UNICAMP

UNIVERSIDADE ESTADUAL DE
CAMPINAS

Instituto de Matemática, Estatística e
Computação Científica

JONATAN ANDRES GOMEZ PARADA

Polynomial Identities and Cocharacters of Matrix Algebras, and the Embedding Problem

Identidades Polinomiais e Cocaracteres de Álgebras de Matrizes, e o Problema do Mergulho

Campinas

2024

Jonatan Andres Gomez Parada

Polynomial Identities and Cocharacters of Matrix Algebras, and the Embedding Problem

Identidades Polinomiais e Cocaracteres de Álgebras de Matrizes, e o Problema do Mergulho

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Supervisor: Plamen Emilov Kochloukov

Este trabalho corresponde à versão final da Tese defendida pelo aluno Jonatan Andres Gomez Parada e orientada pelo Prof. Dr. Plamen Emilov Kochloukov.

Campinas

2024

Ficha catalográfica
Universidade Estadual de Campinas (UNICAMP)
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

G586p Gomez Parada, Jonatan Andres, 1996-
Polynomial identities and cocharacters of matrix algebras, and the embedding problem / Jonatan Andres Gomez Parada. – Campinas, SP : [s.n.], 2024.

Orientador: Plamen Emilov Kochloukov.
Tese (doutorado) – Universidade Estadual de Campinas (UNICAMP), Instituto de Matemática, Estatística e Computação Científica.

1. Identidade polinomial. 2. PI-álgebras. 3. Álgebra de matrizes. 4. Álgebra de Grassmann. 5. Co-caracter. I. Kochloukov, Plamen Emilov, 1958-. II. Universidade Estadual de Campinas (UNICAMP). Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações Complementares

Título em outro idioma: Identidades polinomiais e cocaracteres de álgebras de matrizes, e o problema do mergulho
Palavras-chave em inglês:
Polynomial identity
PI-algebras
Matrix algebra
Grassmann algebra
Cocharacter
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Plamen Emilov Kochloukov [Orientador]
Diogo Diniz Pereira da Silva e Silva
Lucia Satie Ikemoto Murukami
Lucio Centrone
Viviane Ribeiro Tomaz da Silva
Data de defesa: 19-08-2024
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0000-0002-0405-4378
- Currículo Lattes do autor: http://lattes.cnpq.br/6340632206752512

Tese de Doutorado defendida em 19 de agosto de 2024 e aprovada

pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). PLAMEN EMILOV KOCHLOUKOV

Prof(a). Dr(a). LUCIA SATIE IKEMOTO MURAKAMI

Prof(a). Dr(a). VIVIANE RIBEIRO TOMAZ DA SILVA

Prof(a). Dr(a). DIOGO DINIZ PEREIRA DA SILVA E SILVA

Prof(a). Dr(a). LUCIO CENTRONE

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

To my parents: Maria and Ramon

Acknowledgements

First and foremost, I want to thank God for the opportunities I have had along my journey.

To my parents, Maria and Ramon, and my brother, David, for their faith and support in my decisions.

I am grateful to the Institute of Mathematics, Statistics, and Scientific Computing for the opportunity to pursue my doctorate, as well as to each of the professors who contributed to my education.

I extend my thanks to my advisor, Plamen Kochloukov, for his guidance, teachings, and companionship during my doctoral studies.

To my friends, for the moments we shared, the joy they brought, and the support they offered during difficult times; thanks to them, this journey has been easier.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001

"In the end,
it's only a passing thing,
this shadow.
Even darkness must pass.
A new day will come.
And when the sun shines,
it'll shine out the clearer."
Samwise Gamgee

Resumo

Considere $\mathcal{A}$ a subálgebra de $UT_3(K)$ dada por

$$\mathcal{A} = K(e_{1,1} + e_{3,3}) \oplus Ke_{2,2} \oplus Ke_{1,2} \oplus Ke_{2,3} \oplus Ke_{1,3},$$

onde $e_{i,j}$ denotam as matrizes unitárias. Primeiramente, examinamos as graduações na álgebra $\mathcal{A}$ definidas por um grupo abeliano. Além disso, determinamos uma base para as identidades $\mathbb{Z}_2$-graduadas de $\mathcal{A}$, para as identidades com involução e para as identidades $\mathbb{Z}_2$-graduadas com involução graduada. Também exploramos seus cocaracteres.

Em seguida, consideramos as álgebras $\mathbb{Z}_2$-graduadas $M_{1,1}(K)$, $UT_{1,1}(K)$ e $UT_3(K)_{(0,1,0)}$ com uma superinvolução, juntamente com o produto tensorial graduado com a álgebra de Grassmann E, naturalmente dotada com uma $\mathbb{Z}_2$-graduação e também com uma superinvolução. Consideramos tais produtos tensoriais dotados com uma involução graduada e descrevemos as $*$-identidades polinomiais graduadas junto com os cocaracteres correspondentes.

Finalmente, apresentamos alguns resultados sobre o problema do mergulho para álgebras simples com involução, usando identidades polinomiais standard de grau mínimo e considerando-os como $*$-polinômios.

Palavras-chave: Identidade polinomial, PI-álgebra, álgebra de matrizes, álgebra de Grassmann, cocaracteres, involução.

Abstract

Consider $\mathcal{A}$ the subalgebra of $UT_3(K)$ given by

$$\mathcal{A} = K(e_{1,1} + e_{3,3}) \oplus Ke_{2,2} \oplus Ke_{1,2} \oplus Ke_{2,3} \oplus Ke_{1,3},$$

where $e_{i,j}$'s denote the matrix units. First, we examine the gradings in the algebra $\mathcal{A}$ defined by an abelian group. Then, we determine a basis for the $\mathbb{Z}_2$-graded identities of $\mathcal{A}$, for the identities with involution, and for the $\mathbb{Z}_2$-graded identities with graded involution. We also explore their cocharacters.

We then consider the $\mathbb{Z}_2$-graded algebras $M_{1,1}(K)$, $UT_{1,1}(K)$, and $UT_3(K)_{(0,1,0)}$ with a superinvolution, along with their corresponding graded tensor products with the Grassmann algebra E, naturally endowed with a $\mathbb{Z}_2$-grading and also with a superinvolution. We examine these algebras as endowed with a graded involution and describe the graded $*$-polynomial identities and the corresponding cocharacters.

Finally, we present some results concerning the embedding problem for simple algebras with involution, using standard polynomials identities of minimal degree as a tool, considering them as $*$-polynomials.

Keywords: Polynomial identity, PI-algebra, matrix algebra, Grassmann algebra, cocharacters, involution.

Contents

Introduction

In this thesis we study polynomial identities of certain associative algebras. Let K be a field and let $K\langle X\rangle$ be the free associative algebra freely generated by the countable set of indeterminates X. One can view $K\langle X\rangle$ as the set of all polynomials in the non-commuting variables from the set X. Given an associative algebra A, a polynomial f in $K\langle X\rangle$ is called a *polynomial identity* of A if f evaluates to zero when its variables are substituted with arbitrary elements of A. An algebra satisfying a non-zero polynomial identity is called a *PI-algebra*. The set of all identities for A is denoted by $T(A)$. Clearly $T(A)$ is an ideal in $K\langle X\rangle$. Moreover it is closed under endomorphisms of the free algebra $K\langle X\rangle$. It can be seen that every such ideal coincides with $T(A)$ for some A. Among the algebras that satisfy non-zero polynomial identities, those that have been of greatest interest in the development of the theory of polynomial identities and that will play a significant role in this thesis include the Grassmann algebra, the full matrix algebras $M_n(K)$, and $UT_n(K)$, the algebra of the upper triangular matrices of order n. In general, finite-dimensional algebras and commutative algebras are classical examples of PI-algebras.

One initial problem to be considered in the theory of algebras with polynomial identities, which we will address in this thesis, is determining the set of all identities satisfied by a particular algebra, as well as a generating set for them. In the case of a PI-algebra A over a field of characteristic zero, it is known that the polynomial identities of A follow from the multilinear polynomial identities. Therefore, we can restrict our study to multilinear polynomials.

Given that the space P_n of multilinear polynomials in the variables x_1, x_2, $\ldots$, x_n has the structure of a left S_n-module and $P_n \cap T(A)$ is invariant under this action of S_n, it follows that the vector space $P_n \cap T(A)$ is a submodule of P_n. Studying the multilinear identities of A might be a difficult problem since a well known theorem due to A. Regev, proved in 1972, gives us that $P_n \cap T(A)$ tends to become very large when $n \to \infty$. Hence one is led to study $P_n(A) = P_n/P_n \cap T(A)$, this quotient inherits an induced structure of a left S_n-module. The S_n-character of $P_n(A)$ is referred to as the *nth cocharacter* of A. We emphasize that the study of characters is of interest since, when considering finite groups and algebraically closed fields, finite dimensional representations are determined up to isomorphism by their characters. Decomposing the n-th cocharacter of A into irreducibles, we have that

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated with the partition λ with multiplicity $m \geqslant 0$.

We are interested in considering the infinite dimensional Grassmann algebra $E = \langle 1, e_1, e_2, \ldots \mid e_i e_j = -e_j e_i \rangle$ as a superalgebra, that is, a $\mathbb{Z}_2$-graded algebra. The grading is given by the subspaces E_0 and E_1, of elements of even or odd length respectively. Given a superalgebra $A = A_0 \oplus A_1$, we consider the Grassmann envelope of A defined by $G(A) = (A_0 \otimes E_0) \oplus (A_1 \otimes E_1)$. In general, given A and B two $\mathbb{Z}_2$-graded algebras we can consider the graded tensor product $A \hat{\otimes} B = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$.

Matrix algebras with entries in Grassmann algebras have been the subject of various studies. Of particular interest is determining a basis of their polynomial identities as well as their corresponding cocharacters. Let E be the infinite dimensional Grassmann algebra over K and $M_n(K)$ the matrix algebra of order $n \times n$ over K. According to the theory developed in the 80-ies by A. Kemer, it can be deduced that if $\mathrm{char} K = 0$ the only non-trivial T-prime T-ideals in $K\langle X \rangle$ are $T(M_n(K))$, $T(M_n(E))$, for $n \geqslant 1$; and $T(M_{a,b}(E))$, where $M_{a,b}(E) = M_{a,b}(K) \hat{\otimes} E$ and $a \geqslant b \geqslant 1$. It follows from Kemer's Tensor Product Theorem that if $\mathrm{char} K = 0$, then $M_{1,1}(E)$ and $E \otimes E$ share the same polynomial identities. The theorem further proves that the tensor product of two T-prime algebras is PI-equivalent to a T-prime algebra as well. However, this does not hold in the case of positive characteristic, as shown in [2].

Considering the algebras of upper triangular matrices of order 3, we refer to the algebra $\mathcal{A} = K(e_{1,1} + e_{3,3}) \oplus K e_{2,2} \oplus K e_{1,2} \oplus K e_{2,3} \oplus K e_{1,3}$. Here and in what follows, $e_{i,j}$ is the matrix with an entry 1 at position (i, j) and 0 elsewhere. A basis for the identities of this algebra was described by Gordienko in [23]. It turns out that the algebra $\mathcal{A}$ is PI-equivalent to the generic algebra of $M_{1,1}(E)$ in two generators, which in turn was studied by Koshlukov and de Mello in [25].

In [11], Di Vincenzo and Koshlukov studied the graded identities of the algebra $M_{1,1}(E)$ as an algebra with graded involution, while da Silva in [32] considered the $\mathbb{Z}_2$-graded identities of $UT_2(K) \hat{\otimes} E$. A generalization to $UT_{k,l}(K) \hat{\otimes} E$ was presented by Di Vincenzo and da Silva in [14]. Centrone and da Silva studied in [7] the case of $\mathbb{Z}_2$-graded identities of $UT_2(E)$ in characteristic different from 2. Also, Centrone in [6] considered ordinary and $\mathbb{Z}_2$-graded cocharacters of $UT_2(E)$.

An *involution* (of the first kind) on an algebra A is an antiautomorphism of order two, that is, a linear map $* : A \to A$ satisfying $(ab)^* = b^* a^*$ and $(a^*)^* = a$, for all $a, b \in A$. A G-graded algebra $A = \bigoplus_{g \in G} A_g$ with involution $*$ is called a *graded involution algebra* if $(A_g)^* = A_g$ for all $g \in G$. In this case, we say that $*$ is a *graded involution* on A. If A is a graded involution algebra, we say it is a $(G, *)$-algebra.

Given a superalgebra $A = A_0 \oplus A_1$, a *superinvolution* $*$ on A is a graded linear map of order two such that $(ab)^* = (-1)^{|a||b|} b^* a^*$, for any homogeneous elements a, $b \in A_0 \cup A_1$. Here $|x|$ denotes the homogeneous degree of $x \in A_0 \cup A_1$. The $\mathbb{Z}_2$-graded involutions and the superinvolutions of the upper triangular matrix algebra $UT_n(K)$ were described by Ioppolo and Martino in [24], and the corresponding result about the superinvolutions of $M_{1,1}(K)$ was given by Gomez and Shestakov in [22]. Also, note that on E we can define the superinvolutions i_E and $-i_E$, induced by the identity map on the generators e_i of E. Note that if $\circledast$ and $\diamond$ are a pair of superinvolutions defined on the superalgebras A and B respectively, then the map $*$ defined on $A\hat{\otimes}B = (A_0 \otimes B_0)\bigoplus(A_1 \otimes B_1)$ by putting $(a \otimes b)^* = a^{\circledast} \otimes b^{\diamond}$ is an involution on $A\hat{\otimes}B$.

Let us consider $Y = \{y_{i,g} : i \in \mathbb{N}, g \in G\}$, $Z = \{z_{i,g} : i \in \mathbb{N}, g \in G\}$ two countable disjoint sets of indeterminates. We denote by $\deg_G y_{i,g} = \deg_G z_{i,g} = g$ the G-degree of the variables $Y \cup Z$ with respect to the G-grading. Then $Y_g = \{y_{i,g} : i \in \mathbb{N}\}$, $Z_g = \{z_{i,g} : i \in \mathbb{N}\}$ are homogeneous variables of G-degree $g \in G$.

We can define a $*$-action on the monomials over $Y \cup Z$ by the equalities

$$(x_{i_1} \cdots x_{i_n})^* = x_{i_n}^* \cdots x_{i_n}^*, \text{ where } y_{i,g}^* = y_{i,g},\ z_{i,g}^* = -z_{i,g},\ x_j \in Y \cup Z \tag{1}$$

where the linear extension of this action is an involution on the free associative algebra $K\langle Y, Z\rangle$ generated by the set $Y \cup Z$. The algebra $\mathcal{F} = K\langle Y, Z\rangle$ is G-graded with the grading $\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g$ defined by

$$\mathcal{F}_g = \text{Span}_K\{x_{i_1}x_{i_2} \ldots x_{i_n} : \deg_G x_{i_1} \ldots \deg_G x_{i_1} = g,\ x_j \in Y \cup Z\}.$$

It is clear that the involution (1) is graded. The algebra $\mathcal{F}$ is the free associative graded algebra with involution and its elements are called *graded $*$-polynomials*. We are interested in the group $G = \mathbb{Z}_2$, that is, we consider $\mathbb{Z}_2$-graded identities with involution.

Another problem of interest in PI-theory is the classification of algebras based on their polynomial identities. In particular, we can consider the Isomorphism Problem. Consider A and B to be two associative K-algebras. It is easy to see that in the case of A and B being isomorphic, they satisfy the same set of polynomial identities. Thus, the natural question to consider is: If A and B satisfy the same identities, will they be isomorphic? The answer is No, and the known (easy) counterexamples lead us to focus on the study of central simple algebras over algebraically closed fields. Here we recall that if A is PI then A and $A \oplus A$ satisfy the same identities. Another example can be given by the 2×2 matrices $M_2(\mathbb{R})$ and the real quaternion algebra, they satisfy the same identities but are not isomorphic. That is why one considers central simple algebras over an algebraically closed field. A more general problem than the Isomorphism Problem is the Embedding Problem. Once again, we consider two K-algebras, A and B, and the question to be examined is: If the set of polynomial identities of A is a subset of the identities of B, can we view B as a subalgebra of A? Or of some scalar extension of A?

This thesis is organized as follows.

Chapter 1 introduces the notions necessary for the development of this thesis. Our main focus will be on multilinear polynomials and representations of the symmetric group S_n, as well as their applications to the study of polynomial identities. As a special topic, we will consider the Grassmann algebra, its identities, and cocharacters.

In Chapter 2, we consider the algebra of upper triangular matrices $\mathcal{A} = K(e_{1,1} + e_{3,3}) \oplus Ke_{2,2} \oplus Ke_{1,2} \oplus Ke_{2,3} \oplus Ke_{1,3}$. Initially, we consider its possible gradings and show that these are isomorphic to elementary gradings and compute their graded identities as well as their cocharacters. We also consider identities with involution and graded identities with involution, where the grading is given by the group $\mathbb{Z}_2$. In each case, we also compute the respective cocharacters.

In Chapter 3, we consider the $\mathbb{Z}_2$-graded algebras $M_{1,1}(K)$, $UT_{1,1}(K)$, and $UT_3(K)_{(0,1,0)}$ with a superinvolution, along with their corresponding super tensor products with the Grassmann algebra E, naturally endowed with a $\mathbb{Z}_2$-grading, and also with a superinvolution induced by the identity function i_E and by $-i_E$. We regard the resulting algebras as endowed with a graded involution and describe the graded $*$-polynomial identities and the corresponding cocharacters.

In Chapter 4, we consider the embedding problem for algebras with polynomial identities. We introduce the particular case of the isomorphism problem, as well as its historical context. We present some results concerning the embedding problem for simple algebras with involution, using minimal degree standard polynomial identities as a tool, considering them as $*$-polynomials. Among the main results, we have:

> **Theorem:** Let A and B two finite-dimensional central simple algebras with involution over the algebraically closed field K of characteristic 0, A with involution of orthogonal type and A satisfying the identities with involution of the algebra B. Then, there exists an embedding that preserves the involutions of A into B.

> **Theorem:** Let A and B two finite-dimensional central simple algebras with involution over the algebraically closed field K of characteristic 0, B with involution of symplectic type and A satisfying the identities with involution of the algebra B. Then, there exists an embedding that preserves the involutions of A into B.

1 Preliminaries

In this chapter, we will introduce the main notions that will be used throughout this thesis. Our primary interest lies in introducing the concept of polynomial identities in associative algebras, as well as considering identities in algebras with additional structures such as graded algebras and algebras with involution. Another key topic of study will be the action of the symmetric group and the general linear group on the space of multilinear and multihomogeneous polynomials. To conclude the chapter, we will study the Grassmann algebra, its identities, and cocharacters. We will assume that we are working with unitary algebras over fields of characteristic zero. As main references, we will use [16, 17, 18, 30].

1.1 PI-algebras

Let K be a field and $X = \{x_1, x_2, \ldots\}$ a countable set of non-commutative variables. We denote by $K\langle X\rangle$ the free algebra freely generated by the set X.

Definition 1.1. (*i*) *Let $f = f(x_1, \ldots, x_n) \in K\langle X\rangle$ and let A be an associative algebra. We say that $f = 0$ (or f) is a polynomial identity for A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$.*

(*ii*) *If the associative algebra A satisfies a non-trivial polynomial identity $f = 0$ (i.e. f is a non-zero element of $K\langle X\rangle$), we call A a PI-algebra.*

Example 1.1. (*i*) *The algebra A is commutative if and only if it satisfies the polynomial identity $[x_1, x_2] = x_1x_2 - x_2x_1 = 0$.*

(*ii*) *Let A be a finite dimensional associative algebra and let $\dim A < n$. Then A satisfies the standard identity of degree n*

$$St_n(x_1, \ldots, x_n) = \sum_{\omega\in S_n} (\operatorname{sign}\omega)x_{\omega(1)} \cdots x_{\omega(n)} = 0,$$

where S_n is the symmetric group of degree n.

Here we present the famous Amitsur-Levitzki theorem regarding matrix algebra and standard polynomials.

Theorem 1.1. *The $n \times n$ matrix algebra $M_n(K)$ satisfies the standard identity of degree $2n$.*

As a consequence of the previous theorem, up to a multiplicative constant, the standard identity is the only multilinear polynomial identity for $M_n(K)$ of degree $2n$. Additionally, it is known that $M_n(K)$ does not satisfy identities of lower degree.

Definition 1.2. *An ideal J of $K\langle X\rangle$ is called a T-ideal if $\psi(J) \subseteq J$ for all endomorphisms ψ of $K\langle X\rangle$.*

If A is any algebra, we denote by $T(A)$ (or $\mathrm{Id}(A)$) the T-ideal of the polynomial identities of A. Also, we will denote by $F(A)$ the quotient $K\langle X\rangle/T(A)$. It is the relatively free algebra for A (and its identities).

Definition 1.3. *The polynomial identity $g(x_1, \ldots, x_m) = 0$ is called a consequence of the polynomial identities $f_i(x_1, \ldots, x_m) = 0$, $i \in I$, if any algebra satisfying the identities $f_i(x_1, \ldots, x_m) = 0$ satisfies also $g(x_1, \ldots, x_m) = 0$. We denote by*

$$(f_i(x_1, \ldots, x_m) \mid i \in I)^T$$

the smallest T-ideal U containing all $f_i(x_1, \ldots, x_m)$, $i \in I$. This T-ideal coincides with the set of all consequences of the identities $f_i = 0$, $i \in I$, and its elements have the form

$$\sum u_{iw} f_i(w_1, \ldots, w_{m_i}) v_{iw}, \qquad w_1, \ldots, w_{m_i}, u_{iw}, v_{iw} \in K\langle X\rangle.$$

The generating set $\{f_i(x_1, \ldots, x_m) \mid i \in I\}$ is called a basis of the T-ideal U, even if it is not a minimal generating set. (Any generating set of the T-ideal $T(A)$ is called a basis of the polynomial identities of the algebra A.)

Definition 1.4. *Two sets of polynomial identities are equivalent if they generate the same T-ideal.*

1.2 Multilinear Polynomials

In the case of algebras over fields of characteristic zero, multilinear polynomials play an important role since T-ideals are generated by such polynomials. For this reason, one can study PI-algebras in characteristic 0 through their multilinear identities.

Definition 1.5. *A polynomial $f(x_1, \ldots, x_n)$ in the free associative algebra $K\langle X\rangle$ is called multilinear of degree n if the degree of f with respect to each variable x_i, denoted $\deg_{x_i} f$, is equal to 1 (i.e., f is linear in the variable x_i) for $i = 1, \ldots, n$. We denote by P_n the vector space of all polynomials in $K\langle X\rangle$ which are multilinear of degree n.*

Proposition 1.1. *Let*

$$f(x_1, \ldots, x_n) = \sum_{i=0}^{n} f_i \in K\langle X\rangle,$$

where f_i is the homogeneous component of f of degree i in x_1.

(*i*) *If the base field K contains more than n elements (e.g. K is infinite), then the polynomial identities $f_i = 0$, $i = 0, 1, \ldots, n$, follow from $f = 0$.*

(*ii*) *If the base field is of characteristic 0 (or if char* $K > \deg f$*), then* $f = 0$ *is equivalent to a set of multilinear polynomial identities.*

Remark 1.1. *Let* P_n *be the set of all multilinear polynomials of degree* n *in the free associative algebra* $K\langle X\rangle$*. The following action of the symmetric group* S_n *makes* P_n *a left* S_n*-module, isomorphic to the group algebra* KS_n *considered as a left* S_n*-module:*

$$\sigma\left(\sum \alpha_i x_{i_1}\dots x_{i_n}\right) = \sum \alpha_i x_{\sigma(i_1)}\dots x_{\sigma(i_n)},$$

$\sigma \in S_n$, $\alpha_i \in K$, $x_{i_1}\dots x_{i_n} \in P_n$*. Thus we can consider* P_n *as isomorphic to the regular representation of* S_n*.*

1.3 Graded Algebras

Definition 1.6. *Let* G *be an arbitrary group. We say that an associative algebra* A *over a field* K *is a* G*-graded algebra (or equipped with a* G*-grading) if, for each* $g \in G$*, there exists a subspace* $A_g \subseteq A$ *such that* A *can be written as:*

$$A = \oplus_{g\in G} A_g \text{ and } A_g A_h \subseteq A_{gh} \text{ for all } g, h \in G.$$

The subspaces A_g *are called the homogeneous components of* A*, and the elements of each* A_g *are called the homogeneous elements of* A *with homogeneous degree* g*. A subspace* $V \subseteq A$ *is called a homogeneous (or graded) subspace if* $V = \bigoplus_{g\in G}(V \cap A_g)$*.*

In the case of matrix algebras $M_n(K)$, we can consider certain gradings, known as elementary gradings, defined as follows:

Definition 1.7. *Let* $A = M_n(K)$ *be the algebra of* $n \times n$ *matrices over* K *and let the* $e_{i,j}$*'s be the usual matrix units. Given an* n*-tuple* $\hat{g} = (g_1, \dots, g_n)$ *of an arbitrary group* G *we set* $\deg e_{i,j} = g_i^{-1}g_j$ *and let* $A_g = \mathrm{Span}\{e_{i,j} : g_i^{-1}g_j = g\}$*. Then* $A = \bigoplus_{g\in G} A_g$ *is a* G*-grading of* A *called the elementary* G*-grading defined by the* n*-tuple* $\hat{g}$*.*

Example 1.2. *Consider the matrix algebra* $M_n(K)$ *and integers* k *and* h *such that* $k + h = n$*.* $M_{k,h}(K)$ *denotes* $M_n(K)$ *with* $\mathbb{Z}_2$*-grading given by*

$$(M_{k,h}(K))_0 = \left\{\begin{pmatrix} S & 0 \\ 0 & T\end{pmatrix} : S \in M_k(K),\, T \in M_h(K)\right\},$$

$$(M_{k,h}(K))_1 = \left\{\begin{pmatrix} 0 & Y \\ Z & 0\end{pmatrix} : Y \in M_{k\times h}(K),\, Z \in M_{h\times k}(K)\right\}.$$

Now, for $UT_n(K)$ be the algebra of $n \times n$ upper triangular matrices over K we have the following classification of gradings given by Valenti and Zaicev (see [36]).

Theorem 1.2. *Let G be an arbitrary group and K a field. Suppose that the algebra $UT_n(K) = B = \bigoplus_{g\in G} B_g$ of $n \times n$ upper triangular matrices over the field K is G-graded. Then B, as a G-graded algebra, is isomorphic to $UT_n(K)$ with an elementary G-grading.*

Let G be a group, and consider the set of indeterminates $X = \bigcup_{g\in G} X_g$, where $X_g = \{x_1^{(g)}, x_2^{(g)}, \ldots\}$ is a countable infinite set for each g. We say that the indeterminates in X_g have a homogeneous degree g, and the homogeneous degree of a monomial $x_{i_1}^{(g_{j_1})} \cdots x_{i_m}^{(g_{j_m})} \in K\langle X\rangle$ is given by $g_{j_1} \cdots g_{j_m}$. We will write $K\langle X\rangle^{gr}$ to denote the graded algebra $K\langle X\rangle$ with $X = \bigcup_{g\in G} X_g$.

Definition 1.8. *Let $f(x_{i_1}^{(g_1)}, x_{i_2}^{(g_2)}, \ldots, x_{i_r}^{(g_r)}) \in K\langle X\rangle^{gr}$ be a polynomial. If $A = \bigoplus_{g\in G} A_g$ is a G-graded algebra then f is a G-graded polynomial identity (or simply a G-graded identity) for A if $f(a_{i_1}^{(g_1)}, a_{i_2}^{(g_2)}, \ldots, a_{i_r}^{(g_r)}) = 0$ in A for every homogeneous substitution $a^{(g_t)} \in A_{g_t}$. We denote by $\mathrm{Id}_G(A)$ or $T_G(A)$ the ideal of all graded identities of A in $K\langle X\rangle^{gr}$.*

The ideal $\mathrm{Id}_G(A)$ is closed under all G-graded endomorphisms of $K\langle X\rangle^{gr}$; such ideals are called *G-graded T-ideals*.

A particular case is when the group G is given by $\mathbb{Z}_2$, in this case, we say that an associative algebra is a *superalgebra* if A is a $\mathbb{Z}_2$-graded algebra. While the two terms are synonymous for associative algebras, we draw the readers' attention that, say, in the Lie case, a Lie superalgebra seldom is a Lie algebra, and the same holds for Jordan, alternative, etc., algebras.

Definition 1.9. *Let A and B be two $\mathbb{Z}_2$-graded K-algebras. $A\hat{\otimes}B$ denotes the graded tensor product, i.e,*

$$A\hat{\otimes}B = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1).$$

1.4 Involutions and superinvolutions

Involutions are important in the structure theory of both associative and non-associative algebras. In this section, we introduce the notions of involution and superinvolution, as well as identities with involution. As particular examples of algebras with involution, we consider matrix algebras $M_n(K)$ and upper triangular matrix algebras $UT_n(K)$.

1.4.1 Involutions

Definition 1.10. *Given algebras A_1 and A_2, we say that a linear map $\phi : A_1 \to A_2$, is an anti-homomorphism if $\phi(a_1 a_2) = \phi(a_2)\phi(a_1)$ for all a_1, a_2 in A_1. Moreover, if ϕ is an isomorphism of vector spaces, we call ϕ an anti-isomorphism, and if $A_1 = A_2$, we call ϕ an anti-automorphism of A_1.*

A classical example of an anti-isomorphism is given by the map $a \mapsto a$ from A to A^{op}, the opposite algebra of A.

Definition 1.11. *An involution is an anti-automorphism ϕ such that $\phi^2 = 1$, i.e., $\phi^2(a) = a$ for all $a \in A$.*

We observe that such involutions are known as involutions of the first kind (when the map ϕ is a K-linear transformation). There exist involutions of the second kind which we will not consider here, hence we only give an idea what they are. Take the $n \times n$ matrix algebra over the complex numbers $\mathbb{C}$, then the usual transpose is an example of an involution of the first kind (it is a linear transformation). But it is not that important when considering $M_n(\mathbb{C})$. The map $A \mapsto \overline{A}^t$ that sends every matrix to its transpose and conjugate matrix is more relevant. Formally it does not fall into our definition since it is not $\mathbb{C}$-linear, but it is linear over $\mathbb{R}$. Similar involutions are of the second type.

An important example of algebra with involution is the matrix algebra $M_n(K)$.

Example 1.3. *(i) The transpose involution t, given by the classical transpose of a matrix.*

(ii) The symplectic involution s, defined by $x^s = ax^t a$ for all $x \in M_{2m}(K)$, where $a = \sum_{i=1}^{m}(e_{i,i+m} - e_{i+m,i})$. In other words, partitioning a $2m \times 2m$ matrix A into $m \times m$ blocks A_i, $1 \leqslant i \leqslant 4$, we have

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}^s = \begin{pmatrix} A_4^t & -A_2^t \\ -A_3^t & A_1^t \end{pmatrix},$$

where t is the transpose on $M_m(K)$.

Let us establish some conventions regarding algebras with involution.

We write $*$ to denote a given involution of an algebra; $*$-algebra means *algebra with involution*. Write a^* for the image of a under the involution $*$.

$(A, *)$ will denote the algebra A with involution $*$. We define a $*$*-homomorphism* $\phi : (A_1, *_1) \to (A_2, *_2)$ to be a homomorphism $\phi : A_1 \to A_2$, such that $\phi(a^{*_1}) = \phi(a)^{*_2}$ for all a in A_1; equivalently, we say that ϕ is a $*$*-homomorphism*. Let I be an ideal of A, we say that I is a $*$-*ideal* of A if $I^* \subseteq I$, and we denote that by $I \lhd (A,*)$. If $I \lhd (A,*)$, then $*$ induces an involution on A/I by $(a+I)^* = a^* + I$, and the canonical map $A \to A/I$ is a $*$-homomorphism; conversely, the kernel of every $*$-homomorphism is a $*$-ideal.

Definition 1.12. *An element x of $(A,*)$ is symmetric (resp. anti-symmetric or skew-symmetric) if $x^* = x$ (resp. $x^* = -x$). We shall write A^+ (resp. A^-) to denote the set of symmetric (resp. skew-symmetric) elements of A.*

Example 1.4.

(i) Let A be an algebra. Then $A \oplus A^{op}$ has an involution ex given by $(a_1,a_2)^{ex} = (a_2,a_1)$, which is called the exchange involution.

(ii) If $(A,*)$ is a K-algebra with involution and B is a commutative K-algebra, then $(* \otimes 1)$ is an involution of $A \otimes_K B$.

A proof of the following results concerning algebras with involution can be found in [30].

Proposition 1.2. *Let A be a $*$-algebra. Then $\mathrm{Nil}(A) \lhd (A,*)$ and $\mathrm{Jac}(A) \lhd (A,*)$.*

Definition 1.13. *$(A,*)$ is simple if 0 and A are the only $*$-ideals of $(A,*)$.*

Proposition 1.3. *Suppose $(A,*)$ is simple. Then either A is simple, or A has a simple homomorphic image A_1 such that $(A,*) \cong (A_1 \oplus A_1^{op}, ex)$.*

Definition 1.14. *The center of $(A,*)$, written $Z(A,*)$, is $\{z \in Z(A) \mid z^* = z\}$.*

Proposition 1.4. *$Z(A,*)$ is a a subalgebra of A fixed by $*$. If $(A,*)$ is simple, then $Z(A,*)$ is a field.*

Let $X = \{x_1, x_2, \ldots\}$ be a countable set of non-commutative variables and consider $K\langle X,*\rangle = K\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$, the *free algebra with involution* in X over K. By defining $y_i = x_i + x_i^*$ and $z_i = x_i - x_i^*$ for each $i = 1,2,\ldots$, we consider $K\langle X,*\rangle = K\langle y_1, z_1, y_2, z_2, \ldots\rangle$ as generated by symmetric and skew-symmetric variables. The elements of $K\langle X,*\rangle$ will be called $*$-polynomials. Observe that we can write $x_i = (y_i + z_i)/2$ and $x_i^* = (y_i - z_i)/2$, hence if the base field is of characteristic different from 2, this change of variables (free generators) can be performed.

Definition 1.15. *A $*$-polynomial $f(y_1,\ldots,y_n,z_1,\ldots,z_m) \in K\langle Y \cup Z,*\rangle$ is a $*$-polynomial identity of and algebra with involution $(A,*)$ if*

$$f(u_1,\ldots,u_n,v_1,\ldots,v_m) = 0 \text{ for all } u_i \in A^+ \text{ and } v_j \in A^-.$$

Given an algebra with involution $(A,*)$ we denote by $\mathrm{Id}(A,*)$ or $T(A,*)$ the set of $*$-polynomial identities of A.

Proposition 1.5 ([30], Proposition 2.5.5). *Suppose n is even. Then* $\mathrm{Id}(M_n(K), s) \nsubseteq \mathrm{Id}(M_n(K), t)$ *and* $\mathrm{Id}(M_n(K), t) \nsubseteq \mathrm{Id}(M_n(K), s)$.

In the case of the algebra of upper triangular matrices $UT_n(K)$, the classification of its involutions was presented in [12]. Below, we present the results related to this classification.

Definition 1.16. *For every matrix $A \in UT_n(K)$ define $A^\circ = JA^tJ$ where $A \mapsto A^t$ denotes the usual matrix transpose and J is the following permutation matrix:*

$$J = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}$$

Note that for the matrix units we have $e_{i,j}^\circ = e_{n+1-j,n+1-i}$.

Definition 1.17. *Let $n = 2m$ be an even integer and consider the matrix*

$$D = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \in M_n(K).$$

Define the involution s on $UT_n(K)$ by putting $A^s = DA^\circ D$ for all $A \in UT_n(K)$, s is called the symplectic involution on UT_n.

Proposition 1.6. *Every involution on $UT_n(K)$ is equivalent either to $\circ$ or to s.*

In the same paper, the following results were proved about the $*$-identities of $UT_2(K)$ and $UT_3(K)$.

Proposition 1.7. *The ideal $T(UT_2(K), \circ)$ is generated as a T^*-ideal by the set*

$$[y_1, y_2], \quad [z_1, z_2], \quad [y_1, z_1][y_2, z_2], \quad z_1y_1z_2 - z_2y_1z_1.$$

Proposition 1.8. *The ideal $T(UT_2(K), s)$ is generated as a T^*-ideal by the set*

$$[y_1, y_2], \quad [z_1, y_1], \quad [z_1, z_2][z_3, z_4], \quad z_1z_2z_3 - z_3z_2z_1.$$

Proposition 1.9. *The ideal $T(UT_3(K), \circ)$ is generated as a T^*-ideal by the set*

(*i*) $s_3(z_1, z_2, z_3) = z_1[z_2, z_3] - z_2[z_1, z_3] + z_3[z_1, z_2]$,

(*ii*) $(-1)^{|x_1x_2|}[x_1, x_2][x_3, x_4] - (-1)^{|x_3x_4|}[x_3, x_4][x_1, x_2]$,

(*iii*) $(-1)^{|x_1x_2|}[x_1, x_2][x_3, x_4] - (-1)^{|x_1x_3|}[x_1, x_3][x_2, x_4] + (-1)^{|x_1x_4|}[x_1, x_4][x_2, x_3]$,

(iv) $z_1[x_3, x_4]z_2 + (-1)^{|x_3x_4|}z_2[x_3, x_4]z_1$,

(v) $[x_1, x_2]z_5[x_3, x_4]$,

(vi) $z_1[x_4, x_5]z_2x_3 + (-1)^{|x_3|}x_3z_1[x_4, x_5]z_2$.

Here $|x| = 0$ whenever x is skew-symmetric, and $|x| = 1$ if x is symmetric element.

Another important structure concerning involutions are graded involutions.

Definition 1.18. *An involution $*$ on a G-graded algebra $A = \bigoplus_{g\in G} A_g$ is said to be a graded involution if $A_g^* = A_g$ for all $g \in G$.*

Graded involutions of $M_n(K)$, subject to certain restrictions, were described in [3]. In the context of upper-triangular matrix algebras, graded involutions were described in [37].

1.4.2 Superinvolution

Definition 1.19. *Given a superalgebra $A = A_0 \oplus A_1$, a superinvolution $*$ on A is a graded linear map of order 2 such that*

$$(ab)^* = (-1)^{|a||b|}b^*a^*,$$

for any homogeneous elements $a, b \in A_0 \cup A_1$. Here $|x|$ denotes the homogeneous degree of $x \in A_0 \cup A_1$.

Remark 1.2. *If $\circledast$ and $\diamond$ is a pair of superinvolutions defined on the superalgebras A and B respectively, then the map $*$ defined on $A\hat{\otimes}B = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$ by putting $(a \otimes b)^* = a^{\circledast} \otimes b^{\diamond}$ is an involution on $A\hat{\otimes}B$.*

The superinvolutions on the algebra of upper-triangular matrices were described in [24]. We present some of the main results.

Definition 1.20. *Let $A = A_0 \oplus A_1$ be the upper-triangular matrix superalgebra $UT_n(K)$ endowed with the elementary $\mathbb{Z}_2$-grading given by the n-tuple $(g_1, \ldots, g_n) \in \mathbb{Z}_2^n$. We define $\Phi : A \to A$ such that $\Phi = \Phi_{n-1}$, where $\Phi_0(e_{i,j}) = e_{i,j}$ and for all $k = 1, \ldots, n-1$,*

$$\Phi_k(e_{i,j}) = \begin{cases} \Phi_{k-1}(e_{i,j}) & \text{if } e_{i,j} \notin A_1^{k+1} \\ -\Phi_{k-1}(e_{i,j}) & \text{if } e_{i,j} \in A_1^{k+1} \end{cases},$$

for all $1 \leqslant i \leqslant j \leqslant n$.

Let $\circ$ and s be the reflection and the symplectic involution as in Definition 1.16 and Definition 1.17, respectively.

Definition 1.21. *The superinvolution* $\bar{\circ} : UT_n(K) \to UT_n(K)$*, defined by* $\bar{\circ} = \circ\Phi$*, is called the super-reflection superinvolution.*

Definition 1.22. *The superinvolution* $\bar{s} : UT_n(K) \to UT_n(K)$*, defined by* $\bar{s} = s\Phi$*, is called the super-symplectic superinvolution.*

Theorem 1.3. *Every superinvolution on* $UT_n(K)$ *is equivalent either to* $\bar{\circ}$ *or to* $\bar{s}$*. The superinvolution* $\bar{s}$ *can occur only when* n *is even.*

Example 1.5. *In the case of* 2×2 *upper-triangular matrices we have that the superinvolutions coincide with the involutions and are given by*

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\circ} = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix} \text{ and } \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{s} = \begin{pmatrix} b & -c \\ 0 & a \end{pmatrix}.$$

Example 1.6. *In the case of* 3×3 *upper-triangular matrices it is only possible to define a superinvolution when we consider the elementary* $\mathbb{Z}_2$*-grading defined by the triple* $(0, 1, 0)$*, and we have that*

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}^{\bar{\circ}} = \begin{pmatrix} f & e & -c \\ 0 & d & b \\ 0 & 0 & a \end{pmatrix}.$$

1.5 Proper Polynomials

Definition 1.23. *A polynomial* $f \in K\langle X\rangle$ *is called a proper polynomial, if it is a linear combination of products of commutators*

$$f(x_1, \ldots, x_m) = \sum \alpha_{i,\ldots,j}[x_{i_1}, \ldots, x_{i_p}] \cdots [x_{j_1}, \ldots, x_{j_q}], \quad \alpha_{i,\ldots,j} \in K. \tag{1.1}$$

We assume that 1 *is a product of an empty set of commutators. We denote by* B *the set of all proper polynomials in* $K\langle X\rangle$*, that is, polynomials in the form (1.1). We also define the spaces* B_m *as*

$$B_m = B \cap K\langle x_1, \ldots, x_m\rangle,\ m = 1, 2, \ldots, \quad \Gamma_n = B \cap P_n,\ n = 0, 1, 2, \ldots,$$

i.e. B_m *is the set of the proper polynomials in* m *variables and* Γ_n *is the set of all proper multilinear polynomials of degree* n*.*

If A is a PI-algebra, we denote by $B(A)$, $B_m(A)$ and $\Gamma_n(A)$ the images in $F(A) = K\langle X\rangle/T(A)$ of the corresponding vector subspaces of $K\langle X\rangle$.

The importance of proper polynomials for the study of polynomial identities is presented in the following result which combines the Poincaré–Birkhoff–Witt theorem (or PBW theorem) and the Witt theorems.

Proposition 1.10. (i) *Let us choose an ordered basis of the free Lie algebra* $L(X)$

$$x_1, x_2, \ldots, [x_{i_1}, x_{i_2}], [x_{j_1}, x_{j_2}], \ldots, [x_{k_1}, x_{k_2}, x_{k_3}], \ldots,$$

consisting of the variables x_1, x_2, ... *and some commutators, such that the variables precede the commutators. Then the vector space* $K\langle X\rangle$ *has a basis*

$$x_1^{a_1} \cdots x_m^{a_m} [x_{i_1}, x_{i_2}]^b \cdots [x_{l_1}, \ldots, x_{l_p}]^c,$$

where $a_1, \ldots, a_m, b, \ldots, c \geqslant 0$ *and* $[x_{i_1}, x_{i_2}] < \cdots < [x_{l_1}, \ldots, x_{l_p}]$ *in the ordering of the basis of* $L(X)$*. The basis elements of* $K\langle X\rangle$ *with* $a_1 = \cdots = a_m = 0$ *form a basis for the vector space* B *of the proper polynomials.*

(ii) *If* R *is a unitary PI-algebra over an infinite field* K*, then all polynomial identities of* R *follow from the proper ones (i.e. from those in* $T(R) \cap B$*). If char* $K = 0$*, then the polynomial identities of* R *follow from the proper multilinear identities (i.e. from those in* $T(R) \cap \Gamma_n$ *,* $n = 2, 3, \ldots$ *).*

Proposition 1.11. *A basis of the vector space* Γ_n *of all proper multilinear polynomials of degree* $n \geqslant 2$ *consists of the following products of commutators*

$$[x_{i_1}, \ldots, x_{i_k}] \cdots [x_{j_1}, \ldots, x_{j_l}],$$

where:

(i) *All products are multilinear in the variables* $x_1, \ldots, x_n$*;*

(ii) *Each factor* $[x_{p_1}, x_{p_2}, \ldots, x_{p_s}]$ *is a left normed commutator of length* $\geqslant 2$ *and the maximal index is in the first position, i.e.* $p_1 > p_2, \ldots, p_s$*;*

(iii) *In each product the shorter commutators precede the longer, i.e. in the beginning of the statement of the theorem* $k \leqslant \cdots \leqslant l$*;*

(iv) *If two consecutive factors are commutators of equal length, then the first variable of the first commutator is smaller that the first variable in the second one, i.e.*

$$\cdots [x_{p_1}, x_{p_2}, \ldots, x_{p_s}][x_{q_1}, x_{q_2}, \ldots, x_{q_s}] \cdots$$

satisfies $p_1 < q_1$.

1.5.1 Y-Proper Polynomials

Definition 1.24. *Let* $K\langle Y, Z\rangle$ *be the unitary free algebra, and denote by* $\mathcal{B}(Y)$ *the unitary subalgebra of* $K\langle Y, Z\rangle$ *generated by the elements from* Z *and all non-trivial commutators. The elements of* $\mathcal{B}(Y)$ *are called* Y*-proper polynomials.*

For the following result, we consider the free G-graded algebra $K\langle X^G\rangle$, where for $X^G = \bigcup_{g\in G} X_g$, we consider the decomposition: $X_{1_G} = Y$ and $Z = \bigcup_{g\in G\setminus 1_G} X_g$.

Proposition 1.12. *Let A be a unitary G-graded K-algebra.*

(*i*) *If K is an infinite field, then $\mathrm{Id}_G(A)$ is generated, as a T_G-ideal, by Y-proper polynomials.*

(*ii*) *If K has characteristic zero, then $\mathrm{Id}_G(A)$ is generated, as a T_G-ideal, by multilinear Y-proper polynomials.*

Proof. Let f be a graded identity of A. If K is an infinite field, then we can assume $f = f(y_1, \ldots, y_m, z_1, \ldots, z_m)$ to be multihomogeneous, we can write f in the form

$$f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_m)} \lambda_\alpha y_1^{\alpha_1} \cdots y_m^{\alpha_m} \omega_\alpha(y_1, \ldots, y_m, z_1, \ldots, z_m), \quad \lambda_\alpha \in F,$$

where $\omega_\alpha(y_1, \ldots, y_m, z_1, \ldots, z_m)$ is a linear combination of

$$z_1^{\beta_1} \cdots z_n^{\beta_n} [u_{i_1}, u_{i_2}]^\tau \cdots [u_{l_1}, \ldots, u_{l_p}]^\sigma, \quad u_{i_j} \in X^G.$$

If no variable y appears in f, then f is already Y-proper.

Suppose that y_1 appears in f. Since $f(1 + y_1, y_2, \ldots, y_m, z_1, \ldots, z_m)$ is also a graded polynomial identity of A, we have

$$\begin{aligned} 0 &\equiv f(1 + y_1, y_2, \ldots, y_m, z_1, \ldots, z_m) \\ &= \sum_{\alpha=(\alpha_1,\ldots,\alpha_m)} \lambda_\alpha (y_1 + 1)^{\alpha_1} y_2^{\alpha_2} \cdots y_m^{\alpha_m} \omega_\alpha(y_1 + 1, \ldots, y_m, z_1, \ldots, z_m) \\ &= \sum_{\alpha=(\alpha_1,\ldots,\alpha_m)} \lambda_\alpha \left[\sum_{k}^{\alpha_1} \binom{\alpha_1}{k} y_1^k \right] y_2^{\alpha_2} \cdots y_m^{\alpha_m} \omega_\alpha(y_1, \ldots, y_m, z_1, \ldots, z_m). \end{aligned}$$

The homogeneous component of minimal degree with respect to y_1 is obtained from the summands with α_1 maximal among those with $\lambda_\alpha \neq 0$. Since the T_G-ideal $\mathrm{Id}_G(A)$ is homogeneous, we obtain that

$$\sum_{\alpha_1 \text{ max}} \lambda_\alpha y_2^{\alpha_2} \cdots y_m^{\alpha_m} \omega_\alpha(y_1, \ldots, y_m, z_1, \ldots, z_m) \equiv 0.$$

Multiplying from the left this polynomial identity by $y_1^{\alpha_1}$ and subtracting the product from f, we obtain an identity which is similar to f but involving lower values of α_1.

By induction

$$\sum_{\alpha_1 \text{ fixed}} \lambda_\alpha y_2^{\alpha_2} \cdots y_m^{\alpha_m} \omega_\alpha(y_1, \ldots, y_m, z_1, \ldots, z_m) \in \mathrm{Id}_G(A).$$

Proceeding in the same way with the other variables $y_2, \ldots, y_m$, we conclude that

$$\omega_\alpha(y_1, \ldots, y_m, z_1, \ldots, z_m) \in \mathrm{Id}_G(A).$$

In this way we prove the first statement. The second follows directly by the fact that the base field is of characteristic 0. □

A similar argument can be considered in the case of G-algebras with graded involution. To keep notation simple we consider the case of superalgebras.

Proposition 1.13. *Let A be a unitary $\mathbb{Z}_2$-graded K-algebra with graded involution. Let $K\langle Y_0, Y_1, Z_0, Z_1\rangle$ be the free associative $\mathbb{Z}_2$-graded algebra with involution, where Y_0 is the set of symmetric variables of even degree, Y_1 is the set of symmetric variables of odd degree, Z_0 is the set of skew-symmetric variables of even degree and Z_1 is the set of skew-symmetric variables of odd degree.*

(i) *If K is an infinite field, then $\mathrm{Id}^*_G(A)$ is generated, as a T^*_G-ideal, by Y_0-proper polynomials.*

(ii) *If K has characteristic zero, then $\mathrm{Id}^*_G(A)$ is generated, as a T^*_G-ideal, by multilinear Y_0-proper polynomials.*

1.6 Finite dimensional representations of groups

Our main objective is to consider representations of the symmetric group S_n, as well as the action of the symmetric group on the space of multilinear polynomials in n variables and some consequences applied to T-ideals. Partitions and Young tableaux will play an important role in our study.

Let V be a vector space over a field K and let $GL(V)$ be the group of invertible endomorphisms of V. We will consider that the field K has characteristic zero.

Definition 1.25. *A representation of a group G on V is a homomorphism of groups $\rho : G \to GL(V)$.*

Given ρ a representation of a group G on V, we can consider V as a left G-module in the following way: $gv = \rho(g)(v)$ for all $g \in G$, $v \in V$.

Definition 1.26.

(i) *If $\rho : G \to GL(V)$ and $\rho' : G \to GL(W)$ are two representations of a group G, we say that ρ and ρ' are equivalent, and we write $\rho \sim \rho'$, if V and W are isomorphic as G-modules.*

(ii) A representation $\rho : G \to GL(V)$ is irreducible if V is an irreducible G-module. ρ is completely reducible if V is the direct sum of irreducible submodules.

Theorem 1.4 (Maschke)**.** *Let G be a finite group. Then the group algebra KG is semisimple and*

$$KG \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

where $D_1, \ldots, D_k$ are finite dimensional division algebras over K.

Maschke's theorem holds in a more general situation: when the characteristic p of the field K does not divide $|G|$.

Considering G a finite group, we have that KG is a finite dimensional semisimple algebra, then every one-side ideal of KG is generated by an idempotent. If $a \in A$ is such that $a^2 = \alpha a$ for some $\alpha \neq 0$, we say that a is an *essential idempotent* of A.

Definition 1.27. *An idempotent is minimal if it generates a minimal one-sided ideal.*

Proposition 1.14. *If M is an irreducible representation of G, then $M \cong J_i$, a minimal left ideal of $M_{n_i}(D_i)$, for some $i \in 1, \ldots, k$. Hence there exists a minimal idempotent $e \in KG$ such that $M \cong KGe$.*

Definition 1.28. *Let $\rho : G \to GL(V)$ be a representation of G. Then the map*

$$\chi_\rho : G \to K, \quad \chi_\rho(g) = \mathrm{tr}(\rho(g)),$$

is called the character of the representation ρ and $\dim V = \deg \chi_\rho$ is called the degree of the character χ_ρ. The character χ_ρ is irreducible if ρ is irreducible.

Remark 1.3. *If ϕ and ψ are two finite dimensional representations of the group G then*

$$\chi_{\phi \oplus \psi} = \chi_\phi + \chi_\psi \ \text{ and } \ \chi_{\phi \otimes \psi} = \chi_\phi \cdot \chi_\psi.$$

The knowledge of characters is crucial as it reveals a wealth of information about representations. Remarkably, the number of irreducible representations, a key property in ring theory, is uniquely determined by a fundamental group property.

Theorem 1.5. *Let G be a finite group and let the field K be algebraically closed.*

(i) Every finite dimensional representation of G is determined up to isomorphism by its character.

(ii) The number of the non-isomorphic irreducible representations of G is equal to the number of conjugacy classes of G.

1.6.1 S_n-representations

Now, we shift our focus to representations of the symmetric group S_n. We introduce the concepts of Young Diagrams and Tableaux, highlighting their significance in the analysis of S_n-modules. A particular emphasis will be placed on the study of PI-algebras, where we study the space of multilinear polynomials endowed with an S_n-module structure and examine their corresponding characters.

Definition 1.29. *A partition of the non-negative integer m (notation $\lambda \vdash m$ or $|\lambda| = m$) is a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that*

$$\lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0 \qquad \textit{and} \qquad \lambda_1 + \cdots + \lambda_r = m.$$

We assume that two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ are equal if, for some k

$$\lambda_1 = \mu_1, \ldots, \lambda_k = \mu_k,\ \lambda_{k+1} = \cdots = \lambda_r = \mu_{k+1} = \cdots = \mu_s = 0.$$

When $\lambda = (\lambda_1, \ldots, \lambda_{k_1+\cdots+k_p})$ and

$$\lambda_1 = \cdots = \lambda_{k_1} = \mu_1, \ldots, \lambda_{k_1+\cdots+k_{p-1}+1} = \cdots = \lambda_{k_1+\cdots+k_p} = \mu_p,$$

we adopt the notation

$$\lambda = (\mu_1^{k_1}, \ldots, \mu_p^{k_p}).$$

Given $\sigma \in S_n$, we have a unique decomposition of the form

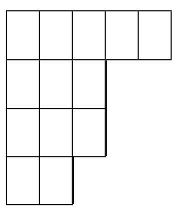
$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_t$$

where $\sigma_1, \sigma_2, \ldots, \sigma_t$ are independent cycles of lengths $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 1$, respectively. Furthermore, since the conjugacy class of σ in S_n is determined by these cycle lengths, the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ determines the conjugacy class of σ.

Definition 1.30. *The Young diagram $[\lambda]$ of the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the set of all knots (points, or square boxes) $(i, j) \in \mathbb{Z}^2$, such that $1 \leqslant j \leqslant \lambda_i$, $i = 1, \ldots, r$.*

It is convenient to represent the Young diagrams graphically as follows. We replace the knots with square boxes such that the first coordinate i (the index of the row) increases from top to bottom and the second coordinate j (the index of the column) increases from left to right. For example, the diagram of the partition $\lambda = (5, 3^2, 2)$ is given in the figure below.

$[\lambda] =$



Definition 1.31. *(i) A Young tableau T_λ of the diagram $[\lambda]$ with m boxes is a filling of the boxes of $[\lambda]$ with the positive integers 1,2,...,m without repetitions. If λ is a partition of m and $\tau \in S_m$, we denote by $T_\lambda(\tau)$ the tableau such that its first column contains the integers $\tau(1)$, ..., $\tau(k_1)$ written in this order from top to bottom, the second column contains consequently written $\tau(k_1+1), \ldots, \tau(k_1+k_2)$, etc.*

(ii) The tableau T_λ is called standard, if the integers written in each column and each row increase, respectively, from top to bottom and from left to right.

For example, for $\lambda = (4,3,1)$,

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 8 & 2 & 1 & 6 & 7 \end{pmatrix}, \qquad \tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 6 & 2 & 4 & 5 & 7 & 8 \end{pmatrix},$$

the tableau $T_\lambda(\tau)$ is not standard and the tableau $T_\lambda(\tau_1)$ is standard:

$$T_\lambda(\tau) = \begin{array}{|c|c|c|c|} \hline 3 & 8 & 1 & 7 \\ \hline 4 & 2 & 6 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} \qquad T_\lambda(\tau_1) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 7 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$$

Given any tableau T_λ of shape $\lambda \vdash n$, we denote by $T_\lambda = D_\lambda(a_{i,j})$, where $a_{i,j}$ is the integer in the (i,j) box. Then, we define the row and column stabilizers as follows.

Definition 1.32. *The row stabilizer of T_λ is the subgroup $R(T_\lambda)$ of all permutations ρ in S_m, such that i and $\rho(i)$ are in the same row of T_λ, $i = 1, \ldots, m$. That is,*

$$R(T_\lambda) = S_{\lambda_1}(a_{1,1}, a_{1,2}, \ldots, a_{1,\lambda_1}) \times \cdots \times S_{\lambda_r}(a_{r,1}, a_{r,2}, \ldots, a_{r,\lambda_r})$$

where $S_{\lambda_i}(a_{i,1}, a_{i,2}, \ldots, a_{i,\lambda_i})$ denotes the symmetric group acting on the following symbols: $a_{i,1}, a_{i,2}, \ldots, a_{i,\lambda_i}$.

Definition 1.33. *The column stabilizer of T_λ is the subgroup $C(T_\lambda)$ of all permutations ρ in S_m, such that i and $\rho(i)$ are in the same column of T_λ, $i = 1, \ldots, m$. That is,*

$$C(T_\lambda) = S_{\lambda'_1}(a_{1,1}, a_{2,1}, \ldots, a_{\lambda'_1,1}) \times \cdots \times S_{\lambda'_s}(a_{1,\lambda_1}, a_{2,\lambda_1}, \ldots, a_{\lambda'_s,\lambda_1})$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the conjugate partition of λ.

Definition 1.34. *For a given tableau T_λ, define*

$$e_{T_\lambda} = \sum_{\rho \in R(T_\lambda)} \sum_{\gamma \in C(T_\lambda)} (sign\ \gamma)\rho\gamma.$$

It can be shown that $e_{T_\lambda}^2 = a e_{T_\lambda}$, where $a = \frac{n!}{d_\lambda} = \prod_{i,j} h_{ij}$ is a non-zero integer. So, e_{T_λ} is an essential idempotent of KS_n.

For each partition λ of m we denote by $M(\lambda)$ and χ_λ the corresponding irreducible S_m-module and its character, respectively.

Theorem 1.6. *Let K be any field of characteristic 0 and let $\tau \in S_m$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of m, let $T = T_\lambda(\tau)$ be the corresponding Young tableau, and let $R(T)$ and $C(T)$ be, respectively, the row and column stabilizers of T. Consider the element of the group algebra KS_m*

$$e_{T_\lambda} = \sum_{\rho \in R(T_\lambda)} \sum_{\gamma \in C(T_\lambda)} (sign\ \gamma)\rho\gamma.$$

(i) Up to a multiplicative constant the element of KS_m, e_{T_λ} is a minimal idempotent which generates a submodule of KS_m isomorphic to $M(\lambda)$.

(ii) The sum of all left S_m-modules $KS_m e_{T_\lambda}$, where T_λ runs over the set of standard λ-tableaux, is direct. It is equal to the minimal two-sided ideal $I(\lambda)$ of KS_m corresponding to λ, and

$$KS_m = \bigoplus_{\lambda \vdash m} I(\lambda).$$

(iii) The dimension $d_\lambda = \dim M(\lambda)$ of $M(\lambda)$ is given by the hook formula

$$\dim M(\lambda) = \frac{m!}{\prod(\lambda_i + \lambda'_j - i - j + 1)},$$

where $\lambda'_1, \ldots, \lambda'_r$, are the lengths of the columns of $[\lambda]$ and the product in the denominator is on all boxes of $[\lambda]$. The dimension $\dim M(\lambda)$ is equal also to the number of standard λ-tableaux $T_\lambda(\tau)$, $\tau \in S_m$.

For example, if $\lambda = (m)$, then the diagram of λ has one row only and for any (m)-tableau T

$$R(T) = S_m, \quad C(T) = 1.$$

Hence the one-dimensional trivial S_m-module $M(\lambda)$ is spanned by the element

$$e_{T_\lambda} = \sum_{\rho \in S_m} \rho.$$

In the other extreme case $\lambda = (1^m)$ we have $R(T) = 1$, $C(T) = S_m$. The one-dimensional S_m-module $M(1^m)$ corresponds to the sign representation and is spanned by

$$e_{T_\lambda} = \sum_{\gamma \in S_m} (\text{sign } \gamma)\gamma.$$

Lemma 1.1. *Let M be an irreducible left S_n-module with character $\chi(M) = \chi_\lambda$, $\lambda \vdash n$. Then M can be generated as an S_n-module by an element of the form $e_{T_\lambda} f$ for some $f \in M$ and some Young tableau T_λ of shape λ. Moreover, for any Young tableau T^*_λ of shape λ there exist $f' \in M$ such that $M = KS_n e_{T^*_\lambda} f'$.*

Lemma 1.2. *Let T_λ be a Young tableau corresponding to $\lambda \vdash n$ and let M be an S_n-module such that $M = M_1 \oplus \cdots \oplus M_m$ where $M_1, \ldots, M_m$ are irreducible S_n-submodules with character χ_λ. Then m is equal to the maximal number of linearly independent $g \in M$ such that $\sigma g = g$ for all $\sigma \in R_{T_\lambda}$.*

All the above considerations hold, formally speaking, for K an algebraically closed field of characteristic 0. But it is well known that the irreducible representations of S_n over the rational numbers $\mathbb{Q}$ are absolutely irreducible (that is they remain irreducible under field extensions). This means the above statements concerning the representations of S_n hold for every field of characteristic 0.

Now, we consider A a PI-algebra and $\mathrm{Id}(A)$ its T-ideal of identities. We know that in characteristic zero, the T-ideal $\mathrm{Id}(A)$ is determined by its multilinear polynomials.

Define the map

$$\phi : KS_n \to P_n, \quad \sum_{\sigma \in S_n} \alpha_\sigma \sigma \mapsto \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

ϕ is a linear isomorphism, so we use the same notation for an element $f \in KS_n$ and its image in P_n.

Let $x_{i_1} x_{i_2} \cdots x_{i_n}$ a monomial and $\tau \in S_n$, and consider the right action

$$(x_{i_1} x_{i_2} \cdots x_{i_n})\tau^{-1} = x_{i_{\tau(1)}} x_{i_{\tau(2)}} \cdots x_{i_{\tau(n)}}.$$

For example, if $n = 4$, $x_{i_1} x_{i_2} x_{i_3} x_{i_4} = x_2 x_3 x_1 x_4$ and $\tau = (1234)$, then

$$(x_2 x_3 x_1 x_4)\tau^{-1} = (x_{i_1} x_{i_2} x_{i_3} x_{i_4})\tau^{-1} = x_{i_2} x_{i_3} x_{i_4} x_{i_1} = x_3 x_1 x_4 x_2.$$

We also can consider a left action given by

$$\sigma(x_{i_1} x_{i_2} \cdots x_{i_n}) = x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_n)}.$$

Since T-ideals are invariant under permutations of the variables, we obtain that $P_n \cap \mathrm{Id}(A)$ is a left S_n-submodule of P_n. Thus,

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

has a structure of left S_n-module.

Definition 1.35. *For $n \geqslant 1$, the S_n-character of $P_n(A) = P_n/(P_n \cap \mathrm{Id}(A))$ is called the n-th cocharacter of A and is denoted by $\chi_n(A)$.*

Decomposing the n-th cocharacter into irreducibles, we obtain

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ and $m_\lambda \geqslant 0$ is the corresponding multiplicity. For details about cocharacters, see [16].

Theorem 1.7. *Let A be a PI-algebra with nth cocharacter $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$. For a partition $\mu \vdash n$, the multiplicity m_μ is equal to zero if and only if for any Young tableau T_μ of shape μ and for any polynomial $f = f(x_1, \ldots, x_n) \in P_n$, the algebra A satisfies the identity $e_{T_\mu} f \equiv 0$.*

1.6.2 $GL_m(K)$-representations

We now shift our focus to representations of the group $GL_m(K)$, specifically examining its action on the free associative algebra of rank m. Our objective is to study the equivalence between representations of the symmetric group and the general linear group, acting on the space of multilinear polynomials and homogeneous polynomials, respectively.

Fix the vector space V_m with basis $\{x_1, \ldots, x_m\}$ and with the canonical action of $GL_m(K)$, and consider $K\langle V_m\rangle = K\langle x_1, \ldots, x_m\rangle$.

Definition 1.36. *Let ϕ be a finite dimensional representation of the general linear group $GL_m(K)$, i.e. $\phi : GL_m(K) \to GL_s(K)$ for some s. The representation ϕ is polynomial if the entries $(\phi(g))_{pq}$ of the $s \times s$ matrix $\phi(g)$ are polynomials of the entries a_{kl} of g for $g \in GL_m(K)$, $k, l = 1, \ldots, m$, $p, q = 1, \ldots, s$. The polynomial representation ϕ is homogeneous of degree d if the polynomials $(\phi(g))_{pq}$ are homogeneous of degree d. The $GL_m(K)$-module W is called polynomial if the corresponding representation is polynomial. Similarly one introduces homogeneous polynomial modules.*

Theorem 1.8. *(i) Every polynomial representation of $GL_m(K)$ is a direct sum of irreducible homogeneous polynomial subrepresentations.*

(ii) Every irreducible homogeneous polynomial $GL_m(K)$-module of degree $n \geqslant 0$ is isomorphic to a submodule of $(K\langle V_m\rangle)^{(n)}$.

The irreducible homogeneous polynomial representations of degree n of $GL_m(K)$ are characterized by partitions of n with at most m parts and Young diagrams.

Theorem 1.9. *(i) The non-isomorphic irreducible homogeneous polynomial $GL_m(K)$-representations of degree $n \geqslant 0$ are in 1-1 correspondence with the partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ of n. We denote by $W_m(\lambda)$ the irreducible $GL_m(K)$-module related to λ.*

(ii) Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n. The $GL_m(K)$-module $W_m(\lambda)$ is isomorphic to a submodule of $(K\langle V_m\rangle)^{(n)}$. The $GL_m(K)$-module $(K\langle V_m\rangle)^{(n)}$ has a decomposition

$$(K\langle V_m\rangle)^{(n)} \cong \sum d_\lambda W_m(\lambda),$$

where d_λ is the dimension of the irreducible S_n-module $M(\lambda)$ and the summation runs over all partitions λ of n in not more than m parts.

(iii) *As a subspace of* $(K\langle V_m\rangle)^{(n)}$*, the vector space* $W_m(\lambda)$ *is multihomogeneous. The dimension of its multihomogeneous component* $W_m^{(n_1,\ldots,n_m)}$ *is equal to the number of semistandard* λ*-tableaux of content* $(n_1,\ldots,n_m)$*.*

Let $\lambda = (\lambda_1,\ldots,\lambda_m)$ be a partition of n and let $q_1,\ldots,q_k$ be the lengths of the columns of the diagram $[\lambda]$. Denote by $s_\lambda(x_1,\ldots,x_q)$, $q = q_1$, the polynomial of $K\langle V_m\rangle$

$$s_\lambda(x_1,\ldots,x_q) = \prod_{j=1}^{k} St_{q_j}(x_1,\ldots,x_{q_j}),$$

where $St_p(x_1,\ldots,x_p)$ is the standard polynomial.

Theorem 1.10. *Let* $\lambda = (\lambda_1,\ldots,\lambda_m)$ *be a partition of* n *in not more than* m *parts and let* $(K\langle V_m\rangle)^{(n)}$ *be the homogeneous component of degree* n *in* $K\langle V_m\rangle$*.*

(i) *The element* $s_\lambda(x_1,\ldots,x_q)$*, defined above, generates an irreducible* $GL_m(K)$*-submodule of* $(K\langle V_m\rangle)^{(n)}$ *isomorphic to* $W_m(\lambda)$*.*

(ii) *Every* $W_m(\lambda) \subset K\langle V_m\rangle^{(n)}$ *is generated by a non-zero element*

$$\omega_\lambda(x_1,\ldots,x_q) = s_\lambda(x_1,\ldots,x_q)\sum_{\sigma\in S_n}\alpha_\sigma\sigma, \quad \alpha_\sigma \in K.$$

The element $\omega_\lambda(x_1,\ldots,x_q)$ *is called the highest weight vector of* $W_m(\lambda)$*. It is unique up to a multiplicative constant and is contained in the one-dimensional vector space of the multihomogeneous elements of degree* $(\lambda_1,\ldots,\lambda_m)$ *in* $W_m(\lambda)$*.*

(iii) *If the* $GL_m(K)$*-submodules* W' *and* W'' *of* $(K\langle V_m\rangle)^{(n)}$ *are isomorphic to* $W_m(\lambda)$ *and have highest weight vectors* ω' *and* ω''*, respectively, then the mapping* $\phi_\alpha : \omega' \mapsto \alpha\omega''$*,* $0 \neq \alpha \in K$*, can be uniquely extended to a* $GL_m(K)$*-module isomorphism. Every isomorphism* $W' \cong W''$ *is obtained in this way.*

Proposition 1.15. *Let* $\lambda = (\lambda_1,\ldots,\lambda_m)$ *be a partition of* n *and let* $W_m(\lambda) \subset (K\langle V_m\rangle)^{(n)}$*. The highest weight vector* ω_λ *of* $W_m(\lambda)$ *can be expressed uniquely as a linear combination of the polynomials* $w_\sigma = s_\lambda\sigma^{-1}$*, where the* σ*'s are such that the* λ*-tableaux* $T(\sigma)$ *are standard.*

Proof. We know that

$$(K\langle V_m\rangle)^{(n)} = \sum_{\lambda\vdash n} d_\lambda W_m(\lambda)$$

and d_λ is equal to the number of standard λ-tableaux. On the other hand, the homogeneous component of degree $\lambda = (\lambda_1,\ldots,\lambda_m)$ of each of the d_λ copies of $W_m(\lambda)$ is one-dimensional. Hence the λ-homogeneous component of the direct sum is of dimension equal to d_λ. Now, we will see that the polynomials ω_σ, with $T(\sigma)$ standard, are linearly independent. We consider the lexicographic ordering on $K\langle V_m\rangle$ assuming that $x_1 > \cdots > x_m$. If $T(\sigma)$ is a standard λ-tableau, then its entries increase from top to bottom and from left to right, then the leading term of ω_σ is $x_1^{\lambda_1} \cdots x_r^{\lambda_r} \sigma^{-1}$. Hence the polynomials ω_σ have pairwise different leading terms for different $T(\sigma)$ and are linearly independent. □

Theorem 1.11. *Let A be a PI-algebra and let*

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda(A)\chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

be the cocharacter of the T-ideal of A, and consider the relative free algebra $F_m(A)$. If $m \geqslant n$ and

$$F_m^{(n)}(A) \cong \sum_{\lambda \vdash n} \eta_\lambda(A) W_m(\lambda),$$

for some $\eta_\lambda(A)$, then $m_\lambda = \eta_\lambda$.

Example 1.7 ([15], Theorem 3.1)**.** *The cocharacter sequence of the T-ideal $T(M_2(K))$ is*

$$\chi_n(M_2(K)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and

(i) $m_{(n)} = 1$;

(ii) $m_{(\lambda_1,\lambda_2)} = (\lambda_1 - \lambda_2 + 1)\lambda_2$, *if* $\lambda_2 > 0$;

(iii) $m_{(\lambda_1,1,1,\lambda_4)} = \lambda_1(2 - \lambda_4) - 1$;

(iv) $m_\lambda = (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)$ *for all other partitions.*

1.6.3 The action of $S_{n_1} \times S_{n_2}$ on $P_{n_1,n_2}(A)$.

In the following chapters, we will explore the polynomial identities of certain $\mathbb{Z}_2$-graded algebras. For this reason, we introduce the action of the group $S_{n_1} \times S_{n_2}$ on the space of multilinear polynomials P_{n_1,n_2}.

Since we are going to consider $G = \mathbb{Z}_2$, we write $x_i^{(0)} = y_i$ and $x_i^{(1)} = z_i$. Let us consider $P_{m,n}$ the space of multilinear polynomials in the variables $y_1, \ldots, y_m$ and $z_1, \ldots, z_n$. Given a $\mathbb{Z}_2$-graded algebra A, denote by $P_{m,n}(A)$ the quotient space

$$P_{m,n}(A) = \frac{P_{m,n}}{P_{m,n} \cap \mathrm{Id}_{\mathbb{Z}_2}(A)}.$$

Consider $n_1, n_2 \geqslant 0$ and the action of the group $S_{n_1} \times S_{n_2}$ on P_{n_1,n_2} given by

$$(\omega, \tau) f(y_1, \ldots, y_{n_1}, z_1, \ldots, z_{n_2}) = f(y_{\omega(1)}, \ldots, y_{\omega(n_1)}, z_{\tau(1)}, \ldots, z_{\tau(n_2)}),$$

where $(\omega, \tau) \in S_{n_1} \times S_{n_2}$ and $f(y_1, \ldots, y_{n_1}, z_1, \ldots, z_{n_2}) \in P_{n_1,n_2}$.

It is known that the irreducible $S_{n_1} \times S_{n_2}$-characters are obtained from the outer tensor product of irreducible characters of S_{n_1} and S_{n_2}. Therefore, we have that there is a bijective correspondence between the irreducible characters of $S_{n_1} \times S_{n_2}$ and the pairs of partitions (λ, μ), where $\lambda \vdash s$ and $\mu \vdash t$. We denote by $\chi_\lambda \otimes \chi_\mu$ the irreducible $S_{n_1} \times S_{n_2}$-character associated to the pair of partitions (λ, μ).

Given a $\mathbb{Z}_2$-graded algebra A, the space $P_{n_1,n_2}(A)$ is a $S_{n_1} \times S_{n_2}$-module, and its character $\chi_{n_1,n_2}(A)$ is called the (n_1, n_2)*-th (graded)-cocharacter of* A. Therefore,

$$\chi_{n_1,n_2}(A) = \sum_{(\lambda,\mu)\vdash(n_1,n_2)} m_{\lambda,\mu}\chi_\lambda \otimes \chi_\mu, \tag{1.2}$$

where $m_{\lambda,\mu}$ is the multiplicity of $\chi_\lambda \otimes \chi_\mu$.

Similarly, if A is a $*$-algebra we define the (n_1, n_2)*-th* $(*)$*-cocharacter of* A.

Theorem 1.12 ([10], Lemma 2, Lemma 5)**.** *Let $\mathcal{I}$ be the T_2-ideal of graded identities of $M_{1,1}(K)$, then*

(1) $\mathcal{I}$ is generated by $y_1y_2 - y_2y_1$ and $z_1z_2z_3 - z_3z_2z_1$,

(2) Let $\chi_{n_1,n_2}(M_{1,1}(K)) = \sum_{(\lambda_1,\lambda_2)\vdash(n_1,n_2)} m_{\lambda_1,\lambda_2}\chi_{\lambda_1} \otimes \chi_{\lambda_2}$ be the (n_1, n_2)-cocharacter of $M_{1,1}(K)$. Then

(i) $m_{\lambda_1,\varnothing} = 1$, if $\lambda_1 = (n_1) \vdash n_1$;

(ii) for $n_2 > 0$, $0 \leqslant r \leqslant \left[\frac{n_1}{2}\right]$ and $0 \leqslant s \leqslant \left[\frac{n_2}{2}\right]$, $m_{\lambda_1,\lambda_2} = n_1 + 1 - 2r$, if $\lambda_1 = (n_1 - r, r) \vdash n_1$, $\lambda_2 = (n_2 - s, s) \vdash n_2$.

1.6.4 The action of $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ on $P_{n_1,n_2,n_3,n_4}(A)$.

We consider the characters of $\mathbb{Z}_2$-graded algebras with graded involutions. So we can consider P_{n_1,n_2,n_3,n_4} the space of multilinear polynomials in symmetry variables of degree 0, skew-symmetry variables of degree 0, symmetry variables of degree 1 and skew-symmetry variables of degree 1.

Given a $\mathbb{Z}_2$-graded algebra A with a graded involution, denote by $P_{n_1,n_2,n_3,n_4}(A)$ the quotient space

$$P_{n_1,n_2,n_3,n_4}(A) = \frac{P_{n_1,n_2,n_3,n_4}}{P_{n_1,n_2,n_3,n_4} \cap \mathrm{Id}^*_{\mathbb{Z}_2}(A)}.$$

We examine the action of $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ on $P_{n_1,n_2,n_3,n_4}(A)$.

Consider n_1, n_2, n_3, $n_4 \geqslant 0$ and the action of the group $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ on P_{n_1,n_2,n_3,n_4} given by

$$(\omega, \sigma, \tau, \rho)f(y_1, \dots, y_{n_1}, u_1, \dots, u_{n_2}, z_1, \dots, z_{n_3}, v_1, \dots, v_{n_4})$$
$$= f(y_{\omega(1)}, \dots, y_{\omega(n_1)}, u_{\sigma(1)}, \dots, u_{\sigma(n_2)}, z_{\tau(1)}, \dots, z_{\tau(n_3)}, v_{\rho(1)}, \dots, v_{\rho(n_4)}),$$

where $(\omega, \sigma, \tau, \rho) \in S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ and $f \in P_{n_1,n_2,n_3,n_4}$.

As in the case of $S_{n_1} \times S_{n_2}$, the $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$-characters are obtained from the outer tensor product of irreducible characters of S_{n_1}, S_{n_2}, S_{n_3} and S_{n_4}, and we have 1-1 correspondence between the irreducible characters of $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$ and the 4-tuples of partitions $(\omega, \sigma, \tau, \rho)$, where $\omega \vdash n_1$, $\sigma \vdash n_2$, $\tau \vdash n_3$ and $\rho \vdash n_4$. We denote by $\chi_\omega \otimes \chi_\sigma \otimes \chi_\tau \otimes \chi_\rho$ the irreducible $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$-character associated to the 4-tuple of partitions $(\omega, \sigma, \tau, \rho)$.

Let A be a $\mathbb{Z}_2$-graded algebra with graded involution. The space $P_{n_1,n_2,n_3,n_4}(A)$ is a $S_{n_1} \times S_{n_2} \times S_{n_3} \times S_{n_4}$-module, and its character $\chi_{n_1,n_2,n_3,n_4}(A)$ is the (n_1, n_2, n_3, n_4)*-th cocharacter of* A. Thus,

$$\chi_{n_1,n_2,n_3,n_4}(A) = \sum_{(\omega,\sigma,\tau,\rho)\vdash(n_1,n_2,n_3,n_4)} m_{\omega,\sigma,\tau,\rho}\chi_\omega \otimes \chi_\sigma \otimes \chi_\tau \otimes \chi_\rho, \tag{1.3}$$

where $m_{\omega,\sigma,\tau,\rho}$ is the multiplicity of $\chi_\omega \otimes \chi_\sigma \otimes \chi_\tau \otimes \chi_\rho$.

1.7 Grassmann algebras

Definition 1.37. *Let V be a vector space with ordered basis $\{e_i : i \in I\}$, with I an ordered set of index. The Grassmann (or exterior) algebra $E(V)$ of V is the associative algebra generated by $\{e_i : i \in I\}$ and with defining relations*

$$e_ie_j + e_je_i = 0,\ i, j \in I,$$

(and $e_i^2 = 0$ if $\mathrm{char}K = 2$). Then $E(V)$ is isomorphic to the algebra $K\langle X\rangle/J$, where $X = \{x_i : i \in I\}$ and the ideal J is generated by $x_ix_j + x_jx_i$, $i, j \in I$. If $\dim V$ is countable, we assume that V has a basis $\{e_1, e_2, \dots\}$ and denote $E(V)$ by E.

The polynomial identities for the Grassmann algebra E were described by Krakowski and Regev [27], and its cocharacters by Olsson and Regev [29].

Theorem 1.13. *Let $\mathrm{char}K = 0$ and let E be the Grassmann algebra of an infinite dimensional vector space. The T-ideal $T(E)$ is generated by $[x_1, x_2, x_3]$.*

Theorem 1.14. *Let E be the Grassmann algebra of an infinite dimensional vector space. Then the cocharacter sequence of the polynomial identities of E is*

$$\chi_n(E) = \sum_{k=0}^{n-1} \chi_{(n-k,1^k)}.$$

We are especially interested in the Grassmann algebra as a $\mathbb{Z}_2$-graded algebra considering the canonical $\mathbb{Z}_2$-grading given by $E = E_0 \oplus E_1$, where E_0 is the set generated by all monomials in the variables e_i of even length and E_1 is the set generated by all monomials in the variables e_i of odd length.

Definition 1.38. *Denote by i_E and $-i_E$ the superinvolutions on E induced by the identity map on the generators e_i of E as follows:*

- $e_i^{i_E} = e_i$ *and* $e_i^{-i_E} = -e_i$,
- $(ab)^{i_E} = (-1)^{|a||b|} b^{i_E} a^{i_E}$ *and* $(ab)^{-i_E} = (-1)^{|a||b|} b^{-i_E} a^{-i_E}$, *for a, b homogeneous elements of E.*

Remark 1.4. *Note that for $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ generator elements of E we have*

$$e_{i_n} e_{i_{n-1}} \cdots e_{i_1} = \lambda_n e_{i_1} e_{i_2} \cdots e_{i_n}$$

where

$$\lambda_n = \begin{cases} 1 & \text{if } n = 2n_1 \text{ or } n = 2n_1 + 1, \text{ with } n_1 \text{ even}, \\ -1 & \text{if } n = 2n_1 \text{ or } n = 2n_1 + 1, \text{ with } n_1 \text{ odd}. \end{cases}$$

Also,

$$(e_{i_1} e_{i_2} \cdots e_{i_n})^{i_E} = \rho_n e_{i_n} e_{i_{n-1}} \cdots e_{i_1} \quad \text{where} \quad \rho_2 = \rho_3 = -1, \, \rho_n = -\rho_{n-2} \text{ for } n \geqslant 4.$$

Therefore, $\rho_n = \lambda_n$ and the superinvolution i_E is equal to the identity function on E. In the case of $-i_E$ we have

$$-i_E(a) = a, \, -i_E(c) = -c, \quad \text{for} \quad a \in E_0, \, c \in E_1.$$

The results regarding the identities and characters of E as a superalgebra were given in [21].

Theorem 1.15. *For the canonical $\mathbb{Z}_2$-grading of the Grassmann algebra, it holds that:*

(i) $\mathrm{Id}_{\mathbb{Z}_2}(E) = \langle [y_1, y_2], [y_1, z_1], z_1 z_2 + z_2 z_1 \rangle_T$;

(ii) $\chi_{r,n-r} = \chi_{(r)} \otimes \chi_{(1^{n-r})}$, *for every* $r \geqslant 0$.

In [13], other $\mathbb{Z}_2$-gradings for the Grassmann algebra were considered, and their identities along with their corresponding characters were computed. In the case of the algebra $M_{1,1}(E)$, its graded identities and cocharacters were described by Di Vincenzo in [10] as follows:

Theorem 1.16. *Let $\mathcal{J}$ be the $T_{\mathbb{Z}_2}$-ideal of graded identities of $M_{1,1}(E)$, then*

(1) $\mathcal{J}$ *is generated by* $y_1 y_2 - y_2 y_1$ *and* $z_1 z_2 z_3 + z_3 z_2 z_1$,

(2) Let

$$\chi_{n_1,n_2}(M_{1,1}(E)) = \sum_{(\lambda_1,\lambda_2) \vdash (n_1,n_2)} m_{\lambda_1,\lambda_2} \chi_{\lambda_1} \otimes \chi_{\lambda_2}$$

be the (n_1, n_2)-cocharacter of $M_{1,1}(E)$. Then *(i)* $m_{\lambda_1,\varnothing} = 1$, *if* $\lambda_1 = (n_1) \vdash n_1$;

(ii) *for* $n_2 > 0$, $0 \leqslant r \leqslant \left[\dfrac{n_1}{2}\right]$ *and* $0 \leqslant s \leqslant \left[\dfrac{n_2}{2}\right]$, $m_{\lambda_1,\lambda_2} = n_1 + 1 - 2r$, *if* $\lambda_1 = (n_1 - r, r) \vdash n_1$, $\lambda_2 = (2^s, 1^{n_2-2s}) \vdash n_2$.

2 Identities of an algebra of upper triangular matrices

Consider A the subalgebra of $UT_3(K)$ given by

$$A = K(e_{1,1} + e_{3,3}) \oplus Ke_{2,2} \oplus Ke_{1,2} \oplus Ke_{2,3} \oplus Ke_{1,3},$$

where $e_{i,j}$ denote the matrix units, that is, A consists of the matrices $\begin{pmatrix} d & a & c \\ 0 & g & b \\ 0 & 0 & d \end{pmatrix}$.

The polynomial identities of A were described by Gordienko in [23]. Furthermore, in [25] the authors considered the generic algebra of $M_{1,1}(E)$ in two generators, and it was shown that its polynomial identities are the same as the identities of A. We are interested in the identities of the algebra A when considering additional structures (grading, involution, etc.), with the aim of determining a new PI-equivalence like in the case of traditional polynomial identities. Here we point out that the research in [25] was motivated by a question posed by A. Berele, about the centre of the generic algebra of $M_{1,1}(E)$ in $d \geqslant 2$ generators. It is well known that the algebra generated by d generic matrices for $M_n(K)$ is a (noncommutative) domain. hence its centre is a commutative domain, and can me embedded into its field of fractions. Several very important questions in PI theory could be solved by using this simple trick; we will not enter into details about these as this thesis is not related to such problems.

The algebra A, with involution, is also considered in [28]. Here we first study the gradings on the algebra A, given by an abelian group. Additionally, we determine a basis for the $\mathbb{Z}_2$-graded identities of A, for the identities with involution, and for the $\mathbb{Z}_2$-graded identities with graded involution. We also determine its cocharacters sequence.

2.1 Gradings on A

We want to characterize the gradings on A starting from elementary gradings. As the main result of this section, we have that the gradings on the algebra A are equivalent to elementary gradings.

Definition 2.1. *Given a triple $\hat{g} = (g_1, g_2, g_3)$ of elements of an arbitrary group G and $1 \leqslant i, j \leqslant 3$, we set the degrees for the unitary matrices as $\deg e_{i,j} = g_i^{-1} g_j$ for $i \neq j$ or $i = j = 2$, and $\deg(e_{1,1} + e_{3,3}) = 1_G$. Given $g \in G$ let $A_g = \operatorname{Span}\{e_{i,j} \mid g_i^{-1} g_j = g\}$ for $g \neq 1_G$ and $A_{1_G} = \operatorname{Span}\left(\{e_{i,j} \mid g_i^{-1} g_j = 1_G\} \cup \{e_{1,1} + e_{3,3}\}\right)$. Then $A = \bigoplus_{g \in G} A_g$ is a G-grading of A called the elementary G-grading defined by the 3-tuple $\hat{g}$.*

We start considering the idempotents of the algebra A.

If $e = \begin{pmatrix} x & a & c \\ 0 & y & b \\ 0 & 0 & x \end{pmatrix} \neq \mathbf{0}$ and $e^2 = e$ then $\begin{pmatrix} x^2 & a(x+y) & 2xc+ab \\ 0 & y^2 & b(x+y) \\ 0 & 0 & x^2 \end{pmatrix} = \begin{pmatrix} x & a & c \\ 0 & y & b \\ 0 & 0 & x \end{pmatrix}$ which implies that $x, y \in \{0,1\}$ where x and y are not simultaneously zero. So, we consider the following cases:

- If $x = 0$, $y = 1$ then $e = \begin{pmatrix} 0 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 0 \end{pmatrix}$. Let $q = \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ and note that $qeq^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

- If $x = 1$, $y = 0$ then $e = \begin{pmatrix} 1 & a & -ab \\ 0 & 0 & b \\ 0 & 0 & 1 \end{pmatrix}$. Let $q = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix}$, so $qeq^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- If $x = 1$, $y = 1$ then $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Therefore we have the following result.

Proposition 2.1. *If $e \in A$ is an idempotent element, then e is conjugated with a diagonal element of A.*

Remark 2.1. *Note that if we have two orthogonal idempotent elements, then each one of these is conjugated to an element of the set $\{(e_{1,1} + e_{3,3}), e_{2,2}\}$.*

Remark 2.2. *Let $A = \bigoplus_{g \in G} A_g$ be graded by an abelian group. Since for g, $h \in G$, $[A_g, A_h] \subset A_{gh} + A_{hg} \subset A_{gh}$, it follows that the commutator subalgebra $[A, A]$ is a nilpotent non-zero graded ideal of A.*

Lemma 2.1. *Let $A = \bigoplus_{g \in G} A_g$ be graded by an abelian group G with identity element $1_G \in G$. Then A_{1_G} contains 2 orthogonal idempotents.*

Proof. Let E be the identity element of A. Since $E \in A_{1_G}$, there exists a non trivial maximal semisimple subalgebra B of A_{1_G}. Let C be one of the simple summands of B and let e be its unit element.

We know that e is conjugated to a diagonal idempotent. Hence either e and $E - e$ are two orthogonal idempotents or $e = E$ and $C = B = \operatorname{Span} E$.

Consider the case $e = E$ and $B = \operatorname{Span} E$, we want to show that this case is not possible.

First, note that any homogeneous element of A is either nilpotent or invertible.

In fact, suppose that $a \in A_g$ is not nilpotent. For m large enough the elements $a, a^2, \dots, a^m$ are linearly dependent and homogeneous. It follows that g must have finite order k and $a^k \in A_{1_G}$. Moreover being not nilpotent the element a^k does not lie in the Jacobson radical $J(A_{1_G})$ of A_{1_G}. Since $A_{1_G}/J(A_{1_G}) \cong F \cdot E$, it follows that $a^k - \lambda E$ is nilpotent for some $\lambda \in F$, $\lambda \neq 0$. This means that a^k and hence also a is invertible.

Next we prove that the Jacobson radical $J(A)$ of A, i.e. the subalgebra of all strictly upper-triangular matrices does not contain non-zero homogeneous elements. $J(A)$ is homogeneous in the G-grading since $\varphi(J(A)) = J(A)$ for any $\varphi \in \operatorname{Aut} A$.

In fact suppose by contradiction that $0 \neq a \in A_g$ is nilpotent and consider its left annihilator $L_a = \{x \in A \colon xa = 0\}$ and right annihilator $R_a = \{x \in A \colon ax = 0\}$, these are graded subspaces of A. Then, as the elements of L_a and R_a are zero divisors they are not invertible, hence they are nilpotent. By our hypothesis a is nilpotent, $a = \begin{pmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}$ and $a^2 = \begin{pmatrix} 0 & 0 & a_1a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that, if $a^2 \neq 0$, $a^2 \in A_{gg}$ and then $e_{2,2} \in L_{a^2}$, a contradiction. Now, if $a^2 = 0$ then $a = \begin{pmatrix} 0 & a_1 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ or $a = \begin{pmatrix} 0 & 0 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}$, so $e_{2,2} \in L_a$ or $e_{2,2} \in R_a$, again a contradiction. Therefore, $J(A)$ has no homogeneous elements. But this cannot happen because $[A, A] \subset J(A)$.

Thus, we have e and $E - e$ are two orthogonal idempotents belonging to A_{1_G}. □

Lemma 2.2. *Let $A = \bigoplus_{g \in G} A_g$ be G-graded. Then the grading is elementary if and only if all matrix units $e_{i,j}$, $1 \leqslant i < j \leqslant 3$, $e_{2,2}$ and $(e_{1,1} + e_{3,3})$ are homogeneous.*

Proof. If the G-grading is elementary then those matrices are homogeneous by definition. Suppose that the said matrices are homogeneous. If we set $g_1 = 1_G$, $g_2 = \deg e_{1,2}$ and $g_3 = g_2 \deg e_{2,3}$ then the triple (g_1, g_2, g_3) satisfies the conditions for the grading to be elementary. □

Lemma 2.3. *Let $A = \bigoplus_{g \in G} A_g$ be G-graded. Then the grading is elementary if and only if the matrices $e_{2,2}$ and $(e_{1,1} + e_{3,3})$ belong to A_{1_G}.*

Proof. It is clear that if the grading is elementary then $e_{2,2}, (e_{1,1} + e_{3,3}) \in A_{1_G}$.

Now, assume that $e_{2,2}, (e_{1,1} + e_{3,3}) \in A_{1_G}$ and note that

- $e_{1,2} \in (e_{1,1} + e_{3,3})Ae_{2,2}$ with $\dim(e_{1,1} + e_{3,3})Ae_{2,2} = 1$,
- $e_{2,3} \in e_{2,2}A(e_{1,1} + e_{3,3})$ with $\dim e_{2,2}A(e_{1,1} + e_{3,3}) = 1$,
- $e_{1,3} \in (e_{1,1} + e_{3,3})A(e_{1,1} + e_{3,3}) = K(e_{1,1} + e_{3,3}) \bigoplus Ke_{1,3}$.

Since $(e_{1,1}+e_{3,3})Ae_{2,2}$, $(e_{1,1}+e_{3,3})A(e_{1,1}+e_{3,3})$ and $e_{2,2}A(e_{1,1}+e_{3,3})$ are graded subalgebras then we can conclude that the elements $e_{1,2}$, $e_{1,3}$ and $e_{1,2}$ are homogeneous. Thus, by Lemma 2.2 the grading is elementary. □

Lemma 2.4. *Let* $A = \bigoplus_{g\in G} A_g$ *be* G*-graded. Then there exist two orthogonal idempotents that are simultaneously diagonalizable and belonging to* A_{1_G}.

Proof. Suppose A is G-graded, then the identity matrix E is homogeneous.

Also J, the Jacobson radical of A is homogeneous. So is $J^2 = span(e_{13})$. By Wedderburn-Malcev theorem, $A = K + K + J$, direct sum of vector spaces, where $K + K$ is a subalgebra. Hence we have two orthogonal idempotents. We show we can choose these homogeneous and simultaneously diagonalizable.

First take the intersection of the left and right annihilators of J^2, it is homogeneous. But this is exactly the span $span(e_{1,2}, e_{1,3}, e_{2,3}, e_{2,2})$. There exists an idempotent in it, and all idempotents in it are of the form $\begin{pmatrix} 0 & a & ac \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix}$.

Suppose the element above is the homogeneous idempotent, call it t. Now, considering A acting in a 3-dimensional vector spaces with base $\{e_1, e_2, e_3\}$, then

$$te_1 = 0, \quad te_2 = ae_1 + e_2, \quad te_3 = ace_1 + ce_2.$$

Now consider the basis f_1, f_2, f_3 such that $tf_1 = 0$, $tf_2 = f_2$, $tf_3 = 0$. Such basis can be obtained as

$$f_1 = e_1, \quad f_2 = ae_1 + e_2, \quad f_3 = -ce_2 + e_3.$$

(f_1 and f_3 are a basis of the kernel of t, and f_2 of its image.)

The change of basis (matrix) is $P = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$.

Form $I - t$, this is an idempotent which is orthogonal to t, and clearly the same P diagonalizes it. As t is homogeneous and I is, then $I - t$ is homogeneous. □

Theorem 2.1. *Let G be an abelian group and K a field. Suppose that the K-algebra $A = \bigoplus_{g\in G} A_g$ is G-graded. Then A, as a G-graded algebra, is isomorphic to A with an elementary G-grading.*

Proof. Let $A = \bigoplus_{g\in G} A_g$ be G-graded. By Lemma 2.1 the homogeneous component A_{1_G} contains two orthogonal idempotents. Now, by Remark 2.1 one of this idempotents is conjugate to $e_{2,2}$ by some $h \in A$. So as a G-graded algebra A is isomorphic to $A' = \bigoplus_{g\in G} A'_g$ where $A'_g = h^{-1}A_g h$. Note that in A' the unit matrix $e_{2,2}$ and the matrix identity E lie in A'_1. Therefore, by Lemma 2.3 A' has an elementary G-grading. □

Another possible proof of Theorem 2.1 can be given as follows.

Proof. Let $A = \bigoplus_{g\in G} A_g$ be G-graded. By Lemma 2.4 the homogeneous component A_{1_G} contains two orthogonal idempotents simultaneously diagonalizable by a element P. So as a G-graded algebra A is isomorphic to $A' = \bigoplus_{g\in G} A'_g$ where $A'_g = P^{-1}A_g P$. Note that in A' the unit matrix $e_{2,2}$ and the matrix identity E lie in A'_1. Therefore, by Lemma 2.3 A' has an elementary G-grading. □

Next, we will consider the graded identities of the algebra A, where the grading is given by the group $\mathbb{Z}_2$. We can view A as a $\mathbb{Z}_2$-graded algebra with gradings

$$A_0 = \left\{ \begin{pmatrix} d & 0 & c \\ 0 & g & 0 \\ 0 & 0 & d \end{pmatrix} \right\}, \qquad A_1 = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2.1}$$

or

$$A_0 = \left\{ \begin{pmatrix} d & a & 0 \\ 0 & g & 0 \\ 0 & 0 & d \end{pmatrix} \right\}, \qquad A_1 = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right\}, \tag{2.2}$$

or

$$A_0 = \left\{ \begin{pmatrix} d & 0 & 0 \\ 0 & g & b \\ 0 & 0 & d \end{pmatrix} \right\}, \qquad A_1 = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \tag{2.3}$$

Denote by $\mathcal{A}^1$ the graded algebra A with grading given by (2.1), $\mathcal{A}^2$ the graded algebra A with grading given by (2.2) and $\mathcal{A}^3$ the graded algebra A with grading given by (2.3).

Remember that, given a group G, for each $g \in G$, we consider the set of variables $X_g = \{x_1^{(g)}, x_2^{(g)}, \dots\}$ of homogeneous variables of degree g, $X^G = \bigcup_{g\in G} X_g$, and the free associative algebra $K\langle X^G\rangle$ as a G-graded algebra. We are going to consider $G = \mathbb{Z}_2$ and we write $x_i^{(0)} = y_i$ and $x_i^{(1)} = z_i$. Let us consider $P_{m,n}$ the space of multilinear polynomials in the variables $y_1, \dots, y_m$ and $z_1, \dots, z_n$. Given a $\mathbb{Z}_2$-graded algebra A, denote by $P_{m,n}(A)$ the quotient space

$$P_{m,n}(A) = \frac{P_{m,n}}{P_{m,n} \cap \mathrm{Id}_{\mathbb{Z}_2}(A)}.$$

2.2 Graded polynomial identities of $\mathcal{A}^1$

It is easy to see that the followings are graded identities of $\mathcal{A}^1$.

(a) $[y_1, y_2] = 0$,

(b) $z_1 z_2 z_3 = 0$,

(c) $[y_1, z_1 z_2] = 0$.

The Identity (c) is a direct consequence of Identity (a).

Note that $P_{m,n}(\mathcal{A}^1) = 0$ if $n \geqslant 3$. By the identity (a) it is clear that

$$P_{m,0}(\mathcal{A}^1) = \mathrm{Span}\{y_1 y_2 \cdots y_m\}. \tag{2.4}$$

Case $n = 1$

Given a monomial μ in $P_{m,1}(\mathcal{A}^1)$, from (a) we can reorder the variables y_i such that modulo $\mathrm{Id}(\mathcal{A}^1)$

$$\mu = y_{i_1} y_{i_2} \cdots y_{i_s} z_1 y_{j_1} y_{j_2} \cdots y_{j_t} \tag{2.5}$$

where $i_1 < i_2 \cdots < i_s$, $j_1 < j_2 \cdots < j_t$ and $s + t = m$, $s, t \geqslant 0$.

Proposition 2.2. *The monomials of the form (2.5) are linearly independent modulo* $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$.

Proof. Let f be a sum of monomials of the form (2.5)

$$f = \sum_{I,J} \alpha_{I,J} y_{i_1} y_{i_2} \cdots y_{i_s} z_1 y_{j_1} y_{j_2} \cdots y_{j_t}.$$

Note that if $y_{i_k} = \beta_{i_k}(e_{1,1} + e_{3,3})$, $z_1 = \lambda e_{1,2}$ and $y_{j_l} = \eta_{j_l} e_{2,2}$ then

$$f = \sum_{I,J} \alpha_{I,J} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_s} \lambda \eta_{j_1} \eta_{j_2} \cdots \eta_{j_t} \cdot e_{1,2}.$$

Suppose that f is a polynomial identity of $\mathcal{A}^1$ and that there exist $\alpha_{I_0,J_0} \neq 0$. Consider the evaluation $y_{i_k} = e_{1,1} + e_{3,3}$ for $i_k \in I_0$, $y_{j_k} = e_{2,2}$ for $j_k \in J_0$ and $z_1 = e_{1,2}$, then $\alpha_{I_0,J_0} = 0$, a contradiction. So, the monomials in the form (2.5) are linearly independent. □

Case $n = 2$

Given a monomial μ in $P_{m,2}(\mathcal{A}^1)$, again from (a) we can reorder the variables y_i such that modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$

$$\mu = y_{i_1} y_{i_2} \cdots y_{i_s} z_{l_1} y_{j_1} y_{j_2} \cdots y_{j_t} z_{l_2} y_{h_1} y_{h_2} \cdots y_{h_r}$$

where $i_1 < i_2 \cdots < i_s$, $j_1 < j_2 \cdots < j_t$, $h_1 < h_2 \cdots < h_r$ and $s + t + r = m$, $s, t, r \geqslant 0$.

As a consequence of (c) we have $z_{l_1} y_j z_{l_2} y_h = y_h z_{l_1} y_j z_{l_2}$. Therefore, $P_{m,2}(\mathcal{A}^1)$ is spanned by monomials in the form

$$y_{i_1} y_{i_2} \cdots y_{i_s} z_{l_1} y_{j_1} y_{j_2} \cdots y_{j_t} z_{l_2} \tag{2.6}$$

where $i_1 < i_2 \cdots < i_s$, $j_1 < j_2 \cdots < j_t$, and $s + t = m$, $s, t \geqslant 0$.

Proposition 2.3. *The monomials in the form (2.6) are linearly independent modulo* $\mathrm{Id}(\mathcal{A}^1)$.

Proof. Note that if $y_{h_k} = \begin{pmatrix} a_{h_k} & 0 & c_{h_k} \\ 0 & b_{h_k} & 0 \\ 0 & 0 & a_{h_k} \end{pmatrix}$ and $z_{l_i} = \begin{pmatrix} 0 & d_{l_i} & 0 \\ 0 & 0 & e_{l_i} \\ 0 & 0 & 0 \end{pmatrix}$ then

$$y_{i_1} y_{i_2} \cdots y_{i_s} z_{l_1} y_{j_1} y_{i_2} \cdots y_{j_t} z_{l_2} = a_{i_1} \cdots a_{i_s} d_{l_1} b_{j_1} \cdots b_{j_t} d_{l_2} \cdot e_{1,3}.$$

Let f be a sum of monomials in the form (2.6)

$$f = \sum_{I,J,L} \alpha_{I,J,L} y_{i_1} y_{i_2} \cdots y_{i_s} z_{l_1} y_{j_1} y_{j_2} \cdots y_{j_t} z_{l_2}$$

$$= \sum_{I,J,L} \alpha_{I,J,L} a_{i_1} \cdots a_{i_s} d_{l_1} b_{j_1} \cdots b_{j_t} e_{l_2} \cdot e_{1,3}.$$

Suppose that f is a polynomial identity of $\mathcal{A}^1$ and that there exist $\alpha_{I_0,J_0,L_0} \neq 0$. Consider the evaluation $y_{i_k} = e_{1,1} + e_{3,3}$ for $i_k \in I_0$, $y_{j_k} = e_{2,2}$ for $j_k \in J_0$, $z_{l_1} = e_{1,2}$ and $z_{l_2} = e_{2,3}$. Then, $\alpha_{I_0,J_0,L_0} = 0$, a contradiction. So, the monomials in the form (2.6) are linearly independent. □

As a consequence of (2.4) Proposition 2.2 and Proposition 2.3, we have the following result.

Theorem 2.2. *The identities (a) and (b), form a basis for the $\mathbb{Z}_2$-graded identities for the algebra $\mathcal{A}^1$.*

2.3 Graded polynomial identities of $\mathcal{A}^2$

Consider some products of homogeneous elements

$$\begin{pmatrix} x_1 & a_1 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & a_2 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 & x_1a_2 + a_1y_2 & 0 \\ 0 & y_1y_2 & 0 \\ 0 & 0 & x_1x_2 \end{pmatrix},$$

$$\begin{pmatrix} x & a & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix}\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & xc + ab \\ 0 & 0 & yb \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x & a & 0 \\ 0 & y & 0 \\ 0 & 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 & xc \\ 0 & 0 & xb \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & b_1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & c_2 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence modulo $\mathrm{Id}(\mathcal{A}^2)$ we have that

$$z_{j_1}z_{j_2} = 0 \text{ and } [y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}] = 0 \tag{2.7}$$

Again, our idea is considering the space $P_{m,n}$ of multilinear polynomials in the variables $y_{i_1}, y_{i_2}, \ldots, y_{i_m}$, $z_{j_1}, z_{j_2}, \ldots, z_{j_n}$. Note that since $z_{j_1}z_{j_2} = 0$, $P_{m,n} = 0$ if $n \geqslant 2$. Thus we consider the cases $n = 0$ and $n = 1$.

Case $n = 0$

By (2.7) we have that $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}] = 0$, then considering proper polynomials, we have only sums of commutators. Also from (2.7) and since

$$[[y_{i_1}, y_{i_2}], [y_{i_3}, y_{i_4}]] = [y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}] - [y_{i_1}, y_{i_2}, y_{i_4}, y_{i_3}],$$

given a commutator we can reorder the variables in the way $[y_k, y_l, y_{i_1}, y_{i_2}, \ldots, y_{i_{m-2}}]$ where $i_1 < i_2 < \cdots < i_{m-2}$, and by Jacobi identity we can only consider the case $[y_{i_1}, y_{i_2}, \ldots, y_{i_m}]$ where $i_1 > i_2 < i_3 < \cdots < i_{m-2}$.

Also, note that the subalgebra $\mathcal{A}_0^2$ is isomorphic to the algebra $UT_2(K)$. So, the polynomial $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}]$ is a basis of the identities in the variables y_i.

Case $n = 1$

For this case, we consider proper polynomials, that is, sums of products of left normed commutators.

Note that since $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}] = 0$ and we have only one z_j then we can consider sums of commutators with products of two commutators.

Now, since $[y_{i_1}, y_{i_2}] = \nu e_{1,2}$, then

$$z_{j_1}[y_{i_1}, y_{i_2}] = 0. \tag{2.8}$$

hence in the case when we have products of two commutators we get that z_{j_1} lies in the second commutator.

By direct computation, we also have the following identity:

$$[[y_{i_1}, y_{i_2}]z_{j_1}, y_{i_3}] = 0, \tag{2.9}$$

Note that from (2.8) $[y_{i_1}, y_{i_2}, z_{j_1}] = [y_{i_1}, y_{i_2}]z_{j_1}$ and using the identity (2.9) we have $[y_{i_1}, y_{i_2}, z_{j_1}, y_{i_3}] = 0$. Therefore, if z_{j_1} lies in a commutator ω, that commutator has the form $\omega = [z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$ or $\omega = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}, z_{j_1}] = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}]z_{j_1}$.

Let us consider the case $\omega = [z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$. Note that by Jacobi identity $[z_{j_1}, y_{i_1}, y_{i_2}] + [y_{i_1}, y_{i_2}, z_{j_1}] + [y_{i_2}, z_{j_1}, y_{i_1}] = 0$, then $[z_{j_1}, y_{i_2}, y_{i_1}] = [z_{j_1}, y_{i_1}, y_{i_2}] + [y_{i_1}, y_{i_2}]z_{j_1}$. So we can write the variables y_i in any order in ω.

If $\omega = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}, z_{j_1}] = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}]z_{j_1}$, as in the case $P_{m,0}$, we can reorder the variables $y_{i_1}, y_{i_2}, \ldots, y_{i_k}$ such that $[y_{i_1}, y_{i_2}, \ldots, y_{i_k}]z_{j_1}$ is a linear combination of polynomials $[y_{i_l}, y_{i_1}, y_{i_2}, \ldots, \widehat{y_{i_l}}, \ldots, y_{i_s}]z_{j_1}$ where $i_l > i_1 < i_2 < \cdots < i_k$.

At this point, we have that if $\mathcal{B}$ is the space of proper polynomials in the variables z_{j_1}, y_{i_1}, y_{i_2}, $\ldots$, y_{i_k}, modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^2)$, $\mathcal{B}$ is spanned by

- $[z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$, $i_1 < i_2 < \cdots < i_k$,
- $[y_{i_l}, y_{i_1}, y_{i_2}, \ldots, \widehat{y_{i_l}}, \ldots, y_{i_k}]z_{j_1}$, $i_l > i_1 < i_2 < \cdots < i_k$,
- $[y_{i_l}, y_{i_1}, y_{i_2}, \ldots, \widehat{y_{i_l}}, \ldots, y_{i_s}][z_{j_1}, y_{h_1}, y_{h_2}, \ldots, y_{h_t}]$, $i_l > i_1 < i_2 < \cdots < i_k$, $h_1 < h_2 < \cdots < h_s$.

Proposition 2.4. *The polynomial*

$$[y_{i_1}, y_{i_2}][z_{j_1}, y_{i_3}] + [y_{i_1}, y_{i_2}, y_{i_3}]z_{j_1} \tag{2.10}$$

is a polynomial identity of the algebra $\mathcal{A}^2$.

Proof. Note that

$$\begin{aligned}[y_{i_1}, y_{i_2}][z_{j_1}, y_{i_3}] &= [y_{i_1}, y_{i_2}]z_{j_1}y_{i_3} - [y_{i_1}, y_{i_2}]y_{i_3}z_{j_1} \\ \text{by (2.9)} \qquad &= y_{i_3}[y_{i_1}, y_{i_2}]z_{j_1} - [y_{i_1}, y_{i_2}]y_{i_3}z_{j_1} \\ &= [y_{i_3}, [y_{i_1}, y_{i_2}]]z_{j_1} \\ &= -[y_{i_1}, y_{i_2}, y_{i_3}]z_{j_1}.\end{aligned}$$

□

Hence from Proposition 2.4 we consider linear combinations of

- $[z_{j_1}, y_{i_1}, y_{i_2}, \dots, y_{i_k}]$, $i_1 < i_2 < \cdots < i_k$,
- $[y_{i_l}, y_{i_1}, y_{i_2}, \dots, \widehat{y_{i_l}}, \dots, y_{i_k}]z_{j_1}$, $i_l > i_1 < i_2 < \cdots < i_k$.

Note that, if $y_{i_k} = \begin{pmatrix} d_{i_k} & a_{i_k} & 0 \\ 0 & g_{i_k} & 0 \\ 0 & 0 & d_{i_k} \end{pmatrix}$ and $z_{j_1} = \begin{pmatrix} 0 & 0 & c_{j_1} \\ 0 & 0 & b_{j_1} \\ 0 & 0 & 0 \end{pmatrix}$, then

$$[z_{j_1}, y_{i_1}, y_{i_2}, \dots, y_{i_m}] = (-1)^{m+1} b_{j_1} \prod_{k=1}^{m-1} (d_{i_k} - g_{i_k}) a_{i_m} \cdot e_{1,3} + b_{j_1} \prod_{k=1}^{m} (d_{i_k} - g_{i_k}) \cdot e_{2,3}$$

and

$$[y_{i_l}, y_{i_1}, y_{i_2}, \dots, \widehat{y_{i_l}}, \dots, y_{i_k}]z_{j_1} = \left\{ [a_{i_1}(d_{i_l} - g_{i_l}) + a_{i_l}(g_{i_l} - d_{i_l})] \prod_{k \neq l}^{m} (g_{i_k} - d_{i_k}) \right\} b_{j_1} \cdot e_{2,3}$$

Theorem 2.3. *The following identities form a basis for the $\mathbb{Z}_2$-graded identities of the algebra $\mathcal{A}^2$:*

(*i*) $z_{i_1} z_{i_2}$,

(*ii*) $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}]$,

(*iii*) $z_{i_1}[y_{i_1}, y_{i_2}]$,

(*iv*) $[[y_{i_1}, y_{i_2}]z_{j_1}, y_{i_3}]$.

Proof. We want to prove that the polynomials $[z_{j_1}, y_{i_1}, y_{i_2}, \dots, y_{i_m}]$ with $i_1 < i_2 < \cdots < i_m$, and $[y_{h_l}, y_{h_1}, y_{h_2}, \dots, \widehat{y_{h_l}}, \dots, y_{h_m}]z_{j_1}$ with $h_l > h_1 < h_2 < \cdots < h_m$ are linearly independent modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^2)$. Consider f the polynomial given by

$$f = \alpha[z_{j_1}, y_{i_1}, y_{i_2}, \dots, y_{i_m}] + \sum_{l=2}^{m} \beta_l [y_{h_l}, y_{h_1}, y_{h_2}, \dots, \widehat{y_{h_l}}, \dots, y_{h_m}]z_{j_1}$$

and suppose that $f \in \mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^2)$. Now, considering the evaluation $z_{j_1} = e_{2,3}$ and $y_i = (e_{1,1} + e_{3,3})$ for $i = 1, \dots, m$ one has $\alpha e_{2,3} = 0$, so $\alpha = 0$. Then,

$$f = \sum_{l=2}^{m} \beta_l [y_{h_l}, y_{h_1}, y_{h_2}, \dots, \widehat{y_{h_l}}, \dots, y_{h_m}]z_{j_1} = 0.$$

Suppose that there exists l_0 such that $\beta_{l_0} \neq 0$, and consider the evaluation $y_{h_{l_0}} = e_{1,2}$, $y_{h_i} = e_{2,2}$ for $i \neq l_0$ and $z_{j_1} = e_{2,3}$. Then, $\beta_{l_0} e_{1,3} = 0$, therefore $\beta_{l_0} = 0$, a contradiction. Thus, we have the desired result. □

2.4 Graded polynomial identities of $\mathcal{A}^3$

Consider $\mathcal{A}^3$, the algebra A with $\mathbb{Z}_2$-grading given by

$$A^{(0)} = \left\{ \begin{pmatrix} d & 0 & 0 \\ 0 & g & b \\ 0 & 0 & d \end{pmatrix} \right\}, \qquad A^{(1)} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

By direct computation we have that the following polynomials are identities of $\mathcal{A}^3$:

(a) $z_{j_1} z_{j_2}$,

(b) $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}]$,

(c) $[y_{i_1}, y_{i_2}] z_{j_1}$,

(d) $[z_{j_1}[y_{i_1}, y_{i_2}], y_{i_3}]$.

Proposition 2.5. *The polynomial*

$$[z_{j_1}, y_{i_3}][y_{i_1}, y_{i_2}] + z_{j_1}[y_{i_1}, y_{i_2}, y_{i_3}] \tag{2.11}$$

is a polynomial identity of the algebra $\mathcal{A}^3$.

Proof. Note that

$$\begin{aligned} [z_{j_1}, y_{i_3}][y_{i_1}, y_{i_2}] &= z_{j_1} y_{i_3} [y_{i_1}, y_{i_2}] - y_{i_3} z_{j_1} [y_{i_1}, y_{i_2}] \\ \text{by (d)} \qquad &= z_{j_1} y_{i_3} [y_{i_1}, y_{i_2}] - z_{j_1} [y_{i_1}, y_{i_2}] y_{i_3} \\ &= z_{j_1} [y_{i_3}, [y_{i_1}, y_{i_2}]] \\ &= -z_{j_1} [y_{i_1}, y_{i_2}, y_{i_3}]. \end{aligned}$$

□

Consider the vector space $P_{m,n}$ of multilinear polynomials in the variables y_{i_1}, $y_{i_2}, \ldots, y_{i_m}, z_{j_1}, z_{j_2}, \ldots, z_{j_n}$. Since $z_{j_1} z_{j_2} = 0$, $P_{m,n} = 0$ if $n \geqslant 2$. So, we consider the cases $n = 0$ and $n = 1$.

Case $n = 0$

Since $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}] = 0$, then considering proper polynomials, we have only sums of commutators, and by Jacobi identity we can reorder the variables of that commutator and consider only the case $[y_{i_1}, y_{i_2}, \ldots, y_{i_m}]$ where $i_1 > i_2 < i_3 < \cdots < i_{m-2}$.

Proposition 2.6. *The polynomials* $[y_{h_l}, y_{h_1}, y_{h_2}, \ldots, \widehat{y_{h_l}}, \ldots, y_{h_m}]$ *with* $h_l > h_1 < h_2 < \cdots < h_m$ *are linearly independent modulo* $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^3)$.

Proof. Consider f the polynomial given by

$$f = \sum_{l=2}^{m} \beta_l [y_{h_l}, y_{h_1}, y_{h_2}, \ldots, \widehat{y_{h_l}}, \ldots, y_{h_m}]$$

and suppose that $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^3)$. Suppose there exists l_0 such that $\beta_{l_0} \neq 0$, and consider the evaluation $y_{h_{l_0}} = e_{2,3}$ and $y_{h_i} = (e_{1,1} + e_{3,3})$ for $i \neq l_0$. Then, $\beta_{l_0} e_{2,3} = 0$, therefore $\beta_{l_0} = 0$, a contradiction. Thus, we have the desired result. □

Case $n = 1$

Once again we consider proper polynomials. Since $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}] = 0$ and we have only one z_j, then we only consider sums of commutators with products of two commutators.

As $[y_{i_1}, y_{i_2}]z_{j_1} = 0$, when we have products of two commutators then z_{j_1} is in the first commutator.

From the identity (c) $[y_{i_1}, y_{i_2}, z_{j_1}] = -z_{j_1}[y_{i_1}, y_{i_2}]$ and using the identity (d) we have $[y_{i_1}, y_{i_2}, z_{j_1}, y_{i_3}] = 0$. Therefore, if z_{j_1} lies in a commutator ω, that commutator has the form $\omega = [z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$ or $\omega = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}, z_{j_1}] = -z_{j_1}[y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$.

Let us consider the case $\omega = [z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$. Note that by Jacobi identity $[z_{j_1}, y_{i_1}, y_{i_2}] + [y_{i_1}, y_{i_2}, z_{j_1}] + [y_{i_2}, z_{j_1}, y_{i_1}] = 0$, so $[z_{j_1}, y_{i_1}, y_{i_2}] - z_{j_1}[y_{i_1}, y_{i_2}] - [z_{j_1}, y_{i_2}, y_{i_1}] = 0$ and $[z_{j_1}, y_{i_2}, y_{i_1}] = [z_{j_1}, y_{i_1}, y_{i_2}] - z_{j_1}[y_{i_1}, y_{i_2}]$. Therefore, we can write the variables y_i in any order in ω.

If $\omega = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}, z_{j_1}] = -z_{j_1}[y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$, as in the case $P_{m,0}$, we can reorder the variables $y_{i_1}, y_{i_2}, \ldots, y_{i_k}$ such that $z_{j_1}[y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$ is a linear combination of polynomials $z_{j_1}[y_{i_l}, y_{i_1}, y_{i_2}, \ldots, \widehat{y_{i_l}}, \ldots, y_{i_s}]$ where $i_l > i_1 < i_2 < \cdots < i_k$.

A nonzero product of commutators has the form $[z_{j_1}, y_{i_1}, \ldots, y_{i_s}][y_{h_1}, y_{h_2}, \ldots, y_{h_t}]$ and by Proposition 2.5 we reduce it to the form $z_{j_1}[y_{i_1}, y_{i_2}, \ldots, y_{i_{s+t}}]$. Then, if $\mathcal{B}$ is the space of proper polynomials in the variables z_{j_1}, y_{i_1}, y_{i_2}, $\ldots$, y_{i_k}, modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^3)$, $\mathcal{B}$ is spanned by

- $[z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$, $i_1 < i_2 < \cdots < i_k$,
- $z_{j_1}[y_{i_l}, y_{i_1}, y_{i_2}, \ldots, \widehat{y_{i_l}}, \ldots, y_{i_k}]$, $i_l > i_1 < i_2 < \cdots < i_k$.

Theorem 2.4. *The following identities form a basis for the* $\mathbb{Z}_2$*-graded identities for the algebra* $\mathcal{A}^3$*:*

(*i*) $z_{i_1}z_{i_2}$,

(*ii*) $[y_{i_1}, y_{i_2}][y_{i_3}, y_{i_4}]$,

(*iii*) $[y_{i_1}, y_{i_2}]z_{i_1}$,

(*iv*) $[z_{j_1}[y_{i_1}, y_{i_2}], y_{i_3}]$.

Proof. Let us show that the polynomials $[z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_m}]$ with $i_1 < i_2 < \cdots < i_m$, and $z_{j_1}[y_{h_l}, y_{h_1}, y_{h_2}, \ldots, \widehat{y_{h_l}}, \ldots, y_{h_m}]$ with $h_l > h_1 < h_2 < \cdots < h_m$ are linearly independent modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^3)$. Consider f the polynomial given by

$$f = \alpha[z_{j_1}, y_{i_1}, y_{i_2}, \ldots, y_{i_m}] + \sum_{l=2}^{m} \beta_l z_{j_1}[y_{h_l}, y_{h_1}, y_{h_2}, \ldots, \widehat{y_{h_l}}, \ldots, y_{h_m}]$$

and suppose that $f \in \mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^3)$. Now, considering the evaluation $z_{j_1} = e_{1,2}$ and $y_i = (e_{1,1} + e_{3,3})$ for $i = 1, \ldots, m$ one has $\alpha e_{1,2} = 0$, so $\alpha = 0$. Then,

$$f = \sum_{l=2}^{m} \beta_l z_{j_1}[y_{h_l}, y_{h_1}, y_{h_2}, \ldots, \widehat{y_{h_l}}, \ldots, y_{h_m}] = 0.$$

Suppose there exists l_0 such that $\beta_{l_0} \neq 0$, and consider the evaluation $y_{h_{l_0}} = e_{2,3}$, $y_{h_i} = (e_{1,1} + e_{3,3})$ for $i \neq l_0$ and $z_{j_1} = e_{1,2}$. Then, $\beta_{l_0} e_{1,3} = 0$, therefore $\beta_{l_0} = 0$, a contradiction. Thus, we have the desired result. □

2.5 $S_m \times S_n$-characters of $\mathcal{A}^1$

In this section, we consider the cocharacters of the $\mathbb{Z}_2$-graded algebra $\mathcal{A}^1$ with the grading given by (2.1). We recall that a basis for the graded identities of $\mathcal{A}^1$ is given by the polynomials $[y_1, y_2]$ and $z_1z_2z_3$.

Let S_m act on the variables $y_1, \ldots, y_m$ and let S_n act on $z_1, \ldots, z_n$. Then $P_{m,n}(\mathcal{A}^1)$ becomes a left $S_m \times S_n$-module. Let $\chi_{m,n}(\mathcal{A}^1)$ be its character. The irreducible $S_m \times S_n$ characters are obtained by taking the outer tensor product of S_m and S_n irreducible characters, respectively. Then, we can write

$$\chi_{m,n}(\mathcal{A}^1) = \sum_{(\lambda,\mu)\vdash(m,n)} m_{\lambda,\mu}\chi_\lambda \otimes \chi_\mu,$$

where $m_{\lambda,\mu} \geqslant 0$ are the corresponding multiplicities.

Let $\lambda \vdash m$, $\mu \vdash n$, and let $W_{\lambda,\mu}$ be a left irreducible $S_m \times S_n$-module. If T_λ is a tableau of shape λ and T_μ a tableau of shape μ, then $W_{\lambda,\mu} \cong F(S_m \times S_n)e_{T_\lambda}e_{T_\mu}$ with S_m and S_n acting on disjoint sets of integers.

From the identity $[y_1, y_2] = 0$ we have $m_{(m),\varnothing} = 1$ and $m_{\lambda,\varnothing} = 0$ for $\lambda \neq (m)$. Also, from $[y_1, y_2] = 0$, we obtain $m_{\lambda,\mu} = 0$ for $h(\lambda) > 2$.

Let us consider the case $n = 1$. Since $P_{m,1}(\mathcal{A}^1) = \text{Span}\{y_{i_1}y_{i_2}\cdots y_{i_s}zy_{j_1}y_{j_2}\cdots y_{i_t}\}$, we consider the case $\lambda \vdash m$ with $\lambda = (p+q, p)$.

For every $i = 0, \ldots, q$ define the following two tableaux: $T_\lambda^{(i)}$ is the tableau

$i+1$	$i+2$	$\cdots$	$i+p$	1	2	$\cdots$	i	$i+2p+2$	$\cdots$	$2p+q+1$
$i+p+2$	$i+p+3$	$\cdots$	$i+2p+1$							

,

$$T_\mu^{(i)} = \boxed{i+p+1}.$$

The associated polynomial is

$$a_{p,q}^{(i)}(y_1, y_2, z) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} z \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i}, \tag{2.12}$$

where $^-$ and $^\sim$ mean alternation on the corresponding elements. That is,

$$f(\overline{x}_1, \ldots, \overline{x}_i, x_{i+1}, \ldots, x_n) = \sum_{\sigma \in S_i} (-1)^\sigma f(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, x_{i+1}, \ldots, x_n).$$

Proposition 2.7. *Modulo* $\text{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$*, the polynomials* $a_{p,q}^{(i)}(y_1, y_2, z)$ *as in (2.12) are linearly independent,* $i = 0, \ldots, q$.

Proof. The proof is similar to that of Theorem 3 in [35]. Suppose that the polynomials $a_{p,q}^{(i)}$ are linearly dependent. Modulo $\text{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$, there exist α_i, $i = 0, \ldots, q$ such that

$$\sum_{i=0}^{q} \alpha_i a_{p,q}^{(i)}(y_1, y_2, z) = 0.$$

Let $t = \max\{i \colon \alpha_i \neq 0\}$. Then

$$\alpha_t a_{p,q}^{(t)}(y_1, y_2, z) + \sum_{i<t} \alpha_i a_{p,q}^{(i)}(y_1, y_2, z) = 0. \tag{2.13}$$

Considering $y_1 = y_1 + y_3$ in (2.13), we have that

$$\begin{aligned} &\alpha_t (y_1+y_3)^t \,\overline{(y_1+y_3)} \cdots (\widetilde{y_1+y_3}) z \bar{y}_2 \cdots \tilde{y}_2 (y_1+y_3)^{q-t} \\ &\quad + \sum_{i<t} \alpha_i (y_1+y_3)^i \,\overline{(y_1+y_3)} \cdots (\widetilde{y_1+y_3}) z \bar{y}_2 \cdots \tilde{y}_2 (y_1+y_3)^{q-i} = 0. \end{aligned}$$

Let us consider the homogeneous component of degree $t+p$ in the variable y_1 and degree $q-t$ in the variable y_3,

$$\alpha_t y_1^t \bar{y}_1 \cdots \tilde{y}_1 z \bar{y}_2 \cdots \tilde{y}_2 y_3^{q-t} + \cdots = 0.$$

Substituting $y_1 = e_{1,1} + e_{3,3}$, $y_2 = y_3 = e_{2,2}$ and $z = e_{1,2}$, we obtain $\alpha_t \cdot e_{1,2} = 0$, so $\alpha_t = 0$ and we have the desired result. $\square$

Proposition 2.8. $m_{\lambda,(1)} = q + 1$*, if* $\lambda = (p+q, p) \vdash m$.

Proof. For every i, $e_{T^{(i)}_\lambda} e_{T^{(i)}_\mu}(y_1, \ldots, y_m, z)$ is the complete linearization of $a^{(i)}_{p,q}(y_1, y_2, z)$. By Proposition 2.7 $m_{\lambda,(1)} \geqslant q + 1$ if $\lambda = (p + q, p) \vdash m$. Also, given T_λ and T_μ two tableaux and $f = e_{T^{(i)}_\lambda} e_{T^{(i)}_\mu}(y_1, \ldots, y_m, z) \notin \mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$, any two alternating variables in f must lie on different sides of z. Since f is a linear combination (modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$) of polynomials, each alternating on p pairs of y_i's, we have that f is a linear combination (modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$) of the polynomials $e_{T^{(i)}_\lambda} e_{T^{(i)}_\mu}$. Hence $m_{\lambda,(1)} = q + 1$, if $\lambda = (p + q, p) \vdash m$. □

Now, consider the case $n = 2$, $P_{m,2}(\mathcal{A}^1) = \mathrm{Span}\{y_{i_1} y_{i_2} \cdots y_{i_s} z_{l_1} y_{j_1} y_{j_2} \cdots y_{i_t} z_{l_2}\}$. For $\mu \vdash 2$ we have the possibilities $\mu = (2)$ and $\mu = (1, 1)$. Define the tableaux $T^{(i)}_\lambda$ as

$i+1$	$i+2$	$\cdots$	$i+p$	1	2	$\cdots$	i	$i+2p+2$	$\cdots$	$2p+q+1$
$i+p+2$	$i+p+3$	$\cdots$	$i+2p+1$							

and

$$T^{(i)}_{(2)} = \begin{array}{|c|c|}\hline i+p+1 & 2p+q+2 \\ \hline\end{array}, \qquad T^{(i)}_{(1,1)} = \begin{array}{|c|}\hline i+p+1 \\ \hline 2p+q+2 \\ \hline\end{array}$$

Then, the associated polynomials are

$$a^{(i)}_{p,q}(y_1, y_2, z) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} z \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i} z, \tag{2.14}$$

and respectively

$$a^{(i)}_{p,q}(y_1, y_2, z_1, z_2) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} \check{z}_1 \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i} \check{z}_2, \tag{2.15}$$

We consider the case of the polynomials (2.14).

Proposition 2.9. *Modulo* $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$*, the polynomials* $a^{(i)}_{p,q}(y_1, y_2, z)$ *as in (2.14) are linearly independent,* $i = 0, \ldots, q$.

Proof. Suppose that the polynomials $a^{(i)}_{p,q}$ are linearly dependent. So, modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$ there exist α_i, $i = 0, \ldots, q$ such that

$$\sum_{i=0}^{q} \alpha_i a^{(i)}_{p,q}(y_1, y_2, z) = 0.$$

Let $t = \max\{i \mid \alpha_i \neq 0\}$. Then,

$$\alpha_t a^{(t)}_{p,q}(y_1, y_2, z) + \sum_{i<t} \alpha_i a^{(i)}_{p,q}(y_1, y_2, z) = 0. \tag{2.16}$$

Considering $y_1 = y_1 + y_3$ in (2.16), we have that

$$\begin{aligned} &\alpha_t (y_1 + y_3)^t\, \overline{(y_1 + y_3)} \cdots \widetilde{(y_1 + y_3)} z \bar{y}_2 \cdots \tilde{y}_2 (y_1 + y_3)^{q-t} z \\ &\quad + \sum_{i<t} \alpha_i (y_1 + y_3)^i\, \overline{(y_1 + y_3)} \cdots \widetilde{(y_1 + y_3)} z \bar{y}_2 \cdots \tilde{y}_2 (y_1 + y_3)^{q-i} z = 0. \end{aligned}$$

We consider the homogeneous component of degree $t+p$ in y_1 and of degree $q-t$ in y_3,

$$\alpha_t y_1^t \bar{y}_1 \cdots \tilde{y}_1 z \bar{y}_2 \cdots \tilde{y}_2 y_3^{q-t} z + \cdots = 0.$$

Substituting $y_1 = e_{1,1} + e_{3,3}$, $y_2 = y_3 = e_{2,2}$ and $z = e_{1,2} + e_{2,3}$, we obtain $\alpha_t \cdot e_{1,3} = 0$, so $\alpha_t = 0$ and we have the result. □

From Proposition 2.9 and with a similar argument to Proposition 2.8, we obtain

Proposition 2.10. *$m_{\lambda,(2)} = q+1$, if $\lambda = (p+q,p) \vdash m$.*

Finally, we consider the case of the polynomial (2.15),

$$a_{p,q}^{(i)}(y_1, y_2, z_1, z_2) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} \check{z}_1 \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i} \check{z}_2.$$

Proposition 2.11. *Modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$, the polynomials $a_{p,q}^{(i)}(y_1, y_2, z_1, z_2)$ as in (2.15) are linearly independent, $i = 0, \ldots, q$.*

Proof. If we consider the evaluation $z_1 = e_{1,2}$ and $z_2 = e_{2,3}$, then

$$a_{p,q}^{(i)}(y_1, y_2, z_1, z_2) = y_1^i \underbrace{\bar{y}_1 \cdots \tilde{y}_1}_{p} z_1 \underbrace{\bar{y}_2 \cdots \tilde{y}_2}_{p} y_1^{q-i} z_2.$$

Suppose the polynomials $a_{p,q}^{(i)}$ are linearly dependent. Modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$ there exist α_i, $i = 0, \ldots, q$ such that

$$\sum_{i=0}^{q} \alpha_i a_{p,q}^{(i)}(y_1, y_2, z_1, z_2) = 0.$$

Let $t = \max\{i \mid \alpha_i \neq 0\}$. Then,

$$\alpha_t a_{p,q}^{(t)}(y_1, y_2, z_1, z_2) + \sum_{i<t} \alpha_i a_{p,q}^{(i)}(y_1, y_2, z_1, z_2) = 0. \tag{2.17}$$

Considering $y_1 = y_1 + y_3$ in (2.17), we have that

$$\begin{aligned}&\alpha_t \, (y_1+y_3)^t \, \overline{(y_1+y_3)} \cdots (\widetilde{y_1+y_3}) \check{z}_1 \bar{y}_2 \cdots \tilde{y}_2 \, (y_1+y_3)^{q-t} \, \check{z}_2 \\ &\quad + \sum_{i<t} \alpha_i \, (y_1+y_3)^i \, \overline{(y_1+y_3)} \cdots (\widetilde{y_1+y_3}) \check{z}_1 \bar{y}_2 \cdots \tilde{y}_2 \, (y_1+y_3)^{q-i} \, \check{z}_2 = 0.\end{aligned}$$

Consider the homogeneous component of degree $t+p$ in y_1 and of degree $q-t$ in y_3,

$$\alpha_t y_1^t \bar{y}_1 \cdots \tilde{y}_1 \check{z}_1 \bar{y}_2 \cdots \tilde{y}_2 y_3^{q-t} \check{z}_2 + \cdots = 0.$$

Substituting $y_1 = e_{1,1} + e_{3,3}$, $y_2 = y_3 = e_{2,2}$, $z_1 = e_{1,2}$ and $z_2 = e_{2,3}$, we obtain $\alpha_t \cdot e_{1,3} = 0$, so $\alpha_t = 0$ and we are done. □

Proposition 2.12. *$m_{\lambda,\mu} = q+1$, if $\lambda = (p+q,p) \vdash m$ and $\mu = (1,1)$.*

Proof. For every i, $e_{T_\lambda^{(i)}}e_{T_\mu^{(i)}}(y_1,\dots,y_m,z_1,z_2)$ is the complete linearisation of the element $a_{p,q}^{(i)}(y_1,y_2,z_1,z_2)$. By Proposition 2.11, $m_{\lambda,\mu}\geqslant q+1$ if $\lambda=(p+q,p)\vdash m$ and $\mu=(1,1)$. Also, given T_λ and T_μ two tableaux, and $f=e_{T_\lambda^{(i)}}e_{T_\mu^{(i)}}(y_1,\dots,y_m,z_1,z_2)\notin \mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$, any two alternating variables in f must lie on different sides of z_1 or z_2, or the two alternating variables are z_1 and z_2. Since f is a linear combination (modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$) of polynomials, each alternating on p pairs of y_i's, and alternating on z_i's, we have that f is a linear combination (modulo $\mathrm{Id}_{\mathbb{Z}_2}(\mathcal{A}^1)$) of the polynomials $e_{T_\lambda^{(i)}}e_{T_\mu^{(i)}}$. Hence $m_{\lambda,(1,1)}=q+1$, if $\lambda=(p+q,p)\vdash m$. □

Finally, based on the previous results, we have the following theorem on the graded cocharacters of the algebra $\mathcal{A}^1$.

Theorem 2.5. *Let*

$$\chi_{m,n}(\mathcal{A}^1)=\sum_{(\lambda,\mu)\vdash(m,n)} m_{\lambda,\mu}\chi_\lambda\otimes\chi_\mu$$

be the (m,n)-cocharacter of $\mathcal{A}^1$. Then

(1) $m_{\lambda,\varnothing}=1$, if $\lambda=(m)\vdash m$;

(2) $m_{\lambda,(1)}=q+1$, if $\lambda=(p+q,p)\vdash m$;

(3) $m_{\lambda,\mu}=q+1$, if $\lambda=(p+q,p)\vdash m$ and $\mu\vdash 2$.

In all remaining cases $m_{\lambda,\mu}=0$.

2.6 Involutions on A

Now, we are interested in studying the identities of the algebra A, considering A as an algebra with involution. We can define on the algebra A the involution $*$ obtained by reflecting a matrix along its secondary diagonal, i.e.

$$\begin{pmatrix} d & a & c\\ 0 & g & b\\ 0 & 0 & d\end{pmatrix}^* = \begin{pmatrix} d & b & c\\ 0 & g & a\\ 0 & 0 & d\end{pmatrix}.$$

Then,

$$A^+=\begin{pmatrix} d & a & c\\ 0 & g & a\\ 0 & 0 & d\end{pmatrix} \text{ and } A^-=\begin{pmatrix} 0 & a & 0\\ 0 & 0 & -a\\ 0 & 0 & 0\end{pmatrix}.$$

This involution was studied in [28]. On the other hand, the identities with involution for the algebra $UT_3(F)$ were found in [12].

Denote the free algebra with involution by $K\langle X, *\rangle = K\langle Y \cup Z\rangle$ generated by symmetric and skew elements, that is

$$K\langle Y \cup Z\rangle = K\langle y_1, z_1, y_2, z_2, , \dots \rangle$$

where y_i stands for a symmetric variable and z_i for a skew-symmetric variable.

Given $f(y_1, \dots, y_m, z_1, \dots, z_s) \in F\langle Y \cup Z\rangle$, we say f is a $*$-polynomial identity of A, if for every $u_1, \dots, u_m \in A^+$ and $v_1, \dots, v_s \in A^-$,

$$f(u_1, \dots, u_m, v_1, \dots, v_s) = 0,$$

and we denote by $\mathrm{Id}^*(A) = \{f \in F\langle Y \cup Z\rangle : f \equiv 0 \text{ on } A\}$ the T^*-ideal of $*$-identities of A.

If in a polynomial it is allowed some variable to be either y_i or z_i, we denote it as x_i, and we set $|x_i| = 0$ if $x_i = z_i$ and $|x_i| = 1$ when $x_i = y_i$. Put $|x_i x_j| = 0$ when the commutator $[x_i, x_j]$ is skew-symmetric, and $|x_i x_j| = 1$ if the commutator $[x_i, x_j]$ is symmetric.

The following theorem describes a basis of $\mathrm{Id}^*(A)$.

Theorem 2.6. *The T^*-ideal $\mathrm{Id}^*(A)$ is generated by the following polynomials:*

1. $z_1y_1z_2y_2z_3$ *2.* $[z_1, z_2]$ *3.* $[z_1y_1z_2, y_2]$ *4.* $z_1y_1z_2 - z_2y_1z_1$
5. $(-1)^{|x_1x_2|}[x_1, x_2][x_3, x_4] - (-1)^{|x_3x_4|}[x_3, x_4][x_1, x_2]$
6. $(-1)^{|x_1x_2|}[x_1, x_2][x_3, x_4] - (-1)^{|x_1x_3|}[x_1, x_3][x_2, x_4] + (-1)^{|x_1x_4|}[x_1, x_4][x_2, x_3]$
7. $[x_1, x_2]z_5[x_3, x_4]$

The proof of Theorem 2.6 will be a consequence of the following propositions.

Let $\mathcal{I}$ be the T^*-ideal generated by the identities (1) to (7) of Theorem 2.6.

Proposition 2.13. *The following is an identity modulo $\mathcal{I}$*

$$y_{\sigma(i_1)}y_{\sigma(i_2)}\cdots y_{\sigma(i_s)}z_1y_{\rho(j_1)}y_{\rho(j_2)}\cdots y_{\rho(j_t)}z_2 = y_{i_1}y_{i_2}\cdots y_{i_s}z_1y_{j_1}y_{j_2}\cdots y_{j_t}z_2, \tag{2.18}$$

for $\sigma \in S_s$ and $\rho \in S_t$.

Proof. Consider the case $y_{i_2}y_{i_1}z_1y_{j_2}y_{j_1}z_2$ and note that $[y_1, y_2] \in A^-$. Then,

$$\begin{aligned} y_{i_2}y_{i_1}z_1y_{j_2}y_{j_1}z_2 &= y_{i_1}y_{i_2}z_1y_{j_2}y_{j_1}z_2 + [y_{i_2}, y_{i_1}]z_1y_{j_2}y_{j_1}z_2 \\ &= y_{i_1}y_{i_2}z_1y_{j_2}y_{j_1}z_2 \\ &= y_{i_1}y_{i_2}z_1y_{j_1}y_{j_2}z_2 + y_{i_1}y_{i_2}z_1[y_{j_2}, y_{j_1}]z_2 \\ &= y_{i_1}y_{i_2}z_1y_{j_1}y_{j_2}z_2. \end{aligned}$$

It follows that

$$y_{\sigma(i_1)}y_{\sigma(i_2)}\cdots y_{\sigma(i_s)}z_1y_{\rho(j_1)}y_{\rho(j_2)}\cdots y_{\rho(j_t)}z_2 = y_{i_1}y_{i_2}\cdots y_{i_s}z_1y_{j_1}y_{j_2}\cdots y_{j_t}z_2,$$

for $\sigma \in S_s$ and $\rho \in S_t$. □

Let $P_{n,m}$ be the set of multilinear polynomials in n symmetric variables and m skew-symmetric variables. By Identity 1., $P_{n,m} = 0$ if $m > 2$. Then, we consider the cases $m = 0$, 1, 2 separately.

Case $m = 0$

Let $\Gamma_{n,0}(\mathcal{I})$ be the subspace of the Y-proper multilinear polynomials in the variables $y_1, \ldots, y_m$ of the relatively free algebra $F\langle Y, Z, *\rangle/\mathcal{I}$.

Since $[x_1, x_2][x_3, x_4][x_5, x_6]$ belongs to $\mathcal{I}$, the vector space $\Gamma_{n,0}(\mathcal{I})$ is spanned by the proper polynomials

$$[x_{j_1}, \ldots, x_{j_s}][x_{k_1}, \ldots, x_{k_t}],$$

where s, $t \geqslant 0$, $s \neq 1$, $t \neq 1$, $j_1 > j_2 < \cdots < j_s$, and $k_1 > k_2 < \cdots < k_t$.

Note that $A^+ = UT_3(K)^+$. We use a result of [12].

Definition 2.2. *A polynomial f is called S_3-standard if f is either of type $[y_{j_1}, \ldots, y_{j_n}]$ or $[y_{j_1}, \ldots, y_{j_{n-2}}][y_{k_1}, y_{k_2}]$, where the commutator $[y_{i_1}, \ldots, y_{i_s}]$ satisfies $i_1 > i_2 < \cdots < i_s$, and if f is of the second type we have that $j_1 > k_1$, $j_2 > k_2$.*

As a particular case of Proposition 5.8 and Lemma 6.2 of [12] we obtain the next result.

Proposition 2.14. *The following statements hold:*

(*i*) *The S_3-standard polynomials span the vector space $\Gamma_{n,0}(A)$.*

(*ii*) *The S_3-standard polynomials in $\Gamma_{n,0}$ are linearly independent modulo the $\mathrm{Id}^*(A)$.*

Case $m = 1$

First, we describe a spanning set of $P_{n,1}(A)$.

Proposition 2.15. *$P_{n,1}(A)$ is spanned by elements of the form*

(*i*) $y_{i_1}y_{i_2}\cdots y_{i_s}zy_{i_{s+1}}y_{i_{s+2}}\cdots y_{i_n}$, *with* $i_1 < \cdots < i_s$ *and* $i_{s+1} < \cdots < i_n$;

(*ii*) $y_{i_1}y_{i_2}\cdots y_{i_s}zy_{i_{s+1}}y_{i_{s+2}}\cdots y_{i_{n-2}}[y_{i_{n-1}}, y_{i_n}]$, *with* $i_1 < \cdots < i_s$, $i_{s+1} < \cdots < i_{n-1}$ *and* $i_n < i_{n-1}$.

Proof. Let $\omega \in P_{n,1}(A)$ be a monomial. Then, $\omega = y_{i_1}y_{i_2}\cdots y_{i_s}zy_{j_1}y_{j_2}\cdots y_{j_t}$ with $s + t = n$.

Suppose $\omega = \omega_1 z\omega_2$, there are the following two cases for ω_2:

Case 1. ω_2 *with indices in non-increasing order.*

Consider $\omega = \omega_1 z \omega_2$ with $\omega_2 = w_{2,1} y_{l_2} y_{l_1} w_{2,2}$ and y_{l_2}, y_{l_1} the first pair of variables such that $l_2 > l_1$.

We can rewrite ω_2 as

$$\omega_2 = w_{2,1} y_{l_1} y_{l_2} w_{2,2} + w_{2,1} [y_{l_2}, y_{l_1}] w_{2,2},$$

then

$$\omega = \omega_1 z w_{2,1} y_{l_1} y_{l_2} w_{2,2} + \omega_1 z w_{2,1} [y_{l_2}, y_{l_1}] w_{2,2} = \omega_1 z \overline{\omega}_2 + \omega_1 w_{2,2} z w_{2,1} [y_{l_2}, y_{l_1}].$$

Note that by Proposition 2.13, in the element $\omega_1 w_{2,2} z w_{2,1} [y_{l_2}, y_{l_1}]$ we can reorder the variables of $\omega_1 w_{2,2}$ and $w_{2,1}$ (separately) with the condition that if $w_{2,1} = y_{k_1} \dots y_{k_p}$, then $y_{k_1} < \cdots < y_{k_p} < y_{l_2}$ and $y_{l_1} < y_{l_2}$.

We can repeat the process for $\omega_1 z \overline{\omega}_2$ until getting an ascending order in the indices of the y's to the right side of z in the element without commutator.

Thus, $\omega = \omega_1 z \omega_2$ can be written as

$$\omega = \omega_1 z y_{h_1} \cdots y_{h_r} + \sum_{I,J} \alpha_{I,J} y_{i_1} \cdots y_{i_s} z y_{j_1} \cdots y_{j_{t-2}} [y_{j_{t-1}}, y_{j_t}]$$

with $h_1 < \cdots < h_r$, $i_1 < \cdots < i_s$, $j_1 < \cdots < y_{j_{t-1}}$, and $j_t < j_{t-1}$.

If the indices of the variables of ω_1 are in increasing order, then we have ω in the desired form.

Case 2. ω_2 *with indices in increasing order.*

Let $\omega = \omega_1 z \omega_2$ with $\omega_1 = w_{1,1} y_{l_2} y_{l_1} w_{1,2}$, and let y_{l_2}, y_{l_1} be the first pair of variables such that $l_2 > l_1$ and the indices of the variables of ω_2 increase.

We rewrite ω_1 as

$$\omega_1 = w_{1,1} y_{l_1} y_{l_2} w_{1,2} + w_{1,1} [y_{l_2}, y_{l_1}] w_{1,2},$$

then

$$\omega = w_{1,1} y_{l_1} y_{l_2} w_{1,2} z \omega_2 + w_{1,1} [y_{l_2}, y_{l_1}] w_{1,2} z \omega_2 = \overline{\omega}_1 z \omega_2 + w_{1,1} \omega_2 z w_{1,2} [y_{l_2}, y_{l_1}].$$

We apply once again Proposition 2.13 to the element $w_{1,1} \omega_2 z w_{1,2} [y_{l_2}, y_{l_1}]$. Thus we reorder the variables of $w_{1,1} \omega_2$ and $w_{1,2}$ (separately). Then,

$$\omega = \overline{\omega}_1 z \omega_2 + \varpi_1 z \varpi_2 [y_{l_2}, y_{l_1}],$$

with the indices of the variables of ω_2, ϖ_1 and ϖ_2 in increasing order, respectively.

Note that there is no relation between the indices of the variables of ϖ_2 and those of the variables of $[y_{l_2}, y_{l_1}]$. Take

$$\omega = \overline{\omega}_1 z \omega_2 + \varpi_1 z \varpi_2 y_{l_2} y_{l_1} - \varpi_1 z \varpi_2 y_{l_1} y_{l_2}.$$

For $\varpi_1 z \varpi_2 y_{l_2} y_{l_1}$ and $\varpi_1 z \varpi_2 y_{l_1} y_{l_2}$ we have two options:

- The indices of the variables of $\varpi_2 y_{l_2} y_{l_1}$ or $\varpi_2 y_{l_2} y_{l_1}$ are not in increasing order. In this case, we proceed as in Case 1 for $\varpi_1 z \varpi_2 y_{l_2} y_{l_1}$ or $\varpi_1 z \varpi_2 y_{l_2} y_{l_1}$.
- The indices of the variables of $\varpi_2 y_{l_2} y_{l_1}$ are in increasing order. In this case, we have the desired order for $\varpi_1 z \varpi_2 y_{l_1} y_{l_2}$.

Finally, if necessary, we repeat the process for $\overline{\omega}_1 z \omega_2$.

This process of rearranging the variables using cases 1. and 2., has finitely many steps. □

Now, we show that the polynomials of Proposition 2.15 are linearly independent.

Proposition 2.16. *The polynomials of the form*

(*I*) $y_{i_1} y_{i_2} \cdots y_{i_s} z y_{i_{s+1}} y_{i_{s+2}} \cdots y_{i_n}$, *with* $i_1 < \cdots < i_s$ *and* $i_{s+1} < \cdots < i_n$;

(*II*) $y_{i_1} y_{i_2} \cdots y_{i_s} z y_{i_{s+1}} y_{i_{s+2}} \cdots y_{i_{n-2}} [y_{i_{n-1}}, y_{i_n}]$, *with* $i_1 < \cdots < i_s$, $i_{s+1} < \cdots < i_{n-1}$ *and* $i_n < i_{n-1}$,

are linearly independent modulo $\mathrm{Id}^*(A)$.

Proof. Let f be a linear combination of elements of the form (I) and (II), such that f is a $*$-polynomial identity of A.

Suppose that not all elements of f of the form (I) have non-zero coefficients, and consider

$$m = y_{l_1} \cdots y_{l_s} z y_{l_{s+1}} \cdots y_{l_n}.$$

Consider the evaluation

$$y_{l_1} = \cdots = y_{l_s} = e_{1,1} + e_{3,3}, \quad z = e_{1,2} - e_{2,3}, \quad y_{l_{s+1}} = \cdots = y_{l_n} = e_{2,2}. \tag{2.19}$$

Since $e_{1,1} + e_{3,3}$ and $e_{2,2}$ commute, the elements with commutator in f vanish.

Now, suppose there exists $\overline{m} = y_{j_1} \cdots y_{j_t} z y_{j_{t+1}} \cdots y_{j_n}$ of the form (I) such that $\overline{m}$ is non-zero on the evaluation (2.19). Then, in $\overline{m}$ the $e_{2,2}$'s can only be substituted on one side of z and similarly for the $(e_{1,1} + e_{3,3})$'s. Then we have two possibilities: $\{j_1, \ldots, j_t\} = \{l_1, \ldots l_s\}$ and $\{j_{t+1}, \ldots, j_n\} = \{l_{s+1}, \ldots l_n\}$, or $\{j_1, \ldots, j_t\} = \{l_{s+1}, \ldots l_n\}$, and $\{j_{t+1}, \ldots, j_n\} = \{l_1, \ldots l_s\}$. In the first case $m = \overline{m}$ and m has zero coefficient in f. In the second case the evaluations of m and $\overline{m}$ are $e_{1,2}$ and $e_{2,3}$, respectively. Therefore, m and $\overline{m}$ have coefficient zero.

Suppose not all elements with one commutator have zero coefficient, and consider

$$m = y_{i_1} y_{i_2} \cdots y_{i_k} z y_{i_{k+1}} y_{i_{k+2}} \cdots y_{i_{n-2}} [y_{i_{n-1}}, y_{i_n}]$$

with the following properties:

(i) its coefficient is non-zero,

(ii) the number $n-k-2$ of variables between z and the commutator is the largest among all elements with non-zero coefficient,

(iii) i_{n-1} is the least of all elements such that the properties (i) and (ii) hold.

Consider the evaluation

$$y_{i_1} = \cdots = y_{i_k} = E,\ z = e_{1,2} - e_{2,3},\ y_{i_{k+1}} = \cdots = y_{i_{n-1}} = e_{2,2},\ y_{i_n} = e_{1,2} + e_{2,3}. \tag{2.20}$$

Let $\overline{m} = y_{j_1} \cdots y_{j_s} z y_{j_{s+1}} \cdots y_{j_{n-2}}[y_{j_{n-1}}, y_{j_n}]$ be a polynomial of the form (II), such that $\overline{m}$ does not vanish under the substitution (2.20). We look for the conditions $\overline{m}$ must satisfy.

Since E and $e_{2,2}$ commute, the elements to be substituted in the commutator will be $e_{1,2}+e_{2,3}$ and $e_{2,2}$. Also $e_{2,2}$ can only be substituted on the right side of $z = (e_{1,2} - e_{2,3})$, and since $n-k-2 \geqslant n-s-2$, then the E's only can be substituted on the left side of z. Then $s = k$ and $\{j_1, \ldots, j_s\} = \{i_1, \ldots i_k\}$.

If $j_{n-1} \notin \{i_{n-1}, i_n\}$ then $j_{n-1} \in \{i_{k+1}, \ldots, i_{n-2}\}$, then $j_{n-1} < i_{n-1}$, but this contradicts the hypothesis of minimality of i_{n-1}. So $j_{n-1} \in \{i_{n-1}, i_n\}$, and since $i_n < i_{n-1}$ we have $j_{n-1} = i_{n-1}$ and $j_n = i_n$. We conclude $\{i_{k+1}, \ldots, i_{n_2}\} = \{j_{k+1}, \ldots, j_{n_2}\}$, $m = \overline{m}$ and the coefficient of m is zero. □

Case $m = 2$

By Proposition 2.13 and the identities 3. and 4., $P_{n,2}(A)$ is spanned by monomials of the form

$$y_{i_1} y_{i_2} \cdots y_{i_s} z_1 y_{j_1} y_{j_2} \cdots y_{j_t} z_2$$

with $i_1 < i_2 \cdots < i_s$, $j_1 < j_2 \cdots < j_t$ and $s + t = n$.

Proposition 2.17. *The polynomials*

$$y_{i_1} y_{i_2} \cdots y_{i_s} z_1 y_{j_1} y_{j_2} \cdots y_{j_t} z_2 \tag{2.21}$$

with $i_1 < i_2 \cdots < i_s$, $j_1 < j_2 \cdots < j_t$ *and* $s + t = n$, *are linearly independent modulo* $\mathrm{Id}(A)$.

Proof. Suppose that

$$f = \sum_{I,J} \alpha_{I,J} y_{i_1} y_{i_2} \cdots y_{i_s} z_1 y_{j_1} y_{j_2} \cdots y_{j_t} z_2$$

is a $*$-identity of A. Choose $\alpha_{I',J'} \neq 0$, such that $|J'| = t'$ is the largest possible. Now, consider the substitution $y_{i_1} = \cdots = y_{i_{s'}} = E$, $y_{j_1} = \cdots = y_{j_{t'}} = e_{2,2}$ and $z_1 = z_2 = (e_{1,2} - e_{2,3})$. Considering the order of the indices and that $|J'|$ is the largest, the only non-zero term of f after the substitution will be the one with coefficient $\alpha_{I',J'}$. Hence $\alpha_{I',J'} = 0$, a contradiction. Thus, the polynomials in Equation (2.21) are linearly independent. □

Finally, by Proposition 2.14, Proposition 2.15, Proposition 2.16 and Proposition 2.17 we have that Theorem 2.6 holds.

2.7 Graded involution

In this section we consider A as a graded algebra with the grading as in (2.1), and the involution from Section 2.6. Then $*$ is a graded involution and

$$A_0^+ = A_0, \qquad A_0^- = \{0\},$$

$$A_1^+ = K(e_{1,2} + e_{2,3}), \qquad A_1^- = K(e_{1,2} - e_{2,3}).$$

2.7.1 Graded $*$-polynomial identities of A

Consider the free algebra with involution $K\langle Y \cup Z, *\rangle$, generated by symmetric and skew elements of even and odd degree, that is

$$K\langle Y \cup Z, *\rangle = K\langle y_1^+, y_1^-, z_1^+, z_1^-, y_2^+, y_2^-, z_2^+, z_2^-, \dots\rangle$$

where y_i^+ stands for a symmetric variable of even degree, y_i^- for a skew variable of even degree, z_i^+ for a symmetric variable of odd degree and z_i^- for a skew variable of odd degree.

Given $f(y_1^+, \dots, y_m^+, y_1^-, \dots, y_n^-, z_1^+, \dots, z_s^+, z_1^-, \dots, z_t^-) \in F\langle Y \cup Z, *\rangle$, then f is a *graded $*$-polynomial identity* for A, if for all $u_1^+, \dots, u_m^+ \in A_0^+$, $u_1^-, \dots, u_n^- \in A_0^-$, $v_1^+, \dots, v_s^+ \in A_1^+$ and $v_1^-, \dots, v_t^- \in A_1^-$,

$$f(u_1^+, \dots, u_m^+, u_1^-, \dots, u_n^-, v_1^+, \dots, v_s^+, v_1^-, \dots, v_t^-) = 0.$$

We denote by $\mathrm{Id}_{\mathbb{Z}_2}^*(A) = \{f \in K\langle Y \cup Z, *\rangle : \ f \equiv 0 \text{ on } A\}$ the T_2^*-ideal of graded $*$-identities of A.

The following theorem gives a basis of $\mathrm{Id}_{\mathbb{Z}_2}^*(A)$.

Theorem 2.7. *The T_2^*-ideal $\mathrm{Id}_{\mathbb{Z}_2}^*(A)$ is generated, as a T_2^*-ideal, by the following polynomials:*

1. y^-	*2.* $[y_1^+, y_2^+]$	*3.* $[z_1^+, z_2^+]$
4. $[z_1^-, z_2^-]$	*5.* $z_1^+ \circ z_1^-$	*6.* $[y^+, z_1 z_2]$
7. $[y_1^+, z_1 y_2^+ z_2]$	*8.* $z_1 z_2 z_3$	*9.* $z_1^+ y^+ z_2^+ - z_2^+ y^+ z_1^+$
10. $z_1^- y^+ z_2^- - z_2^- y^+ z_1^-$	*11.* $z_1^- y^+ z_2^+ + z_2^+ y^+ z_1^-$	

Proof. One easily sees that the polynomials 1. - 11. are graded $*$-identities. Let $P_{n_1,n_2,n_3,n_4}(A)$ be the set of multilinear graded $*$-polynomials modulo $\mathrm{Id}_{\mathbb{Z}_2}^*(A)$ in n_1 symmetric variables of degree 0, n_2 skew variables of degree 0, n_3 symmetric variables of degree 1 and n_1 skew variables of degree 1. From identities 1. and 8., we have that $P_{n_1,n_2,n_3,n_4}(A) = \{0\}$ if $n_2 > 0$ or $n_3 + n_4 > 2$.

(i) *Case* $n_3 + n_4 = 0$. From identity 2., it is easy to see that

$$P_{n_1,0,0,0}(A) = \text{Span}\{y_1^+ y_2^+ \cdots y_{n_1}^+\}.$$

(ii) *Case* $n_3 + n_4 = 1$. We consider the case $n_3 = 1$ (the case $n_4 = 1$ is analogous). Again, by identity 2., we see that

$$P_{n_1,0,1,0}(A) = \text{Span}\{y_{i_1}^+ \cdots y_{i_s}^+ z^+ y_{j_1}^+ \cdots y_{j_t}^+\}, \tag{2.22}$$

with $s + t = n_1$, $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_t$.

But the monomials $y_{i_1}^+ \cdots y_{i_s}^+ z^+ y_{j_1}^+ \cdots y_{j_t}^+$ are linearly independent. Indeed, let

$$f = \sum_{I,J} \alpha_{I,J} y_{i_1}^+ \cdots y_{i_s}^+ z^+ y_{j_1}^+ \cdots y_{j_t}^+.$$

Suppose $f \equiv 0$ modulo $Id^*_{\mathbb{Z}_2}(A)$ and that there are sets of indices I_0 and J_0, such that $\alpha_{I_0,J_0} \neq 0$. Considering the evaluation $y_i^+ = e_{1,1} + e_{3,3}$, $y_j^+ = e_{2,2}$, $z^+ = e_{1,2} + e_{2,3}$, for $i \in I_0$, $j \in J_0$, we obtain $\alpha_{I_0,J_0} e_{1,2} = 0$. So, $\alpha_{I_0,J_0} = 0$, a contradiction. Then, the polynomials in (2.22) are linearly independent.

(iii) *Case* $n_3 + n_4 = 2$. Consider the case $n_3 = 2$ (the cases $n_4 = 2$ and $n_3 = n_4 = 1$ are analogous). From the identities 2., 3., 6., 7. and 9., we conclude

$$P_{n_1,0,2,0}(A) = \text{Span}\{y_{i_1}^+ \cdots y_{i_s}^+ z_1^+ y_{j_1}^+ \cdots y_{j_t}^+ z_2^+\}, \tag{2.23}$$

with $s + t = n_1$, $i_1 < \cdots < i_s$ and $j_1 < \ldots j_t$.

In order to see that the polynomials in (2.23) are linearly independent modulo $Id^*_{\mathbb{Z}_2}(A)$, suppose that

$$\sum_{I,J} \alpha_{I,J} y_{i_1}^+ \cdots y_{i_s}^+ z_1^+ y_{j_1}^+ \cdots y_{j_t}^+ z_2^+ = 0 \qquad \text{mod } \text{Id}^*_{\mathbb{Z}_2}(A)$$

and that there are index sets I_0 and J_0, such that $\alpha_{I_0,J_0} \neq 0$. Considering the evaluation $y_i^+ = e_{1,1} + e_{3,3}$, $y_j^+ = e_{2,2}$, $z_l^+ = e_{1,2} + e_{2,3}$, for $i \in I_0$, $j \in J_0$, we obtain $\alpha_{I_0,J_0} e_{2,3} = 0$. So, $\alpha_{I_0,J_0} = 0$, a contradiction. Then, the polynomials in (2.23) are linearly independent.

Therefore, from items (i), (ii) and (iii), we conclude that the set of polynomials 1.-11. determines a basis for the graded $*$-identities of the algebra A. □

2.7.2 Cocharacters of $(A, gr, *)$

Let $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$ be a multipartition of (n_1, n_2, n_3, n_4), i.e. $\lambda(i) \vdash n_i$. We are interested in computing the $(n_1, \ldots, n_4)$-th cocharacter of $(A, gr, *)$,

$$\chi_{n_1,n_2,n_3,n_4}(A) = \sum_{\langle\lambda\rangle\vdash(n_1,n_2,n_3,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\cdots\otimes\chi_{\lambda(4)}. \tag{2.24}$$

Since $\dim(\mathcal{A}^1)^+ = \dim(\mathcal{A}^1)^- = 1$, then $m_{\langle\lambda\rangle} = 0$ if $h(\lambda(i)) > 1$ for $i = 3,\ 4$. Also, by the graded $*$-identities of A, we have $m_{\langle\lambda\rangle} = 0$ if $h(\lambda(1)) > 2$.

In the following results, we consider only the case $h(\lambda(i)) \leqslant 1$ for $i = 3,\ 4$. First, we consider the case of even degree variables only or odd degree variables only.

Proposition 2.18. *If either $\langle\lambda\rangle = ((n_1), \varnothing, \varnothing, \varnothing)$ with $n_1 > 0$ or $\langle\lambda\rangle = (\varnothing, \varnothing, (n_3), (n_4))$ with $0 < n_3 + n_4 \leqslant 2$, then $m_{\langle\lambda\rangle} = 1$ in (2.24).*

Proof. Let $\langle\lambda_1\rangle = ((n_1), \varnothing, \varnothing, \varnothing)$ and $\langle\lambda_2\rangle = (\varnothing, \varnothing, (n_3), (n_4))$ as in the statement. Then $\omega_1 = (y_1^+)^{n_1}$ and $\omega_2 = (z_1^+)^{n_3}(z_1^-)^{n_4}$ are highest weight vectors corresponding to the multipartitions $\langle\lambda_1\rangle$ and $\langle\lambda_2\rangle$ respectively. Since ω_1 and ω_2 are not polynomials identities of A then $m_{\langle\lambda_i\rangle} \geqslant 1$. By the identities in Theorem 2.7 we conclude that ω_1 and ω_2 are the only (up to a scalar) highest weight vectors corresponding to $\langle\lambda_1\rangle$ and $\langle\lambda_2\rangle$. Therefore, $m_{\langle\lambda_i\rangle} = 1$ for $i = 1,\ 2$. □

Before dealing with the case $n_3 + n_4 = 1$, we state a technical result (similar results can be found in [8, 19]).

Proposition 2.19. *Modulo $Id^*_{\mathbb{Z}_2}(A)$, the following equality holds:*

$$\begin{aligned}&\underbrace{\bar{y}_1^+\cdots\tilde{y}_1^+}_{p}(y_1^+)^{i_1-p}z^+(y_1^+)^{i_2-p}\underbrace{\bar{y}_2^+\cdots\tilde{y}_2^+}_{p}\\ &=\sum_{j=0}^{p}(-1)^j\binom{p}{j}(y_1^+)^{i_1-j}(y_2^+)^j z^+(y_1^+)^{i_2-p+j}(y_2^+)^{p-j},\end{aligned} \tag{2.25}$$

where $i_1,\ i_2 \geqslant 0$ and $p \geqslant 1$.

Proof. We prove it by induction on p. The case $p = 1$ is a straightforward computation. Let $p > 1$ and let ω be the polynomial (2.25).

$$\begin{aligned}\omega &= \underbrace{\bar{y}_1^+\cdots\tilde{y}_1^+}_{p}(y_1^+)^{i_1-p}z^+(y_1^+)^{i_2-p}\underbrace{\bar{y}_2^+\cdots\tilde{y}_2^+}_{p}\\ &= \underbrace{\bar{y}_1^+\cdots\tilde{y}_1^+}_{p-1}(y_1^+)^{i_1-(p-1)}z^+(y_1^+)^{i_2-1-(p-1)}y_2^+\underbrace{\bar{y}_2^+\cdots\tilde{y}_2^+}_{p-1}\\ &\quad-\underbrace{\bar{y}_1^+\cdots\tilde{y}_1^+}_{p-1}(y_1^+)^{i_1-1-(p-1)}y_2^+z^+(y_1^+)^{i_2-(p-1)}\underbrace{\bar{y}_2^+\cdots\tilde{y}_2^+}_{p-1}.\end{aligned}$$

Applying induction to $p-1$ we obtain

$$\begin{aligned}\omega = &\sum_{j=0}^{p-1}(-1)^j\binom{p-1}{j}(y_1^+)^{i_1-j}(y_2^+)^j z^+ (y_1^+)^{(i_2-1)-(p-1)+j}(y_2^+)^{(p-1)-j+1}\\ &-\sum_{j=0}^{p-1}(-1)^j\binom{p-1}{j}(y_1^+)^{(i_1-1)-j}(y_2^+)^{j+1} z^+ (y_1^+)^{i_2-(p-1)+j}(y_2^+)^{(p-1)-j}.\end{aligned} \tag{2.26}$$

The j-th monomial of the second summand is similar to the $j+1$-th monomial of the first summand, for every $j = 0, \ldots, p-2$. Considering the sum of these monomials, for the corresponding coefficients we have

$$(-1)^{j+1}\binom{p-1}{j+1} - (-1)^j\binom{p-1}{j} = (-1)^{j+1}\left(\binom{p-1}{j+1}+\binom{p-1}{j}\right) = (-1)^{j+1}\binom{p}{j+1}.$$

Thus the sum of the similar monomials of (2.26) corresponds to the $j+1$-th monomial of

$$\omega_1 = \sum_{j=0}^{p}(-1)^j\binom{p}{j}(y_1^+)^{i_1-j}(y_2^+)^j z^+ (y_1^+)^{i_2-p+j}(y_2^+)^{p-j},$$

for $j = 0, \ldots, p-2$. But the first monomial of the first summand of (2.26) is equal to the first monomial of ω_1, whereas the $(p-1)$-th monomial of the second summand of (2.26) is equal to the p-monomial of ω_1. Thus, we have the desired equality. □

Proposition 2.20. *If* $\langle\lambda\rangle = ((p+q,p),\varnothing,(1),\varnothing)$ *or* $\langle\lambda\rangle = ((p+q,p),\varnothing,\varnothing,(1))$*, where* p*,* $q \geqslant 0$*, then* $m_{\langle\lambda\rangle} = (q+1)$ *in (2.24).*

Proof. We deal with the case $\langle\lambda\rangle = ((p+q,p),\varnothing,(1),\varnothing)$. The case $\langle\lambda\rangle = ((p+q,p),\varnothing,\varnothing,(1))$ is analogous.

Consider Young diagrams of shape $\langle\lambda\rangle$ filled in the standard way. From the identity $[y_1^+, y_2^+]$, fixed the tableaux $T_{\lambda(3)} = \boxed{t_1}$ for an integer t_1, then t_1 must be larger than all the integers lying in the first p positions of the first row of $T_{\lambda(1)}$ and less than the ones lying in the second row. Otherwise, the corresponding highest weight vector will be a polynomial identity of A.

Thus, the possibilities for the standard Young tableaux, such that the corresponding highest weight vectors are linearly independent, are given by

$$T_{\lambda(1)} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & p & \cdots & t_1-1 & t_1+1 & \cdots & t_2 \\ \hline t_2+1 & t_2+2 & \cdots & n_1 \\ \cline{1-4}\end{array},$$

$$T_{\lambda(3)} = \boxed{t_1}, \qquad T_{\lambda(2)} = T_{\lambda(4)} = \varnothing,$$

and the corresponding highest weight vectors are

$$\omega_{t_1} = \overline{y}_1^+ \cdots \widetilde{y}_1^+ (y_1^+)^{t_1-1-p} z^+ (y_1^+)^{t_2-t_1} \overline{y}_2^+ \cdots \widetilde{y}_2^+,$$

where $t_1 = p + 1, \ldots, p + q + 1$. We can rewrite the polynomials ω_{t_1} as

$$\omega_{t_1} = \overline{y}_1^+ \cdots \widetilde{y}_1^+ (y_1^+)^{t_1-p} z^+ (y_1^+)^{t_2-p} \overline{y}_2^+ \cdots \widetilde{y}_2^+,$$

where $t_1 = p, \ldots, p + q$ and $t_1 + t_2 = n_1$. By Equation (2.25)

$$\omega_{t_1} = \sum_{j=0}^{p} (-1)^j \binom{p}{j} (y_1^+)^{t_1-j} (y_2^+)^j z^+ (y_1^+)^{t_2-p+j} (y_2^+)^{p-j}.$$

Let us see that the elements of the set $\{\omega_{t_1} : t_1 = p, \ldots, p + q\}$ are linearly independent. Consider the evaluations $z^+ = e_{1,2} + e_{2,3}$, $y_i^+ = \alpha_i(e_{1,1} + e_{3,3}) + \beta_i e_{2,2}$. Then,

$$\begin{aligned}
\omega_{t_1} &= \left[\sum_{j=0}^{p} (-1)^j \binom{p}{j} \alpha_1^{t_1-1} \alpha_2^j \beta_1^{t_2-p+j} \beta_2^{p-j} \right] e_{1,2} + [*] e_{2,3} \\
&= \left[\sum_{j=0}^{p} (-1)^j \binom{p}{j} \beta_1^j \alpha_2^j \beta_2^{p-j} \alpha_1^{p-j} \alpha_1^{t_1-p} \beta_1^{t_2-p} \right] e_{1,2} + [*] e_{2,3} \\
&= \left[(\beta_2\alpha_1 - \beta_1\alpha_2)^p \alpha_1^{t_1-p} \beta_1^{t_2-p} \right] e_{1,2} + [*] e_{2,3}.
\end{aligned}$$

Now, $(\beta_2\alpha_1 - \beta_1\alpha_2)^p$ appears in each evaluation of ω_{t_1}, so we consider just the part $\alpha_1^{t_1-p}\beta_1^{t_2-p} e_{1,2}$. Suppose that there are γ_i's such that

$$\sum_{t_1=p}^{p+q} \gamma_{t_1} \omega_{t_1} \equiv 0.$$

Then, for all $\alpha_1, \beta_1 \in F$

$$\sum_{t_1=p}^{p+q} \gamma_{t_1} \alpha_1^{t_1-p} \beta_1^{t_2-p} = 0.$$

It follows $\gamma_i = 0$ for each i. Therefore, the highest weight vectors ω_{t_1} are linearly independent for $t_1 = p, \ldots, p + q$ and $m_{\langle\lambda\rangle} = (q + 1)$. □

Now, we consider the case $n_3 + n_4 = 2$.

Proposition 2.21. *If* $\langle\lambda\rangle = ((p + q, p), \varnothing, (n_3), (n_4))$, *where* $p, q \geqslant 0$ *and* $n_3 + n_4 = 2$, *then* $m_{\langle\lambda\rangle} = (q + 1)$ *in (2.24).*

Proof. We prove the case $\langle\lambda\rangle = ((p+q, p), \varnothing, (2), \varnothing)$. The cases $\langle\lambda\rangle = ((p+q, p), \varnothing, (1), (1))$ and $\langle\lambda\rangle = ((p + q, p), \varnothing, \varnothing, (2))$ are analogous. Consider Young diagrams of shape $\langle\lambda\rangle$ filled in the standard way.

As in the proof of Proposition 2.20, fixed an integer t_1 in the tableau

$$T_{\lambda(3)} = \begin{array}{|c|c|}\hline t_1 & \\ \hline \end{array},$$

t_1 must be larger than the integers lying in the first p position of the first row of $T_{\lambda(1)}$ and less than all the ones lying in the second row. Also, by the identities $[y^+, z_1 z_2]$, $[y_1^+, z_1 y_2^+ z_2]$, and $z_1^+y^+z_2^+ - z_2^+y^+z_1^+$, we can fix the integer n_1+2 in the second block of the tableaux $T_{\lambda(3)}$.

Thus, the possibilities for the standard Young tableaux, such that the corresponding highest weight vectors are linearly independent, are given by

$$T_{\lambda(1)} = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & \cdots & p & \cdots & t_1-1 & t_1+1 & \cdots & t_2 \\ \hline t_2+1 & t_2+2 & \cdots & n_1+1 \\ \cline{1-4} \end{array},$$

$$T_{\lambda(3)} = \begin{array}{|c|c|} \hline t_1 & n_1+2 \\ \hline \end{array}, \qquad T_{\lambda(2)} = T_{\lambda(4)} = \varnothing,$$

and the corresponding highest weight vectors are

$$\omega_{t_1} = \overline{y}_1^+ \cdots \widetilde{y}_1^+ (y_1^+)^{t_1-p} z_1^+ (y_1^+)^{t_2-p} \overline{y}_2^+ \cdots \widetilde{y}_2^+ z_1^+,$$

where $t_1 = p, \ldots, p+q$ and $t_1+t_2 = n_1$. By Equation (2.25)

$$\omega_{t_1} = \sum_{j=0}^{p} (-1)^j \binom{p}{j} (y_1^+)^{t_1-j} (y_2^+)^j z_1^+ (y_1^+)^{t_2-p+j} (y_2^+)^{p-j} z_1^+.$$

We prove the elements of the set $\{\omega_{t_1} : t_1 = p, \ldots, p+q\}$ are linearly independent. Consider the evaluations $z_1^+ = e_{1,2} + e_{2,3}$, $y_i^+ = \alpha_i(e_{1,1} + e_{3,3}) + \beta_i e_{2,2}$. Then,

$$\begin{aligned} \omega_{t_1} &= \left[\sum_{j=0}^{p} (-1)^j \binom{p}{j} \alpha_1^{t_1-1} \alpha_2^j \beta_1^{t_2-p+j} \beta_2^{p-j} \right] e_{1,3} \\ &= \left[\sum_{j=0}^{p} (-1)^j \binom{p}{j} \beta_1^j \alpha_2^j \beta_2^{p-j} \alpha_1^{p-j} \alpha_1^{t_1-p} \beta_1^{t_2-p} \right] e_{1,3} \\ &= \left[(\beta_2\alpha_1 - \beta_1\alpha_2)^p \alpha_1^{t_1-p} \beta_1^{t_2-p} \right] e_{1,3}. \end{aligned}$$

Since $(\beta_2\alpha_1 - \beta_1\alpha_2)^p$ appears in each evaluation of ω_{t_1}, we consider only the part $\alpha_1^{t_1-p}\beta_1^{t_2-p}e_{1,2}$. Suppose there exist γ_i's such that

$$\sum_{t_1=p}^{p+q} \gamma_{t_1} \omega_{t_1} \equiv 0.$$

Then, for all $\alpha_1, \beta_1 \in F$

$$\sum_{t_1=p}^{p+q} \gamma_{t_1} \alpha_1^{t_1-p} \beta_1^{t_2-p} = 0.$$

It follows that $\gamma_i = 0$ for each i. Therefore, the highest weight vectors ω_{t_1} are linearly independent for $t_1 = p, \ldots, p+q$ and $m_{\langle\lambda\rangle} = (q+1)$. □

Hence we obtain the following theorem.

Theorem 2.8. *Let*

$$\chi_{n_1,n_2,n_3,n_4}(A) = \sum_{\langle\lambda\rangle \vdash (n_1,n_2,n_3,n_4)} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)}$$

be the (n_1, n_2, n_3, n_4)*-cocharacter of* A. *Then*

(1) $m_{\langle\lambda\rangle} = 1$, *if* $\langle\lambda\rangle = ((n_1), \varnothing, \varnothing, \varnothing)$ *or* $\langle\lambda\rangle = (\varnothing, \varnothing, (n_3), (n_4))$, *where* $n_1 > 0$ *and* $0 < n_3 + n_4 \leqslant 2$.

(2) $m_{\langle\lambda\rangle} = q + 1$, *if* $\lambda = ((p + q, p), \varnothing, (n_3), (n_4))$, *where* $p, q \geqslant 0$ *and* $n_3 + n_4 = 1$;

(3) $m_{\langle\lambda\rangle} = q + 1$, *if* $\lambda = ((p + q, p), \varnothing, (n_3), (n_4))$, *where* $p, q \geqslant 0$ *and* $n_3 + n_4 = 2$.

In all remaining cases $m_{\langle\lambda\rangle} = 0$.

3 Matrices over the Grassmann algebra

Let $E = \langle 1, e_1, e_2, \dots \mid e_ie_j = -e_je_i \rangle$ be the infinite dimensional Grassmann algebra over K with its natural $\mathbb{Z}_2$-grading $E = E_0 \oplus E_1$, given by the subspaces E_0 and E_1, of elements of even or odd length respectively. One can define on E the superinvolutions i_E and $-i_E$ induced by the identity map on the generators e_i of E, as presented in Definition 1.38 and Remark 1.4.

Matrix algebras with entries in Grassmann algebras have been subject of extensive studies, of particular interest is determining a basis of their polynomial identities as well as the corresponding cocharacters. Given a superalgebra $A = A_0 \oplus A_1$, we consider the Grassmann envelope given by $G(A) = (A_0 \otimes E_0) \oplus (A_1 \otimes E_1)$. Furthermore, if $\circledast$ and $\diamond$ are superinvolutions defined on the superalgebras A and B respectively, the map $*$ defined on $A\hat{\otimes}B = (A_0 \otimes B_0) \bigoplus (A_1 \otimes B_1)$ by putting $(a \otimes b)^* = a^{\circledast} \otimes b^{\diamond}$ is an involution on $A\hat{\otimes}B$.

We consider the $\mathbb{Z}_2$-graded algebras $M_{1,1}(K)$, $UT_{1,1}(K)$, and $UT_3(K)_{(0,1,0)}$ with a superinvolution, along with their corresponding super tensor products with the Grassmann algebra E, naturally endowed with a $\mathbb{Z}_2$-grading and also with a superinvolution. We regard the resulting algebras as endowed with a graded involution and describe the graded $*$-polynomial identities and the corresponding cocharacters.

In this chapter we will consider graded $*$-polynomials in the following way: Let us consider $Y = \{y_{i,g} \colon i \in \mathbb{N}, g \in G\}$, $Z = \{z_{i,g} \colon i \in \mathbb{N}, g \in G\}$ two countable sets of pairwise different indeterminates. We denote by $\deg_G y_{i,g} = \deg_G z_{i,g} = g$ the G-degree of the variables $Y \cup Z$ with respect to the G-grading. Then $Y_g = \{y_{i,g} \colon i \in \mathbb{N}\}$, $Z_g = \{z_{i,g} \colon i \in \mathbb{N}\}$ are homogeneous variables of G-degree $g \in G$. We can define the $*$-action on monomials over $Y \cup Z$ by the equalities

$$(x_{i_1} \cdots x_{i_n})^* = x_{i_n}^* \cdots x_{i_1}^*, \text{ where } y_{i,g}^* = y_{i,g},\ z_{i,g}^* = -z_{i,g},\ x_j \in Y \cup Z, \tag{3.1}$$

the linear extension of this action is an involution on the free associative algebra $K\langle Y, Z\rangle$ generated by the set $Y \cup Z$. The algebra $\mathcal{F} = K\langle Y, Z\rangle$ is G-graded with the grading $\mathcal{F} = \bigoplus_{g\in G} \mathcal{F}_g$ defined by

$$\mathcal{F}_g = \mathrm{Span}_K\{x_{i_1}x_{i_2} \dots x_{i_n} \colon \deg_G x_{i_1} \dots \deg_G x_{i_1} = g,\ x_j \in Y \cup Z\}.$$

3.1 Graded $*$-identities of $M_{1,1}(E)$

By [22], we have two superinvolutions on $M_{1,1}(K)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\circ} = \begin{pmatrix} d & b \\ -c & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\bullet} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}.$$

Consider the algebra $M_{1,1}(E) = M_{1,1}(K)\hat{\otimes}E = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in E_0,\, b, c \in E_1\right\}$. On $M_{1,1}(E)$ can be considered the graded involutions:

- $*$ induced by $(\circ, i_E)$,
- $*_1$ induced by $(\circ, -i_E)$,
- $*_2$ induced by $(\bullet, i_E)$,
- $*_3$ induced by $(\bullet, -i_E)$,

where $* = *_3$, $*_1 = *_2$ and these involutions are defined as follows. If $a, b \in E_0$, $c \in E_1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{*_1} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$$

and note that the linear map ψ define by $\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ is an isomorphism of the algebras with involution $(M_{1,1}(E), *)$ and $(M_{1,1}(E), *_1)$.

Let R be the algebra $M_{1,1}(E)$ endowed with the involution $*$ induced by the pair $(\circ, i_E)$, that is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}.$$

Note that $R_0^+ = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in E_0\right\}$, $R_0^- = \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in E_0\right\}$,
$R_1^+ = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in E_1\right\}$ and $R_1^- = \left\{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in E_1\right\}$.

We consider the free graded algebra with involution $K\langle Y, Z\rangle$, where $Y = Y_0 \cup Y_1$ is the set of symmetric variables of degree 0 and 1, and $Z = Z_0 \cup Z_1$ is the set of skew-symmetric variables of degree 0 and 1. So, $y_{i,0}$ and $y_{i,1}$ ($z_{i,0}$ and $z_{i,1}$) denote symmetric (skew-symmetric) variables of degree 0 and degree 1, respectively.

One sees directly that the following polynomials in $K\langle Y, Z\rangle$ are identities on $M_{1,1}(E)$.

1. $y_{i,1}y_{j,1}$,
2. $z_{i,1}z_{j,1}$,
3. $z_{i,1} \circ z_{j,0}$,
4. $z_{i,0} \circ y_{j,1}$,
5. $[z_{i,0}, z_{j,0}]$,
6. $[y_{i,1}, y_{j,0}]$,
7. $[z_{i,0}, y_{j,1}][z_{k,0}, y_{l,1}]$,
8. $[z_{i,1}, z_{j,0}][z_{k,1}, z_{l,0}]$,
9. $y_{1,1}z_{1,1}y_{2,1} + y_{2,1}z_{1,1}y_{1,1}$,
10. $z_{1,1}y_{1,1}z_{2,1} + z_{2,1}y_{1,1}z_{1,1}$,
11. $[y_{i,0}, x]$ for all $x \in Y \cup Z$.

Our goal is to prove the following theorem, which provides a basis for the graded $*$-identities of $M_{1,1}(E)$. To this end, we make use of Y_0-proper polynomials.

Theorem 3.1. *Let $(M_{1,1}(E), *)$ be the algebra of 2×2 matrices with entries in Grassmann algebra and with the canonical $\mathbb{Z}_2$-grading and endowed with the involution $*$ given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}.$$

Then, its T_2^*-ideal of identities is generated, as a T_2^*-ideal, by the identities 1-11.

From now on, we denote by $\mathcal{H}$ the T_2^*-ideal of $K\langle Y, Z\rangle$ generated by the identities 1-11. Note that for the last two identities, we can only consider products of $z_{i,1}$ and $y_{j,1}$ in the order $y_{1,1}z_{1,1}y_{2,1}z_{2,1}\cdots y_{1,k}z_{1,k}$ and $z_{1,1}y_{1,1}z_{2,1}y_{2,1}\cdots z_{1,k}y_{1,k}$. Also, it is useful to note that

$$[R_1^-, R_0^-] \subseteq R_1^-, \quad [R_1^+, R_0^-] \subseteq R_1^+, \quad [R_1^+, R_1^-] \subseteq R_0^+.$$

The following three propositions are easy to deduce. The first of them follows directly from Id. (3) and Id. (4) together with the inclusions above, the second can be deduced from the first and again Id. (3) and Id. (4), while the third is a direct consequence of Id. (2) and Id. (1).

Proposition 3.1. *The polynomials*

$$[y_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}] - 2^k y_{1,1}z_{i_1,0}z_{i_2,0}\cdots z_{i_k,0},$$
$$[z_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}] - 2^k z_{1,1}z_{i_1,0}z_{i_2,0}\cdots z_{i_k,0}$$

belong to the ideal $\mathcal{H}$.

Proposition 3.2. *The polynomials*

$$[y_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}, z_{1,1}] = (-1)^k[z_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}, y_{1,1}].$$

belong to the ideal $\mathcal{H}$.

Proposition 3.3. *The polynomials*

$$y_{1,1}z_{1,1}[y_{2,1}, z_{2,1}] - y_{1,1}z_{1,1}y_{2,1}z_{2,1} \text{ and } z_{1,1}y_{1,1}[y_{2,1}, z_{2,1}] + z_{1,1}y_{1,1}z_{2,1}y_{2,1}$$

belong to the ideal $\mathcal{H}$.

3.1.1 Y_0-proper polynomials for $M_{1,1}(E)$

Since 1_R is in R_0^+, every proper polynomial is a linear combination of products of variables $y_{i,1}$, $z_{i,0}$, $z_{k,1}$ followed by a product of commutators. We consider the spaces $\Gamma_{n,l,m,k}(R)$ where n, l, m, $k \geqslant 0$ of all multilinear Y_0-proper polynomials on $M_{1,1}(E)$ in the variables $y_{1,0}, \ldots, y_{n,0}, y_{1,1}, \ldots, y_{l,1}, z_{1,0}, \ldots, z_{m,0}, z_{1,1}, \ldots, z_{k,1}$ in the free algebra $K\langle Y \cup Z\rangle$.

By $[y_{i,0}, x] = 0$ we have $\Gamma_{n,l,m,k}(R) = 0$ if $n > 0$. Hence we consider the possibilities for $\Gamma_{0,l,m,k}(R)$. It is clear that by the identities in $\mathcal{H}$ we have that:

- $\Gamma_{0,0,0,k}(R) = 0$, if $k > 1$;
- $\Gamma_{0,0,m,0}(R) = \text{Span}\{z_{1,0}\cdots z_{m,0}\}$;
- $\Gamma_{0,0,m,1}(R) = \text{Span}\{z_{1,0}\cdots z_{m,0}z_{1,1}\}$ and $\Gamma_{0,0,m,k}(R) = 0$, if $k > 1$;
- $\Gamma_{0,1,m,0}(R) = \text{Span}\{z_{i_1,0}\cdots z_{i_m,0}y_{1,1}\}$ and $\Gamma_{0,l,m,0}(R) = 0$, if $k > 1$.

Consider now $\Gamma_{0,l,0,k}$. By Proposition 3.3 we have that $\Gamma_{0,l,0,l}(R)$ is generated by polynomials in the forms $z_{1,1}y_{1,1}z_{2,1}y_{2,1}\cdots z_{l,1}y_{l,1}$, $y_{1,1}z_{1,1}y_{2,1}z_{2,1}\cdots y_{l,1}z_{l,1}$ and $[y_{1,1}, z_{1,1}]\cdots[y_{l,1}, z_{l,1}]$. Furthermore, since $[y_{1,1}, z_{1,1}] = y_{1,1}z_{1,1} - z_{1,1}y_{1,1}$ we obtain that $\Gamma_{0,l,0,l}(R) = \text{Span}\{z_{1,1}y_{1,1}z_{2,1}y_{2,1}\cdots z_{l,1}y_{l,1},\ y_{1,1}z_{1,1}y_{2,1}z_{2,1}\cdots y_{l,1}z_{l,1}\}$.

The remaining possible cases are $\Gamma_{0,l+1,0,l}(R)$ and $\Gamma_{0,l,0,l+1}(R)$, and in these cases we have that

- $\Gamma_{0,l+1,0,l}(R)$ is spanned by $y_{1,1}z_{1,1}y_{2,1}z_{2,1}\cdots y_{l,1}z_{l,1}y_{l+1,1}$
- $\Gamma_{0,l,0,l+1}(R)$ is spanned by $z_{1,1}y_{1,1}z_{2,1}y_{2,1}\cdots z_{l,1}y_{l,1}z_{l+1,1}$.

$\Gamma_{0,l,m,k}(R)$

In each commutator, at most one $y_{i,1}$ and one $z_{j,1}$ can appear. The non zero commutators are of the form $[y_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}]$, $[z_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}]$, and $[y_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}, z_{1,1}]$. Also $\Gamma_{0,l,m,k}(R) = 0$ if $|l-k| > 1$.

Consider the polynomials

- $p_l = y_{i_1,1}z_{j_1,1}y_{i_2,1}z_{j_2,1}\cdots y_{i_l,1}z_{j_l,1}$
- $p_l^- = y_{i_1,1}z_{j_1,1}y_{i_2,1}z_{j_2,1}\cdots y_{i_l,1}$
- $q_l = z_{j_1,1}y_{i_1,1}z_{j_2,1}y_{i_2,1}\cdots z_{j_l,1}y_{i_l,1}$
- $q_l^- = z_{j_1,1}y_{i_1,1}z_{j_2,1}y_{i_2,1}\cdots z_{j_l,1}$

$\Gamma_{0,k,m,k}(R)$

We can generate $\Gamma_{0,k,m,k}(R)$ by products of $z_{i,0}$ and the polynomials p_l, q_l, and commutators $[y_{j,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}, z_{t,1}]$. Now, for the commutators we have that

$$[y_{1,1}, z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_k,0}, z_{1,1}] = 2^k z_{i_1,0}z_{i_2,0}\cdots z_{i_k,0}\left((-1)^k y_{j,1}z_{t,1} - z_{t,1}y_{j,1}\right).$$

Thus, by identities 5, 7, 10 and 11 we conclude that $\Gamma_{0,k,m,k}$ is spanned, modulo $\mathcal{H}$, by the monomials

- $\prod_{i=1}^{m} z_{i,0}y_{1,1}z_{1,1}\cdots y_{1,k}z_{1,k}$
- $\prod_{i=1}^{m} z_{i,0}z_{1,1}y_{1,1}\cdots z_{1,k}y_{1,k}$

$\Gamma_{0,k,m,k+1}(R)$ and $\Gamma_{0,k+1,m,k}(R)$

Now, considering p_l^- and q_l^- and extending the analysis of case $\Gamma_{0,k,m,k}(R)$ to $\Gamma_{0,k+1,m,k}(R)$ and $\Gamma_{0,k,m,k+1}(R)$ we can conclude that

- $\Gamma_{0,k+1,m,k}(R)$ is generated by $\prod_{i=1}^{m} z_{i,0}y_{1,1}z_{1,1}\cdots y_{1,k}z_{1,k}y_{1,k+1}$,
- $\Gamma_{0,k,m,k+1}(R)$ is generated by $\prod_{i=1}^{m} z_{i,0}z_{1,1}y_{1,1}\cdots z_{1,k}y_{1,k}z_{1,k+1}$.

Note that modulo $\mathcal{H}$ the elements of the subspaces $\Gamma_{n,l,m,k}$ are linearly independent. Therefore, Theorem 3.1 has been proven.

3.1.2 Cocharacters of $(M_{1,1}(E), *)$

Now, we study the cocharacters of $(M_{1,1}(E), *)$. Consider the vector spaces $P_{n,l,m,k}(R)$, n, l, m, $k \geqslant 0$ of all multilinear polynomials for $R = M_{1,1}(E)$ in the sets of variables $y_{1,0}, \ldots, y_{n,0}, y_{1,1}, \ldots, y_{l,1}, z_{1,0}, \ldots, z_{m,0}, z_{1,1}, \ldots, z_{k,1}$ in the free algebra $K\langle Y \cup Z\rangle = K\langle y_{1,0}, y_{1,1}, z_{1,0}, z_{1,1}, \ldots\rangle$.

Let $\langle\lambda\rangle$ be a multipartition of n, $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, where $\lambda(i) \vdash n_i$, $1 \leqslant i \leqslant 4$. Let us consider the (n_1, n_2, n_3, n_4)-th cocharacter of $R = (M_{1,1}(E), *)$,

$$\chi_{n_1,n_2,n_3,n_4}(R) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)}. \tag{3.2}$$

Theorem 3.2. *Let*

$$\chi_{n_1,n_2,n_3,n_4}(R) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)}$$

be the (n_1, n_2, n_3, n_4)-th cocharacter of $R = (M_{1,1}(E), *)$.

(i) If $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ with $n_1 + n_3 \geqslant 1$, then $m_{\langle\lambda\rangle} = 1$;

(ii) If $\langle\lambda\rangle = ((n_1), (1^{n_2}), (n_3), (1^{n_2}))$ with $n_2 \geqslant 1$, then $m_{\langle\lambda\rangle} = 2$;

(iii) If $\langle\lambda\rangle = ((n_1), (1^{n_2+1}), (n_3), (1^{n_2}))$ with $n_2 \geqslant 0$, then $m_{\langle\lambda\rangle} = 1$;

(iv) If $\langle\lambda\rangle = ((n_1), (1^{n_2}), (n_3), (1^{n_2+1}))$ with $n_2 \geqslant 0$, then $m_{\langle\lambda\rangle} = 1$;

In all remaining cases $m_{\langle\lambda\rangle} = 0$.

Proof. From the identities in Theorem 3.1, $m_{\langle\lambda\rangle} = 0$ if:

- $h(\lambda(1)) > 1$,
- $h(\lambda(3)) > 1$,
- $|n_2 - n_4| > 1$.

Given a multipartition $\langle\lambda\rangle$ we consider the corresponding Young diagrams filled in a standard way.

If $T_{\lambda(1)}, T_{\lambda(2)}, T_{\lambda(3)}$ and $T_{\lambda(4)}$ are the corresponding tableaux, due to the identities of Theorem 3.1 we can fill $T_{\lambda(1)}$ with the integers 1, 2, …, n_1, and $T_{\lambda(3)}$ with $n_1 + 1$, $n_1 + 2$, …, $n_1 + n_3$. Also, by the identities $y_{i,1}y_{j,1} = 0$ and $z_{i,1}z_{j,1} = 0$ we can't write consecutive integers in the same tableaux $T_{\lambda(2)}$ or $T_{\lambda(4)}$. So, we consider the tableaux $T_{\lambda(2)}$ and $T_{\lambda(4)}$ with the integers $n_1 + n_3 + 1$, …, $n_1 + n_2 + n_3 + n_4$. For convenience, we consider the tableaux $T_{\lambda(2)}$ and $T_{\lambda(4)}$ with the integers 1, 2, …, m.

Let us consider first $\langle\lambda\rangle = ((n_1), (1^{n_2}), (n_3), (1^{n_2}))$ with $n_2 > 0$. For it, considering the standard tableaux, for $T_{\lambda(2)}$ and $T_{\lambda(4)}$ we have only the possibilities:

$$T_{\lambda(2)} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \vdots \\ \hline m-1 \\ \hline \end{array}, \qquad T_{\lambda(4)} = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \vdots \\ \hline m \\ \hline \end{array} \quad \text{or} \quad \overline{T}_{\lambda(2)} = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \vdots \\ \hline m \\ \hline \end{array}, \qquad \overline{T}_{\lambda(4)} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \vdots \\ \hline m-1 \\ \hline \end{array}$$

The corresponding highest weight vectors are given by

$$\omega = \sum_{\sigma,\tau \in S_p} (\operatorname{sign}\sigma)(\operatorname{sign}\tau) y_{\sigma(1),1} z_{\tau(1),1} \cdots y_{\sigma(p),1} z_{\tau(p),1}$$

and

$$\overline{\omega} = \sum_{\sigma,\tau \in S_p} (\operatorname{sign}\sigma)(\operatorname{sign}\tau) z_{\tau(1),1} y_{\sigma(1),1} \cdots z_{\tau(p),1} y_{\sigma(p),1}.$$

Modulo the identities satisfied by $(M_{1,1}(E), *)$, we can rewrite the above polynomials as $\omega = (p!)^2 y_{1,1} z_{1,1} \cdots y_{p,1} z_{p,1}$ and $\overline{\omega} = (p!)^2 z_{1,1} y_{1,1} \cdots z_{p,1} y_{p,1}$.

For $\langle\lambda\rangle = ((n_1), (1^{n_2+1}), (n_3), (1^{n_2}))$ with $n_2 \geqslant 0$ we have

$$T_{\lambda(2)} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \vdots \\ \hline m-1 \\ \hline m+1 \\ \hline \end{array}, \qquad T_{\lambda(4)} = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \vdots \\ \hline m \\ \hline \end{array}$$

where the corresponding highest weight vector

$$\omega^+ = \sum_{\sigma,\tau} (\operatorname{sign}\sigma)(\operatorname{sign}\tau) y_{\sigma(1),1} z_{\tau(1),1} \cdots y_{\sigma(p),1} z_{\tau(p),1} y_{\sigma(p+1),1}$$

can be rewritten as $\omega^+ = (p+1)!p! y_{1,1} z_{1,1} \cdots y_{p,1} z_{p,1} y_{p+1,1}$.

Finally, for $\langle\lambda\rangle = ((n_1), (1^{n_2}), (n_3), (1^{n_2+1}))$ with $n_2 \geqslant 0$ we have

$$T_{\lambda(2)} = \begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline \vdots \\ \hline m \\ \hline \end{array}, \qquad T_{\lambda(4)} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \vdots \\ \hline m-1 \\ \hline m+1 \\ \hline \end{array}$$

where the corresponding highest weight vector is

$$\overline{\omega}^+ = \sum_{\sigma,\tau} (\operatorname{sign}\sigma)(\operatorname{sign}\tau) z_{\tau(1),1} y_{\sigma(1),1} \cdots z_{\tau(p),1} y_{\sigma(p),1} z_{\tau(p+1),1},$$

which can be rewritten as $\overline{\omega}^+ = (p+1)!p! z_{1,1} y_{1,1} \cdots z_{p,1} y_{p,1} z_{p+1,1}$.

The highest weight vectors are not polynomial identities of A, and ω, $\overline{\omega}$ are linearly independent. Therefore, since $\dim P_{n_1,n_2,n_3,n_2}(R) = 2$ with $n_2 > 0$, and $\dim P_{n_1,n_2+1,n_3,n_2}(R) = \dim P_{n_1,n_2,n_3,n_2+1}(R) = 1$ with $n_2 \geqslant 0$, we conclude that the multiplicities $m_{\langle\lambda\rangle}$ are as in the statement. □

3.2 Graded $*$-identities of $UT_{1,1}(E)$

The algebra of upper triangular matrices of size two, $UT_2(K)$, as graded algebra has only two possible grading (up to isomorphism), the trivial grading and the canonical $\mathbb{Z}_2$-grading given by $UT_2(K) = (UT_2(K))_0 \oplus (UT_2(K))_1$; where $(UT_2(K))_0 = Ke_{1,1} + Ke_{2,2}$ and $(UT_2(K))_1 = Ke_{1,2}$ (see [35]). We denote by $UT_{1,1}(K)$ the algebra $UT_2(K)$ with the canonical $\mathbb{Z}_2$-grading.

The superinvolutions on $UT_{1,1}$ coincide with the graded involutions, and the only graded involutions (up to equivalence) on $UT_{1,1}$ are given by

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{\circ} = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}, \qquad \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{s} = \begin{pmatrix} b & -c \\ 0 & a \end{pmatrix}$$

(see [24]).

We consider the algebra $UT_{1,1}(E) = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} : a, d \in E_0,\ b, c \in E_1 \right\}$. One can consider on $UT_{1,1}(E) = UT_{1,1}(K) \hat{\otimes} E$ the following graded involutions:

- $*$ induced by $(\circ, i_E)$,
- $*_1$ induced by (s, i_E),
- $*_2$ induced by $(\circ, -i_E)$,
- $*_3$ induced by $(s, -i_E)$,

where $* = *_3$, $*_1 = *_2$ and these involutions are defined as follows. If $a, b \in E_0$, $c \in E_1$

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}, \qquad \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^{*_1} = \begin{pmatrix} b & -c \\ 0 & a \end{pmatrix}.$$

Let $A = UT_{1,1}(E)$ be endowed with the involution $*$ given by

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}.$$

Then,

$$A_0^+ = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in E_0 \right\}, \quad A_0^- = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in E_0 \right\},$$
$$A_1^+ = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} : c \in E_1 \right\}, \quad A_1^- = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

We have the following relations in $K\langle Y, Z\rangle$ modulo $\mathrm{Id}^*(UT_{1,1}(E))$

(a) $[y_{i,0}, x] = 0$, (b) $y_{i,1}y_{j,1} = 0$, (c) $z_{i,1} = 0$,
(d) $[z_{i,0}, z_{j,0}] = 0$, (e) $z_{i,0} \circ y_{j,1} = 0$.

3.2.1 Y_0-proper polynomials on $UT_{1,1}(E)$

Since 1_A is in A_0^+, every proper polynomial is a linear combination of products of variables $y_{i,1}$, $z_{i,0}$, $z_{k,1}$ followed by a product of commutators. Consider the vector spaces $\Gamma_{n,l,m,k}(A)$ $(n, l, m, k \geqslant 0)$ of all multilinear Y_0-proper polynomials on $A = UT_{1,1}(E)$ in the sets of variables $y_{1,0}, \ldots, y_{n,0}, y_{1,1}, \ldots, y_{l,1}, z_{1,0}, \ldots, z_{m,0}, z_{1,1}, \ldots, z_{k,1}$ in the free algebra $K\langle Y \cup Z\rangle$ and denote by $\mathcal{I}$ the T_2^*-ideal of $K\langle Y \cup Z\rangle$ generated by the corresponding polynomials in the relations (a)-(e). The identity (e) gives us the following proposition.

Proposition 3.4. *The polynomial* $[y_{1,1}, z_{\sigma(i_1),0}, \ldots, z_{\sigma(i_k),0}] - (-2)^k z_{i_1,0}, \ldots, z_{i_k,0} y_{1,1}$ *belongs to* $\mathcal{I}$.

Note that modulo $\mathcal{I}$:

- if $k > 0$, then by $z_{1,1} = 0$, $\Gamma_{n,l,m,k}(A) = \{0\}$,
- if $n > 0$, then by $[y_{1,0}, x] = 0$, $\Gamma_{n,l,m,k}(A) = \{0\}$.

Thus, we only need to consider $\Gamma_{0,l,m,0}(A)$ with $l, m \geqslant 0$. Since $z_{i,0} \circ y_{j,1} = 0$ and $y_{i,1}y_{j,1} = 0$ then $\Gamma_{0,l,m,0}(A) = \{0\}$ if $l > 1$, and for the remaining cases we have:

- $\Gamma_{0,1,0,0}(A) = \mathrm{Span}\{y_{1,1}\}$,
- $\Gamma_{0,0,m,0}(A) = \mathrm{Span}\{z_{1,0}z_{2,0}\cdots z_{m,0}\}$, $m \leqslant 1$
- $\Gamma_{0,1,m,0}(A) = \mathrm{Span}\{z_{1,0}z_{2,0}\cdots z_{m,0}y_{1,1}\}$, $m \leqslant 1$

Modulo $\mathcal{I}$ the elements of the subspaces $\Gamma_{n,l,m,k}(A)$ are linearly independent, this proves the following theorem:

Theorem 3.3. *Let $(UT_{1,1}(E), *)$ be the algebra of 2×2 upper-triangular matrices over the Grassmann algebra, with the canonical $\mathbb{Z}_2$-grading and endowed with the involution $*$ given by*

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}.$$

Then its T_2^*-ideal of identities is generated, as a T_2^*-ideal, by

(i) $[y_{i,0}, x]$, *(ii)* $y_{i,1}y_{j,1}$, *(iii)* $z_{i,1}$,
(iv) $[z_{i,0}, z_{j,0}]$, *(v)* $z_{i,0} \circ y_{j,1}$.

3.2.2 Cocharacters of $(UT_{1,1}(E), *)$

Let $\langle\lambda\rangle$ be a multipartition of n, $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$, where $\lambda(i) \vdash n_i$, $1 \leqslant i \leqslant 4$. Let us consider the (n_1, n_2, n_3, n_4)-th cocharacter of $A = (UT_{1,1}(E), *)$,

$$\chi_{n_1,n_2,n_3,n_4}(A) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)}. \tag{3.3}$$

Theorem 3.4. *Let*

$$\chi_{n_1,n_2,n_3,n_4}(A) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)}$$

be the (n_1, n_2, n_3, n_4)-th cocharacter of $A = (UT_{1,1}(E), *)$.

(i) If $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ with $n_1 + n_3 \geqslant 1$, then $m_\lambda = 1$;

(i) If $\langle\lambda\rangle = ((n_1), (1), (n_3), \varnothing)$, then $m_\lambda = 1$.

In all remaining cases $m_\lambda = 0$.

Proof. From the identities in Theorem 3.3, $m_{\langle\lambda\rangle} = 0$ if $h(\lambda(1)) > 1$, or $h(\lambda(3)) > 1$, or $n_4 > 0$, or $n_2 > 1$. In this way we have to consider the cases $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ and $\langle\lambda\rangle = ((n_1), (1), (n_3), \varnothing)$. We deal with $\langle\lambda\rangle = ((n_1, \varnothing, (n_3), \varnothing)$ with $n_1 + n_3 > 0$. To this end, consider the tableaux

$$T_{\lambda(1)} = \boxed{1} \boxed{2} \boxed{\cdots} \boxed{n_1}\,, T_{\lambda(3)} = \boxed{n_1+1} \boxed{n_1+2} \boxed{\cdots} \boxed{n_1+n_3}\,, T_{\lambda(2)} = T_{\lambda(4)} = \varnothing.$$

The corresponding highest weight vector $\omega = (y_{1,0})^{n_1}(z_{1,0})^{n_3}$ is not an identity of A. Therefore, $m_{\langle\lambda\rangle} \geqslant 1$. By the identities of A, there is only one linearly independent highest weight vector corresponding to the multipartition $\langle\lambda\rangle$. Thus, $m_{\langle\lambda\rangle} = 1$.

For $\langle\lambda\rangle = ((n_1), (1), (n_3), \varnothing)$ consider the tableaux

$$T_{\lambda(1)} = \boxed{1} \boxed{2} \boxed{\cdots} \boxed{n_1}\,, \quad T_{\lambda(3)} = \boxed{n_1+1} \boxed{n_1+2} \boxed{\cdots} \boxed{n_1+n_3}\,,$$

$$T_{\lambda(2)} = \boxed{n_1+n_3+1}\,, \quad T_{\lambda(4)} = \varnothing.$$

The corresponding highest weight vector $\omega = (y_{1,0})^{n_1}(z_{1,0})^{n_3}y_{1,1}$ is not an identity of A and from the identities of A, it is the only one. Therefore, $m_{\langle\lambda\rangle} = 1$. □

3.3 Graded $*$-identities of $UT_{(0,1,0)}(E)$

Consider $UT_3(K)$ the algebra of upper triangular matrices of size three. As in [24] we consider $C = UT_3(K)_{(0,1,0)}$ the algebra $UT_3(K)$ with the $\mathbb{Z}_2$-grading given by

$$C_0 = Ke_{1,1} \oplus Ke_{2,2} \oplus Ke_{3,3} \oplus Ke_{1,3}, \qquad C_1 = Ke_{1,2} \oplus Ke_{2,3}$$

In [24] it was shown that, up to an isomorphism, the only superinvolution on C is given by

$$\begin{pmatrix} a & f & d \\ 0 & b & g \\ 0 & 0 & c \end{pmatrix}^{\circ} = \begin{pmatrix} c & g & -d \\ 0 & b & f \\ 0 & 0 & a \end{pmatrix}.$$

Then, on $G(UT_3) = UT_3(E)_{(0,1,0)}\hat{\otimes}E$ we consider the following graded involutions:

- $*$ induced by $(\circ, i_E)$,
- $*_1$ induced by $(\circ, -i_E)$.

Let B be the algebra $G(UT_3)$ with involution $*$. That is, if $a, b, c, d \in E_0$, $f, g \in E_1$,

$$\begin{pmatrix} a & f & d \\ 0 & b & g \\ 0 & 0 & c \end{pmatrix}^{*} = \begin{pmatrix} c & g & -d \\ 0 & b & f \\ 0 & 0 & a \end{pmatrix}.$$

Then,

$$B_0^+ = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & a \end{pmatrix} : a, b \in E_0 \right\}, \quad B_0^- = \left\{ \begin{pmatrix} d & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix} : c, d \in E_0 \right\}$$

$$B_1^+ = \left\{ \begin{pmatrix} 0 & g & 0 \\ 0 & 0 & g \\ 0 & 0 & 0 \end{pmatrix} : g \in E_1 \right\}, \quad B_1^- = \left\{ \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & -f \\ 0 & 0 & 0 \end{pmatrix} : f \in E_1 \right\}$$

Let P_{n_1,n_2,n_3,n_4} be the multilinear polynomials in the variables $y_{1,0}, \ldots, y_{n_1,0}, y_{1,1}, \ldots, y_{n_2,1}$, $z_{1,0}, \ldots, z_{n_3,0}, z_{1,1}, \ldots, z_{n_4,1}$ where $y_{i,j}$ is symmetric of degree j and $z_{i,j}$ is skew symmetric of degree j. The following polynomials in $K\langle Y, Z\rangle$ are identities of B.

(i) $[y_{i,0}, y_{j,0}]$,
(ii) $[y_{i,0}, z_{j,0}]$
(iii) $y_{i,1} \circ y_{j,1}$,
(iv) $[y_{i,1}, z_{j,1}]$,
(v) $z_{i,1} \circ z_{j,1}$,
(vi) $y_{i,1}y_{j,0}y_{k,1} + y_{k,1}y_{j,0}y_{i,1}$,
(vii) $z_{i,1}y_{j,0}z_{k,1} + z_{k,1}y_{j,0}z_{i,1}$,
$(viii)$ $y_{i,1}y_{j,0}z_{k,1} - z_{k,1}y_{j,0}y_{i,1}$,
(ix) $z_{i,0}z_{j,0}z_{k,0} - z_{k,0}z_{j,0}z_{i,0}$,
(x) $[z_{i,0}, z_{j,0}][z_{k,0}, z_{l,0}]$,
(xi) $x_{k,1}[z_{i,0}, z_{j,0}]$, $\quad x \in \{y, z\}$,
(xii) $[z_{i,0}, z_{j,0}]x_{k,1}$, $\quad x \in \{y, z\}$,
$(xiii)$ $z_{i,0}x_{k,1}z_{j,0}$, $\quad x \in \{y, z\}$,
(xiv) $x_{i,1}z_{k,0}w_{j,1}$, $\quad x, w \in \{y, z\}$,
(xv) $x_{i,1}x_{j,1}x_{k,1}$, $\quad x_{*,1} \in \{y_{*,1}, z_{*,1}\}$.

Let $\mathcal{J}$ be the T_2^*-ideal of $K\langle Y, Z\rangle$ generated by these polynomials. We prove

Theorem 3.5. *Let $B = (G(UT_3), *)$ be the Grassmann envelope of the algebra of 3×3 upper-triangular matrices, with the $\mathbb{Z}_2$-grading induced by the tuple $(0, 1, 0)$, and endowed with the involution $*$ given by*

$$\begin{pmatrix} a & f & d \\ 0 & b & g \\ 0 & 0 & c \end{pmatrix}^* = \begin{pmatrix} c & g & -d \\ 0 & b & f \\ 0 & 0 & a \end{pmatrix}.$$

Then its T_2^*-ideal is generated, as a T_2^*-ideal, by the identities $(i) - (xv)$.

Since $B_0 \cdot B_1$, $B_1 \cdot B_0 \subset B_1$, from the identity (xv) we obtain that if $n_2 + n_4 > 2$ then $P_{n_1,n_2,n_3,n_4}(B) = 0$. Hence, we consider the following three cases:

(1) $n_2 = 0$ and $n_4 = 0$.

Considering the multilinear Y_0-proper polynomials $\Gamma_{n_1,0,n_3,0}(B)$, since $[B_0^-, B_0^-] \in B_0^-$, then $\Gamma_{n_1,0,n_3,0}(B) = 0$ if $n_1 > 1$. From the identity $z_{i,0}z_{j,0}z_{k,0} - z_{k,0}z_{j,0}z_{i,0} = 0$ we have $[z_{i,0}, z_{j,0}, z_{k,0}] = -2z_{k,0}[z_{i,0}, z_{j,0}]$. Thus $\Gamma_{0,0,n_3,0}(B) = \text{Span}\{[z_{i_1,0}, z_{i_2,0}, \ldots, z_{i_{n_3},0}]\}$.

By $[[x_1, x_2], [x_3, x_4]] = [x_1, x_2, x_3, x_4] - [x_1, x_2, x_4, x_3]$, $[z_{i,0}, z_{j,0}][z_{k,0}, z_{l,0}] = 0$, and the Jacobi Identity, we conclude that modulo $\mathcal{J}$,

$$\Gamma_{0,0,n_3,0}(B) = \text{Span}\{[z_{l,0}, z_{1,0}, \ldots, \widehat{z_{l,0}}, \ldots, z_{n_3,0}]\},$$

where $2 \leqslant l \leqslant n_3$ and $\widehat{z_{l,0}}$ means that the variable $z_{l,0}$ can be omitted.

Suppose

$$\sum_{l=2}^{n_3} \alpha_l [z_{l,0}, z_{1,0}, \ldots, \widehat{z_{l,0}}, \ldots, z_{n_3,0}] = 0 \qquad \mathrm{mod}\ (\mathrm{Id}(B)),$$

and that there exists $\alpha_l \neq 0$, then by the evaluation $z_{l,0} = e_{1,3}$ and $z_{i,0} = e_{1,1} - e_{3,3}$, $i \in \{1, 2, \ldots, n_3\} \backslash \{l\}$, we obtain $\alpha_l (-2)^{n_3 - 1} e_{1,3} = 0$, a contradiction. Thus, the commutators $[z_{l,0}, z_{1,0}, \ldots, \widehat{z_{l,0}}, \ldots, z_{n_3,0}]$ $(2 \leqslant l \leqslant n_3)$ are linearly independent.

(2) $n_2 = 1$ and $n_4 = 0$ (the case $n_2 = 0$, $n_4 = 1$ is analogous).

By $[y_{i,0}, y_{j,0}] = 0$, $[y_{i,0}, z_{j,0}] = 0$, $z_{i,0} y_{1,1} z_{j,0} = 0$ and the identities $z_{j_1,0} \cdots z_{j_{n_3},0} y_{1,1} = z_{1,0} \cdots z_{n_3,0} y_{1,1}$ and $y_{1,1} z_{j_1,0} \cdots z_{j_{n_3},0} = y_{1,1} z_{1,0} \cdots z_{n_3,0}$ we obtain that $P_{n_1,1,n_3,0}$ is spanned modulo $\mathcal{J}$ by the monomials

$$y_{i_1,0} \cdots y_{i_m,0} z_{1,0} \cdots z_{n_3,0} y_{1,1} y_{k_1,0} \cdots y_{k_r,0}, \quad y_{i_1,0} \cdots y_{i_m,0} y_{1,1} y_{k_1,0} \cdots y_{k_r,0} z_{1,0} \cdots z_{n_3,0},$$

where $m + r = n_1$, $i_1 < i_2 < \cdots < i_m$, $k_1 < k_2 < \cdots < k_r$.

To prove these monomials are linearly independent we consider the polynomial

$$\begin{aligned} f = &\sum_{I,K} \alpha_{I,K} y_{i_1,0} \cdots y_{i_m,0} z_{1,0} \cdots z_{n_3,0} y_{1,1} y_{k_1,0} \cdots y_{k_r,0} \\ &+ \sum_{J,H} \beta_{J,H} y_{j_1,0} \cdots y_{j_s,0} y_{1,1} y_{h_1,0} \cdots y_{h_t,0} z_{1,0} \cdots z_{n_3,0} \end{aligned}$$

with $m + r = s + t = n_1$, $I = \{i_1, \ldots, i_m\}$, $K = \{k_1, \ldots, k_r\}$, $J = \{j_1, \ldots, j_s\}$, $H = \{h_1, \ldots, h_t\}$, and $i_1 < \cdots < i_m$, $k_1 < \cdots < k_r$, $j_1 < \cdots < j_s$, $h_1 < \cdots < h_t$.

Suppose $f \in \mathrm{Id}(B)$ and that there exists $\alpha_{I,K} \neq 0$ or $\beta_{I,K} \neq 0$ for some I and K. Now, if we consider $y_{i_l,0} = e_{1,1} + e_{3,3}$ for $l = 1, \ldots, m$, $y_{k_c,0} = e_{2,2}$ for $c = 1, \ldots, r$, $y_{1,1} = e_1(e_{1,2} + e_{2,3})$ and $z_{j,0} = e_{1,1} - e_{3,3}$ for $j = 1, \ldots, n_3$, we obtain $\alpha_{I,K} e_1 e_{1,2} \pm \beta_{I,K} e_1 e_{2,3} = 0$. Therefore, $\alpha_{I,K} = \beta_{I,K} = 0$, a contradiction.

(3) $n_2 = 2$ and $n_4 = 0$ (the cases $n_2 = 0$, $n_4 = 2$ and $n_2 = 1$, $n_4 = 1$ are analogous).

Since $z_{2,0} y_{1,1} y_{2,1} z_{1,0} = z_{1,0} y_{1,1} y_{2,1} z_{2,0}$ and $y_{2,0} y_{1,1} y_{2,1} y_{1,0} = y_{1,0} y_{1,1} y_{2,1} y_{2,0}$ (because $B_1^+ \cdot B_1^+ \subset B_0^-$), by the identities above we obtain $P_{n_1,2,n_3,0}$ is spanned, modulo $\mathcal{J}$, by

$$y_{i_1,0} \cdots y_{i_m,0} z_{1,0} \cdots z_{l,0} y_{1,1} y_{j_1,0} \cdots y_{j_s,0} y_{2,1} y_{i_{m+1},0} \cdots y_{i_{n_1-s},0} z_{l+1,0} \cdots z_{n_3,0},$$

with $i_1 < i_2 < \cdots < i_{n_1 - s}$, $j_1 < j_2 < \cdots < j_s$. These monomials are linearly independent modulo $\mathcal{J}$: consider the polynomial

$$f = \sum_{I,J,l} \alpha_{I,J}^l y_{i_1,0} \cdots y_{i_m,0} z_{1,0} \cdots z_{l,0} y_{1,1} y_{j_1,0} \cdots y_{j_s,0} y_{2,1} y_{i_{m+1},0} \cdots y_{i_{n_1-s},0} z_{l+1,0} \cdots z_{n_3,0}$$

where $I = \{i_1, \ldots, i_{n_1-s}\}$, $J = \{j_1, \ldots, j_s\}$, $i_1 < \cdots < i_{n_1-s}$, $j_1 < \cdots < j_s$.

Suppose $f \in \mathrm{Id}(B)$ and there exists $\alpha_{I,J}^l \neq 0$ for some I, K, l. Considering the evaluation $y_{i_n,0} = e_{1,1} + e_{3,3}$ for $n = 1, \ldots, n_1 - s$, $y_{j_k} = e_{2,2}$ for $k = 1, \ldots, s$, $y_{j,1} = e_j(e_{1,2} + e_{2,3})$ for $j = 1, 2$, and $z_{c,0} = e_{1,1} - e_{3,3}$ for $c = 1, \ldots, n_3$ we obtain $\alpha^l_{I,J} e_1 e_2 e_{1,3} = 0$. Thus, $\alpha^l_{I,J} = 0$, a contradiction.

As a consequence of (1)-(3) we have the proof of the Theorem 3.5.

3.3.1 Cocharacters of B

Consider the spaces $P_{n_1,n_2,n_3,n_4}(B)$ $(n_1, n_2, n_3, n_4 \geqslant 0)$ of all multilinear polynomials on B in the variables $y_{1,0}, \ldots, y_{n_1,0}, y_{1,1}, \ldots, y_{n_2,1}, z_{1,0}, \ldots, z_{n_3,0}, z_{1,1}, \ldots, z_{n_4,1}$ in the free algebra $K\langle Y \cup Z\rangle = K\langle y_{1,0}, y_{1,1}, z_{1,0}, z_{1,1}, \ldots\rangle$. Let $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4))$ be a multipartition of $n = n_1 + n_2 + n_3 + n_4$, where $\lambda(i) \vdash n_i$, $1 \leqslant i \leqslant 4$, and consider the (n_1, n_2, n_3, n_4)-th cocharacter of B,

$$\chi_{n_1,n_2,n_3,n_4}(B) = \sum_{\langle\lambda\rangle\vdash(n_1,\ldots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \chi_{\lambda(2)} \otimes \chi_{\lambda(3)} \otimes \chi_{\lambda(4)}. \tag{3.4}$$

Our next aim is to compute the multiplicities of the cocharacters, starting with the case when we only have variables of degree zero, that is $n_2 + n_4 = 0$.

In order to find the cocharacters of B, we consider Young diagrams of shape $\langle\lambda\rangle = (\lambda(1), \lambda(2), \lambda(3), \lambda(4)) \vdash (n_1, n_2, n_3 n_4)$ and the corresponding standard tableaux.

Let ω be the highest weight vector associated to $\langle\lambda\rangle = (\lambda(1), \varnothing, \lambda(3), \varnothing)$. By the identities 1, 2, 9 and 10 we conclude that $\omega = 0$ in the following situations:

$$h(\lambda(1)) > 1, \quad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \subseteq T_{\lambda(3)}, \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \subseteq T_{\lambda(3)}.$$

Therefore, we consider the cases $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ and
$\langle\lambda\rangle = ((n_1), \varnothing, (n_3 - 1, 1), \varnothing)$.

Proposition 3.5. *If* $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ *with* $n_1 + n_3 > 0$ *or* $\langle\lambda\rangle = ((n_1), \varnothing, (n_3 - 1, 1), \varnothing)$ *for* $n_3 > 1$, *then* $m_{\langle\lambda\rangle} = 1$ *in the cocharacter (3.4).*

Proof. Since $P_{n_1,0,n_3,0}(B)$ is spanned by the monomial $y_{1,0} \cdots y_{n_1,0} z_{1,0} \cdots z_{n_3,0}$ we have that the only (linearly independent) highest weight vector associated to $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ is $\omega = (y_{1,0})^{n_1}(z_{1,0})^{n_3}$ which is not an identity of A. Thus, if $\langle\lambda\rangle = ((n_1), \varnothing, (n_3), \varnothing)$ then $m_{\langle\lambda\rangle} = 1$.

Now we examine the case $\langle\lambda\rangle = ((n_1), \varnothing, (n_3 - 1, 1), \varnothing)$. The tableaux with possibly linearly independent highest weight vectors are

$$T_{\lambda(3)} = \begin{array}{|c|c|c|c|c|c|c|}\hline n_1+1 & n_1+2 & \cdots & n_1+i-1 & n_1+i+1 & \cdots & n_1+n_3 \\ \hline n_1+i \\ \cline{1-1} \end{array}$$

$$T_{\lambda(1)} = \boxed{1} \boxed{2} \boxed{\cdots} \boxed{n_1}, \quad T_{\lambda(2)} = \varnothing, \qquad T_{\lambda(4)} = \varnothing,$$

Then, from the identity (ix) for each $i = 2, \ldots, n_3$ the corresponding highest weight vector has the form

$$\begin{aligned} \omega_i &= (y_{1,0})^{n_1} \hat{z}_{1,0}(z_{1,0})^k \hat{z}_{2,0}(z_{1,0})^l, \qquad k + l = n_3 - 2 \\ &= \begin{cases} 0, & k \text{ odd} \\ (y_{1,0})^{n_1}[z_{1,0}, z_{2,0}](z_{1,0})^{n_3-2}, & k \text{ even.} \end{cases} \end{aligned}$$

Thus, if $\langle\lambda\rangle = ((n_1), \varnothing, (n_3 - 1, 1), \varnothing)$ then $m_{\langle\lambda\rangle} = 1$. □

Next, we consider the cases when $n_2 + n_3 = 1$.

Proposition 3.6. *If $\langle\lambda\rangle = ((p + q, p), (1), (n_3), \varnothing)$, where p, $q \geqslant 0$ and $n_3 > 0$, then $m_{\langle\lambda\rangle} = 2(q + 1)$ in the cocharacter (3.4).*

Proof. Let $\langle\lambda\rangle = ((p + q, p), (1), (n_3), \varnothing)$. We determine the linearly independent highest weight vectors associated to the standard Young tableaux of shape $\langle\lambda\rangle$ which are not identities of B. Let $T_{\lambda(i)}$ be the tableau associated to $\lambda(i) \vdash n_i$ in $\langle\lambda\rangle$.

A basis for $P_{n_1,1,n_3,0}(B)$ is given by

$$y_{i_1,0} \cdots y_{i_m,0} z_{1,0} \cdots z_{n_3,0} y_{1,1} y_{k_1,0} \cdots y_{k_r,0}, \quad y_{i_1,0} \cdots y_{i_m,0} y_{1,1} y_{k_1,0} \cdots y_{k_r,0} z_{1,0} \cdots z_{n_3,0},$$

where $m + r = n_1$, $i_1 < i_2 < \cdots < i_m$, $k_1 < k_2 < \cdots < k_r$. Given a positive integer t_1 and $T_{\lambda(2)} = \boxed{t_1}$, we fill $T_{\lambda(3)}$ with positive integers, all larger than t_1 or all less than t_1. Also, since $[y_{i,0}, y_{j,0}] = 0$, we need t_1 to be larger than the integers in the first p positions of the first row of $T_{\lambda(1)}$ and less than the integers in the second row of $T_{\lambda(1)}$, or vice versa.

These remarks reduce our study to the following cases.

The first case is

$T_{\lambda(1)} =$

$n_3 + 1$	$n_3 + 2$	$\cdots$	$n_3 + p$	$\cdots$	$t_1 - 1$	$t_1 + 1$	$\cdots$	t_2
$t_2 + 1$	$t_2 + 2$	$\cdots$	n					

$$T_{\lambda(2)} = \boxed{t_1}, \quad T_{\lambda(3)} = \boxed{1} \boxed{2} \boxed{\cdots} \boxed{n_3}, \quad T_{\lambda(4)} = \varnothing,$$

where $n_3 < t_1 < t_2 < n$, and the corresponding highest weight vector is

$$f_{t_1}^1 = (z_{1,0})^{n_3} \underbrace{\bar{y}_{1,0} \cdots \tilde{y}_{1,0}}_{p} (y_{1,0})^{i_1-p} y_{1,1} \underbrace{\bar{y}_{2,0} \cdots \tilde{y}_{2,0}}_{p} (y_{1,0})^{i_2-p}.$$

The second case is

$T_{\lambda(1)} =$

1	2	$\cdots$	p	$\cdots$	$t_1 - 1$	$t_1 + n_3 + 1$	$\cdots$	t_2
$t_2 + 1$	$t_2 + 2$	$\cdots$	n					

$$T_{\lambda(2)} = \boxed{t_1}, \quad T_{\lambda(3)} = \boxed{t_1+1 \mid t_1+2 \mid \cdots \mid t_1+n_3}, \quad T_{\lambda(4)} = \varnothing,$$

where $t_1 < t_2 < n$, and the corresponding highest weight vector is

$$f^2_{t_1} = \underbrace{\bar{y}_{1,0}\cdots\tilde{y}_{1,0}}_{p}(y_{1,0})^{i_1-p}y_{1,1}(z_{1,0})^{n_3}\underbrace{\bar{y}_{2,0}\cdots\tilde{y}_{2,0}}_{p}(y_{1,0})^{i_2-p}.$$

We focus on the first case. For $i_1 = p,\ p+1,\ \ldots,\ p+q$, consider the generic highest weight vector

$$f^1_{i_1} = (z_{1,0})^{n_3}\underbrace{\bar{y}_{1,0}\cdots\tilde{y}_{1,0}}_{p}(y_{1,0})^{i_1-p}y_{1,1}\underbrace{\bar{y}_{2,0}\cdots\tilde{y}_{2,0}}_{p}(y_{1,0})^{i_2-p},$$

$i_1 + i_2 = n_1$. Similarly as in Proposition 2.19, $f^1_{i_1}$ can be written as

$$f^1_{i_1} = \sum_{j=0}^{p}(-1)^j\binom{p}{j}(z_{1,0})^{n_3}(y_{1,0})^{i_1-j}(y_{2,0})^j y_{1,1}(y_{1,0})^{i_2-p+j}(y_{2,0})^{p-j}.$$

We consider the evaluation $y_{1,0} = \alpha_1(e_{1,1}+e_{3,3}) + \beta_1 e_{2,2}$, $y_{2,0} = \alpha_2(e_{1,1}+e_{3,3}) + \beta_2 e_{2,2}$, $z_{1,0} = \zeta(e_{1,1}-e_{3,3}) + \delta_1 e_{1,3}$ and $y_{1,1} = \gamma e_1(e_{1,2}+e_{2,3})$. Then,

$$f^1_{i_1} = \left[\zeta^{n_3}\gamma\alpha_1^{i_1-p}\beta_1^{i_2-p}e_1(\alpha_1\beta_2-\alpha_2\beta_1)^p\right]e_{1,2}.$$

Notice that the coefficient $\zeta^{n_3}\gamma(\alpha_1\beta_2-\alpha_2\beta_1)^p e_1\cdot e_{1,2}$ is present in every evaluation of $f^1_{i_1}$, for $i_1 = p,\ p+1,\ \ldots,\ p+q$. Therefore, to prove linear independence, we consider only $\alpha_1^{i_1-p}\beta_1^{i_2-p}$.

If there exist scalars μ_{i_1}'s such that $\sum_{i_1=p}^{p+q}\mu_{i_1}f^1_{i_1} = 0$, then for every $\alpha_1,\ \beta_1 \in K$, we have

$$\sum_{i_1=p}^{p+q}\mu_{i_1}\alpha_1^{i_1-p}\beta_1^{i_2-p} = 0.$$

Therefore, μ_{i_1} vanish for all i_1, and $\{f^1_{i_1} : i_1 = p, p+1, \ldots, p+q\}$ are linearly independent.

Similarly the set $\{f^2_{i_1} : i_1 = p, p+1, \ldots, p+q\}$ is linearly independent. Since $f^1_{i_1} = \alpha e_{1,2}$ and $f^2_{i_1} = \beta e_{2,3}$, then $m_{\langle\lambda\rangle} \geqslant 2(q+1)$. We have considered all possibilities for the highest weight vectors being linearly independent, as well as generic evaluations, and thus we conclude that $m_{\langle\lambda\rangle} = 2(q+1)$. □

Similarly, in the case of one skew-symmetric variable of degree 1, we have

Proposition 3.7. *If* $\langle\lambda\rangle = ((p+q,p),\varnothing,(n_3),(1))$*, where* $p,q \geqslant 0$ *and* $n_3 > 0$*, then* $m_{\langle\lambda\rangle} = 2(q+1)$ *in the cocharacter (3.4).*

In the cases $\langle\lambda\rangle = ((p+q,p),(1),(n_3),\varnothing)$ and $\langle\lambda\rangle = ((p+q,p),\varnothing,(n_3),(1))$, the division of the two families of highest weight vectors depends on the presence of $z_{1,0}$, therefore, we consider separately the case when $n_3 = 0$.

Proposition 3.8. *If* $\langle\lambda\rangle = ((p+q,p),(1),\varnothing,\varnothing)$*, where* p*,* $q \geqslant 0$*, then* $m_{\langle\lambda\rangle} = q+1$ *in the cocharacter (3.4).*

Proof. Considering the corresponding tableaux and the identities of B, we have that given $T_{\lambda(2)} = \boxed{t_1}$, then we need t_1 to be larger than the integers in the first p positions of the first row of $T_{\lambda(1)}$ and less than the integers in the second row of $T_{\lambda(1)}$, or vice versa. Taking into account standard tableaux and looking for highest weight vectors that are linearly independent, we find that the options are determined by

$$T_{\lambda(1)} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & p & \cdots & t_1-1 & t_1+1 & \cdots & t_2 \\ \hline t_2+1 & t_2+2 & \cdots & n \\ \cline{1-4}\end{array}$$

$$T_{\lambda(2)} = \boxed{t_1}, \quad T_{\lambda(3)} = \varnothing, \quad T_{\lambda(4)} = \varnothing,$$

where $t_1 < t_2 < n$, and the corresponding highest weight vector is

$$f_{i_1} = \underbrace{\bar{y}_{1,0}\cdots\tilde{y}_{1,0}}_{p}(y_{1,0})^{i_1-p}y_{1,1}\underbrace{\bar{y}_{2,0}\cdots\tilde{y}_{2,0}}_{p}(y_{1,0})^{i_2-p}.$$

Through calculations similar to those in the proof of Proposition 3.6, we establish that the highest weight vectors f_{i_1} are linearly independent for $i_1 = p, p+1, \ldots, p+q$. Considering that we encompass all possibilities, we conclude that $m_{\langle\lambda\rangle} = q+1$. ☐

Analogously, we have the case of one skew-symmetric variable of degree 1.

Proposition 3.9. *If* $\langle\lambda\rangle = ((p+q,p),\varnothing,\varnothing,(1))$*, where* p*,* $q \geqslant 0$*, then* $m_{\langle\lambda\rangle} = q+1$ *in the cocharacter (3.4).*

Let us now consider the case where $n_2+n_4 = 2$. First, we recall that $P_{n_1,2,n_3,0}(A)$ is spanned by polynomials of the form

$$y_{i_1,0}\cdots y_{i_m,0}z_{1,0}\cdots z_{l,0}y_{1,1}y_{j_1,0}\cdots y_{j_s,0}y_{1,1}y_{i_{m+1},0}\cdots y_{i_{n_1-s},0}z_{l+1,0}\cdots z_{n_3,0}.$$

Let $\langle\lambda\rangle = ((p+q,p),(2),(n_3),\varnothing)$, where p, q, $n_3 \geqslant 0$, then the corresponding highest weight vectors that are not zero (considering the identities of B) will be of the form

$$\underbrace{\bar{y}_{1,0}\cdots\tilde{y}_{1,0}}_{p}(y_{1,0})^{i_1-p}(z_{1,0})^{j_1}y_{1,1}\underbrace{\bar{y}_{2,0}\cdots\tilde{y}_{2,0}}_{p}(y_{1,0})^{i_2-p}y_{1,1}(y_{1,0})^{i_3}(z_{1,0})^{j_2}.$$

Consider separately the parts

$$m = y_{1,1}y_{k_1,0}\cdots y_{k_p,0}(y_{1,0})^{i_2-p}y_{1,1}$$

where $k_i \in \{1,2\}$, and a generic evaluation $y_{i,0} = a_i(e_{1,1}+e_{3,3}) + b_ie_{2,2}$, $y_{1,1} = c(e_{1,2}+e_{2,3})$ where a_i, $b_i \in E_0$ and $c \in E_1$. Then $m = cbc \cdot e_{1,3} = 0$. Therefore, if $\langle\lambda\rangle = ((p+q,p),(2),(n_3),\varnothing)$ then $m_{\langle\lambda\rangle} = 0$. Similarly for $\langle\lambda\rangle = ((p+q,p),\varnothing,(n_3),(2))$.

Next, we consider the case when $\langle\lambda\rangle = ((p+q,p),(1),(n_3),(1))$.

Proposition 3.10. *If* $\langle\lambda\rangle = ((p+q,p),(1),(n_3),(1))$, *where* $p,q \geqslant 0$, *then* $m_{\langle\lambda\rangle} = n_3(q+1)$ *in the cocharacter (3.4).*

Proof. Observe that $P_{n_1,1,n_3,1}(A)$ is spanned by polynomials of the form

$$y_{i_1,0}\cdots y_{i_m,0}z_{1,0}\cdots z_{l,0}y_{1,1}y_{j_1,0}\cdots y_{j_s,0}z_{1,1}y_{i_{m+1},0}\cdots y_{i_{n_1-s},0}z_{l+1,0}\cdots z_{n_3,0}.$$

The possible highest weight vectors associated to standard tableaux of shape $\langle\lambda\rangle$, which are not identities of B, are described by

$$f_{i_1,i_2,j_1} = \underbrace{\bar{y}_{1,0}\cdots\tilde{y}_{1,0}}_{p}(y_{1,0})^{i_1-p}(z_{1,0})^{j_1}y_{1,1}\underbrace{\bar{y}_{2,0}\cdots\tilde{y}_{2,0}}_{p}(y_{1,0})^{i_2-p}z_{1,1}(y_{1,0})^{i_3}(z_{1,0})^{j_2},$$

where $i_1+i_2+i_3 = n_1$, $p \leqslant i_1, i_2 \leqslant p+q$ and $j_1+j_2 = n_3$.

Again, we rewrite f_{i_1,i_2,j_1} in the following manner

$$f_{i_1,i_2,j_1} = \sum_{k=0}^{p}(-1)^k\binom{p}{k}(y_{1,0})^{i_1-k}(y_{2,0})^k(z_{1,0})^{j_1}y_{1,1}(y_{1,0})^{i_2-p+k}(y_{2,0})^{p-k}z_{1,1}(y_{1,0})^{i_3}(z_{1,0})^{j_2},$$

and considering the evaluation $y_{1,0} = \alpha_1(e_{1,1}+e_{3,3})+\beta_1e_{2,2}$, $y_{2,0} = \alpha_2(e_{1,1}+e_{3,3})+\beta_2e_{2,2}$, $z_{1,0} = \zeta(e_{1,1}-e_{3,3})+\delta_1e_{1,3}$, $y_{1,1} = e_1(e_{1,2}+e_{2,3})$ and $z_{1,1} = e_2(e_{1,2}-e_{2,3})$ we obtain

$$f_{i_1,i_2,j_1} = \left[\zeta^{n_3}(\alpha_1\beta_2-\alpha_2\beta_1)^pe_1e_2\right]\left[(-1)^{j_2}\alpha_1^{n_1-i_2-p}\beta_1^{i_2-p}\right]e_{1,3}.$$

Since the term $\zeta^{n_3}(\alpha_1\beta_2-\alpha_2\beta_1)^pe_1e_2e_{1,3}$ is common to each f_{i_1,i_2,j_1}, we can reduce to considering only the terms $(-1)^{j_2}\alpha_1^{n_1-i_2-p}\beta_1^{i_2-p}$, which depend on i_2 and j_2, where $p \leqslant i_2 \leqslant p+q$ and $0 \leqslant j_2 \leqslant n_3$.

Suppose that there exist $\mu_{i_2,j_2} \in F$ such that

$$\sum_{i_2,j_2}\mu_{i_2,j_2}(-1)^{j_2}\alpha_1^{n_1-i_2-p}\beta_1^{i_2-p} = 0,$$

for all $\alpha_1,\beta_1 \in F$. Then,

$$\sum_{i_2=p}^{p+q}\left(\mu_{i_2,0}-\mu_{i_2,1}+\cdots+(-1)^{n_3}\mu_{i_2,n_3}\right)\alpha_1^{n_1-i_2-p}\beta_1^{i_2-p} = 0,$$

so for each i_2

$$\mu_{i_2,0}-\mu_{i_2,1}+\cdots+(-1)^{n_3}\mu_{i_2,n_3} = 0.$$

We conclude that fixing i_2 we have n_3 linearly independent variables $\mu_{i_2,k}$. Therefore, $m_{\langle\lambda\rangle} \geqslant n_3(q+1)$. As we consider generic evaluations, we get $m_{\langle\lambda\rangle} = n_3(q+1)$. □

Similarly to the case when $n_2+n_4 = 1$, we have some differences between the cases $n_3 > 0$ and $n_3 = 0$. Next, we consider the case $n_2 = n_4 = 1$ and $n_3 = 0$.

Proposition 3.11. *If $\langle\lambda\rangle = ((p+q,p),(1),\varnothing,(1))$, where p, $q \geqslant 0$, then $m_{\langle\lambda\rangle} = q+1$ in the cocharacter (3.4).*

Proof. Similarly to the previous proposition, we have that the possible linearly independent highest weight vectors that are not polynomial identities of B are represented by

$$\begin{aligned} f_{i_1,i_2,j_1} &= \underbrace{\bar{y}_{1,0}\cdots\tilde{y}_{1,0}}_{p}(y_{1,0})^{i_1-p}y_{1,1}\underbrace{\bar{y}_{2,0}\cdots\tilde{y}_{2,0}}_{p}(y_{1,0})^{i_2-p}z_{1,1}(y_{1,0})^{i_3} \\ &= \sum_{k=0}^{p}(-1)^k\binom{p}{k}(y_{1,0})^{i_1-k}(y_{2,0})^k y_{1,1}(y_{1,0})^{i_2-p+k}(y_{2,0})^{p-k}z_{1,1}(y_{1,0})^{i_3} \end{aligned}$$

where $i_1+i_2+i_3 = n_1$ and $p \leqslant i_1, i_2 \leqslant p+q$. Considering $y_{1,0} = \alpha_1(e_{1,1}+e_{3,3})+\beta_1 e_{2,2}$, $y_{2,0} = \alpha_2(e_{1,1}+e_{3,3})+\beta_2 e_{2,2}$, $y_{1,1} = e_1(e_{1,2}+e_{2,3})$ and $z_{1,1} = e_2(e_{1,2}-e_{2,3})$,

$$f_{i_1,i_2,j_1} = \left[(\alpha_1\beta_2-\alpha_2\beta_1)^p e_1 e_2\right]\left[\alpha_1^{n_1-i_2-p}\beta_1^{i_2-p}\right]e_{1,3}.$$

Similarly to the previous propositions, we conclude that $m_{\langle\lambda\rangle} = q+1$. □

Based on the above presented results and considering the graded $*$-identities of A, we have the description of the cocharacters of B.

Theorem 3.6. *Let*

$$\chi_{n_1,n_2,n_3,n_4}(B) = \sum_{\langle\lambda\rangle\vdash(n_1,\dots,n_4)} m_{\langle\lambda\rangle}\chi_{\lambda(1)}\otimes\chi_{\lambda(2)}\otimes\chi_{\lambda(3)}\otimes\chi_{\lambda(4)}$$

be the (n_1,n_2,n_3,n_4)-cocharacter of B. Then

(*i*) $m_{\langle\lambda\rangle} = 1$, *if* $\langle\lambda\rangle = ((n_1),\varnothing,(n_3),\varnothing)$, *where* $n_1+n_3 > 0$ *or if* $\langle\lambda\rangle = ((n_1),\varnothing,(n_3-1,1),\varnothing)$ *and* $n_3 > 1$.

(*ii*) $m_{\langle\lambda\rangle} = q+1$, *if* $\langle\lambda\rangle = ((p+q,p),(1),\varnothing,\varnothing)$ *or* $\langle\lambda\rangle = ((p+q,p),\varnothing,\varnothing,(1))$, *where* $n_3 > 0$ *and* $n_1 = 2p+q$.

(*iii*) $m_{\langle\lambda\rangle} = 2(q+1)$, *if* $\langle\lambda\rangle = ((p+q,p),(1),(n_3),\varnothing)$ *or* $\langle\lambda\rangle = ((p+q,p),\varnothing,(n_3),(1))$, *where* $n_3 > 0$ *and* $n_1 = 2p+q$.

(*iv*) $m_{\langle\lambda\rangle} = n_3(q+1)$, *if* $\langle\lambda\rangle = ((p+q,p),(1),(n_3),(1))$, *where* $n_3 > 0$ *and* $n_1 = 2p+q$.

(*v*) $m_{\langle\lambda\rangle} = q+1$, *if* $\langle\lambda\rangle = ((p+q,p),(1),\varnothing,(1))$, *where* $n_1 = 2p+q$.

In all remaining cases $m_{\langle\lambda\rangle} = 0$.

4 The embedding problem

Let A and B be two K-algebras. A natural question arises: If $\mathrm{Id}(A) = \mathrm{Id}(B)$, then is $A \cong B$? In general, this is not true. For example, for any algebra A, it holds that $\mathrm{Id}(A) = \mathrm{Id}(A \oplus A)$. Another example to consider is $\mathrm{Id}(\mathbb{H}_{\mathbb{R}}) = \mathrm{Id}(M_2(\mathbb{R}))$, but the matrix algebra $M_2(\mathbb{R})$ and the Quaternion algebra $\mathbb{H}_{\mathbb{R}}$ are not isomorphic.

Therefore, our strategy is restricting ourselves to the case of central simple algebras over algebraically closed fields. In this case, we have a positive answer for:

- Finite-dimensional associative algebras
- Finite-dimensional Lie algebras by Kushkulei and Razmyslov (1983)
- Finite-dimensional Jordan algebras by Drensky and Racine (1992)
- Finite-dimensional algebras by Shestakov and Zaicev (2011).

The case of simple associative algebras graded by an abelian group was solved by Koshlukov and Zaicev (see [26]), and the result was extended by Aljadeff and Haile to arbitrary groups (see [1]). For finite-dimensional algebras graded by a semigroup, the positive answer was given by Bahturin and Yasumura (see [4]). In the latter paper very general results were obtained concerning the isomorphism of two algebraic systems provided they satisfy the same identical relations.

A more general problem is the *embedding problem*: Consider A and B two K-algebras, such that A satisfies the polynomial identities of B, then is possible to see the algebra A as a subalgebra of B?

Now, concerning the question about the embedding of simple algebras over an algebraically closed field, we have positive results for:

- Finite-dimensional associative algebras.
- Algebras graded by an abelian group over a field of characteristic zero, [9].

We consider K to be an algebraically closed field of characteristic different from 2, and recall that in the case of matrices $M_n(K)$, there are two types of involutions. For the readers' convenience we recall it once again here:

- The *transpose involution*, denoted by t:

$$(a_{ij})^t = (a_{ji}), \quad \text{where} \quad (a_{ij}) \in M_n(K).$$

- In the case where $n = 2k$ is even, the *symplectic involution* denoted by s is defined as follows:
$$a^s = Ta^tT^{-1}, \text{ for all } a \in M_{2k}(K),$$
where $T = \sum_{i=1}^{k}(e_{i,i+k} - e_{i+k,i})$.
That is, if $n = 2k$ and $B \in M_{2k}(K)$, we consider B as a block matrix of size $k \times k$, and thus
$$\begin{bmatrix} R & S \\ P & Q \end{bmatrix}^s = \begin{bmatrix} Q^t & -S^t \\ -P^t & R^t \end{bmatrix}$$

We consider the embedding problem for algebras with involution. In the case of simple algebras with involution, we have the following classification:

Theorem 4.1 ([34], Lemma 4)**.** *Let K be an algebraically closed field. Any finite dimensional $*$-simple K-algebra with involution A is isomorphic as a $*$-algebra to one of the following types:*

- *$(M_k(K), t)$ - the full matrix algebra with the transpose involution,*
- *$(M_k(K), s)$ - the full matrix algebra with the symplectic involution, ($k \in 2\mathbb{Z}$),*
- *$(M_k(K) \oplus M_k(K)^{op}, ex)$ - the direct product of the full matrix algebra and its opposite algebra with the exchange involution $*$.*

Denote by $(M_n(K), *)$ the full matrix algebra with involution $*$.

Definition 4.1. *$\mathcal{L}(n, d, k, *)$ stands for the set of all*
$$St_d(X_1 + X_1^*, \ldots, X_k + X_k^*, X_{k+1} - X_{k+1}^*, \ldots, X_d - X_d^*),$$
that are identities for $(M_n(K), *)$. Here $St_d(x_1, \ldots, x_d)$ is the standard polynomial.

A possible approach to the embedding problem can be given based on standard polynomials, as in the case of simple algebras. Hence we look for the minimum degree such that a standard polynomial becomes an involution identity for the matrix algebra.

By Amitsur-Levitzki $\mathcal{L}(n, d, k, *)$ holds for $d \geqslant 2n$. On the other hand, if $* = t$ (the transpose involution), we have the following classification (see [30]).

Theorem 4.2. *$\mathcal{L}(n, 2n-2, 0, t)$, $\mathcal{L}(n, 2n-1, 0, t)$ and $\mathcal{L}(n, 2n-1, 1, t)$ hold for all n; $\mathcal{L}(n, 2n-2, 1, t)$ holds for all odd n. All other $\mathcal{L}(n, d, k, t)$ do not hold whenever $d < 2n$.*

Corollary 4.1. *The minimum degree of a standard identity in skew-symmetric variables for $(M_n(K), t)$ is $2n - 2$.*

Consider now $(M_n(K), t)$ and $(M_m(K), t)$ with $m > 1$, and suppose that $\operatorname{Id}(M_n(K), t) \subseteq \operatorname{Id}(M_m(K), t)$. By Corollary 4.1, we know that it satisfies $\mathcal{L}(n, 2n-2, 0, t)$. Therefore, by assumption, we also have $\mathcal{L}(m, 2n-2, 0, t)$, and once again, by Corollary 4.1, we have $2n - 2 \geqslant 2m - 2$. Therefore, $n \geqslant m$.

Thus, we have the injective K-homomorphism $\iota : M_m(K) \to M_n(K)$, given by

$$\iota\left((a_{kl})\right) = (b_{kl}), \text{ where } b_{kl} = \begin{cases} a_{kl} & \text{if } 1 \leqslant k, l \leqslant m \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Therefore, $\iota\left((a_{kl})^t\right) = \iota\left((a_{kl})\right)^t$, and thus $(M_m(K), t)$ embeds into $(M_n(K), t)$. In the case where $m = 1$, it is evident that $M_m(K)$ embeds into $M_n(K)$.

Now, let us consider the case of the symplectic involution.

Theorem 4.3 ([20], Lemma 4.1)**.** *Let $(M_{2k}(K), s)$ be the algebra of $2k \times 2k$ matrices endowed with the symplectic involution. Then the polynomial $St_{4k}(x_1, \ldots, x_{4k})$ is a standard $*$-identity of minimal degree in skew variables.*

In Rowen's notation, we have that $\mathcal{L}(2n, 4n, 0, s)$ is satisfied, and we do not have $\mathcal{L}(2n, d, 0, s)$ for $d < 4n$.

Let us now consider $(M_{2n}(K), s)$ and $(M_{2m}(K), s)$ with $m \geqslant 1$, and suppose that $\operatorname{Id}(M_{2n}(K), s) \subseteq \operatorname{Id}(M_{2m}(K), s)$. By Theorem 4.3, we have that $m \leqslant n$. Furthermore, if $m \leqslant n$, for $A \in M_m(K)$, we consider $\overline{A} = \iota(A) \in M_n(K)$ as in (4.1), and we define

$$\varphi : M_{2m}(K) \to M_{2n}(K), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}.$$

For example,

$$\varphi : M_2(K) \to M_4(K), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

φ is an injective homomorphism that preserves the symplectic involution. Therefore, if $\operatorname{Id}(M_{2n}(K), s) \subseteq \operatorname{Id}(M_{2m}(K), s)$, then $(M_{2m}(K), s) \hookrightarrow (M_{2n}(K), s)$.

In the case where $\operatorname{Id}((M_m(K), t)) \subseteq \operatorname{Id}((M_{2n}(K), s))$, by Corollary 4.1 and Theorem 4.3, we have $2m - 2 \geqslant 4n$. Therefore, it follows that $m > m - 1 \geqslant 2n$, and as a result, $M_{2n}(K) \hookrightarrow M_m(K)$.

For standard identities in symmetric variables we have the following results.

Theorem 4.4 ([33], Proposition 2)**.** *Let $(M_n(K), t)$ be the algebra of $n \times n$ matrices endowed with the transpose involution. Then the polynomial $St_{2n}(x_1, \ldots, x_{2n})$ is a standard $*$-identity of minimal degree in symmetric variables.*

Theorem 4.5 ([31], Theorem 3). *The standard polynomial $S_{4k-2}(y_1, \ldots, y_{4k-2})$ is a $*$-identity of $(M_{2k}(K), s)$ in symmetric variables for all $k \geqslant 1$.*

Now, we consider some relations between $\mathrm{Id}\left((M_{2n}(K), s)\right)$ and $\mathrm{Id}\left((M_m(K), t)\right)$.

If $\mathrm{Id}\left((M_{2n}(K), s)\right) \subseteq \mathrm{Id}\left((M_m(K), t)\right)$, by Theorem 4.4 and Theorem 4.5, we have $4n - 2 \geqslant 2m$. Thus, $2n \geqslant m$ and $M_m(K) \hookrightarrow M_{2n}(K)$.

Proposition 4.1 ([5], Proposition 4.4). *Let m be a positive integer.*

$$\Psi : (M_m(K), t) \to (M_{2m}(K), s), \qquad \alpha \mapsto \begin{pmatrix} \alpha & \mathbf{0} \\ \mathbf{0} & \alpha \end{pmatrix}$$

is a homomorphism of K-algebras with involution.

Corollary 4.2. $\mathrm{Id}(M_{2m}(K), s) \subset \mathrm{Id}(M_m(K), t)$.

Proposition 4.2 ([30], Corollary 2.5.12). *$(M_n(K), t) \nsubseteq (M_{2m}(K), s)$ for all $m < n$.*

Even though Rowen's result specifically references algebras, in its proof, it is shown that if $m < n$, then $\mathrm{Id}(M_{2m}(K), s) \nsubseteq \mathrm{Id}(M_n(K), t)$. From Proposition 4.2, if $\mathrm{Id}\left((M_{2m}(K), s)\right) \subseteq \mathrm{Id}\left((M_n(K), t)\right)$, then $n \leqslant m$. Proposition 4.1, implies

$$(M_n(K), t) \hookrightarrow (M_m(K), t) \hookrightarrow (M_{2m}(K), s)\,.$$

Based on the facts described above, we have the following result.

Proposition 4.3. *Consider the algebras with involution $(M_{n_1}(K), *_1)$ and $(M_{n_2}(K), *_2)$, where n_i is even if $*_i = s$.*

(a) If $\mathrm{Id}\left((M_{n_1}(K), *_1)\right) \subseteq \mathrm{Id}\left((M_{n_2}(K), *_2)\right)$, then $M_{n_2}(K) \hookrightarrow M_{n_1}(K)$.

(b) If $\mathrm{Id}\left((M_{n_1}(K), *)\right) \subseteq \mathrm{Id}\left((M_{n_2}(K), *)\right)$, then $(M_{n_2}(K), *) \hookrightarrow (M_{n_1}(K), *)$.

(c) If $\mathrm{Id}\left((M_{n_1}(K), s)\right) \subseteq \mathrm{Id}\left((M_{n_2}(K), *)\right)$, then $(M_{n_2}(K), *) \hookrightarrow (M_{n_1}(K), s)$.

As a consequence of the previous result, we have the following theorems:

Theorem 4.6. *Let A and B two finite-dimensional central simple algebras with involution over the algebraically closed field K of characteristic 0, A with involution of orthogonal type and A satisfying the identities with involution of the algebra B. Then, there exists an embedding that preserves the involutions of A into B.*

Theorem 4.7. *Let A and B two finite-dimensional central simple algebras with involution over the algebraically closed field K of characteristic 0, B with involution of symplectic type and A satisfying the identities with involution of the algebra B. Then, there exists an embedding that preserves the involutions of A into B.*

Note that for $k \leqslant l$

$$(M_k(K) \times M_k(K)^{op}, ex) \to (M_l(K) \times M_l(K)^{op}, ex), \quad (A, B) \mapsto (\overline{A}, \overline{B})$$

and

$$(M_k(K) \times M_k(K)^{op}, ex) \to (M_{2k}(K), s), \quad (A, B) \mapsto \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & B^t \end{pmatrix}$$

are embeddings of $*$-algebras.

Therefore, if $\mathrm{Id}\left((M_{2k}(K), s)\right) \subseteq \mathrm{Id}\left((M_l(K) \times M_l(K)^{op}, ex)\right)$ considering standard polynomials in symmetric variables, we have $4k - 2 \geqslant 2l$. Thus, $2k \geqslant l$, and we have an embedding that preserves the involutions from $(M_l(K) \times M_l(K)^{op}, ex)$ to $(M_{4k}(K), s)$.

Corollary 4.3. *Let A be a finite-dimensional simple algebra with involution over the algebraically closed field K of characteristic 0 such that A satisfies the identities with involution of the matrix algebra $(M_n(K), s)$.*

- *If A is central, there exists an embedding of A into $(M_n(K), s)$ that preserves the involutions.*
- *If A is not central, there exists an embedding of A into $(M_{2n}(K), s)$ that preserves the involutions.*

Bibliography

[1] E. Aljadeff and D. Haile, *Simple G-graded algebras and their polynomial identities*, Transactions of the American Mathematical Society **366** (2014), no. 4, 1749–1771.

[2] S. Azevedo, M. Fidelis, and P. Koshlukov, *Tensor product theorems in positive characteristic*, Journal of Algebra **276** (2004), no. 2, 836–845.

[3] Y. Bahturin, I. Shestakov, and M. Zaicev, *Gradings on simple Jordan and Lie algebras*, Journal of Algebra **283** (2005), no. 2, 849–868.

[4] Y. Bahturin and F. Yasumura, *Graded polynomial identities as identities of universal algebras*, Linear Algebra and its Applications **562** (2019), 1–14.

[5] K. J. Becher, N. Grenier-Boley, and J. Tignol, *Involutions and stable subalgebras*, Journal of Algebra **493** (2018), 381–409.

[6] L. Centrone, *Ordinary and $\mathbb{Z}_2$-graded cocharacters of $UT_2(E)$*, Communications in Algebra **39** (2011), no. 7, 2554–2572.

[7] L. Centrone and V. R. T. da Silva, *On $\mathbb{Z}_2$-graded identities of $UT_2(E)$ and their growth*, Linear Algebra and its Applications **471** (2015), 469–499.

[8] A. Cirrito and F. Martino, *Ordinary and graded cocharacter of the Jordan algebra of 2×2 upper triangular matrices*, Linear Algebra and its Applications **451** (2014), 246–259.

[9] O. David, *Graded embeddings of finite dimensional simple graded algebras*, Journal of Algebra **367** (2012), 120–141.

[10] O. M. Di Vincenzo, *On the graded identities of $M_{1,1}(E)$*, Israel journal of mathematics **80** (1992), 323–335.

[11] O. M. Di Vincenzo and P. Koshlukov, *On the $*$-polynomial identities of $M_{1,1}(E)$*, Journal of Pure and Applied Algebra **215** (2011), no. 3, 262–275.

[12] O. M. Di Vincenzo, P. Koshlukov, and R. La Scala, *Involutions for upper triangular matrix algebras*, Advances in Applied Mathematics **37** (2006), no. 4, 541–568.

[13] O. M. Di Vincenzo and V. R. T. da Silva, *On $\mathbb{Z}_2$-graded polynomial identities of the Grassmann algebra*, Linear algebra and its applications **431** (2009), no. 1-2, 56–72.

[14] ______, *On $\mathbb{Z}_2$-graded identities of the generalized Grassmann envelope of the upper triangular matrices $UT_{k,l}(F)$*, Journal of Pure and Applied Algebra **218** (2014), no. 2, 285–296.

[15] V. Drensky, *Codimensions of t-ideals and hilbert series of relatively free algebras*, Journal of Algebra **91** (1984), no. 1, 1–17.

[16] ______, *Free algebras and PI-algebras: Graduate course in algebra*, Springer, Singapore, 2000.

[17] V. Drensky and E. Formanek, *Polynomial identity rings*, Birkhäuser, 2012.

[18] . Giambruno and M. Zaicev, *Polynomial identities and asymptotic methods*, no. 122, American Mathematical Soc., 2005.

[19] A. Giambruno, A. Ioppolo, and D. La Mattina, *Superalgebras with involution or superinvolution and almost polynomial growth of the codimensions*, Algebras and Representation Theory **22** (2019), no. 4, 961–976.

[20] A. Giambruno, A. Ioppolo, and F. Martino, *Standard polynomial and matrices with superinvolutions*, Linear Algebra Appl **504** (2016), 272–291.

[21] A. Giambruno, S. Mishchenko, and M. Zaicev, *Polynomial identities on superalgebras and almost polynomial growth*, Communications in Algebra **29** (2001), no. 9, 3787–3800.

[22] C. Gómez-Ambrosi and I. Shestakov, *On the Lie structure of the skew elements of a simple superalgebra with superinvolution*, Journal of Algebra **208** (1998), no. 1, 43–71.

[23] A. S. Gordienko, *Regev's conjecture and codimensions of PI algebras*, Acta applicandae mathematicae **108** (2009), no. 1, 33–55.

[24] A. Ioppolo and F. Martino, *Superinvolutions on upper-triangular matrix algebras*, Journal of Pure and Applied Algebra **222** (2018), no. 8, 2022–2039.

[25] P. Koshlukov and T. C. de Mello, *On the polynomial identities of the algebra $M_{1,1}(E)$*, Linear Algebra and its Applications **438** (2013), no. 11, 4469–4482.

[26] P. Koshlukov and M. Zaicev, *Identities and isomorphisms of graded simple algebras*, Linear Algebra and its applications **432** (2010), no. 12, 3141–3148.

[27] D. Krakowski and A. Regev, *The polynomial identities of the Grassmann algebra*, Transactions of the American mathematical Society **181** (1973), 429–438.

[28] S. Mishchenko and A. Valenti, *A star-variety with almost polynomial growth*, Journal of Algebra **223** (2000), no. 1, 66–84.

[29] J. B. Olsson and A. Regev, *Colength sequence of some T-ideals*, Journal of Algebra **38** (1976), no. 1, 100–111.

[30] L. H. Rowen, *Polynomial identities in ring theory*, Academic Press, 1980.

[31] ______, *A simple proof of kostant's theorem and an analogue for the symplectic involution*, Contemp. Math **13** (1982), 207–215.

[32] V. R. T. da Silva, *On $\mathbb{Z}_2$-graded identities of the super tensor product of $UT_2(F)$ by the Grassmann algebra*, Israel Journal of Mathematics **188** (2012), no. 1, 441–462.

[33] A. M. Slin'ko, *Special varieties of Jordan algebras*, Mathematical notes of the Academy of Sciences of the USSR **26** (1979), no. 3, 661–665.

[34] I. Sviridova, *Finitely generated algebras with involution and their identities*, Journal of Algebra **383** (2013), 144–167.

[35] A. Valenti, *The graded identities of upper triangular matrices of size two*, Journal of Pure and Applied Algebra **172** (2002), no. 2-3, 325–335.

[36] A. Valenti and M. Zaicev, *Group gradings on upper triangular matrices.*, Archiv der Mathematik **89** (2007), no. 1.

[37] ______, *Graded involutions on upper-triangular matrix algebras*, Algebra Colloquium, vol. 16, World Scientific, 2009, pp. 103–108.