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Instituto de Matemática, Estatística e Computação Científica

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Entropy *in* Piecewise Smooth Dynamics *and* Symbolic Dynamics

Entropia em Dinâmica Suave por Partes e Dinâmica Simbólica

Campinas
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Supervisor: José Régis Azevedo Varão Filho

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> Ciência e indústria, saber e aplicação, descoberta e realização prática conducente a novas descobertas, trabalho cerebral e trabalho manual, — pensamento e obra braçal — tudo se liga. Cada descoberta, cada progresso, cada aumento da riqueza da humanidade tem sua origem no conjunto do trabalho manual e cerebral do passado e do presente.
>
> PIOTR KROPOTKIN (1892)

Agradeço à minha mãe, *Sandra*, ao meu pai, *Fernando*, às minhas avós, *Nena* e *Celeste*, e ao meu avô, *Benedito* (*in memoriam*), por me criarem e me darem amor em todos momentos. Vocês me ensinaram a ler e escrever, a ser crítico e curioso, e me apresentaram às riquezas da humanidade: a matemática, a linguística, a música e a arte, entre tantas outras.

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I guess I'm just currently feeling that mathematics remains a human activity, because if I were to truly check that the computer did the right thing, I would have to unravel the whole proof in my head and convince myself that it all adds up.

PETER SCHOLZE (2022)

If we take in our hand any volume; of divinity or school metaphysics, for instance; let us ask, *Does it contain any abstract reasoning concerning quantity or number?* No. *Does it contain any experimental reasoning concerning matter of fact and existence?* No. Commit it then to the flames: For it can contain nothing but sophistry and illusion.

DAVID HUME (1748)

Resumo

Neste trabalho estudamos os conceitos de entropia topológica e entropia de medida e os aplicamos à teoria de sistemas dinâmicos oriundos de campos vetoriais suaves por partes e de deslocamentos bissimbólicos.

Os deslocamentos bissimbólicos, também chamados *deslocamentos zip*, são uma variação dos deslocamentos de Bernoulli bilaterais que possibilitam codificar dinâmicas não invertíveis, como a transformação do padeiro 2 para 1. Neste trabalho, exibimos uma desintegração da medida no espaço do deslocamento bissimbólico e a usamos para calcular a entropia de dobra do deslocamento. Além disso, calculamos a entropia de medida do deslocamento e a relacionamos com a entropia de dobra.

No contexto de campos vetoriais suaves por partes em variedades Riemannianas, consideramos o conjunto de todas as órbitas possíveis do sistema, estabelecidas pela convenção de Filippov. Nesse *espaço de órbitas* definimos um fluxo contínuo induzido pela dinâmica original no *espaço base* dos campos vetoriais suaves por partes. Definimos também uma distância no espaço de órbitas a partir da distância Riemanniana da variedade base e mostramos que o espaço métrico resultante, sob algumas hipóteses, herda propriedades topológicas da variedade base: ele é separável, completo e não tem pontos isolados. Mostramos então que, se a dinâmica no espaço de órbitas é transitiva, então a dinâmica no espaço base também é, e a recíproca vale para uma classe específica de campos vetoriais suaves por partes cujos pontos de tangência são suficientemente conectados entre si.

Nos restringimos então ao contexto de variedades 2-dimensionais e estudamos algumas propriedades clássicas de caoticidade adaptadas ao caso de campos vetoriais suaves por partes. Sob certas hipóteses, mostramos que sistemas transitivos são caóticos no sentido de existir um conjunto denso de órbitas periódicas e ter dependência sensível a condições iniciais. Finalmente, comentamos sobre como definir entropia topológica para campos vetoriais suaves por partes usando o espaço de órbitas, e obtemos que a entropia topológica desses sistemas 2-dimensionais transitivos é estritamente positiva. Isso reforça o caráter caótico dessas dinâmicas.

Palavras-chave Sistemas dinâmicos · Teoria ergódica · Entropia · Dinâmica simbólica · Campos vetoriais suaves por partes.

Abstract

In this work we study the concepts of topological entropy and measure entropy and apply them to the theory of dynamical systems originated by piecewise smooth vector fields and by bisymbolic dynamics.

Bisymbolic shifts, also called *zip shifts*, are a variation of bilateral Bernoulli shifts that make it possible to encode non-invertible dynamics, like the 2-to-1 baker's transformation. In this work, we exibit a disintegration of the measure on the bisymbolic shift space and use it to calculate the folding entropy of the shift. Besides that, we calculate the measure entropy of the shift and relate it to the folding entropy.

In the context of piecewise smooth vector fields on Riemannian manifolds, we consider the set of all possible orbits of the system, established by Filippov's convention. On this *orbit space* we define a continuous flow induced by the original dynamics on the *base space* of the piecewise smooth vector field. We also define a distance on the orbit space from the Riemannian distance on the base manifold and show that the resulting metric space, under certain hypotheses, inherits the topological properties of the base manifold: it is separable, complete and has no isolated points. We then show that, if the dynamics on the orbit space is transitive, then the dynamics on the base space is also transitive, and the converse is valid for a specific class of piecewise smooth vector fields whose tangency points are sufficiently connected among them.

We then restrict ourselves to the context of 2-dimensional manifolds and study some classic properties of chaoticity adapted to the case of piecewise smooth vector fields. Under certain hypotheses, we show that transitive systems are chaotic in the sense that there exists a dense set of periodic orbits and they have sentive dependence on initial conditions. Finally, we comment on how to define topological entropy for piecewise smooth vector fields using the orbit space, and obtain that the topological entropy of these transitive 2-dimensional systems is strictly positive. This reinforces the chaotic character of these dynamics.

Keywords Dynamical systems · Ergodic theory · Entropy · Symbolic dynamics · Piecewise smooth vector fields.

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Introduction

Symbol	Definition
$\vee$, $\bigvee$	logical disjunction, existential quantifier
$\wedge$, $\bigwedge$	logical conjunction, universal quantifier
$:=$	equality by definition
$\subseteq$, $\subset$	subset partial order, strict partial order
$\#X$	cardinality of a set X
$\wp(X)$	power set of a set X
$\cup$, $\bigcup$	union
$\cap$, $\bigcap$	intersection
$\setminus$	relative complement
$\overline{X}$	complement of a set X
$\triangle$	symmetric difference
$f\colon X \longrightarrow X'$	function from a set X to a set X'
$\mathrm{I}_X\colon X \longrightarrow X$	identity function of a set X
$\circ$	composition of functions
$\mathbb{0}$	empty set
$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integers
$[n]$	the set $\{0, \ldots, n-1\}$ of natural numbers less than $n \in \mathbb{N}$
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_{\geq 0}$	positive real numbers
$\mathbb{R}_{\leq 0}$	negative real numbers
$\mathbb{R}_{>0}$	strictly positive real numbers
$\mathbb{R}_{<0}$	strictly negative real numbers
$\mathbb{S}^d$	d-dimensional sphere
$\lvert\cdot\rvert$	absolute value
$\leq$	standard order

Symbol	**Definition**
$\sup$	supremum
$\inf$	infimum
$(x_n)_{n\in\mathbb{N}}$	sequence
$\lim_{n\to\infty} x_n$	limit of a sequence $(x_n)_{n\in\mathbb{N}}$
$\overline{\lim}_{n\to\infty} x_n$	limit superior of a sequence $(x_n)_{n\in\mathbb{N}}$
$\underline{\lim}_{n\to\infty} x_n$	limit inferior of a sequence $(x_n)_{n\in\mathbb{N}}$
e^x	exponential of a real number x
$\log x$	logarithm of a real number x
$\mathcal{T}$	topology
S°	interior of a set S
$S^\bullet$	closure of a set S
d	distance function of a metric space
$\Theta(S)$	diameter of a set S on a metric space
$\mathrm{B}_d(c;r)$	open ball of center c and radius r of the distance d
$\mathrm{B}^\bullet_d(c;r)$	closed ball of center c and radius r of the distance d
$\langle\!\langle f \rangle\!\rangle$	distortion of a Lipschitz continuous transformation f
$\mathcal{C}$	cover of a set
$\mathcal{M}$	σ-algebra of measurable sets
m	measure on a σ-algebra
$f_\vdash(\mathcal{M})$	pushforward σ-algebra
$f_\vdash(m)$	pushforward measure
$\mathring{\subseteq}$	almost superset relation of measurable sets
$\mathring{=}$	almost equality of measurable sets
$\int_M fm$	integral of a function f on a measure space $(M, \mathcal{M}, m)$
$\int_{x\in M} f(x)m(\mathrm{d}x)$	integral with variable x explicitly denoted
$\mathcal{P}$	partition of a set
$\pi_\mathcal{P}\colon X \longrightarrow \mathcal{P}$	natural projection of a partition $\mathcal{P}$ of a set X
$\wp_1(X), \epsilon$	atomic partition of a set X
$\mathcal{P}\vert_S$	induced partition on a subset S
$f^\dashv(\mathcal{P}')$	pullback partition
$\preceq$	refinement order of partitions
$\curlyvee, \bigcurlyvee$	correfinement of partitions
$\mathcal{M}_\mathcal{P}$	quotient σ-algebra on a partition $\mathcal{P}$
$m_\mathcal{P}$	quotient measure on partition $\mathcal{P}$
$(m_P)_{P\in\mathcal{P}}$	disintegration of a measure m with respect to a partition $\mathcal{P}$
$\Phi^{tV}(p)$	flow of a vector field V, at a point p after time t

Entropy

Symbol	Definition
$\mathbb{\Delta}^d$	d-dimensional unit simplex
u_d	uniform distribution of $\mathbb{\Delta}^d$
In	information function
$\bigoplus_{i\in[n]} p_i q^i$	p-weighted sum of distributions $q^i \in \mathbb{\Delta}^{k_i-1}$
$\mathrm{H}_{\mathbb{\Delta}^n}(p)$	(Shannon) entropy of a distribution $p \in \mathbb{\Delta}^n$
$\mathrm{In}_{\mathcal{P}}$	information of a partition $\mathcal{P}$
$\mathrm{H}_m(\mathcal{P})$	measure entropy of a partition $\mathcal{P}$
$\mathrm{In}_{\mathcal{P}\mid\mathcal{P}'}$	relative information of a partition $\mathcal{P}$ with respect to $\mathcal{P}'$
$\mathrm{H}_m(\mathcal{P} \mid \mathcal{P}')$	conditional measure entropy of a partition $\mathcal{P}$ with respect to $\mathcal{P}'$
$\mathcal{P}_f^n$	n-th f-dynamical correfinement of a partition $\mathcal{P}$
$\mathcal{P}_f^{\pm n}$	n-th bilateral f-dynamical correfinement of a partition $\mathcal{P}$
$\mathrm{h}_m(f,\mathcal{P})$	measure entropy of dynamics f with respect to a partition $\mathcal{P}$
$\mathrm{h}_m(f)$	measure entropy of dynamics f
$\mathcal{F}(f)$	folding entropy of dynamics f
d_f^n	dynamical distance of order n relative to dynamics f and distance d
$\mathrm{B}_f^n(c;r)$	dynamical ball of order n, center c and radius r relative to f (and d)
$\bar{\mathrm{H}}_d^n(f,K,\varepsilon)$	minimal cardinality of an (n,ε)-generator of a compact set K
$\mathrm{h}_d(f,K,\varepsilon)$	ε-imprecise topological entropy of dynamics f with respect to K
$\mathrm{h}_d(f,K)$	topological entropy of f with respect to K
$\mathrm{h}_d(f)$	topological entropy of f

Symbolic dynamics

Symbol	Definition
Σ_n^+	unilateral n-symbolic space
Σ_n	bilateral n-symbolic space
$\sigma\colon \Sigma_n \longrightarrow \Sigma_n$	(left) shift on Σ_n^+ or Σ_n
$C_{i_0,\dots,i_{n-1}}^{s_0,\dots,s_{n-1}}$	basic cylinder of indices i_j and symbols s_j
Σ_{n_-,n_+}	(n_-,n_+)-bisymbolic space
$\phi\colon [n_+] \longrightarrow [n_-]$	transition function
$\sigma_\phi\colon \Sigma_{n_-,n_+} \longrightarrow \Sigma_{n_-,n_+}$	bisymbolic shift with transition function ϕ
$C_i^{\phi^{-1}(s)}$	extended basic cylinder of index i and symbol s
p^+	probability distribution on $[n_+]$
p^-	probability distribution on $[n_-]$ pushforward of p^+ by ϕ

Symbol	Definition
m_{p^+}	probability measure on Σ_{n_-,n_+} induced by p^+
q^{s^-}	quotient distribution on $\phi^{-1}(s^-)$ induced by p^+

Piecewise smooth dynamics

Symbol	Definition
M_i	regular region of a switch-partitioned manifold M
$\Sigma_{i,i'}, \Sigma$	switching region of M
$h_{i,i'}$	switching function of $\Sigma_{i,i'}$
F_i	regular vector field on the regular region M_i
$\Sigma^{\mathrm{c}}_{i,i'}, \Sigma^{\mathrm{c}}$	crossing region of M
$\Sigma^{i\mathrm{c}}_{i,i'}, \Sigma^{i\mathrm{c}}$	i-ward crossing region
$\Sigma^{i'\mathrm{c}}_{i,i'}, \Sigma^{i'\mathrm{c}}$	i'-ward crossing region
$\Sigma^{\mathrm{t}}_{i,i'}, \Sigma^{\mathrm{t}}$	tangent region of M
$\Sigma^{\mathrm{s}}_{i,i'}, \Sigma^{\mathrm{s}}$	sliding region of M
$\Sigma^{\mathrm{us}}_{i,i'}, \Sigma^{\mathrm{us}}$	unstable sliding (or escaping) region of M
$\Sigma^{\mathrm{ss}}_{i,i'}, \Sigma^{\mathrm{ss}}$	stable sliding (or accessing) region of M
$F^{\mathrm{s}}_{i,i'}, F^{\mathrm{s}}$	sliding vector field
Φ^t_i	flow of the regular vector field F_i
Φ^t_{s}	flow of the sliding vector field F^{s}
$\gamma\gamma'$	concatenation of two trajectories γ and γ'
e_j	j-th escaping point of an orbit of a piecewise smooth vector field F
a_j	j-th accessing point of an orbit of a piecewise smooth vector field F
$\Vert F\Vert$	supremum norm of a piecewise smooth vector field F
$\tilde{M}$	orbit space of the piecewise smooth system (M, F)
$\tilde{\Phi}$	flow on the orbit space $\tilde{M}$ induced by F
$\tilde{d}$	integral distance on the orbit space $\tilde{M}$
$\tilde{d}_{\mathrm{sup}}$	supremum distance on the orbit space $\tilde{M}$
$\mathrm{h}(F)$	topological entropy of a piecewise smooth vector field F

Contents

Chapter 1

Introduction

In this chapter, we introduce the structure of this thesis and basic preliminary definitions and propositions, as well as fix notation, that will be used throughout this work. The style is intentionally terse, as this is intended as more of a reference to definitions rather than a detailed exposition. Set theory can be found in SECTION 1.2, topology in SECTION 1.3, measure theory in SECTION 1.4 and dynamics in SECTION 1.5.

1.1 Structure of the thesis

This thesis in divided into 4 chapters. CHAPTER 1 is this introductory chapter, listing basic preliminary notions relevant for the following chapters.

CHAPTER 2 presents a brief history of entropy in physics and mathematics, and some basic definitions and propositions related to the measure-theoretic and topological entropies.

In CHAPTER 3, we present basic notation on symbolic dynamics, define the extended symbolic shift dynamics and compute the folding and measure entropies of this extended shift.

In CHAPTER 4, we present basic definitions of piecewise smooth dynamics, develop the orbit spaces theory of this dynamics, and later use it to define and compute the topological entropy of a class of piecewise smooth systems.

Comment on typography Internal references to sections, pages and notes, definitions and propositions, and any variant of these (but not bibliographical references), as well as external references, are typeset using SMALL CAPITALS and TEXT FIGURES (0123456789) to create emphasis. On the digital version of this document, these references are links.

1.2 Sets

We will use $\wedge$ for *logical conjunction* and $\vee$ for *logical disjunction*, $\bigwedge$ for the *universal quantifier* and $\bigvee$ for the *existential quantifier*. We shall avoid these as much as possible, using normal English language words on text (e.g. 'and', 'or', 'for all', 'for some'), and the logic symbols only for set-builder notation. We state *definitions* by equality in formulas with the symbol $:=$ to indicate that the left side of the expression (relative to the symbol) is to be defined by the right side of the expression.

We denote the *subset* partial order by $\subseteq$, so that $X \subseteq X'$ means that every element of X is an element of X', and $\subset$ is the *strict subset* relation.

Let X be a set. We denote its *cardinality* by $\#X$ and its *power set* (the set of all its subsets) by $\wp(X)$. Let S and S' be subsets of X. We denote their *union* by $S \cup S'$, their intersection by $S \cap S'$, and their *relative complement* by $S \smallsetminus S'$. The *complement* of S is $\overline{S} := X \smallsetminus S$, the *symmetric difference* of sets S, S' by $S \triangle S' := (S \smallsetminus S') \cup (S' \smallsetminus S)$. We denote the *identity function* on X by I_X (or I, when there is no ambiguity).

We will denote the *empty set* by $\mathbb{0}$, the set of *natural numbers* by $\mathbb{N} = \{0, 1, 2, \dots, \}$, the set of *integers* by $\mathbb{Z}$, the set of positive integers less than $n \in \mathbb{N}$ by $[n] = \{0, \dots, n-1\}$, the set of *rational numbers* by $\mathbb{Q}$ and the set of *real numbers* by $\mathbb{R}$. The sets of *positive*, *negative*, *strictly positive* and *strictly negative* real numbers are denoted by $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{\leq 0}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$, respectively (and analogous notation is used for the other number sets). The d-dimensional sphere is denoted $\mathbb{S}^d$.

We denote the *absolute value* on $\mathbb{R}$ by $|\cdot|$, the *standard order* on $\mathbb{R}$ by $\leq$, the *supremum* and *infimum* relative to it by sup and inf, respectively. The *limit* of a sequence $(x_n)_{n \in \mathbb{N}}$ is denoted by $\lim_{n\to\infty} x_n$, its *limit superior* by $\overline{\lim}_{n\to\infty} x_n$ and its *limit inferior* by $\underline{\lim}_{n\to\infty} x_n$. We denote the *exponential* function by $\mathrm{e}^{(\cdot)}$ and the *logarithm* function by log.

1.3 Topological spaces and metric spaces

1.3.1 Basic definitions for topological spaces

Definition 1.1. Let X be a set. A *topology* on X is a set $\mathcal{T} \subseteq \wp(X)$ of subsets of X that satisfies

1. $\mathbb{0} \in \mathcal{T}$ and $X \in \mathcal{T}$;
2. For every family $(A_i)_{i \in I}$ of sets $A_i \in \mathcal{T}$, $\bigcup_{i \in I} A_i \in \mathcal{T}$;
3. For every finite sequence $(A_i)_{i \in [n]}$ of sets $A_i \in \mathcal{T}$, $\bigcap_{i \in I} A_i \in \mathcal{T}$.

The sets $A \in \mathcal{T}$ are the *open sets* of $\mathcal{T}$. A pair $(X, \mathcal{T})$ is a *topological space*. A *closed* set is a set $S \subseteq X$ whose complement $\overline{C}$ is open.

Example 1.1. *Let X be a set.*

1. *The set* $\{\mathbb{0}, X\}$ *is the* trivial topology *on* X*;*
2. *The set* $\wp(X)$ *is the* discrete topology *on* X*.*

Definition 1.2. Let $\boldsymbol{X} = (X, \mathfrak{T})$ and $\boldsymbol{X'} = (X', \mathfrak{T}')$ be topological spaces. A *continuous function* from $\boldsymbol{X}$ to $\boldsymbol{X'}$ is a function $f \colon X \longrightarrow X'$ such that, for every open set $A' \in \mathfrak{T}'$, its inverse image by f is open: $f^{-1}(A') \in \mathfrak{T}$.

A *homeomorphism* from $\boldsymbol{X}$ to $\boldsymbol{X'}$ is a continuous function from $\boldsymbol{X}$ to $\boldsymbol{X'}$ whose inverse is continuous.

Special subsets of topological spaces

Definition 1.3. Let $\boldsymbol{X} = (X, \mathfrak{T})$ be a topological space and $S \subseteq X$. The *interior* of S is the set

$$S^\circ := \bigcup \{A \subseteq S \mid A \in \mathfrak{T}\}.$$

The *closure* of S is the set

$$S^\bullet := \bigcap \{C \subseteq S \mid \overline{C} \in \mathfrak{T}\}.$$

Definition 1.4. Let $\boldsymbol{X} = (X, \mathfrak{T})$ be a topological space. A *dense* set is a set $D \subseteq X$ whose closure is the whole space: $D^\bullet = X$. A *residual* set is a set that is a countable intersection of sets whose interior is dense. A *meager* set is a set that is a countable union of sets whose closure has empty interior.

A residual set is the complement of a meager set. Residual sets represent a way to express the idea of "almost all" using only topology, since meager sets form a class of negligible sets in a topological space.

Definition 1.5. Let $\boldsymbol{X} = (X, \mathfrak{T})$ be a topological space. An *isolated point* is a point $x \in X$ such that $\{x\}$ is open. A *perfect* set is a set $S \subseteq X$ that has no isolated points.

1.3.2 Basic definitions for metric spaces

Distance, diameter and balls

Definition 1.6. Let M be a set. A *distance function* (or *metric*) on M is a function $d \colon M \times M \longrightarrow \mathbb{R}_{\geq 0}$ such that

1. For every $p, p' \in M$, $d(p, p') = 0$ if, and only if, $p = p'$;
2. For every $p, p' \in M$, $d(p, p') = d(p', p)$;
3. For every $p, p', p'' \in M$, $d(p, p'') \leq d(p, p') + d(p', p'')$.

A pair (M, d) is a *metric space.*

Definition 1.7. Let $\boldsymbol{M} = (M, d)$ a metric space and $S \subseteq M$ a subset. The *diameter* of S is

$$\Theta(S) := \sup\{d(p, p') \mid p, p' \in S\}.$$

Topology, continuity and uniform continuity

Definition 1.8. Let $\boldsymbol{M} = (M, d)$ a metric space, $c \in M$ a point and $r \in \mathbb{R}_{>0}$ a strictly positive real number. The *open ball* of center c and radius r of $\boldsymbol{M}$ is the set

$$\mathrm{B}_d(c; r) := \{p \in M \mid d(c, p) < r\}.$$

The *closed ball* of center c and radius r of $\boldsymbol{M}$ is the set

$$\mathrm{B}^{\bullet}_d(c; r) := \{p \in M \mid d(c, p) \leq r\}.$$

A metric space has a natural topological space structure induced by its distance function.

Definition 1.9. Let $\boldsymbol{M} = (M, d)$ a metric space. An *open set* of $\boldsymbol{M}$ is a set $A \subseteq M$ such that, for every $p \in A$ there is a $r \in \mathbb{R}_{>0}$ such that $\mathrm{B}(p; r) \subseteq A$. The *topology* of $\boldsymbol{M}$ is the set of all its open sets.

It can be shown that a function between metric spaces is continuous with respect to this topological structure if, and only if, it is continuous in the standard ε–δ formulation of real analysis.

Proposition 1.1. *Let $\boldsymbol{M} = (M, d)$ and $\boldsymbol{M'} = (M', d')$ be metric spaces and $f\colon M \longrightarrow M'$ a function. Then f is continuous if, and only if, for every $p \in M$ and every* error *$\varepsilon \in \mathbb{R}_{>0}$, there exists a* difference *$\delta \in \mathbb{R}_{>0}$ such that, for every $q \in M$, the condition $d(p, q) < \delta$ implies the condition $d'(f(p), f(q)) < \varepsilon$.*

In PROPOSITION 1.1, the values of the errors ε and differences δ depend on each point $p \in M$. If we can find these values independently of the point, we have a stronger notion of continuity.

Definition 1.10. Let $\boldsymbol{M} = (M, d)$ and $\boldsymbol{M'} = (M', d')$ be metric spaces. A *uniformly continuous* function from $\boldsymbol{M}$ to $\boldsymbol{M'}$ is a function $f\colon M \longrightarrow M'$ that satisfies: for every *error* $\varepsilon \in \mathbb{R}_{>0}$, there exists a *difference* $\delta \in \mathbb{R}_{>0}$ such that, for every $p, q \in M$, the condition $d(p, q) < \delta$ implies the condition $d'(f(p), f(q)) < \varepsilon$.

A *uniform homeomorphism* from $\boldsymbol{M}$ to $\boldsymbol{M'}$ is a uniformly continuous function from $\boldsymbol{M}$ to $\boldsymbol{M'}$ whose inverse is uniformly continuous.

Every uniformly continuous function is continuous. Another important type of continuity in metric spaces is when the variation of the function is bounded by a number in the following sense.

Definition 1.11. Let $\boldsymbol{M} = (M, d)$ and $\boldsymbol{M'} = (M', d')$ be metric spaces. A *Lipschitz continuous* function from $\boldsymbol{M}$ to $\boldsymbol{M'}$ is a function $f\colon M \longrightarrow M'$ for which there exists a number $c \in \mathbb{R}_{\geq 0}$ such that, for every $p, q \in M$,

$$d'(f(p), f(q)) \leq c\, d(p, q).$$

The infimum of all such c is the *Lipschitz distortion* of f, denoted $\langle\!\langle f \rangle\!\rangle$.

A *Lipschitz homeomorphism* from $\boldsymbol{M}$ to $\boldsymbol{M'}$ is a Lipschitz continuous function from $\boldsymbol{M}$ to $\boldsymbol{M'}$ whose inverse is Lipschitz continuous.

Equivalence of distance functions

Definition 1.12. Let M be a set and d and d' distance functions on M.

1. The distance functions d and d' are *topologically equivalent* when the identity $\mathrm{I}\colon (M, d) \longrightarrow (M, d')$ is a homeomorphism.
2. The distance functions d and d' are *uniformly equivalent* when the identity $\mathrm{I}\colon (M, d) \longrightarrow (M, d')$ is a uniform homeomorphism.
3. The distance functions d and d' are *Lipschitz equivalent* when the identity $\mathrm{I}\colon (M, d) \longrightarrow (M, d')$ is a Lipschitz homeomorphism.

It can be shown that all these are equivalence relations. Lipschitz equivalence implies uniform equivalence, and uniform equivalence implies topological equivalence, but neither of the reverse implications hold.

1.3.3 Norms, inner products, metrics, and distances on manifolds

We shall generally denote *norms* by $\|\cdot\|$ and *inner products* by $\langle\cdot\,,\cdot\rangle$. The class of *r-differentiable functions* (for $r \geq 0$) between normed vector spaces or manifolds will be denoted by $\mathscr{C}^r$. We denote the tangent space of a manifold $\boldsymbol{M}$ by $\mathrm{T}\boldsymbol{M}$ and the derivative of a trajectory γ on a manifold by $\dot{\gamma}$. Given a Riemannian manifold $\boldsymbol{M}$, we denote its *metric* at a point $p \in M$ by $\langle\cdot\,,\cdot\rangle_p$ and its *norm* at p by $\|\cdot\|_p$. With the metric we can define the *length* of a piecewise continuously differentiable trajectory $\gamma\colon [a, b] \longrightarrow M$ by

$$\ell(\gamma) := \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)}\, \mathrm{d}t$$

and hence the metric induces a distance function on M by considering, for every pair of points $p, p' \in M$, the set $D_{p,p'}$ of all piecewise continuously differentiable trajectories from p to p', and then setting

$$d(p, p') := \inf_{\gamma \in D_{p,p'}} \ell(\gamma).$$

1.3.4 Covers

Cover and subcovers

Definition 1.13 (Cover and subcover)**.** Let X be a set. A *cover* of X is a set $\mathcal{C} \subseteq \wp(X)$ such that $X = \bigcup_{C \in \mathcal{C}} C$. A *subcover* of a cover $\mathcal{C}$ is a cover $\mathcal{C}'$ of X such that $\mathcal{C}' \subseteq \mathcal{C}$. A *cover of a subset* $S \subseteq X$ is a set $\mathcal{C} \subseteq \wp(X)$ such that $S \subseteq \bigcup_{C \in \mathcal{C}} C$.

Example 1.2. *Let X be a set.*

1. *The set $\{X\}$ is the* trivial cover *of X;*
2. *The set $\wp(X)$ is the* total cover *of X;*
3. *The set*
$$\wp_1(X) = \{C \in \wp(X) \mid \#C = 1\} = \{\{x\} \mid x \in X\}$$
is the atomic cover *of X.*

Definition 1.14 (Open cover)**.** Let $\boldsymbol{X} = (X, \mathcal{T})$ be a topological space. An *open cover* of $\boldsymbol{X}$ is a cover $\mathcal{C}$ of X such that every $C \in \mathcal{C}$ is open ($C \in \mathcal{T}$).

The subcovers of an open cover are always open.

Example 1.3. *Let $\boldsymbol{X} = (X, \mathcal{T})$ be a topological space. The sets $\{X\}$ and $\mathcal{T}$ are open covers of $\boldsymbol{X}$.*

Definition 1.15. Let $\boldsymbol{X} = (X, \mathcal{T})$ be a topological space. A *compact* set of $\boldsymbol{X}$ is a set $K \subseteq M$ such that, for every open cover $\mathcal{C}$ of K, there exists a finite subcover $\mathcal{C}' \subseteq \mathcal{C}$ of K.

1.4 Measure spaces

1.4.1 Basic notation and definitions

Measurable spaces and measure spaces

Definition 1.16. Let X be a set. A *σ-algebra* on X is a set $\mathcal{M} \subseteq \wp(X)$ of subsets of X that satisfies

1. $\mathbb{0} \in \mathcal{M}$;

2. For every $M \in \mathcal{M}$, $\overline{M} \in \mathcal{M}$;
3. For every sequence $(M_i)_{i\in\mathbb{N}}$ of sets in $\mathcal{M}$, $\bigcup_{i\in\mathbb{N}} M_i \in \mathcal{M}$.

The elements of $\mathcal{M}$ are *measurable* sets. A pair $(X, \mathcal{M})$ is a *measurable space*.

Definition 1.17. Let $\boldsymbol{X} = (X, \mathcal{M})$ and $\boldsymbol{X'} = (X', \mathcal{M}')$ be measurable spaces. A *measurable function* from $\boldsymbol{X}$ to $\boldsymbol{X'}$ is a function $f : X \longrightarrow X'$ such that, for every measurable set $M' \in \mathcal{M}'$, its inverse by f is measurable: $f^{-1}(M') \in \mathcal{M}$.

Definition 1.18. Let $\boldsymbol{X} = (X, \mathcal{M})$ be a measurable space. A *measure* on $\boldsymbol{X}$ is a function $m : \mathcal{M} \longrightarrow \mathbb{R}_{\geq 0}$ such that

1. $m(\mathbb{0}) = 0$;
2. For every pairwise disjoint sequence $(M_i)_{i\in\mathbb{N}}$ of measurable sets, $m(\bigcup_{i\in\mathbb{N}} M_i) = \sum_{i\in\mathbb{N}} m(M_i)$.

The triple $(X, \mathcal{M}, m)$ is a *measure space*.

A *probability measure* is a measure m on $\boldsymbol{X}$ such that $m(X) = 1$ and a *probability space* is a measure space whose measure is a probability measure.

Definition 1.19. Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ and $\boldsymbol{X'} = (X', \mathcal{M}, m)$ be measure spaces. A *measure-preserving function* from $\boldsymbol{X}$ to $\boldsymbol{X'}$ is a measurable function $f : X \longrightarrow X'$ such that, for every measurable set $M' \in \mathcal{M}'$, $m(f^{-1}(M')) = m'(M')$.

Induced σ-algebras and measures

Definition 1.20. Let $\boldsymbol{X} = (X, \mathcal{M})$ be a measurable space, X' a set and $f : X \longrightarrow X'$ a function. The *σ-algebra pushed-forward by f on X'* is

$$f_{\vdash}(\mathcal{M}) := \{M' \subseteq X' \mid f^{-1}(M') \in \mathcal{M}\}.$$

It is easy to show that $f_{\vdash}(\mathcal{M})$ is in fact a σ-algebra and that $f : X \longrightarrow X'$ is measurable with respect to it.

Definition 1.21. Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a measure space, $(X', \mathcal{M}')$ a measurable space and $f : X \longrightarrow X'$ a measurable function. The *measure pushed-forward by f on $(X', \mathcal{M}')$* is the function

$$\begin{aligned} f_{\vdash}m : \mathcal{M}' &\longrightarrow \mathbb{R} \\ M' &\longmapsto f_{\vdash}m(M') := m(f^{-1}(M')). \end{aligned}$$

It is also easy to show that $f_{\vdash}m$ is a measure on $(X', \mathcal{M}')$ and that $f : X \longrightarrow X'$ is measure-preserving with respect to it.

Almost equality of sets

When we have a measure space $\boldsymbol{X} = (X, \mathcal{M}, m)$ we can define what it means for two measurable sets $M, M' \in \mathcal{M}$ to be almost equal: the elements they do not have in common have measure zero. The elements they do not have in common are expressed mathematically as the symmetric difference of the two sets, given by

$$M \triangle M' = (M \smallsetminus M') \cup (M' \smallsetminus M).$$

In the same vein, we can say a set is almost contained in another, and use this to define almost equality.

Definition 1.22. Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a measure space and $M \in \mathcal{M}$ a measurable set. An *almost subset* of M is a measurable set $M' \in \mathcal{M}$ such that $m(M \smallsetminus M') = 0$. This is denoted by $M' \mathrel{\mathring{\subseteq}} M$, and M is an *almost superset* of M'. Two *almost equal* sets are two measurable sets $M, M' \in \mathcal{M}$ that satisfy $M' \mathrel{\mathring{\subseteq}} M$ and $M \mathrel{\mathring{\subseteq}} M'$ (that is, $m(M \triangle M') = 0$). This is denoted by $M \mathrel{\mathring{=}} M'$.

The almost equality relation $\mathring{=}$ is an equivalence relation on $\mathcal{M}$, and the almost subset relation $\mathring{\subseteq}$ becomes a partial order when we take the quotient of $\mathcal{M}$ under this equivalence relation.

Integration

Let $(X, \mathcal{M}, m)$ be a measure space. We will denote the *integral* of an integrable function $f\colon X \longrightarrow \mathbb{R}$ over a measurable set $M \in \mathcal{M}$ with respect to m by

$$\int_M fm,$$

or, if we must make the variable x of f explicit,

$$\int_{x \in M} f(x) m(\mathrm{d}x).$$

This notation makes it more natural to transition between integration and summation.

1.4.2 Partitions

Partitions and the natural projection

A partition is a cover of X by disjoint sets that does not contain the empty set.

Definition 1.23. Let X be a set. A *partition* of X is a set $\mathcal{P} \subseteq \wp(X)$ such that

1. (Cover) $X = \bigcup_{P \in \mathcal{P}} P$;
2. (Non-emptiness) $\mathbb{0} \notin \mathcal{P}$;
3. (Non-coincidence) For every $P, P' \in \mathcal{P}$, if $P \neq P'$, then $P \cap P' = \mathbb{0}$.

The sets $P \in \mathcal{P}$ are the *parts* of the partition $\mathcal{P}$. The set of the partitions of X is denoted $\mathfrak{P}(X)$.

Example 1.4. *Let X be a set.*

1. *The set $\{X\}$ is the* trivial partition *of X.*
2. *The set*
$$\wp_1(X) = \{C \in \wp(X) \mid \#C = 1\} = \{\{x\} \mid x \in X\}$$
is the atomic partition *of X.*

As a consequence of the definition of partitions, every point of the partitioned set belongs to a unique part of the partition. This defines a unique surjective function onto the partition.

Definition 1.24 (Natural projection of a partition)**.** Let X be a set and $\mathcal{P}$ a *partition.* The *natural projection* of $\mathcal{P}$ is the unique function

$$\pi_{\mathcal{P}} \colon X \longrightarrow \mathcal{P}$$

such that, for every $x \in X$, $x \in \pi_{\mathcal{P}}(x)$.

Induced and pulled-back partitions

We define here some partitions that are formed from partitions we already have.

Definition 1.25 (Partition induced on a subset)**.** Let X be a set, $\mathcal{P}$ a partition of X and $S \subseteq X$ a subset. The *partition of S induced by* $\mathcal{P}$ is

$$\mathcal{P}|_S := \{P \cap S \mid P \in \mathcal{P}\}.$$

Definition 1.26. Let X and X' be sets, $f \colon X \longrightarrow X'$ a function and $\mathcal{P}'$ a partition of X'. The *partition of X pulled-back by f* is

$$f^{\dashv}(\mathcal{P}') := \{f^{-1}(P') \mid P' \in \mathcal{P}'\}.$$

Is is easy to show these are in fact partitions.

Proposition 1.2. *Let X, X' and X'' be sets, $f \colon X \longrightarrow X'$ and $f' \colon X' \longrightarrow X''$ functions and $\mathcal{P}''$ a partition of X''. Then*

$$(f' \circ f)^{\dashv}(\mathcal{P}'') = f^{\dashv}(f'^{\dashv}(\mathcal{P}'')).$$

Refinements and correfinements

We define a partial order on the set of all partitions of a set X.

Definition 1.27 (Refinement of partition)**.** Let X be a set and $\mathcal{P}$ a partition of X. A *refinement* of $\mathcal{P}$ is a partition $\mathcal{P}'$ of X that satisfies: for every part $P' \in \mathcal{P}'$, there is a part $P \in \mathcal{P}$ such that $P' \subseteq P$. The partition $\mathcal{P}'$ is said to be *finer* than $\mathcal{P}$, and $\mathcal{P}$ is said to be *coarser* than $\mathcal{P}'$. We denote $\mathcal{P} \preceq \mathcal{P}'$.

Proposition 1.3. *Let X be a set. The refinement relation is a partial order relation on the set of partitions $\mathfrak{P}(X)$.*

Proof. Reflexivity and transitivity are immediate. We will show antisymmetry. Let $\mathcal{P}, \mathcal{P}' \in \mathfrak{P}(X)$ be partitions such that $\mathcal{P} \preceq \mathcal{P}'$ and $\mathcal{P}' \preceq \mathcal{P}$. Take $P \in \mathcal{P}$. Since $\mathcal{P} \preceq \mathcal{P}'$, there is a $P' \in \mathcal{P}'$ such that $P' \subseteq P$, and, since $\mathcal{P}' \preceq \mathcal{P}$, there is $Q \in \mathcal{P}$ such that $Q \subseteq P'$. This implies that $Q \subseteq P$; since $\mathcal{P}$ is a partition, then $Q \nparallel \mathbb{0}$, hence $Q = P$. Thus $P = P' \in \mathcal{P}'$, so $\mathcal{P} \subseteq \mathcal{P}'$. By changing $\mathcal{P}$ and $\mathcal{P}'$ the opposite containment follows, so $\mathcal{P} = \mathcal{P}'$. ∎

Using this order, we can define joins and meets. We are only interested on joins, which we define as follows.

Definition 1.28. Let X be a set and $(\mathcal{P}_i)_{i \in I}$ a family of partitions of X. The *correfinement* of $(\mathcal{P})_{i \in I}$ is the smallest partition (relative to the refinement order $\preceq$) that is a finer than every partition $\mathcal{P}_i$. We denote it by $\bigcurlyvee_{i \in I} \mathcal{P}_i$. For a finite family of partitions $\mathcal{P}_0, \dots, \mathcal{P}_{n-1}$, we denote

$$\mathcal{P}_0 \curlyvee \cdots \curlyvee \mathcal{P}_{n-1}.$$

The correfinement can be shown to satisfy the following proposition.

Proposition 1.4. *Let X be a set and $(\mathcal{P}_i)_{i \in I}$ a family of partitions of X. The correfinement of $(\mathcal{P})_{i \in I}$ is the set*

$$\bigcurlyvee_{i \in I} \mathcal{P}_i := \{\bigcap_{i \in I} P_i \mid \bigwedge_{i \in I} \; P_i \in \mathcal{P}_i\} \smallsetminus \{\mathbb{0}\}.$$

Measurable partitions

In this section we define measurable partitions and countably generated measurable partitions. The first are used to define entropy, while the second are needed for some results related to disintegration of measures and thus to the folding entropy.

We start by defining a partition of a measure space. We call these partitions *measurable partitions* because they are simply a partition of the space in measurable sets, but in the literature they are generally called just *partitions* (see, for instance, [Rok67, Section 1.3, p. 3; Wal00, Section 4.1, Definition 4.1, p. 75; VO16, Section 5.1.1, p. 143] for the definitions, with slight variations). We require that the sets, besides being measurable, satisfy the properties of a generic partition of sets: cover the space, be non-empty and have empty intersection.

Definition 1.29. Let $\boldsymbol{X} = (X, \mathcal{M})$ be a measurable space. A *measurable partition* of $\boldsymbol{X}$ is a partition $\mathcal{P}$ of X such that every part $P \in \mathcal{P}$ is a measurable set.

Notice that the expression *measurable partitions* is usually used for what we will call *countably generated measure partitions* (DEFINITION 1.31): a measurable partition that is, in a technical sense, generated by a sequence of the partitions of DEFINITION 1.29 (see [VO16, Section 9.1.2, p. 245; Rok67, Section 1.3, p. 4] for precise definitions). These concepts are related, as we shall see ahead, but are not the same.

A classification of correfinements of measurable partitions, analogous to PROPOSITION 1.4, only works for countable families of partitions, since the σ-algebras are only closed under countable intersections. Because of this, we generally restrict ourselves to countable (of even finite) measurable partitions. The countably generated measurable partitions of DEFINITION 1.31 give a way to work with uncountable measurable partitions using countable ones.

When we have a measure space, we can identify different measurable partitions if their parts differ only on measure zero sets. The following definition formalizes this.

Definition 1.30. Let $\boldsymbol{X}$ be a measure space and $\mathcal{P}$ a measurable partition of $\boldsymbol{X}$. An *almost-refinement* of $\mathcal{P}$ is an measurable partition $\mathcal{P}'$ of $\boldsymbol{X}$ that satisfies: for every part $P' \in \mathcal{P}'$ there exists a part $P \in \mathcal{P}$ such that $P' \mathring{\subseteq} P$. The partition $\mathcal{P}'$ is said to be *almost-finer* than $\mathcal{P}$, and $\mathcal{P}$ is said to be *almost-coarser* than $\mathcal{P}'$. We denote $\mathcal{P} \mathring{\preceq} \mathcal{P}'$. The measurable partitions $\mathcal{P}$ and $\mathcal{P}'$ are *almost equal* when $\mathcal{P} \mathring{\preceq} \mathcal{P}'$ and $\mathcal{P}' \mathring{\preceq} \mathcal{P}$. We denote $\mathcal{P} \mathring{=} \mathcal{P}'$.

In the following definition, an *increasing* sequence of measurable partitions $(\mathcal{P}_n)_{n\in\mathbb{N}}$ is increasing with respect to the partition refinement order: $\mathcal{P}_0 \mathring{\preceq} \mathcal{P}_1 \mathring{\preceq} \cdots$.

Definition 1.31. Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a measure space. A *countably generated measurable partition* of $\boldsymbol{X}$ is a measurable partition $\mathcal{P}$ of $(X, \mathcal{M})$ for which there exists an increasing sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of countable measurable partitions of $(X, \mathcal{M})$ such that

$$\mathcal{P} \mathring{=} \bigvee_{n\in\mathbb{N}} \mathcal{P}_n.$$

These partitions are called *measurable partitions* in the literature [VO16, Section 9.1.2, p. 245; Rok67, Section 1.3, p. 4]. In [Rok67], they are defined using a *basis*, and the following proposition shows that our definition is equivalent to that in terms of a basis. The sets $(M_n)_{n\in\mathbb{N}}$ in PROPOSITION 1.5 are such a basis.

Proposition 1.5. *Let $\boldsymbol{X} = (M, \mathcal{M}, m)$ be a measure space. A m-measure partition $\mathcal{P}$ (of $\boldsymbol{X}$) is countably generated if, and only if, there is a sequence of measurable sets $(M_n)_{n\in\mathbb{N}}$ such that*

$$\mathcal{P} \mathrel{\dot{=}} \bigvee_{n\in\mathbb{N}} \{M_n, \overline{M_n}\}.$$

1.4.3 Disintegration of measure

We begin by defining a measure space structure on a partition of a measure space. In this section we will only consider probability spaces.

Definition 1.32. Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space and $\mathcal{P}$ a measurable partition. The *quotient σ-algebra* of $\mathcal{P}$ is the σ-algebra $\mathcal{M}_{\mathcal{P}} := \pi_{\mathcal{P}\vdash}\mathcal{M}$ pushed-forward by the natural projection $\pi_{\mathcal{P}} \colon X \longrightarrow \mathcal{P}$. The *quotient measure* on $(\mathcal{P}, \mathcal{M}_{\mathcal{P}})$ is the measure $m_{\mathcal{P}} := (\pi_{\mathcal{P}})_{\vdash} m$ pushed-forward by the natural projection.

By this definition, the measurable sets $\mathcal{Q} \in \mathcal{M}_{\mathcal{P}}$ are those $\mathcal{Q} \subseteq \mathcal{P}$ such that $\pi_{\mathcal{P}}^{-1}(\mathcal{Q}) \in \mathcal{M}$, and its quotient measure is

$$m_{\mathcal{P}}(\mathcal{Q}) = m(\pi_{\mathcal{P}}^{-1}(\mathcal{Q})).$$

Definition 1.33 (Disintegration of measure)**.** Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space and $\mathcal{P}$ a countably generated measure partition of $\boldsymbol{X}$. A *disintegration* of m with respect to $\mathcal{P}$ is a family of probability measures $(m_P)_{P\in\mathcal{P}}$ such that

1. For almost every $P \in \mathcal{P}$, $m_P(P) = 1$;
2. For every measurable set $M \in \mathcal{M}$, the function

$$\begin{aligned} m_{(\cdot)}(M) \colon \mathcal{P} &\longrightarrow \mathbb{R} \\ P &\longmapsto m_P(M) \end{aligned}$$

   is measurable;
3. For every measurable set $M \in \mathcal{M}$,

$$m(M) = \int_{P\in\mathcal{P}} m_P(M) m_{\mathcal{P}}(\mathrm{d}P). \tag{1.1}$$

The measure m_P is the *conditional probability* of m in P relative to $\mathcal{P}$.

If $f \colon X \longrightarrow \mathbb{R}$ is an integrable function, then FORMULA 1.1 implies that

$$\int_{x\in X} f(x) m(\mathrm{d}x) = \int_{P\in\mathcal{P}} \int_{x\in P} f(x) m_P(\mathrm{d}x) m_{\mathcal{P}}(\mathrm{d}P). \tag{1.2}$$

Conversely, FORMULA 1.1 can be reobtained from FORMULA 1.2 by integrating the indicator function of M. The next proposition states that this disintegration is $m_{\mathcal{P}}$-essentially unique. We omit the proof.

Proposition 1.6 ([VO16, Proposition 5.1.7, p. 145])**.** *Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space, $\mathcal{M}$ countably generated, and $\mathcal{P}$ a measurable partition of $\boldsymbol{X}$. If $(m_P)_{P\in\mathcal{P}}$ and $(m'_P)_{P\in\mathcal{P}}$ are disintegrations of m with respect to $\mathcal{P}$, then, for $m_{\mathcal{P}}$-almost every $P \in \mathcal{P}$,*

$$m_P = m'_P.$$

The disintegration of a probability measure with respect to a countable measurable partition $\mathcal{P}$ has a particularly simple form.

Proposition 1.7. *Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space and $\mathcal{P}$ is a countable measurable partition. The family of conditional measures $(m_P)_{P\in\mathcal{P}}$ defined, for each measurable set $M \in \mathcal{M}$, by*

$$m_P := \begin{cases} \dfrac{m(M \cap P)}{m(P)} & m(P) > 0 \\ 0 & m(P) = 0 \end{cases}$$

is the disintegration of m with respect to $\mathcal{P}$.

Proof. By definition of the induced σ-algebra $\mathcal{M}_{\mathcal{P}}$, for each $P \in \mathcal{P}$ the atomic set $\{P\} \subseteq \mathcal{P}$ is measurable, since $\pi_{\mathcal{P}}^{-1}(\{P\}) = P$ and P is measurable. Since $\mathcal{P}$ is countable, $\mathcal{M}_{\mathcal{P}}$ is the discrete σ-algebra. The quotient measure is then given for each $\mathcal{Q} \subseteq \mathcal{P}$ by

$$m_{\mathcal{P}}(\mathcal{Q}) = \sum_{P\in\mathcal{Q}} m(P). \tag{1.3}$$

In particular, $m_{\mathcal{P}}(\{P\}) = m(P)$.

Let $\mathcal{N} \subseteq \mathcal{P}$ be the set of parts of $\mathcal{P}$ that have m-measure 0. Then it follows from FORMULA 1.3 that

$$m_{\mathcal{P}}(\mathcal{N}) = \sum_{N\in\mathcal{N}} m(N) = 0. \tag{1.4}$$

Besides that, for every $N \in \mathcal{N}$ and every $M \in \mathcal{M}$, $m(M \cap N) = 0$, so

$$m_{\mathcal{P}}(\mathcal{N}) = \sum_{N\in\mathcal{N}} m(N) = 0. \tag{1.5}$$

Then it follows that, for every $M \in \mathcal{M}$,

$$\begin{aligned} m(M) &= \sum_{P \in \mathcal{P}} m(M \cap P) \\ &= \sum_{P \in \mathcal{P} \smallsetminus \mathcal{N}} m(M \cap P) \\ &= \sum_{P \in \mathcal{P} \smallsetminus \mathcal{N}} \frac{m(M \cap P)}{m(P)} m(P) \\ &= \sum_{P \in \mathcal{P} \smallsetminus \mathcal{N}} m_P(M) m(P) \\ &= \int_{P \in \mathcal{P} \smallsetminus \mathcal{N}} m_P(M) m_{\mathcal{P}}(\mathrm{d}P) \\ &= \int_{P \in \mathcal{P}} m_P(M) m_{\mathcal{P}}(\mathrm{d}P). \end{aligned}$$ ■

The second inequality follows from FORMULA 1.5, and the last one follows from FORMULA 1.4.

1.5 Dynamics

1.5.1 Discrete dynamics and flows

Definition 1.34. A *discrete-time dynamical system* is a pair $\boldsymbol{X} = (X, f)$ such that X is a set, the *phase space* of $\boldsymbol{X}$, and $f \colon X \longrightarrow X$ is a function, the *dynamics* of $\boldsymbol{X}$. The system $\boldsymbol{X}$ is *invertible* when f is invertible.

On a discrete-time dynamical system, we can iterate the dynamics: we consider $f^0 := \mathrm{I}$, the identity in X, and, for each $n \in \mathbb{Z}_{>0}$, the composition $f^n := f \circ f^{n-1}$. This gives a transformation (which we also denote by f)

$$\begin{aligned} f \colon \mathbb{Z}_{\geq 0} \times X &\longrightarrow X \\ (n, x) &\longmapsto f^n(x) \end{aligned}$$

that satisfies 1. $f^0 = \mathrm{I}$; 2. For every $n, n' \in \mathbb{Z}_{\geq 0}$, $f^{n'} \circ f^n = f^{n+n'}$. This can be shortly stated as saying that f induces a monoid action of $\mathbb{Z}_{\geq 0}$ on the space X. When the dynamics f is invertible, this action can be extended to a group action of $\mathbb{Z}$ by defining, for every $n \in \mathbb{Z}_{>0}$, $f^{-n} := (f^{-1})^n$.

Besides defining discrete-time dynamics, we can also consider flows. These are obtained by substituting $\mathbb{R}$ for $\mathbb{Z}$ on the discrete case just mentioned.

Definition 1.35. Let X be a set. A *flow* on X is a function

$$\begin{aligned} f \colon \mathbb{R} \times X &\longrightarrow X \\ (t, x) &\longmapsto f^t(x) \end{aligned}$$

that satisfies

1. $f^0 = \mathrm{I}$;
2. For every $t, t' \in \mathbb{R}$, $f^{t'} \circ f^t = f^{t+t'}$.

We can also analogously define *semi-flows* on X, whose domain is restricted to $\mathbb{R}_{\geq 0} \times X$.

Definition 1.36. A *continuous-time dynamical system* is a pair $\boldsymbol{X} = (X, f)$ such that X is a set, the *phase space* of $\boldsymbol{X}$, and $f\colon \mathbb{R} \times X \longrightarrow X$ is a *flow*, the *dynamics* of $\boldsymbol{X}$.

Both discrete-time and continuous-time dynamical systems can be defined on other categories of spaces besides sets, such as measure spaces, topological spaces, metric spaces, normed spaces and manifolds. In these cases the dynamics must preserve the relevant structure, being measure-preserving, continuous, differentiable etc. On metric spaces, we highlight the following definition, based on DEFINITION 1.10.

Definition 1.37. Let $\boldsymbol{M} = (M, d)$ be a metric space. A *uniformly continuous* flow on $\boldsymbol{M}$ is a flow $f\colon \mathbb{R} \times M \longrightarrow M$ that satisfies: for every $t \in \mathbb{R}_{>0}$ and every *error* $\varepsilon \in \mathbb{R}_{>0}$, there exists *difference* $\delta \in \mathbb{R}_{>0}$ such that, for every $s \in [-t, t]$ and every $p, q \in M$, the condition $d(p, q) < \delta$ implies the condition $d(f^s(p), f^s(q)) < \varepsilon$.

More generally, continuous-time dynamical systems can arise from smooth vector fields on manifolds, whose partial flow defines the dynamics on the space. (In CHAPTER 4, we will consider flows induced by vector field that are not smooth.)

Let us consider a manifold M modeled on $\mathbb{R}^d$. Given an open set $A \subseteq M$ and a smooth vector field $V\colon A \longrightarrow \mathrm{T}M$, there exists a set $\mathrm{dom}(\Phi^V) \subseteq \mathbb{R} \times A$, called the *maximal flow domain*, and a unique *maximal flow*

$$\begin{aligned} \Phi^V\colon \mathrm{dom}(\Phi^V) &\longrightarrow A \\ (t, p) &\longmapsto \Phi^{tV}(p). \end{aligned}$$

We adopt this notation because the flow Φ^V satisfies the derivation formula

$$\dot{\Phi}^{tV}(p) = \frac{\mathrm{d}}{\mathrm{d}t} \Phi^{tV}(p)|_{t=0} = V \circ \Phi^{tV}(p),$$

which is analogous to the derivation formula for the exponential function (the latter is in fact a consequence of the former).

When the vector field $V\colon M \longrightarrow \mathrm{T}M$ is Lipschitz, controlled by a constant $\langle\!\langle V \rangle\!\rangle$, that is, for every pair of points $p, q \in M$,

$$d(V(p), V(q)) \leq \langle\!\langle V \rangle\!\rangle\, d(p, q),$$

then the flow Φ^V of V is controlled above by $\mathrm{e}^{\langle\!\langle V \rangle\!\rangle}$ [Col12, Section 9.2] and below by $\mathrm{e}^{-\langle\!\langle V \rangle\!\rangle}$, in the sense that, for every pair of points $p, q \in M$ and every $t_0, t \in \mathbb{R}$ that is in the interval of definition of the flow for these points,

$$\mathrm{e}^{-|t-t_0|\langle\!\langle V \rangle\!\rangle}\, d(p,q) \leq d(\Phi^{(t-t_0)V}(p), \Phi^{(t-t_0)V}(q)) \leq \mathrm{e}^{|t-t_0|\langle\!\langle V \rangle\!\rangle}\, d(p,q). \tag{1.6}$$

1.5.2 Topological transitivity

The action of a group on a set is called *transitive* when there is a unique orbit, that is, every point can be taken to any other point by the action of the group. This is equivalent to saying that the set does not contain a proper invariant subset.

When we consider a topological dynamical system, *topological* notions of transitivity can be considered by using the topological structure of the space to approximate points (see [AC12] for a thorough discussion of many notions of topological transitivity). The concept of topological transitivity goes back to Birkhoff[1] [GH55].

We consider the following concepts of topological transitivity. We define them on discrete-time dynamical systems, but analogous definitions can be stated for flows.

Definition 1.38. Let X be a topological space and $f\colon X \longrightarrow X$ a continuous transformation.

- the dynamics f is *topologically transitive* when, for every pair of non-empty open sets U and V in X, there is a time t such that $f^t(U) \cap V \neq \emptyset$.
- the dynamics f is *topologically point-transitive* when there exists a point $x \in X$ whose orbit is dense in X.

Note that topological transitivity is equivalent to: for every non-empty open set $U \subseteq X$, the set $\bigcup_t f^t(U)$ is dense in X.

Such different concepts of transitivity are related in the following result (see [AC12; GH55]).

Theorem 1.8 (Birkhoff transitivity theorem)**.** *Let M be a separable, complete and perfect metric space and f a continuous transformation on it. Then f is topologically transitive if, and only if it is topologically point-transitive.*

In CHAPTER 4 we will discuss the concept of topological transitivity for piecewise smooth dynamical systems.

1. George David Birkhoff (1884–1944), American mathematician.

Chapter 2

Entropy

In this chapter we present the concept of *entropy* in dynamical systems. We will state the definitions and propositions (and some of the proofs) relevant to understanding the results of entropy related to symbolic dynamics in CHAPTER 3 and piecewise smooth systems in CHAPTER 4. In SECTION 2.1 we present a brief history of the concept in the natural sciences and Mathematics; in SECTION 2.2, we discuss the concept of information and entropy for finite probability distributions; in SECTION 2.3, we define the concept of entropy of dynamics on measure spaces and state some basic results that will be used later; in SECTION 2.4 we do the same for dynamics on topological and metric spaces.

2.1 History of entropy

The word ENTROPY was coined by Rudolf Clausius[1] in 1865 [Cla65], in German, from Ancient Greek ἐν (*en*, 'in') and τροπή (*tropé*, 'transformation'), with the meaning of *content of transformation* or *internal transformation* of a system [Bai92]. Clausius created it in analogy with the word ENERGY, from Ancient Greek ἐνέργεια (*enérgeia*, 'activity, vigor'), from the roots ἐν (*en*, 'in') and ἔργον (*érgon*, 'work'), adopted by Thomas Young[2] in 1802 [Smi98].

But more important than the etymology of the word, to understand the concept we must understand how it has been used in the technical literature. Initially, the concept of entropy was used in physics, specifically in thermodynamics, to refer to the ratio between an infinitesimal quantity of heat and the instantaneous temperature of a system. In 1872, still in the context of physics, but now in

1. RUDOLF JULIUS EMANUEL CLAUSIUS (1822 – 1888), German physicist and mathematician.
2. THOMAS YOUNG (1773 – 1829), English polymath.

statistical mechanics, Boltzmann[3] introduced his 'H-Theorem'[4], which related the concept of entropy with the macroscopic and microscopic states of a system; he explained entropy as the measure of the number of microstates of a system [Bol72].

A change of focus occurred in 1948, with the article "A Mathematical Theory of Communication" by Shannon[5], in which the author introduced the concept of information entropy [Sha48]. This concept of entropy was not strictly related to physics anymore[6] and had a fuzzy relationship with the concept of entropy in statistical mechanics, but the same word was used by a suggestion of von Neumann[7]. Shannon reports:

> My greatest concern was what to call it. I thought of calling it 'information', but the word was overly used, so I decided to call it 'uncertainty'. When I discussed it with John von Neumann, he had a better idea. Von Neumann told me, "You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage". [TM71, p. 180]

A decade later, Kolmogorov[8] brought the concept of entropy to mathematics, introducing the metric entropy for dynamical systems [Kol58] in a similar formulation to the information entropy of Shannon and, in the following year, Sinai[9] perfected the definition to basically what is currently used [Sin59].

Some years later, in analogy to the definition of metric entropy, topological entropy was introduced by Adler, Konheim, and McAndrew for topological spaces [AKM65], and formulated for metric spaces by Dinaburg [Din70] and Bowen[10] [Bow71]. For more detailed history and further developments, check [Kat07].

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3. Ludwig Eduard Boltzmann (1844–1906), Austrian physicist.
4. Boltzmann first used the letter 'E' for entropy, probably because it is the first letter of the word, but later adopted 'H' [Cha37]. The uppercase Latin letter 'H' (aitch) may have initially been an uppercase Greek letter 'H' (eta), as was used by Gibbs [Gib02] and Zermelo in the first decade of the 20th century [Hja77].
5. Claude Elwood Shannon (1916–2001), American mathematician and computer scientist.
6. This assertion may be contested by physicists because of discussions about how "information is physical" (see, for instance, the homonymous article [Lan91]), but what is meant here is that the concept gained an abstract formulation that did not depend anymore on thermodynamics and statistical mechanics.
7. John von Neumann (Hungarian: Neumann János Lajos) (1903–1957), Hungarian mathematician and physicist.)
8. Andrei Nikolaevich Kolmogorov (Russian: Андре́й Никола́евич Колмого́ров) (1903–1987), Russian mathematician.
9. Iakov Grigorevitch Sinai (Russian: Я́ков Григо́рьевич Сина́й) (1935–), Russian mathematician.
10. Robert Edward "Rufus" Bowen (1947–1978), American mathematician.

2.2 Entropy of finite probability distributions

The definition of entropy in information theory depends on *finite probability distributions*, that is, an n-uple $p = (p_0, \dots, p_{n-1}) \in \mathbb{R}^n_{\geq 0}$ such that

$$\sum_{k=0}^{n-1} p_k = 1.$$

This can be identified with a probability measure ρ on $[n]$ with the discrete σ-algebra $\wp([n])$ by setting, for every $i \in [n]$,

$$\rho(\{i\}) = p_i. \tag{2.1}$$

Every probability distribution is an element of some $\mathbb{R}^n$, and the set of all of them, for a fixed dimension, is a geometric object known as a *simplex* (plural *simplexes* or *simplices*) and is also relevant in some others areas of Mathematics, e.g. for triangulation of topological spaces in Algebraic Topology [Hat01] and as a parameter set in Convex Analysis [BV04; Roc15]. The dimension of the simplex is always one less than the dimension of the space it is embedded in (see FIGURE 2.1 for some examples). Therefore, in order to have a d-dimensional simplex, in the next definition we consider probability distributions with $d + 1$ entries.

Definition 2.1 (Simplex)**.** Let $d \in \mathbb{N}$. The *d-dimensional unit simplex*[11] (or *d-simplex*) is the set

$$\Delta^d := \left\{ p \in \mathbb{R}^{d+1}_{\geq 0} \mid \sum_{k=0}^{d} p_k = 1 \right\}.$$

A *vertex* of Δ^d is a point $p \in \Delta^d$ such that, for some $i \in [d+1]$, $p_i = 1$. A *boundary point* of Δ^d is a point $b \in \Delta^d$ such that, for some $i \in [d+1]$, $b_i = 0$. The *uniform distribution* (or *center point*) of Δ^d is the point

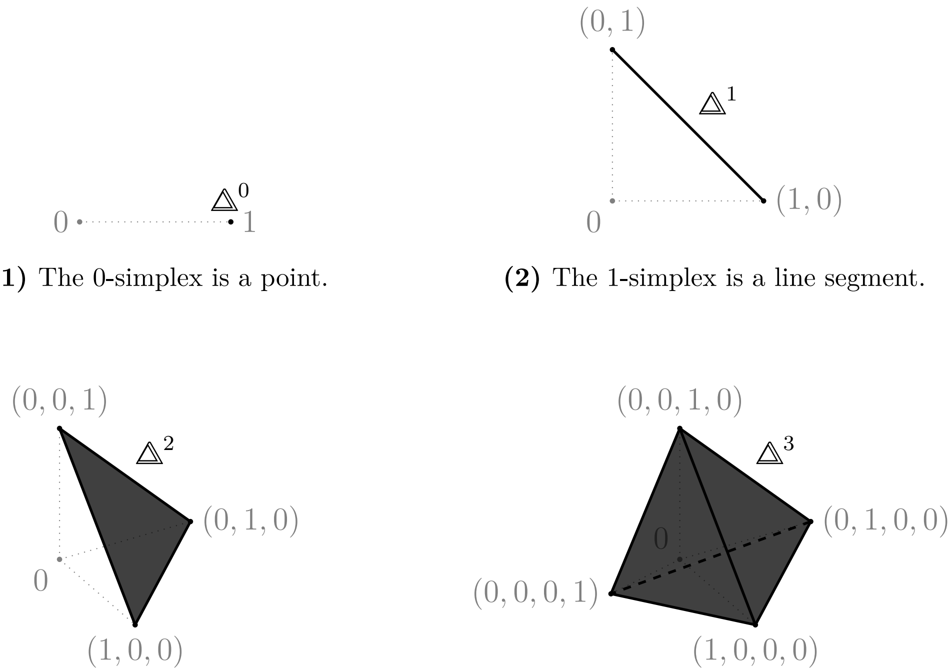
$$\mathrm{u}_d := \left(\frac{1}{d+1}, \cdots, \frac{1}{d+1} \right).$$

Before defining the entropy of a distribution in Δ^d, we will briefly talk about the *information function.* We wish to have a function $\mathrm{In}\colon]0,1] \longrightarrow \mathbb{R}$ that satisfies the following property: for every $p, p' \in]0,1]$,

$$\mathrm{In}(pp') = \mathrm{In}(p) + \mathrm{In}(p').$$

(Notice that p is now a real number, not a distribution.) This is because, if $\mathrm{In}(p)$ represents the amount of information of performing an experiment that has

11. Also called *probability simplex* or *standard simplex.*



(1) The 0-simplex is a point.

(2) The 1-simplex is a line segment.

(3) The 2-simplex is a triangle.

(4) The 3-simplex is a tetrahedron.

Figure 2.1. Unit simplexes $\triangle^0$, $\triangle^1$, $\triangle^2$ and $\triangle^3$. The axes are represented in gray as a referential frame. The simplexes are represented in black and, in the cases of dimension 2 and 3, their faces are have opacity in order to make the axis visible.

probability p, then the information we gain by performing an experiment with probability p followed by an (independent) experiment with probability p' is the sum of the information of each experiment. Besides that, we also suppose that In is continuous and positive. Then we have the following result.

Proposition 2.1. *Let* $\operatorname{In}\colon]0,1] \longrightarrow \mathbb{R}_{\geq 0}$ *be a continuous function that satisfies:*

1. *For every* $p, p' \in]0,1]$,

$$\operatorname{In}(pp') = \operatorname{In}(p) + \operatorname{In}(p').$$

Then there is $c \in \mathbb{R}_{\geq 0}$ *such that*

$$\operatorname{In}(p) = -c\log(p).$$

Proof. Define the function

$$\begin{aligned} f\colon [0,\infty[&\longrightarrow \mathbb{R} \\ p &\longmapsto \operatorname{In}(\mathrm{e}^{-p}). \end{aligned}$$

Then, for every $x, x' \in [0, \infty[$, it follows from PROPERTY 1 that

$$f(x + x') = \mathrm{In}(\mathrm{e}^{-(x+x')}) = \mathrm{In}(\mathrm{e}^{-x}\mathrm{e}^{-x'}) = \mathrm{In}(\mathrm{e}^{-x}) + \mathrm{In}(\mathrm{e}^{-x'}) = f(x) + f(x'),$$

so f is additive. It is a standard fact from analysis that any continuous additive function on $\mathbb{R}$ is linear, so there is a $c \in \mathbb{R}$ such that $f(p) = cp$. This implies that $\mathrm{In}(p) = f(-\log(p)) = -c\log(p)$. Since In is positive, it follows that $c \in \mathbb{R}_{\geq 0}$. ■

This shows that the negative of the logarithm is the only function that satisfies the properties we would want for an information function, up to a constant multiple c which amounts to a choice of unit information, or equivalently a change of basis for the logarithm. In what follows, we will also classify the entropy function according to some properties, and this will imply that it is the mean of the information of each entry of a distribution. This will be used to define the entropy of a partition in SECTION 2.3.1.

In PROPOSITION 2.2 we will consider the following function, whose graph for $c = 1$ is depicted in FIGURE 2.2.

$$\begin{aligned} f\colon [0,1] &\longrightarrow \mathbb{R} \\ p &\longmapsto \begin{cases} -cp\log(p) & x > 0 \\ 0 & x = 0. \end{cases} \end{aligned}$$

The logarithm is not defined at 0, but since $\lim_{p\searrow 0} p\log p = 0$, this function is continuous on $[0, 1]$. Because of this, from now on, we will always consider that $0\log 0 = 0$.

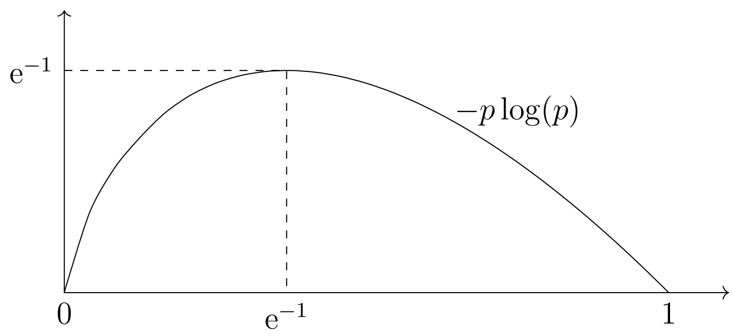


Figure 2.2. Graph of the function $-p\log p$ on the interval $[0, 1]$.

The entropy of a probability distribution $p \in \mathbb{\Delta}^d$ is a positive real number that measures, in a sense, the amount this distribution deviates from the center of the simplex $\mathbb{\Delta}^d$, given by the uniform distribution u_d.

We will not present here a detailed explanation of this function, only a proposition that classifies it completely according to some properties. In order to state the main property of entropy and to simplify calculation, we will define a 'sum' between discrete probability distributions. Given $n \in \mathbb{N}$ probability distributions $q^i = (q^i_0, \ldots, q^i_{k_i-1}) \in \mathbb{\Delta}^{k_i-1}$ and a probability distribution $p \in \mathbb{\Delta}^{n-1}$, we can consider a new distribution defined as the concatenation of the distributions q_i, but weighted by p so that the sum of its entries equals 1. This distribution is denoted

$$(p_i q^i_j)_{i,j} = (p_0 q^0_0, \ldots, p_0 q^0_{k_0-1}, p_1 q^1_0, \ldots, p_{n-1} q^{n-1}_{k_{n-1}-1})$$

and we can easily check that $(p_i q^i_j)_{i,j} \in \mathbb{\Delta}^{k_0+\cdots+k_{n-1}-1}$, since

$$\sum_{i=0}^{n-1} \sum_{j=0}^{k_i-1} p_i q^i_j = \sum_{i=0}^{n-1} p_i \sum_{j=0}^{k_i-1} q^i_j = \sum_{i=0}^{n-1} p_i = 1.$$

This distribution can also be interpreted as each q^i giving weights according to which we divide the entry p_i of p.

Definition 2.2. Let $p \in \mathbb{\Delta}^{n-1}$ and, for each $i \in [n]$, $q^i \in \mathbb{\Delta}^{k_i-1}$. The *$p$-weighted sum of $q^0, \ldots, q^{n-1}$* is

$$\bigoplus_{i\in[n]} p_i q^i := p_0 q^0 \oplus \cdots \oplus p_{n-1} q^{n-1} := (p_i q^i_j)_{i,j} \in \mathbb{\Delta}^{k_0+\cdots+k_{n-1}-1}.$$

Proposition 2.2. *Let $\{\mathrm{H}_{\mathbb{\Delta}^n} : \mathbb{\Delta}^n \longrightarrow \mathbb{R}_{\geq 0}\}_{n\in\mathbb{N}}$ be a family of continuous functions that satisfy*

1. *(Boundary restriction) For every $p \in \mathbb{\Delta}^n$, $\mathrm{H}_{\mathbb{\Delta}^n}(p) = \mathrm{H}_{\mathbb{\Delta}^{n+1}}(p_0, \ldots, p_n, 0)$;*
2. *(Center maximality) For every $n \in \mathbb{N}$, $\mathrm{H}_{\mathbb{\Delta}^n}(\mathrm{u}_n) = \max(\mathrm{H}_{\mathbb{\Delta}^n})$.*
3. *(Additivity) For every $p \in \mathbb{\Delta}^{n-1}$ and, for each $i \in [n]$, $q^i \in \mathbb{\Delta}^{k_i-1}$,*

$$\mathrm{H}_{\mathbb{\Delta}^{k_0+\cdots+k_{n-1}-1}}\Big(\bigoplus_{i\in[n]} p_i q^i\Big) = \mathrm{H}_{\mathbb{\Delta}^{n-1}}(p) + \sum_{i=0}^{n-1} p_i \mathrm{H}_{\mathbb{\Delta}^{k_i-1}}(q^i).$$

Then there is a positive real number $c \in \mathbb{R}_{\geq 0}$ such that, for every $p \in \mathbb{\Delta}^n$,

$$\mathrm{H}_{\mathbb{\Delta}^n}(p) = -c \sum_{i=0}^{n} p_i \log(p_i).$$

Proof. Let us define, for each $n \in \mathbb{N} \setminus \{0\}$ and $\mathrm{u}_{n-1} = (\frac{1}{n}, \ldots, \frac{1}{n}) \in \mathbb{\Delta}^{n-1}$, the number

$$L(n) := \mathrm{H}_{\mathbb{\Delta}^{n-1}}(\mathrm{u}_{n-1}).$$

We will first show that, for some real number $c \in \mathbb{R}_{\geq 0}$

$$L(n) = c \log(n). \tag{2.2}$$

By PROPERTIES 1 and 2,

$$L(n) = \mathrm{H}_{\triangle^{n-1}}(\mathrm{u}_{n-1}) = \mathrm{H}_{\triangle^{n}}\left(\frac{1}{n},\dots,\frac{1}{n},0\right) \le \mathrm{H}_{\triangle^{n}}(\mathrm{u}_{n}) = L(n+1), \tag{2.3}$$

which shows that L is a monotone increasing function.

Let $n \in \mathbb{N}$ and $k \in \mathbb{N} \smallsetminus \{0\}$ and consider the uniform distributions $\mathrm{u}_{n^{k-1}-1} = (\frac{1}{n^{k-1}})_{i\in[n^{k-1}]}$ and $\mathrm{u}_{n-1} = (\frac{1}{n})_{j\in[n]}$. Since

$$\bigoplus_{i\in[n^{k-1}]} \frac{1}{n^{k-1}}\mathrm{u}_{n-1} = \left(\frac{1}{n^{k-1}}\frac{1}{n},\dots,\frac{1}{n^{k-1}}\frac{1}{n}\right) = \left(\frac{1}{n^{k}},\dots,\frac{1}{n^{k}}\right) = \mathrm{u}_{n^k-1},$$

it follows from PROPERTY 3 that

$$\begin{aligned}\mathrm{H}_{\triangle^{n^k-1}}(\mathrm{u}_{n^k-1}) &= \mathrm{H}_{\triangle^{n^k-1}}\Big(\bigoplus_{i\in[n^{k-1}]} \frac{1}{n^{k-1}}\mathrm{u}_{n-1}\Big)\\ &= \mathrm{H}_{\triangle^{n^{k-1}-1}}(\mathrm{u}_{n^{k-1}-1}) + \sum_{i=0}^{n^{k-1}-1} \frac{1}{n^{k-1}}\mathrm{H}_{\triangle^{n-1}}(\mathrm{u}_{n-1})\\ &= \mathrm{H}_{\triangle^{n^{k-1}-1}}(\mathrm{u}_{n^{k-1}-1}) + \mathrm{H}_{\triangle^{n-1}}(\mathrm{u}_{n-1}).\end{aligned}$$

This shows that

$$L(n^k) = L(n^{k-1}) + L(n). \tag{2.4}$$

We will show, by induction on k, that $L(n^k) = kL(n)$: the relation holds for the base case $k = 1$, since $L(n^1) = 1L(n)$, and, assuming the induction hypothesis $L(n^{k-1}) = (k-1)L(n)$ is valid, it follows from FORMULA 2.4 that

$$L(n^k) = L(n^{k-1}) + L(n) = (k-1)L(n) + L(n) = kL(n). \tag{2.5}$$

Now take $n_0, n_1, k_0 \in \mathbb{N} \smallsetminus \{0\}$, and choose $k_1 \in \mathbb{N} \smallsetminus \{0\}$ such that

$${n_1}^{k_1} \le {n_0}^{k_0} < {n_1}^{k_1+1}.$$

Then it follows that $k_1 \log(n_1) \le k_0 \log(n_0) < (k_1+1)\log(n_1)$, therefore

$$\frac{k_1}{k_0} \le \frac{\log(n_0)}{\log(n_1)} < \frac{k_1}{k_0} + \frac{1}{k_0}. \tag{2.6}$$

Since L is increasing (FORMULA 2.3), it also follows that

$$L({n_1}^{k_1}) \le L({n_0}^{k_0}) \le L({n_1}^{k_1+1}),$$

so from this and FORMULA 2.5, it follows that $k_1 L(n_1) \leq k_0 L(n_0) \leq (k_1+1)L(n_1)$, therefore

$$\frac{k_1}{k_0} \leq \frac{L(n_0)}{L(n_1)} < \frac{k_1}{k_0} + \frac{1}{k_0}. \tag{2.7}$$

From these inequalities between fractions (FORMULAS 2.6 and 2.7), we obtain

$$\left|\frac{L(n_0)}{L(n_1)} - \frac{\log(n_0)}{\log(n_1)}\right| \leq \frac{1}{k_0}$$

and, taking the limit as $k_0 \to \infty$,

$$\frac{L(n_0)}{L(n_1)} = \frac{\log(n_0)}{\log(n_1)}.$$

Because n_0 e n_1 are arbitrary, we conclude that FORMULA 2.2 is valid for some constant $c \in \mathbb{R}$. Since L is increasing (FORMULA 2.3), we must have $c \geq 0$.

Consider now a probability distribution $p \in \Delta^{n-1}$ with only rational entries: for each $i \in [n]$, $p_i \in \mathbb{Q}$. Let $g_i \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$ be such that $p_i = \frac{g_i}{k}$ and define

$$q^i := \mathrm{u}_{g_i-1} = \left(\frac{1}{g_i}, \ldots, \frac{1}{g_i}\right).$$

Observe that, since $p_i \frac{1}{g_i} = \frac{1}{k}$,

$$\bigoplus_{i\in[n]} p_i q^i = (p_i q^i_j)_{i,j} = \left(\frac{1}{k}, \ldots, \frac{1}{k}, \ldots, \frac{1}{k}, \ldots, \frac{1}{k}\right) = \mathrm{u}_{k-1}.$$

So by using PROPERTY 3, it follows that

$$\mathrm{H}_{\Delta^{k-1}}(\mathrm{u}_{k-1}) = \mathrm{H}_{\Delta}\Big(\bigoplus_{i\in[n]} p_i q^i\Big) = \mathrm{H}_{\Delta^{n-1}}(p) + \sum_{i=0}^{n-1} p_i \mathrm{H}_{\Delta^{g_i-1}}(\mathrm{u}_{g_i-1}).$$

It then follows from $g_i = p_i k$ and FORMULA 2.2 that

$$\begin{aligned} c\log(k) &= \mathrm{H}_{\Delta^{k-1}}(\mathrm{u}_{k-1}) \\ &= \mathrm{H}_{\Delta^{n-1}}(p) + \sum_{i=0}^{n-1} p_i c \log(g_i) \\ &= \mathrm{H}_{\Delta^{n-1}}(p) + c\sum_{i=0}^{n-1} p_i \log(p_i) + c\log(k), \end{aligned}$$

so we obtain, for every p with rational entries, that

$$\mathrm{H}_{\triangle^{n-1}}(p) = -c\sum_{i=0}^{n-1} p_i \log(p_i).$$

From the continuity of $\mathrm{H}_{\triangle^{k-1}}$, it finally follows that this formula is valid for every $p \in \triangle^{n-1}$. ■

This is basically the demonstration presented by Khinchin[12] [Khi57, p. 9, Theorem 1], with some modifications such that PROPERTY 3 does not depend upon defining schemes or partitions, and some adaptation of the statement and notation used by Peter Walters [Wal00, p. 77, Theorem 4.1].

Some other interesting properties of $\mathrm{H}_{\triangle^n}$ that follow from PROPOSITION 2.2 are

- (Symmetry) For every $n \in \mathbb{N}$, $\mathrm{H}_{\triangle^n}$ is symmetric;
- (Vertex minimality) $\mathrm{H}_{\triangle^n}(p) = 0$ if, and only if, p is a vertex of $\triangle^n$.

Faddeev showed [Fad56] that it is sufficient to assume symmetry, continuity and the following simplified additivity:

- (Faddeev additivity) For every $p \in \triangle^{n-1}$ and $t \in [0, 1]$

$$\mathrm{H}_{\triangle^n}(p_0, \ldots, p_{n-2}, (1-t)p_{n-1}, tp_{n-1}) = \mathrm{H}_{\triangle^{n-1}}(p) + p_{n-1}\mathrm{H}_{\triangle^1}((1-t), t).$$

A categorical approach can be found in [BFL11].

A point of $\triangle^1$ has the form $(x, 1-x)$, with $x \in [0, 1]$. If we take $c = 1$, the entropy $\mathrm{H}_{\triangle^1}$ has the form

$$\mathrm{H}_{\triangle^1}(x, 1-x) = -x\log(x) - (1-x)\log(1-x).$$

The graph of this function is represented in FIGURE 2.3. Notice that the function is symmetric, is 0 at the vertices, and attains its maximum at the center $\frac{1}{2}$. The graph of $\mathrm{H}_{\triangle^2}$ has this form on the boundaries of $\triangle^2$.

2.3 Measure entropy

2.3.1 Entropy of a partition

When we have a probability space $\boldsymbol{X} = (X, \mathcal{M}, m)$ and a finite measure partition $\mathcal{P} = (P_k)_{k=0}^{n-1}$ of this space, we can consider the probability distribution whose entries are the measures of the parts of the partition $\mathcal{P}$:

$$(m(P_0), \ldots, m(P_{n-1})) \in \triangle^{n-1}.$$

12. Aleksandr Yakovlevich Khinchin (Russian: Алекса́ндр Я́ковлевич Хи́нчин) (1894–1959), Russian mathematician.

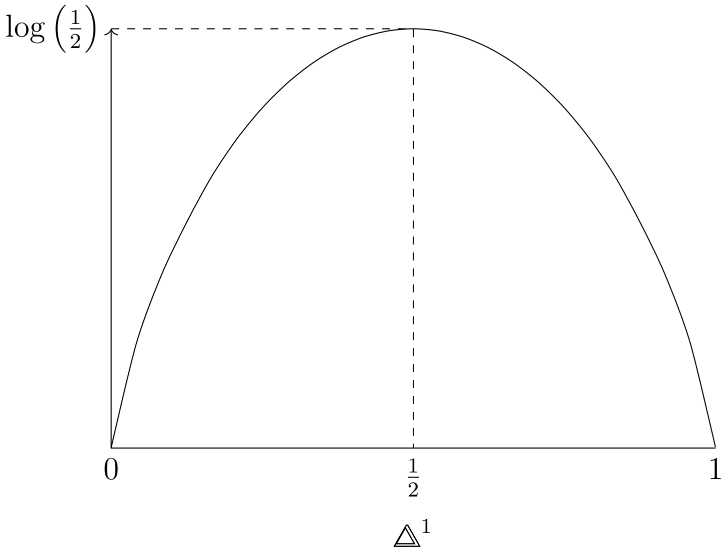


Figure 2.3. Graph of the function $\mathrm{H}_{\triangle^1}$.

The entropy of this distribution is given by

$$\mathrm{H}_{\triangle}(m(P_0), \ldots, m(P_{n-1})) = -\sum_{k=0}^{n-1} m(P_k) \log(m(P_k)). \tag{2.8}$$

Thus we define the entropy of the partition $\mathcal{P}$. Nevertheless, in the case the partition is not finite, but countable, we can still define its entropy, but using a more general definition in terms of the information function. In the following definition, we denote by $\pi_{\mathcal{P}}(x)$ the part of $\mathcal{P}$ the point $x \in X$ belongs to. By the definition of $\mathcal{P}$, this is not always a function properly defined on the whole X, but it *is* defined on a set with full measure, which is sufficient.

Definition 2.3 (Information function of a partition)**.** Let $\boldsymbol{X}$ be a probability space and $\mathcal{P}$ a measure partition of $\boldsymbol{X}$. The *information* of $\mathcal{P}$ is the function

$$\begin{aligned} \mathrm{In}_{\mathcal{P}} \colon X &\longrightarrow \mathbb{R} \\ x &\longmapsto -\log(m \circ \pi_{\mathcal{P}}(x)). \end{aligned}$$

From this definition of information, we can define the entropy as the integral of the information (considering always that $0 \log 0 = 0$). Since the measure of the space is 1, because we are considering probability measures, the entropy is the mean information of a partition [Rok67, Section 4.1, p. 13].

Definition 2.4 (Entropy of a partition)**.** Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space and $\mathcal{P}$ a measure partition of $\boldsymbol{X}$. The *entropy* of $\mathcal{P}$ is

$$\mathrm{H}_m(\mathcal{P}) := \int_X \mathrm{In}_{\mathcal{P}}\, m = \int_X -\log(m \circ \pi_{\mathcal{P}}) m.$$

Making the variable x explicit, we have

$$\mathrm{H}_m(\mathcal{P}) := \int_{x\in X} -\log(m(\pi_{\mathcal{P}}(x)))m(\mathrm{d}x).$$

In the case $\mathcal{P}$ is finite or countable, this definition implies that

$$\begin{aligned}\mathrm{H}_m(\mathcal{P}) &= \sum_{P\in\mathcal{P}} \int_{x\in P} -\log(m(\pi_{\mathcal{P}}(x)))m(\mathrm{d}x) \\ &= \sum_{P\in\mathcal{P}} \int_{x\in P} -\log(m(P))m(\mathrm{d}x) \qquad (2.9)\\ &= \sum_{P\in\mathcal{P}} -m(P)\log m(P),\end{aligned}$$

which coincides with FORMULA 2.8 in the beginning of this section.

This definition via integral of the information function does not exclude the possibility that the value of the entropy of some partition be infinite. In fact, there are examples of this [VO16, p. 246, Example 9.1.4]. From this point forward, we shall only consider partitions that have finite entropy and, occasionally, we might restrict ourselves further to the case of finite partitions.

Besides the entropy of a partition, we will also define the conditional entropy of a partition with respect to another partition. DEFINITION 2.6 is based on [Rok67, Section 5.1], but we present here the concept of conditional information in order to motivate it, and to draw a parallel between the expositions of entropy and conditional entropy via the integral of the respective information functions.

Consider two measure partitions $\mathcal{P}$ and $\mathcal{P}'$, and a disintegration $(m_{P'})_{P'\in\mathcal{P}'}$ of m with respect to $\mathcal{P}'$. For each $P' \in \mathcal{P}'$, we can consider the induced partition $\mathcal{P}|_{P'}$ on P', given by the sets $P \cap P'$ with $P \in \mathcal{P}$. The conditional information of $\mathcal{P}$ with respect to $\mathcal{P}'$ is the information obtained from conducting an experiment to test if a point belongs to a part of $\mathcal{P}$ after having already conduced the experiment for $\mathcal{P}'$. In more concrete terms, if we already know that a point $x \in X$ belongs to P', and we learn that it also belongs to P, we want this extra information to have the value of the information of $x \in P \cap P'$ in the induced partition $\mathcal{P}|_{P'}$, which is

$$\mathrm{In}_{\mathcal{P}|_{P'}}(x) = -\log m_{P'}(P\cap P') = -\log m_{P'}(P).$$

Since $P = \pi_{\mathcal{P}}(x)$ and $P' = \pi_{\mathcal{P}'}(x)$, we define the conditional information as follows.

Definition 2.5 (Conditional information)**.** Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space and $\mathcal{P}, \mathcal{P}'$ measurable partitions, and let $(m_{P'})_{P'\in\mathcal{P}}$ be a disintegration of m with respect to $\mathcal{P}'$. The *conditional information* of $\mathcal{P}$ with respect to $\mathcal{P}'$ is the function

$$\begin{aligned}\mathrm{In}_{\mathcal{P}|\mathcal{P}'}\colon X &\longrightarrow \mathbb{R}\\ x &\longmapsto -\log(m_{\pi_{\mathcal{P}'}(x)} \circ \pi_{\mathcal{P}}(x)).\end{aligned}$$

Notice that, because of the remarks before the definition, for every $x \in P'$ we have $\mathrm{In}_{\mathcal{P}|\mathcal{P}'}(x) = \mathrm{In}_{\mathcal{P}|_{P'}}(x)$. Analogously to DEFINITION 2.4, we define the conditional entropy as the mean of the conditional information.

Definition 2.6 (Conditional entropy)**.** Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space and $\mathcal{P}, \mathcal{P}'$ measurable partitions, and let $(m_{P'})_{P' \in \mathcal{P}}$ be a disintegration of m with respect to $\mathcal{P}'$. The *conditional entropy* of $\mathcal{P}$ with respect to $\mathcal{P}'$ is

$$\mathrm{H}_m(\mathcal{P} \mid \mathcal{P}') := \int_X \mathrm{In}_{\mathcal{P}|\mathcal{P}'}\, m = \int_{x \in X} -\log(m_{\pi_{\mathcal{P}'}(x)} \circ \pi_{\mathcal{P}}(x)) m(\mathrm{d}x).$$

From this definition, it is not at all clear how to compute this value, but from FORMULA 1.2 we obtain that

$$\begin{aligned} \mathrm{H}_m(\mathcal{P} \mid \mathcal{P}') &= \int_{x \in X} \mathrm{In}_{\mathcal{P}|\mathcal{P}'}(x) m(\mathrm{d}x) \\ &= \int_{P' \in \mathcal{P}'} \int_{x \in P'} \mathrm{In}_{\mathcal{P}|\mathcal{P}'}(x) m_{P'}(\mathrm{d}x) m_{\mathcal{P}'}(\mathrm{d}P) \\ &= \int_{P' \in \mathcal{P}'} \mathrm{H}_{m_{P'}}(\mathcal{P}|_{P'}) m_{\mathcal{P}'}(\mathrm{d}P'), \end{aligned} \tag{2.10}$$

which is formula 12 presented in [Rok67, Section 5.1, p. 15]. If $\mathcal{P}$ and $\mathcal{P}'$ are countable, the conditional measures $m_{P'}$ are given by $m_{P'}(M) = \frac{m(M \cap P')}{m(P')}$ (PROPOSITION 1.7) and the quotient measure $m_{\mathcal{P}'}$ is given by $m_{\mathcal{P}'}(\{P'\}) = m(P')$, so

$$\begin{aligned} \mathrm{H}_{m_{P'}}(\mathcal{P}|_{P'}) &= \sum_{P \in \mathcal{P}} -m_{P'}(P \cap P') \log m_{P'}(P \cap P') \\ &= \sum_{P \in \mathcal{P}} -\frac{m(P \cap P')}{m(P')} \log\left(\frac{m(P \cap P')}{m(P')}\right), \end{aligned}$$

which implies that

$$\begin{aligned} \mathrm{H}_m(\mathcal{P} \mid \mathcal{P}') &= \sum_{P' \in \mathcal{P}'} \mathrm{H}_{m_{P'}}(\mathcal{P}|_{P'}) m_{\mathcal{P}'}(P') \\ &= \sum_{P \in \mathcal{P}} -\frac{m(P \cap P')}{m(P')} \log\left(\frac{m(P \cap P')}{m(P')}\right) m(P') \\ &= \sum_{P \in \mathcal{P}} \sum_{P' \in \mathcal{P}'} -m(P \cap P') \log\left(\frac{m(P \cap P')}{m(P')}\right). \end{aligned}$$

This is the simplified formula presented in [VO16, Section 9.1.2, p. 247].

2.3.2 Measure entropy of a system

In this section, we will only consider countable measurable partitions with finite entropy.

Dynamical correfinement

Definition 2.7. Let $\boldsymbol{X}$ be a probability space, $f\colon X \longrightarrow X$ a measure-preserving transformation, $\mathcal{P}$ a measurable partition and $n \in \mathbb{N} \setminus \{0\}$. The n-th *dynamical correfinement* of $\mathcal{P}$ by f is the partition

$$\mathcal{P}_f^n := \bigvee_{i=0}^{n-1} (f^i)^{\dashv}(\mathcal{P})$$

The n-th *bilateral dynamical correfinement* of $\mathcal{P}$ by f is the partition

$$\mathcal{P}_f^{\pm n} := \bigvee_{i=-n}^{n-1} (f^i)^{\dashv}(\mathcal{P})$$

An element of $P \in \mathcal{P}_f^{[n]}$ is of the form

$$P = \bigcap_{i=0}^{n-1} f^{-i}(P_i),$$

in which $P_i \in \mathcal{P}$ for every $i \in [n]$. This means that, if $x \in P$, then, for every $i \in [n]$, $f^i(x) \in P_i$. This gives the $\mathcal{P}$-itinerary of the point x up to n iterates.

Entropy of the system

It is possible to show that the sequence of entropies $(\mathrm{H}_m(\mathcal{P}_f^n))_{n\in\mathbb{N}}$ is increasing and subadditive [VO16, Section 9.1.3, p. 250]. This implies that the limit

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{H}_m(\mathcal{P}_f^n)$$

exists and is equal to the infimum of the sequence. This leads to the following definition.

Definition 2.8 (Measure entropy of a system)**.** Let $\boldsymbol{X}$ be a probability space, $f\colon X \longrightarrow X$ a measure-preserving transformation and $\mathcal{P}$ a measurable partition of $\boldsymbol{X}$ with finite entropy. The *measure entropy* of f with respect to $\mathcal{P}$ is

$$\mathrm{h}_m(f, \mathcal{P}) := \lim_{n\to\infty} \frac{1}{n} \mathrm{H}_m\left(\mathcal{P}_f^n\right).$$

Notice that the bilateral dynamical correfinement $\mathcal{P}_f^{\pm n}$ is not used in the definition of the measure entropy of the system relative to the partition $\mathcal{P}$. It is used in PROPOSITION 2.6 to simplify the calculation of the entropy of an invertible system.

The entropy $\mathrm{h}_m(f, \mathcal{P})$ is an increasing function of the partition $\mathcal{P}$ (with respect to the almost-refinement order $\mathring{\preceq}$) [VO16, Section 9.1.3, p. 250].

Definition 2.9. Let $\boldsymbol{X}$ be a probability space and $f\colon X \longrightarrow X$ a measure-preserving transformation. The *measure entropy*[13] of f is

$$\mathrm{h}_m(f) := \sup_{\mathcal{P},\ \mathrm{H}_m(\mathcal{P})<\infty} \mathrm{h}_m(f, \mathcal{P}).$$

It is possible to show that taking the supremum over all finite measurable partitions $\mathcal{P}$ of $\boldsymbol{X}$ gives the same value of $\mathrm{h}_m(f)$.

Proposition 2.3. *Let $\boldsymbol{X}$ be a probability space and $f\colon X \longrightarrow X$ a measure-preserving transformation.*

1. *For every $k \in \mathbb{N}$, $\mathrm{h}_m(f^k) = k\mathrm{h}_m(f)$;*
2. *If f is invertible, then, for every $k \in \mathbb{Z}$, $\mathrm{h}_m(f^k) = |k|\mathrm{h}_m(f)$.*

The calculation of the measure entropy of a system involves taking the supremum over all measurable partitions with finite entropy. This is not feasible in practice for most cases. Calculating the entropy with respect to a partition is generally much easier. The following theorem allows us to calculate the entropy of the system if we can find an increasing sequence of partitions (with respect to $\mathring{\preceq}$) that generate the σ-algebra (up to measure zero). The proof can be found in [VO16, Section 9.2, p. 254].

Theorem 2.4 (Kolmogorov–Sinai [VO16, Theorem 9.2.1])**.** *Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space, $f\colon X \longrightarrow X$ a measure-preserving transformation and $(\mathcal{P}_k)_{k\in\mathbb{N}}$ an increasing sequence of measurable partitions with finite entropy such that $\bigcup_{k\in\mathbb{N}} \mathcal{P}_k$ generates the σ-algebra $\mathcal{M}$ (up to measure zero). Then*

$$\mathrm{h}_m(f) = \lim_{k\in\mathbb{N}} \mathrm{h}_m(f, \mathcal{P}_k).$$

This is a general version of the Kolmogorov–Sinai theorem, which has some corollaries that are sometimes also called by the same name. We state the corollaries here to avoid any confusion. The proofs can be found in [VO16, Section 9.2.1, p. 256]

Proposition 2.5. *Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a probability space, $f\colon X \longrightarrow X$ a measure-preserving transformation and $\mathcal{P}$ a measurable partition with finite entropy such that $\bigcup_{k\in\mathbb{N}\smallsetminus\{0\}} \mathcal{P}_f^k$ generates the σ-algebra $\mathcal{M}$ (up to measure zero). Then*

$$\mathrm{h}_m(f) = \mathrm{h}_m(f, \mathcal{P}).$$

If f is invertible, there is also another version of the corollary.

13. This is usually called *metric entropy* in the literature. We avoid the term *metric* because it may lead to confusion with the *topological entropy* defined on a metric space.

Proposition 2.6. *Let* $\boldsymbol{X} = (X, \mathcal{M}, m)$ *be a probability space,* $f\colon X \longrightarrow X$ *an invertible measure-preserving transformation and* $\mathcal{P}$ *a measurable partition with finite entropy such that* $\bigcup_{k\in\mathbb{N}\smallsetminus\{0\}} \mathcal{P}_f^{\pm k}$ *generates the* σ*-algebra* $\mathcal{M}$ *(up to measure zero). Then*

$$\mathrm{h}_m(f) = \mathrm{h}_m(f, \mathcal{P}).$$

2.3.3 Folding entropy

Definition 2.10. Let $\boldsymbol{X}$ be a probability space and $f\colon X \longrightarrow X$ a measure-preserving transformation. The *atomic partition* of X is

$$\epsilon := \wp_1(X) = \{\{x\} \mid x \in X\}$$

and the *dynamical pullback of* ϵ is

$$f^{\dashv}(\epsilon) = \{f^{-1}(x) \mid x \in \Sigma\}.$$

In [Rue96] the author introduces the *folding entropy* for $\mathscr{C}^1$ transformations. It can be defined [Liu03; WZ21] as the conditional entropy of the atomic partition ϵ with respect to its dynamical pullback.

Definition 2.11. Let $\boldsymbol{X}$ be a probability space and $f\colon X \longrightarrow X$ a measure-presering transformation. The *folding entropy* of f with respect to m is

$$\mathcal{F}(f) := \mathrm{H}_m(\epsilon \mid f^{\dashv}(\epsilon)).$$

2.4 Topological entropy

The *topological entropy* of a compact topological space was first defined by Adler, Konheim, and McAndrew in [AKM65], motivated by the measure entropy for measure spaces, but substituting open covers for measurable partitions. We will not present this construction here, only a later development for metric spaces that is equivalent to the general definition for compact topological spaces.

2.4.1 Topological entropy for metric spaces

In this section we study the topological entropy of a metric space that was first defined by Bowen and Dinaburg [Bow71; Din70]. It is defined in any metric space with a uniformly continuous transformation and coincides with the topological entropy of [AKM65] in the adequate context. For that reason, this entropy is also called *topological entropy*, but it is possible to generalize the definition in metric spaces to uniform spaces (as done by Hood in [Hoo74]) and show that they coincide when one considers the uniform structure on the metric space that is generated by the distance function. This suggests the topological entropy of Bowen–Dinaburg is a kind of *uniform entropy*, a notion that is reinforced by the fact that they are invariant by uniformly continuous conjugation and coincident for two distance functions of a metric that are uniformly equivalent (PROPOSITION 2.8).

The definition by dynamical balls presented here makes it clear that the topological entropy can be interpreted as the *rate of exponential growth of the number of essentially different orbit of the dynamics*. It is also more general in the sense that it is defined for non-compact spaces.

Dynamical balls

Definition 2.12. Let $\boldsymbol{M} = (M, d)$ be a metric space, $f\colon M \longrightarrow M$ a continuous transformation and $n \in \mathbb{Z}_{>0}$. The *dynamical distance* of order n relative to f and d in M is

$$\begin{aligned} d_f^n\colon M \times M &\longrightarrow \mathbb{R} \\ (p, p') &\longmapsto d_f^n(p, p') := \max_{i \in [n]} d(f^i(p), f^i(p')). \end{aligned}$$

It is a straightforward exercise to show that $d_f^n\colon M \times M \longrightarrow \mathbb{R}$ is a distance function on M and that, for every $p, p' \in M$, the sequence $(d_f^n(p, p'))_{n \in \mathbb{Z}_{>0}}$ is increasing.

Definition 2.13. Let $\boldsymbol{M} = (M, d)$ be a metric space, $f\colon M \longrightarrow M$ a continuous transformation, $n \in \mathbb{Z}_{>0}$, $c \in M$ and $r \in \mathbb{R}_{>0}$. The *dynamical ball* of order n, center c and radius r relative to f and d is the ball of center c and radius r of (M, d_f^n), denoted

$$\mathrm{B}_f^n(c; r) := \{p \in M \mid d_f^n(c, p) < r\}.$$

When there is no ambiguity, we shall suppress the subindex f to simplify notation, resulting in $d^n(c, p)$ and $\mathrm{B}^n(c; r)$. It is also a straightforward exercise to show that

$$\mathrm{B}^n(c; r) = \bigcap_{i \in [n]} f^{-i}(\mathrm{B}(f^i(c); r)),$$

which implies that $\mathrm{B}^n(c; r)$ is open in $\boldsymbol{M}$ and that the sequence of sets $(\mathrm{B}^n(c; r))_{n \in \mathbb{Z}_{>0}}$ is decreasing.

Generating sets

Definition 2.14. Let $\boldsymbol{M} = (M, d)$ be a metric space, $f\colon M \longrightarrow M$ a continuous transformation, $K \subseteq M$ a compact set, $n \in \mathbb{Z}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$. A (n, ε)*-generator* of K is a set $G \subseteq M$ such that

$$K \subseteq \bigcup_{p \in G} \mathrm{B}^n(p; \varepsilon).$$

In this case, G (n, ε)*-generates* K. The *minimal cardinality* of an (n, ε)-generator of K is denoted

$$\bar{\mathrm{H}}^n_d(f, K, \varepsilon) := \min\{\#G \mid G \subseteq M \ (n, \varepsilon)\text{-generates } K\}.$$

Since $\{\mathrm{B}^n(p; \varepsilon) \mid p \in K\}$ is an open cover of K, it follows by compacity that there exists a finite (n, ε)-generator of K, so $\bar{\mathrm{H}}^n_d(f, \varepsilon, K)$ is always finite.

There is a dual notion of a generator set, called a *separated set*. This is used to ease calculations and proofs related to the topological entropy of a metric space. We will not define this dual notion here, but the details can be found in [VO16, Section 10.1.2, p. 305].

Entropy of the system

Definition 2.15. Let $\boldsymbol{M} = (M, d)$ be a metric space, $f\colon M \longrightarrow M$ a continuous transformation, $K \subseteq M$ a compact set and $\varepsilon \in \mathbb{R}_{>0}$. The ε*-imprecise topological entropy* of f with respect to K is

$$\mathrm{h}_d(f, K, \varepsilon) := \overline{\lim_{n \to \infty}} \frac{1}{n} \log \bar{\mathrm{H}}^n_d(f, K, \varepsilon).$$

It is possible to show that $\mathrm{h}_d(f, K, \varepsilon)$ is a decreasing function of ε. As a consequence, the limit for $\varepsilon \to 0$ exists.

Definition 2.16. Let $\boldsymbol{M} = (M, d)$ be a metric space, $f\colon M \longrightarrow M$ a continuous transformation and $K \subseteq M$ a compact set. The *topological entropy* of f with respect to K is

$$\mathrm{h}_d(f, K) := \lim_{\varepsilon \to 0} \mathrm{h}_d(f, K, \varepsilon).$$

The *topological entropy* of f is

$$\mathrm{h}_d(f) := \sup_{\text{compact } K \subseteq M} \mathrm{h}_d(f, K).$$

The topological entropy satisfies an exponent law analogous to that for measure entropy of PROPOSITION 2.3.

Proposition 2.7. *Let* $\boldsymbol{M} = (M, d)$ *be a metric space and* $f\colon M \longrightarrow M$ *a uniformly continuous transformation.*

1. *For every* $k \in \mathbb{N}$, $\mathrm{h}_d(f^k) = k\mathrm{h}_d(f)$*;*

2. If f is a homeomorphism and M is compact, then, for every $k \in \mathbb{Z}$, $\mathrm{h}_d(f^k) = |k|\mathrm{h}_d(f)$.

Proposition 2.7 is not true in general for non-compact metric spaces, as can be easily be checked by showing that $f(x) = 2x$ in $\mathbb{R}$ (with the standard distance function) has topological entropy greater than or equal to $\log 2$, but f^{-1} has topological entropy equal to 0.

To finish this section, we present a proof that two distance functions that are uniformly equivalent give the same topological entropy.

Proposition 2.8. *Let M be a set, d and d' uniformly equivalent distance functions on M, and $f\colon M \longrightarrow M$ a continuous transformation on (M, d). Then*

$$\mathrm{h}_d(f) = \mathrm{h}_{d'}(f).$$

Proof. We first show that, for every $n \in \mathbb{Z}_{>0}$, the dynamical distances d^n and d'^n are uniformly equivalent. Since d and d' are uniformly equivalent, for every $\varepsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that, for every $p, q \in M$, $d(p, q) < \delta$ implies $d'(p, q) < \varepsilon$. If $d^n(p, q) < \delta$ then, for every $i \in [n]$, $d(f^i(p), f^i(q)) < \delta$, so it follows from the uniform equivalence of the distance functions that $d'(f^i(p), f^i(q)) < \varepsilon$, which implies that $d'^n(p, q) < \varepsilon$. The opposite implication follows from exchanging d and d', and we conclude that d^n and d'^n are uniformly equivalent.

Now let $K \subseteq M$ be a compact set, $\varepsilon' \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$. By the uniform equivalence of the dynamical distance functions, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\varepsilon \leq \varepsilon'$, for every $p, q \in M$, $d^n(p, q) < \varepsilon$ implies $d'^n(p, q) < \varepsilon'$, which is equivalent to having $\mathrm{B}^n(p; \varepsilon) \subseteq \mathrm{B}'^n(p; \varepsilon')$.

Let $G_{K,\varepsilon,n} \subseteq M$ be a set such that $\bar{\mathrm{H}}^n_d(f, K, \varepsilon) = \#G_{K,\varepsilon,n}$. Then it follows that

$$K \subseteq \bigcup_{p \in G_{K,\varepsilon,n}} \mathrm{B}^n(p; \varepsilon) \subseteq \bigcup_{p \in G_{K,\varepsilon,n}} \mathrm{B}'^n(p; \varepsilon'),$$

which shows that

$$\bar{\mathrm{H}}^n_{d'}(f, K, \varepsilon') \leq \#G_{K,\varepsilon,n} = \bar{\mathrm{H}}^n_d(f, K, \varepsilon).$$

By taking a sequence $(\varepsilon'_m)_{m \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $\lim_{m \to \infty} \varepsilon'_m = 0$, there exists a sequence $(\varepsilon_m)_{m \in \mathbb{N}}$ in $\mathbb{R}_{>0}$ such that $\lim_{m \to \infty} \varepsilon_m = 0$ and

$$\bar{\mathrm{H}}^n_{d'}(f, K, \varepsilon'_m) \leq \bar{\mathrm{H}}^n_d(f, K, \varepsilon_m),$$

so it follows that

$$\begin{aligned}
\mathrm{h}_{d'}(f) &= \sup_K \lim_{m \to \infty} \overline{\lim_{n \to \infty}} \frac{1}{n} \bar{\mathrm{H}}^n_{d'}(f, K, \varepsilon'_m) \\
&\leq \sup_K \lim_{m \to \infty} \overline{\lim_{n \to \infty}} \frac{1}{n} \bar{\mathrm{H}}^n_d(f, K, \varepsilon_m) \\
&= \mathrm{h}_d(f).
\end{aligned}$$

The opposite inequality follows from exchanging d and d', and we conclude that $\mathrm{h}_d(f) = \mathrm{h}_{d'}(f)$. ■

2.4.2 Topological entropy of flows

Topological entropy can be easily defined for a continuous flow $\phi\colon \mathbb{R} \times M \longrightarrow M$ in a metric space $\boldsymbol{M}$, based on the same approach for continuous transformations on $\boldsymbol{M}$ [VO16, Section 10.2.3, p. 318]. The *dynamical distance* of order $t \in \mathbb{R}_{>0}$ is defined for $p, p' \in M$ as

$$d_\phi^t(p, p') := \sup_{s \in [0,t[} d(\phi^s(p), \phi^s(p')),$$

the *dynamical ball* of order $t \in \mathbb{R}_{>0}$, center $c \in M$ and radius $r \in \mathbb{R}_{>0}$ is defined as

$$\mathrm{B}_\phi^t(p; r) := \{p \in M \mid d_\phi^t(c, p) < r\}$$

and, for each compact set $K \subseteq M$, a set $G \subseteq M$ (t, ε)-*generates* K when

$$K \subseteq \bigcup_{p \in G} \mathrm{B}^t(p; \varepsilon).$$

Then the *topological entropy* of ϕ can be defined as

$$\mathrm{h}_d(\phi) := \sup_{\text{compact } K \subseteq M} \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \frac{1}{t} \log \#(G_{\phi,K,\varepsilon}^{(t)}),$$

in which $G_{\phi,K,\varepsilon}^{(t)} \subseteq K$ is a set whose cardinality is minimal among those that (t, ε)-generates K.

The topological entropy of a flow can be shown to be equivalent to the topological entropy of the time-1 map $\phi^1\colon M \longrightarrow M$ when the flow is uniformly continuous [VO16, Proposition 10.2.7, p. 319]. The exponent law of entropy is also valid for flows.

Proposition 2.9. *Let $\boldsymbol{M} = (M, d)$ be a compact metric space and $\phi\colon \mathbb{R} \times M \longrightarrow M$ a uniformly continuous flow. Then, for every $k \in \mathbb{Z}$,*

$$\mathrm{h}_d(\phi^k) = |k| \mathrm{h}_d(\phi).$$

Chapter 3

Symbolic dynamics

In this chapter we present results for bisymbolic shift spaces, a theory of extended symbolic shift spaces called *zip shifts* in the literature. In SECTION 3.1 we present an introduction to unilateral shift, as well as a result in disintegration of measures on these spaces, to motivate the work on bisymbolic shifts, which is exposed in SECTION 3.2. In this section we defined the bisymbolic shift and compute its measure entropy and its folding entropy. The work of this section was developed in [MMV24].

3.1 Unilateral shift

3.1.1 Space and dynamics

The shift dynamical system is a model used to study the dynamical behavior of different systems by considering the itinerary of points relative to a finite number of parts the original system is divided in. When we have a finite partition $\mathcal{P}$ of a system $f\colon X \longrightarrow X$, we can determine, for each point p of the system, the part that p belongs to in $\mathcal{P}$, say P_0, then the part $f(p)$ belongs to, say P_1, and so on. Since $\mathcal{P}$ is a partition, each point of the system belong to exactly one part of $\mathcal{P}$. This results in a sequence

$$(P_0, P_1, \ldots) \tag{3.1}$$

on the set $\mathcal{P}$ which is indexed by $\mathbb{Z}_{\geq 0}$.

To consider another perspective, every point of $P \in \mathcal{P}$ is going to have an encoding sequence that has 0-th entry P. Moreover, the points of the set $f^{-1}(P)$ are the points p whose image by the dynamics is in P, that is, $f(p) \in P$. This means that their encoding sequence must have their 1-st entry P. These facts combined are equivalent to this: for every $P_0, P_1 \in \mathcal{P}$, the elements of $P_0 \cap f^{-1}(P_1)$ must have the beginning of their encoding sequence be (P_0, P_1); generalizing for $n \in \mathbb{N} \smallsetminus \{0\}$, we obtain that, for every $P_0, \ldots, P_{n-1} \in \mathcal{P}$, the elements of $\bigcap_{i=0}^{n-1} f^{-i}(P_i)$ must have the beginning of their encoding sequence be

$$(P_0, P_1, \ldots, P_{n-1}).$$

If we proceed in this for larger values of n, we obtain the limiting case of the set

$$\bigcap_{i=0}^{\infty} f^{-i}(P_i) \tag{3.2}$$

whose points must be associated to the (infinite) sequence in FORMULA 3.1. To obtain a full equivalence of points of the space and sequences on $\mathcal{P}$, every point of the space must belong to the set in FORMULA 3.2, and every such set must consist of a single point. This is not always the case for the systems we encode using a partition but, for those that this does happen, we can understand their dynamics simply by understanding the dynamics of the space of sequences, which is much easier to work with in general.

Nevertheless, to encode the original dynamics f on X with a new dynamics on the abstract space of all sequences on a partition, we must also determine what the dynamics on the space of sequences is. For that, notice the itinerary sequence of $f(p)$ is

$$(P_1, P_2, \ldots);$$

that is, the itinerary sequence of $f(p)$ is the same as the itinerary sequence of p in FORMULA 3.1, except for its first entry, for the orbit of $f(p)$ is the same as that of p, except for the first point. These objects are formalized in the following definitions. Remember that we denote $[n] := \{0, \ldots, n-1\}$.

Definition 3.1. Let $n \in \mathbb{N} \smallsetminus \{0\}$. The *unilateral n-symbolic shift dynamical system* is the pair (Σ_n^+, σ) in which

1. the *unilateral n-symbolic space* is the set $\Sigma_n^+ := [n]^{\mathbb{Z}_{\geq 0}}$ of all *unilateral n-symbolic sequences*; that is, $\mathbb{Z}_{\geq 0}$-sequences

$$\begin{aligned} x\colon \mathbb{Z}_{\geq 0} &\longrightarrow [n] \\ i &\longmapsto x_i, \end{aligned}$$

   which are also denoted $x = (x_0, x_1, \ldots)$.
2. the *unilateral shift on* Σ_n^+ is the function

$$\begin{aligned} \sigma\colon \Sigma_n^+ &\longrightarrow \Sigma_n^+ \\ x &\longmapsto \sigma(x)\colon \mathbb{Z}_{\geq 0} \longrightarrow [n] \\ &\qquad\qquad i \longmapsto x_{i+1}. \end{aligned}$$

That is, the function σ shifts each entry of the sequence $(x_0, x_1, \ldots)$ to the left, deleting the first symbol x_0 in the process, resulting in the sequence $(x_1, x_2, \ldots)$. This implies that each sequence x has n preimages, which (in the common case $n > 1$) implies that σ is not invertible. The dynamical system (Σ_n^+, σ) is known as the *unilateral Bernoulli*[1] *shift*.

3.1.2 Measure structure of unilateral shifts

To define the measure structure of Σ_n^+, we will first define basic sets that generate this structure.

Definition 3.2. Let $n \in \mathbb{N} \setminus \{0\}$, $i \in \mathbb{Z}_{\geq 0}$ and $s \in S$. The *basic cylinder* of index i and symbol s in Σ_n^+ is the set

$$C_i^s := \{x \in \Sigma_n^+ \mid x_i = s\}.$$

Let $i_0, \ldots, i_{k-1} \in \mathbb{Z}_{\geq 0}$ and $s_0, \ldots, s_{k-1} \in [n]$. The *cylinder* of indices $i_0, \ldots, i_{k-1}$ and symbols $s_0, \ldots, s_{k-1}$ in Σ_n^+ is the set

$$C_{i_0,\ldots,i_{k-1}}^{s_0,\ldots,s_{k-1}} := \bigcap_{j=0}^{k-1} C_{i_j}^{s_j} = \{x \in \Sigma_n^+ \mid \textstyle\bigwedge_{j=0}^{k-1} x_{i_j} = s_j\}.$$

The σ-algebra of Σ_n^+, denoted $\mathcal{M}_{\Sigma_n^+}$, is the measurable (Borel) σ-algebra generated by the the cylinder sets.

Considering the discrete σ-algebra on $[n] = \{0, \ldots, n-1\}$, any probability measure ρ on $[n]$ can be identified with a probability distribution $p = (p_0, \ldots, p_{n-1}) \in \mathbb{\Delta}^{n-1}$ by denoting $p_i := \rho(\{i\})$. We define the probability measure m_p^+ on $(\Sigma_n^+, \mathcal{M}_{\Sigma_n^+})$ as the product measure of p on $\Sigma_n^+ = [n]^{\mathbb{Z}_{\geq 0}}$, which is given by its values on the cylinders as follows:

$$m_p^+(C_i^s) := p_s.$$

Thus $(\Sigma_S^+, \mathcal{M}_{\Sigma_n^+}, m_p^+)$ is a probability space.

3.1.3 Disintegration of measure for unilateral shifts

We will present the disintegration (see DEFINITION 1.33) of the measure m_p^+ of unilateral shifts in order to ease the understanding of the case of bisymbolic shifts (PROPOSITION 3.13) we will present ahead in SECTION 3.2.

The *atomic partition* of Σ_n^+ is $\epsilon := \{\{x\} \mid x \in \Sigma_n^+\}$, the partition into points. We will disintegrate the measure m_p^+ with respect to the *dynamical pullback partition*

1. Jacob Bernoulli (1655–1705), Swiss mathematician.

$\sigma^{-1}(\epsilon) = \{\sigma^{-1}(\{x\}) \mid x \in \Sigma_n^+\}$. Also, for each symbol $s \in [n]$ and sequence $x \in \Sigma_n^+$, we denote the sequence formed by their juxtaposition by

$$sx := (s, x_0, x_1, \ldots).$$

In this way, we have $\hat{x} := \sigma^{-1}(\{x\}) = \{sx \mid s \in [n]\}$. We will denote the measure m_p^+ by m for simplicity.

Proposition 3.1. *Let $n \in \mathbb{N} \setminus \{0\}S$, (Σ_n^+, σ) be the unilateral n-symbolic shift space, $p \in \triangle^{n-1}$ a probability distribution and, for each $\hat{x} \in \sigma^{-1}(\epsilon)$, $m_{\hat{x}}$ be the probability measure on $\hat{x}$ given on each $sx \in \hat{x}$ by $m_{\hat{x}}(\{sx\}) = p_s$.*

The family $\{m_{\hat{x}}\}$ is the disintegration of m relative to $\sigma^{-1}(\epsilon)$.

Proof. It suffices to prove that, for each basic cylinder C_i^s, it holds that

$$m(C_i^s) = \int_{\hat{x} \in \sigma^{-1}(\epsilon)} m_{\hat{x}}(C_i^s) \hat{m}(\mathrm{d}\hat{x}).$$

First, we describe a base for the σ-algebra of $\sigma^{-1}(\epsilon)$, that is the pushforward algebra by the projection $\pi \colon \Sigma_n^+ \longrightarrow \sigma^{-1}(\epsilon)$. For each basic cylinder C, we define the set

$$\hat{C} := \{\hat{x} \mid x \in C\},$$

and notice that its inverse image by π is the union of the sets $sC := \{sx \mid x \in C\}$; that is

$$\pi^{-1}(C) = \bigcup_{s \in S} sC.$$

The sets $\hat{C}$ are measurable because their inverse image is a union of cylinders, hence measurable. Also, every subset must be of the form $\hat{C}$ for some set C, since the elements of $\sigma^{-1}(\epsilon)$ are of the form $\{sx \mid s \in [n]\}$ for some x. It can be checked that they are indeed a base for the σ-algebra.

The projected measure of $\hat{C}$ in $\sigma^{-1}(\epsilon)$ is

$$\hat{m}(\hat{C}) = m(\pi^{-1}(C)) = m(\bigcup_{s \in [n]} sC) = \sum_{s \in [n]} p_s m(C) = m(C).$$

Now we consider 2 cases:

- $(i = 0)$ We have the cylinder C_0^s. Calculating $m_{\hat{x}}(C_0^s)$ is the same calculating $m_{\hat{x}}(C_0^s \cap \hat{x})$. But $C_0^s \cap \hat{x} = \{sx\}$, since sx is the only element of $\hat{x}$ that is an element of C_0^s. Hence

$$m_{\hat{x}}(C_0^s) = m_{\hat{x}}(C_0^s \cap \hat{x}) = m_{\hat{x}}(\{sx\}) = p_s.$$

  Since $m_{\hat{x}}(C_0^s)$ has the same value for every $\hat{x}$, it follows that

$$\int_{\hat{x} \in \sigma^{-1}(\epsilon)} m_{\hat{x}}(C_0^s) \, \mathrm{d}\hat{m}(\hat{x}) = \int_{\hat{x} \in \sigma^{-1}(\epsilon)} p_s \, \mathrm{d}\hat{m}(\hat{x}) = p_s = m(C_0^s).$$

- $(i > 0)$ Notice that $\bigcup_{s\in S} sC^s_{i-1} = C^s_i$. We want to calculate $m_{\hat{x}}(C^s_i)$, or $m_{\hat{x}}(C^s_i \cap \hat{x})$. Consider 2 cases: 1. in the first case, $\hat{x} \in \hat{C}^s_{i-1}$, which is equivalent to $x \in C^s_{i-1}$, hence it holds that $C^s_i \cap \hat{x} = \hat{x}$ and $m_{\hat{x}}(C^s_i) = m_{\hat{x}}(\hat{x}) = 1$; 2. in the second case, $\hat{x} \notin \hat{C}^s_{i-1}$, which is equivalent to $x \notin C^s_{i-1}$, hence it holds that $C^s_i \cap \hat{x} = \mathbb{0}$ and $m_{\hat{x}}(C^s_i) = m_{\hat{x}}(\mathbb{0}) = 0$. Since $\hat{m}(\hat{C}^s_{i-1}) = m(C^s_{i-1}) = p_s$, this implies that

$$\begin{aligned}\int_{\hat{x}\in\sigma^{-1}(\epsilon)} m_{\hat{x}}(C^s_i)\hat{m}(\mathrm{d}\hat{x}) &= \int_{\hat{x}\in\hat{C}^s_{i-1}} m_{\hat{x}}(C^s_i)\hat{m}(\mathrm{d}\hat{x}) + \int_{\hat{x}\notin\hat{C}^s_{i-1}} m_{\hat{x}}(C^s_i)\hat{m}(\mathrm{d}\hat{x}) \\ &= \int_{\hat{x}\in\hat{C}^s_{i-1}} 1\hat{m}(\mathrm{d}\hat{x}) + \int_{\hat{x}\notin\hat{C}^s_{i-1}} 0\hat{m}(\mathrm{d}\hat{x}) \\ &= p_s + 0 \\ &= m(C^s_i).\end{aligned}$$

■

3.2 Bisymbolic shift

3.2.1 Space and dynamics

Analogously to the way we defined the unilateral shift in SECTION 3.1, we can define a bilateral shift in a space X with invertible dynamics $f\colon X \longrightarrow X$ and finite partition $\mathcal{P}$. For each point $p \in X$, the 'forward' sequence $(P_0, P_1, \ldots)$ represents the same as in the unilateral shift case — each part P_i is the one the point $f^i(p)$ belongs to — but now the point $f^{-1}(p)$ exists, so we can also determine the part of $\mathcal{P}$ the point $f^{-1}(p)$ belongs to, say $\mathcal{P}_{-1}$, and likewise for $f^{-2}(p)$ a part P_{-2}, and so on and so forth, establishing in this manner a 'backwards' sequence

$$(\ldots, P_{-2}, P_{-1}).$$

Concatenating both 'forward' and 'backward' sequences, we obtain a 'bilateral' sequence[2]

$$(\ldots, P_{-2}, P_{-1}; P_0, P_1, \ldots).$$

This leads to the construction of the *bilateral n-symbolic shift dynamical system* (Σ_n, σ) by defining $\Sigma_n := [n]^{\mathbb{Z}}$ and the shift σ is defined as

$$\begin{aligned}\sigma\colon \Sigma_n &\longrightarrow \Sigma_n \\ x &\longmapsto \sigma(x)\colon \mathbb{Z} \longrightarrow [n] \\ &\qquad\qquad\quad i \longmapsto x_{i+1},\end{aligned}$$

2. The semicolon (;) is used to indicate the separation of the 'forward' and 'backward' sequences.

that is, σ shifts the sequence $(\ldots, x_{-2}, x_{-1}; x_0, x_1, \ldots)$ to the sequence

$$(\ldots, x_{-1}, x_0; x_1, x_2, \ldots)$$

(notice the semicolon). In contrast to the unilateral case, the symbol x_0 is not lost in the process, and the function σ is invertible.

There is a well-developed theory of symbolic dynamics for these spaces, but here we intend to present a different construction based on these standard shift spaces. Instead of considering only one finite partition $\mathcal{P}$, we may consider two partitions $\mathcal{P}^+$ and $\mathcal{P}^-$, and use the former to encode the orbit of the point p going forward, and the latter to encode its backwards orbit. In this way, we obtain for each point $p \in X$ a sequence

$$(\ldots, P^-_{-2}, P^-_{-1}; P^+_0, P^+_1, \ldots) \tag{3.3}$$

in which the entries P^-_i (for $i \in \mathbb{Z}_{<0}$) are parts of $\mathcal{P}^-$ and the entries P^+_i (for $i \in \mathbb{Z}_{\geq 0}$) are parts of $\mathcal{P}^+$. The shift function may be defined for every entry as usual by shifting it to the left, except for the 0-th entry. If the usual procedure were to be followed, the resulting shifted sequence would have an element of the partition $\mathcal{P}^+$ in its (-1)-st position, and hence this sequence would not be of the same type as the one in FORMULA 3.3. To decide how to solve this problem (or even to better argue why these different types of sequences must not be allowed), we must understand the dynamical reason the bisymbolic shift spaces are relevant.

In order to do that, let us consider a motivating example (check [MM22]). Consider the unit square $Q := [0, 1] \times [0, 1]$. The 2-*to*-1 *baker's transformation*[3] on Q is the transformation

$$f(x, y) = \begin{cases} (4x, \frac{1}{2}y), & (x, y) \in [0, \frac{1}{4}[\times [0, 1] \\ (4x - 1, \frac{1}{2}y + \frac{1}{2}), & (x, y) \in [\frac{1}{4}, \frac{1}{2}[\times [0, 1] \\ (4x - 2, \frac{1}{2}y), & (x, y) \in [\frac{1}{2}, \frac{3}{4}[\times [0, 1] \\ (4x - 3, \frac{1}{2}y + \frac{1}{2}), & (x, y) \in [\frac{3}{4}, 1] \times [0, 1]. \end{cases}$$

$$f(x, y) = \begin{cases} (4x, \frac{1}{2}y), & (x, y) \in V_0 \\ (4x - 1, \frac{1}{2}y + \frac{1}{2}), & (x, y) \in V_1 \\ (4x - 2, \frac{1}{2}y), & (x, y) \in V_2 \\ (4x - 3, \frac{1}{2}y + \frac{1}{2}), & (x, y) \in V_3. \end{cases}$$

The transformation f dilates the square Q, cuts the dilated bands and then glues them on top or above the others (check FIGURE 3.1 for detailed visual explanation of the action of the transformation). An analogous transformation can be defined for the n-to-1 baker's transformation.

3. The name comes from the classic baker's transformation that is encoded by the bilateral shift, which comes from an analogy with the action of a baker kneading bread dough: stretching it, cutting the dough in parts and placing some parts over the others, then repeating the process.

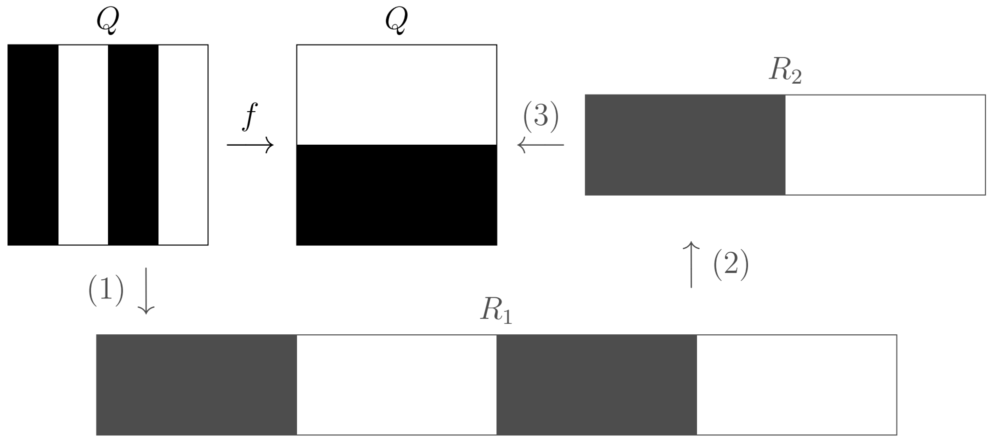


Figure 3.1. The 2-to-1 baker's transformation. On STEP 1, the square Q is dilated by 4 in the horizontal direction and by $\frac{1}{2}$ in the vertical direction, resulting in a rectangle R_1. On STEP 2, the right-most bands of the rectangle R_1 are glued over the left-most ones, resulting in a smaller rectangle R_2. On STEP 3, the left right band of the rectangle R_2 is glued above the left band, resulting again in the square Q.

The square Q has 2 natural partitions (check FIGURE 3.2) which come from the definition of the baker's transformation itself: the *vertical partition* $\mathcal{V}$ is composed of the the vertical bands

$$\begin{aligned} V_0 &:= \left[0, \frac{1}{4}\right[\times [0,1] \\ V_1 &:= \left[\frac{1}{4}, \frac{1}{2}\right[\times [0,1] \\ V_2 &:= \left[\frac{1}{2}, \frac{3}{4}\right[\times [0,1] \\ V_3 &:= \left[\frac{3}{4}, 1\right] \times [0,1] \end{aligned}$$

and the *horizontal partition* $\mathcal{H}$ is composed of the horizontal bands

$$\begin{aligned} H_a &:= [0,1] \times [0, \frac{1}{2}[\\ H_b &:= [0,1] \times [\frac{1}{2}, 1[. \end{aligned}$$

These sets are subjected to the important relations

$$f(V_0) = f(V_2) = H_a \qquad \text{and} \qquad f(V_1) = f(V_3) = H_b. \tag{3.4}$$

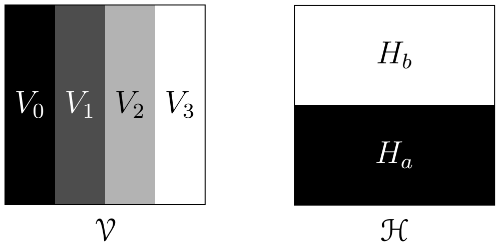


Figure 3.2. The vertical partition $\mathcal{V}$ and horizontal partition $\mathcal{H}$ of the 2-to-1 baker's transformation.

Using these partitions, we can encode the forward and backward orbits of a point p in the square Q under the transformation f. For the entries of the encoding sequences, we will use the symbol sets $S^+ := \{0, 1, 2, 3\}$ and $S^- := \{a, b\}$ instead of the horizontal and vertical bands themselves. This will simplify notation.

Take a point $p \in Q$ and let us describe its encoding sequence x. For each $i \in \mathbb{Z}_{\geq 0}$, the entry x_i of the sequence is defined to be the unique symbol $s \in S^+$ such that $f^i(p) \in V_s$; for each $i \in \mathbb{Z}_{<0}$, the entry x_i is defined to be the unique symbol $s \in S^-$ such that $f^{i+1}(p) \in H_s$. As an example, suppose such a sequence is given by

$$x = (\ldots, b, b, a, b; 0, 3, 2, 4, \ldots). \tag{3.5}$$

Notice that the forward iterations of the horizontal bands are thinner horizontal bands (check FIGURE 3.3), so in the limit the backwards sequence determines the horizontal position of the point p, while the backward iterations of the vertical bands are thinner vertical bands (check FIGURE 3.3), so in the limit the forwards sequence determines the vertical position of the point p. In this way, the sequence determines (almost every) point.

Let us understand what the dynamical behavior of a point encoded by such a sequence is. Consider a point p with encoding sequence x as in FORMULA 3.5. The backwards part of the sequence encodes that

$$p \in H_b \cap f(H_a) \cap f^2(H_b) \cap f^3(H_b) \cap \cdots,$$

so then $f(p) \in f(H_b) \cap f^2(H_a) \cap f^3(H_b) \cap f^4(H_b) \cap \cdots$, which means that the backwards part of the encoding sequence y of $f(p)$ is $(\ldots, b, b, a, b, y_{-1})$. The (-1)-st entry y_{-1} is not determined by this sequence.

Likewise, the forwards part of the sequence x in FORMULA 3.5 encodes that

$$p \in V_0 \cap f^{-1}(V_3) \cap f^{-2}(V_2) \cap f^{-3}(V_4) \cap \cdots,$$

so then $f(p) \in f(V_0) \cap V_3 \cap f^{-1}(V_2) \cap f^{-2}(V_4) \cap \cdots$. Let us separate this in the formulas $f(p) \in f(V_0)$ and $f(p) \in V_3 \cap f^{-1}(V_2) \cap f^{-2}(V_4) \cap \cdots$. The second formula

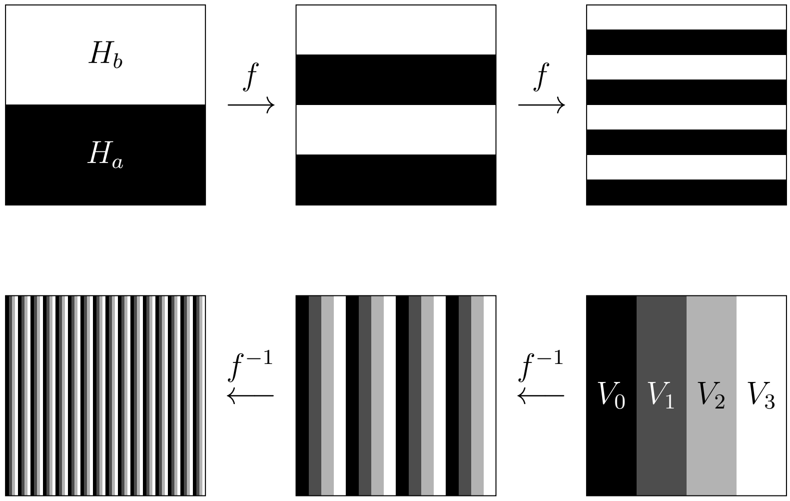


Figure 3.3. The first 2 iterations of the horizontal partition $\mathcal{H}$ and the vertical partition $\mathcal{V}$ of the 2-to-1 baker's transformation. In the limit, the iterations of the horizontal partition is the partition of the square by horizontal line, and the iteration of the vertical partition is the partition by vertical lines.

means that the forwards part of the encoding sequence y of $f(p)$ is $(3, 2, 4, \ldots)$. Since $f(V_0) = H_a$ (check FORMULA 3.4), the first formula means that the entry y_{-1} of the backwards sequence of $f(p)$ is a, which was the only entry yet to be determined, so the encoding sequence of $f(p)$ is

$$y = (\ldots, b, a, b, a; 3, 2, 4, \ldots).$$

In general, whatever the encoding sequence of p is, we can shift it to the left to obtain the backwards part of the encoding sequence of $f(p)$, except for the (-1)-st entry, and the forwards part, while the (-1)-st entry is determined by the relations in FORMULA 3.4. Based on these relations, we can define a function $\phi\colon S^+ \longrightarrow S^-$ by

$$\phi(0) = \phi(2) = a \qquad \text{and} \qquad \phi(1) = \phi(3) = b.$$

This function ϕ is surjective.

This construction can be formalized in the following definition. In the previous discussion, we used numbers and letters to emphasize the distinction between 'forward' and 'backward' symbols, but in the definition we used two sets of numbers.

Definition 3.3. Let $n_-, n_+ \in \mathbb{N} \smallsetminus \{0\}$ and $\phi\colon [n_+] \longrightarrow [n_-]$ a surjective function. The (n_-, n_+)-*bisymbolic ϕ-shift dynamical system* is the pair $(\Sigma_{n_-,n_+}, \sigma_\phi)$ in which

1. the (n_-, n_+)-*bisymbolic space* is the set

$$\Sigma_{n_-,n_+} := \{x = (\dots, x_{-1}; x_0, x_1, \dots) \mid \bigwedge_{i<0} x_i \in [n_-], \bigwedge_{i\geq 0} x_i \in [n_+]\}.$$

2. the *bisymbolic shift* with *transition function* ϕ is the function

$$\begin{aligned} \sigma_\phi \colon \Sigma_{n_-,n_+} &\longrightarrow \Sigma_{n_-,n_+} \\ x &\longmapsto \sigma_\phi(x) \colon \mathbb{Z} \longrightarrow [n_-] \cup [n_+] \\ &\qquad\qquad i \longmapsto \begin{cases} x_{i+1} & i \neq -1 \\ \phi(x_0) & i = -1. \end{cases} \end{aligned}$$

To simplify notation, we may denote $\sigma := \sigma_\phi$. Definition 3.3 determines the shift σ to take a sequence $(\dots, x_{-1}; x_0, x_1, \dots) \in \Sigma_{n_-,n_+}$ to the sequence

$$(\dots, x_{-1}, \phi(x_0); x_1, \dots) \in \Sigma_{n_-,n_+},$$

as we found analyzing the example of the 2-to-1 baker's transformation. As a consequence of ϕ being surjective, we have that $n_+ \geq n_-$.

3.2.2 Measurable structure

The σ-algebra $\mathcal{B}$ of the space Σ_{n_-,n_+} is the one generated by cylinder sets: for each $(s, i) \in [n_-] \times \mathbb{Z}_{<0}$ or $(s, i) \in [n_+] \times \mathbb{Z}_{\geq 0}$, we define the cylinder

$$C_i^s := \{x \in \Sigma_{n_-,n_+} \mid x_i = s\}.$$

We also define the *extended cylinder*

$$C_i^{\phi^{-1}(s)} := \bigcup_{s' \in \phi^{-1}(s)} C_i^{s'}.$$

The next proposition shows how the dynamics acts backwards and forwards on cylinders.

Proposition 3.2 ([MMV24]). *Let $k \in \mathbb{N}$ and $s \in [n_-] \cup [n_+]$. Then*

$$\sigma^{-k}(C_i^s) = \begin{cases} C_{i+k}^s & i \notin [-k, -1] \cap \mathbb{Z} \\ C_{i+k}^{\phi^{-1}(s)} & i \in [-k, -1] \cap \mathbb{Z}. \end{cases}$$

and

$$\sigma^k(C_i^s) = \begin{cases} C_{i-k}^s & i \notin [0, k-1] \cap \mathbb{Z} \\ C_{i-k}^{\phi(s)} & i \in [0, k-1] \cap \mathbb{Z}. \end{cases}$$

Proof. For the inverse image, it holds that

$$\sigma^{-1}(C_i^s) = \begin{cases} C_{i+1}^s & i \neq -1 \\ C_0^{\phi^{-1}(s)} & i = -1. \end{cases}$$

hence it holds that $x \in C_i^s$ if, and only if, $\sigma^{-1}(x)_{s'} \in C_{i+1}^s$ (such that $s' = s$ if $i = -1$). Then, by induction, we obtain that, for every $k \in \mathbb{N}$,

$$\sigma^{-k}(C_i^s) = \begin{cases} C_{i+k}^s & i \notin [-k,-1] \cap \mathbb{Z} \\ C_{i+k}^{\phi^{-1}(s)} & i \in [-k,-1] \cap \mathbb{Z}. \end{cases}$$

For the direct image, it holds that

$$\sigma(C_i^s) = \begin{cases} C_{i-1}^s & i \neq 0 \\ C_{-1}^{\phi(s)} & i = 0. \end{cases}$$

Then, by induction, we obtain that, for every $k \in \mathbb{N}$,

$$\sigma^k(C_i^s) = \begin{cases} C_{i-k}^s & i \notin [0,k-1] \cap \mathbb{Z} \\ C_{i-k}^{\phi(s)} & i \in [0,k-1] \cap \mathbb{Z}. \end{cases}$$ ■

3.2.3 Measure structure

In order to define a measure on $(\Sigma_{n_-,n_+}, \mathcal{B})$, it is sufficient to define it on the basic cylinders C_i^s. We start with a probability distribution $p^+ \in \mathbb{\Delta}^{n_+-1}$, which can be identified with a probability measure ρ^+ on $[n_+]$ (with the discrete σ-algebra) by setting (as in FORMULA 2.1), for each $s^+ \in [n_+]$, $\rho^+(\{s^+\}) := p^+_{s^+}$.

Using the surjective transition function $\phi\colon [n_+] \longrightarrow [n_-]$, we can define a probability distribution $p^- \in \mathbb{\Delta}^{n_--1}$ by setting, for each $s^- \in [n_-]$,

$$p^-_{s^-} := \sum_{s^+ \in \phi^{-1}(s^-)} p^+_{s^+}.$$

This is the probability distribution associated to the pushforward measure $\phi_{\vdash} p^+$, which can be calculated by considering the partition $\{\phi^{-1}(s^-)\}_{s^- \in [n_-]}$ of $[n_+]$ by the inverse images of elements of $[n_-]$ (this is a partition because ϕ is a surjective function). The pushforward measure of $\{s^-\} \subseteq [n_-]$ is then the sum of the measure of all the elements of $\phi^{-1}(s^-)$ on $[n_+]$, given for each $s^- \in [n_-]$ by

$$p^-(\{s^-\}) = \phi_{\vdash} p^+(\{s^-\}) = p^+(\phi^{-1}(\{s^-\})) = \sum_{s^+ \in \phi^{-1}(s^-)} p^+(\{s^+\}).$$

Then, for a basic cylinder C_i^s, we can define its measure as p_s^+ if $i \geq 0$ and p_s^- if $i < 0$.

Definition 3.4. Let $(\Sigma_{n_-,n_+}, \sigma_\phi)$ be a bisymbolic shift dynamical system, and $p^+ \in \mathbb{A}^{n_+-1}$ a discrete probability distribution. The *probability measure* on $(\Sigma_{n_-,n_+}, \sigma_\phi)$ induced by p^+ is the probability measure $m_{p^+} \colon \mathcal{B} \longrightarrow [0,1]$ defined on basic cylinders by

$$m_{p^+}(C_i^s) := \begin{cases} p_s^-, & i < 0 \\ p_s^+, & i \geq 0 \end{cases} = \begin{cases} \sum\limits_{s' \in \phi^{-1}(s)} p_{s'}^+, & i < 0 \\ p_s^+, & i \geq 0. \end{cases}$$

For simplicity, we denote $m := m_{p^+}$. From the way we defined the distribution $p^- \in \mathbb{A}^{n_--1}$ by the pushforward, it is easy to show that the shift dynamics is measure-preserving. We just need to be careful considering the different cases.

Proposition 3.3 ([MMV24])**.** *Let $(\Sigma_{n_-,n_+}, \sigma_\phi)$ be a bisymbolic shift dynamical system and $p^+ \in \mathbb{A}^{n_+-1}$. The dynamics σ_ϕ preserves the measure m_{p^+}.*

Proof. It suffices to show that, for every basic cylinder C_i^s,

$$m(\sigma^{-1}(C_i^s)) = m(C_i^s).$$

We consider 3 cases:

1. $(i \geq 0)$ In this case, $\sigma^{-1}(C_i^s) = C_{i+1}^s$ (PROPOSITION 3.2). Since $i+1 \geq 1$, if follows from DEFINITION 3.4 that
$$m(\sigma^{-1}(C_i^s)) = m(C_{i+1}^s) = p_s^+ = m(C_i^s).$$
2. $(i < -1)$ In this case, it also holds that $\sigma^{-1}(C_i^s) = C_{i+1}^s$ (PROPOSITION 3.2). Since $i+1 < 0$, i follows from DEFINITION 3.4 that
$$m(\sigma^{-1}(C_i^s)) = m(C_{i+1}^s) = p_s^- = m(C_i^s).$$
3. $(i = -1)$ In this case, $\sigma^{-1}(C_i^s) = C_0^{\phi^{-1}(s)} = \bigcup_{s' \in \phi^{-1}(s)} C_0^{s'}$ (PROPOSITION 3.2). Since $i+1 = 0$, it follows from DEFINITION 3.4 that
$$m(\sigma^{-1}(C_i^s)) = m\Big(\bigcup_{s' \in \phi^{-1}(s)} C_0^{s'}\Big) = \sum_{s' \in \phi^{-1}(s)} m(C_0^{s'}) = \sum_{s' \in \phi^{-1}(s)} p_{s'}^+ = m(C_i^s). \quad \blacksquare$$

3.2.4 Measure entropy

In this section we calculate the metric entropy of $(\Sigma_{n_-,n_+}, \sigma_\phi)$ and relate it to the entropy of the probability distributions p^+ and p^-.

Partitions by cylinders

We start by defining partitions by cylinders, since all our calculations depend on them.

Definition 3.5. Let $i \in \mathbb{Z}$. The *partition by cylinders of index i* is the partition

$$\mathcal{C}_i := \begin{cases} \{C_i^s \mid s \in [n_+]\} & i \geq 0 \\ \{C_i^s \mid s \in [n_-]\} & i < 0. \end{cases}$$

Let $n, n' \in \mathbb{Z}$. The *partition by cylinders of indices from n to n'* is the partition

$$\mathcal{C}_{n,\ldots,n'} := \bigvee_{i=n}^{n'} \mathcal{C}_i.$$

The following simple lemma sums up how the dynamics of the σ acts on these partitions.

Lemma 3.4 ([MMV24])**.** *For every $i \geq 0$,*

1. $\sigma^i(\mathcal{C}_0) = \mathcal{C}_{-i}$;
2. $\sigma^{-i}(\mathcal{C}_0) = \mathcal{C}_i$;
3. $\sigma^{-i}(\mathcal{C}_{-(i+1)}) = \mathcal{C}_{-1}$;
4. $\mathcal{C}_0^n = \mathcal{C}_{0,\ldots,n-1}$;
5. $\mathcal{C}_0^{\pm n} = \mathcal{C}_{-n,\ldots,n-1}$.

Proof. This is a consequence of PROPOSITION 3.2.

1. Since $\sigma(C_0^s) = C_{-1}^{\phi(s)}$ and ϕ is surjective, it follows that $\sigma(\mathcal{C}_0) = \mathcal{C}_{-1}$. By induction, $\sigma^i(\mathcal{C}_0) = \mathcal{C}_{-i}$.
2. Since $\sigma^{-1}(C_0^s) = C_1^s$, it follows that $\sigma^{-1}(\mathcal{C}_0) = \mathcal{C}_1$. By induction, $\sigma^{-i}(\mathcal{C}_0) = \mathcal{C}_i$.
3. Since $\sigma^{-1}(C_{-s}^s) = C_{-1}^s$, it follows that $\sigma^{-1}(\mathcal{C}_{-2}) = \mathcal{C}_{-1}$. By induction, $\sigma^{-i}(\mathcal{C}_{-(i+1)}) = \mathcal{C}_{-1}$.
4. It follows that
$$\mathcal{C}_0^n = \bigvee_{i=0}^{n-1} \sigma^{-i}(\mathcal{C}_0) = \bigvee_{i=0}^{n-1} \mathcal{C}_i.$$
5. It follows that
$$\mathcal{C}_0^{\pm n} = \bigvee_{i=-n}^{n-1} \sigma^{-i}(\mathcal{C}_0) = \bigvee_{i=-n}^{n-1} \mathcal{C}_i. \qquad \blacksquare$$

Entropy of partitions by cylinders

Let us calculate the metric entropy of the partitions $\mathcal{C}_0$ and $\mathcal{C}_{-1}$.

Lemma 3.5 ([MMV24]). *Let $(\Sigma_{n_-,n_+}, \sigma_\phi)$ be an extended shift system with measure m defined by a probability distribution $p^+ \in \triangle^{n_+-1}$. Then*

1. $\mathrm{H}_m(\mathcal{C}_0) = \mathrm{H}_{\triangle^{n_+-1}}(p^+)$;
2. $\mathrm{H}_m(\mathcal{C}_{-1}) = \mathrm{H}_{\triangle^{n_--1}}(p^-)$;

Proof. The measure entropy of $\mathcal{C}_0$ follows from the simple calculation

$$\mathrm{H}_m(\mathcal{C}_0) = -\sum_{0\leq s<n_+} m(C_0^s)\log(m(C_0^s)) = -\sum_{0\leq s<n_+} p_s^+ \log(p_s^+) = \mathrm{H}_{\triangle^{n_+-1}}(p^+).$$

and, likewise, the measure of $\mathcal{C}_{-1}$ follows from

$$\mathrm{H}_m(\mathcal{C}_{-1}) = -\sum_{0\leq s<n_-} m(C_{-1}^s)\log(m(C_{-1}^s)) = -\sum_{0\leq s<n_-} p_s^- \log(p_s^-) = \mathrm{H}_{\triangle^{n_--1}}(p^-).$$

■

This shows, as could be expected, that the entropy of the partition $\mathcal{C}_0$ is the entropy of p^+, the measure of the positive part of the extended shift Σ_{n_-,n_+}, while the entropy of the partition $\mathcal{C}_{-1}$ is the entropy of p^-, the measure of the negative part of Σ_{n_-,n_+}. We can now calculate the measure entropy of a partition by cylinders other than the basic $\mathcal{C}_0$ and $\mathcal{C}_{-1}$.

Lemma 3.6 ([MMV24]). $\mathrm{H}_m(\mathcal{C}_{-n,\dots,0,\dots,n'-1}) = n\mathrm{H}_m(\mathcal{C}_{-1}) + n'\mathrm{H}_m(\mathcal{C}_0)$.

Proof. For every $i \geq 1$, it holds that $\mathcal{C}_i = \sigma^{-i}(\mathcal{C}_0)$ and $\sigma^{-i}(\mathcal{C}_{-(i+1)}) = \mathcal{C}_{-1}$ (LEMMA 3.4). Since σ preserves the measure m (PROPOSITION 3.3), it follows that $\mathrm{H}_m(\mathcal{C}_i) = \mathrm{H}_m(\mathcal{C}_0)$ and $\mathrm{H}_m(\mathcal{C}_{-(i+1)}) = \mathrm{H}_m(\mathcal{C}_{-1})$.

Besides that, for any integers $i < i'$, the partitions $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ are independent, because $C_i^s \cap C_{i'}^{s'} = C_{i,i'}^{s,s'}$ and $m(C_{i,i'}^{s,s'}) = m(C_i^s)m(C_{i'}^{s'})$. Thus it follows that

$$\mathrm{H}_m(C_{-n,\dots,0,\dots,n'-1}) = \mathrm{H}_m\left(\bigvee_{i=-n}^{n'-1} \mathcal{C}_i\right) = \sum_{i=-n}^{n'-1} \mathrm{H}_m(\mathcal{C}_i) = n\mathrm{H}_m(\mathcal{C}_{-1}) + n'\mathrm{H}_m(\mathcal{C}_0). \quad ■$$

In particular, since $\mathcal{C}_0^n = \mathcal{C}_{0,\dots,n-1}$ (LEMMA 3.4), this implies that

$$\mathrm{h}_m(\sigma, \mathcal{C}_0) = \lim_{n\to\mathbb{N}} \frac{1}{n}\mathrm{H}_m(\mathcal{C}_0^n) = \lim_{n\to\mathbb{N}} \frac{1}{n} n\mathrm{H}_m(\mathcal{C}_0) = \mathrm{H}_m(\mathcal{C}_0).$$

Entropy of the system

To calculate the measure entropy of the system, we will use the Kolmogorov–Sinai theorem (THEOREM 2.4). To that end we define a sequence of partitions.

Definition 3.6. $\mathcal{P}_n := \mathcal{C}_0^{\pm n} = \mathcal{C}_{-n,\ldots,n-1}$.

We will eventually need to use the measure entropy of $\mathcal{P}_n^k$, the kth dynamical correfinement of the partition $\mathcal{P}_n$, so the following lemma shows that it is just a partition by cylinders. The proof is trickier than would be expected.

Lemma 3.7 ([MMV24])**.** *Let $n \geq 1$ and $k \geq 2n$. Then $\mathcal{P}_n^k = \mathcal{C}_{-n,\ldots,n+k-2}$.*

Proof. The dynamical correfinement of $\mathcal{P}_n$ is defined by $\mathcal{P}_n^k = \bigvee_{j=0}^{k-1} \sigma^{-j}(\mathcal{P}_n)$, so let us first calculate a generic element of the pulled-back partition

$$\sigma^{-j}(\mathcal{P}_n) = \{\sigma^{-j}(C) \mid C \in \mathcal{P}_n\}.$$

Each cylinder of $\mathcal{P}_n = \mathcal{C}_{-n,\ldots,n-1}$ has the form

$$C_{-n,\ldots,n-1}^{s_{-n},\ldots,s_{n-1}} = \bigcap_{i=-n}^{n-1} C_i^{s_i},$$

with $s_i \in [n_-]$ if $i < 0$ and $s_i \in [n_+]$ if $i \geq 0$. Then

$$\sigma^{-j}(C_{-n,\ldots,n-1}^{s_{-n},\ldots,s_{n-1}}) = \sigma^{-j}(\bigcap_{i=-n}^{n-1} C_i^{s_i}) = \bigcap_{i=-n}^{n-1} \sigma^{-j}(C_i^{s_i}).$$

Based on PROPOSITION 3.2, we can separate this in 3 intersections[4] as follows:

$$\begin{aligned} \sigma^{-j}(C_{-n,\ldots,n-1}^{s_{-n},\ldots,s_{n-1}}) &= \bigcap_{i=-n}^{-(j+1)} \sigma^{-j}(C_i^{s_i}) \cap \bigcap_{i=-j}^{-1} \sigma^{-j}(C_{-1}^{s_{-1}}) \cap \bigcap_{i=0}^{n-1} \sigma^{-j}(C_i^{s_i}) \\ &= \bigcap_{i=-n}^{-(j+1)} C_{i+j}^{s_i} \cap \bigcap_{i=-j}^{-1} C_{i+j}^{\phi^{-1}(s_i)} \cap \bigcap_{i=0}^{n-1} C_{i+j}^{s_i}. \end{aligned} \tag{3.6}$$

Notice that in FORMULA 3.6, for $-n \leq i \leq -(j+1)$ and $0 \leq i \leq n-1$ we have basic cylinders of the form $C_{i+j}^{s_i}$ and, for $-j \leq i \leq -1$, we have extended cylinders (unions of cylinders) of the form

$$C_{i+j}^{\phi^{-1}(s_i)} = \bigcup_{s \in \phi^{-1}(s_i)} C_{i+j}^{s}.$$

4. In order to simplify notation, we define that intersections that have the top index strictly smaller than the bottom index should be considered to be the whole space Σ, so that they can be ignored. In FORMULA 3.6, this happens for the first intersection in the case $j > n-1$ (or equivalently $-(j+1) < -n$) and for the second intersection in the case $j = 0$ (or equivalently $-1 < -j$).

This shows that $\sigma^{-j}(\mathcal{P}_n)$ is not a partition by cylinders (unless ϕ is bijective and hence the sets $\phi^{-1}(s_i^j)$ are singletons, but this is just a regular shift, not the usual case for extended shifts).

We must now calculate a generic element of $\mathcal{P}_n^k = \bigvee_{j=0}^{k-1} \sigma^{-j}(\mathcal{P}_n)$. To that end, for each $0 \leq j \leq k-1$ we take cylinders $C^j \in \mathcal{P}_n$, defined by

$$C^j := C_{-n,\dots,n-1}^{s_{-n}^j,\dots,s_{n-1}^j} = \bigcap_{i=-n}^{n-1} C_i^{s_i^j}$$

with $s_i^j \in [n_-]$ if $i < 0$ and $s_i^j \in [n_+]$ if $i \geq 0$. An element of $\mathcal{P}_n^k$ is a non-empty set of the form $\bigcap_{j=0}^{k-1} \sigma^{-j}(C^j)$. From FORMULA 3.6, it follows that this set is given by

$$\bigcap_{j=0}^{k-1} \sigma^{-j}(C_{-n,\dots,n-1}^{s_{-n}^j,\dots,s_{n-1}^j}) = \bigcap_{j=0}^{k-1} \bigcap_{i=-n}^{-(j+1)} C_{i+j}^{s_i^j} \cap \bigcap_{j=0}^{k-1} \bigcap_{i=-j}^{-1} C_{i+j}^{\phi^{-1}(s_i^j)} \cap \bigcap_{j=0}^{k-1} \bigcap_{i=0}^{n-1} C_{i+j}^{s_i^j}. \tag{3.7}$$

This shows that a generic element of $\mathcal{P}_n^k$ (as in FORMULA 3.7) is an intersection of basic cylinders and extended cylinders (which are unions of basic cylinders). These cylinders on the right-hand side of FORMULA 3.7 are indexed by $l := i + j$, which varies between $-n$ and $n - k - 2$ since j varies between 0 and $k - 1$, and i varies between $-n$ and $n - 1$.

We wish to find conditions on the symbols s_i^j that guarantee the intersections in FORMULA 3.7 is non-empty. For that, we will reorganize the intersections based on the indices l and j. Define B_l to be the intersection of every cylinder and extended cylinder in FORMULA 3.7 that has index l. Thus

$$\bigcap_{j=0}^{k-1} \sigma^{-j}(C_{-n,\dots,n-1}^{s_{-n}^j,\dots,s_{n-1}^j}) = \bigcap_{l=-n}^{n+k-2} B_l, \tag{3.8}$$

and each set B_l is an intersection that depends on a range of values of j.

Since the intersection of a cylinder or extended cylinder with another cylinder or extended cylinder is non-empty if they have different indices, the intersection on the right-hand side of FORMULA 3.8 is non-empty if, and only if, each $B_l \neq \emptyset$. In what follows we shall determine the range of j for each l and find conditions on the symbols s_i^j. We separate our analysis in many cases.

1. ($-n \leq l \leq -1$) In this case $0 \leq j \leq l + n$ and no extended cylinder occurs. In order to have $B_l \neq \emptyset$, all the relations in TABLE 3.1 must be satisfied, and hence

$$B_l = \bigcap_{j=0}^{l+n} C_l^{s_{l-j}^j} = C_l^{s_l^0}. \tag{3.9}$$

2. $(0 \leq l \leq n-1)$ In this case, when $0 \leq j \leq l$ we have basic cylinders and when $l+1 \leq j \leq l+n$ we have extended cylinders. In order to have $B_l \neq \mathbb{0}$, all the relations in TABLE 3.1 must be satisfied, and hence

$$B_l = \bigcap_{j=0}^{l} C_l^{s^j_{l-j}} \cap \bigcap_{j=l+1}^{l+n} C_l^{\phi^{-1}(s^j_{l-j})} = C_l^{s^l_0}. \tag{3.10}$$

3. $(n \leq l \leq k-n-1)$ In this case, when $l-n+1 \leq j \leq l$ we have basic cylinders and when $l+1 \leq j \leq l+n$ we have extended cylinders. In order to have $B_l \neq \mathbb{0}$, all the relations in TABLE 3.1 must be satisfied, and hence

$$B_l = \bigcap_{j=l-n+1}^{l} C_l^{s^j_{l-j}} \cap \bigcap_{j=l+1}^{l+n} C_l^{\phi^{-1}(s^j_{l-j})} = C_l^{s^l_0}. \tag{3.11}$$

4. $(k-n \leq l \leq k-2)$ In this case, when $l-n+1 \leq j \leq l$ we have basic cylinders and when $l+1 \leq j \leq k-1$ we have extended cylinders. In order to have $B_l \neq \mathbb{0}$, all the relations in TABLE 3.1 must be satisfied, and hence

$$B_l = \bigcap_{j=l-n+1}^{l} C_l^{s^j_{l-j}} \cap \bigcap_{j=l+1}^{k-1} C_l^{\phi^{-1}(s^j_{l-j})} = C_l^{s^l_0}. \tag{3.12}$$

5. $(k-1 \leq l \leq k+n-2)$ In this case $l-n+1 \leq j \leq k-1$ and no extended cylinder occurs. In order to have $B_l \neq \mathbb{0}$, all the relations in TABLE 3.1 must be satisfied, and hence

$$B_l = \bigcap_{j=l-n+1}^{k-1} C_l^{s^j_{l-j}} = C_l^{s^{k-1}_{l-k+1}}. \tag{3.13}$$

Thus using FORMULAS 3.9 to 3.13 on FORMULA 3.8, if follows that

$$\bigcap_{j=0}^{k-1} \sigma^{-j}(C_{-n,\dots,n-1}^{s^j_{-n},\dots,s^j_{n-1}}) = \bigcap_{l=-n}^{-1} C_l^{s^0_l} \cap \bigcap_{l=0}^{k-2} C_l^{s^l_0} \cap \bigcap_{l=k-1}^{n-1+k-1} C_l^{s^{k-1}_l},$$

that is, a generic element of $\mathcal{P}_n^k$ is a cylinder of $\mathcal{C}_{-n,\dots,n+k-2}$, and every such cylinder can be formed in this way because the symbols $s^0_{-n}, \dots, s^0_0, \dots, s^{k-1}_0, \dots, s^{k-1}_{n-1}$ can be chosen arbitrarily, so we conclude that $\mathcal{P}_n^k = \mathcal{C}_{-n,\dots,n+k-2}$. ■

It is now trivial to conclude the following last results.

Lemma 3.8 ([MMV24]). $\mathrm{h}_m(\sigma, \mathcal{P}_n) = \mathrm{H}_m(\mathcal{C}_0)$.

l	Relations		
$-n$	s^0_{-n}		
$-n+1$	$s^0_{-n+1} = s^1_{-n}$		
$\vdots$	$\vdots$		
-1	$s^0_{-n} = \cdots = s^{n-1}_{-n}$		
0	s^0_0	$\in$	$\phi^{-1}(s^1_{-1}) = \cdots = \phi^{-1}(s^n_{-n})$
1	$s^0_1 = s^1_0$	$\in$	$\phi^{-1}(s^2_{-1}) = \cdots = \phi^{-1}(s^{n+1}_{-n})$
$\vdots$		$\vdots$	
$n-1$	$s^0_{n-1} = \cdots = s^{n-1}_0$	$\in$	$\phi^{-1}(s^n_{-1}) = \cdots = \phi^{-1}(s^{2n-1}_{-n})$
n	$s^1_{n-1} = \cdots = s^n_0$	$\in$	$\phi^{-1}(s^{n+1}_{-1}) = \cdots = \phi^{-1}(s^{2n}_{-n})$
$\vdots$		$\vdots$	
$k-1-n$	$s^{k-2n}_{n-1} = \cdots = s^{k-1-n}_0$	$\in$	$\phi^{-1}(s^{k-n}_{-1}) = \cdots = \phi^{-1}(s^{k-1}_{-n})$
$k-n$	$s^{k-2n+1}_{n-1} = \cdots = s^{k-n}_0$	$\in$	$\phi^{-1}(s^{k-n+1}_{-1}) = \cdots = \phi^{-1}(s^{k-1}_{-(n-1)})$
$\vdots$		$\vdots$	
$k-2$	$s^{k-1-n}_{n-1} = \cdots = s^{k-2}_0$	$\in$	$\phi^{-1}(s^{k-1}_{-1})$
$k-1$	$s^{k-n}_{n-1} = \cdots = s^{k-1}_0$		
k	$s^{k-n+1}_{n-1} = \cdots = s^{k-1}_1$		
$\vdots$	$\vdots$		
$n+k-2$	s^{k-1}_{n-1}		

Table 3.1. Relations between the symbols $s^j_i = s^j_{l-j}$ from FORMULA 3.7 for $-n \le l \le n+k-2$. For each l, the symbol in red determines every other symbol in that line.

Proof. From LEMMAS 3.6 and 3.7 it follows that

$$\mathrm{H}_m(\mathcal{P}^k_n) = \mathrm{H}_m(\mathcal{C}_{-n,\dots,n-1+k-1}) = n\mathrm{H}_m(\mathcal{C}_{-1}) + (n+k-1)\mathrm{H}_m(\mathcal{C}_0),$$

therefore

$$\begin{aligned}\mathrm{h}_m(\sigma, \mathcal{P}_n) &= \lim_{k\to\infty} \frac{1}{k}\mathrm{H}_m(\mathcal{P}^k_n)\\ &= \lim_{k\to\infty} \frac{1}{k}(n\mathrm{H}_m(\mathcal{C}_{-1}) + (n+k-1)\mathrm{H}_m(\mathcal{C}_0))\\ &= \mathrm{H}_m(\mathcal{C}_0).\end{aligned}$$

■

Theorem 3.9 ([MMV24])**.** $\mathrm{h}_m(\sigma) = \mathrm{H}_m(\mathcal{C}_0)$.

Proof. The sequence of partitions $\mathcal{P}_n = \mathcal{C}_{-n,\dots,n-1}$ $(n \in \mathbb{N})$ is increasing relative to the refinement order:

$$\mathcal{P}_0 \preceq \mathcal{P}_1 \preceq \cdots \preceq \mathcal{P}_n \preceq \cdots .$$

Besides that, the union of $\mathcal{P}_n$ generates the σ-algebra of the space Σ_{n_-,n_+}. Finally, the entropy of $\mathcal{P}_n$ is finite, because the entropy of $\mathcal{C}_{-1}$ and $\mathcal{C}_0$ are finite. Therefore, by the Kolmogorov–Sinai theorem (THEOREM 2.4), the measure entropy of the system is

$$\mathrm{h}_m(\sigma) = \lim_{n\to\infty} \mathrm{h}_m(\sigma, \mathcal{P}_n).$$

We thus have to calculate $\mathrm{h}_m(\sigma, \mathcal{P}_n)$, which is, by definition,

$$\mathrm{h}_m(\sigma, \mathcal{P}_n) := \lim_{k\to\infty} \frac{1}{k}\mathrm{H}_m(\mathcal{P}_n^k),$$

which shows we have to calculate $\mathrm{H}_m(\mathcal{P}_n^k)$.

This finally implies that

$$\mathrm{h}_m(\sigma) = \lim_{n\to\infty} \mathrm{h}_m(\sigma, \mathcal{P}_n) = \mathrm{H}_m(\mathcal{C}_0). \qquad \blacksquare$$

3.2.5 Folding entropy

In this section we calculate the folding entropy of the extended shift (see SECTION 2.3.3 for the definition of the folding entropy).

As a consequence of ϕ being surjective, we have that $n_+ \geq n_-$. When $n_+ > n_-$, the shift σ is not invertible and, for any given $x \in \Sigma_{n_-,n_+}$, the set $\sigma^{-1}(x)$ has more than one element. In the following discussion, we shall need a way the refer to each element of $\sigma^{-1}(x)$, so, for each $s \in \phi^{-1}(x_{-1})$, we define[5]

$$\hat{x}(s) := (\ldots, x_{-2}; s, x_0, \ldots). \tag{3.14}$$

and

$$\hat{x} := \sigma^{-1}(x) = \{\hat{x}(s) \mid s \in \phi^{-1}(x_{-1})\}. \tag{3.15}$$

From DEFINITION 2.11, the folding entropy of σ is given by

$$\mathcal{F}(\sigma) = \mathrm{H}_m(\epsilon \mid \sigma^{-1}(\epsilon))$$

and, from FORMULA 2.10, the conditional entropy of the atomic partition ϵ with respect to the dynamical pullback $\sigma^{-1}(\epsilon) = \{\hat{x} \mid x \in \Sigma_{n_-,n_+}\}$ can be calculated by

$$\mathrm{H}_m(\epsilon \mid \sigma^{-1}(\epsilon)) = \int_{\hat{x}\in\sigma^{-1}(\epsilon)} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}})\hat{m}(\mathrm{d}\hat{x}),$$

in which $(m_{\hat{x}})_{\hat{x}\in\sigma^{-1}(\epsilon)}$ is the disintegration of m with respect to $\sigma^{-1}(\epsilon)$ and $\hat{m} := m_{\sigma^{-1}(\epsilon)}$ is the quotient measure of $\sigma^{-1}(\epsilon)$.

So in order to calculate the folding entropy of σ, we need to find the quotient measure $\hat{m}$ and to disintegrate the measure m with respect to the dynamical pullback $\sigma^{-1}(\epsilon)$ of the atomic partition ϵ of Σ_{n_-,n_+}.

5. A possibly more descriptive, but longer, alternative notation is $\sigma^{-1}(x^-); sx^+$.

The quotient measure of the dynamical pullback of the atomic partition

Let us first determine the quotient σ-algebra $\hat{\mathcal{B}}$, which is the pushforward of the cylinders σ-algebra of Σ_{n_-,n_+} by the natural projection π.

Definition 3.7. Let $(\Sigma_{n_-,n_+}, \sigma)$ be a bisymbolic shift dynamical system. For each $X \subseteq \Sigma$, the *quotient projection* of X onto $\sigma^{-1}(\epsilon)$ is the set

$$\hat{X} := \{\hat{x} \mid x \in X\} = \{\sigma^{-1}(x) \mid x \in X\} \subseteq \sigma^{-1}(\epsilon).$$

Since $\hat{x} = \sigma^{-1}(x)$, it may be confusing to understand the difference between the sets $\hat{X}$ and $\sigma^{-1}(X)$. To better understand the notation, it is worth noticing that, if $x \in X$, then $\hat{x} = \sigma^{-1}(x) \subseteq \sigma^{-1}(X)$; that is, for each $s \in \phi \in (x_{-1})$, we have $\hat{x}(s) \in \sigma^{-1}(X)$. This shows that the elements of the set $\hat{x}$ (which is an element of $\hat{X}$) do not belong to the set $\hat{X}$, but instead to $\sigma^{-1}(X)$. To further avoid confusion, consider this example. Suppose $x, y \in \Sigma_{n_-,n_+}$, $\hat{x} = \{\hat{x}(0), \hat{x}(1)\}$ and $\hat{y} = \{\hat{y}(0), \hat{y}(1)\}$. If $X = \{x, y\}$, then

$$\hat{X} = \{\hat{x}, \hat{y}\} = \{\{\hat{x}(0), \hat{x}(1)\}, \{\hat{y}(0), \hat{y}(1)\}\},$$

while $\sigma^{-1}(X) = \{\hat{x}(0), \hat{x}(1), \hat{y}(0), \hat{y}(1)\}$. The next proposition shows explicitly how the two sets are related. Let us denote the natural projection with respect to the partition $\sigma^{-1}(\epsilon)$ by $\pi \colon \Sigma_{n_-,n_+} \longrightarrow \sigma^{-1}(\epsilon)$.

Proposition 3.10 ([MMV24])**.** *Let $(\Sigma_{n_-,n_+}, \sigma)$ be a bisymbolic shift dynamical system and $\pi \colon \Sigma_{n_-,n_+} \longrightarrow \sigma^{-1}(\epsilon)$ the natural projection induced by the dynamical pullback of the atomic partition. For every set $X \subseteq \Sigma_{n_-,n_+}$,*

$$\pi^{-1}(\hat{X}) = \sigma^{-1}(X).$$

Proof. This follows directly from

$$\pi^{-1}(\hat{X}) = \bigcup \hat{X} = \bigcup \{\sigma^{-1}(x) \mid x \in X\} = \sigma^{-1}(X). \qquad \blacksquare$$

In particular, it is worth noting that, for a cylinder C_i^s,

$$\pi^{-1}(\hat{C}_i^s) = \sigma^{-1}(C_i^s) = \begin{cases} C_{i+1}^s & i \neq -1 \\ \bigcup_{s' \in \phi^{-1}(s)} C_0^{s'} & i = -1. \end{cases}$$

Proposition 3.11 ([MMV24])**.** *Let $(\Sigma_{n_-,n_+}, \sigma)$ be a bisymbolic shift dynamical system. The quotient σ-algebra $\hat{\mathcal{B}}$ is generated by the projected cylinder sets $\hat{C}$ ($C \in \mathcal{B}$ is a cylinder).*

Proof. Now that $\mathcal{Q} \subseteq \sigma^{-1}(\epsilon)$. Since each element of $\sigma^{-1}(\epsilon)$ is of the form $\hat{x}$ for some $x \in \Sigma$, there exists a set $X \subseteq \Sigma_{n_-,n_+}$ such that $\mathcal{Q} = \{\hat{x} \mid x \in X\} = \hat{X}$. This implies that its inverse image by the projection is of the form $\pi^{-1}(\mathcal{Q}) = \pi^{-1}(\hat{X}) = \sigma^{-1}(X)$. This shows that $\hat{\mathcal{B}}$ is generated by sets $\hat{C}$ such that $\sigma^{-1}(C) \in \mathcal{B}$ is a cylinder, which means that C is also a cylinder. ■

The quotient measure $\hat{m} := \pi_{\llcorner} m$ on $\sigma^{-1}(\epsilon)$ is the pushforward of m by the natural projection $\pi \colon \Sigma_{n_-,n_+} \longrightarrow \sigma^{-1}(\epsilon)$ of the dynamical pullback of the atomic partition. The next proposition shows how we can easily calculate it using the original measure m.

Proposition 3.12 ([MMV24]). *Let (Σ, σ) be a bisymbolic shift dynamical system. For every measurable set $M \subseteq \Sigma_{n_-,n_+}$,*

$$\hat{m}(\hat{M}) = m(M).$$

Proof. Since $\pi^{-1}(\hat{M}) = \sigma^{-1}(M)$ (PROPOSITION 3.10) and σ is measure-preserving (PROPOSITION 3.3), it follows that

$$\hat{m}(\hat{M}) = m(\pi^{-1}(\hat{M})) = m(\sigma^{-1}(M)) = m(M). \qquad ■$$

Disintegration of the measure with respect to the dynamical pullback of the atomic partition

We wish to disintegrate the measure $m = m_{p^+}$ with respect to the pullback partition $\sigma^{-1}(\epsilon)$. In order to do that, we must find the quotient measure $\hat{m}$ on $\sigma^{-1}(\epsilon)$ and, for each $\hat{x} \in \sigma^{-1}(\epsilon)$, the conditional measures $m_{\hat{x}}$ on Σ_{n_-,n_+}, in such a way that, for every measurable set $M \in \mathcal{B}$, it holds that

$$m(M) = \int_{\hat{x} \in \sigma^{-1}(\epsilon)} m_{\hat{x}}(M) \hat{m}(\mathrm{d}\hat{x}).$$

To define the conditional measures on $\hat{x}$, remember that $\hat{x} = \{\hat{x}(s) \mid s \in \phi^{-1}(x_{-1})\}$ and that the conditional measure is supported on $\hat{x}$, so, for each measurable set $M \in \mathcal{B}$, it is given by $m_{\hat{x}}(M) = m_{\hat{x}}(M \cap \hat{x})$. Thus, since $\hat{x}$ is finite, we can define it on each atom $\{\hat{x}(s)\}$.

Based on the probability distribution $p^+ \in \mathbb{A}^{n_+-1}$ we have described how to induce a probability distribution $p^- \in \mathbb{A}^{n_--1}$ by taking the pushforward of p^+ by the transition function ϕ (SECTION 3.2.3). Using the two distributions p^+ and p^-, we can define, for each $s^- \in S^-$, a new probability measure q^{s^-} on $\phi^{-1}(s^-)$ by setting, for each $s^+ \in \phi^{-1}(s^-)$

$$q^{s^-}_{s^+} := \frac{p^+_{s^+}}{p^-_{s^-}}.$$

This is a probability measure because, for each $s^- \in [n_-]$

$$\sum_{s^+\in\phi^{-1}(s^-)} q^{s^-}_{s^+} = \sum_{s^+\in\phi^{-1}(s^-)} \frac{p^+_{s^+}}{p^-_{s^-}} = \frac{\sum\limits_{s^+\in\phi^{-1}(s^-)} p^+_{s^+}}{p^-_{s^-}} = 1.$$

It is important to notice that, as a direct consequence of this definition,

$$p^+ = (p^+_{s^+})_{s^+\in[n_+]} = ((p^-_{s^-} q^{s^-}_{s^+})_{s^+\in\phi^{-1}(s^-)})_{s^-\in[n_-]} = \bigoplus_{0\le s^- < n_-} p^-_{s^-} q^{s^-}. \tag{3.16}$$

We use these measures q^{s^-} to define the conditional measures as follows, by identifying the set $\hat{x}$ with the preimage $\phi^{-1}(x_{-1})$.

Definition 3.8. Let $(\Sigma_{n_-,n_+}, \sigma)$ be a bisymbolic shift dynamical system, $p^+ \in \mathbb{\Delta}^{n_+-1}$ a probability distribution and $\hat{x} \in \sigma^{-1}(\epsilon)$. The *conditional measure $m_{\hat{x}}$ on $\hat{x}$* is the probability measure defined, for each $s \in \phi^{-1}(x_{-1})$, by

$$m_{\hat{x}}(\{\hat{x}(s)\}) := q^{x_{-1}}_s = \frac{p^+_s}{p^-_{x_{-1}}}.$$

Now we show this is the disintegration of m.

Proposition 3.13 ([MMV24])**.** *Let $(\Sigma_{n_-,n_+}, \sigma)$ be a bisymbolic shift dynamical system. The family $\{m_{\hat{x}}\}_{\hat{x}\in\sigma^{-1}(\epsilon)}$ is the disintegration of m with respect to $\sigma^{-1}(\epsilon)$.*

Proof. It suffices to show that, for each basic cylinder C^s_i, it holds that

$$m(C^s_i) = \int_{\hat{x}\in\sigma^{-1}(\epsilon)} m_{\hat{x}}(C^s_i \cap \hat{x}) \hat{m}(\mathrm{d}\hat{x}).$$

First let us calculate the sets $C^s_i \cap \hat{x}$. For any set $C \subseteq \Sigma_{n_-,n_+}$, it holds that $x \in C$ if, and only if, $\hat{x} \subseteq \sigma^{-1}(C)$. Because of this, we must consider the cases $\hat{x} \in \widehat{\sigma(C^s_i)}$ and $\hat{x} \notin \widehat{\sigma(C^s_i)}$; or equivalently, $x \in \sigma(C^s_i)$ and $x \notin \sigma(C^s_i)$. According to PROPOSITION 3.2, the expression for $\sigma(C^s_i)$ depends on the value for i, so we consider 2 cases:

- ($i \neq 0$) In this case, we have $\sigma(C^s_i) = C^s_{i-1}$, hence

$$C^s_i \cap \hat{x} = \begin{cases} \hat{x} & \hat{x} \in \hat{C}^s_{i-1} \\ \mathbb{0} & \hat{x} \notin \hat{C}^s_{i-1}. \end{cases}$$

Since $m_{\hat{x}}(\hat{x}) = 1$ e $m_{\hat{x}}(\mathbb{0}) = 0$, it follows that

$$\begin{aligned} m(C_i^s) &= m(C_{i-1}^s) \\ &= \hat{m}(\hat{C}_{i-1}^s) \\ &= \int_{\hat{x}\in\hat{C}_{i-1}^s} 1\hat{m}(\mathrm{d}\hat{x}) + \int_{\hat{x}\in\sigma^{-1}(\epsilon)\setminus\hat{C}_{i-1}^s} 0\hat{m}(\mathrm{d}\hat{x}) \\ &= \int_{\hat{x}\in\hat{C}_{i-1}^s} m_{\hat{x}}(\hat{x})\hat{m}(\mathrm{d}\hat{x}) + \int_{\hat{x}\in\sigma^{-1}(\epsilon)\setminus\hat{C}_{i-1}^s} m_{\hat{x}}(\mathbb{0})\hat{m}(\mathrm{d}\hat{x}) \\ &= \int_{\hat{x}\in\sigma^{-1}(\epsilon)} m_{\hat{x}}(C_i^s \cap \hat{x})\hat{m}(\mathrm{d}\hat{x}). \end{aligned}$$

- ($i = 0$) In this case, we have that

$$C_0^s \cap \hat{x} = \begin{cases} \{\hat{x}(s)\} & \hat{x} \in \hat{C}_{-1}^{\phi(s)} \\ \mathbb{0} & \hat{x} \notin \hat{C}_{-1}^{\phi(s)}. \end{cases}$$

Since $m_{\hat{x}}(\{\hat{x}(s)\}) = q_s^{x_{-1}}$ and $m_{\hat{x}}(\mathbb{0}) = 0$ (and, for each $x \in C_{-1}^{\phi(s)}$, it holds that $x_{-1} = \phi(s)$), it follows that

$$\begin{aligned} m(C_0^s) &= q_s^{\phi(s)} m(C_{-1}^{\phi(s)}) \\ &= q_s^{\phi(s)} \hat{m}(\hat{C}_{-1}^{\phi(s)}) \\ &= \int_{\hat{x}\in\hat{C}_{-1}^{\phi(s)}} q_s^{x_{-1}}\hat{m}(\mathrm{d}\hat{x}) + \int_{\hat{x}\in\sigma^{-1}(\epsilon)\setminus\hat{C}_{-1}^{\phi(s)}} 0\hat{m}(\mathrm{d}\hat{x}) \\ &= \int_{\hat{x}\in\hat{C}_{-1}^{\phi(s)}} m_{\hat{x}}(C_0^s \cap \hat{x})\hat{m}(\mathrm{d}\hat{x}) + \int_{\hat{x}\in\sigma^{-1}(\epsilon)\setminus\hat{C}_{-1}^{\phi(s)}} m_{\hat{x}}(C_0^s \cap \hat{x})\hat{m}(\mathrm{d}\hat{x}) \\ &= \int_{\hat{x}\in\sigma^{-1}(\epsilon)} m_{\hat{x}}(C_0^s \cap \hat{x})\hat{m}(\mathrm{d}\hat{x}). \end{aligned}$$ ■

Calculation of the folding entropy

We are finally ready to prove our main result on the folding entropy.

Theorem 3.14 ([MMV24])**.** *Let $(\Sigma_{n_-,n_+}, \sigma_\phi)$ be a bisymbolic shift dynamical system with measure m induced by the probability distribution p^+. Then*

$$\mathcal{F}(\sigma_\phi) = \sum_{0\le s<n_-} \mathrm{H}_{\triangle}(q^s)p_s^- = \mathrm{H}_m(\mathcal{C}_0) - \mathrm{H}_m(\mathcal{C}_{-1}).$$

Proof. As discussed in the beginning of this section (SECTION 3.2.5), it follows from DEFINITION 2.11 and FORMULA 2.10 that the folding entropy of σ is given by

$$\mathcal{F}(\sigma) = \mathrm{H}_m(\epsilon \mid \sigma^{-1}(\epsilon)) = \int_{\hat{x}\in\sigma^{-1}(\epsilon)} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}})\hat{m}(\mathrm{d}\hat{x}).$$

Now notice that

$$\epsilon|_{\hat{x}} = \{\{y\} \cap \hat{x} \mid \{y\} \in \epsilon\} = \{\{\hat{x}(s^+)\} \mid s^+ \in \phi^{-1}(x_{-1})\},$$

hence, from FORMULA 2.9 and DEFINITION 3.8, it follows that

$$\mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}}) = \sum_{s^+\in\phi^{-1}(x_{-1})} -m_{\hat{x}}(\{\hat{x}(s^+)\}) \log m_{\hat{x}}(\{\hat{x}(s^+)\}) = \sum_{s^+\in\phi^{-1}(x_{-1})} -q_{s^+}^{x_{-1}} \log q_{s^+}^{x_{-1}} = \mathrm{H}_{\triangle}(q^{x_{-1}}).$$

This shows that this value depends only on x_{-1}, so it is constant on each set $\hat{C}_{-1}^{s^-}$. The set

$$\hat{\mathcal{C}}_{-1} := \{\hat{C}_{-1}^{s^+} \mid s \in S^-\}$$

is a partition of $\sigma^{-1}(\epsilon)$, since (1) $\hat{C}_{-1}^{s^-} \neq \emptyset$, (2) $\hat{C}_{-1}^{s^-} \cap \hat{C}_{-1}^{r^-} = \emptyset$ when $s^- \neq r^-$, and (3) $\sigma^{-1}(\epsilon) = \bigcup_{s^-\in S^-} \hat{C}_{-1}^{s^-}$.

Besides that, it follows from PROPOSITION 3.12 and DEFINITION 3.4 that $\hat{m}(\hat{C}_{-1}^{s^-}) = m(C_{-1}^{s^-}) = p_{s^-}^-$. Thus the folding entropy of σ is

$$\begin{aligned}
\mathcal{F}(\sigma) &= \mathrm{H}_m(\epsilon \mid \sigma^{-1}(\epsilon)) \\
&= \int_{\hat{x}\in\sigma^{-1}(\epsilon)} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}}) \hat{m}(\mathrm{d}\hat{x}) \\
&= \sum_{0\le s^-<n_-} \int_{\hat{x}\in\hat{C}_{-1}^{s^-}} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}}) \hat{m}(\mathrm{d}\hat{x}) \\
&= \sum_{0\le s^-<n_-} \mathrm{H}_{\triangle}(q^{s^-}) p_{s^-}^-.
\end{aligned} \tag{3.17}$$

This proves the first equality of THEOREM 3.14.

Since $p^+ = \bigoplus_{s^-\in[n_-]} p_{s^-}^- q^{s^-}$ (FORMULA 3.16), it follows from the additivity of $\mathrm{H}_{\triangle}$ (PROPOSITION 2.2, PROPERTY 3) and from FORMULA 3.17 that

$$\mathrm{H}_{\triangle}(p^+) = \mathrm{H}_{\triangle}(\bigoplus_{0\le s^-<n_-} p_{s^-}^- q^{s^-}) = \mathrm{H}_{\triangle}(p^-) + \sum_{0\le s^-<n_-} p_{s^-}^- \mathrm{H}_{\triangle}(q^{s^-}).$$

Finally, since $\mathrm{H}_m(\mathcal{C}_0) = \mathrm{H}_{\triangle}(p^+)$ and $\mathrm{H}_m(\mathcal{C}_{-1}) = \mathrm{H}_{\triangle}(p^-)$ (LEMMA 3.5), we conclude that

$$\mathcal{F}(\sigma) = \mathrm{H}_m(\mathcal{C}_0) - \mathrm{H}_m(\mathcal{C}_{-1}). \qquad \blacksquare$$

In particular, since the measure entropy is given by $\mathrm{h}_m(\sigma) = \mathrm{H}_m(\mathcal{C}_0)$ (THEOREM 3.9), then

$$\mathrm{h}_m(\sigma) = \mathcal{F}(\sigma) + \mathrm{H}_m(\mathcal{C}_{-1}).$$

Chapter 4

Piecewise smooth dynamics

In this chapter we present some preliminary definitions about piecewise smooth vector fields and later the main results. We first introduce, in SECTION 4.1, the problems approached in this chapter. In SECTION 4.2 we define piecewise smooth vector fields, the sliding, escaping and tangent regions, the sliding vector field and what are the orbits of the system. Then, in SECTION 4.3 we present the concept of the orbit space, which we use to introduce new notions on piecewise smooth dynamics, by relating the orbit space to the base space. We define a metric space structure on the orbit and its dynamics, and prove some topological properties of these spaces before investigating some conditions under which topological transitivity of the dynamics of the base and of the orbit space dynamics are related. The work of this section was developed in [GMV23]. Finally, in SECTION 4.4 we restrict ourselves to the 2-dimensional case and study how classical properties of chaotic dynamics are related in the piecewise smooth case. We show topological transitivity implies the other properties of Devaney chaos and also that these systems must have strictly positive topological entropy. The work of this section was developed in [EMV24].

4.1 Introduction

It is widely known that ordinary differential equations (ODEs) are one of the most relevant tools for modeling problems emerging from real-world applications. Nevertheless, in the last decades, the assumption that solutions of ODEs may experience a discontinuity has gained traction within the theory of dynamical systems. Some examples of discontinuous ODEs includes control theory, the dynamics of a bouncing ball, foraging predators, the anti-lock braking system (ABS), mechanical devices in which components collide with each other, problems with friction, sliding or squealing, etc. See the referred applications and others in

[BJV01; Bro99; RD12; Dix95; GPK07; JT12; KKU+00; LN04; BCB+08], as well as references therein. In light of this, there is a natural interest in understanding the dynamics associated with them, which is very complicated and presents fascinating behaviors. These systems are called non-smooth systems or piecewise smooth systems.

The novelties coming from piecewise smooth vector fields compared to the continuous or smooth case resides on a proper or convenient way to define a solution and its behavior when two distinct vector fields meet (namely at the *switching manifold* or *discontinuity manifold*). There are several ways to define the solutions of a piecewise smooth vector field, and each of them gives rise to a new dynamics [Utk77; Utk92]; nevertheless, there is a special interest in the dynamics given by Filippov's convention [Fil88], since it provides very accurate models for different kinds of problems.

Filippov's solutions may experience non-uniqueness so non-typical behavior eventually emerges in relatively familiar scenarios such as planar phase portraits. Other approaches for dealing with discontinuous ODEs are nicely described in [Cor08]. The non-uniqueness of solutions is an important feature of piecewise smooth systems, and is related to the existence of some regions of the switching manifold known as stable and unstable sliding regions. Many orbits of a piecewise smooth system can visit the same point, which is a behavior that does not occur for continuous vector fields.

In this chapter we consider two main problems in piecewise smooth dynamics: (1) defining topological transitivity in a novel way which takes into account the non-uniqueness of solutions resulting from Filippov's convention; and (2) defining non-deterministic chaos for these systems and studying how topological transitivity implies chaos in the 2-dimensional case.

The first problem is dealt with in SECTION 4.3. Many works have studied the dynamics of piecewise smooth vector fields, but one of the goals of this work is to propose a new way to understand these systems. Due to the nature of piecewise smooth vector fields having different orbits going through the same points, it may cause a lack of understanding of the true meaning of the system as a whole. Therefore in this work we provide a way to understand how the non-uniqueness of solutions impacts the dynamics of a piecewise smooth vector field, and we do it so by introducing an associated dynamical system, the orbit space, in which we are able to restore the uniqueness property. This approach is motivated by the study of *inverse limits* of endomorphisms [CV23; Prz76; QXZ09].

The second problem is analyzed in SECTION 4.4. There we address the problem of determining the chaotic behavior of piecewise smooth systems defined on 2-dimensional manifolds. A well-accepted definition of chaos in the context of smooth deterministic dynamical systems is due to Devaney [Dev89]. A system which is topologically transitive, sensitive with respect to initial conditions and has density of periodic orbits is, in the classical dynamical system scenario, called chaotic or Devaney chaotic. A refinement of those conditions can be found in [BBC+92], where it is shown that topological transitivity and density of periodic solutions are sufficient for a system to be chaotic. We define an analogue of this definition for piecewise smooth systems. Since piecewise smooth system are non-deterministic in the sense that there may exist more than one distinct forward orbit starting at the same point, in this paper we are in fact considering *non-deterministic chaos*. The reader may see more results on that in [BCE18; CJ11; EJ20; NPV17], for instance.

4.2 Piecewise smooth vector fields

4.2.1 Switching manifold

We will consider a d-dimensional Riemannian $\mathscr{C}^r$-manifold ($r > 1$) M (possibly with boundary) with metric $\langle\cdot,\cdot\rangle$. For each $m \in \mathbb{N}$, we denote $[m] := \{0, \dots, m-1\}$ to simplify notation. The following definitions are based on [BCB+08, p. 71, Definition 2.18].

Definition 4.1. Let M be a d-dimensional $\mathscr{C}^r$-manifold ($r > 1$) and $l \in \mathbb{N}$. A *switching m-partition* of M is a partition $\{M_0, \dots, M_{m-1}, \Sigma\}$ of M in which

1. For each $i \in [m]$, the *regular region* M_i is an open d-dimensional submanifold of M;
2. The *switching region* Σ is a union $\Sigma =: \bigcup_{i,i' \in [m]} \Sigma_{i,i'}$, such that, for each $i, i' \in [m]$, $\Sigma_{i,i'} := M_i^\bullet \cap M_{i'}^\bullet$ is either empty or is an embedded 1-codimensional connected submanifold of M that is included in the boundaries ∂M_i and $\partial M_{i'}$;
3. For each $i, i' \in [m]$, there is a $\mathscr{C}^r$-differentiable function $h_{i,i'}\colon M_i^\bullet \cup M_{i'}^\bullet \longrightarrow \mathbb{R}$ that has 0 as a regular value, and $\Sigma_{i,i'} = h_{i,i'}^{-1}(0)$, $h_{i,i'}^{-1}(\mathbb{R}_{>0}) = M_i$ and $h_{i,i'}^{-1}(\mathbb{R}_{<0}) = M_{i'}$.

We refer to the pair (M, Σ) as a *switch-partitioned manifold*.[1]

Definition 4.2. Let (M, Σ) be a switch-partitioned manifold. A *piecewise smooth vector field* (*PSVF*) on M is a vector field $F\colon M \longrightarrow \mathrm{T}M$ such that, for each $i \in [m]$, the restriction $F|_{M_i}\colon M_i \longrightarrow \mathrm{T}M_i$ can be extended to a smooth[2] vector field $F_i\colon M_i^\bullet \longrightarrow \mathrm{T}M_i^\bullet$.

1. This is not a standard name. We use it here in order to easily reference it in the definitions.
2. Here we follow the standard definition that a vector field defined on a closed set is smooth when it can be extended to a smooth vector field on a open neighborhood of the closed set.

Notice that the definition of F on the switching manifold Σ is irrelevant; only its definition on the M_i matters. To study the behavior of the dynamics on the switching manifold, we will have to define another vector field on it. This will be done by a criterion called *Filippov's convention* [Fil88], which we shall define ahead.

4.2.2 Regions of the switching manifold

To start, we define different parts of the switching manifold with respect to the vector field F. We seek to partition each $\Sigma_{i,i'}$ according to how the smooth vector fields F_i and $F_{i'}$ meet there. For this we will consider the Lie derivative of $h_{i,i'}$ with respect to each smooth field.

Remember that the Lie derivative of a function h with respect to a vector field is the measure of how h varies in relation to the flow of F, and that, on a Riemannian manifold, its value at a point p is given by

$$Fh(p) = \langle \nabla\, h(p), F(p) \rangle_p.$$

Successive applications of the Lie derivative result in high-order Lie derivatives, defined recursively for vector fields $F_0, \ldots, F_{n-1}$ by

$$F_{n-1} \cdots F_0 h(p) := \left\langle F_{n-1}(p), \nabla F_{n-2} \cdots F_0 h(p) \right\rangle_p .$$

and denoted $F^n h(p)$ when $F_0 = \cdots = F_{n-1} = F$.

Definition 4.3. Let (M, Σ) be a switch-partitioned manifold and F a piecewise smooth vector field on M.

1. The *crossing region* of $\Sigma_{i,i'}$ is the set

   $$\Sigma^{\mathrm{c}}_{i,i'} := \{p \in \Sigma_{i,i'} \mid F_i h_{i,i'}(p) F_{i'} h_{i,i'}(p) > 0\}.$$

   The *crossing region* of Σ is $\Sigma^{\mathrm{c}} := \bigcup_{i,i' \in [m]} \Sigma^{\mathrm{c}}_{i,i'}$. The crossing region $\Sigma^{\mathrm{c}}_{i,i'}$ can be further divided:

   1.1. The *i-ward crossing region* (or *crossing region towards M_i*) of $\Sigma_{i,i'}$ is the set

   $$\Sigma^{i\mathrm{c}}_{i,i'} := \{p \in \Sigma^{\mathrm{c}} \mid F_i h_{i,i'}(p) > 0\}.$$

   The *crossing region towards M_i* of Σ is $\Sigma^{i\mathrm{c}} := \bigcup_{i' \in [m]} \Sigma^{i\mathrm{c}}_{i,i'}$.

   1.2. The *i'-ward crossing region* (or *crossing region towards $M_{i'}$*) of $\Sigma_{i,i'}$ is the set

   $$\Sigma^{i'\mathrm{c}}_{i,i'} := \{p \in \Sigma^{\mathrm{c}} \mid F_i h_{i,i'}(p) < 0\}.$$

   The *crossing region towards $M_{i'}$* of Σ is $\Sigma^{i'\mathrm{c}} := \bigcup_{i \in [m]} \Sigma^{i'\mathrm{c}}_{i,i'}$.

2. The *tangent region* of $\Sigma_{i,i'}$ is the set

$$\Sigma^{\mathrm{t}}_{i,i'} := \{p \in \Sigma_{i,i'} \mid F_i h_{i,i'}(p) F_{i'} h_{i,i'}(p) = 0\}.$$

   The *tangent region* of Σ is $\Sigma^{\mathrm{t}} := \bigcup_{i,i' \in [m]} \Sigma^{\mathrm{t}}_{i,i'}$.
3. The *sliding region* of $\Sigma_{i,i'}$ is the set

$$\Sigma^{\mathrm{s}}_{i,i'} := \{p \in \Sigma_{i,i'} \mid F_i h_{i,i'}(p) F_{i'} h_{i,i'}(p) < 0\}.$$

   The *sliding region* of Σ is $\Sigma^{\mathrm{s}} := \bigcup_{i,i' \in [m]} \Sigma^{\mathrm{s}}_{i,i'}$. The sliding region $\Sigma^{\mathrm{s}}_{i,i'}$ can be further divided:
   3.1. The *unstable*[3] *sliding region* (or *escaping region*) of $\Sigma_{i,i'}$ is the set

$$\Sigma^{\mathrm{us}}_{i,i'} := \{p \in \Sigma^{\mathrm{s}} \mid F_i h_{i,i'}(p) > 0\}.$$

   The *unstable sliding region* (or *escaping region*) of Σ is $\Sigma^{\mathrm{us}} := \bigcup_{i,i' \in [m]} \Sigma^{\mathrm{us}}_{i,i'}$.
   3.2. The *stable sliding region* (or *accessing region*[4]) of $\Sigma_{i,i'}$ is the set

$$\Sigma^{\mathrm{ss}}_{i,i'} := \{p \in \Sigma^{\mathrm{s}} \mid F_i h_{i,i'}(p) < 0\}.$$

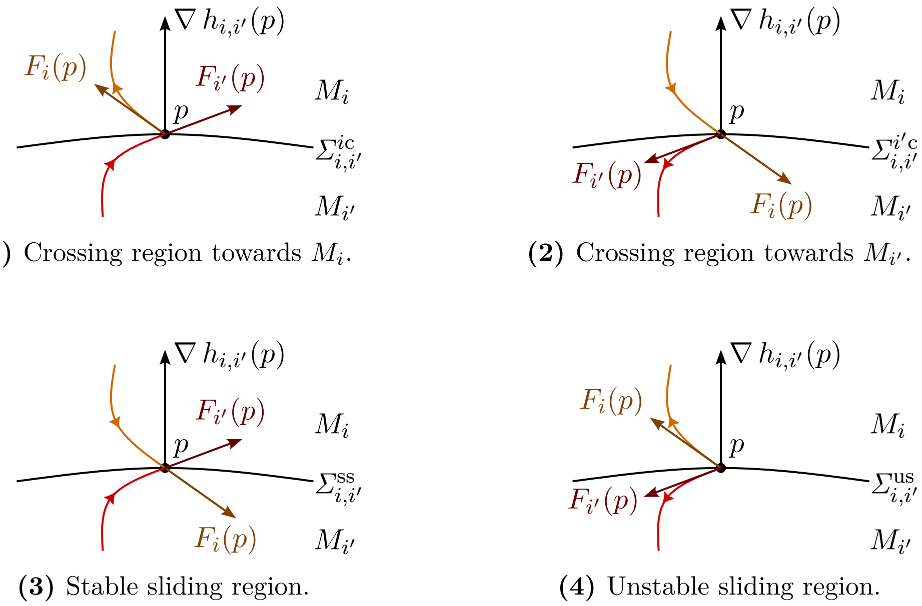
   The *stable sliding region* (or *accessing region*) of Σ is $\Sigma^{\mathrm{us}} := \bigcup_{i,i' \in [m]} \Sigma^{\mathrm{us}}_{i,i'}$.

The points that belong to these regions are called *crossing*, *tangent*, and *sliding* points, respectively.

Notice that the sets Σ^{c}, Σ^{us} and Σ^{ss} are relative open in Σ, and that Σ^{t} consists of the boundary point of these sets.

3. Here we follow [KRG03, Section 2.1, p. 6] in adopting the nomenclature *unstable* and *stable* for the regions of Σ^{s}.

4. This is not a standard name for this set. It is usually also called the *sliding region*, but this is ambiguous with the naming of Σ^{s}. The name *accessing* has been chosen in analogy to *escaping*, as these sets have dual roles in the theory. As we shall see in SECTION 4.2.4, Σ^{us} admits orbits leaving or *escaping* it at any point, while, dually, Σ^{ss} admits orbits entering or, one could say, *accessing* it. We adopt the name *sliding* for the whole set $\Sigma^{\mathrm{s}} = \Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{us}}$ in accordance with the standard naming of the field induced on Σ^{s} by Filippov's convention as the *sliding* vector field (SECTION 4.2.3).



(1) Crossing region towards M_i. **(2)** Crossing region towards $M_{i'}$. **(3)** Stable sliding region. **(4)** Unstable sliding region.

Figure 4.1. Regions of the switching manifold Σ.

In order to classify tangency points we use higher order Lie derivatives of $h_{i,i'}$. We say that a tangency point $p \in \Sigma_{i,i'}$ has *finite multiplicity* if there exist natural numbers $n, n' \geq 2$ such that $F_i^n h_{i,i'}(p) \neq 0$ and $F_{i'}^{n'} h_{i,i'}(p) \neq 0$. In this case, there is only a finite number of regular trajectories of F_i and $F_{i'}$ that arrive at such tangency point or depart from it.

We will from now on assume that the tangency set Σ^{t} is a finite set and every tangency point has finite multiplicity.

4.2.3 The sliding vector field

We are now going to define a vector field on the sliding region $\Sigma^{\mathrm{s}}_{i,i'}$ by following Filippov's convention [Fil88]. The motivation for this definition is that, at a point p on the sliding region, the smooth vector fields F_i and $F_{i'}$ point to opposite directions of Σ, so we may consider, among all possible values of convex combinations of the vectors $F_i(p)$ and $F_{i'}(p)$, the only one that is tangent to the manifold $\Sigma_{i,i'}$.

The convex combinations of $F_i(p)$ and $F_{i'}(p)$ are given by

$$f(p, \lambda) := \frac{1+\lambda}{2} F_i(p) + \frac{1-\lambda}{2} F_{i'}(p),$$

for $\lambda \in [-1, 1]$. Such a vector that is tangent to $\Sigma_{i,i'}$ must satisfy

$$\langle \nabla h_{i,i'}(p), f(p, \lambda) \rangle_p = 0.$$

This implies that

$$\begin{aligned}0 &= \langle \nabla h_{i,i'}(p), f(p,\lambda)\rangle_p \\ &= \langle \nabla h_{i,i'}(p), \frac{1+\lambda}{2}F_i(p) + \frac{1-\lambda}{2}F_{i'}(p)\rangle_p \\ &= \frac{1+\lambda}{2}\langle \nabla h_{i,i'}(p), F_i(p)\rangle_p + \frac{1-\lambda}{2}\langle \nabla h_{i,i'}(p), F_{i'}(p)\rangle_p \\ &= \frac{1+\lambda}{2}F_i h_{i,i'}(p) + \frac{1-\lambda}{2}F_{i'} h_{i,i'}(p).\end{aligned}$$

Solving for λ, we obtain

$$\lambda(p) = \frac{F_{i'}h_{i,i'}(p) + F_i h_{i,i'}(p)}{F_{i'}h_{i,i'}(p) - F_i h_{i,i'}(p)}.$$

Defining $\alpha := -\frac{F_i h_{i,i'}(p)}{F_{i'}h_{i,i'}(p)}$, we have $\alpha > 0$ and $-1 < \lambda = \frac{1-\alpha}{1+\alpha} < 1$.

Substituting this value for λ in the convex combination results in the vector

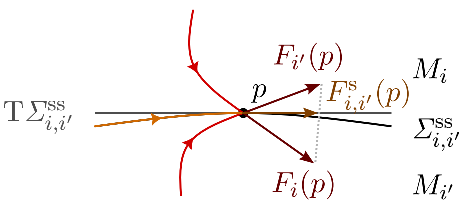
$$\begin{aligned}f(p,\lambda(p)) &= \frac{1 + \frac{F_{i'}h_{i,i'}(p)+F_i h_{i,i'}(p)}{F_{i'}h_{i,i'}(p)-F_i h_{i,i'}(p)}}{2}F_i(p) + \frac{1 - \frac{F_{i'}h_{i,i'}(p)+F_i h_{i,i'}(p)}{F_{i'}h_{i,i'}(p)-F_i h_{i,i'}(p)}}{2}F_{i'}(p) \\ &= \frac{F_{i'}h_{i,i'}(p)}{F_{i'}h_{i,i'}(p) - F_i h_{i,i'}(p)}F_i(p) - \frac{F_i h_{i,i'}(p)}{F_{i'}h_{i,i'}(p) - F_i h_{i,i'}(p)}F_{i'}(p).\end{aligned}$$

This justifies the next definition.

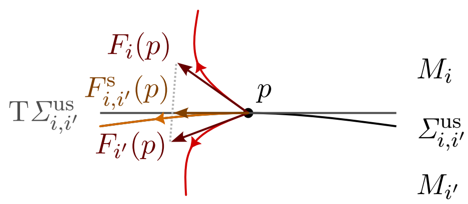
Definition 4.4. Let (M, Σ) be a switch-partitioned manifold and F a piecewise smooth vector field on M. The *sliding vector field* on Σ^{s} is the vector field $F^{\mathrm{s}}\colon \Sigma^{\mathrm{s}} \longrightarrow \mathrm{T}\Sigma^{\mathrm{s}}$ defined on each $\Sigma^{\mathrm{s}}_{i,i'}$ and $p \in \Sigma^{\mathrm{s}}_{i,i'}$ by

$$F^{\mathrm{s}}_{i,i'}(p) := \frac{F_{i'}h_{i,i'}(p)}{F_{i'}h_{i,i'}(p) - F_i h_{i,i'}(p)}F_i(p) - \frac{F_i h_{i,i'}(p)}{F_{i'}h_{i,i'}(p) - F_i h_{i,i'}(p)}F_{i'}(p).$$

Since the components $\Sigma_{i,i'}$ do not intersect, we can define F^{s} separately on each component. Notice that $F^{\mathrm{s}}_{i,i'}$ is always well-defined since its denominator is always non-zero in Σ^{s}. In some cases $F^{\mathrm{s}}_{i,i'}$ can be extended to $\Sigma^{\mathrm{s}\bullet}$, but its value on Σ^{t} (or Σ^{c}) is not of matter.



(1) $F^{\mathrm{s}}_{i,i'}$ on the stable sliding region.



(2) $F^{\mathrm{s}}_{i,i'}$ on the unstable sliding region.

Figure 4.2. The sliding vector field $F^{\mathrm{s}}_{i,i'}$ on the sliding region $\Sigma^{\mathrm{s}}_{i,i'}$ of the switching manifold Σ.

4.2.4 Orbits of a piecewise smooth vector field

In order to define what are the solutions of the piecewise smooth vector field F, we will first define its regular and sliding solutions. Each smooth vector field $F_i\colon M_i \longrightarrow \mathrm{T}M_i$ has a smooth flow Φ^{tF_i} which we will denote simply by Φ_i^t. The sliding vector field F^s also has a smooth flow $\Phi^{tF^{\mathrm{s}}}$ on $\varSigma^{\mathrm{s}\bullet}$ which we will denote by Φ_{s}^t

Our approach here is to define a solution to be a concatenation of regular and sliding solutions, as has been done in [KRG03]. This is different from the definitions adopted in [GST11], in which the authors consider only the regular and slding solutions and do not allow for loss of uniqueness.

This approach may give rise to a lack of uniqueness of solutions because different trajectories on M_i, $\varSigma^{\mathrm{s}}_{i,i'}$ and $M_{i'}$ may pass through the same point, hence a flow Φ cannot be defined for the whole piecewise smooth vector field F. In light of this, we introduce in SECTION 4.3 a new flow $\tilde{\Phi}$ defined on a larger space related to the piecewise smooth vector field which will be able to restore the idea of uniqueness lost in this scenario.

Definition 4.5. Let $(M, \varSigma)$ be a switch-partitioned manifold and F a piecewise smooth vector field on M. A *regular orbit* (or *regular solution*) of F is a trajectory that is an integral solution of some F_i. A *sliding orbit* (or *sliding solution*) of F is a trajectory that is an integral solution of F^{s}.

Definition 4.6. Let M be a manifold. The *concatenation* of two trajectories $\gamma\colon]a, b[\longrightarrow M$ and $\gamma'\colon]a', b'[\longrightarrow M$ such that $\lim_{t\to b}\gamma(t) = \lim_{t\to a'}\gamma'(t)$ is the trajectory

$$\gamma\gamma'\colon]a, b + b' - a'[\longrightarrow X$$
$$t \longmapsto \begin{cases} \gamma(t) & a < t < b, \\ \lim_{t\to b}\gamma(t) = \lim_{t\to a'}\gamma'(t) & t = b \\ \gamma'(t - b + a') & b < t < b + b' - a'. \end{cases}$$

The limiting point $\lim_{t\to ba}\gamma(t) = \lim_{t\to a'}\gamma'(t)$ is the *concatenation point* of γ and γ'.

The concatenation of a maximal regular orbit with either another maximal regular orbit or with a sliding orbit must always have a concatenation point in $\varSigma$, since otherwise the regular orbit could be extended in the regular region and would not be maximal. They can be of the following types:

1. If the concatenation point is in a crossing region $\varSigma^{\mathrm{c}}_{i,i'}$, the concatenation is between regular orbits of F_i and $F_{i'}$;

2. If the concatenation point is in an unstable sliding region $\Sigma^{\mathrm{us}}_{i,i'}$, the concatenation is between a sliding orbit of $F^{\mathrm{s}}_{i,i'}$, and a regular orbit of either F_i or $F_{i'}$. The first case is called a trajectory *escaping* to M_i, and in the second, *escaping* to $M_{i'}$;
3. If the concatenation point is in a stable sliding region $\Sigma^{\mathrm{ss}}_{i,i'}$, the concatenation is between a regular orbit of either F_i or $F_{i'}$, and a sliding orbit of $F^{\mathrm{s}}_{i,i'}$. The first case is called a trajectory *accessing* from M_i, and in the second, *accessing* from $M_{i'}$;
4. If the concatenation point is in a tangent region $\Sigma^{\mathrm{t}}_{i,i'}$, they are in the closure of a crossing or sliding region, and the concatenations follow the previous cases accordingly.

With these definitions, we can define what an orbit of the piecewise smooth vector field is.

Definition 4.7 (Orbit of a PSVF)**.** Let (M, Σ) be a switch-partitioned manifold and F a piecewise smooth vector field on M. An *orbit of* F is a (finite or infinite) concatenation of regular and sliding orbits of F.

A *maximal orbit* is an orbit that cannot be extended. A *periodic* orbit of F with *period* $s \in \mathbb{R}$ is an orbit γ such that, for every $t \in \mathbb{R}$, $\gamma(t) = \gamma(s + t)$.

At a tangent point $p \in \Sigma^{\mathrm{t}}$ to which regular or sliding orbits can be extended, an orbit through p is a concatenation of such orbits; otherwise, the orbit is stationary at p.

Let $\gamma\colon I \longrightarrow M$ be an orbit of F and take any $t_0 \in I$. The definition of γ as a concatenation of regular and sliding orbits implies that there are $k^-, k^+ \in \mathbb{N} \cup \{\infty\}$ and, for each $-k^- \leq j \leq k^+$, a (concatenation of) regular orbit $\gamma_j\colon [t_j^-, t_j^+] \longrightarrow M_{i_j}^{\bullet}$ (possibly passing through crossing regions) and a sliding orbit $\gamma_i^{\mathrm{s}}\colon [t_j^+, t_{j+1}^-] \longrightarrow \Sigma^{\mathrm{s}\bullet}$ such that[5] $t_{-1}^- \leq t_0 \leq t_0^+$, $\gamma_j(t_j^+) = \gamma_j^{\mathrm{s}}(t_j^+)$, $\gamma_j^{\mathrm{s}}(t_{j+1}^-) = \gamma_j(t_j^+)$ and

$$\gamma = \gamma_{k^-}\gamma^{\mathrm{s}}_{k^-} \cdots \gamma^{\mathrm{s}}_{-1}\gamma_0\gamma^{\mathrm{s}}_0 \cdots \gamma_{k^+}\gamma^{\mathrm{s}}_{k^+}.$$

With this notation in mind, we define the nomenclature of accessing and escaping points (see FIGURE 4.3) that will be essential in the proof of PROPOSITION 4.9.

Definition 4.8 (Σ-sequence of an orbit)**.** Let (M, Σ) be a switch-partitioned manifold, F a piecewise smooth vector field on M and $\gamma\colon I \longrightarrow M$ an orbit of F such that $0 \in I$ and

$$\gamma = \gamma_{k^-}\gamma^{\mathrm{s}}_{k^-} \cdots \gamma^{\mathrm{s}}_{-1}\gamma_0\gamma^{\mathrm{s}}_0 \cdots \gamma_{k^+}\gamma^{\mathrm{s}}_{k^+}.$$

5. The orbits $\gamma^{\mathrm{s}}_{k^+}$ and γ_{k^-} may not exist, in the case γ is a regular orbit in the forward extreme and a sliding orbit in the backward extreme, respectively.

1. The *j-th escaping point* of γ is
$$e_j := \gamma_i(t_j^-) \in \Sigma^{\mathrm{us}\bullet}$$
and the *jth escaping region* of γ is i_j. The *Σ-escaping sequence* of γ is the sequence of pairs (e_j, i_j).
2. The *j-th accessing point* of γ is
$$a_j := \gamma_i(t_j^+) \in \Sigma^{\mathrm{ss}\bullet}$$
and the *jth accessing region* of γ is i_j. The *Σ-accessing sequence* of γ is the sequence of pairs (a_j, i_j).

In PROPOSITION 4.9, we will use to positive part of the Σ-escaping sequence and the negative part of the Σ-accessing sequence of an orbit γ to approximate with one of a countable number of orbits of the system.

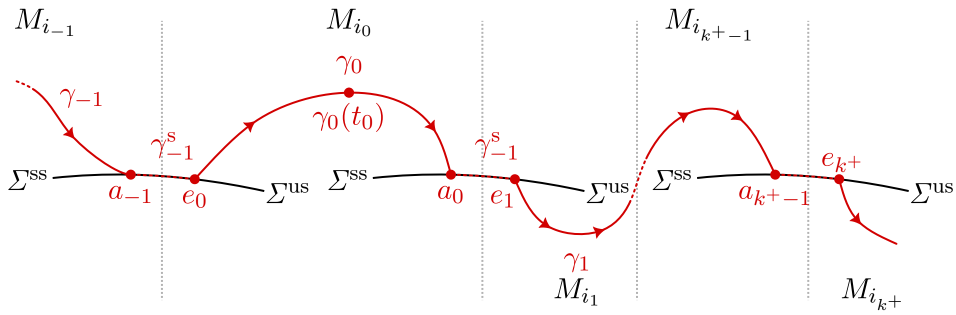


Figure 4.3. The Σ-escaping sequence of an orbit γ is defined by taking the points e_j where the component orbit γ_j escapes $\Sigma^{\mathrm{us}\bullet}$. The values i_j indicate the regular region M_{i_j} to which it escaped $\Sigma^{\mathrm{us}\bullet}$. The Σ-accessing sequence of an orbit γ is defined by taking the points a_j where the component orbit γ_j accesses $\Sigma^{\mathrm{ss}\bullet}$. The values i_j indicate the regular region M_{i_j} from which they accessed $\Sigma^{\mathrm{ss}\bullet}$.

Singularities

A special attention must be paid to the singularities of a piecewise smooth vector field F. Since we consider a new way to define solutions, we must distinguish some points of Σ which will also behave as singularities in a certain way. A point $p \in \Sigma$ is said to be a *Σ-singularity* of F provided that p is either a point of $\Sigma^{\mathrm{t}}_{i,i'}$ such that $F_i(p), F_{i'}(p) \neq 0$, an equilibrium of F_i or $F_{i'}$, or an equilibrium of $F^{\mathrm{s}}_{i,i'}$ (known as *pseudo-equilibrium* of F). A point $p \in \Sigma$ which is not a Σ-singularity of F is also referred to as a *regular-regular* point of F. We say that an orbit γ is a *regular orbit* of F if it is a piecewise smooth trajectory such that $\gamma \cap M_i$ and $\gamma \cap M_{i'}$ are unions of regular orbits of F_i and $F_{i'}$, respectively, and $\gamma \cap \Sigma \subseteq \Sigma^{\mathrm{c}}$. More details on the classification of Σ-singularities can be found in [GST11; KRG03]

4.3 Orbit spaces

Among all properties of smooth dynamical systems, *topological transitivity* has always been one of the most fundamental ones. In the literature, this term has been used for different properties of the dynamics, mainly the existence of a dense orbit or the density of the orbit of any non-empty open set (see SECTION 1.5.2 for definitions and references). In our context, these definitions are equivalent. This is an interesting concept by itself but it is worth mentioning that topological transitivity is also one of the ingredients of chaos. Despite the importance of piecewise smooth vector fields, there is still a lack of results about its global dynamics. We may claim that this is due to the scenario of non-uniqueness of solutions, which we propose to properly overcome in the present work.

In [BCE16; EJV22], the authors provided some chaotic examples of 2-dimensional piecewise smooth vector fields displaying a dense orbit in the phase space. These two articles explore intensively the non-uniqueness of solution, in particular the escaping regions. Escaping regions produce so many different orbits that one may have a feeling that topological transitivity is forcibly more common in the PSVF scenario. This is in some sense true, but one should see it as due to the richness of the dynamics. For instance, in [EJV22] the authors provide a first example of topological transitivity of a piecewise smooth vector field on a 2-sphere $\mathbb{S}^2$, which is interesting and shows the richness of PSVFs, since a continuous topologically transitive flow on $\mathbb{S}^2$ is long known not to exist.

In this work we propose to understand the topological transitivity of a PSVF in a more appropriate space. This space will be called the *orbit space*, which is in fact the space of all orbits. This is inspired by the technique known as *inverse limit* widely studied by those which deal with endomorphisms[CV23; Prz76; QXZ09]. An endomorphism is a non-invertible transformation and the inverse limit is simply the space of all possible orbits. There is a very natural dynamics in this space which is just the translation of the original dynamics to this space. We apply this idea to the PSVF (see DEFINITION 4.10) since it is a non-invertible system. The orbit space then carries all the complexity of the system, but without the issues of having multiple orbits going through the same place.

Our goal is to propose the use of the orbit space in piecewise smooth systems. So our first main result is in fact a series of results with the purpose of establishing a well-defined theory of “orbit spaces” for PSVFs. In this section, we define the orbit space and its dynamics, provide two topologically equivalent distance functions for this space, and provide some results that help the use of such space for piecewise smooth analysis.

Following this, we prove a result that establishes when a PSVF is topologically transitive in the orbit space, the space where we understand it to be natural to consider transitivity, and we show that the examples presented in [BCE16; EJV22] are also topologically transitive in our new context. The content of this section was developed in [GMV23].

4.3.1 Definition of the orbit space

In this section we still consider a d-dimensional Riemannian $\mathscr{C}^r$-manifold ($r > 1$) M (possibly with boundary), a switching submanifold Σ and a piecewise smooth vector field F on M. But now we assume M is complete and all F_i are bounded, that is, its *supremum norm*

$$\|F\| := \sup_{p \in M} \|F(p)\|$$

is finite. We also assume $\|F\| > 0$. Notice that this implies all F_i are bounded and, as a consequence of DEFINITION 4.4, that all $F^{\mathrm{s}}_{i,i'}$ are bounded as well.

Definition 4.9. Let (M, Σ) be a switch-partitioned manifold and F a piecewise smooth vector field on M. The *orbit space* of the system is the set $\tilde{M}$ of all maximal orbits of F on M. In this context, we call M the *phase space* or *base space* of the system.

Notice that $\tilde{M}$ does not depend only on the manifold M, but mainly on the vector field F, but the notation $\tilde{M}$ is simpler, express a better analogy with M and also is not ambiguous since we will not consider more than one piecewise smooth vector field on the same manifold M.

Using the piecewise smooth vector field F we can induce a flow on $\tilde{M}$. For the next definition, for each $\gamma \in \tilde{M}$ we denote the maximal interval it is defined on by I_γ, and the backwards translation of this interval by $t \in \mathbb{R}$ as

$$I_\gamma - t = \{s - t \mid s \in I_\gamma\}. \tag{4.1}$$

Definition 4.10. Let (M, Σ) be a switch-partitioned manifold, F a piecewise smooth vector field on M and $\tilde{M}$ the orbit space of the system. The *flow domain* on $\tilde{M}$ is the set

$$D_{\tilde{\Phi}} := \bigcup_{\gamma \in \tilde{M}} I_\gamma \times \{\gamma\} \subseteq \mathbb{R} \times \tilde{M}.$$

The *flow on $\tilde{M}$ induced by F* is the transformation

$$\begin{aligned} \tilde{\Phi}\colon D_{\tilde{\Phi}} &\longrightarrow \tilde{M} \\ (t, \gamma) &\longmapsto \tilde{\Phi}^t(\gamma)\colon I_\gamma - t \longrightarrow M \\ &\qquad\qquad\qquad\qquad s \longmapsto \gamma(s + t). \end{aligned}$$

When we consider spaces whose orbits are defined on the whole $\mathbb{R}$, the definition of the flow simplifies to

$$\begin{aligned}\tilde{\Phi}\colon \mathbb{R}\times\tilde{M} &\longrightarrow \tilde{M}\\ (t,\gamma) &\longmapsto \tilde{\Phi}^t(\gamma)\colon \mathbb{R}\longrightarrow M\\ &\qquad\qquad s\longmapsto \gamma(s+t).\end{aligned}$$

We will show that this results in a continuous dynamical system on the orbit space, but for that we must at first introduce the structure of a metric space on $\tilde{M}$.

4.3.2 Metric structure of orbit spaces

The orbit distance function

In this section, we assume M is complete. The distance function on M is the distance induced from the Riemannian metric on M, and is denoted by $d\colon M\times M\longrightarrow\mathbb{R}$. Given two orbits $\gamma,\gamma'\in\tilde{M}$, we wish to define the distance between them. We assume at first that $I_\gamma = I_{\gamma'} = \mathbb{R}$. We could try and compare the distance of points of these orbits on the base system. A general strategy for defining new distance functions is to consider a converging weighted sum of previously know distance functions. In our case, we consider the quantity

$$\sum_{i\in\mathbb{Z}}\frac{1}{2^{|i|}}\int_i^{i+1} d(\gamma_0(t),\gamma_1(t))\,\mathrm{d}t.$$

This accounts for every point of both orbits, and the weights $\frac{1}{2^{|i|}}$ guarantee convergence, as we shall see ahead. But this is only defined for orbits that have the whole $\mathbb{R}$ as a domain. Sometimes this may not be the case, for instance when an orbit reaches the boundary of the base space M, or when it reaches a pseudo-equilibrium point on the switching manifold Σ. Hence let us show how to extend the definition of the distance in the case the orbits are only defined for finite time.

Let $\gamma\in\tilde{M}$ be some orbit whose maximal domain I_γ is an interval with endpoints $\omega_- < \omega_+$. Since we assume M is a complete space, this implies that, if $\omega_+\in\mathbb{R}$, then there is a point $x_{\omega_+}\in M$ such that $\gamma(t_n)\to x_{\omega_+}$ for any increasing sequence of times $(t_n)_{n\in\mathbb{N}}$ such that $t_n\to\omega_+$, and also that $\omega_+\in I_\gamma$. Analogously, if $\omega_-\in\mathbb{R}$, then there is a point $x_{\omega_-}\in M$ such that $\gamma(t_n)\to x_{\omega_-}$ for any decreasing sequence of times $(t_n)_{n\in\mathbb{N}}$ such that $t_n\to\omega_-$, and also $\omega_-\in I_\gamma$.

In each of these cases, we can extended γ to a new trajectory with maximal domain $\mathbb{R}$, which we will denote $\overline{\gamma}:\mathbb{R}\to M$, that is identical to γ in the maximal domain of γ, and is a stationary orbit $\overline{\gamma}(t) = x_{\omega_-}$ for all $t\le\omega_-$ (if $\omega_-\in\mathbb{R}$) and $\overline{\gamma}(t) = x_{\omega_+}$ for all $t\ge\omega_+$ (if $\omega_+\in\mathbb{R}$). We define the distance between orbits

$\gamma_0, \gamma_1 \in \tilde{M}$ as the distance between their extended counterparts:

$$\tilde{d}(\gamma_0, \gamma_1) := \tilde{d}(\overline{\gamma}_0, \overline{\gamma}_1) := \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} d(\overline{\gamma}_0(t), \overline{\gamma}_1(t))\, \mathrm{d}t.$$

It is important to understand that this is done only for the sake of calculation of the distance, and the way it is done we still have a clear indication concerning if one orbit is close or not to another one. The extended trajectories $\overline{\gamma}$ are not elements of $\tilde{M}$ (unless $\omega_- = -\infty$ and $\omega_+ = \infty$, in which case they coincide with γ). In order to keep the notation simple, we will not use the extended trajectories when calculating the summation in the definition of the distance.

Definition 4.11 (Integral distance)**.** Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. The (*integral*[6]) *distance on $\tilde{M}$ induced by d* is the function

$$\begin{aligned} \tilde{d}\colon \tilde{M} \times \tilde{M} &\longrightarrow \mathbb{R} \\ (\gamma_0, \gamma_1) &\longmapsto \tilde{d}(\gamma_0, \gamma_1) := \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} d(\gamma_0(t), \gamma_1(t))\, \mathrm{d}t. \end{aligned}$$

The orbit space $\tilde{M}$ with this distance function becomes a metric space. This has already been shown for compact spaces [ACV23], but here we assume only that F is bounded. We must show that the sum in the definition of the distance is always finite and that $\tilde{d}$ is in fact a distance. First we prove the following lemma, which assumes the field F is bounded and shows that specific suprema related to the distance of trajectory points must be bounded.

Lemma 4.1 ([GMV23])**.** *Let M be a Riemannian manifold and F a bounded piecewise smooth vector field on M. For every $\gamma_0, \gamma_1 \in \tilde{M}$ and $n \in \mathbb{Z}$,*

$$\sup_{n \le t < n+1} d(\gamma_0(t), \gamma_1(t)) \le d(\gamma_0(0), \gamma_1(0)) + 2\|F\|(1 + |n|). \tag{4.2}$$

Proof. We will first prove that, for every $i \in \mathbb{Z}$,

$$\sup_{i \le t < i+1} d(\gamma_0(t), \gamma_1(t)) \le \sup_{i-1 \le t < i} d(\gamma_0(t), \gamma_1(t)) + 2\|F\|. \tag{4.3}$$

For every $t \in [i, i+1[$ it follows from the triangle inequality that

$$d(\gamma_0(t), \gamma_1(t)) \le d(\gamma_0(t), \gamma_0(i)) + d(\gamma_0(i), \gamma_1(i)) + d(\gamma_1(i), \gamma_1(t)).$$

6. We use the name *integral distance* to differentiate it from the *supremum* distance we will introduce ahead.

Since the field is bounded by $\|F\|$, it follows from the mean value inequality that $d(\gamma_0(t), \gamma_0(i)) \leq \|F\||t-i|$ and $d(\gamma_1(i), \gamma_1(t)) \leq \|F\||i-t|$, hence

$$\begin{aligned}\sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_1(t)) &\leq \sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_0(i)) + d(\gamma_0(i), \gamma_1(i)) + \sup_{i\leq t<i+1} d(\gamma_1(i), \gamma_1(t)) \\ &\leq \sup_{i\leq t<i+1} \|F\||t-i| + d(\gamma_0(i), \gamma_1(i)) + \sup_{i\leq t<i+1} \|F\||i-t| \\ &\leq \sup_{i-1\leq t<i} d(\gamma_0(t), \gamma_1(t)) + 2\|F\|.\end{aligned}$$

Analogously, for every $i \in \mathbb{Z}$ we can also obtain that

$$\sup_{i-1\leq t<i} d(\gamma_0(t), \gamma_1(t)) \leq \sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_1(t)) + 2\|F\|. \tag{4.4}$$

By induction on n, using FORMULA 4.3 for positive n and FORMULA 4.4 for negative n, we conclude that

$$\sup_{n\leq t<n+1} d(\gamma_0(t), \gamma_1(t)) \leq \sup_{0\leq t<1} d(\gamma_0(t), \gamma_1(t)) + 2\|F\||n|. \tag{4.5}$$

Finally, for every $t \in [0, 1]$,

$$\begin{aligned}d(\gamma_0(t), \gamma_1(t)) &\leq d(\gamma_0(t), \gamma_0(0)) + d(\gamma_0(0), \gamma_1(0)) + d(\gamma_1(0), \gamma_1(t)) \\ &\leq \|F\||-t| + d(\gamma_0(0), \gamma_1(0)) + \|F\||t| \\ &= d(\gamma_0(0), \gamma_1(0)) + 2\|F\||t|,\end{aligned}$$

so $\sup_{0\leq t<1} d(\gamma_0(t), \gamma_1(t)) \leq d(\gamma_0(0), \gamma_1(0)) + 2\|F\|$. From this formula and FORMULA 4.5 we obtain FORMULA 4.2. ■

Proposition 4.2 ([GMV23])**.** *Let M be a Riemannian manifold and F a bounded piecewise smooth vector field on M. The function $\tilde{d}$ is a distance function on $\tilde{M}$.*

Proof. The most important part is to prove the function is well defined in the sense that the infinite sum always has a finite value. For this we will use the fact that the field is bounded. Let $\gamma_0, \gamma_1 \in \tilde{M}$. From the fact that $\sum_{i=n}^{\infty} \frac{i}{2^i} = \frac{n+1}{2^{n-1}}$ and from LEMMA 4.1, it follows that

$$\begin{aligned}\tilde{d}(\gamma_0, \gamma_1) &= \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} d(\gamma_0(t), \gamma_1(t))\,\mathrm{d}t \\ &\leq \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_1(t)) \\ &\leq \sum_{i\in\mathbb{Z}} \frac{d(\gamma_0(0), \gamma_1(0)) + 2\|F\|(1+|i|)}{2^{|i|}} \\ &= 3\,d(\gamma_0(0), \gamma_1(0)) + 14\|F\| < \infty.\end{aligned}$$

This shows that $\tilde{d}$ is well-defined.

Now we show the properties of a distance function. If $\gamma_0 = \gamma_1$, then $\tilde{d}(\gamma_0, \gamma_1) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} 0 = 0$; if $\tilde{d}(\gamma_0, \gamma_1) = 0$, then $\int_i^{i+1} d(\gamma_0(t), \gamma_1(t))\,\mathrm{d}t = 0$ for every $i \in \mathbb{Z}$, hence $\gamma_0(t) = \gamma_1(t)$ for every $t \in [i, i+1[$, therefore $\gamma_0 = \gamma_1$. Finally, symmetry and the triangle inequality of $\tilde{d}$ follow directly from these properties for d. ■

The summation of suprema that appeared in the preceding proposition to prove that the distance $\tilde{d}$ is finite motivates the definition of another distance function on $\tilde{M}$.

Definition 4.12 (Supremum distance)**.** Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. The *supremum distance on* $\tilde{M}$ *induced by* d is the function

$$\begin{aligned} \tilde{d}_{\sup} \colon \tilde{M} \times \tilde{M} &\longrightarrow \mathbb{R} \\ (\gamma_0, \gamma_1) &\longmapsto \tilde{d}_{\sup}(\gamma_0, \gamma_1) := \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i\le t<i+1} d(\gamma_0(t), \gamma_1(t)). \end{aligned}$$

This function can be proven to be a distance function in the same way that was done for the integral distance function $\tilde{d}$. These distance functions are topologically equivalent, as the following proposition shows, and thus will be used interchangeably when analyzing topological properties of the orbit space $\tilde{M}$.

Proposition 4.3 ([GMV23])**.** *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. The distance functions $\tilde{d}$ e $\tilde{d}_{\sup}$ are topologically equivalent.*

Proof. Let us denote the $\tilde{d}$ and $\tilde{d}_{\sup}$ balls respectively as B and $B_{\sup}$, and their topologies as $\mathcal{T}$ and $\mathcal{T}_{\sup}$. We will prove that each topology is finer than the other.

- ($\mathcal{T} \subseteq \mathcal{T}_{\sup}$) For every $\gamma_0, \gamma_1 \in \tilde{M}$ and every $i \in \mathbb{Z}$,

  $$\int_i^{i+1} d(\gamma_0(t), \gamma_1(t))\,\mathrm{d}t \le \int_i^{i+1} \sup_{i\le t<i+1} d(\gamma_0(t), \gamma_1(t))\,\mathrm{d}t = \sup_{i\le t<i+1} d(\gamma_0(t), \gamma_1(t)),$$

  hence it follows that

  $$\begin{aligned} \tilde{d}(\gamma_0, \gamma_1) &= \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} d(\gamma_0(t), \gamma_1(t))\,\mathrm{d}t \\ &\le \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i\le t<i+1} d(\gamma_0(t), \gamma_1(t)) \\ &= \tilde{d}_{\sup}(\gamma_0, \gamma_1). \end{aligned}$$

  This implies that every ball of $\tilde{d}_{\sup}$ is contained in the ball of $\tilde{d}$ with same center and radius, hence that the topology generated by $\tilde{d}_{\sup}$ is finer than the one generated by $\tilde{d}$.

- ($\mathcal{T}_{\sup} \subseteq \mathcal{T}$) Take $\gamma \in \tilde{M}$ and $r' > 0$ and consider the ball $B_{\sup}(\gamma, r')$ with center point γ and radius r'. We must find $r > 0$ such that $B(\gamma, r) \subseteq B_{\sup}(\gamma, r')$. Suppose, for the sake of contradiction, that such r did not exist. In that case, there would exist a sequence $(r_n)_{n\in\mathbb{N}}$ of positive real numbers such that $r_n \to 0$ and, for every $n \in \mathbb{N}$, an orbit $\gamma_n \in \tilde{M}$ such that $\gamma_n \in B(\gamma, r_n)$ and $\gamma_n \notin B_{\sup}(\gamma, r')$, which means that

$$\tilde{d}(\gamma, \gamma_n) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} d(\gamma(t), \gamma_n(t))\, \mathrm{d}t < r_n, \tag{4.6}$$

and

$$\tilde{d}_{\sup}(\gamma, \gamma_n) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t)) \geq r'. \tag{4.7}$$

From this it would follow that, for every $i \in \mathbb{Z}$, each term $\int_i^{i+1} d(\gamma(t), \gamma_n(t))\, \mathrm{d}t$ of the summation in FORMULA 4.6 would converge to 0 as $n \to \infty$. Therefore, by continuity of the orbits in $\tilde{M}$, each term $\sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t))$ of the summation in FORMULA 4.7 would also converge to 0 as $n \to \infty$, while the summation in FORMULA 4.7 would be bounded below by $r' > 0$. This would lead to the following contradiction.
Since the field is bounded by $\|F\|$, it would follow from LEMMA 4.1 that, for every $n \in \mathbb{N}$ and every $i_0 \in \mathbb{N}$ large enough,

$$\sum_{|i|>i_0} \frac{1}{2^{|i|}} \sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t)) \leq \frac{d(\gamma(0), \gamma_n(0)) + 2\|F\|(i_0+3)}{2^{i_0-1}} \leq \frac{r'}{2},$$

so that

$$\sum_{|i|\le i_0} \frac{1}{2^{|i|}} \sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t)) \geq \frac{r'}{2} > 0.$$

But, for every $|i| \leq i_0$, $\sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t)) \to 0$ as $n \to \infty$, so there would be $n_0 \in \mathbb{N}$ large enough such that, for every $n \geq n_0$ and every $|i| \leq i_0$,

$$\sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t)) < \frac{r'}{2(2i_0+1)},$$

and so

$$\sum_{|i|\le i_0} \frac{1}{2^{|i|}} \sup_{i\le t<i+1} d(\gamma(t), \gamma_n(t)) < (2i_0+1)\frac{r'}{2(2i_0+1)} = \frac{r'}{2}.$$

This contradiction shows that $\gamma' \in B_{\sup}(\gamma, r')$, hence that the topology generated by $\tilde{d}$ is finer than the one generated by $\tilde{d}_{\sup}$. ■

Besides topological equivalence, if we assume the piecewise smooth vector field F is Lipschitz continuous (which is true when M is compact), then we can also show that $\tilde{d}$ and $\tilde{d}_{\text{sup}}$ are Lipschitz equivalent, hence uniformly equivalent. This will be relevant in SECTION 4.4.2 when we calculate the topological entropy of our system, since then the value of the entropy is the same whether we use $\tilde{d}$ or $\tilde{d}_{\text{sup}}$.

Proposition 4.4. *Let M be a complete Riemannian manifold (possibly with boundary) and F a Lipschitz continuous and bounded piecewise smooth vector field on M. The distance functions $\tilde{d}$ e $\tilde{d}_{\text{sup}}$ are Lipschitz equivalent.*

Proof. We already have that $\tilde{d} \leq \tilde{d}_{\text{sup}}$, which implies that $\mathrm{I}\colon (\tilde{M}, \tilde{d}_{\text{sup}}) \longrightarrow (\tilde{M}, \tilde{d})$ is Lipschitz continuous.

Let $\langle\!\langle F \rangle\!\rangle$ be the Lipschitz distortion of F. For every $\gamma_0, \gamma_1 \in \tilde{M}$ and every $i \in \mathbb{Z}$, let $t_i \in [i, i+1]$ be such that

$$s_i := \sup_{t \in [i, i+1[} d(\gamma_0(t), \gamma_1(t)) = d(\gamma_0(t_i), \gamma_1(t_i)).$$

Then it follows from FORMULA 1.6 that, for every $t \in [i, i+1]$,

$$d(\gamma_0(t_i), \gamma_1(t_i)) \leq \mathrm{e}^{\langle\!\langle F \rangle\!\rangle} d(\gamma_0(t), \gamma_1(t)).$$ ■

Properties of the orbit distance function

It is very important to understand the intuitive meaning of the metric $\tilde{d}$ (or $\tilde{d}_{\text{sup}}$). Two orbits being close in the orbit space should be the same as the points of these orbits being sufficiently close for a sufficient amount of time (see FIGURE 4.4). The following lemmas provide a precise meaning of these ideas. We first show that if the points of two orbits are close for enough time in the base space, then the orbits are close in the orbit space. LEMMAS 4.5 and 4.7 provide a precise meaning to these ideas.

Lemma 4.5 ([GMV23])**.** *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. For every $\varepsilon > 0$, there exist $\tau > 0$ and $\delta > 0$ such that, for every $\gamma_0, \gamma_1 \in \tilde{M}$, if $d(\gamma_0(t), \gamma_1(t)) < \delta$ for every $t \in [-\tau, \tau]$, then $\tilde{d}(\gamma_0, \gamma_1) < \varepsilon$.*

Proof. Let $\varepsilon > 0$, choose an integer[7] $\tau \geq 0$ such that $\frac{\|F\|(\tau+3)}{2^{\tau-1}} < \varepsilon$, and define $\delta := \frac{1}{3}\left(\varepsilon - \frac{\|F\|(\tau+3)}{2^{\tau-1}}\right) > 0$. Using LEMMA 4.1, it follows that, for every $\gamma_0, \gamma_1 \in \tilde{M}$,

7. If τ is not an integer, we can substitute $\lfloor \tau \rfloor$ for τ in the following calculations.

if $d(\gamma_0(t), \gamma_1(t)) < \delta$ for every $t \in [-\tau, \tau]$, then

$$\begin{aligned}
\tilde{d}(\gamma_0, \gamma_1) &\leq \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_1(t)) \\
&= \sum_{|i|\leq\tau} \frac{1}{2^{|i|}} \sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_1(t)) + \sum_{|i|>\tau} \frac{1}{2^{|i|}} \sup_{i\leq t<i+1} d(\gamma_0(t), \gamma_1(t)) \\
&\leq \sum_{|i|\leq\tau} \frac{1}{2^{|i|}}\delta + \sum_{|i|>\tau} \frac{1}{2^{|i|}} \|F\|(1+2|i|) \\
&= \delta\left(3 - \frac{1}{2^{\tau-1}}\right) + \|F\|\frac{\tau+3}{2^{\tau-1}} \\
&< 3\delta + \frac{\|F\|(\tau+3)}{2^{\tau-1}} \\
&= \varepsilon.
\end{aligned}$$

■

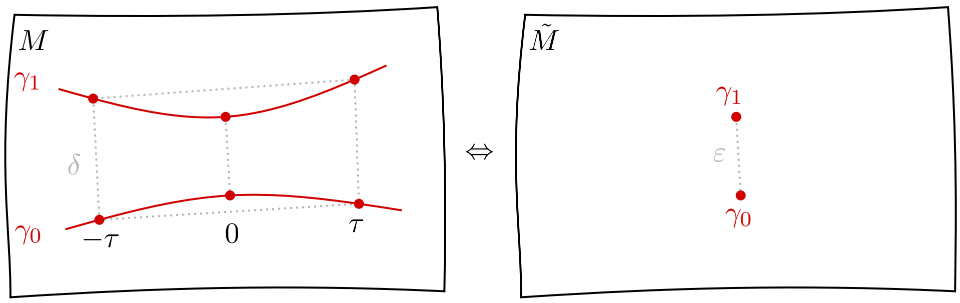


Figure 4.4. Graphical representation of LEMMAS 4.5 and 4.7. Orbits γ_0 and $\gamma_1 \in \tilde{M}$ are ε close in the orbit space if, and only if, their image points are δ close in the base space M.

To show that if two orbits are close in the orbit space, then the points of the orbits are close for enough time in the base space, we must first prove the following lemma (see FIGURE 4.5).

Lemma 4.6 ([GMV23])**.** *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. Let $\delta > 0$, $\alpha > 0$, $\gamma_0, \gamma_1 \in \tilde{M}$ and $t_0 \in \mathbb{R}$. If $d(\gamma_0(t_0), \gamma_1(t_0)) \geq \delta$, then, for every $t \in \left[t_0 - \frac{\alpha}{2\|F\|}, t_0 + \frac{\alpha}{2\|F\|}\right]$,*

$$d(\gamma_0(t), \gamma_1(t)) \geq \delta - \alpha.$$

Proof. Since F is bounded by $\|F\|$, it follow from the mean value inequality that, for every $\gamma \in \tilde{M}$ and every $t, t' \in \mathbb{R}$,

$$d(\gamma(t), \gamma(t')) \leq \|F\||t' - t|.$$

Then, for every $\gamma_0, \gamma_1 \in \tilde{M}$ and every $t \in \left[t_0 - \frac{\alpha}{2\|F\|}, t_0 + \frac{\alpha}{2\|F\|}\right]$,

$$\begin{aligned}
\delta &\leq d(\gamma_0(t_0), \gamma_1(t_0)) \\
&\leq d(\gamma_0(t_0), \gamma_0(t)) + d(\gamma_0(t), \gamma_1(t)) + d(\gamma_1(t), \gamma_1(t_0)) \\
&\leq \|F\||t - t_0| + d(\gamma_0(t), \gamma_1(t)) + \|F\||t_0 - t| \\
&\leq 2\|F\|\frac{\alpha}{2\|F\|} + d(\gamma_0(t), \gamma_1(t)),
\end{aligned}$$

therefore $\delta - \alpha \leq d(\gamma_0(t), \gamma_1(t))$. ■

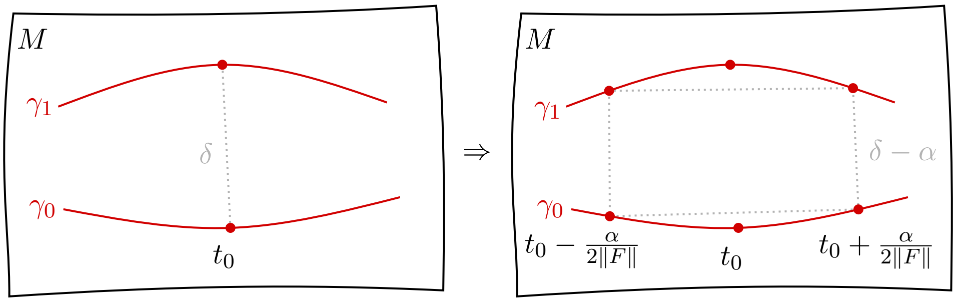


Figure 4.5. Graphical representation of LEMMA 4.6. If the orbit points of 2 orbits γ_0 and $\gamma_1 \in \tilde{M}$ at a time t_0 are at least δ apart, then their orbit points must be $\delta - \alpha$ apart for at least $\frac{\alpha}{2\|F\|}$ units of time forwards and backwards.

The next result has also appeared in [ACV23, Proposition 4.2]; we state it and also prove it here since we understand that it is important for the full comprehension of the ideas we are presenting.

Lemma 4.7 ([GMV23])**.** *Let M be a complete Riemannian manifold (possibly with boundary) and F be a bounded piecewise smooth vector field on M. For every $\tau > 0$ and $\delta > 0$, there exists $\varepsilon > 0$ such that, for every $\gamma_0, \gamma_1 \in \tilde{M}$, if $\tilde{d}(\gamma_0, \gamma_1) < \varepsilon$ then $d(\gamma_0(t), \gamma_1(t)) < \delta$ for every $t \in \,]-\tau, \tau[$.*

Proof. We will prove this by contradiction. Suppose some $\tau > 0$ and some $\delta > 0$ satisfy that, for every $\varepsilon > 0$, there are $\gamma_0^\varepsilon, \gamma_1^\varepsilon \in \tilde{M}$ and a time $t_\varepsilon \in \,]-\tau, \tau[$ such that $\tilde{d}(\gamma_0^\varepsilon, \gamma_1^\varepsilon) < \varepsilon$ and $d(\gamma_0^\varepsilon(t_\varepsilon), \gamma_1^\varepsilon(t_\varepsilon)) \geq \delta$. Taking $\alpha = \frac{\delta}{2}$ in LEMMA 4.6, and defining $I_\varepsilon := \left[t_\varepsilon - \frac{\delta}{4\|F\|}, t_\varepsilon + \frac{\delta}{4\|F\|}\right]$, it follows that, for every $t \in I_\varepsilon$,

$$d(\gamma_0^\varepsilon(t), \gamma_1^\varepsilon(t)) \geq \delta - \frac{\delta}{2} = \frac{\delta}{2} > 0. \tag{4.8}$$

Notice that the size of I_ε is independent of ε, since $|I_\varepsilon| = \frac{\delta}{2\|F\|}$, and that $I_\varepsilon \subseteq \left[-\tau - \frac{\delta}{4\|F\|}, \tau + \frac{\delta}{4\|F\|}\right]$. Therefore it follows that, for every $\varepsilon > 0$,

$$\varepsilon > \tilde{d}(\gamma_0^\varepsilon, \gamma_1^\varepsilon) > \frac{1}{2^{\lceil \tau + \frac{\delta}{4\|F\|}\rceil}} \int_{I_\varepsilon} d(\gamma_0^\varepsilon(t), \gamma_1^\varepsilon(t))\, \mathrm{d}t \geq \frac{1}{2^{\lceil \tau + \frac{\delta}{4\|F\|}\rceil}} \delta |I_\varepsilon| = \frac{\delta^2}{2^{\lceil \tau + \frac{\delta}{4\|F\|}\rceil} 2\|F\|},$$

So choosing $\varepsilon \leq \frac{\delta^2}{2^{\lceil \tau + \frac{\delta}{4\|F\|}\rceil} 2\|F\|}$ leads to a contradiction. ■

Topological properties of the orbit space

The first use we make of LEMMAS 4.5 and 4.7 is to prove the continuity of $\tilde{\Phi}$.

Proposition 4.8 ([GMV23]). *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. The flow $\tilde{\Phi}$ is continuous.*

Proof. We are going to assume the domain of the flow $\tilde{\Phi}$ is $\mathbb{R} \times \tilde{M}$ and also assume that the distance function on the product space $\mathbb{R} \times \tilde{M}$ is given by the maximum of the distances on $\mathbb{R}$ and $\tilde{M}$.

Take $(s_0, \gamma_0) \in \mathbb{R} \times \tilde{M}$ and $\varepsilon > 0$. We must find $\delta > 0$ such that, for every $(s, \gamma) \in \mathbb{R} \times \tilde{M}$, if $|s - s_0| < \delta$ and $\tilde{d}(\gamma_0, \gamma) < \delta$, then $\tilde{d}(\tilde{\Phi}^{s_0}(\gamma_0), \tilde{\Phi}^{s}(\gamma)) < \varepsilon$.

First notice that, since F is bounded by $\|F\|$, it follows from the mean value inequality that, for every $t \in \mathbb{R}$,

$$\begin{aligned} d(\gamma_0(t+s_0), \gamma(t+s)) &\leq d(\gamma_0(t+s_0), \gamma(t+s_0)) + d(\gamma(t+s_0), \gamma(t+s)) \\ &\leq d(\gamma_0(t+s_0), \gamma(t+s_0)) + \|F\||s - s_0|. \end{aligned}$$

Then, for every $i \in \mathbb{Z}$,

$$\sup_{i \leq t < i+1} d(\gamma_0(t+s_0), \gamma(t+s)) \leq \sup_{i \leq t < i+1} d(\gamma_0(t+s_0), \gamma(t+s_0)) + \|F\||s - s_0|,$$

so, by summing over all integers $i \in \mathbb{Z}$,

$$\tilde{d}(\tilde{\Phi}^{s_0}(\gamma_0), \tilde{\Phi}^{s}(\gamma)) = \tilde{d}(\tilde{\Phi}^{s_0}(\gamma_0), \tilde{\Phi}^{s_0}(\gamma)) + 3\|F\||s - s_0|. \tag{4.9}$$

Now notice that, for every $\varepsilon' > 0$, there is a $\delta' > 0$ such that, if $\tilde{d}(\gamma_0, \gamma) < \delta'$, then $\tilde{d}(\tilde{\Phi}^{s_0}(\gamma_0), \tilde{\Phi}^{s_0}(\gamma)) < \varepsilon'$. This is the case since, from LEMMA 4.5, there are $\tau_0 > 0$ and $\delta_0 > 0$ such that, if $d(\tilde{\Phi}^{s_0}(\gamma_0)(t), \tilde{\Phi}^{s_0}(\gamma)(t)) < \delta_0$ for every $t \in [-\tau_0, \tau_0]$, then $\tilde{d}(\tilde{\Phi}^{s_0}(\gamma_0), \tilde{\Phi}^{s_0}(\gamma)) < \varepsilon'$. But this hypothesis is equivalent to having $d(\gamma_0(t), \gamma(t)) < \delta_0$ for every $t \in [-\tau_0 + s_0, \tau_0 + s_0]$. So by choosing some $\tau_1 > 0$ such that $[-\tau_0 + s_0, \tau_0 + s_0] \subseteq [-\tau_1, \tau_1]$, it follows from LEMMA 4.7 that there is a $\delta' > 0$ such that, if $\tilde{d}(\gamma_0, \gamma) < \delta'$, then $d(\gamma_0(t), \gamma(t)) < \delta_0$ for every $t \in [-\tau_1, \tau_1]$.

Taking $\varepsilon' := \frac{\varepsilon}{2}$, using the respective δ' of the last paragraph, and defining $\delta := \min\{\delta', \frac{\varepsilon}{6\|F\|}\}$, we conclude from FORMULA 4.9 that, if $|s - s_0| < \delta$ and $\tilde{d}(\gamma_0, \gamma) < \delta$, then

$$\tilde{d}(\tilde{\Phi}^{s_0}(\gamma_0), \tilde{\Phi}^{s}(\gamma)) < \frac{\varepsilon}{2} + 3\|F\|\frac{\varepsilon}{6\|F\|} = \varepsilon. \qquad \blacksquare$$

We now study the topological structure of the orbit space. Our manifold is a separable (has a countable dense subset) and complete metric space. We show that the orbit space inherits these properties, and also has no isolated points.

Proposition 4.9 ([GMV23])**.** *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. If Σ^{t} is finite and each tangency point has finite multiplicity, then the orbit space $\tilde{M}$ is separable.*

Proof. The proof is divided in 5 steps.

1. (Augmenting the countable dense subset) The first step of the proof is to augment a countable dense subset of M to work better with respect to tangents. For each tangency point $p \in \Sigma^{\mathrm{t}}_{i,i'}$ (see FIGURE 4.6), there are (up to) four[8] orbit segments starting at p: (1) orbit segment $\gamma^{\mathrm{u}}_i \colon [0, t^{\mathrm{u}}_i] \longrightarrow M$ goes forwards from p, leaving $\Sigma^{\mathrm{us}}_{i,i'} \cup \Sigma^{\mathrm{t}}_{i,i'}$ at $\gamma^{i}_{+}(0) = p$ to the region M_i, and stops the first time the orbit reaches $\Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{t}}$ after that (if it does not, we take $t^{\mathrm{u}}_{=}\infty$ and consider the interval $[0, +\infty[$ as its domain); (2) orbit segment $\gamma^{\mathrm{u}}_{i'} \colon [0, t^{i'}_{-}] \longrightarrow M$ is defined analogously, but leaves at p to the region $M_{i'}$; (3) orbit segment $\gamma^{\mathrm{s}}_i \colon [-t^{\mathrm{s}}_i, 0] \longrightarrow M$ goes backwards from p, leaving $\Sigma^{\mathrm{ss}}_{i,i'} \cup \Sigma^{\mathrm{t}}_{i,i'}$ at $\gamma^{\mathrm{s}}_i(0) = p$ to region M_i, and stops the first time it reaches $\Sigma^{\mathrm{us}} \cup \Sigma^{\mathrm{t}}$ after that; and, finally, (4) orbit segment $\gamma^{\mathrm{s}}_{i'} \colon [-t^{\mathrm{s}}_{i'}, 0] \longrightarrow M$ is defined analogously to γ^{s}_i, but leaving p to the region $M_{i'}$.
   The image of each of these orbit segments is in a regular region, apart possibly from their endpoints, so they are homeomorphic to a real interval and thus we can take a countable dense subset of them. The union of all these sets is also a countable set, which we denote as E^{t}_p, and finally the set $E^{\mathrm{t}} := \bigcup_{p \in \Sigma^{\mathrm{t}}} E^{\mathrm{t}}_p$ is also a countable set, since Σ^{t} is finite. Finally, let E_0 be a countable dense subset of M and define $E_1 := E_0 \cup E^{\mathrm{t}}$.

8. This is a consequence of the finite multiplicity of the tangency points.

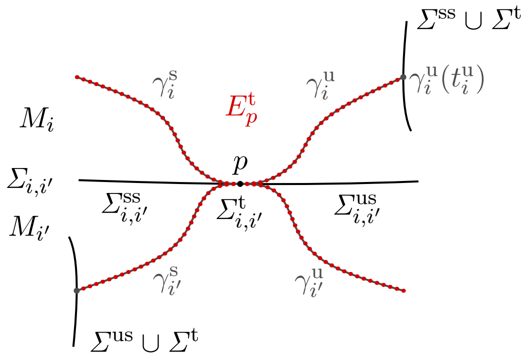


Figure 4.6. A dense set of points E^{t} is taken on the orbit segments γ_i^{u}, $\gamma_{i'}^{u}$, γ_i^{s} and $\gamma_{i'}^{s}$ arriving and leaving each tangency point p.

2. (Defining a dense subset of $\Sigma^{s} \cup \Sigma^{t}$) For each $p \in E_1$, we define points $p^{ss} \in \Sigma^{ss} \cup \Sigma^{t}$ and $p^{us} \in \Sigma^{us} \cup \Sigma^{t}$ as follows (see FIGURE 4.7). We consider an orbit γ of F such that $\gamma(0) = p$ and take $p^{ss} := \gamma(t^{ss})$, where $t^{ss} \geq 0$ is the smallest positive time t such that $\gamma(t) \in \Sigma^{ss} \cup \Sigma^{t}$ (that is, the first point in the orbit that reaches $\Sigma^{ss} \cup \Sigma^{t}$ going forward); if the orbit never enters $\Sigma^{ss} \cup \Sigma^{t}$, this point is left undefined. This is not dependent on the choice of γ since, before entering $\Sigma^{ss} \cup \Sigma^{t}$ for the first time, the orbit of p is unique, hence all such orbits coincide. Likewise, we take $p^{us} := \gamma(-t^{us})$, where $t^{us} \geq 0$ is the smallest positive time t such that $\gamma(-t) \in \Sigma^{us} \cup \Sigma^{t}$ (that is, the first point in the orbit that reaches $\Sigma^{us} \cup \Sigma^{t}$ going backward); if the orbit never reaches $\Sigma^{us} \cup \Sigma^{t}$, this point is left undefined. We define sets $E^{ss} := \{p^{ss} \mid p \in E_1\}$ and $E^{us} := \{p^{us} \mid p \in E_1\}$. As a consequence of the definition of E^{t}, all tangency points belong to these sets.
   Notice that the set E^{ss} is dense in $\Sigma^{ss} \cup \Sigma^{t}$ and the set E^{us} is dense in $\Sigma^{us} \cup \Sigma^{t}$. If it were otherwise, there would be a neighborhood of a point $p \in \Sigma^{ss}$ (respectively Σ^{us}) without any points of E^{ss} (respectively E^{us}), which could be translated backwards (resp. forwards) along the orbits to create a tubular neighborhood of a point of M without any point of E_1, which would contradict the fact that E_1 is dense in M. Finally, define $E := E_1 \cup E^{ss} \cup E^{us}$.

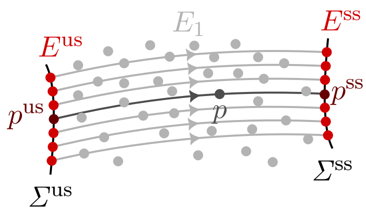


Figure 4.7. The sets of points E^{ss} and E^{us} (dense on Σ^{ss} and Σ^{us}, respectively) are taken by flowing the points $p \in E_1$ along their orbits until they reach the sliding region at the points p^{ss} and p^{us}.

3. (Classifying orbit behavior through $\Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{t}}$) For each orbit $\gamma \in \tilde{M}$ and each $\tau \geq 0$, we consider the restriction $\gamma|_{[-\tau,\tau]}$. The orbit $\gamma|_{[-\tau,\tau]}$ may enter and leave Σ only a finite number of times. Let $k^+ \in \mathbb{N} \cup \{0\}$ be the number of times it leaves $\Sigma^{\mathrm{us}} \cup \Sigma^{\mathrm{t}}$ going forward and $k^- \in \mathbb{N}$ be number of times $\gamma|_{[-\tau,\tau]}$ enters $\Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{t}}$ going backwards. We consider (see DEFINITION 4.8) the Σ-escaping sequence of $\gamma|_{[-\tau,\tau]}$ going forwards,

$$(e_1, i_1), \ldots, (e_{k^+}, i_{k^+}),$$

and the Σ-accessing sequence of $\gamma|_{[-\tau,\tau]}$ going backwards,

$$(a_{-k^-}, i_{-k^-}), \ldots, (a_{-1}, i_{-1}).$$

In the case $k^+ = 0$ or $k_- = 0$, we just take the empty sequence to represent the orbit's travel through Σ.
These sequences represent the points γ looses uniqueness going forwards (see FIGURE 4.8) and backwards (see FIGURE 4.9). For each $j \in \{1, \ldots, k^+\}$, $e_j \in \Sigma^{\mathrm{us}} \cup \Sigma^{\mathrm{t}}$ is the point through which $\gamma|_{[-\tau,\tau]}$ escapes $\Sigma^{\mathrm{us}} \cup \Sigma^{\mathrm{t}}$ for the j-th time, and i_j indicates it escapes to the region M_{i_j}. Analogously, for each $j \in \{-k^-, \ldots, -1\}$, $a_j \in \Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{t}}$ is the point through which $\gamma|_{[-\tau,\tau]}$ accesses $\Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{t}}$ for the $(-j)$-th time, and i_{-j} indicates it came from the region $M_{i_{-j}}$. The whole sequence is going to be called the Σ-sequence of γ in $[-\tau, \tau]$.

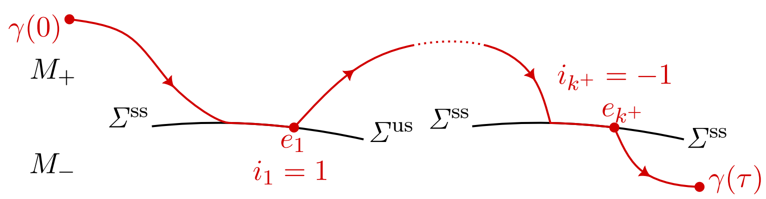


Figure 4.8. The Σ-escaping sequence of γ going forwards gives the points e_j where γ escapes Σ, and the indices i_j give the region M_{i_j} they escaped to.

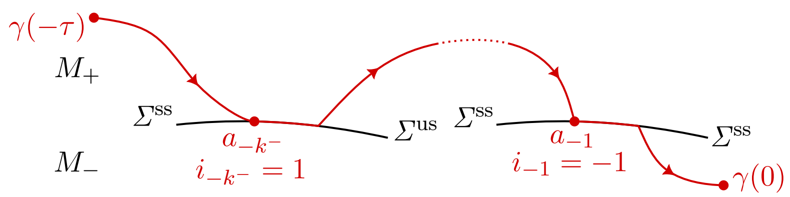


Figure 4.9. The Σ-accessing sequence of γ going backwards gives the points a_j where γ acesses Σ, and the indices i_j give the region M_{i_j} they accesses from.

4. (Constructing the countable dense subset of orbits) For each $p \in E$, we will construct a countable set of orbits in $\tilde{M}$ (see FIGURE 4.10). The union of theses orbits for all $p \in E$ will be our countable set. Let $p \in E$ and, for each $n \in \mathbb{N}$, define the subset $\Gamma_p^n \subseteq \tilde{M}$ of all orbits γ such that $\gamma(0) = p$ and, in the time interval $[-n, n]$, their Σ-sequence satisfies $a_{k^-}, \dots, a_{-1} \in E^{\mathrm{ss}}$ and $e_1, \dots, e_{k^+} \in E^{\mathrm{us}}$. We define an equivalence relation in Γ_p^n by determining that 2 orbits are equivalent if their Σ-sequence is equal in $[-n, n]$, and take $\tilde{E}_p^n \subseteq \Gamma_p^n$ to be a set of orbits with 1 representative of each equivalence class. This set is countable, since in the finite interval $[-n, n]$ there is a maximum for all possible k^+, a minimum for all k^-, the points a_j and e_j belong to the countable set $E^{\mathrm{ss}} \cup E^{\mathrm{us}}$, and the indices i_j belong to the finite set $[m] = \{0, \dots, m-1\}$. We define

$$\tilde{E} := \bigcup_{p \in E} \bigcup_{n \in \mathbb{N}} \tilde{E}_p^n.$$

This is going to be our countable set dense on $\tilde{M}$.

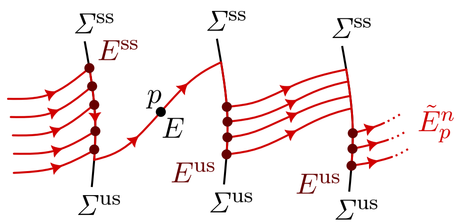


Figure 4.10. A countable set $\tilde{E}_p^n$ of orbits is chosen, each orbit starting at a point $p \in E$ and, for a set time interval $[-n, n]$, entering Σ^{ss} through points of E^{ss} on negative time and leaving Σ^{us} through points of E^{us} on positive time. The set is chosen such that all possible sequences of points of E^{ss} and E^{us} are taken into account.

5. (Approximating orbits) Let $\gamma \in \tilde{M}$ and $\varepsilon > 0$. We must find an orbit $\alpha \in \tilde{E}$ such that $\tilde{d}(\gamma, \alpha) < \varepsilon$ (see FIGURE 4.11). For this ε, we take $\delta > 0$ and $\tau > 0$ as in LEMMA 4.5. Define $n := \lfloor \tau \rfloor$. For each $j \in \{-k^-, \dots, k^+\}$, we will choose points $p_j \in M$ and numbers $\delta_j > 0$ as follows. If $\gamma(0) \in \Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{t}}$, we choose $p_0 \in E^{\mathrm{ss}} \cup E^{\mathrm{us}}$; if $\gamma(0) \in M \setminus (\Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{t}})$ and the first time forwards that γ reaches $\Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{t}}$ is at a tangency point, or if the first time backwards that it reaches $\Sigma^{\mathrm{us}} \cup \Sigma^{\mathrm{t}}$ is at a tangency point, we take $p_0 \in E_p^{\mathrm{t}}$, otherwise we take $p_0 \in E$. In all cases we choose p_0 close enough to $\gamma(0)$, such that $d(\gamma(0), p_0) < \delta_0$. Now, for each $j \in \{1, \dots, k^+\}$, we choose $p_j \in E^{\mathrm{us}}$ close enough to e_j, such that $d(e_j, p_j) < \delta_j$, considering that, if the first time γ reaches $\Sigma^{\mathrm{ss}} \cup \Sigma^{\mathrm{t}}$ after leaving through e_j is at a tangency point, we must

take $p_j = e_j$. The same procedure must be carried on for the negative part of the orbit, choosing, for each $j \in \{-k^-, \ldots, -1\}$, $p_j \in E^{\mathrm{ss}}$ such that $d(a_j, p_j) < \delta_j$. Since $p_0 \in E$, $p_j \in E^{\mathrm{us}}$ for positive j and $p_j \in E^{\mathrm{ss}}$ for negative j, there is a representative $\alpha \in \tilde{E}^n_p$. Choosing all δ_j small enough, we can guarantee that, for all $t \in [-\tau, \tau]$, we have $d(\gamma(t), \alpha(t)) < \delta$, so it follows by LEMMA 4.5 that $\tilde{d}(\gamma, \alpha) < \varepsilon$. ■

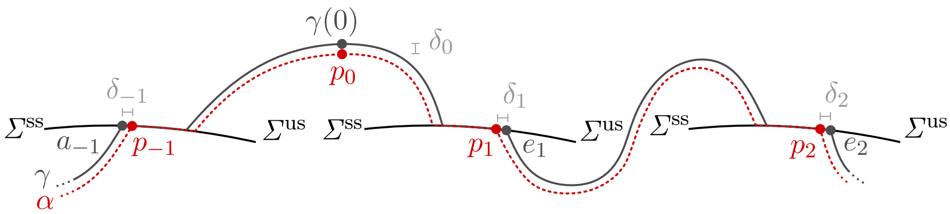


Figure 4.11. For a given orbit γ, we chose an orbit $\alpha \in \tilde{E}$ that is close to it each time it enters Σ^{ss} on negative time, and leaves Σ^{us} on positive time.

Proposition 4.10 ([GMV23]). *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. If Σ^{t} is finite, then the orbit space $\tilde{M}$ is a complete metric space.*

Proof. Let $(\gamma_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in the orbit space $\tilde{M}$. For each $t \in \mathbb{R}$, the sequence $(\gamma_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence in M by LEMMA 4.7. Since M is complete, there is a limit point $\gamma_\infty(t) \in M$. We must show the function

$$\begin{aligned} \gamma_\infty \colon \mathbb{R} &\longrightarrow M \\ t &\longmapsto \gamma_\infty(t) \end{aligned}$$

is an orbit of the piecewise smooth vector field system.

Take $t_0 \in \mathbb{R}$. If $\gamma_\infty(t_0) \in M \setminus (\Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{t}})$, the behavior of γ_∞ around $\gamma_\infty(t_0)$ is the same as that of points in a regular system, so in this case γ_∞ is an orbit of the system in the neighborhood of t_0. We must study the behavior of γ_∞ when $\gamma_\infty(t_0)$ belongs to Σ^{ss}, Σ^{us} or Σ^{t}.

Let us first consider $\gamma_\infty(t_0) \in \Sigma^{\mathrm{ss}}$. Take a tubular neighborhood $V \subseteq M$ of $\gamma_\infty(t_0)$. Since $\gamma_n(t_0) \to \gamma_\infty(t_0)$ as $n \to \infty$, there is a natural number $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\gamma_n(t_0) \in V$. For each of these n, take τ_n to be the smallest time $t \geq 0$ such that $\gamma_n(t_0 + \tau_n) \in \Sigma^{\mathrm{ss}}$. Since $\gamma_n(t_0) \to \gamma_\infty(t_0) \in \Sigma^{\mathrm{ss}}$, we must have $\tau_n \to 0$ as $n \to \infty$. Now, since $\gamma_n(t_0 + \tau_n)$ is on Σ^{ss}, its flow is given by the slide flow Φ_{s}, so there is a neighborhood $I =]c, d[$ of 0 such that, for all $t \in [0, d[$,

$$\gamma_n(t_0 + \tau_n + t) = \Phi^t_{\mathrm{s}}(\gamma_n(t_0 + \tau_n)).$$

Taking the limit as $n \to \infty$, we conclude that, for all $t \in [0, d[$,

$$\gamma_\infty(t_0 + t) = \Phi^t_{\mathrm{s}}(\gamma_\infty(t_0)).$$

For negative t, there are 2 cases. (1) If there is a neighborhood $I =]a', b[$ of 0 such that, for all $t \in I$, $\gamma_\infty(t_0 + t) \in \Sigma^{\mathrm{ss}}$; taking some $a \in]a', 0[$ and apply the previous proof for $\gamma_\infty(t_0 + a)$, which implies that $\gamma_\infty(t_0 + t) = \Phi^t_{\mathrm{s}}(\gamma_\infty(t_0))$ for all $t \in [a, 0[$. (2) In the other case, there is a neighborhood $I =]a, b[$ of 0 such that, for all $t \in]a, 0[$, the limit points $\gamma_\infty(t_0 + t)$ belong to the regular region $M \setminus \Sigma$ of the system, hence

$$\gamma_\infty(t_0 + t) = \Phi^t_i(\gamma_\infty(t_0)).$$

This proves that, in both cases, γ_∞ is an orbit around $\gamma_\infty(t_0)$. The behavior around $\gamma_\infty(t_0) \in \Sigma^{\mathrm{us}}$ is the same as the previous case in Σ^{ss}, just with the direction of the orbits inverted.

Finally, consider $\gamma_\infty(t_0) \in \Sigma^{\mathrm{t}}$. Notice that Σ^{t} is finite, so the tangency points are isolated. Since γ_∞ is an orbit around all points other than the tangency points, it follows that in a neighborhood of $\gamma_\infty(t_0)$ all points of γ_∞ are orbit points, so by continuity $\gamma_\infty(t_0)$ is also an orbit point. ■

Proposition 4.11 ([GMV23]). *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise smooth vector field on M. The orbit space $\tilde{M}$ is perfect.*

Proof. We must prove $\tilde{M}$ has no isolated points. Given any orbit $\gamma \in \tilde{M}$, we can apply the flow to it for every $t \in \mathbb{R}$ to obtain a family of orbits $\{\tilde{\Phi}^t(\gamma)\}_{t \in \mathbb{R}}$. To measure the distance between the original orbit and one of its translations, let us fix $\tau \in \mathbb{R}$. Since the piecewise smooth vector field is bounded by $\|F\|$ and each orbit is differentiable by parts, then from the mean value inequality it follows that, for any interval $I \subseteq \mathbb{R}$ and any orbit $\alpha \in \tilde{M}$, we have $d(\alpha(t), \alpha(t')) \leq \|F\||t' - t|$. Then

$$\tilde{d}_{\sup}(\gamma, \tilde{\Phi}^\tau(\gamma)) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i \leq t < t+1} d(\gamma(t), \gamma(\tau + t)) \leq 3\|F\||\tau|.$$

Therefore we conclude that, as τ approaches 0, $\tilde{\Phi}^\tau(\gamma)$ approaches γ, so γ is not an isolated point of $\tilde{M}$. ■

4.3.3 Topological transitivity in the orbit space

We now investigate the topological transitivity property. The first proposition is an immediate consequence of the topological properties proven in the last subsection.

Proposition 4.12 (Topological transitivity equivalence [GMV23]). *Let M be a complete Riemannian manifold (possibly with boundary) and F a bounded piecewise* *smooth vector field on M. For the flow $\tilde{\Phi}$ on $\tilde{M}$, topological transitivity is equivalent to topological point-transitivity.*

Proof. The orbit space $\tilde{M}$ is a separable complete metric space without isolated points (PROPOSITIONS 4.9 to 4.11), so this follows from Birkhoff's transitivity (THEOREM 1.8). ■

Because of this equivalence, we will refer to either property as 'topological transitivity'. It is easy to show that topological transitivity in the orbit space implies topological transitivity in the base space, as the next proposition shows.

Proposition 4.13 ([GMV23])**.** *Let F be a piecewise smooth vector field on M and $\tilde{\Phi}$ its respective flow over the orbit space system $\tilde{M}$. If $\tilde{\Phi}$ is topologically transitive, then F is also topologically transitive.*

Proof. To show F is topologically transitive, we must find an orbit $\gamma \in \tilde{M}$ that is dense in M. Let $p \in M$ and $\delta > 0$, and take $\alpha \in \tilde{M}$ such that $\alpha(0) = p$. Since the flow $\tilde{\Phi}$ is topologically transitive, there is a dense orbit Γ in the orbit space $\tilde{M}$. Define $\gamma := \Gamma(0)$. For the given δ and any $\tau > 1$, take $\varepsilon > 0$ as in LEMMA 4.7. Since Γ is dense in $\tilde{M}$, there is $t_\alpha > 0$ such that $\tilde{d}(\alpha, \tilde{\Phi}^{t_\alpha}(\gamma)) < \varepsilon$, which implies by the choice of ε that $d(\alpha(t), \tilde{\Phi}^{t_\alpha}(\gamma)(t)) < \delta$ for every $|t| \leq 1$. In particular,

$$d(p, \gamma(t_\alpha)) = d(\alpha(0), \tilde{\Phi}^{t_\alpha}(\gamma)(0)) < \delta,$$

so γ is dense in M. ■

On the other hand, to prove the converse — that topological transitivity in the base space implies its validity for the orbit space — we restrict our analysis to a class of systems that have enough connection between tangency points.

Theorem 4.14 ([GMV23])**.** *Let M be a connected Riemannian manifold and F be a piecewise smooth vector field on M. Suppose the set of tangency points Σ^{t} of the system is finite and there is a subset $\mathcal{T} \subseteq \Sigma^{\mathrm{t}}$ such that:*

1. *For every $T, T' \in \mathcal{T}$, there is a F-orbit segment $\theta\colon [-s, s] \to M$ connecting T to T'. If $T = T'$ we can consider $s = 0$;*
2. *For every orbit γ, there are a tangency point $T_\gamma \in \mathcal{T}$ and time $s_\gamma > 0$ such that $\gamma(s_\gamma) = T_\gamma$;*
3. *For every orbit γ, there are a tangency point $T_\gamma \in \mathcal{T}$ and time $s_\gamma < 0$ such that $\gamma(s_\gamma) = T_\gamma$;*

Then the flow $\tilde{\Phi}$ is topologically transitive on the orbit space $\tilde{M}$.

Proof. Let $\alpha, \beta \in \tilde{M}$ be two orbits and $\varepsilon > 0$, a real number. To prove the flow $\tilde{\Phi}$ is topologically transitive, we must find an orbit $\gamma \in \tilde{M}$ such that $\tilde{d}(\alpha, \gamma) < \varepsilon$ and, for some $\tau > 0$, $\tilde{d}(\beta, \tilde{\Phi}^{\tau}(\gamma)) < \varepsilon$. Since the flow $\tilde{\Phi}$ is invertible, this is the equivalent to finding $\tau_\alpha, \tau_\beta > 0$ such that $\tilde{d}(\alpha, \tilde{\Phi}^{-\tau_\alpha}(\gamma)) < \varepsilon$ and $\tilde{d}(\beta, \tilde{\Phi}^{\tau_\beta}(\gamma)) < \varepsilon$.

To construct the orbit γ, we first cut orbit α at a tangency point $T_\alpha \in \mathfrak{T}$ and orbit β at a tangency point $T_\beta \in \mathfrak{T}$, and take an intermediary orbit segment θ connecting T_α to T_β (using hypothesis 1). Then, we glue the negative orbit segment of α to the beginning of θ, and the end of θ to the positive orbit segment of β. The resulting curve γ is an orbit of the system, since it is made up of orbit segments connected at tangency points.

Take $\delta > 0$ and $\tau > 0$ with respect to the chosen ε, such as in LEMMA 4.5. We can take the smallest time t_α such that $t_\alpha \geq \tau > 0$ and $\alpha(t_\alpha) = T_\alpha$, for some $T_\alpha \in \mathfrak{T}$ (by 2), and the smallest time t_β such that $-t_\beta \leq -\tau < 0$ and $\beta(-t_\beta) = T_\beta$, for some $T_\beta \in \mathfrak{T}$ (by 3).

Let $\theta\colon [-s, s] \to M$ be an orbit segment connecting T_α to T_β (by 1) and define orbit γ to be

$$\gamma(t) := \begin{cases} \alpha(t + s + t_\alpha), & -\infty < t < -s \\ \theta(t), & -s < t < s \\ \beta(t - s - t_\beta), & s < t < \infty. \end{cases} \tag{4.10}$$

Since $\tau \leq t_\alpha$, this means $\tilde{\Phi}^{-t_\alpha - s}(\gamma)$ coincides with α on the interval $]-\infty, \tau]$, so

$$d(\alpha(t), \tilde{\Phi}^{-t_\alpha - s}(\gamma)(t)) = 0 < \delta$$

for every $|t| \leq \tau$, and likewise

$$d(\alpha(t), \tilde{\Phi}^{t_\beta + s}(\gamma)(t)) = 0 < \delta$$

for every $|t| \leq \tau$. By the choice of δ and τ from LEMMA 4.5, this implies that $\tilde{d}(\alpha, \tilde{\Phi}^{-t_\alpha - s}(\gamma)) < \varepsilon$ and $\tilde{d}(\beta, \tilde{\Phi}^{t_\beta + s}(\gamma)) < \varepsilon$. ■

In particular, THEOREM 4.14 implies that both the bean model of [BCE16] and the sphere model of [EJV22] have topologically transitive orbit spaces. For the bean model (see FIGURE 4.12) this is so because the system has 1 central tangency point p and every orbit passes through it infinite times going forwards and backwards, so the hypothesis is satisfied.

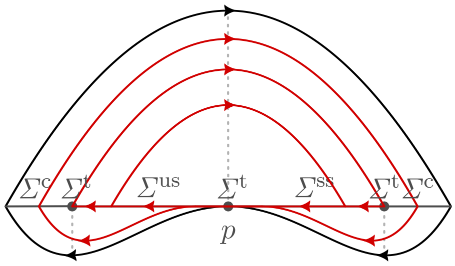


Figure 4.12. The bean model.

The dynamics of the sphere model (see FIGURE 4.13) has 4 tangency points; we choose 2 of them, say T_1^+ and T_2^-, to be set $\mathcal{T}$ of THEOREM 4.14. There is a closed orbit passing through both of them, which implies there are segments connecting one of them to the other, so the hypothesis is satisfied.

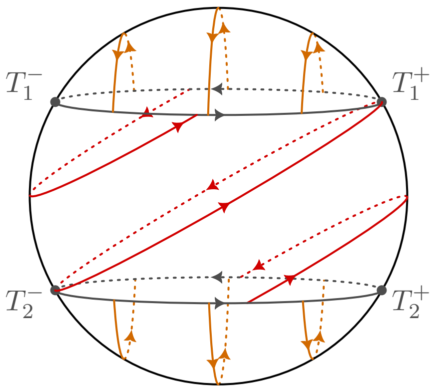


Figure 4.13. The sphere model.

PROPOSITION 4.13 shows that transitivity in the orbit space implies transitivity in the base space, and THEOREM 4.14 is a partial converse for a specific class of systems that have enough connections between tangency points. Therefore, we pose the following question.

Question 1. Is there an example of a piecewise smooth vector field which is topologically transitive, but whose orbit space is not topologically transitive?

The above question is an interesting and important one for the theory. It impacts, in spirit, how we understand PSVFs. If the answer is 'yes', then we might have to use the orbit space to justify a more natural topological transitivity. If the answer is 'no', then both notions of topological transitivity are the same and hence the effect of escaping regions in creating topological transitivity is not actually an "artificial" one.

4.4 Non-deterministic chaos

In this section we address the problem of determining the chaotic behavior of piecewise smooth vector fields. We define chaos for these systems in DEFINITION 4.13, based on the well-accepted definition of chaos for smooth systems by Devaney [Dev89], according to which a chaotic system is one that is topologically transitive, has sensitive dependence on initial conditions, and has density of periodic orbits. In fact, topological transitivity and density of periodic solutions are sufficient for a system to be chaotic [BBC+92]. The term *non-deterministic* is used because there is non-uniqueness of solutions on a piecewise smooth system.

We restrict ourselves to 2-dimensional manifolds and piecewise smooth vector fields with a finite number of tangency points and show in THEOREM 4.18 that, in this setting, topological transitivity implies both density of periodic points and sensitive dependence on initial conditions. After that we make some comments about finding a residual set of periodic orbits before proceeding to define and estimate the topological entropy of our system.

In our context, topological transitivity implies positive entropy (THEOREM 4.22). Frequently one uses entropy to determine some level of chaoticity of a given dynamics. That is because entropy is a number which may be understood as measuring the creation of new orbits — more precisely, the exponential growth rate of the number of different orbits of the system — hence a system with many "genuinely" distinct orbits (i.e. with positive entropy) might be a complex or "chaotic" system. The content of this section was developed in [EMV24].

4.4.1 Topological transitivity implies non-deterministic chaos

We start with a definition of chaoticity for piecewise smooth vector fields. The definition is based on the classical definition of Devaney for chaos (see [Dev89; BBC+92]), but adapted to our context.

Definition 4.13. Let M be a Riemannian manifold and F a piecewise smooth vector field on M.

1. F is *topologically transitive* if given any two non-empty open sets U and V of M, there exists an orbit from a point of U to a point of V.
2. F has *sensitive dependence on initial conditions* if there exists a fixed $\delta > 0$ such that, for any non-empty open set U, there exist points $x, y \in U$, orbits γ_x and γ_y which start at x and y, respectively, and some time t such that $d(\gamma_x(t), \gamma_y(t)) > \delta$.
3. F is *chaotic* if it is topologically transitive, has sensitive dependence on initial conditions and the union of all periodic orbits is a dense set.

We also define the following sets.

Definition 4.14. Let M be a Riemannian manifold and F a piecewise smooth vector field on M.

1. E_+ is the set of points $p \in M$ that first loose uniqueness going forward in Σ^{ss} (that is, there is $t_{\mathrm{s}} > 0$ such that $\gamma(t_{\mathrm{s}}) \cap \Sigma^{\mathrm{ss}} \neq \emptyset$ and, for every $0 < t < t_{\mathrm{s}}$, $\gamma(t)$ is in a regular region R_i or crossing region).
2. E_- is the set of points of M that first loose uniqueness going backward in Σ^{us}.
3. D_+ is the set of points $p \in M$ for which there is some orbit γ starting at p that reaches Σ^{ss} going forward (that is, there is $t_{\mathrm{s}} > 0$ such that $\gamma(t_{\mathrm{s}}) \cap \Sigma^{\mathrm{ss}} \neq \emptyset$).
4. D_- is the set of points $p \in M$ for which there is some orbit starting at p that reaches Σ^{us} going backward.

We shall prove a number of important lemmas that will be needed for the proof of our results. First we show how topological transitivity leads to the existence of a connection of points on the sliding region.

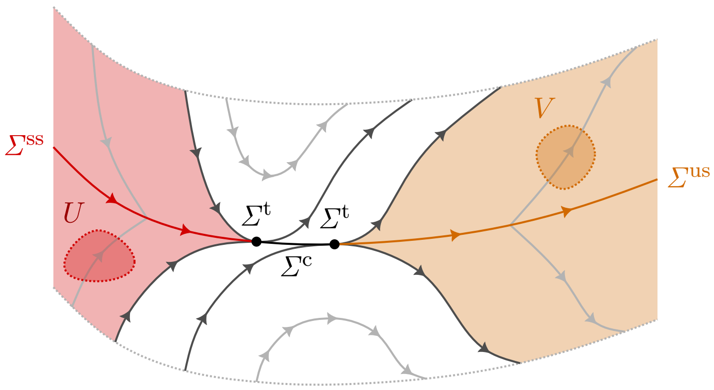


Figure 4.14. An open set U that reaches Σ^{ss} flowing forward and an open set V that reaches Σ^{us} flowing backwards.

Lemma 4.15 ([EMV24])**.** *Let M be a 2-dimensional Riemannian manifold and F a topologically transitive piecewise smooth vector field on M. For every pair of points q_0, $q_1 \in \Sigma^{\mathrm{s}}$, there is an orbit segment from q_0 to q_1.*

Proof. The points q_0 and q_1 can be in either Σ^{ss} or Σ^{us}, so there are 4 possibilities of connections of q_0 and q_1. We will describe how to connect them in all cases. If $q_0 \in \Sigma^{\mathrm{ss}}$, since Σ^{s} is relatively open in Σ we can choose a point q_0' before q_0 (that is, q_0' is in the same connected component of Σ^{ss} as q_0 and it reaches q_0 through an orbit segment of the sliding vector field on Σ^{ss}) and an open set $U_0 \subset M$ around

q_0' (that does not contain q_0) in such a way that every point of U_0 flows to Σ^{ss} and reaches q_0 (see FIGURES 4.14 and 4.15).

If $q_0 \in \Sigma^{us}$, we can choose an open set U_0 in an analogous way, but now inverting the direction of the flow of time: we take a point q_0' after q_0 and an open set U_0 around it such that every point of U_0 must have come from Σ^{us} and, consequently, passed through q_0 (see FIGURE 4.15). In the same way we choose an open set $U_1 \subset M$ for q_1.

By topological transitivity, there is an orbit segment from a point $p_0 \in U_0$ to a point of $p_1 \in U_1$. In the case that $q_0 \in \Sigma^{ss}$, by the choice of U_0 the orbit segment must have passed through q_0, so it can be restricted to an orbit segment from q_0 to p_1; in the case that $q_0 \in \Sigma^{us}$, by the choice of U_0 the point p_0 must have flowed away from q_0, so the orbit segment can also be extended to one from q_0 to p_1.

On the other end, the situation is inverted. If $q_1 \in \Sigma^{ss}$, the point p_1 must flow into Σ^{ss} and pass through q_1, so the orbit segment may be extended to one from q_0 to q_1; if $q_1 \in \Sigma^{ss}$, the point p_1 must have passed through q_1, so the orbit segment may also be extended to one from q_0 to q_1. ■

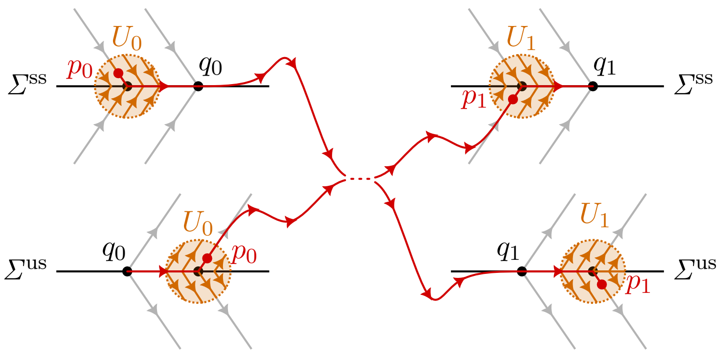


Figure 4.15. All 4 possibilities of connection of 2 points q_0 and q_1 on the sliding region Σ^s. The dashed orbit in the center of the figure represents that each orbit segment on the left can connect to each one on the right.

The next lemma establishes some properties of the sets described in DEFINITION 4.14.

Lemma 4.16 ([EMV24])**.** *Let M be a 2-dimensional Riemannian manifold and F a topologically transitive piecewise smooth vector field on M.*

1. *Suppose $\Sigma^{ss} \neq \emptyset$. Then E_+ is open and D_+ is dense.*
2. *Suppose $\Sigma^{us} \neq \emptyset$. Then E_- is open and D_- is dense.*

Proof. We prove the first item, since the second is the same but with the direction of orbits inverted. Let $p \in E_+$ and $q \in \Sigma^{ss}$ be the point where p first looses uniqueness going forward. Since Σ^{ss} is open in Σ and all points of the orbit from p to q are regular, we can find an open set around p that first looses uniqueness in Σ^{ss}. This shows E_+ is open.

Showing D_+ is dense is the same as showing $\overline{D_+}$ has empty interior. For this, suppose for the sake of contradiction that there is a non-empty open set $U \subset \overline{D_+}$. Since $\Sigma^{ss} \neq \emptyset$, there is an open set V such that all of its points reach Σ^{ss} (see FIGURE 4.14). By topological transitivity, there is an orbit from U to V, and so this orbit can be extended to reach Σ^{ss}, which contradicts the fact that U is a set of points that do not reach Σ^{ss} by any orbit. ■

In some examples of transitive piecewise smooth vector field in the literature like the bean model and the sphere model [EJV22; BCE16; ACV23], it is easy to check that the sets E_+ and E_- are dense, so in the following results (LEMMA 4.17, THEOREM 4.18) we will assume this in order to obtain dense periodic orbits (for further discussion, check SECTION 4.4.1).

Lemma 4.17 ([EMV24])**.** *Let M be a 2-dimensional Riemannian manifold, F a topologically transitive piecewise smooth vector field on M, and assume $\Sigma^s \neq \emptyset$. Let $q_0 \in \Sigma^s$ and $U \subseteq M$ be a non-empty open set. Then there is a periodic orbit segment through q_0 that intersects U.*

Proof. We assume that $q_0 \in \Sigma^{ss}$. The proof for $\Sigma^{us} \neq \emptyset$ is the same after inverting the direction of orbits. Since q_0 is in the sliding region, we may choose an open set $U_0 \subseteq M$ such that all of its points pass through q_0 going forwards, as done in LEMMA 4.15 (see FIGURE 4.16). Since the set E_+ is open and we assume it is dense, take $p \in E_+ \cap U$ and an open neighborhood $V \subseteq E_+$ of p.

By topological transitivity, we choose an orbit segment γ_0 from U_0 to $U \cap V$. Since all the points of U_0 pass through q_0, γ_0 can be restricted to start at q_0, and since γ_0 passes through $V \subseteq E_+$, it can be extended to reach a point $q_1 \in \Sigma^{ss}$. Now from LEMMA 4.15, there is an orbit segment γ_1 from q_1 to q_0. Concatenating the orbit segments γ_0 and γ_1, we obtain a periodic orbit segment starting at q_0 that intersects U. ■

Finally, we can prove the theorem.

Theorem 4.18 ([EMV24])**.** *Let M be a 2-dimensional Riemannian manifold, F a topologically transitive piecewise smooth vector field on M, Σ^t is finite and $\Sigma^s \neq \emptyset$.*

1. *There is a dense set Δ such that, for every $x \in \Delta$,*
   1.1. *there is a dense orbit through x;*
   1.2. *the periodic orbits through x form a dense set;*
2. *The system has sensitive dependence on initial conditions.*

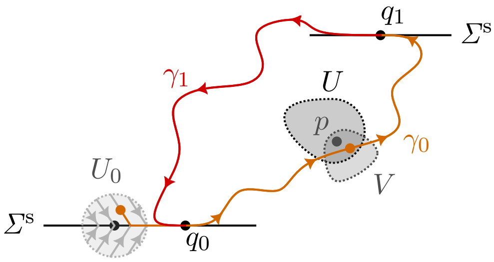


Figure 4.16. Given a point $q_0 \in \Sigma^{s}$ and an open set U, we can find a periodic orbit segment starting at q_0 that intersects U. In the figure we depict the case in which $\Sigma^{ss} \neq \emptyset$.

Proof. We prove each item separately.

1. Take $x \in \Sigma^{s}$.
   1.1. Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable base of M. By LEMMA 4.17, for each U_i there is a periodic orbit segment γ_i that starts at x and intersects U_i. Define γ to be the concatenation of the orbit segments γ_i in order of index, backwards and forwards; i.e., define

$$\gamma := \cdots\gamma_1\gamma_0\gamma_0\gamma_1\cdots.$$

   Now for every non-empty open set $U \subseteq M$, there is an open set U_i contained in U, so γ intersects U because γ_i intersects U_i. This shows γ is dense in M.
   1.2. Now let P be the union of (the image of) all the periodic orbits of F through x. By LEMMA 4.17 it follows that, for every non-empty open set $U \subseteq M$, there is a periodic orbit segment γ that passes through x and intersects U, hence a periodic orbit through x and U which means that $P \cap U \neq \emptyset$, so P is dense in M.

   We now take Δ as the union of all the points on periodic orbits as constructed in the preceding items.
2. We will assume that $\Sigma^{ss} \neq \emptyset$, but the proof for $\Sigma^{us} \neq \emptyset$ is analogous. First, we take $q \in \Sigma^{ss}$ and two open sets U_0 and U_1, each one on each side of the connected component of Σ^{ss} on which q is (see FIGURE 4.17), and not intersecting Σ^{ss}. Now we choose two different periodic orbits β_0 and β_1 starting at q, similarly to what was done in LEMMA 4.17), but in this case we force each orbit to enter Σ^{ss} on the open sets U_0 and U_1, respectively, before they reach q. Denote their periods respectively by b_0 and b_1. Since Σ^{ss} is open, we may disturb these orbits slightly in order to have the ratio of their periods, $\frac{b_1}{b_0}$, be an irrational number. For each $i \in \{0,1\}$, we choose a point $p_i \in U_0$ through which orbit β_i passes, and define a restricted orbit β_i'

from q to p_i, and denote its period b'_i. Denote $\delta := d(U_0, U_1)$, the infimum of the distance between any point of U_0 and any point of U_1. Since each open set has been taken on one side of Σ^{ss}, we have $\delta > 0$.
Now let U be any non-empty open set. Since our set Δ as defined in the preceding item of this proof is dense, we may take $x, y \in \Delta \cap U$. Then there exist orbits $\alpha_x : [0, a_x] \to M$ and $\alpha_y : [0, a_y] \to M$ starting at x and y, respectively, and ending at q. Now we define orbit γ_x as the concatenation of α_x and $k_x \in \mathbb{N}$ orbits β_0, followed by the restricted orbit β'_0; that is, $\gamma_x := \alpha_x(\beta_0)^{k_x}\beta'_0$. Likewise, define $\gamma_y := \alpha_y(\beta_1)^{k_y}\beta'_1$, for $k_y \in \mathbb{N}$. We denote $c_x := a_x + k_x b_0 + b'_0$ and $c_y := a_y + k_y b_0 + b'_0$, which represent the time x (or y) takes to traverse γ_x (resp. γ_y) and reach p_0 (resp. p_1). Since the ratio $\frac{b_1}{b_0}$ is irrational, the integers k_x and k_y may be chosen in order that the c_x and c_y are as close as needed, such that (supposing $c_x \leq c_y$ without loss of generality) $\gamma_x(c_x) = p_0$ and $\gamma_y(c_y) \in U_1$, close to p_1. Defining $t := c_x$, this implies that $d(\gamma_x(t), \gamma_x(t)) > \delta$. ■

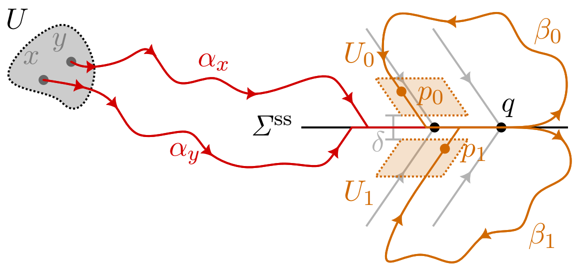


Figure 4.17. For any non-empty open set U, there are points x and y and orbits γ_x and γ_y such that, following these orbits, the two points eventually become δ apart. The orbits are constructed by connecting the points to a sliding region and then following periodic orbits for some time.

Comments about finding a residual set

In the classical case of smooth vector fields, transitivity is equivalent to the existence of a residual set of points through which there is a dense orbit. One good question that we are not able to prove so far is the following.

Question 2. Let M be a 2-dimensional manifold and F a piecewise smooth vector field with Σ^{t} finite. Does the existence of a dense orbit implies the existence of a residual set with dense orbit?

In our setting, we have found a dense set Δ instead. As mentioned after the proof of LEMMA 4.16, for the bean model and the sphere model [BCE16; EJV22] both sets E_+ and E_- are dense, hence open dense sets. So we ask the following question:

Question 3. Let M be a 2-dimensional manifold and F a piecewise smooth vector field with Σ^{t} finite. Are the sets E_+ and E_- dense?

If both[9] E_+ and E_- were open dense sets, this would imply that their intersection $E := E_+ \cap E_-$ is a residual set (and hence a dense set by Baire category theorem). The proof of LEMMA 4.17 can be easily adapted to be valid for points x in this residual set E (FIGURE 4.18). This can be done by starting with a point $x \in E$ instead of $q_0 \in \Sigma^{\mathrm{s}}$, and then connecting x to Σ^{s} forward and backwards, which is possible by definition of E. After that, the final details of the proof would be almost the same as in LEMMA 4.17.

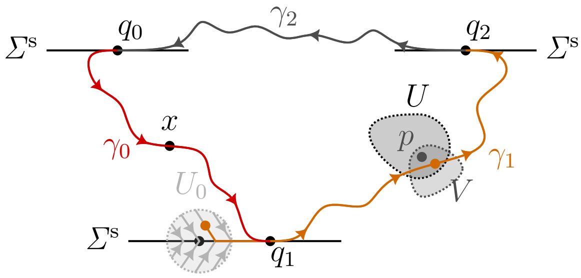


Figure 4.18. Given a point $x \in \Delta$ and an open set U, we could find a periodic orbit segment starting at x that intersects U.

4.4.2 Entropy of piecewise smooth dynamical systems

Topological entropy for a piecewise smooth vector field was first defined in [ACV23] by use of orbit spaces. Here we follow this approach and use it to calculate the topological entropy of topologically transitive piecewise smooth vector fields on 2-dimensional manifolds (THEOREM 4.22).

Definition of topological entropy

In the classical context, we can define the topological entropy a continuous flow Φ^t ($t \in \mathbb{R}$) in a compact metric space K using dynamical balls and generating sets, and its value is the same as the topological entropy of the time 1 map $\Phi^1 : K \longrightarrow K$ of the flow (SECTION 2.4.2). Because of this, the topological entropy of Φ^1 is sometimes taken as the definition of topological entropy for the flow Φ. This is the motivation for the definition of topological entropy for piecewise smooth systems which we will use. We will assume our space M is compact.

9. Notice we are assuming here that both Σ^{ss} and Σ^{us} are non-empty.

Definition 4.15. Let M be a compact Riemannian manifold and F a piecewise smooth vector field on M. The *topological entropy* of F is the topological entropy of the time 1 map $\tilde{\Phi}^1 \colon \tilde{M} \longrightarrow \tilde{M}$ on the orbit space $\tilde{M}$, denoted

$$\mathrm{h}(F) := \mathrm{h}(\tilde{\Phi}^1).$$

In what follows, we will estimate the entropy of our topologically transitive system F to show it is strictly positive. We will use the orbit space $\tilde{M}$ and its flow $\tilde{\Phi}^t$, induced by F. By DEFINITION 4.15, the entropy of F is the entropy of the time 1 map $\tilde{\Phi}^1 \colon \tilde{M} \longrightarrow \tilde{M}$. We will define a subset $\tilde{\Gamma}_p$ of $\tilde{M}$ and consider the induced flow on it to show that its entropy is strictly positive, which implies that the entropy of $\tilde{M}$ is also positive since the entropy of a subsystem is always smaller than the entropy of the system [VO16].

To define the subset $\tilde{\Gamma}_p$, we first need to have two different periodic orbits of the piecewise smooth system F which have the same period and have a point in common. We start with a simple lemma that proves this is the case for our setting.

Lemma 4.19 ([EMV24])**.** *Let M be a 2-dimensional Riemannian manifold and F a topologically transitive piecewise smooth vector field on M. If $\Sigma^{\mathrm{s}} \neq \emptyset$, then there exist 2 distinct periodic orbit segments $\gamma_0, \gamma_1 \colon [0, \alpha] \longrightarrow M$ of F with period $\alpha \in \mathbb{R}_{>0}$ and initial point $p = \gamma_0(0) = \gamma_0(\alpha) = \gamma_1(0) = \gamma_1(\alpha) \in M$.*

Proof. We can find two distinct periodic orbit segments η_0 and η_1 that have an initial point $p \in \Sigma^{\mathrm{ss}}$, in the same way it was done in LEMMAS 4.15 and 4.17, and they can be forced to differ by making each one pass through each different side of the connected component of Σ^{s} being considered (see FIGURE 4.14). Since these orbit segments may have different periods, we can concatenate η_0 and η_1 in each possible order to obtain the orbit segments $\gamma_0 := \eta_0\eta_1$ and $\gamma_1 := \eta_1\eta_0$ (see FIGURE 4.19), which have the same period (the sum of the periods of γ_0 and γ_1). ∎

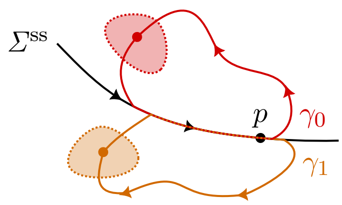


Figure 4.19. Distinct periodic orbits with the same period (Σ^{ss} case).

Construction of the subsystem

Now we present the construction of the subsystem $\tilde{\Gamma}_p$ of the orbit space $\tilde{M}$. We take the periodic orbit segments γ_0 and γ_1 as in LEMMA 4.19. Let $\Gamma_0 := \gamma_0([0,\alpha])$ and $\Gamma_1 := \gamma_1([0,\alpha])$ denote their images, two curves on the base space M that are distinct and have their initial point p in common[10]. Denote the union of these curves as $\Gamma := \Gamma_0 \cup \Gamma_1$. The set $\tilde{\Gamma}_p$ is the set of all the possible integral trajectories of F that start at p and whose image lies on Γ.

All trajectories of $\tilde{\Gamma}_p$ must start at p and follow either Γ_0 or Γ_1, returning to p after the period α has passed. Then, it must again follows either of the two curves and so on forwards and backwards in time. We can more formally describe this as follows.

We consider the bilateral shift system in 2 symbols (Σ_2, σ), with $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ and $\sigma(x)_i = x_{i+1}$. For each sequence $x \in \Sigma_2$, we can define an orbit γ_x of F (which starts at p and whose image lies on Γ) as the concatenation of the orbit segments γ_0 and γ_1 according to the entries of x: for each $i \in \mathbb{Z}$ and each $t \in [i\alpha, (i+1)\alpha[$, we have $\gamma_x(t) = \gamma_{x_i}(t - i\alpha)$. We will denote this by[11]

$$\gamma_x(t) = \Big\{\gamma_{x_i}(t - i\alpha), \quad i\alpha \leq t < (i+1)\alpha.$$

This gives the characterization $\tilde{\Gamma}_p = \{\gamma_x \mid x \in \Sigma_2\}$.

To define the dynamics on $\tilde{\Gamma}_p$, we take the map $\tilde{\Phi}^\alpha \colon \tilde{\Gamma}_p \longrightarrow \tilde{\Gamma}_p$ given on each $\gamma \in \tilde{\Gamma}_p$ by $\tilde{\Phi}^\alpha(\gamma)(t) = \gamma(\alpha + t)$. This is the flow map $\tilde{\Phi}^\alpha \colon \tilde{M} \longrightarrow \tilde{M}$ (DEFINITION 4.10) restricted to $\tilde{\Gamma}_p$. The restriction is well defined because $\tilde{\Phi}^\alpha(\gamma_x) = \gamma_{\sigma(x)} \in \tilde{\Gamma}_p$ for every $x \in \Sigma_2$, which follows from the calculation

$$\begin{aligned}
\tilde{\Phi}^\alpha(\gamma_x)(t) &= \gamma_x(t + \alpha) \\
&= \Big\{\gamma_{x_i}(t + \alpha - i\alpha), \quad i\alpha \leq t + \alpha < (i+1)\alpha \\
&= \Big\{\gamma_{x_i}(t - (i-1)\alpha), \quad (i-1)\alpha \leq t < i\alpha \\
&= \Big\{\gamma_{x_{i+1}}(t - i\alpha), \quad i\alpha \leq t < (i+1)\alpha \\
&= \gamma_{\sigma(x)}(t).
\end{aligned}$$

This shows that $\tilde{\Phi}^\alpha \colon \tilde{\Gamma}_p \longrightarrow \tilde{\Gamma}_p$ is a subsystem of $\tilde{\Phi}^\alpha \colon \tilde{M} \longrightarrow \tilde{M}$.

Entropy calculation

We are ready to calculate the entropy. Let us define the constant

$$\mu := \sup_{0 \leq t < \alpha} d(\gamma_0(t), \gamma_1(t)),$$

10. We do not need to assume the only intersection of the curves is p.
11. The use of curly braces is motivated by the similar notation used in the definition of functions by cases, for instance FORMULA 4.10.

which is the maximum distance points of γ_0 and γ_1 may be from each other at the same time. Since γ_0 and γ_1 are not equal for all times, $\mu > 0$. This will be used in the estimates that follow.

Also, we define the quantity

$$N(\gamma_x, \gamma_{x'}) := N(x, x') := \min\{|i| \mid x_i \neq x'_i\}.$$

between two orbits γ_x and $\gamma_{x'}$. The quantity $N(x, x')$ is related to the distance function for the symbolic space Σ_2, and $N(\gamma_x, \gamma_{x'})$ serves a similar purpose in helping to estimate orbital distances, as the next lemma shows.

Lemma 4.20 ([EMV24]). *Let $m \in \mathbb{N}$ and $x, x' \in \Sigma_2$. If $N(\gamma_x, \gamma_{x'}) \leq m$, then*

$$\tilde{d}_{\sup}(\gamma_x, \gamma_{x'}) \geq \mu 2^{-(m+1)\alpha}.$$

Proof. Denote $N := N(\gamma_x, \gamma_{x'})$. By definition of N, the orbits γ_x and $\gamma_{x'}$ are different on the interval $[N\alpha, (N+1)\alpha[$ or on the interval $[-N\alpha, -(N-1)\alpha[$. Denote $u_i := \sup_{i \leq t < i+1} d(\gamma_x(t), \gamma_{x'}(t))$. Then $u_j = \mu$ for some $j \in \mathbb{N}$ such that $\lfloor N\alpha \rfloor \leq j < \lfloor (N+1)\alpha \rfloor$, so

$$\tilde{d}_{\sup}(\gamma_x, \gamma_{x'}) = \sum_{i \in \mathbb{Z}} 2^{-|i|} u_i \geq \sum_{i=\lfloor N\alpha \rfloor}^{\lfloor (N+1)\alpha \rfloor} 2^{-i} u_i \geq 2^{-j}\mu \geq 2^{-(m+1)\alpha}\mu. \qquad \blacksquare$$

Finally, we calculate the entropy of the subsystem $\tilde{\Phi}^\alpha \colon \tilde{\Gamma}_p \longrightarrow \tilde{\Gamma}_p$.

Proposition 4.21 ([EMV24]). $\mathrm{h}(\tilde{\Phi}^\alpha|_{\tilde{\Gamma}_p}) \geq \log(2)$.

Proof. To simplify notation, we will just write $\tilde{\Phi}^\alpha$ instead of $\tilde{\Phi}^\alpha|_{\tilde{\Gamma}_p}$ inside this proof. We will first show that $\bar{\mathrm{H}}^n(\tilde{\Phi}^\alpha, \mu 2^{-(m+1)\alpha}) \geq 2^{2m+n}$. Let $E \subseteq \tilde{\Gamma}_p$ be a set of orbits γ_y such that $\#E \leq 2^{2m+n} - 1$. Consider the set of $(2m+n)$-tuples

$$F := \{(y_{-m}, \ldots, y_{m+n-1}) \in \{0,1\}^{2m+n} \mid \gamma_y \in E\}.$$

Since there are at most $2^{2m+n} - 1$ elements in E, then $\#F \leq 2^{2m+n} - 1$; and since $\#(\{0,1\}^{2m+n}) = 2^{2m+n}$, there is at least one $(2m+n)$-tuple $(s_{-m}, \ldots, s_{m+n-1}) \in \{0,1\}^{2m+n} \smallsetminus F$. Take a sequence $x \in \Sigma_2$ such that

$$(x_{-m}, \ldots, x_{m+n-1}) = (s_{-m}, \ldots, s_{m+n-1}).$$

By the choice of the s_i, it follows that, for every $\gamma_y \in E$,

$$(y_{-m}, \ldots, y_{m+n-1}) \neq (x_{-m}, \ldots, x_{m+n-1}).$$

So there exist $l_y \in \{-m, \dots, m\}$ and $k_y \in \{0, \dots, n-1\}$ such that $y_{l_y+k_y} \nparallel x_{l_y+k_y}$, which is the same as $\sigma^{k_y}(y)_{l_y} \nparallel \sigma^{k_y}(x)_{l_y}$. Since $|l_y| \leq m$, this implies that $N(\tilde{\Phi}^{\alpha k_y}(\gamma_y), \tilde{\Phi}^{\alpha k_y}(\gamma_x)) = N(\sigma^{k_y}(y), \sigma^{k_y}(x)) \leq m$, so it follows from LEMMA 4.20 that the dynamical distance $[\tilde{d}_{\sup}]^n_{\tilde{\Phi}^\alpha}$ satisfies

$$[\tilde{d}_{\sup}]^n_{\tilde{\Phi}^\alpha}(\gamma_y, \gamma_x) \geq \tilde{d}_{\sup}(\tilde{\Phi}^{\alpha k_y}(\gamma_y), \tilde{\Phi}^{\alpha k_y}(\gamma_x)) \geq \mu 2^{-(m+1)\alpha};$$

that is, $\gamma_x \notin \mathrm{B}^n_{\tilde{\Phi}^\alpha}(\gamma_y; \mu 2^{-(m+1)\alpha})$. Since this is valid for every $\gamma_y \in E$, we conclude that $\gamma_x \notin \bigcup_{\gamma_y \in E} \mathrm{B}^n_{\tilde{\Phi}^\alpha}(\gamma_y; \mu 2^{-(m+1)\alpha})$ and, because E is an arbitrary set with $\#E \leq 2^{2m+n} - 1$, it follows that $\bar{\mathrm{H}}^n(\tilde{\Phi}^\alpha, \mu 2^{-(m+1)\alpha}) > 2^{2m+n} - 1$.

Finally, to estimate the entropy we note that, since $\mu 2^{-(m+1)\alpha} \to 0$ as $m \to \infty$, and

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \log 2^{2m+n} = \lim_{m\to\infty} \overline{\lim_{n\to\infty}} \frac{2m+n}{n} \log(2) = \log(2),$$

so it follows from $\bar{\mathrm{H}}^n(\tilde{\Phi}^\alpha, \mu 2^{-(m+1)\alpha}) \geq 2^{2m+n}$ that $\mathrm{h}(\tilde{\Phi}^\alpha) \geq \log(2)$. ■

We can now prove the main entropy theorem.

Theorem 4.22 ([EMV24]). *Let M be a 2-dimensional Riemannian manifold, F is a transitive piecewise smooth vector field on M, and Σ^{t} is finite. If the sliding or escaping regions are non-empty, then F has positive topological entropy.*

Proof. From LEMMA 4.19 we know that our system F has two periodic orbit segments γ_0 and γ_1, so the construction of the $\tilde{\Gamma}_p$ can be done. From PROPOSITION 4.21, we have that $\mathrm{h}(\tilde{\Phi}^\alpha|_{\tilde{\Gamma}_p}) \geq \log(2)$. Since $\tilde{\Phi}^\alpha : \tilde{\Gamma}_p \to \tilde{\Gamma}_p$ is a subsystem of $\tilde{\Phi}^\alpha : \tilde{M} \to \tilde{M}$, this means that $\mathrm{h}(\tilde{\Phi}^\alpha) \geq \mathrm{h}(\tilde{\Phi}^\alpha|_{\tilde{\Gamma}_p})$. From the fact that $\mathrm{h}(\tilde{\Phi}^\alpha) = \alpha \mathrm{h}(\tilde{\Phi}^1)$ (the exponent property of entropy, PROPOSITION 2.9), it follows that

$$\mathrm{h}(F) = \mathrm{h}(\tilde{\Phi}^1) = \alpha^{-1}\mathrm{h}(\tilde{\Phi}^\alpha) \geq \alpha^{-1} \log(2) > 0.$$ ■

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