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Random Perturbation of dynamics: Shadowing and Statistical Properties

Perturbação Aleatória da Dinâmica: Sombreamento e propriedades estatísticas

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática Aplicada.

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Resumo

Na primeira parte do trabalho, estudamos a estabilidade estocástica de um sistema dinâmico sujeito a pequenas perturbações aleatórias. Mostramos que medidas estacionárias convergem para medidas físicas quando o nível de perturbação aleatória vai para zero se o sistema dinâmico tem a propriedade de sombreamento. Na segunda parte, provamos que um atrator aleatório de um sistema perturbado, em termos de uma familia de difeomorfismos aleatórios parametrizados, é estocasticamente estável. Finalmente, provamos o teorema da existência e finititude de atratores aleatórios.

Palavras-chave: Propriedade de sombreamento, estabilidade estocástica , cadeia de Markov, dinâmica aleatória, atratores.

Abstract

In the first part of the work, we study stochastic stability of dynamical system subject to small random perturbation. We show that stationary measures converge to physical measures when the level of random perturbation goes to zero if the dynamical system has a shadowing property. In the second part, we prove that random attractor of dynamical system perturbed, in terms of a family of parameterised random diffeomorphisms, is stochastically stable. Finally, we prove the theorem of existence and finiteness of random attractors.

Keywords: Shadowing property, stochastic stability, Markov chain, random dynamics, attractors.

List of symbols

$\mathscr{A}$	The Borel σ-algebra on M
A^c	The complement $X \backslash A$ of each subset A
$\mathcal{B}(\mu)$	Basin of the measure μ
$\mathcal{B}$	Set of bounded continuous function on M
$C(M)$	Set of bounded continuous function on M
d	Distance induced by a Riemannian metric
δ_x	Dirac delta probability measure supported at x
M	Compact metric space
$\mathscr{M}(M)$	Denote the space of Borel probability measures on M
$P_\varepsilon(.\|x)$	Transition probabilities of family of Markov chains
$\mathbb{R}$	Set of real numbers
$W^s(\Lambda)$	The basin of attraction of the Λ
Z_μ	Set of shadowable point μ
$\mathbb{Z}_0^+$	Denote non-negative integers
$M^{\mathbb{N}}$	Denote the sequence from natural number to M
Λ	Denote an attractor
Λ^ε	Denote a random attractor
Δ_ε	Denote the perturbation space
$\\|.\\|_p$	The norm on space L^p

Contents

Introduction

The study of statistical properties of Dynamical Systems is currently present in almost all fields of science. In its essence, one wants to classify long term behaviour of processes undergoing time transformation from a statistical point of view, and understand how stable they are. In this context, invariant attracting sets are some of the most important objects whose properties are studied in Dynamical Systems. They are directly related not only to the asymptotic properties of dynamics, but also to its stability, and most of the theory of classical dynamical systems has to do with this.

In the case of dynamical systems evolving under random perturbations, their characterisation can be an even more cumbersome task when compared with deterministic dynamics, from the perspective of fundamental Mathematics as well as of applied modelling. It is important to distinguish between bounded and unbounded perturbations though. In this thesis, we focus on the former case.

When the dynamics evolves under bounded perturbations, there might coexist many invariant measures, the noise blurs the dynamics and fine scale characteristics may not be detected, and several emergent features may take place when the intensity of the noise is changed (JSWLR15; CSRK14). Bounded random perturbation of dynamics is often claimed as being more realistic in terms of modelling, given that oscillations in natural processes are in general limited. In spite of that, the mathematical basis to tackle such systems remain surprisingly unexplored, which contrasts with the case of unbounded randomness.

For processes under unbounded stochastic perturbations (Stratonovich, Itō, Martingales, etc.) one can use tools from stochastic analysis, heavily dependent on the analytical properties of such perturbations in order to build up a general theory; see Arnold (Arn98) and references therein. In the case of bounded perturbations the study of statistical properties of the dynamics is usually carried out in terms of Markov chains and/or perturbations on the space of maps (JJR15; JJR19).

Although several contributions regarding statistical properties of such dynamical systems have been made on specific cases by the work of Kifer (Kif86; Kif88), L.-S. Young (You86), Keller (Kel82), Araújo (Ara00), Alves (Alv03), Viana (BV06; Via97), among others, even characterising attracting sets or bifurcations become difficult in such context. The main challenge is that this requires strong assumptions on the properties of the perturbations themselves, on the existence of nearby orbits, on the space where it takes place, and on the classes of systems.

In (JJR15) the authors have considered what happens to the support of invariant measures in order to formulate a characterisation for bifurcation in randomly perturbed dynamical systems. One of the key observations to this aim is that the focus is shifted from a statistical perspective to the dynamics of sets. This is the approach we want to consider in part of this work. In this thesis we shall consider an appropriate definition for attracting sets for randomly perturbed dynamics and study some of their properties in terms of set dynamics combining methods from Ergodic Theory.

Alternatively, a way to characterise robust behaviour when the dynamical system evolves under small random perturbations is by means of the so called *stochastic stability*. We say that a system is stochastically stable if the stationary measure for the randomly perturbed dynamics is close, in some sense, to the invariant physical or Sinai-Ruelle-Bowen (SRB) measure, which in turn encodes important dynamical information. A more precise definition will be timely presented.

Since the 1980's, several results have been obtained under different hypotheses leading to proving stochastic stability of some classes of systems. To mention a few, Kifer has first proved stochastic stability of diffeomorphisms having (uniformly) hyperbolic attractors and expanding maps (Kif86; Kif88). Under slightly different technical assumptions L.-S. Young then showed that the stability of (uniformly) hyperbolic attractors of C^2- diffeomorphisms follows from the persistence of hyperbolic structures (You86). Viana has shown how to use spectral properties of transfer operators to infer stochastic stability of several classes of maps (Via97). Later Araújo has proved the existence and finiteness of physical measures for randomly perturbed dynamics considering some special type of perturbations (Ara00; Ara01). Then Alves and Araújo, based on some construction already present in (Ara00), gave sufficient and necessary conditions for stochastic stability of some non-uniformly expanding maps (AA03). More recently, Benedicks and Viana have proved the stochastic stability of Hénon like maps (BV06), and Alves, Araújo, and Vásquez the stochastic stability of a type of non-uniformly hyperbolic diffeomorphisms with dominating splitting (JFAV07).

Another common approach to characterise robustness of dynamics is through the idea of *structural stability*. A dynamical system is said to be structurally stable if it is equivalent to some other dynamical system in its vicinity, choosing an appropriate topology. Among different consequences of structural stability, one may investigate the *shadowing property*. Vaguely speaking, the shadowing property says that every pseudo-orbit, i.e. every sequence of points whose elements are close enough to an orbit of a dynamical system, can be described by a real orbit itself. Several authors have studied different types of shadowing under different technical assumptions, mainly, because of its close connection to structural stability (Bow75; Kat80; Pil99; PT10). See (Pil99) for an extensive historical account.

These two apparently distinct approaches to stability lead us to an important question: **how structural properties of dynamical systems are related to their statistical ones?**

The concept of shadowing is clearly related to stochastic stability because the orbits evolving under random perturbations are pseudo-orbits. In this context, proving stochastic stability equals to proving that most random orbits mimics the behaviour of real orbits (the unperturbed ones). The connection of shadowing with stochastic stability has never been satisfactorily addressed. This problem has also been raised by Bonatti, Díaz and Viana in (CBV04)[Problem D10]. To the best of our knowledge, the only attempt to study this connection was presented in the manuscript (Tod14), and in (WBV14) where the authors impose several hypotheses and much more restrictive contexts than ours.

In this thesis, we study stochastic stability of a dynamical system with shadowing property, which evolves under small random perturbation. We define time averages with respect to continuous observables along the trajectory which converges to their averages with respect to an invariant measure. Using this framework we study time averages along the pseudo-trajectory that converge with respect to stationary measure for the randomly perturbed dynamics. Finally, we proof that stationary measures converge to physical measures when the noise level goes to zero if dynamical system has a shadowing property.

The thesis is organised as follows. In Chapter 1, we introduce basic concepts of probability theory, invariant measure and ergodicity, existence of invariant measure and theorems that will be useful in this work. In Chapter 2, we present some result of shadowing property and shadowable point following (Mor16). Furthermore, the orbit of systems subject to small perturbation is called random orbit and it is given by the sequence of random variables with certain probability distribution. This family of Markov chain will be called small random perturbation of systems. In Chapter 3, we present the mean result of the thesis where we prove that the dynamical system with shadowing property is stochastically stable. Finally, In Chapter 4, we prove that existence and finiteness of random attractors, and a random attractor is stochastically stable.

1 Preliminaries

In this chapter we give some preliminary topics for development of this thesis. As it follows, in the first section we address some basic concept of probability theory. Next we present about invariant measure and ergodicity. In the latter section we remember of theorem that guarantees the existence of invariant measures for continuous maps.

1.1 Introduction measure theory

In this section we start to giving a brief introduction about the measure theory, and then we give some theorems and results which will be useful throughout the work. We shall use (CBV04), (MV16), (Wal00) and (Mañ12) as main references.

Let M be a nonempty set equipped with a σ-algebra $\mathscr{A}$. A measure space is a triple $(M, \mathscr{A}, \mu)$, where $(M, \mathscr{A})$ is a measurable space, and μ is a measure on $(M, \mathscr{A})$. Moreover, if $\mu(M) = 1$ then μ is called a probability measure and $(M, \mathscr{A}, \mu)$ is called a probability space. In addition, let $(M, \mathscr{A})$ be any measure space. For $x \in M$, define the measure $\delta_x : \mathscr{A} \to [0, \infty)$ by $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$ otherwise. We call δ_x the Dirac delta probability measure supported at x.

Let M be a compact metric space and let $\mathscr{B}$ be the Borel σ-algebra on M. Let μ be a measure on $(M, \mathscr{B})$. Consider the set

$$\mathscr{U} := \bigcup \{U : U \quad \text{is a open} \quad , \mu(U) = 0\}$$

That is, $\mathscr{U}$ is the largest open set with zero measure. The complement of $\mathscr{U}$ is called the **support** of μ.

Definition 1. *If $(M, \mathscr{A})$ and $(N, \mathscr{A}')$ are measurable spaces, a function $f : M \to N$ is measurable if*

$$f^{-1}(A) = \{x \in M : f(x) \in A\} \in \mathscr{A}, \quad \textit{for every} \quad A \in \mathscr{A}'$$

Definition 2. *Let $(M, \mathscr{A}, \mu)$ be a probability space and $(N, \mathscr{A}')$ be a measurable space. Every measurable function $g : M \to N$ induces a probability measure on N of the following way*

$$\mu_g(A) = \mu(g^{-1}(A)) = g_\star\mu(A) = \mu\{x \in M : g(x) \in A\}$$

*for $A \in \mathscr{A}'$. We will call that μ_g is **law or distribution** of g where $g_\star\mu$ denote push forward of μ by g defined by $g_\star\mu(A) = \mu(g^{-1}(A))$ for $A \in \mathscr{A}'$.*

Now we denote by $\mathscr{M}(M)$ the set of probability measure on the Borel σ-algebra of M. For the next definition, consider $C(M)$ the space of continuous functions from M to $\mathbb{R}$

with sup-norm defined by $||f||_\infty = \sup_{x\in M} |f(x)|$. In addition, we denote by $\mathcal{B}$ the set of bounded continuous function on M.

Definition 3. *Given a sequence of probability measure $\{\mu_i\}_{i\in\mathbb{N}}$ and μ on $\mathscr{M}(M)$. We say that μ_i converge to, in the $weak^\star$-topology sense, a measure μ if $\int \varphi d\mu_i \to \int \varphi d\mu$ when $i \to \infty$ for any function $\varphi \in C(M)$.*

1.2 Invariant measure and ergodicity

The study of invariant measure may give important information about the long time behavior dynamical of systems. For instance, Poincaré's recurrence theorem states that the orbit of almost every point, relative to any finite invariant measure, returns arbitrarily close to its initial state.

Definition 4. *Let $(M, \mathscr{A}, \mu)$ and $(N, \mathscr{A}', \nu)$ two probability spaces. A function $f : M \to N$ is measure-preserving or μ is said to be a f-invariant measure if f is measurable and $\mu(f^{-1}(A)) = \nu(A)$ for all $A \in \mathscr{A}'$.*

Example 1. *Let $f : [0,1] \to [0,1]$ be given by $f(x) = 2x(mod 1)$. Then f preserves the Lebesgue measure m on the Borel sets $[0,1]$. In fact, let I be the family of subintervals of $[0,1]$. Given any $J = [a,b] \in I$, we have that $f^{-1}(J)$ is made of two disjoint intervals $I_1 = \left[\frac{a}{2}, \frac{b}{2}\right]$ and $I_2 = \left[\frac{a+1}{2}, \frac{b+1}{2}\right]$ with $m(I_1) = m(I_2) = \frac{1}{2}m(J)$. Thus*

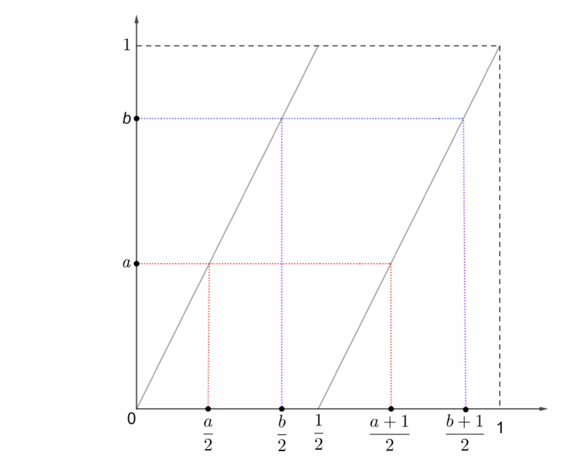

Figure 1 – The pre-image of an interval under the map f

$$f_\star m(J) = m(f^{-1}(I)) = m(I_1 \bigcup I_2) = m(I_1) + m(I_2) = \frac{1}{2}m(J) + \frac{1}{2}m(J) = m(J)$$

This show that f preserves Lebesgue measure.

Definition 5. *We say that f is ergodic if every f-invariant subset A of M has measure 0 or 1. Recall that a set $A \in \mathscr{A}$ is called f-invariant if $f^{-1}(A) = A$.*

Example 2. *Let $f : X \to X$ be a measurable function. Then the Dirac delta, δ_x is f invariant and ergodic where x is a fixed point for f. That is, let $A \subset X$ be a measurable set. Firstly, we suppose that $x \in A$, then $\delta_x(A) = 1$. Since $x = f(x)$, we have that $x \in f^{-1}(A)$. Thus $\delta_x(f^{-1}(A)) = 1$. For the second case, $x \notin A$, then $\delta_x(A) = 0$ so we also have $x \notin f^{-1}(A)$. On the other hand, if we did have $x \in f^{-1}(A)$, then by definition imply that $f(A) \in A$, it contradicts our assumption. Therefore, $\delta_x(f^{-1}(A)) = 0$. Thus it is invariant measure. Finally, ergodicity is trivial.*

Theorem 1. *(Birkhoff Ergodic Theorem) If $f : M \to M$ preserves a probability measure μ and $\varphi : M \to \mathbb{R}$ is integrable, then there is $\varphi^\star : M \to \mathbb{R}$ integrable such that , for μ-a.e $x \in M$*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(x)) = \varphi^\star(x)$$

Moreover, if μ is ergodic, then $\varphi^\star(x) = \int \varphi d\mu$ for almost every $x \in M$.

We say that a linear functional $L : C(M) \to \mathbb{R}$ is positive if $L(\varphi) \geqslant 0$ for every function $\varphi \in C(M)$ with $\varphi(x) \geqslant 0$ for every $x \in M$. In that case, the following theorem relates elements of $\mathscr{M}(M)$ to linear functional on $C(M)$.

Theorem 2. *(Riesz-Markov Kakutani Representation Theorem) Let M be a compact metric space. Consider any positive linear functional $L : C(M) \to \mathbb{R}$. Then there exists a unique finite Borel measure μ on M such that*

$$L(\varphi) = \int \varphi d\mu \quad \textit{for every} \quad \varphi \in C(M)$$

1.3 Existence of invariant measures for continuous maps

In this section we present the theorem that assure existence of invariant measures for continuous maps on compact metric space. Furthermore, we give two examples that theorem does not hold when the function do not is continuous or the space do not is compact.

Theorem 3. *(Krylov-Bogulyobov theorem) If M is a compact metric space and $f : M \to M$ is a continuous function, then there exists at least one f-invariant probability measure on M.*

Proof. The reader can see the proof of theorem on page 45 (MV16). □

Now, this theorem fails when the space is not compact. Before showing the examples we present the following theorem:

Theorem 4. *(Poincaré's Recurrence Theorem) Let $f : M \to M$ measure preserving function of a probability space $(M, \mathscr{A}, \mu)$. Then for any $A \in \mathscr{A}$ with positive measure $\mu(B) > 0$, μ- almost every point $x \in B$ returns to B.*

Example 3. *Let $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x + 1$. We assume that f has an invariant probability measure μ. Let A a subset of $\mathbb{R}$ with positive measure and $a = sup(A)$, then note that A is bounded. Now, let $x \in A$. Since f is a expansion on $\mathbb{R}$, then there exist $\alpha \in \mathbb{N}$ such that $f^i(x) > a$. However this contradicts the Poincare Recurrent Theorem. Thus, there are not exist invariant probability measure if the space does not is compact.*

The theorem (3) that is not true for discontinuous function. See the following example.

Example 4. *Let $f : [0,1] \to [0,1]$ defined by $f(x) = \dfrac{x}{2}$ if $x \neq 0$ and $f(0) = 1$. Note that f is discontinuous at 0. We assume that δ_0 is f invariant. Let $A = \{0\}$, then $\delta_0(A) > 0$ and $f^i(0) = \dfrac{1}{2^{i-1}}$ for each $i \in \mathbb{N}$, then by Poincare Recurrent Theorem there exist $\{i_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $f^{i_k} \in A$, that is $f^i(0) = \dfrac{1}{2^{i-1}} = 0$ for each $i_k \in \mathbb{N}$. So this is contradiction. Thus δ_0 is not invariant measure for f.*

1.4 Hausdorff distance

To discuss random attractors in Chapter (4), we need first to introduce fundamental concepts following (Fol99). In a metric space (M, d) we can define the following concepts:

The ε-neighborhood of a point $x_0 \in M$ and radius $\varepsilon > 0$ by $B_\varepsilon(x_0) := \{x \in M : d(x, x_0) < \varepsilon\}$ and we also define the diameter of the set $A \subset M$ which is given as

$$diam(A) := \sup\{d(x,y) : x, y \in A\}$$

Moreover, for any arbitrary nonempty sets $A, B \subset M$ and $x \in M$. We can define the distance from a point to a set as follows,

$$d(x, A) := \inf\{d(x,y) : y \in A\}$$

and also the distance between two sets:

$$d(A, B) := \sup\{d(x, B) : x \in A\}$$

Definition 6. *Let (M, d) be a metric space and let A, B be nonempty subsets of M. The Hausdorff distance between A and B is defined as follows:*

$$h(A, B) := \max\{d(A, B), d(B, A)\}.$$

Denote by $\mathcal{H}(M)$ the collection of all nonempty compact subsets of a metric space (M, d). In addition, following (Bar11) it can be shown that h defines a metric on $\mathcal{H}(M)$. Therefore when equipped with the Hausdorff distance h, is also a metric space $(\mathcal{H}(M), h)$.

Lemma 1. *Let A and B be nonempty subsets of M then:*

1. $dist(A, B) = 0$ *if and only if* $A \subset \overline{B}$;
2. $h(A, B) = 0$ *if and only if* $\overline{A} = \overline{B}$.

Proof.

1. First, assume that $dist(A, B) = 0$, then $\sup\{dist(x, B) : x \in A\} = 0$ and $d(x, B) = 0$ for all $x \in A$. Then, for any $\varepsilon > 0$ there exists $y \in B$ such that $d(x, y) < \varepsilon$, then $B(x, \varepsilon) \bigcap B \neq \emptyset$. As a consequence $x \in \overline{B}$, then $A \subset \overline{B}$.

   Conversely, suppose that $A \subset \overline{B}$, given any $x \in A$ we have $x \in \overline{B}$. Then for all $\varepsilon > 0$, there exists $y \in B$ such that $d(x, y) < \varepsilon$. Thus $0 \leqslant dist(x, B) = \inf\{d(x, y) : y \in A\} \leqslant d(x, y) < \varepsilon$ which implies $0 \leqslant dist(x, B) < \varepsilon$. Hence $dist(x, B) = 0$, which implies that $dist(A, B) = \sup\{dist(x, B) : x \in A\} = 0$. Thus $dist(A, B) = 0$

2. $h(A, B) := \max\{dist(A, B), dist(B, A)\} = 0 \Leftrightarrow dist(A, B) = 0$ and $dist(B, A) = 0 \Leftrightarrow A \subset \overline{B}$ and $B \subset \overline{A}$. Thus $\overline{A} = \overline{B}$.

□

2 Shadowing property and random perturbation

In this section we introduce some results of shadowing property and random perturbation, for that we shall use bibliography of (Pil99), (Mor16), (Kif88), and (CBV04). Let (M, d) be a metric space and $f : M \to M$ be a continuous map. First, we consider that a pseudo-trajectory of a dynamical system f is a result of a small random perturbation of f. A random perturbation is understood that a particle jumps from x to $f(x)$ and disperses randomly close to $f(x)$ with some distribution. On the other hand, the shadowing property means there are real trajectories close to trajectories of a system randomly perturbed.

2.1 Shadowing property

The shadowing theory is an important part of the study stability of dynamical systems. In addition, shadowing property means that close to pseudo-orbit there exists an exact or real orbit. Now, let M be a metric space endowed by the induced Riemannian metric d and let $f : M \to M$ be a continuous function. Furthermore, for our goal we consider the dynamics generated by the iteration the function $f : M \to M$, given by $z_j = f^j(z_0) = \underbrace{f \circ \cdots \circ f}_{\text{j times}}(z_0)$ for any initial condition $z_0 \in M$. Hereafter we set $f^0(z) = z$, for every $z \in M$. Additionally, let a sequence of points, $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$, in M where are close enough to trajectory of any $z \in M$.

Definition 7. *Given $\delta > 0$, we say that a sequence of points $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ in M is a δ-**pseudo-trajectory** of f if*

$$d\left(f(x_j), x_{j+1}\right) \leqslant \delta, \textit{ for all } j \geqslant 0. \tag{2.1}$$

Note that the set of δ-pseudo orbits or δ-pseudo trajectory are similar to real trajectories. Hence, in the next definition we should give the meaning of an orbit of f is **approximate** by a pseudo-orbit.

Definition 8. *We say that a dynamical system generated by f has the **shadowing property** if, for every $\varepsilon > 0$, we can find $\delta > 0$, such that, for each δ-pseudo-trajectory $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ of f, there exists a point $z \in M$, such that,*

$$d(f^j(z), x_j) < \varepsilon, \textit{ for all } j \geqslant 0.$$

In this case, we say that $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ is ε-shadowed by some $z \in M$.

Example 5. *Let $T : [0, 1] \to [0, 1]$ be the tent function, defined as $T(x) = 1 - |1 - 2x|$. Let $X = \{(0, 0)\} \cup \bigcup \{\frac{1}{k}\} \times [0, \frac{1}{k}]$ be endowed with metric by the Euclidean metric and we define* *the following function*

$$f_T(x,y) = \begin{cases} (\frac{1}{k}, \frac{1}{k}T(kx)) & \text{for} k > 0 \\ (0,0) & \text{otherwise} \end{cases}$$

If we fix $\varepsilon > 0$ *then there is* $\delta > 0$ *such that if* $(\frac{1}{k}, x), (\frac{1}{p}, y) \in X$ *and* $\frac{1}{k} > \sqrt{\varepsilon}$ *then either* $k = p$ *or* $|\frac{1}{k} - \frac{1}{p}| > \delta$*. Thus, for any* δ*-Pseudo Orbit* $\mathbf{x} = \{(\frac{1}{k_n}, x_n)\}_{n=o}^{\infty}$ *we either have that* $\mathbf{x} \subset [0, \sqrt{\varepsilon}] \times [0, \sqrt{\varepsilon}]$ *or* $k_n = k_{n+1}$ *for every* $n > 0$*. Thus,* f *has shadowing property.*

Example 6. *In general, we consider the family of tent maps, that is, the piecewise linear maps* $f_s : [0,2] \to [0,2]$ *for every* $\sqrt{2} \leqslant s \leqslant 2$ *defined by*

$$f_s(x) = \begin{cases} sx & ,0 \leqslant x \leqslant 1 \\ s(2-x) & ,1 \leqslant x \leqslant 2 \end{cases}$$

Following (EMCY98), this family of tent maps have the shadowing property for almost all parameters s, for every $s \in [\sqrt{2}, 2]$.

2.1.1 Shadowable point

Now, we introduce the concept of shadowable point, which has been extensively studied by Morales (Mor16). In his article "Shadowable points", Morales proved that if $f : M \to M$ is a homeomorphism of a compact metric space M, then f has the shadowing property if only if every point in M is shadowable. Moreover, in the next lemma we will prove the previous statement requiring only that f be a continuous map.

Definition 9. *We say that a point* $y \in M$ *is* ***shadowable*** *if for every* $\varepsilon > 0$ *there is* $\delta > 0$*, such that, every* δ*-pseudo orbit* $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$*, with* $x_0 = y$ *can be* ε*-shadowed.*

Lemma 2. *Let* $f : M \to M$ *be a continuous function of a compact metric space* M*. Then,* f *has the shadowing property if, and only if, every point in* M *is shadowable.*

Proof. ($\Rightarrow$) Since f has the shadowing property, for every $\varepsilon > 0$ and each $y \in M$, there exists $\delta > 0$ such that, for every δ-pseudo trajectory $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ with $x_0 = y$ there exists a point $z \in M$ satisfying

$$d(f^j(z), x_j) \leqslant \varepsilon, \text{ for all } j \geqslant 0.$$

Hence, y is shadowable.

($\Leftarrow$) Conversely, if every point in M is shadowable, then for any $\varepsilon > 0$ and every point $w \in M$ there exists $\delta_w > 0$, such that every δ_w-pseudo trajectory $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$, with $x_0 \in B_{\delta_w}[w]$, the closed ball centered at w with radius δ_w, is ε-shadowed by some point in M. For each $w \in M$, consider the open ball $B_{\delta_w}(w)$ centered at w with radius δ_w. Then $\{B_{\delta_w}(w)\}_{w \in M}$ the collection of such balls centered at each $w \in M$ provides an open cover for M. Since M is a compact metric space, we can extract a finite subcover $\{B_{\delta_{w_i}}(w_i)\}_{i=1}^n$ from $\{B_{\delta_w}(w)\}_{w\in M}$. Let δ be the smallest radius from the balls $\{B_{\delta_{w_i}}(w_i)\}_{i=1}^n$, and consider any δ-pseudo orbit $\mathbf{y} = (y_j)_{j\in\mathbb{Z}_0^+}$ of f. Then $y_0 \in B_{\delta_{w_i}}[w_i]$ for some i. Since $\delta \leqslant \delta_{w_i}$ we have $\mathbf{y} = (y_j)_{j\in\mathbb{Z}_0^+}$ is ε- shadowed by some point in M. □

2.2 Random perturbation

In this section, we study a random perturbation of a dynamics. One of the key problems in dynamical system is studying the stability of the dynamical behavior under random perturbation. We consider that random perturbation is represented by a sequence of random variables following a specific probability distribution. In addition, this sequence is considered a realization of a family of Markov chain.

2.2.1 Markov chain

First, let us recall the definition of a stochastic process. Let $(M, \mathscr{A}, \mathcal{P})$ be a probability space, and let T be an arbitrary set called the index set. In general, $T = \mathbb{R}, [0, \infty >, \mathbb{N}$ or $\mathbb{Z}$. Then a stochastic process is a collection of random variables $\{X_t : t \in T\}$ defined on the probability space $(M, \mathscr{A}, \mathcal{P})$ with T as the indice set. Here, we are interested in a special class of stochastic process called Markov chains, where $T = \mathbb{Z}_0^+$ and the random variable X_n take values in a state space M. Below, we provide the formal definition of a Markov chain.

Definition 10. *Let $(M, \mathscr{A}, \mathcal{P})$ be a probability space. A sequence of random variables $\{X_n\}_{n\geqslant 0}$ taking values in M is said satisfying the Markov property if*

$$\mathcal{P}(X_{n+1} \in B|X_0, X_1, ..., X_n) = \mathcal{P}(X_{n+1} \in B|X_n), \forall B \in \mathscr{A}, n = 0, 1, 2, ...$$

such a sequence is called a Markov chain.

In addition, for each $x \in M$ and $B \in \mathscr{A}$, let

$$P(B|x) := \mathcal{P}(X_{n+1} \in B|X_n = x)$$

This defines a Stochastic Kernel on M, which means that

- $P(.|x)$ is a probability measure on $\mathscr{A}$ for each fixed $x \in M$, and
- $P(B|.)$ is a measurable measure on M for each fixed $B \in \mathscr{A}$.

The stochastic kernel $P(B|x)$ is also known as a Markov transition probability function.

In this work, following (Kif88) we consider perturbation of dynamical systems thought a family of Markov chains X_n^ε, $n = 0, 1, 2, ...$ with transition probabilities

$$P_\varepsilon(B|x) = \mathcal{P}(X_{n+1}^\varepsilon \in B|X_n^\varepsilon = x)$$

defined for any $x \in M$ and a Borel set $B \subset M$.

Throughout the work, we assume that every realization of Markov chain is a δ-pseudo orbit for any $\delta > 0$. Moreover, we call **small random perturbation** of the dynamical to the family of Markov chain and the orbit of dynamical subject to small perturbation we will call **random orbits**.

Definition 11. *A Borel probability measure μ_ε on M is a stationary measure for the Markov chain if*

$$\mu_\varepsilon(G) = \int P_\varepsilon(G|x) d\mu_\varepsilon$$

for every $x \in M$ and every Borel set $G \subset M$.

Definition 12. *Let M be a compact metric space. We denote by $M^{\mathbb{N}}$ the sequence of $\theta : \mathbb{N} \to M$, endowed with the product measure. The left shift map $\sigma : M^{\mathbb{N}} \to M^{\mathbb{N}}$ is the map defined by*

$$\sigma(\mathbf{x}) = (x_1, x_2, ...)$$

where $\mathbf{x} = (x_0, x_1, x_2, ...) \in M^{\mathbb{N}}$

3 Shadowing and stochastic stability

Stochastic stability has been studied by many authors, see (MV16) and (Kif88). In this work, we introduce a small random perturbation to a deterministic dynamical systems through Markov chain. In this context, stochastic stability means that the invariant measure is close, in some sense of the stationary measures associated to the Markov chain. Next we will state the main theorem and its demonstration.

3.1 Ergodicity and physical measures

Given a compact metric space (M, d), consider a continuous function $f : M \to M$. and considering the same dynamics defined in section(2.1). We define the following probability measures:

First, we can define the following probability measure for each $z \in M$, which we denote by $S_n^f(z)$, that is

$$S_n^f(z) = S_n\left(\{f^j(z)\}_{j=0}^{\infty}\right) = \frac{1}{n+1}\sum_{j=0}^{n}\delta_{f^j(z)}. \tag{3.1}$$

Since M is a compact metric space and f is continuous, the Krylov-Bogolyubov Theorem, guarantees the existence of at least one f-invariant probability measure μ on M. If μ is such an f-invariant probability measure and $\varphi \in L^1(\mu)$ then, by the Birkhoff Ergodic Theorem, the following limit exists for μ-almost every $z \in M$:

$$\tilde{\varphi} = \lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\varphi\left(f^j(z)\right)$$

Moreover, for any $\varphi \in L^1(M)$, from (3.1) we have

$$\begin{aligned}\int \varphi dS_n^f(z) &= \int \varphi d\left(\frac{1}{n+1}\sum_{j=0}^{n}\delta_{f^j(z)}\right)\\ &= \frac{1}{n+1}\sum_{j=0}^{n}\int \varphi d\delta_{f^j(z)}\\ &= \frac{1}{n+1}\sum_{j=0}^{n}\varphi\left(f^j(z)\right).\end{aligned}$$

Thus, there exists the limit

$$E_z(\varphi) \equiv \lim_{n\to\infty}\int \varphi dS_n^f(z)$$

for μ-almost every $z \in M$. On the other hand, we also have that the map $\varphi \mapsto E(\varphi) = E_z(\varphi)$, for each $z \in B \subset M$, where B is some set of μ positive measure, defines a non-negative linear functional on the space $L^1(M)$. Therefore, by the Riesz-Markov-Kakutani Representation Theorem, there exists a unique Radon measure μ on M, such that, for every $\varphi \in L^1(M)$

$$\int \varphi d\mu = E(\varphi) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right). \tag{3.2}$$

Hence, a measure μ is ergodic if for μ-almost every point $z \in M$, it holds that

$$S_n^f(z) \to \mu, \text{ as } n \to \infty$$

in the $weak^{\star}$-topology on the space of probability measures on M. These considerations naturally lead to the concept of physical measures.

Definition 13. *A physical measure, or SRB measure, of a continuous function on M is an invariant probability measure μ on M, such that the time average of every continuous function $\varphi : M \to \mathbb{R}$ coincides with the corresponding space-average with respect to*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(x)) = \int \varphi d\mu \tag{3.3}$$

for a set of initial points z with positive Lebesgue measure. We call this set the basin of μ, and denoted by $\mathcal{B}(\mu)$

Lemma 3. *The measure μ is ergodic if for μ-a.e point $z \in M$, it holds that $S_n^f(z) \to \mu$, as $n \to \infty$. Then, for every $\varphi \in \mathcal{B}$, and for each $\varepsilon > 0$ sufficiently small, there exists $n_0 = n_0(\mu, \varphi, z) \in \mathbb{N}$, such that,*

$$\left| \int \varphi dS_n^f(z) - \int \varphi d\mu \right| \leqslant \varepsilon, \textit{ for all } n \geqslant n_0(\mu, \varphi, z).$$

Proof. Suppose that $S_n^f(z) \to \mu$ in M and take $\varphi \in \mathcal{B}$. Given $\varepsilon > 0$, consider the neighborhood

$$N_{\varepsilon,\varphi}(\mu) = \left\{ \nu : \left| \int \varphi d\nu - \int \varphi d\mu \right| < \varepsilon \right\}.$$

By hypothesis, there exists $n_0 = n_0(\mu, \varphi, z) \in \mathbb{N}$, such that, for all $n \geqslant n_0$, we have $S_n^f(z) \in N_{\varepsilon,\varphi}(\mu)$. That is, $\left| \int \varphi dS_n^f(z) - \int \varphi d\mu \right| < \varepsilon$, for all $n \geqslant n_0$. □

Finally, let $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ be a sequence of points in M indexed by the non-negative integers. Then similarly, we can define the following probability measure for each sequence in M, which we denote by $S_n(\mathbf{x})$. That is,

$$S_n(\mathbf{x}) = \frac{1}{n+1} \sum_{j=0}^{n} \delta_{x_j} \tag{3.4}$$

We can define a shift map $F : M \times M^{\mathbb{N}} \to M \times M^{\mathbb{N}}$, $(x_0, \mathbf{x}) \mapsto (x_1, \sigma(\mathbf{x}))$, where σ is the left shift on the space of random orbit, such that, $\sigma(\mathbf{x}) = (x_1, x_2, ...)$ for any

$\mathbf{x} = (x_0, x_1, x_2, \dots)$ and x_0. The stationary measure defined above is equivalent to the skew-product measure $\mu_\varepsilon \times P_\varepsilon{}^{\mathbb{N}}$, given by

$$d(\mu_\varepsilon \times P_\varepsilon{}^{\mathbb{N}})(x_0, x_1, \dots, x_j, \dots) = \mu_\varepsilon(dx_0)P_\varepsilon(dx_1|x_0)...P_\varepsilon(dx_j|x_{j-1})$$

to be invariant under the shift map. Since M is a compact metric space, by Tychonoff theorem, the space $M \times M^{\mathbb{N}}$ is also compact. Moreover, F is a continuous map, then by the Krylov-Bogolyubov Theorem, there exists at least one F-invariant probability measure on $M \times M^{\mathbb{N}}$. So, if $\mu_\varepsilon \times P_\varepsilon{}^{\mathbb{N}}$ is such an F-invariant probability measure and ψ is an integrable function, then, by the Birkhoff Ergodic Theorem,

$$\widetilde{\psi}(\mathbf{x}) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \psi(x_j) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} (\psi \circ \pi_1)(F^j(\mathbf{x}))$$

the limit above exists for $\mu_\varepsilon \times P_\varepsilon{}^{\mathbb{N}}$-a.e $(x_0, \mathbf{x}) \in M \times M^{\mathbb{N}}$, where $\pi_1 : M \times M^{\mathbb{N}} \to M$ is the projection onto the first coordinate. Thus, there exists the limit below

$$E_{x_0}(\psi) := \lim_{n\to\infty} \int \psi dS_n(\mathbf{x})$$

for μ_ε-almost every $x_0 \in M$. Moreover, the mapping $\psi \mapsto E(\psi)$, for any $x_0 \in B \subset M$, where B is a set of positive measure, defines a non-negative linear functional on the space $L^1(M)$. Thus, by the Riesz-Markov Kakutani Representation Theorem, there exists a unique Radon measure μ_ε on M. Such that,

$$\int \psi d\mu_\varepsilon = E(\psi) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \psi(x_j).$$

Hence, the measure μ_ε is ergodic if for μ_ε-almost every point $x_0 \in M$, it holds that

$$S_n(\mathbf{x}) \to \mu_\varepsilon, \text{ as } n \to \infty \tag{3.5}$$

in the $weak^\star$-topology sense in the space of probability measures on M.

Lemma 4. *The measure μ_ε is ergodic, for $\mu_\varepsilon \times P_\varepsilon{}^{\mathbb{N}}$-a.e point $(x_0, \mathbf{x}) \in M \times M^{\mathbb{N}}$, if it holds that $S_n(\mathbf{x}) \to \mu_\varepsilon$, as $n \to \infty$. Then, for every $\varphi \in \mathcal{B}$, and for each $\delta(\varepsilon) > 0$, there exists $n_0 \in \mathbb{N}$, such that,*

$$\left| \int \varphi dS_n(\mathbf{x}) - \int \varphi d\mu_\varepsilon \right| \leqslant \delta(\varepsilon), \forall n \geqslant n_0$$

Proof. Suppose that $S_n(\mathbf{x}) \to \mu_\varepsilon$ in M and take $\varphi \in \mathcal{B}$. Given $\delta(\varepsilon) > 0$, consider the neighborhood $N_{\varepsilon,\varphi}(\mu_\varepsilon) = \left\{ \nu_\varepsilon : \left| \int \varphi d\nu_\varepsilon - \int \varphi d\mu_\varepsilon \right| < \delta(\varepsilon) \right\}$. By hypothesis, there exists $n_0 \in \mathbb{N}$, such that, for all $n \geqslant n_0$ we have $S_n(\mathbf{x}) \in N_{\varepsilon,\varphi}(\mu)$, that is $\left| \int \varphi dS_n(\mathbf{x}) - \int \varphi d\mu_\varepsilon \right| < \delta(\varepsilon)$, for all $n \geqslant n_0$. □

We can define the stochastic stability stability for random perturbation systems as following.

Definition 14. *The system (f, μ) is stochastically stable under the small perturbation scheme $\{P_\varepsilon(\cdot|x) : x \in M, \varepsilon > 0\}$ if μ_ε converges, in the weak* sense, to the physical measure μ, when the noise level ε goes to zero.*

Definition 15 (Typical shadowing)**.** *We say that a dynamical system generated by f with shadowing property has the μ-typical shadowing if, for every $\varepsilon > 0$, we can find $\delta > 0$ such that, for each δ-pseudo-trajectory $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ of f there exist μ-typical points $z \in M$, such that*

$$d(f^j(z), x_j) \leqslant \varepsilon \textit{ for all } j \geqslant 0,$$

Thus, in this case, $\mathbf{x} = (x_j)_{j \in \mathbb{Z}_0^+}$ is ε-shadowing by μ-typical $z \in M$.

3.2 Main result about shadowing property and stochastic stability

Theorem 5. *Consider the dynamics generated by the continuous map $f : M \to M$, and suppose that it has a unique absolutely continuous with respect to the Lebesgue measure, invariant, ergodic probability μ and the μ-typical shadowing property. Furthermore, suppose that under the perturbation scheme $\{P_\varepsilon(\cdot|x) : x \in M, \varepsilon > 0\}$, with $P_\varepsilon(\cdot|x)$ absolutely continuous with respect to the Lebesgue measure and $supp(P_\varepsilon(\cdot|x)) \subset B_\varepsilon(f(x))$ for every $x \in M$, there is a unique stationary probability measure μ_ε for every small ε. Then, the dynamics is stochastically stable.*

Proof. **Step 1**: Given any small $\varepsilon > 0$, note that the realization of Markov chain defines, for each $x_0 \in M$ and for each $\delta(\varepsilon) > 0$, a $\delta(\varepsilon)$-pseudo-trajectory of f. That is

$$d(f(x_j), x_{j+1}) \leqslant \delta(\varepsilon) \text{ for all } j \geqslant 0,$$

where each x_{j+1} has probability distribution $P_\varepsilon(\cdot|x_j)$. Denote by $B_{\delta,\varepsilon}$ a set of realizations of the Markov chain. By hypothesis, f has the shadowing property, which means that, for every $\varepsilon > 0$, we can find a $\delta(\varepsilon) > 0$, such that, for any $\delta(\varepsilon)$-pseudo-trajectory, $\mathbf{x} \in B_{\delta,\varepsilon}$, of f, there exist $z \in M$, such that,

$$d(f^j(z), x_j) < \varepsilon, \text{ for all } j \geqslant 0. \tag{3.6}$$

Furthermore, since M is compact, by Lemma 2, every point in M is shadowable. Therefore, any $x \in M$ can be chosen as $x = x_0$ for a random orbit $(x_j)_{j \geqslant 0}$, such that, there exists $z(x_0) \in M$ satisfying $d(f^j(z), x_j) < \varepsilon,$ for all $j \geqslant 0$.

Step 2: We want to ensure that the points z can be taken μ-typical. By hypothesis, f has the shadowing property. Then, by Lemma (2), every point in M is shadowable. Given $x \in M$, an arbitrary shadowable point, let $\varepsilon > 0$ be sufficiently small. Then there exists $\delta(\varepsilon) > 0$, such that for each $\delta(\varepsilon)$-pseudo-trajectory $\mathbf{x} \in B_{\delta,\varepsilon}$ of f with $x_0 = x$, there exist $z \in M$ such that,

$$d(f^j(z), x_j) < \varepsilon, \text{ for all } j \geqslant 0. \tag{3.7}$$

Moreover, μ is ergodic if, for μ-a.e point $z \in M$, the sequence $S_n^f(z) \to \mu$, as $n \to \infty$. For such a measure, we consider the following set of points

$$Z_\mu := \left\{ z \in M : \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(z)) \to \int \varphi d\mu, \text{ when } n \to \infty \right\}, \tag{3.8}$$

defined for every continuous function $\varphi \in C(M)$. Since $C(M)$ is separable, we can take a dense sequence $(\phi_k)_k$ in $C(M)$. By the Birkhoff Ergodic Theorem, for each ϕ_k, there is a set M_k with $\mu(M_k) = 1$, such that for $z \in M_k$, it holds that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \phi_k(f^j(z)) = \int \phi_k d\mu.$$

Defining $X = \bigcap_k M_k$ we have $\mu(X) = \mu\left(\bigcap_k M_k\right) = 1$, since the countable intersection of full measure sets is a full measure set. Furthermore, on X we have for every ϕ_k

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \phi_k(f^j(z)) = \int \phi_k d\mu.$$

Let $\varphi \in C(M)$, then there is a subsequence $(\psi_k)_k$ of $(\phi_k)_k$ such that for every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ so that for every $k > K$ we have $\|\varphi - \psi_k\|_\infty \leqslant \varepsilon$. Thus, for every $z \in X$ we have

$$\begin{aligned} \left| \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right) \right. &- \left. \int \varphi d\mu \right| \\ &= \left| \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right) - \frac{1}{n+1} \sum_{j=0}^{n} \psi_k\left(f^j(z)\right) \right. \\ &\quad \left. + \frac{1}{n+1} \sum_{j=0}^{n} \psi_k\left(f^j(z)\right) - \int \psi_k d\mu + \int (\psi_k - \varphi) d\mu \right|. \end{aligned}$$

Note that,

$$\left| \int (\psi_k - \varphi) d\mu \right| \leqslant \varepsilon \mu(M),$$

$$\left| \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right) - \frac{1}{n+1} \sum_{j=0}^{n} \psi_k\left(f^j(z)\right) \right| \leqslant \varepsilon,$$

and

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \psi_k\left(f^j(z)\right) - \int \psi_k d\mu = 0.$$

Then, by the triangle inequality we obtain,

$$\limsup_n \left| \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right) - \int \varphi d\mu \right| \leqslant \varepsilon(1 + \mu(M)),$$

which is valid for every ε. Thus,

$$\limsup_n \left| \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right) - \int \varphi d\mu \right| = 0.$$

Hence,

$$\lim_n \left| \frac{1}{n+1} \sum_{j=0}^{n} \varphi\left(f^j(z)\right) - \int \varphi d\mu \right| = 0.$$

Therefore, we conclude that $\mu(Z_\mu) = 1$.

Note also that, since μ is absolutely continuous with respect to the Lebesgue measure m, having $\mu(Z_\mu) = 1$ implies $m(Z_\mu) > 0$, so the measure μ is physical. We do not require each random trajectory to be shadowed by a unique point z.

Step 3: Considering a given random trajectory, we can prove that for every fixed $\varepsilon > 0$, for an observable function φ in the set of bounded continuous function $\mathcal{B}$, and for any sufficiently large $N \in \mathbb{N}$, there exists a constant C_φ, such that

$$\left| \int \varphi dS_n^f(z) - \int \varphi dS_n(\mathbf{x}) \right| \leqslant C_\varphi \varepsilon, \text{ for all } n > N, \tag{3.9}$$

holds. Indeed,

$$\begin{aligned} &\left| \int \varphi dS_n^f(z) - \int \varphi dS_n(\mathbf{x}) \right| \\ &\quad = \left| \int \varphi d\left(\frac{1}{n+1} \sum_{j=0}^{n} \delta_{f^j(z)} \right) - \int \varphi d\left(\frac{1}{n+1} \sum_{j=0}^{n} \delta_{x_j} \right) \right| \\ &\quad \leqslant \frac{1}{n+1} \sum_{j=0}^{n} \left| \int \varphi d\delta_{f^j(z)} - \int \varphi d\delta_{x_j} \right| \\ &\quad = \frac{1}{n+1} \sum_{j=0}^{n} \left| \varphi(f^j(z)) - \varphi(x_j) \right| \\ &\quad \leqslant \frac{1}{n+1} \sum_{j=0}^{n} \sup_{z \in M} \left| \varphi(f^j(z)) - \varphi(x_j) \right|. \end{aligned} \tag{3.10}$$

Since $\varphi \in \mathcal{B}$, there exists a constant $C_\varphi > 0$, such that,

$$\sup_{z \in M} |\varphi(z)| \leqslant C_\varphi.$$

Note that, we are dealing with continuous functions on compact sets, then by Weierstrass approximation theorem they can be approximated by polynomials. So, we can bound such functions by their Lipschitz constants. By abuse of notation, we can write from the equation (3.10) for every $j \geqslant 0$,

$$\sup_{z \in M} \left| \varphi(f^j(z)) - \varphi(x_j) \right| \leqslant \sup_{z \in M} |\varphi(z)| d(f^j(z), x_j).$$

Thus, we have

$$\left|\int \varphi dS_n^f(z) - \int \varphi dS_n(\mathbf{x})\right| \leqslant C_\varphi \varepsilon, \text{ for all } n \geqslant N.$$

Step 4: Finally we show stochastic stability. In fact, by applying the triangle inequality, we obtain the following result:

$$\begin{aligned}\left|\int \varphi d\mu - \int \varphi d\mu_\varepsilon\right| &\leqslant \left|\int \varphi d\mu - \int \varphi dS_n^f(z)\right| + \left|\int \varphi dS_n(\mathbf{x}) - \int \varphi d\mu_\varepsilon\right| \\ &+ \left|\int \varphi dS_n^f(z) - \int \varphi dS_n(\mathbf{x})\right| \\ &\leqslant (C_\varphi + 1)\varepsilon + \delta(\varepsilon).\end{aligned}$$

Observe that the first term from the right-hand side of the inequality follows from Lemma 3, which, as shown in Step 2, can be applied to almost all orbits of f, with z chosen μ-almost surely. The second term follows from Lemma 4, which can be applied to almost all random trajectories. The last term is a consequence from (3.9). Moreover, since $B_{\delta,\varepsilon}$ is a set of realizations of Markov chain, and the measure $S_n(\mathbf{x})$ is defined as (3.4), from (3.5) and ergodicity assumption we conclude that in the limit $\mu_\varepsilon(B_{\delta,\varepsilon}) \to 1$ as $N \to \infty$. This completes the proof. □

4 Random Attractor

In this chapter, we explore random attractors in systems subject to perturbations, specifically those affected by noise. There are several methods to introduce such perturbations. In this section, the dynamics are perturbed by a family of parameterized random diffeomorphisms. First, we will prove the stochastic stability of the the random attractor. Then, we will establish the theorem regarding the existence and finiteness of random attractors. Additionally, we assume throughout this chapter that M is a finite-dimensional compact metric space with a distance $d : M \times M \to \mathbb{R}$, induced by some Riemannian structure, and that $f : M \to M$ is a diffeomorphim.

4.1 Iteration of random maps and random attractor

In this section following (Ara00), we describe the perturbation acting on the dynamics as a family of parameterized random diffeomorphisms. At each iteration, we chose the maps randomly in the neighborhood of the original one. Let $F : M \times B_\varepsilon(0) \to M$, $(x, t) \mapsto f_t(x) = f(x, t)$, be C^1-map where every $t \in B_\varepsilon(0)$ is chosen randomly according to the probability law ν_ε on $B_\varepsilon(0)$. Furthermore, we assume that $f_t : M \to M$, $x \mapsto f(x, t)$ is a diffeomorphism for every $t \in B_\varepsilon(0)$, where $B_\varepsilon(0)$ is a ball of radius $\varepsilon > 0$ on $\mathbb{R}^n$. In addition, the unperturbed map f must be contained at the origin of the family of diffeomorphisms, that is $f_0 \equiv f$.

Moreover, for a given $\varepsilon > 0$, we define the perturbation space

$$\Delta_\varepsilon = \{\underline{t} = (t_j)_{j \geqslant 1}^{\infty} : t_j \in B_\varepsilon(0)\},$$

with the product topology and the measure infinite product probability measure ν_ε^∞ on Δ_ε, where ν_ε is the normalized Lebesgue volume measure restricted to the closure of $B_\varepsilon(0)$. Specifically, for sets $B_1, ...B_k$ from the Borel σ- algebra of family $B_\varepsilon^n(0)$, we define as follows

$$\nu_\varepsilon^\infty(B_1 \times ... \times B_k \times \overline{B_\varepsilon^n}(0)^{\mathbb{N}}) = \nu_\varepsilon(B_1)...\nu_\varepsilon(B_k).$$

and, if $A \subset B_\varepsilon^n(0)$, then

$$\nu_\varepsilon(A) = \frac{Leb(A)}{Leb(\overline{B_\varepsilon^n}(0))}$$

where $Leb(A)$ will mean the Lebesgue volume measure of A.

On the other hand, the perturbed iterate of $x \in M$ is defined by the perturbed vector $\underline{t} \in \Delta_\varepsilon$, as

$$f_{\underline{t}}^j(x) = f^j(x, \underline{t}) = f_{t_j} \circ \cdots \circ f_{t_1}(x), \quad for \quad j \geqslant 1, \quad \underline{t} \in \Delta_\varepsilon \quad and \quad f^0(x, \underline{t}) = x.$$

Furthermore, we shall use the convention

$$f_V^k(U) = f^k(U,V) = \{f_{\underline{t}}^k(x) : \underline{t} \in V, \quad x \in U\}, \quad U \subset M, \quad V \subset \Delta_\varepsilon \tag{4.1}$$

for every $k \geqslant 1$, and we remark that a different sequence $\underline{t}$ is considered, referred to as$\{f_{\underline{t}}^j(x)\}_{j=1}^{\infty}$, the t-orbit of x, for which we also write $\mathcal{O}(x,\underline{t})$, for all $x \in U$. The next definitions which follows partly the papers of (Ara01; Rul81) we shall introduce concepts of attractor and random attractor.

Definition 16. *A compact set Λ is an attractor for the system f if Λ is f-invariant, and there is an open neighborhood U_Λ of Λ, called trapping region, such that*

$$f(\overline{U}_\Lambda) \subset U_\Lambda \quad \textit{and} \quad \Lambda = \bigcap_{n \geqslant 0} f^n(\overline{U}_\Lambda).$$

Moreover, the basin of attraction of Λ is the set

$$W^s(\Lambda) = \{x \in M : \lim_{n \to +\infty} dist(f^k(x), \Lambda) = 0\}$$

We say that an attractor is a *random attractor* if it is an attractor for the perturbed system $F : M \times B_\varepsilon(0) \to M$. The random attractor will be denoted by Λ^ε and is defined as follows:

Definition 17. *A compact set Λ^ε is a random attractor for the system F if Λ^ε is f_t-invariant, and there is an open neighborhood U_{Λ^ε} of Λ^ε called trapping region, such that*

$$f_t(\overline{U}_{\Lambda^\varepsilon}) \subset U_{\Lambda^\varepsilon} \quad \textit{and} \quad \Lambda^\varepsilon = \bigcap_{n \geqslant 0} f_t^n(\overline{U}_{\Lambda^\varepsilon}).$$

We define the probability measures μ and μ_ε on M as follows: First, By Proposition 2.1 (Ara01). Let Λ be an attractor for f, then there are a neighborhood U_Λ of Λ and $\varepsilon_0 \in (0,1)$ such that for $\varepsilon \in (0, \varepsilon_0)$, there is a unique probability measure μ^ε such that $\Lambda \subset \text{supp}\mu^\varepsilon \subset U_\Lambda$ and for all $x \in U_\Lambda$ and ν_ε^∞ almost every $\underline{t} \in \Delta_\varepsilon$, we have for each continuous function $\varphi : M \to \mathbb{R}$, the following holds:

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(x, \underline{t})) = \int \varphi d\mu^\varepsilon. \tag{4.2}$$

Second, following the approach outlined (Ara00), we can define a shift operator $S : M \times \Delta_\varepsilon \to M \times \Delta_\varepsilon$, by the rule: $(x, \underline{t}) \mapsto (f_{t_1}(x), \sigma(\underline{t}))$, where σ is the left shift acting on sequences in Δ_ε; defined such that $\sigma(\underline{t}) = \underline{r}$ when $\underline{t} = (t_1, t_2, t_3, ...)$ and $\underline{r} = (t_2, t_3, ...)$. Moreover, since M is a compact metric space and f is continuous, there exists at least one f-invariant probability measure μ on M. This probability measure is said stationary measure if the measure $\mu \times \nu_\varepsilon^\infty$ is invariant under the shift operator S. Additionally, μ is a stationary ergodic probability measure if $\mu \times \nu_\varepsilon^\infty$ is S-ergodic. By Birkhoff's ergodic theorem, we have

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} \psi(S^j(x, \underline{s})) = \int \psi d(\mu \times \nu_\varepsilon^\infty)$$

for $\mu \times \nu_\varepsilon^\infty$-a.e $(x, \underline{s}) \in M \times \Delta_\varepsilon$ and for every $\psi \in C^0(M \times \Delta_\varepsilon)$. Consider the projection on the first component $\pi : M \times \Delta_\varepsilon \to M$, where we take $\psi = \varphi \circ \pi$, with $\varphi \in C^0(M)$, then we have

$$\psi(S^j(x, \underline{s})) = \varphi(f^j(x, \underline{s})), \quad j = 1, 2, ... \quad and \quad \int \psi d(\mu \times \nu_\varepsilon^\infty) = \int \varphi d\mu.$$

Thus, for every continuous function $\varphi : M \to \mathbb{R}$, the following limit exists for $\mu \times \nu_\varepsilon^\infty$-a.e $(x, \underline{s}) \in M \times \Delta_\varepsilon$.

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(x, \underline{s})) = \int \varphi d\mu \tag{4.3}$$

Lemma 5. *Let Λ^ε be a random attractor of F and μ^ε be the probability measure defined above, then $\Lambda^\varepsilon \subset supp(\mu^\varepsilon)$.*

Proof. Let $x \in \Lambda^\varepsilon$. By the definition of random attractor, we know $x \in \bigcap_{k\geqslant 0} f_{\underline{t}}^k(\overline{U}_{\Lambda^\varepsilon})$. Then $x \in f_{\underline{t}}^k(\overline{U}_{\Lambda^\varepsilon})$ for all $k \geqslant 0$. Moreover, using the definition (17), we conclude, $x \in U_{\Lambda^\varepsilon}$. Since Λ^ε is a compact invariant nonempty set and μ^ε is a probability measure, then $\mu^\varepsilon(U_{\Lambda^\varepsilon}) > 0$. Hence $x \in supp(\mu^\varepsilon)$. This shows that

$$\Lambda^\varepsilon \subset supp(\mu^\varepsilon).$$

□

Lemma 6. *Let $f : M \to M$ be a diffeomorphism of a compact finite dimensional metric space M. We assume that a random perturbation F of f is given. Let Λ^ε be a random attractor of F, and Λ be an attractor of f, then $\Lambda^\varepsilon \to \Lambda$ in the Hausdorff topology when $\varepsilon \to 0$.*

Proof. We know that $\Lambda^\varepsilon \subset supp(\mu^\varepsilon)$ for all small $\varepsilon > 0$. Given any neighborhood V of Λ^ε, take $U_{\Lambda^\varepsilon}^k = \bigcap_{i=0}^{k} f_{\underline{t}}^i(U_{\Lambda^\varepsilon})$ such that $dist(U_{\Lambda^\varepsilon} \backslash U_{\Lambda^\varepsilon}^k, \Lambda^\varepsilon) \to 0$ when $k \to \infty$, that is $U_{\Lambda^\varepsilon}^k \subset V$ for big enough k, and $\Lambda^\varepsilon \subset f_t(\overline{U^k}_{\Lambda^\varepsilon}) \subset U_{\Lambda^\varepsilon}^k$ for all $k \geqslant 1$. By continuity, we have $diam(f^i(x, \Delta_\varepsilon)) = sup\{d(f^i(x, \underline{t}), f^i(x, \underline{s})); \underline{t}, \underline{s} \in \Delta_\varepsilon\} \to 0$ when $\varepsilon \to 0$. Therefore, for sufficiently small $\delta_0 > 0$, the open set V is a trapping region with respect to f_t for all $\varepsilon \in (0, \delta_0)$, and every $t \in B_\varepsilon$. Since $f_t(U_{\Lambda^\varepsilon}) \subset U_{\Lambda^\varepsilon}$, then exists some physical probability measure μ^ε with $supp(\mu^\varepsilon) \subset U_{\Lambda^\varepsilon}^k$. By theorem 3.1 (Ara00), for all $\varepsilon \in (0, \delta_0)$ with $\delta(k) \to 0$ as $k \to \infty$. This show that $supp(\mu^\varepsilon) \to \Lambda^\varepsilon$ in the Hausdorff topology when $\varepsilon \to 0$.
Since $supp(\mu^\varepsilon) \to \Lambda^\varepsilon$ and by proposition 2.1 of (Ara00) we have that $supp(\mu^\varepsilon) \to \Lambda$. Then we can write as $h(\Lambda^\varepsilon, \Lambda) \leqslant h(\Lambda^\varepsilon, supp(\mu^\varepsilon)) + h(supp(\mu^\varepsilon), \Lambda)$. Thus, $h(\Lambda^\varepsilon, \Lambda) \to 0$ as $\varepsilon \to 0$. □

By following (Ara01) we introduce the following definition:

Definition 18. *An attractor Λ with respect to the dynamical system f is stochastically stable with respect to a perturbation F if it supports an f-invariant probability measure μ ($supp\mu = \Lambda$) such that μ_ε converges to μ in the weak* sense.*

Proposition 1. *A random attractor is stochastically stable.*

Proof. Let Λ^ε be an attractor of F, and Λ be an attractor with respect to f where F is a random perturbation of f. Then by equation (4.2), there is a unique probability measure μ^ε such that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\varphi(f^j(x,\underline{t}))=\int\varphi d\mu^\varepsilon$$

for each $\varphi\in C(M)$. Moreover, by equation (4.3) we have,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\varphi(f^j(x,\underline{s}))=\int\varphi d\mu$$

Then by equations (4.2) and (4.3) we have,

$$\begin{aligned}\left|\int\varphi d\mu-\int\varphi d\mu^\varepsilon\right| &= \left|\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\varphi(f^j(x,\underline{s})-\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\varphi(f^j(x,\underline{t}))\right|\\ &= \lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\left|\varphi(f^j(x,\underline{s}))-\varphi(f^j(x,\underline{t})\right|.\end{aligned}$$

Since M is a compact metric space and $\varphi\in C(M)$, then φ is bounded. Thus there exists $\sup_{x\in M}|\varphi(x)|<\infty$, and

$$\begin{aligned}\left|\int\varphi d\mu-\int\varphi d\mu^\varepsilon\right| &\leqslant \sup_{x\in M}|\varphi(x)|\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}d(f^j(x,\underline{t}),f^j(x,\underline{s}))\\ &\leqslant \sup_{x\in M}|\varphi(x)|\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\sup_{\underline{t},\underline{s}\in\Delta_\varepsilon}d(f^j(x,\underline{t}),f^j(x,\underline{s}))\\ &\leqslant \sup_{x\in M}|\varphi(x)|\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}diam(f^j(x,\Delta_\varepsilon))\\ &\leqslant \sup_{x\in M}|\varphi(x)|\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\sup_{j=0,1,\ldots,n}\{diam(f^j(x,\Delta_\varepsilon))\}\\ &\leqslant \sup_{x\in M}|\varphi(x)|\sup_{j=0,\ldots,n}\{diam(f^j(x,\Delta_\varepsilon))\}\end{aligned}$$

Since F is continuous, the $diam(f^j(x,\Delta_\varepsilon))\to 0$ when $\varepsilon\to 0$. Hence $\left|\int\varphi d\mu^\varepsilon-\int\varphi d\mu\right|\to 0$, when $\varepsilon\to 0$. ☐

4.2 Invariant domains

In this section, following (Ara00), we introduce the concept of invariant domains. Those domains will play a key rule in demonstrating the uniqueness of random attractor.

Definition 19. *An invariant domain for $F : M \times B_\varepsilon(0) \to M$ is defined as a finite collection $\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}$ of pairwise disjoint open sets, i.e; $\overline{\mathcal{U}}_i \cap \overline{\mathcal{U}}_j = \varnothing$ for $i \neq j$, such that $f^k(\mathcal{U}_0, \Delta_\varepsilon) \subset \mathcal{U}_{k \mod r}$ for all $k \geqslant 1$. This collection is denoted by $D = \{\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}\}$, where $r \in \mathbb{N}$ will be referred to as the period of the random attractor.*

Definition 20. *An invariant domain such that*

$$f^k(\mathcal{U}_i, \Delta_\varepsilon) \subset \mathcal{U}_{(k+i) \mod r}, \; for\ all\ k \geqslant 1,$$

where $i \in \{0, \ldots, r-1\}$, is called symmetric invariant.

Remark 1. *Additionally, since the f_t are diffeomorphisms for all $t \in T$ and the collection $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ is F-invariant, then $\overline{D} = (\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1})$ also satisfies $f^k(\mathcal{U}_i, \Delta_\varepsilon) \subset \mathcal{U}_{(k+i) \mod r}$ and conversely: if the closure $\overline{D} = (\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1})$ satisfies $f^k(\mathcal{U}_i, \Delta_\varepsilon) \subset \mathcal{U}_{(k+i) \mod r}$ with $\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}$ pairwise disjoint open sets, then $D = (\mathcal{U}_0, \ldots, \mathcal{U}_{r-1})$ is an F-invariant domain.*

4.3 Existence and Finiteness of Random Attractors

Theorem 6 (Existence and finiteness of random attractors)**.** *Let $f : M \to M$ be a C^r diffeomorphism of class $r \geqslant 1$, of a compact, boundaryless manifold M satisfying:*

A. There exist open sets $U \subset M$, such that $f|U$ is a contraction.

Then the following holds:

1) There is $\varepsilon_0 = \varepsilon_0(L)$, representing the maximum allowable perturbation, such that for all $\varepsilon < \varepsilon_0$, the perturbed system F of f remains a contraction on U;

2) Each collection $D_{\Lambda_i} = \{\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}\}$ serves a trapping region U_{Λ_i} for a unique random attractor Λ_i for any $\varepsilon < \varepsilon_0$;

3) The sets $\{U_{\Lambda_i}\}_{i=1}^N$ are pairwise disjoints;

4) There exists a finite number N of random attractors $\{\Lambda_i\}_{i=1}^N$ on M;

5) Each random attractor Λ_i^ε supports a unique probability measure.

Further, assuming the additional condition:

B. The basins of attractions of the $\{\Lambda_i\}_{i\geqslant 1}$ cover Lebesgue almost every point of M with respect to the Lebesgue measure, that is

$$\bigcup_{i=1}^{N} W^s(\Lambda_i) = M$$

we also prove that,

6) *The time average of Lebesgue almost all orbits under perturbation $\mathcal{O}(x,\underline{t})$ is represented by a finite number of probability measures.*

We now begin the prove of theorem as follows:

1) First item we prove with the following proposition.

Proposition 2. *Let $f : M \to M$, be a map with Lipschitz constant, $Lip(f) := L < 1$. Then for any $\varepsilon < \dfrac{1-L}{2}$, the perturbed map $F : M \times B_\varepsilon \to M$, is a contraction for all $x, y \in M$ such that $d(x, y) > 2\varepsilon$ under any probability law ν_ε.*

Proof. For all $x, y \in M$ and any $t \in B_\varepsilon(0)$, under the probability measure ν_ε we have the following

$$d(F(x,t), F(y,t)) \leqslant d(f(x), f(y)) + 2\varepsilon.$$

By hypothesis, there exist open set $U \subset M$, such that $f|U$ is a contraction. Then

$$d(f(x), f(y)) + 2\varepsilon \leqslant Ld(x, y) + 2\varepsilon, \forall x, y \in U$$

Moreover, we want $F : M \times B_\varepsilon \to M$ to contract distances, we say that

$$Ld(x, y) + 2\varepsilon < d(x, y),$$

thus

$$2\varepsilon < (1 - L)d(x, y).$$

Since $L < 1$, it follows that for every $x, y \in X$, such that $d(x, y) > 2\varepsilon$, and

$$\varepsilon < \frac{1-L}{2}$$

the perturbed system $F : M \times B_\varepsilon \to M$ is a contraction. □

Therefore, if we restrict f on an open set $U \subset M$, there is $\varepsilon_0(L)$, such that $F|U$ is a contraction.

2) Each collection $D_{\Lambda_i^\varepsilon} = \{\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}\}$ is a trapping region U_{Λ_i} of a unique random attractor Λ_i^ε for any $\varepsilon < \varepsilon_0$.

Proof. Suppose that $D = \{\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}\}$ is a trapping region of two different random attractors, Λ_i^ε and Λ_j^ε, where $0 \leqslant i, j \leqslant r - 1$ and $i \neq j$. Since $\Lambda_{i,j}^\varepsilon$ are random attractors, they are f_t-invariants:

$$\Lambda_i^\varepsilon = f(\Lambda_i^\varepsilon, \Delta_\varepsilon) \ \ and \ \ \Lambda_j^\varepsilon = f(\Lambda_j^\varepsilon, \Delta_\varepsilon).$$

Using induction, it follows that

$$\Lambda_i^\varepsilon = f^k(\Lambda_i^\varepsilon, \Delta_\varepsilon)$$

Moreover, D is an invariant domain, it holds that $f^k(\mathcal{U}_i, \Delta_\varepsilon) \subset \mathcal{U}_{(i+k) \mod r}$ according to Remark(1). Since f_t are diffeomorphisms for all $t \in T$, if $D = \{\mathcal{U}_0, \ldots, \mathcal{U}_{r-1}\}$ is F-invariant, then $\overline{D} = \{\overline{\mathcal{U}}_0, \ldots, \overline{\mathcal{U}}_{r-1}\}$ also satisfies definition (20). Thus, $f^k(\overline{\mathcal{U}}_i, \Delta_\varepsilon) \subset \overline{\mathcal{U}}_{(i+k) \mod r}$. Assume $0 \leqslant i < j \leqslant r-1$, then there is exists $k > 0$ such that $j = i + k \mod r$. Since, Λ_i is a closed set, by Remark(1) , we have

$$\Lambda_i^\varepsilon = f^k(\Lambda_i^\varepsilon, \Delta_\varepsilon) \subset \Lambda_{i+k \mod r}^\varepsilon = \Lambda_j^\varepsilon$$

Similarly, we can show that $\Lambda_j^\varepsilon \subset \Lambda_i^\varepsilon$. Thus $\Lambda_i^\varepsilon = \Lambda_j^\varepsilon$. □

3) Suppose that a random perturbation F of f is given and there is a family of pairwise disjoint compact $\{\Lambda_i^\varepsilon\}_{i=1}^N$, $N \in \mathbb{N}$. Then the sets $\{U_{\Lambda_i^\varepsilon}\}_{i=1}^N$ are also pairwise disjoint.

Proof. Let $\{\Lambda_i^\varepsilon\}_{i=1}^N$ be a family of pairwise disjoint compact sets, where $N \in \mathbb{N}$. Then, for each $i \in \{1, \ldots, N\}$ there is an open neighborhood $U_{\Lambda_i^\varepsilon}$ of Λ_i^ε, such that

$$f_t(\overline{U}_{\Lambda_i^\varepsilon}) \subset U_{\Lambda_i^\varepsilon} \ \ and \ \ \Lambda_i^\varepsilon = \bigcap_{n \geqslant 0} f_t^n(\overline{U}_{\Lambda_i^\varepsilon}), \forall i = 1, 2, ..., N.$$

Since $\overline{U}_{\Lambda_i^\varepsilon}$ is compact and $f_t(\overline{U}_{\Lambda_i^\varepsilon}) \subset U_{\Lambda_i^\varepsilon}$, it follows by induction that

$$\overline{U}_{\Lambda_i^\varepsilon} \supset U_{\Lambda_i^\varepsilon} \supset f_t(\overline{U}_{\Lambda_i^\varepsilon}) \supset f_t^2(\overline{U}_{\Lambda_i^\varepsilon}) \supset \cdots \supset f_t^n(\overline{U}_{\Lambda_i^\varepsilon}) \supset, ...$$

Since $\{f_t^n(\overline{U}_{\Lambda_i^\varepsilon})\}_{n \geqslant 0}$ is a nested sequence of a compact sets, it follows that

$$\bigcap_{n \geqslant 0} f_t^n(\overline{U}_{\Lambda_i^\varepsilon}) \neq \varnothing$$

Moreover, by hypothesis the family $\{\Lambda_i^\varepsilon\}_{i=1}^N$ are pairwise disjoint. That is, for any $i \neq j$, it follows that

$$\Lambda_i^\varepsilon \bigcap \Lambda_j^\varepsilon = \varnothing$$

On the other hand, since f is a diffeomorphism we have

$$\varnothing = \Lambda_i^\varepsilon \bigcap \Lambda_j^\varepsilon = \bigcap_{n \geqslant 0} f_t^n(\overline{U}_{\Lambda_i^\varepsilon} \bigcap \overline{U}_{\Lambda_j^\varepsilon})$$

Therefore, since the intersection $\bigcap_{n \geqslant 0} f_t^n(\overline{U}_{\Lambda_i^\varepsilon} \bigcap \overline{U}_{\Lambda_j^\varepsilon})$ are disjoint for $i \neq j$, and the diffeomorphism preserves disjointness, the open neighborhoods $\{U_{\Lambda_i^\varepsilon}\}_{i=1}^N$ are pairwise disjoints.

□

4) There exist a finite number of random attractors $\{\Lambda_i^{\varepsilon}\}_{i=1}^{N}$.

Proof. Let $\{\Lambda_i^{\varepsilon}\}_{i\geqslant 1}$ be a collection of random attractors where $\Lambda_i^{\varepsilon} \subset M$ for all $i \geqslant 1$. Since each $\Lambda_i^{\varepsilon} \subset M$ is a closed set, the collection $\{(\Lambda_i^{\varepsilon})^c\}_{i\geqslant 1}$ forms an open cover of M. Given that M is a compact set, we can extract a finite subcover $\{(\Lambda_i^{\varepsilon})^c\}_{i=1}^{i=N}$. This implies there are exists a finite number of random attractors, $\{\Lambda_i^{\varepsilon}\}_{i=1}^{i=N}$, where $N \in \mathbb{N}$.

Therefore, the number of random attractors is finite. □

5) Each random attractor supports a unique probability measure.

Proof. Let Λ^{ε} be a random attractor, and suppose there exist two distinct probability measure μ_1^{ε} and μ_2^{ε} on M supported on Λ^{ε}, such that

$$\Lambda^{\varepsilon} = supp({\mu_1}^{\varepsilon}) \text{ and } \Lambda^{\varepsilon} = supp({\mu_2}^{\varepsilon})$$

Let μ be an f-invariant probability measure on M, and let $\varphi : M \to \mathbb{R}$ be a bounded continuous function. The total variation distance is given by:

$$\begin{aligned} d(\mu_1^{\varepsilon}, \mu_2^{\varepsilon}) &= sup\{|\int \varphi d\mu_1^{\varepsilon} - \int \varphi d\mu_2^{\varepsilon}| : \varphi \in \mathcal{B}\} \\ &= sup\{|\int_{\Lambda^{\varepsilon}} \varphi d\mu_1^{\varepsilon} - \int_{\Lambda^{\varepsilon}} \varphi d\mu_2^{\varepsilon}| : \varphi \in \mathcal{B}\} \\ &\leqslant sup\{|\int_{\Lambda^{\varepsilon}} \varphi d\mu_1^{\varepsilon} - \int_{\Lambda^{\varepsilon}} \varphi d\mu| : \varphi \in \mathcal{B}\} + sup\{|\int_{\Lambda^{\varepsilon}} \varphi d\mu_2^{\varepsilon} - \int_{\Lambda^{\varepsilon}} \varphi d\mu| : \varphi \in \mathcal{B}\} \end{aligned}$$

By proposition (1), we know that a random attractor is stochastically stable, as a result $d(\mu_1^{\varepsilon}, \mu_2^{\varepsilon}) = 0$ when $\varepsilon \to 0$. Therefore, $\mu_1^{\varepsilon} = \mu_2^{\varepsilon}$

□

[B] The basins of attractions of the Λ_i cover Lebesgue almost every point of M, i.e., $\bigcup_{i=1}^{N} W^s(\Lambda_i) = M$, then we further prove that,

6) The time average of Lebesgue almost all orbits under perturbation, $\mathcal{O}(x, \underline{t})$, is given by a finite number of probability measures.

Proof. We assume that $\bigcup_{i=1}^{N} W^s(\Lambda_i) = M$. Then there are exists a finite number of random attractors $\{\Lambda_i\}_{i=1}^{N}$. In addition, by item b) of proposition 2.1 of (Ara00) there is only one for every probability measure μ_i^{ε} such that $\Lambda_i \subset supp(\mu_i^{\varepsilon}) \subset U_{\Lambda_i}$ and for all $x \in U$ and $\nu_{\varepsilon}^{\infty}$ almost every $\underline{t} \in \Delta_{\varepsilon}$ we have for each continuous $\varphi : M \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(x, \underline{t})) = \int \varphi d\mu_i^{\varepsilon}$$

where $i = 1, 2, \cdots, N$. This mean that, the time average of Lebesgue almost all orbits under perturbation is given by $\mu_i^{\varepsilon}, 0 < i \leqslant N$, □

Proof. Let Λ_i be the random attractors, and assume that the basins of attraction of the Λ_i cover almost every point of M, that is,

$$M = \bigcup_{i=1}^{N} W^s(\Lambda_i)$$

W^s denotes the basin attraction of Λ_i. This implies that are finitely many random attractor $\Lambda_i, ..., \Lambda_N$ whose basin attraction cover M. By proposition 2.1(item b) in (Ara00), for each random attractor Λ_i, there exists a unique probability measure μ_i^ε, such that

$$\Lambda_i \subset supp(\mu_i^\varepsilon) \subset U_{\Lambda_i}$$

where U_{Λ_i} is a neighborhood of Λ_i. Additionally, for almost every M, and for almost every $\underline{\mathrm{t}}$ under the perturbation ν_ε^∞, we have that the time averages of the trajectory of x, denoted by $O(x, \underline{\mathrm{t}})$, converges to the expected value with respect to μ_i^ε:

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \varphi(f^j(x, \underline{\mathrm{t}})) = \int \varphi d\mu_i^\varepsilon$$

for any function $\varphi : M \to \mathbb{R}$. Thus, for Lebesgue almost every orbit under perturbation, time average is given by a finite number of probability measures μ_i^ε corresponding to the random attractors Λ_i's. In conclusion, the time average of almost all orbits under perturbation is determined by the measures $\mu_i^\varepsilon, i = 1, 2, ..., N$. □

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