack{i,j \in I_{l+1} \\ i<j}} (\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-) \oplus \bigoplus_{i \in I_{l+1}} \mathfrak{m}_i \right) \oplus \ldots \oplus \left(\bigoplus_{\substack{i,j \in I_r \\ i<j}} (\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-) \oplus \bigoplus_{i \in I_r} \mathfrak{m}_i \right)& \end{aligned} \tag{4.3.3}$$

for some given $l \in \{0, \ldots, r\}$ and $\epsilon_1^k, \ldots, \epsilon_{n_k-1}^k \in \{-, +\}$ for $1 \leq k \leq l$. The signs $\epsilon_{ij}^k \in \{-, +\}, i < j, i, j \in I_k$ are given as in (4.10). For $s \leq l$ and $n_s = 1$, this notation means that $\mathfrak{u}(1)_{I_s} = \mathbb{R}$ is contained in $\mathfrak{t}$. Moreover, if $l = 0$, then $r \geq 2$ and if $l = r$, then $n_s \geq 2$ for at least one $s \in \{1, \ldots, r\}$.

Proof. Fix some $s \in \{1, \ldots, r\}$. If $s \leq l$, then let $\mathfrak{m}_{I_s} := \bigoplus_{i,j \in I_s, i<j} \mathfrak{m}_{ij}^{\epsilon_{ij}^{sj}}$, otherwise let $\mathfrak{m}_{I_s} := \bigoplus_{i,j \in I_s, i<j} \left(\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-\right) \oplus \bigoplus_{i \in I_s} \mathfrak{m}_i$. In particular, $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ for $s \leq l$ by Lemma 4.2.4. Moreover, 4.3.2 and 3.1.5 imply

$$[\mathfrak{m}_{ij}^{\pm}, \mathfrak{m}_k] = \begin{cases} \mathfrak{m}_{ij}^{\mp}, & k \in \{i, j\} \\ \{0\}, & k \notin \{i, j\} \end{cases} \tag{4.3.4}$$

and

$$[\mathfrak{m}_{ij}^-, \mathfrak{m}_{ij}^+] = \mathfrak{m}_i \oplus \mathfrak{m}_j \tag{4.3.5}$$

Now, 4.2.3, 4.3.4, (4.3.5 and $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$ for $i \neq j$ yield $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ for $s > l$ and $\left[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_{s_s}}\right] = 0$ for $s \neq s'$. Thus, 4.3.3 is a T-subalgebra.

On the other hand, let $\mathfrak{k}$ be any T-subalgebra. As above, let $\{1, \ldots, n\}$ be the vertex set of a graph Γ where i and j are connected by an edge if and only if $\mathfrak{m}_{ij}^+ \subseteq \mathfrak{k}$ or $\mathfrak{m}_{ij}^- \subseteq \mathfrak{k}$. Let $I_1, \ldots, I_r$ be the connected components of Γ and let $s \in \{1, \ldots, r\}$. If $n_s = 1, I_s = \{i\}$, then $\mathfrak{k}$ contains either $\mathfrak{sp}(1)_{I_s}$ or $\mathfrak{u}(1)_{I_s}$ depending on whether $\mathfrak{k}$ contains $\mathfrak{m}_i$ or not. So, assume $n_s \geq 2$. By 4.2.4, if i and j are connected and if j and k are connected then so are i and k. Thus, for all $i, j \in I_s, i < j$, it is $\mathfrak{k}_{ij} \subseteq \mathfrak{k}$ for some $\mathfrak{k}_{ij} \in \left\{\mathfrak{m}_{ij}^-, \mathfrak{m}_{ij}^+, \mathfrak{m}_{ij}^- \oplus \mathfrak{m}_{ij}^+\right\}$. If for all $i, j \in I_s, i < j$, there exists a unique sign $\epsilon_{ij} \in \{-, +\}$ such that $\mathfrak{k}_{ij} = \mathfrak{m}_{ij}^{\epsilon_{ij}}$, then $\mathfrak{m}_i \nsubseteq \mathfrak{k}$ for $i \in I_s$ by 4.3.4 and the uniqueness statement of Lemma 4.2.4 implies that $\mathfrak{t} \oplus \mathfrak{m}_{I_s}$ is of type $\mathfrak{u}(n_s)_{I_s, \ldots, \epsilon_{n_s-1}^s}^{\epsilon_0^s}$. If $\mathfrak{k}_{ij} = \mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-$for some $i, j \in I_s, i < j$, then it follows as in the proof of Proposition 4.2.5 that $\mathfrak{k}_{pq} = \mathfrak{m}_{pq}^+ \oplus \mathfrak{m}_{pq}^-$for all $p, q \in I_s, p < q$. Moreover, by 4.3.5, $\mathfrak{m}_i \subseteq \mathfrak{k}$ for all $i \in I_s$.

Thus, $\mathfrak{t} \oplus \mathfrak{m}_{I_s}$ is of type $\mathfrak{sp}\left(n_s\right)_{I_s}$. It follows that $\mathfrak{k}$ is of type 4.3.3. Furthermore, if $l = 0$, then $r \geq 2$ since $\mathfrak{k} \neq \mathfrak{sp}(n)$ and if $l = r$, then $n_s \geq 2$ for at least one s, since $\mathfrak{k} \neq \mathfrak{t}$.

□

Again, for $\mathfrak{k} = \bigoplus_{i=1}^{l} \mathfrak{u}\left(n_i\right)_{I_i}^{\epsilon_1^i,\ldots,\epsilon_{n_i-1}^i} \oplus \bigoplus_{i=l+1}^{r} \mathfrak{sp}\left(n_i\right)_{I_i}$ one may assume that $I_i = \left\{\sum_{j=0}^{i-1} n_j + 1, \ldots, \sum_{j=0}^{i} n_j\right\}$ for $1 \leq i \leq r, n_0 := 0$, after conjugation with an appropriate element of type $\begin{pmatrix} P_\sigma & 0 \\ 0 & P_\sigma \end{pmatrix} \in Sp(n)$ for some $\sigma \in S_n$. Moreover, for $1 \leq l \leq n$ let

$$P_l := E_{n+l,l} - E_{l,n+l} + \sum_{\substack{k=1 \\ k \neq l}}^{n} E_{kk} + E_{n+k,n+k} \in Sp(n)$$

Then $\mathrm{Ad}(P_l)\left(\mathfrak{m}_{ij}^{\pm}\right) = \mathfrak{m}_{ij}^{\mp}$ for $l \in \{i,j\}$ and $\mathrm{Ad}(P_l)\left(\mathfrak{m}_{ij}^{\pm}\right) = \mathfrak{m}_{ij}^{\pm}$for $l \notin \{i,j\}$. Hence, after conjugation with appropriate elements of type P_l one may assume $\epsilon_j^i = -$ for all $1 \leq i \leq l, 1 \leq j \leq n_i$.

Using the notation $\mathfrak{t} = \oplus_{i=1}^{n} \mathfrak{u}(1)_{\{i\}}$ again, the contractility or non-contractility of $\Delta_{G/H}$ for $G = Sp(n)$ and $H < G$ connected of maximal rank is now given by the following theorem.

Theorem 4.3.2. *[Rau16, p. 53] Let* $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}\left(n_i\right)_{I_i}^{\epsilon_1^i,\ldots,\epsilon_{n_i-1}^i} \oplus \bigoplus_{i=l+1}^{r} \mathfrak{sp}\left(n_i\right)_{I_i}$ *be as in Proposition 4.3.1. Then, the following statement holds:*

1. *If* $l = 0$, *then* $\tilde{H}_{r-3}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.
2. *If* $l = r$, *then* $\tilde{H}_{r-2}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$.
3. *If* $l \notin \{0, r\}$, *then* $\Delta_{G/H}$ *is contractible.*

Proof. First, let $l = 0$, i.e. $2 \leq r \leq n$ and $\mathfrak{h} = \bigoplus_{i=1}^{r} \mathfrak{sp}\left(n_i\right)_{I_i}$. Then the claim follows as in Theorem 4.1.2. In fact, every H-subalgebra is of type $\mathfrak{l} = \oplus_{j=1}^{r'} \mathfrak{sp}\left(m_j\right)_{I_j}$, hence $\mathfrak{l}$ is already determined by the partition $J_1 \dot\cup \ldots \dot\cup J_{r'} = \{1, \ldots, n\}$. Thus, if $\mathfrak{h}' := \mathfrak{s}\left(\oplus_{i=1}^{r} \mathfrak{u}\left(n_i\right)_{I_i}\right)$, then

$$P_{Sp(n)/H} \longrightarrow P_{SU(n)/H'} \ : \ \oplus_{j=1}^{r'} \mathfrak{sp}\left(m_j\right)_{I_j} \mapsto \mathfrak{s}\left(\oplus_{j=1}^{r'} \mathfrak{u}\left(m_j\right)_{I_j}\right)$$

is an isomorphism of posets, i.e., a bijection between posets that preserves the order. Therefore, $\Delta_{Sp(n)/H} \cong \Delta_{SU(n)/H'}$. By Theorem 4.1.2, it follows $\tilde{H}_{r-3}(\Delta_{G/H}, \mathbb{Q}) \cong \tilde{H}_{r-3}(\Delta_{SU(n)/H'}, \mathbb{Q}) \neq 0$.

Now, let $l = r$. If $r = 1$, i.e. $\mathfrak{h} \cong \mathfrak{u}(n)$, then $\Delta_{G/H} = \emptyset$ and $\tilde{H}_{-1}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$. So, let $r \geq 2$ and assume $\mathfrak{h} = \oplus_{i=1}^{r} \mathfrak{u}\left(n_i\right)_{I_i}^{-\dots-}$. For $p \in \{0, \dots, r-2\}$ let

$$\mathfrak{k}_p^q := \mathfrak{u}\left(n_{1,\dots,r-p-1,r-p+q}\right)_{I_{1,\dots,r-p-1,r-p+q}}^{-\dots-} \oplus \bigoplus_{\substack{l=r-p \\ l \neq r-p+q}}^{r} \mathfrak{u}\left(n_l\right)_{I_l}^{-\dots-}, 0 \leq q \leq p, \text{ and}$$

$$\mathfrak{k}_p^{p+1} := \mathfrak{u}\left(n_{1,\dots,r-p-1,r}\right)_{I_{1,\dots,r-p-1,r}}^{-\dots\dots+} \oplus \bigoplus_{l=r-p}^{r-1} \mathfrak{u}\left(n_l\right)_{I_l}^{-\dots-}$$

as in the proof of Theorem 4.2.6, so Again, $\left(\mathfrak{k}_0^0 > \dots > \mathfrak{k}_{r-2}^0\right)$ is a maximal chain of H-subalgebras and with $\mathfrak{k}_{-1}^0 := \mathfrak{g}$ and $\mathfrak{k}_{r-1}^0 := \mathfrak{h}$ it follows

$$\begin{aligned} &\mathfrak{k}_p^q \neq \mathfrak{k}_p^0 \\ &\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0 \end{aligned}$$

for $q \neq 0$ and

$$\left\langle \mathfrak{k}_p^1, \dots, \mathfrak{k}_p^{p+1} \right\rangle = \mathfrak{sp}\left(2n_{1,\dots,\widehat{r-p},\dots,r}\right)_{I_{1,\dots,\widehat{r-p},\dots,r}} \oplus \mathfrak{u}\left(n_{r-p}\right)_{I_{r-p}}^{-\dots-} < \mathfrak{g}$$

for $p \geq 1$. So, $\tilde{H}_{r-2}\left(\Delta_{G/H}, \mathbb{Q}\right) \neq 0$ by Theorem 2.4.13.

Now, let $l \notin \{0, r\}$ and assume $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}\left(n_i\right)_{I_i}^{-\dots-} \oplus \bigoplus_{i=l+1}^{r} \mathfrak{sp}\left(n_i\right)_{I_i}$. The contractibility of $\Delta_{G/H}$ follows as in the proof of 4.2.6. More precisely,

$$\mathfrak{k} := \bigoplus_{i=1}^{r} \mathfrak{sp}\left(n_i\right)_{I_i}$$

is an H-subalgebra. Note, that, different from the case $\mathfrak{g} = \mathfrak{so}(2n)$, no conditions for the indices n_i are needed, since $\mathfrak{u}(1)$ is a proper subalgebra of $\mathfrak{sp}(1)$. Now, if $\mathfrak{l} = \bigoplus_{j=1}^{l'} \mathfrak{u}\left(m_j\right)_{J_j}^{\epsilon_1^j \dots \epsilon_{m_j-1}^j} \oplus \bigoplus_{j=l'+1}^{r'} \mathfrak{sp}\left(m_j\right)_{J_j}$, is any H-subalgebra, then $r' \geq 2$ and

$$\langle \mathfrak{k}, \mathfrak{l} \rangle = \bigoplus_{j=1}^{r'} \mathfrak{sp}\left(m_j\right)_{I_j} < \mathfrak{sp}(n)$$

Hence, there exists no subalgebra in $P_{G/H}$ that together wit $\mathfrak{k}$ generates $\mathfrak{sp}(n)$, so $\Delta_{G/H}$ is contractible by 2.4.9. □

Corollary 4.3.3. *Following the notation from the theorem above, if H is a connected Lie subgroup of maximal rank of $Sp(n)$, then*

1. *If* $H \cong \prod_{i=1}^{r} Sp(n_i)$*, then* $Sp(n)/H$ *admits a* $Sp(n)$*-invariant Einstein metric.*
2. *If* $H \cong \prod_{i=1}^{r} U(n_i)$*, then* $Sp(n)/H$ *admits a* $Sp(n)$*-invariant Einstein metric.*

Proof. It follows from 4.1.2, 2.3.2 and 1.3.14.

Observe that, for the case $l \notin \{0, r\}$, 2.3.2 is inconclusive. □

Examples from 4.1.4 also apply in this case.

5 An application in a real flag manifold

Consider the real flag manifold $G/H := SO(4)/S(O(1) \times O(1) \times O(1) \times O(1))$ [PS15]. We have that $\mathfrak{g} = \mathfrak{so}(4)$ and $\mathfrak{h} = \{0\}$, since $S(O(1)^4)$ is a discrete finite Lie subgroup of $SO(4)$. Observe that H is not connected, then there is possibility to a intermediary subalgebra $\{0\} < \mathfrak{k} < \mathfrak{g}$ to not be a H-subalgebra.

We have that $\mathfrak{m} = \mathfrak{so}(4)$ and $\mathfrak{m}_0 = \{X \in \mathfrak{so}(4) \mid [X, \mathfrak{h}] = 0\} = \mathfrak{so}(4)$. In particular, we are in the case of 2.3.1, different from what we considered before in chapter 4.

So we will describe a $\Delta^T_{G/H}$ and a $P^T_{G/H}$ (2.2.7) of $SO(4)/S(O(1)^4)$ where T is a maximal torus of the Lie subgroup associated to $\mathfrak{m}_0 = \mathfrak{so}(4)$.

From 2.2.6, we just need to consider $\mathfrak{t} := Lie(T)$ that is a maximal abelian subalgebra of $\mathfrak{m}_0 = \mathfrak{so}(4)$ which is given by

$$\mathfrak{t} := \left\{ \left. \begin{pmatrix} 0 & a & & \\ -a & 0 & & \\ & & 0 & b \\ & & -b & 0 \end{pmatrix} \right| a, b \in \mathbb{R} \right\} \tag{5.0.1}$$

as in 4.2.

We need to consider non-trivial subalgebras $\mathfrak{k}$ that are $Ad(H)$-invariant, $ad(\mathfrak{t})$-invariant and minimal non-toral H-subalgebra. In this case, being a non-toral H-subalgebra is equivalent to be a non-abelian non-trivial subalgebra of $\mathfrak{so}(4)$ since $\mathfrak{h} = \{0\}$.

First, we consider the condition of $ad(\mathfrak{t})$-invariance.

The subspaces of $\mathfrak{so}(4)$ invariants by $ad(\mathfrak{t})$ are its roots spaces. As we said in 4.2.1, we have the decomposition in root spaces

$$\mathfrak{so}(4) = \mathfrak{t} \oplus \mathfrak{m}^+_{12} \oplus \mathfrak{m}^-_{12} \tag{5.0.2}$$

where

$$\mathfrak{m}^+_{12} = \left\{ \left. \begin{pmatrix} & & x & y \\ & & y & -x \\ -x & -y & & \\ -y & x & & \end{pmatrix} \right| x, y \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{m}^-_{12} = \left\{ \left. \begin{pmatrix} & & x & y \\ & & -y & x \\ -x & y & & \\ -y & -x & & \end{pmatrix} \right| x, y \in \mathbb{R} \right\}$$

with $\mathfrak{t}$ being the trivial $ad(\mathfrak{t})$-module, so every subspace of $\mathfrak{t}$ is $ad(\mathfrak{t})$-invariant, and $\mathfrak{m}^+_{12}, \mathfrak{m}^-_{12}$ irreducible modules of dimension 2:

If $X := \begin{pmatrix} & & x & y \\ & & y & -x \\ -x & -y & & \\ -y & x & & \end{pmatrix} \in \mathfrak{m}_{12}^{+}$ with $x \neq y$ and $A := \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \in \mathfrak{t}$, then

$$[A, X] = \begin{pmatrix} & & y & -x \\ & & -x & -y \\ -y & x & & \\ x & y & & \end{pmatrix}$$

Then, $[A, X]$ is not a real multiple of X and every not-trivial subspace of $\mathfrak{m}_{12}^{+}$ is not $ad(\mathfrak{t})$-invariant. Hence, $\mathfrak{m}_{12}^{+}$ is an irreducible $ad(\mathfrak{t})$-module. The prove that $\mathfrak{m}_{12}^{-}$ is an irreducible $ad(\mathfrak{t})$-module is completely analogous.

$\mathfrak{m}_{12}^{+}$ and $\mathfrak{m}_{12}^{-}$ are not equivalent as $ad(\mathfrak{t})$-modules:

Using the notations, definitions and results from 4.2.1, we have that

$$\mathfrak{m}_{12}^{+} = ((\mathfrak{so}(4)_{\mathbb{C}})_{\nu_1+\nu_2} \oplus (\mathfrak{so}(4)_{\mathbb{C}})_{-\nu_1-\nu_2}) \cap \mathfrak{so}(4)$$

and

$$\mathfrak{m}_{12}^{-} = ((\mathfrak{so}(4)_{\mathbb{C}})_{\nu_1-\nu_2} \oplus (\mathfrak{so}(4)_{\mathbb{C}})_{-\nu_1+\nu_2}) \cap \mathfrak{so}(4)$$

Suppose that there exists $T : \mathfrak{m}_{12}^{+} \to \mathfrak{m}_{12}^{-}$ isomorphim $ad(\mathfrak{t})$-equivariant. There exists $0 \neq X \in \mathfrak{m}_{12}^{+} \cap (\mathfrak{so}(4)_{\mathbb{C}})_{\nu_1+\nu_2}$ and for $A \in \mathfrak{t} \subseteq \mathfrak{t}_{\mathbb{C}}$ we have

$$[A, TX] = T[A, X] = T((\nu_1 + \nu_2)(A)X) = (\nu_1 + \nu_2)(A)TX$$

Which implies $TX \in \mathfrak{m}_{12}^{-} \cap (\mathfrak{so}(4)_{\mathbb{C}})_{\nu_1+\nu_2} = \{0\}$. This contradicts T being an isomorphism of vector spaces.

For the purpose of finding Lie subalgebras, we emphasise that

$$[\mathfrak{t}, \mathfrak{m}_{12}^{+}] \subseteq \mathfrak{m}_{12}^{+} \quad \text{and} \quad [\mathfrak{t}, \mathfrak{m}_{12}^{+}] \subseteq \mathfrak{m}_{12}^{-}$$

If there is $\mathfrak{k} \in P_{G/H}^{T}$, the $ad(\mathfrak{t})$-invariant implies that it needs to be a sum of $\mathfrak{m}_{12}^{+}, \mathfrak{m}_{12}^{-}$ or a subspace of $\mathfrak{t}$, since 5.0.2 is a isotypical decomposition, which is unique ([BD13, p. 70]).

Second, we consider the condition of $Ad(H)$-invariance.

Let $h \in H$ and $\epsilon_1, \ldots, \epsilon_4 \in O(1) = \{-1, 1\} \cong \mathbb{Z}_2$ with $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$ such that $h = diag(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$. For $i \neq j \neq k \neq l \in \{1, \ldots, 4\}$, we have that $\epsilon_i\epsilon_j = (\epsilon_k\epsilon_l)^{-1} = \epsilon_k\epsilon_l$.

If $X := \begin{pmatrix} 0 & a & c & d \\ -a & 0 & e & f \\ -c & -e & 0 & b \\ -d & -f & -b & 0 \end{pmatrix} \in \mathfrak{so}(4)$, then

$$Ad(h)X = hXh^{-1} = hXh = \begin{pmatrix} 0 & \epsilon_1\epsilon_2 a & \epsilon_1\epsilon_3 c & \epsilon_1\epsilon_4 d \\ -\epsilon_2\epsilon_1 a & 0 & \epsilon_2\epsilon_3 e & \epsilon_2\epsilon_4 f \\ -\epsilon_3\epsilon_1 c & -\epsilon_3\epsilon_2 e & 0 & \epsilon_3\epsilon_4 b \\ -\epsilon_4\epsilon_1 d & -\epsilon_4\epsilon_2 f & -\epsilon_4\epsilon_3 b & 0 \end{pmatrix}$$

Define $\delta_1 := \epsilon_1\epsilon_2 = \epsilon_3\epsilon_4, \delta_2 := \epsilon_1\epsilon_3 = \epsilon_2\epsilon_4, \delta_3 := \epsilon_1\epsilon_4 = \epsilon_2\epsilon_3$.

Suppose that $X \in \mathfrak{t}$, then $c = d = e = f = 0$ and $Ad(h)X = \delta_1 X$. Hence, every subspace of $\mathfrak{t}$ is $Ad(H)$-invariant.

Suppose that $X \in \mathfrak{m}_{12}^+$, then $a = b = 0, f = -c, e = d$ and

$$Ad(h)X = \begin{pmatrix} & & \delta_2 c & \delta_3 d \\ & & \delta_3 d & -\delta_2 c \\ -\delta_2 c & -\delta_3 d & & \\ -\delta_3 d & \delta_2 c & & \end{pmatrix} \in \mathfrak{m}_{12}^+$$

Hence, $\mathfrak{m}_{12}^+$ is $Ad(H)$-invariant.

Suppose that $X \in \mathfrak{m}_{12}^-$, then $a = b = 0, f = c, e = -d$ and

$$Ad(h)X = \begin{pmatrix} & & \delta_2 c & \delta_3 d \\ & & -\delta_3 d & \delta_2 c \\ -\delta_2 c & -\delta_3 d & & \\ \delta_3 d & -\delta_2 c & & \end{pmatrix} \in \mathfrak{m}_{12}^-$$

Hence, $\mathfrak{m}_{12}^-$ is $Ad(H)$-invariant.

We conclude that all possibilities of subalgebras we have considered before, sums of $\mathfrak{m}_{12}^+, \mathfrak{m}_{12}^-$ or a subspace of $\mathfrak{t}$, are $Ad(H)$-invariant.

Now, we will describe all the subalgebras in $P_{G/H}^T$.

Since $(\nu_1 + \nu_2) + (\nu_1 - \nu_2) = 2\nu_1$ and $(\nu_1 + \nu_2) - (\nu_1 - \nu_2) = 2\nu_2$ are not roots, 4.2.2 implies that

$$[\mathfrak{m}_{12}^+, \mathfrak{m}_{12}^-] = 0$$

For any $x, y, z, w \in \mathbb{R}$, we have the following Lie brackets relations:

$$\left[\begin{pmatrix} & & x & y \\ & & y & -x \\ -x & -y & & \\ -y & x & & \end{pmatrix}, \begin{pmatrix} & & z & w \\ & & w & -z \\ -z & -w & & \\ -w & z & & \end{pmatrix} \right] = (2wx - 2yz) \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} \in \mathfrak{t}$$

$$\left[\begin{pmatrix} & & x & -y \\ & & y & x \\ -x & -y & & \\ y & -x & & \end{pmatrix}, \begin{pmatrix} & & z & -w \\ & & w & z \\ -z & -w & & \\ w & -z & & \end{pmatrix}\right] = (2wx - 2yz) \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} \in \mathfrak{t}$$

If $A_{12}^{+} := \left(\begin{smallmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{smallmatrix}\right)$ and $A_{12}^{-} := \left(\begin{smallmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{smallmatrix}\right)$, then

$$[\mathfrak{m}_{12}^{+}, \mathfrak{m}_{12}^{+}] = span_{\mathbb{R}}\{A_{12}^{+}\}$$

and

$$[\mathfrak{m}_{12}^{-}, \mathfrak{m}_{12}^{-}] = span_{\mathbb{R}}\{A_{12}^{-}\}$$

Since $\{A_{12}^{+}, A_{12}^{-}\}$ is linearly independent, $\langle \mathfrak{m}_{12}^{+}, \mathfrak{m}_{12}^{-}\rangle = span_{\mathbb{R}}\{X_{12}^{+}, X_{12}^{-}\} \oplus \mathfrak{m}_{12}^{+} \oplus \mathfrak{m}_{12}^{-} = \mathfrak{t} \oplus \mathfrak{m}_{12}^{+} \oplus \mathfrak{m}_{12}^{-} = \mathfrak{so}(4)$. With this information, a Lie subalgebra of the type we are looking for cannot contain both $\mathfrak{m}_{12}^{+}$ and $\mathfrak{m}_{12}^{-}$.

Hence, the non-abelian non-trivial $ad(\mathfrak{t})$-invariant Lie subalgebras of $\mathfrak{so}(4)$ are

$$\begin{array}{ll} \mathfrak{t} \oplus \mathfrak{m}_{12}^{+} & \mathfrak{t} \oplus \mathfrak{m}_{12}^{-} \\ span_{\mathbb{R}}\{A_{12}^{+}\} \oplus \mathfrak{m}_{12}^{+} & span_{\mathbb{R}}\{A_{12}^{-}\} \oplus \mathfrak{m}_{12}^{-} \end{array}$$

Then $P_{G/H}^{T} = \{span_{\mathbb{R}}\{A_{12}^{+}\} \oplus \mathfrak{m}_{12}^{+},\ span_{\mathbb{R}}\{A_{12}^{-}\} \oplus \mathfrak{m}_{12}^{-}\}$ and $\Delta_{G/H}^{T}$ consists of a 0-dimensional order complex with just two points as vertices. Hence, $\Delta_{G/H}^{T}$ is not-contractible.

We conclude by 2.3.1 that

Theorem 5.0.1. *The real flag manifold* $G/H = SO(4)/S(O(1) \times O(1) \times O(1) \times O(1))$ *admits a* $SO(4)$*-invariant Einstein metric.*

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