UNIVERSIDADE ESTADUAL DE CAMPINAS
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Stability Analysis and Output-Feedback Control for Takagi-Sugeno Fuzzy Systems via Linear Matrix Inequalities

Análise de Estabilidade e Controle por Realimentação de Saída para Sistemas Nebulosos Takagi-Sugeno via Desigualdades Matriciais Lineares

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Supervisor/Orientador: Prof. Dr. Ricardo Coração de Leão Fontoura de Oliveira

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Resumo

Esta tese investiga os problemas de análise de estabilidade e síntese de controladores por realimentação de estados e de saída por meio da Compensação Distribuída Paralela (do inglês, *Parallel Distributed Compensation* — PDC) para sistemas nebulosos Takagi-Sugeno (T-S) de tempo contínuo e discreto. A principal contribuição no domínio do tempo contínuo é o projeto de controladores PDC globais e regionais utilizando funções de Lyapunov polinomiais homogêneas de graus maiores que dois nos estados do sistema, generalizando assim os resultados baseados em funções de Lyapunov quadráticas nos estados. Para sistemas nebulos T-S de tempo discreto, a principal novidade reside em uma modelagem especial das funções de pertinência, que elimina a necessidade de conhecimento de limites nas taxas de variação dessas funções. As condições de síntese para ambos os casos contínuo e discreto são formuladas em termos de algoritmos iterativos localmente convergentes baseados em desigualdades matriciais lineares. Esses algoritmos são capazes de abordar a estabilização, maximizar a taxa de decaimento e fornecer uma estimativa para a região de atração. Uma vantagem chave desta classe de algoritmos é que os ganhos do PDC são considerados como variáveis de otimização, sem exigir a habitual mudança de variáveis. Isto facilita o tratamento de restrições estruturais, tais como limites de magnitude nas entradas dos ganhos e descentralização. Experimentos numéricos são fornecidos para ilustrar as vantagens e aplicabilidade dos resultados propostos.

Palavras-chave: Sistemas não lineares; Sistemas nebulosos Takagi-Sugeno; Estabilização global; Estabilização regional; Realimentação de estados; Realimentação de saída; Região de atração; Desigualdades matriciais lineares.

Abstract

This thesis investigates the problems of stability analysis and synthesis of state- and output-feedback controllers through Parallel Distributed Compensation (PDC) for both continuous-and discrete-time Takagi-Sugeno (T-S) fuzzy systems. The main contribution in the continuous-time domain is the design of global and regional PDC controllers utilizing homogeneous polynomial Lyapunov functions of degrees greater than two on the system states, thus generalizing the results based on quadratic-on-the-state Lyapunov functions. For discrete-time T-S fuzzy systems, the main novelty lies in a special modeling of the membership functions, which eliminates the need for knowing bounds on the variation rates of those functions. The synthesis conditions for both continuous- and discrete-time cases are formulated in terms of locally-convergent iterative algorithms based on linear matrix inequalities. These algorithms are capable of addressing stabilization, maximizing the decay rate, and providing an estimation for the region of attraction. A key advantage of this class of algorithms is that the PDC gains are handled as optimization variables, without requiring the usual change of variables. This facilitates the treatment of structural constraints, such as magnitude bounds on the entries of the gains and decentralization. Numerical experiments are provided to illustrate the advantages and applicability of the proposed results.

Key words: Nonlinear systems; Takagi-Sugeno fuzzy systems; Global stabilization; Regional stabilization; State-feedback; Output-feedback; Region of attraction; Linear Matrix Inequalities.

Notation

Basic Sets

$\mathbb{N}_r$	set of the first r positive integers, i.e., $\mathbb{N}_r = \{1, \ldots, r\}$
$\mathbb{R}$	space of real numbers
$\mathbb{R}^{n \times m}$	set of $n \times m$ real matrices
$\mathbb{S}^n$	set of $n \times n$ real symmetric matrices
I_n	$n \times n$ identity matrix
$0_{m \times n}$	$m \times n$ null matrix
e_j	unity vector of appropriate dimension, i.e., $e_j = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}^\top$
$\mathbb{1}_r$	vector of ones of dimension r, i.e., $\mathbb{1}_r = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^\top \in \mathbb{R}^{r \times 1}$
Λ_r	unit simplex of dimension r, i.e., $\Lambda_r = \left\{ \lambda \in \mathbb{R}^r : \sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i \geq 0,\ i \in \mathbb{N}_r \right\}$

Elementary Functions and Operators

$\star$	block induced symmetry in a square matrix
$X^\top$	transpose of $X \in \mathbb{R}^{m\times n}$
$\mathrm{He}(X)$	$X + X^\top$, with $X \in \mathbb{R}^{n\times n}$
$\mathrm{tr}(X)$	trace of X
$\mathrm{co}\{\mathcal{R}\}$	convex hull of the elements of the set $\mathcal{R}$
$X > 0$	symmetric positive definite matrix $X \in \mathbb{S}^n$, i.e., $X > 0 \leftrightarrow y^\top X y > 0 \quad \forall y \in \mathbb{R}^n, \quad y \neq 0$
$X \geq 0$	symmetric positive semidefinite matrix $X \in \mathbb{S}^n$, i.e., $X \geq 0 \leftrightarrow y^\top X y \geq 0 \quad \forall y \in \mathbb{R}^n$
$\mathcal{A}^\perp$	basis for the null space of $\mathcal{A}$, i.e., $\mathcal{A}\mathcal{A}^\perp = 0$
$X \otimes Y$	Kronecker product of matrices X and Y, i.e., $X \otimes Y = \begin{bmatrix} x_{11}Y & x_{12}Y & \cdots \\ x_{21}Y & x_{22}Y & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$
$X^{[i]}$	i-th Kronecker power, i.e., $X^{[i]} = \begin{cases} X \otimes X^{[i-1]}, & i \geq 1 \\ 1, & i = 0. \end{cases}$
$\lvert\lvert x \rvert\rvert$	2-norm of x
$a \bullet b$	scalar product of a and b
$x^{\{m\}}$	power vector

Contents

Chapter 1

Introduction

From wind turbines [1] to microelectromechanical systems [2], in practice the dynamic behavior of most systems can be considered nonlinear. Nonlinear models provide a valuable and powerful framework for representing the complex dynamics often observed in nonlinear systems. However, features such as non-unique equilibria, complex dynamics leading to bifurcations and chaos, sensitivity to initial conditions, interactions of nonlinearities in different parts of a system, and the absence of analytical solutions, among others, render the theoretical analysis of nonlinear systems a challenging task [3]. In contrast to linear models, where trajectory prediction and precise qualitative analysis can be achieved using Lyapunov stability theory [4], nonlinear models cannot be approached with the same level of precision and generality. Although linear control is a mature and well-established area, with a wide variety of methods and tools with proven success, it presupposes that the dynamic system operates around a small range in which linear behavior can be assumed [5]. Thus, when the sources of nonlinearities that were neglected begin to influence the dynamics of the system, its operation in the linear regime is no longer guaranteed, and a description of the nonlinear dynamics that allows the design of control systems with guarantees of stability, performance and robustness is necessary.

In this case, one option is to use Takagi-Sugeno (T-S) [6] fuzzy systems. Using the sector nonlinearity modeling strategy, a T-S model accurately represents the nonlinear system in a compact region of the state space containing the origin through the fuzzy combination of linear submodels [7, 8]. This combination of linear models can be suitably addressed through convex optimization tools. In fact, problems like robust stability analysis [9–11], design of feedback laws [12–17], and estimation of domains of attraction (DOA) [18–21] for T-S fuzzy models can be treated through quadratic and non-quadratic Lyapunov functions, generally in terms of Linear Matrix Inequalities (LMIs) [22]. Since the first appearance in 1985, T-S fuzzy models have been extensively and successfully used to represent nonlinear dynamics of a variety of dynamic systems [23], such as generators [24], motors [25–27], robots [28–32], vehicles [33–35], missiles [36], heating processes [37–39], microelectromechanical systems [40–42], twin rotors [43], Chua's chaotic electrical circuits [44–47], wind turbines [48], inverted pendulums [49], biological processes [50, 51], medicine [52–54], buildings [55], aircrafts [56], and spacecrafts [57–59]. Concerning control of T-S fuzzy systems, Parallel Distributed Compensation (PDC) allows the design of a linear feedback controller for each local linear model; the resulting controller — which is nonlinear in general — is a fuzzy combination of each individual linear controller [60]. If the stability or synthesis conditions are represented in terms of LMIs, the problem can be numerically solved using convex optimization techniques implemented in computational packages of proven efficiency [22, 61, 62].

Particularly regarding PDC stabilization problem for T-S systems, initially the results based on quadratic Lyapunov functions (quadratic-on-the-state functions with a constant matrix) stand out. However, this approach often yields overly conservative conditions since the same Lyapunov matrix must certify stability for all linear subsystems [63]. To address this issue, researchers have employed techniques such as slack variables, Pólya's relaxations, and also exploited fuzzy summation properties to generate progressively less conservative conditions [15, 64–68]. These efforts date back to the 1990's and have evolved into convergent relaxations based on Pólya's theorem [69–71]. However, due to its conservativeness, it is well known that quadratic functions cannot completely characterize stability and performance of T-S systems. As a matter of fact, polyhedral [72], piecewise quadratic [73, 74], and homogeneous polynomial (of degree larger than 2) on the states Lyapunov functions [75, 76] may be needed. Regarding stability analysis, Homogeneous Polynomial Lyapunov Functions (HPLFs) are certainly the most attractive technique since the corresponding stability conditions can be formulated in terms of convergent LMI relaxations [75–77]. Notwithstanding, this class of Lyapunov functions has still been little explored in the context of controller synthesis, since the known linearization techniques — such as congruence transformations and change of variables — are not useful in this case. As a consequence, this subject certainly warrants further investigation in the context of robust stabilization of T-S fuzzy systems, specially when dealing with output-feedback, where the existing results, based on quadratic-on-the-states Lyapunov functions, are conservative.

The problem becomes more complicated when investigating regional stability and estimating regions of attraction. Regional stability guarantees the existence of a neighborhood around the equilibrium point, known as DOA, where all trajectories originating from any initial condition within the DOA asymptotically converge to the equilibrium point [3]. Quadratic Lyapunov functions are based on a constant Lyapunov matrix (independent of the Membership Functions (MFs)), implying that there is no need to take into account the derivative of the MFs in the stability conditions. Despite the notable success from a theoretical standpoint, stability analysis and control design results based on quadratic Lyapunov functions do not fully explore the key fact that the T-S model is valid only within a compact region of the state-space. In other words, although the interest is only regional, the stability conditions are assured globally, inevitably leading to more conservative results. Less conservative Lyapunov functions, which generally require a treatment of the derivatives of MFs, have also emerged as alternatives to the quadratic function. Fuzzy Lyapunov Functions (FLFs) — i.e., quadratic-on-the-state functions with a Lyapunov matrix that depends on the MFs — can be used to provide less conservative results, applicable to both continuous-time [20, 78–80] and discrete-time [81–84] T-S fuzzy systems.

Considering the regional stability of continuous-time systems, the time-derivative of the MFs appearing in the stability conditions must be taken into account. In this context, FLFs with affine [78, 85] and polynomial [86] dependence on the MFs have emerged as an alternative when the variation rates of the MFs are known or at least bounded. When considering bounds for the time-derivative of the MFs [78, 86–89], it is necessary to verify *a posteriori* the validity of the results. Different strategies based on the imposition of bounds for the time-derivative of the MFs have been proposed (see for instance Guerra et al. [19], Lee, Park, and Joo [20], Bernal and Guerra [90], Pan et al. [91], Pan et al. [92], and Lee and Kim [93]). The application of FLFs has evolved to better account for the locality of the T-S model. Enhanced methodologies were first introduced in Guerra et al. [19], Lee, Park, and Joo [20], Bernal and Guerra [90], Pan et al. [91], Pan et al. [92], Lee and Kim [93], and Guerra and Bernal [94], where the locality of the model is further explored to compute precise bounds for the time-derivative of the MFs with respect to the premise variables. In Lee, Joo, and Tak [95], Gomes et al. [96, 97], and Marinho, Oliveira, and Peres [98], the time-derivative of the MFs is described entirely in terms of the T-S dynamic model by means of a polytopic representation of the gradient of the MFs. These strategies produced less conservative results and improved estimates for the DOA.

In the discrete-time domain, one of the earliest instances of FLFs was introduced in Guerra and Vermeiren [99], being after that improved to incorporate techniques such as relaxations [100], k sample variation approach [101], as well as Lyapunov matrices and control gains depending on multiple instants of time [102–109]. However, most of the aforementioned works predominantly address global stability, not fully exploring the locality of the T-S model when computing the variation of the MFs. Moreover, the study of regional stability conditions and estimation of the DOA in discrete-time T-S fuzzy systems pose additional challenges. This complexity arises from the need to define subsets of the state-space where the MFs remain valid for both the current and next instants of time, enabling the computation of variations of the MFs and of the candidate Lyapunov function. However, this aspect has received comparatively less attention [63, 110, 111], and the use of bounds for the variations of the MFs in the conditions does not exploit all information associated to the MFs that can be used to reduce the conservativeness of the analysis, possibly with significant impact on the size of the estimated DOA.

1.1 Contributions

Considering the scenario where the variation rates of the MFs are unknown, the first proposal of this thesis is to approach the global state- or output-feedback stabilization of continuous-time T-S fuzzy systems through HPLFs of arbitrary degree on the state, generalizing the results based on quadratic stability. As main technical novelty when compared with standard approaches, this thesis proposes stability conditions where the closed-loop matrices appear affinely, following the strategy suggested by Felipe and Oliveira [112] that does not require change of variables. In this way, the extended matrix, a commonly employed representation for stability analysis [75] using HPLFs, can be handled in the context of synthesis. As a result, the control design problem is formulated in terms of a convex optimization procedure based on LMIs, solved iteratively by means of a locally convergent algorithm. Numerical experiments are presented to show the advantage of using HPLFs of higher degrees to reduce conservatism and improve performance in terms of the decay rate in the PDC design for T-S fuzzy systems.

Besides, this thesis introduces enhancements to the regional stability analysis of T-S fuzzy systems, presenting novel synthesis conditions for state-feedback and output-feedback stabilization, as well as for the estimation of the DOA. The innovation lies in extending the previous results to address PDC control (both state- and output-feedback) with two different Lyapunov functions: one that depends polynomially on the MFs and quadratically on the states (FLFs), and another that depends polynomially on both the MFs and the system states, named Homogeneous Polynomial Parameter-Dependent Lyapunov Function (HPPDLF). While quadratic-on-the-states Lyapunov functions have been extensively used for PDC controller design, functions with higher polynomial dependence on the states have primarily been utilized for stability analysis. This is because the well-known linearizing change of variables does not generally apply. To overcome this challenge, a different strategy based on Finsler's lemma is proposed, deriving a design condition where the Lyapunov matrix, the closed-loop dynamic matrix, and the time-derivative of the Lyapunov matrix (which also depends on the closed-loop matrix) appear affinely. Overall, the proposed technique simplifies the design of output-feedback control laws, making the synthesis procedure tractable through a locally convergent iterative LMI-based algorithm. This algorithm also has the objectives of minimizing the decay rate of the trajectories and maximize an estimate of the DOA. Regarding the treatment of the time-derivative of the MFs, an approach similar to those in Gomes et al. [96, 97] and Campos [113] is employed. A polytopic representation is used for the domain of validity of the T-S model as well as for the gradient of the time-derivatives of the MFs, overcoming one of the main difficulties of dealing with fuzzy Lyapunov functions in continuous-time. Using the sector nonlinearity approach, stability conditions that do not require bounds for the time-derivatives of the MFs are derived. Numerical examples demonstrate the advantages of the proposed approach compared to some existing techniques from the literature.

Considering discrete-time T-S systems, this thesis focuses on developing conditions for the stability analysis, stabilization and estimation of DOA using parameter-dependent Lyapunov functions, commonly referred to as FLFs. Building upon the inspiration from previous works for continuous-time systems [96–98], a novel approach is introduced by proposing a polytopic representation for both the domain of validity of the T-S model and the MFs of discrete-time T-S systems. This innovative strategy effectively addresses the primary challenge of dealing with variation rates of the MFs. By utilizing the sector nonlinearity approach, stability conditions that do not rely on the availability of bounds are derived. The stability analysis procedure involves solving a single-parameter minimization problem subject to LMI constraints, thereby maximizing the estimate of the DOA. On the other hand, the feedback stabilization entails solving a locally-convergent iterative algorithm based on LMIs, consisting of two phases: first, ensuring the stability of the closed-loop system, and second, maximizing the estimate of the DOA. Numerical examples are presented to assess the effectiveness of the proposed method, and a comparative analysis with existing techniques from the literature in the context of state-feedback is conducted.

The relevance of this thesis extends beyond its academic and theoretical contributions, as it addresses critical aspects of stability analysis and control design for nonlinear systems. By advancing methods that can be applied to a wide range of systems, such as aircrafts, drones, motors, vehicles, and biological processes, the results of this work have the potential to indirectly enhance the development of cutting-edge technologies. These improvements can lead to more efficient, reliable, and safe systems across various industries, highlighting the societal impact of this research on both technological innovation and practical applications.

1.2 Thesis organization

This work is organized as follows. The main concepts related to LMIs are introduced in Chapter 2, including Finsler's Lemma, a fundamental tool for developing the stabilization conditions proposed in this work. These conditions are based on the Lyapunov stability theory, presented in the same chapter, which addresses both global and regional stability theory, as well as concepts related to the estimation of the DOA. Properties and notations related to homogeneous polynomials, which are essential for defining Lyapunov functions with degrees of dependence on the states higher than 2, are also covered in Chapter 2, where, subsequently, a general LMI-based iterative algorithm is presented. Variations of this algorithm are used to solve the global and regional feedback stabilization problems for both continuous- and discrete-time T-S systems, which are described in Chapter 3, along with concepts involving PDC control. Chapters 4, 5 and 6 present the main results of this thesis, encompassing feedback stabilization, improvement of decay rates and maximization of estimates for the DOA of continuous- and discrete-time T-S systems. Finally, Chapter 7 outlines the conclusions and suggests directions for future work.

Chapter 2

Preliminary Concepts, Definitions and Notations

This chapter introduces fundamental concepts in control theory, focusing on LMIs, Lyapunov stability theory, and homogeneous polynomials. These mathematical tools and techniques are essential to the stability analysis and design of control systems, offering effective, numerically efficient and robust methods for ensuring system stability and performance.

The development and application of LMIs have revolutionized the field of applied mathematics, specially control theory. From the foundational contributions of early 20th-century mathematicians to the sophisticated numerical algorithms available today, LMIs have proven to be an indispensable tool in both theoretical research and practical engineering. The robust mathematical framework provided by LMIs enables the systematic handling of stability analysis and control design methods taking into account uncertainties, time-varying parameters, some types of nonlinearities, hybrid dynamics, stochastic parameters, and performance specifications, making them a cornerstone of modern control theory. The significance of LMIs lies in their ability to convert complex, non-convex problems into tractable convex forms, which can be efficiently solved using modern computational tools [22, 62, 114–116].

Lyapunov stability theory provides a powerful set of tools for analyzing the stability of equilibrium points in dynamical systems. It offers a set of techniques for determining whether the equilibrium points of a system are stable, asymptotically stable, or unstable. By constructing appropriate Lyapunov functions, one can determine both local and global stability conditions and estimate the domain of attraction. The theory's versatility and robustness make it indispensable in modern control theory and dynamical systems analysis [3, 117].

Homogeneous polynomials are defined as polynomials where all the terms (monomials) have the same total degree [77], providing a uniform structure that is advantageous in various analytical contexts [77]. This property makes them particularly useful in simplifying complex mathematical expressions and performing algebraic manipulations essential in fields such as control theory and optimization.

The integration of these concepts — LMIs, Lyapunov stability theory, and homogeneous polynomials — provides a compelling and effective framework for addressing a wide range of analysis and control problems. By exploring the strengths of each approach, one can develop robust and efficient methods for system analysis and controller design. This chapter aims to introduce these concepts, providing the necessary theoretical background, mathematical formulations, and practical applications.

2.1 Linear Matrix Inequalities

The advent of modern control theory in the 1960s and 1970s, with contributions from Lyapunov, Kalman, and others, paved the way for the use of LMIs in system analysis and synthesis of filters, observers and controllers. LMIs are a powerful tool in control theory, optimization, and various fields of engineering and applied mathematics. The origins of LMIs can be traced back to the work on quadratic forms and matrix theory in the early 20th century. Significant contributions came from fields such as optimization and control theory, where LMIs were used to assess system stability and performance [22, 117].

An LMI is an inequality of the form

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0, \tag{2.1}$$

where $F_0, F_1, \cdots, F_n$ are symmetric matrices and $x^\top = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$ is a vector of decision variables. The set of values x that satisfies the LMI (2.1) is called the *feasible set*. Finding a solution to an LMI involves determining whether this feasible set is non-empty and, if so, identifying the values of x within it. The feasibility of LMIs can be efficiently checked using convex optimization techniques, particularly semidefinite programming (SDP) [118].

It is important to emphasize that the previous inequality is related to the concept of *positive definiteness* of the matrix $F(x)$. A matrix $A \in \mathbb{R}^{n\times n}$ is defined as *positive definite* if

$$x^\top A x > 0 \quad \forall x \neq 0.$$

If the strict inequality is relaxed to $x^\top A x \geq 0$, then A is referred to as *positive semidefinite*. Similarly, a matrix is termed *negative definite* or *negative semidefinite* if the inequalities are reversed in the definitions of positive definite and positive semidefinite, or equivalently, if $-A$ is positive definite or positive semidefinite, respectively [119].

The foundational works of control theorists like Lyapunov and Kalman eventually led to the formulation of LMI conditions, setting the stage for the systematic use of LMIs in control and optimization. Kalman's development of the Kalman filter and his criteria for controllability and observability laid the groundwork for the practical application of matrix inequalities in system analysis [120]. Lyapunov's stability theory, which provides conditions for the stability of dynamical systems via the existence of Lyapunov functions, often translates into LMI conditions [117], specially when dealing with linear systems.

In control theory, LMIs are extensively applied, since determining whether a system is stable can be formulated as an LMI problem [120]. The foundational works by Boyd et al. [22] and Gahinet et al. [61] have significantly contributed to the popularity and widespread adoption of LMIs in control applications. LMIs have become an essential tool in robust control design, providing a systematic way to handle system uncertainties and performance specifications [121, 122].

Moreover, LMIs are instrumental in solving problems related to $\mathscr{H}_\infty$ control, model predictive control, and multi-objective optimization. Their versatility and the existence of efficient numerical solvers, such as those discussed by Nesterov and Nemirovskii [123], make LMIs an invaluable tool in modern engineering and applied mathematics. Consequently, the LMI framework has established itself as a cornerstone in both theoretical research and practical applications.

The application of LMIs extends beyond control theory into various domains. In signal processing, LMIs are used to design filters and optimize various performance criteria [124]. In structural design, as in civil and mechanical engineering, LMIs are applied to optimize the design of structures to ensure stability and resilience under various loads [125]. Additionally, LMIs provide a convenient framework for convex optimization problems, enabling efficient solution techniques [123, 126].

A key result in the theory of LMIs, providing a useful condition for the existence of solutions to certain types of matrix inequalities, is Finsler's lemma, presented in sequence (extended to deal with parameter-dependent matrices). This lemma is particularly important in control theory and optimization for its role in handling quadratic forms and matrix inequalities.

Lemma 2.1

Let $x \in \mathbb{R}^n$, $\mathscr{Q}(\alpha) \in \mathbb{S}^n$ and $\mathscr{B}(\alpha) \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(\mathscr{B}(\alpha)) < n$, and consider $\mathscr{B}(\alpha)^\perp$ such that $\mathscr{B}(\alpha)\mathscr{B}(\alpha)^\perp = 0$, where $\alpha \in \Lambda_d$, the unity simplex of appropriate dimension d, i.e., $\Lambda_d = \left\{\lambda \in \mathbb{R}^d : \sum_{i=1}^{d} \lambda_i = 1, \lambda_i \geq 0, i \in \mathbb{N}_d\right\}$. Then the following *statements are equivalent:*

(i) $x^{\perp}\mathscr{Q}(\alpha)x < 0, \forall \mathscr{B}(\alpha)x = 0, x \neq 0, \forall \alpha \in \Lambda_d$.

(ii) $\mathscr{B}(\alpha)^{\perp\top}\mathscr{Q}(\alpha)\mathscr{B}(\alpha)^{\perp} < 0, \forall \alpha \in \Lambda_d$.

(iii) $\exists \mu \in \mathbb{R} : \mathscr{Q}(\alpha) - \mu\mathscr{B}(\alpha)^{\top}\mathscr{B}(\alpha) < 0, \forall \alpha \in \Lambda_d$.

(iv) $\exists \mathscr{X}(\alpha) \in \mathbb{R}^{n\times m} : \mathscr{Q}(\alpha) + \mathrm{He}(\mathscr{X}(\alpha)\mathscr{B}(\alpha)) < 0, \forall \alpha \in \Lambda_d$.

Proof. See [127] for a proof where all matrices are α-independent. □

Finsler's lemma is a technique in control theory that deals with quadratic inequalities and provides a set of equivalent conditions that can be used to check the feasibility of certain matrix inequalities, making it a powerful tool in the analysis and design of control systems [128, 129].

2.2 Lyapunov Stability Theory

Lyapunov stability theory is a cornerstone in the field of control theory and dynamical systems, providing a robust framework for analyzing the stability of equilibrium points in nonlinear systems. The theory, developed by the Russian mathematician Aleksandr Lyapunov in the late 19th century, offers both qualitative and quantitative tools to determine the stability of a system without solving its differential equations explicitly.

The origins of Lyapunov stability theory date back to Aleksandr Lyapunov's seminal work in 1892, titled "The General Problem of the Stability of Motion". Lyapunov's methods extended the classical approaches to stability analysis, which were primarily based on linearization and eigenvalue analysis. His approach introduced the concept of a Lyapunov function, a scalar function that can be used to assess the stability of an equilibrium point for both linear and nonlinear systems. Even though largely unknown in the West until about 1960 [117], Lyapunov's work laid the groundwork for modern stability theory and has influenced numerous developments in control theory, particularly in the analysis and design of robust control systems. The significance of Lyapunov's contributions is evident in various fields, including engineering, physics, and applied mathematics [3, 117].

Consider a continuous-time autonomous dynamical system described by the differential equation

$$\dot{x} = f(x), \tag{2.2}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function. An equilibrium point x_e is a point where $f(x_e) = 0$, and it can be defined as Lyapunov, asymptotically or exponentially stable [3, 117], according to the definition presented in the sequence.

Definition 2.1

The equilibrium point x_e is said to be

- Lyapunov stable *if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $||x(0) - x_e|| < \delta$, then $||x(t) - x_e|| < \varepsilon$ for all $t \geq 0$;*
- unstable *if it is not stable;*
- asymptotically stable *if it is Lyapunov stable and, in addition, $||x(t) - x_e|| \rightarrow 0$ as $t \rightarrow \infty$;*
- exponentially stable *if there exist positive constants c and λ such that $||x(t) - x_e|| \leq c||x(0) - x_e|| \exp(-\lambda t)$ for all $t \geq 0$.*

Therefore, an equilibrium point is stable if all solutions starting at nearby points stay nearby; otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity. If this tendency is exponential, then the equilibrium point is exponentially stable.

To analyze the local stability of an equilibrium point, Lyapunov theory uses a scalar function $V(x(t)) : \mathbb{D} \rightarrow \mathbb{R}$, as defined next [3].

Theorem 2.1

Let $x_e = 0$ be an equilibrium point of the dynamical system (2.2) *and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing x_e. Let $V(x(t)) : \mathbb{D} \rightarrow \mathbb{R}$ be a continuously differentiable function such that:*

1. *$V(x_e) = 0$ and $V(x(t)) > 0$ in $\mathbb{D} - \{x_e\}$*
2. *$\dot{V}(x(t)) = \nabla V \bullet f(x) \leq 0$ in $\mathbb{D}$.*

Then, $V(x(t))$ is a Lyapunov function for the equilibrium point x_e and x_e is stable. Moreover, if $\dot{V}(x(t)) < 0$ in $\mathbb{D} - \{x_e\}$, then $V(x)$ is called a strict *Lyapunov function, and the equilibrium point x_e is asymptotically stable.*

An extension of the previous theorem to address the global case can be immediately obtained by considering $\mathbb{D} = \mathbb{R}^n$ and imposing that $V(x(t))$ is radially unbounded, meaning $\|x(t)\| \rightarrow \infty \Rightarrow V(x(t)) \rightarrow \infty$. Lyapunov stability theory provides conditions for global and regional stability, asymptotic stability and exponential stability [3, 117]. When considering the regional stability, a concept that arises is the domain of attraction (DOA). The DOA of an equilibrium point x_e is the set of all initial conditions that lead to trajectories converging to x_e, as mathematically defined in the next definition.

Definition 2.2

The domain of attraction $\mathscr{D}(x_e)$ of an equilibrium point x_e is defined as

$$\mathscr{D}(x_e) = \{x(0) \in \mathbb{R}^n \,:\, x(t) \to x_e \text{ as } t \to \infty\}\,.$$

Even though finding the exact DOA analytically may be difficult or even impossible, an estimation of the DOA can be obtained by considering a Lyapunov function $V(x)$ that satisfies the following conditions:

1. $V(x)$ is positive definite.
2. $\dot{V}(x)$ is negative definite.

The level sets of $V(x)$, defined as $\Omega_c = \{x \in \mathbb{R}^n \,:\, V(x) \leq c\}$, can provide an estimate of the DOA. If Ω_c is bounded and $\dot{V}(x) < 0$ in $\Omega_c \backslash \{x_e\}$, then Ω_c is an estimate of the DOA [130].

Lyapunov stability theory also applies to discrete-time systems, providing similar tools for analyzing the stability of equilibrium points. As in the continuous-time case, stability analysis does not require explicit solutions to the difference equations, relying instead on the construction of a Lyapunov function.

Consider a discrete-time autonomous system described by the difference equation

$$x(k+1) = f(x(k)), \tag{2.3}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. An equilibrium point x_e satisfies $f(x_e) = x_e$, and, analogous to the continuous-time case, it can be classified as stable if solutions starting near the equilibrium remain nearby, and asymptotically stable if those solutions converge to the equilibrium over time. Exponential stability implies a more rapid, geometric convergence rate [3].

To assess local stability, a discrete-time Lyapunov function $V(x(k)) : \mathbb{D} \to \mathbb{R}$ is used. The following conditions are required [131]:

Theorem 2.2

Let $x_e = 0$ be an equilibrium point of the dynamical system (2.3) *and $\mathbb{D} \subset \mathbb{R}^n$ be a domain* *containing x_e. Let $V(x(k)) : \mathbb{D} \to \mathbb{R}$ be a continuous function such that:*

1. $V(x_e) = 0$ *and* $V(x(k)) > 0$ *in* $\mathbb{D} - \{x_e\}$
2. $\Delta V(x(k)) = V(x(k+1)) - V(x(k)) \leq 0$ *in* $\mathbb{D}$.

Then, $V(x(k))$ is a Lyapunov function for the equilibrium point x_e and x_e is stable. Moreover, if $\Delta V(x(k)) < 0$ in $\mathbb{D} - \{x_e\}$, then $V(x)$ is called a strict *Lyapunov function, and the equilibrium point x_e is asymptotically stable.*

In the previous theorem, if $\mathbb{D} = \mathbb{R}^n$ and $V(x(k))$ is such that $V(x(k)) \to \infty$ as $\|x(k)\| \to \infty$ (i.e., $V(x(k))$ is radially unbounded), then the equilibrium point is globally stable.

2.3 Homogeneous Polynomials

In modern control theory, the use of homogeneous polynomials became prominent with the advent of state-space representations and the development of robust control techniques in the 1960s and 1970s, with significant contributions from researchers like Kalman and Lyapunov, who utilized these polynomials to describe system dynamics and formulate stability criteria [117, 120].

Homogeneous polynomials are defined as polynomials where all the monomials have the same total degree, a property that makes them particularly useful in simplifying complex mathematical expressions and in performing algebraic manipulations that are essential in various fields such as control theory and optimization. For instance, in control theory, they are used to describe the state space of a system and to formulate stability criteria.

Additionally, homogeneous polynomials play a pivotal role in stability analysis, where the concept of an extended matrix (to be defined in Theorem 2.3) arises. Stability analysis often utilizes Lyapunov functions, which can be constructed using homogeneous polynomials to assess the stability of equilibrium points in dynamical systems. The extended matrix, defined in the context of dynamical systems, is computed through linear operations with the entries of the dynamic matrix, particularly via Kronecker products [77]. This method facilitates stability assessments by transforming the system equations into a higher-dimensional space where the stability criteria are easier to verify. The use of Kronecker products in this transformation is crucial as it allows for the efficient handling of the polynomial terms and their interactions. By transforming the original system dynamics into a higher-dimensional space, it becomes easier to apply LMI techniques, which are widely used in modern control theory to derive stability conditions and design controllers that guarantee system robustness.

To provide a formal definition of homogeneous polynomials, this section relies on the concepts and definitions thoroughly described in Chesi et al. [77]. The set of homogeneous polynomials of degree d on n scalar variables is formally defined as

$$\Xi_{n,d} = \left\{ h : \mathbb{R}^n \to \mathbb{R} \;:\; h(x) = \sum_{q \in \mathcal{D}_{n,d}} a_q x^q \right\},$$

where $x \in \mathbb{R}^n$, $a_q \in \mathbb{R}$ is the coefficient associated to the monomial x^q, and q belongs to the set

$$\mathcal{D}_{n,d} = \{ d \in \mathbb{N}^n : q_1 + q_2 + \cdots + q_n = d \} .$$

The number of monomials of any polynomial in $\Xi_{n,d}$ is given by the cardinality of $\mathcal{D}_{n,d}$, which is equal to

$$\sigma(n,d) = \frac{(n+d-1)!}{(n-1)!d!}.$$

Therefore, a homogeneous polynomial is a weighted sum of monomials of degree d. For instance, consider $n = 2$ and $d = 3$. In this case one has the set $\mathcal{D}_{2,3} = \{(3,0),(2,1),(1,2),(0,3)\}$ and the resulting homogeneous polynomials is represented by $h(x) = a_{(3,0)}x_1^3 + a_{(2,1)}x_1^2x_2 + a_{(1,2)}x_1x_2^2 + a_{(0,3)}x_2^3$ where $a_{(i,j)}$ are known coefficients.

Polynomials can be represented by vectors which contains their coefficients with respect to an appropriate base. Let $x^{\{d\}} \in \mathbb{R}^{\sigma(n,d)}$ be any vector such that, for all $h \in \Xi_{n,d}$, there exists $g \in \mathbb{R}^{\sigma(n,d)}$ satisfying

$$h(x) = g^\top x^{\{d\}}.$$

Then, $x^{\{d\}}$ is called a *power vector* for $\Xi_{n,d}$. Therefore, the entries of the vector $x^{\{d\}}$ are a finite generating set for $\Xi_{n,d}$ — i.e., every polynomial $h \in \Xi_{n,d}$ can be represented as a linear combination of the elements of $x^{\{d\}}$. Special choices for the power vector are those where each entry is a monomial, a typical one being

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^{\{k\}} = \begin{cases} \begin{bmatrix} x_1 \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}^{\top\{k-1\}} \\ x_2 \begin{bmatrix} x_2 & x_3 & \dots & x_n \end{bmatrix}^{\top\{k-1\}} \\ \vdots \\ x_n \begin{bmatrix} x_n \end{bmatrix}^{\top\{k-1\}} \end{bmatrix}, \text{if } k > 0 \\ 1, \text{otherwise} \end{cases} ,$$

which yields the lexicographical order of the monomials in $x^{\{d\}}$ and is adopted in this thesis.

Homogeneous polynomials of even degrees, $h \in \Xi_{n,2m}$, can be effectively represented utilizing the concept of the power vector $x^{\{m\}} \in \mathbb{R}^{\sigma(n,m)}$. This representation is facilitated through the usage of a matrix $H \in \mathbb{S}^{\sigma(n,m)}$ that satisfies the relation

$$h(x) = x^{\{m\}\top} H x^{\{m\}}. \tag{2.4}$$

Matrix H mentioned in (2.4) assumes the designation of a *Square Matricial Representation* (SMR) of the homogeneous polynomial $h(x)$ concerning the power vector $x^{\{m\}}$. Introduced in Chesi et al. [132] and Chesi et al. [133], the SMR is also known as *Gram matrix* [134].

Given the set defined as

$$\mathcal{E}_{n,m} = \left\{ E \in \mathbb{S}^{\sigma(n,m)} : x^{\{m\}\top} E x^{\{m\}} = 0,\ \forall x \in \mathbb{R}^n \right\},$$

and considering $E(\beta)$ as a linear parametrization of $\mathcal{E}_{n,m}$, specifically

$$E(\beta) = \sum_{i=1}^{\omega(n,m)} \beta_i E_i,$$

where $\omega(n,m) = \sigma(n,m)\,(1+\sigma(n,m))\,/2 - \sigma(n,2m)$ and $\beta \in \mathbb{R}^{\omega(n,m)}$ is a free vector, the expression for $h(x)$ takes the form

$$h(x) = x^{\{m\}\top} \left(H + E(\beta)\right) x^{\{m\}}.$$

This particular representation is known as the *Complete Square Matricial Representation* (complete SMR) of the homogeneous polynomial $h(x)$ concerning the power vector $x^{\{m\}}$. As an example, consider $n = m = 2$ and the quartic homogeneous polynomial $h(x) = x_1^4 + 2x_1^3x_2 + 2x_2^4$. Computing $\sigma = 3$, $\omega = 1$ and the complete SMR, one gets

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad E(\beta) = \begin{bmatrix} 0 & 0 & -\beta \\ 0 & 2\beta & 0 \\ -\beta & 0 & 0 \end{bmatrix}$$

and, consequently,

$$h(x) = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}^\top \begin{bmatrix} 1 & 1 & -\beta \\ 1 & 2\beta & 0 \\ -\beta & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix} = \begin{matrix} x_1^4 + 2x_1^3x_2 + 2x_2^4 \\ +\beta \underbrace{\left(-x_1^2x_2^2 + 2x_1^2x_2^2 - x_1^2x_2^2\right)}_{=0} \end{matrix}$$

A key concept that arises from the use of homogeneous polynomials for stability analysis is the *extended matrix*. Given the system $\dot{x}(t) = Ax(t)$, $A^\#$ is called the extended matrix of A, $m \geq 1$, if the following relation is satisfied

$$\frac{dx^{\{m\}}}{dt} = \frac{dx^{\{m\}}}{dx}\dot{x}(t) = A^\# x^{\{m\}}.$$

The extended matrix can be computed in terms of Kronecker products by means of linear operations with the entries of matrix A, according to the following result.

Theorem 2.3

Let $G_m \in \mathbb{R}^{n^m \times \sigma(n,m)}$ be the (full column rank) matrix satisfying

$$x^{[m]} = G_m x^{\{m\}} \tag{2.5}$$

where $x^{[m]}$ is the i-th Kronecker power of x. Then the extended matrix $A^{\#}$ is given by

$$A^{\#} = \left(G_m^\top G_m\right)^{-1} G_m^\top \left(\sum_{i=0}^{m-1} I_{n^{m-1-i}} \otimes A \otimes I_{n^i}\right) G_m. \tag{2.6}$$

Proof. See Chesi et al. [77]. □

Each of the non-zero entries of matrix G_m holds a unity value, and these coefficients can be determined by comparing the monomials on both sides of (2.5). An example is presented in the sequence to illustrate the construction of the extended matrix considering a second order matrix A and $m = 3$.

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad x^{\{3\}} = \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}, \quad x^{[3]} = \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_1 x_2^2 \\ x_2^3 \end{bmatrix}$$

$$G_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^{\#} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ -6 & -5 & 2 & 0 \\ 0 & -12 & -10 & 1 \\ 0 & 0 & -18 & -15 \end{bmatrix}$$

Pólya's relaxation is a powerful method for addressing polynomial optimization problems, particularly in the context of positivity conditions for polynomials, such as when using parameter-dependent Lyapunov functions in the stability analysis of uncertain linear systems [135, 136]. This approach is grounded in Pólya's theorem, which provides conditions under which a positive homogeneous polynomial with non-negative variables can be expressed with all positive coefficients after a suitable scaling.

This simplifies the problem of verifying positivity by converting it into a form where the polynomial's positivity is easier to check, typically by ensuring that its coefficients are positive after the transformation [137].

Pólya's relaxations are particularly useful for assessing the positivity of homogeneous polynomials where the coefficients of the monomials are matrices. This type of relaxation has found extensive applications in stability analysis and control design for uncertain linear systems with parameters (either time-invariant or time-varying) that lie within the unit simplex. To illustrate this, consider the polynomial matrix inequality

$$T(\alpha) = \alpha_1^2 T_{20} + \alpha_1 \alpha_2 T_{11} + \alpha_2^2 T_{02} > 0, \quad \alpha \in \Lambda_2$$

where Λ_2 refers to the unity simplex of dimension 2, and suppose that the coefficients T_{20}, T_{11} and T_{02} are matrices that depend affinely on optimization variables. One sufficient (though conservative) condition to check the positivity of $T(\alpha)$ is the following set of LMIs

$$T_{20} > 0, \quad T_{11} > 0, \quad T_{02} > 0 \tag{2.7}$$

since α_1 and α_2 are non-negative and sum up one. To improve the accuracy of the positivity test, Pólya's relaxations can be used. For instance, one can test the condition $(\alpha_1 + \alpha_2)T(\alpha) > 0$, which leads to

$$\alpha_1^3 T_{20} + \alpha_1^2 \alpha_2 (T_{20} + T_{11}) + \alpha_1 \alpha_2^3 (T_{02} + T_{11}) + \alpha_2^3 T_{02} > 0,$$

that can be tested through the following set of LMIs

$$T_{20} > 0, \quad T_{20} + T_{11} > 0, \quad T_{02} + T_{11}, \quad T_{02} > 0 \tag{2.8}$$

It can be noted that if the LMIs in (2.7) are feasible, then the set of LMIs in (2.8) also holds, but the converse is not necessarily true. Notably, in (2.8) the term T_{11} does need to be positive definite. The number of relaxations can be generalized to $(\alpha_1 + \alpha_2)^d T(\alpha) > 0$, and if $T(\alpha)$ is positive definite, then for a sufficient large d all coefficient will be positive definite. For more details about the use of Pólya's relaxations in control theory, see [135, 136].

2.4 LMI-based Iterative Algorithm

Motivated by the problem of robust stabilization of uncertain linear systems, Felipe and Oliveira [112] introduced a novel design procedure using Lyapunov stability theory that diverges from traditional methods by not employing the classic change of variables. Instead, the gain is directly treated as an optimization variable. This approach offers a unified treatment for state-feedback, static output-feedback, and decentralized control problems , addressing both continuous- and discrete-time systems. The method formulates an inequality where both the Lyapunov and closed-loop matrices appear affinely, and is solved iteratively using polynomial approximations for the optimization variables and objective function minimization. Despite its local convergence, numerical comparisons indicate this technique can outperform existing methods, suggesting potential for new robust control algorithms for systems with uncertainties [112].

Consider an uncertain linear time-invariant system with the dynamics

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)u(t), \\ y(t) = C(\alpha)x(t), \end{cases} \tag{2.9}$$

where $x(t)$, $u(t)$ and $y(t)$ are respectively the vectors of states, control inputs and measured outputs, matrices $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ belong to a polytope and α is a vector of time-invariant parameters in the unit simplex Λ_r, i.e.,

$$(A, B, C)(\alpha) = \sum_{i=1}^{r} \alpha_i (A, B, C)_i, \quad \alpha \in \Lambda_r.$$

Considering the control law $u(t) = Ky(t)$, and using the Lyapunov theory, the stability conditions for the closed-loop system are given in terms of the existence of a parameter-dependent Lyapunov matrix $P(\alpha) \in \mathbb{S}^n$, $P(\alpha) > 0$, such that

$$\mathrm{He}\left(P(\alpha)A_{cl}(\alpha)\right) < 0,$$

where $A_{cl}(\alpha) = A(\alpha) + B(\alpha)KC(\alpha)$.

In the case of state-feedback, congruence transformations allow this problem to be addressed using a classic change of variables [138], where the product of the Lyapunov matrix and the gain is replaced by a new variable, after fixing $P(\alpha) = P$. Specifically, the product $A_{cl}(\alpha)P = A(\alpha)P + B(\alpha)Z$, where $Z = KP$.

On the other hand, according to statement (ii) of Finsler's lemma, the previous condition can be rewritten as

$$\mathscr{B}_1(\alpha)^{\perp\top}\mathscr{Q}_1(\alpha)\mathscr{B}_1(\alpha)^{\perp} < 0, \quad \mathscr{B}_1(\alpha)^{\perp} = \begin{bmatrix} I \\ A_{cl}(\alpha) \end{bmatrix}, \quad \mathscr{Q}_1(\alpha) = \begin{bmatrix} 0 & \star \\ P(\alpha) & 0 \end{bmatrix}.$$

From the orthogonal property, it is possible to determine $\mathscr{B}_1(\alpha) = \begin{bmatrix} A_{cl}(\alpha) & -I \end{bmatrix}$, which allows to obtain the condition (iv) of Lemma 2.1 with slack variables, i.e.,

$$\mathscr{Q}_1(\alpha) + \mathrm{He}\left(\begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \end{bmatrix} \begin{bmatrix} A_{cl}(\alpha) & -I \end{bmatrix}\right) < 0. \tag{2.10}$$

Even though equation (2.10) has been explored in the robust synthesis of controllers, typically associated with a change of variables, Felipe and Oliveira [112] propose applying Finsler's lemma once more to provide new equivalent conditions with additional slack variables. This innovative approach offers more flexibility and potentially enhances the robustness of the resulting control strategies by introducing additional degrees of freedom in the formulation, since all optimization variables can be made parameter-dependent.

Rewriting (2.10) in the form of statement (ii) of Lemma 2.1, $\mathscr{B}_2(\alpha)^{\perp\top}\mathscr{Q}_2(\alpha)\mathscr{B}_2(\alpha)^{\perp} < 0$, with

$$\mathscr{Q}_2(\alpha) = \begin{bmatrix} \mathscr{Q}_1(\alpha) & \star \\ \mathscr{B}_1(\alpha) & 0 \end{bmatrix}, \quad \mathscr{B}_2(\alpha)^{\perp} = \begin{bmatrix} I & 0 \\ 0 & I \\ X_1(\alpha)^\top & X_2(\alpha)^\top \end{bmatrix},$$

it is possible to obtain

$$\mathscr{B}_2(\alpha) = \begin{bmatrix} Z_1(\alpha) & Z_2(\alpha) & Z_3(\alpha) \end{bmatrix} = -Z_3(\alpha)\begin{bmatrix} X_1(\alpha)^\top & X_2(\alpha)^\top & -I \end{bmatrix}.$$

Therefore, the condition (iv) of Finsler's lemma can be obtained, i.e.,

$$\mathscr{Q}_2(\alpha) + \mathrm{He}\left(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)\right) < 0, \tag{2.11}$$

where

$$\mathscr{Y}(\alpha) = \begin{bmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ Y_3(\alpha) \end{bmatrix}, \quad \mathscr{B}_2(\alpha) = \begin{bmatrix} Z_1(\alpha) & Z_2(\alpha) & Z_3(\alpha) \end{bmatrix}.$$

However, it is important to note that conditions in (2.11) take the form of Bilinear Matrix Inequalities (BMIs) due to the presence of the product $\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)$. To address this challenge, an iterative algorithm with local convergence is proposed in Felipe and Oliveira [112], starting from an initial feasible solution through an appropriate choice of variables $\mathscr{B}_2(\alpha)$. The existence of a feasible solution can be ensured by considering a relaxed stability condition. This involves examining the stability of $A_{cl}(\alpha) - \rho I$, where ρ is an upper bound to the maximum real part of the eigenvalues of $A_{cl}(\alpha)$. The introduced real positive scalar ρ can be viewed as a *relaxation* parameter and it can be minimized as an objective function. If $\rho \leq 0$ is obtained, $A_{cl}(\alpha)$ is robustly stable and K is a robust stabilizing gain. Therefore, ensuring the existence of a feasible initial solution requires selecting a sufficiently large positive value for ρ.

The key feature of condition (2.11), enabling the iterative solution, is that

$$\mathrm{He}(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)) = \mathrm{He}(\mathscr{B}_2(\alpha)^\top\mathscr{Y}(\alpha)^\top).$$

Therefore, any $\mathscr{Y}(\alpha)^\top$ serves as a valid choice for $\mathscr{B}_2(\alpha)$ in the subsequent iteration. The local convergence of the algorithm, with non-increasing ρ, can be demonstrated by assuming the feasibility of (2.11) at iteration *it* and showing it remains feasible at the subsequent iteration $it+1$. This is achieved with specific selections $\mathscr{Q}_{2,it+1}(\alpha) = \mathscr{Q}_{2,it}(\alpha)$ and $\mathscr{B}_{2,it+1}(\alpha) = \mathscr{Y}_{it}(\alpha)^\top$, ensuring $\rho_{it+1} \leq \rho_{it}$.

The described procedure is presented in Algorithm 1. It takes the systems matrices $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ as inputs, as well as the maximum number of iterations $it_{\max}$ and a tolerance ε, which determines the end of the iterative procedure if the progress between two consecutive iterations is not significant.

Algorithm 1 Iterative procedure for robust stabilization

Input: A_i, B_i, C_i, $\bar{\mathscr{B}}_2$, $it_{\max}$, ε;
$it \leftarrow 0$;
While $it < it_{\max}$
  $it \leftarrow it + 1$;
  minimize ρ **subject to** (2.11);
  If $\rho_{it} \leq 0$ **Then**
    $K \leftarrow K_{it}$, $\rho \leftarrow \rho_{it}$;
    Return
  Else If $\rho_{it} - \rho_{it-1} \leq \varepsilon$ **Then**
    break;
  End If
  $\bar{\mathscr{B}}_2(\alpha) \leftarrow \mathscr{Y}(\alpha)^\top$
End While

The algorithm also requires an initial value for $\bar{\mathscr{B}}_2(\alpha)$, which follows a predefined form proposed by Felipe and Oliveira [112], which is specific for the problem of robust stabilization of time-invariant polytopic systems. However, this thesis proposes different initialization techniques in subsequent chapters, specialized for each problem under investigation, that also present variations of the iterative procedure described, particularly when dealing with performance criteria. The iterative algorithm has already proved its advantages in uncertain linear systems and Lur'e systems [139], and the similarity of the description of T-S systems — presented in the next chapter — with the polytopic representation of uncertain linear systems has inspired the development of the main results of this thesis.

Chapter 3

Takagi-Sugeno Fuzzy Systems

Fuzzy logic, introduced by Zadeh [140], extends classical Boolean logic to handle the concept of partial truth, accommodating the inherent vagueness and ambiguity of real-world situations. At its core, fuzzy logic employs fuzzy sets, which are characterized by MFs that assign each element a grade of membership between 0 and 1, in contrast to the binary nature of classical sets. Premise variables, also known as antecedent variables, are the input variables used in fuzzy rule-based systems to define the conditions or premises of the rules. These variables are mapped to fuzzy sets using MFs, which assign a degree of membership to each premise value. The degree of fulfillment (or firing strength) measures the extent to which the premise satisfies the fuzzy condition, and is computed using operations such as the minimum or product of the membership values of the premise variables. Fuzzy systems can be broadly classified into several types based on their rule structures and inference mechanisms [141]. For example, T-S fuzzy systems use linear or affine functions in the consequent part of their rules, making them computationally efficient and well-suited for control problems.

Introduced by Takagi and Sugeno [6] — and also known as Takagi-Sugeno-Kang (TSK) fuzzy systems —, T-S systems were first intended for system identification and control of nonlinear systems, offering a powerful tool for modeling, simulation, and designing robust control strategies, providing a systematic and effective way to handle nonlinear systems. The T-S model structure allowed for the approximation of nonlinear systems through a combination of linear or affine models, making it a powerful method for capturing the dynamics of complex systems. Another primary application was in control systems design. T-S systems provided a framework for designing controllers that could handle nonlinearities more effectively than traditional linear controllers. By using a set of fuzzy rules to blend linear control actions, T-S models enabled more precise and robust control strategies for nonlinear processes. T-S models are composed of a set of fuzzy If-Then rules. Usually, each rule associates a fuzzy condition with a linear (or affine) consequent part. This structure allows for the smooth integration of local linear models into a nonlinear model.

As shown in Tanaka and Wang [7], the linear T-S fuzzy model can be a universal approximator of any smooth nonlinear control system [142]. The T-S model's ability to interpolate between multiple linear models based on the degree of fulfillment of the fuzzy conditions provides a powerful way to handle nonlinearities. This mechanism ensures a smooth transition between different operating regimes of the system. Despite their ability to model complex behavior, T-S systems remain relatively simple and interpretable compared to other nonlinear modeling approaches. The use of linear consequents in the fuzzy rules makes it easier to analyze and understand the model's behavior.

The sector nonlinearity approach [6, 8] is a fundamental concept in fuzzy modeling, especially in the context of T-S fuzzy systems. This methodology aims to represent the nonlinear behavior of a system within a compact region of the state space by combining linear submodels. By defining sectors within which the nonlinearities of the system can be bounded, the sector nonlinearity approach facilitates the construction of T-S fuzzy models that accurately capture the dynamics of the system. This technique enables the application of well-established linear control techniques to nonlinear systems, thereby simplifying stability analysis and controller design.

3.1 Continuous-time T-S Systems

Consider a nonlinear system represented by the state-space model

$$\begin{cases} \dot{x}(t) = f_x(x(t))x(t) + f_g(x(t))u(t), \\ y(t) = f_h(x(t))x(t) \end{cases} \tag{3.1}$$

where $f_x(0) = 0$ for $u = 0$ (the origin is an equilibrium point), and $f_x(\cdot)$, $f_g(\cdot)$ and $f_h(\cdot)$ are bounded and smooth functions in a compact set of the state-space. Associated to this dynamics, the i-th rule of the T-S fuzzy system is given by [7, 60]

$$\begin{array}{ll} \textbf{If} & z_1(t) \text{ is } M_{i1} \text{ and } \ldots \text{ and } z_p(t) \text{ is } M_{ip} \\ \textbf{Then} & \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t), \quad i \in \mathbb{N}_r, \\ y(t) = C_i x(t) \end{cases} \end{array}$$

where M_{ij} denotes the fuzzy set linked to the premise j in the i-th rule, and r is the count of model rules. The vector of premise variables, denoted as

$$z(t)^\top = \begin{bmatrix} z_1(t) & \cdots & z_p(t) \end{bmatrix},$$

can capture the states, external disturbances, and time dependencies. In this thesis, for continuous-time T-S systems, it is assumed that the premise variables $z(t)$ are functions of the state variables, i.e., $z(t) = f_z(x(t))$. This formalism allows for expressing the premise variables in terms of the state variables, thereby facilitating the subsequent analysis and modeling procedures. For feedback control purposes, it is assumed that all the premise variables can be measured or estimated in real time.

In the previous If-Then expression, the vectors $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$, and $y(t) \in \mathbb{R}^s$ respectively represent the state, input, and output of system (3.1). Matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times q}$, and $C_i \in \mathbb{R}^{s\times n}$ correspond to the local model components. Consequently, these If-Then rules encapsulate the local linear input-output relationships inherent within the original system (3.1).

For a given triple $(x(t), u(t), y(t))$, the representation of (3.1) using a fuzzy model is given by

$$\begin{cases} \dot{x}(t) = \dfrac{\sum_{i=1}^{r} w_i(z(t))\,(A_i x(t) + B_i u(t))}{\sum_{i=1}^{r} w_i(z(t))} \\ y(t) = \dfrac{\sum_{i=1}^{r} w_i(z(t)) C_i x(t)}{\sum_{i=1}^{r} w_i(z(t))} \end{cases},$$

where

$$w_i(z(t)) = \prod_{j=1}^{p} M_{ij}(z_j)(t)$$

and $M_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in M_{ij}. Observing that

$$\sum_{i=1}^{r} w_i(z(t)) > 0, \quad w_i(z(t)) \geq 0,\, i \in \mathbb{N}_r,$$

it is possible to write

$$\alpha_i(z(t)) = \frac{w_i(z(t))}{\sum_{i=1}^{r} w_i(z(t))}.$$

In the previous equation, $\alpha_i(z(t))$ represents the MF associated with the i-th rule and, for all $t \geq 0$ and $x(t) \in \mathcal{X}$ (the T-S model validity domain, to be defined in the sequence), the vector of MFs $\alpha(z(t))$ belongs to the simplex, i.e.,

$$\alpha(z(t)) = \begin{bmatrix} \alpha_1(z(t)) & \cdots & \alpha_r(z(t)) \end{bmatrix}^{\top} \in \Lambda_r.$$

Therefore, the ultimate representation of the nonlinear system (3.1) using a fuzzy model is given by

$$\begin{cases} \dot{x}(t) = A(\alpha(z(t)))x(t) + B(\alpha(z(t)))u(t) \\ y(t) = C(\alpha(z(t)))x(t) \end{cases}, \tag{3.2}$$

valid for all $x(t) \in \mathcal{X}$, where matrices $A(\alpha(z(t)))$, $B(\alpha(z(t)))$ and $C(\alpha(z(t)))$, are inferred through a summation called *center-of-gravity defuzzification* process expressed as [7, 60]

$$(A, B, C)(\alpha(z(t))) = \sum_{i=1}^{r} \alpha_i(z(t))(A, B, C)_i. \tag{3.3}$$

The MFs and their associated parameters hold substantial sway over the performance and interpretability of the system. These choices delineate not only which local linear model (consequent) is triggered but also the extent to which it is activated. Central to this process is the computation of the degree of fulfillment or firing strength of each rule, which hinges on the membership values of the premise variables. This parameter assumes a pivotal role in the operation of fuzzy systems, particularly in the aggregation of rule consequents in Takagi-Sugeno (T-S) models, as depicted in equation (3.3), thereby exerting a direct influence on the overall output of the system.

Furthermore, the set $\mathcal{X} \subseteq \mathbb{R}^n$ is characterized as a polytope

$$\mathcal{X} = \left\{ \xi \in \mathbb{R}^n : \xi_l \in [-z_{l,max}, z_{l,max}], \, l \in \mathbb{N}_p \right\},$$

where $z_{l,max} > 0$, $l \in \mathbb{N}_p$, are predefined real numbers. To simplify, $\mathcal{X}$ delineates the region that establishes the limits within the T-S model remains applicable. The polytope $\mathcal{X}$ can also be equivalently described by two different representations that are more useful for the purposes of this thesis. First, the set $\mathcal{X}$ can be represented in terms of linear inequalities [22]

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \, : \, a_k^\top x \leq 1, \, k \in \mathbb{N}_\mu \right\}, \tag{3.4}$$

where the vectors $a_k \in \mathbb{R}^n$, $k \in \mathbb{N}_\mu$ are known and μ is the number of constraints. Second, $\mathcal{X}$ can be expressed by $\mathcal{X} = \mathrm{co}\{h^1, h^2, \ldots, h^\kappa\}$, i.e., by the convex hull of a set of given vectors h^i, $i \in \mathbb{N}_\kappa$ (the computation of h^i from the inequalities (3.4) can be performed through linear programming tools [143]). Thus, any $x \in \mathcal{X}$ can be written as

$$x(\zeta) = \sum_{k=1}^{\kappa} \zeta_k h^k, \quad \zeta \in \Lambda_\kappa. \tag{3.5}$$

From the above discussion, the nonlinear system outlined by equation (3.1) can be effectively portrayed using the T-S fuzzy model as presented in (3.2) [7]. For the sake of notational clarity, the explicit dependency on the time index t is suppressed whenever possible and $\alpha(z(t))$ is simplified to α.

With respect to the T-S fuzzy system as expressed in (3.2), by employing the PDC methodology for the design of output-feedback controllers, each control rule aligns with the corresponding rule of the T-S model [7], i.e.,

If $z_1(t)$ is M_{i1} and $\ldots$ and $z_p(t)$ is M_{ip}
Then $u(t) = K_i y(t), \quad i \in \mathbb{N}_r.$

Consequently, this leads to the derivation of the control law $u(t) = K(\alpha)y(t)$, with

$$K(\alpha) = \sum_{i=1}^{r} \alpha_i K_i,\ K_i \in \mathbb{R}^{q\times s},\ \alpha \in \Lambda_r.$$

By extension, the closed-loop T-S fuzzy system can be described as

$$\dot{x}(t) = A_{cl}(\alpha)x(t),\ \forall x(t) \in \mathcal{X} \tag{3.6}$$

where

$$A_{cl}(\alpha) = A(\alpha) + B(\alpha)K(\alpha)C(\alpha).$$

Hence, distinct from other nonlinear control techniques, the PDC approach for T-S models hinges on determining the local linear gains K_i within the consequent portion of the T-S model rules.

3.2 Discrete-time T-S Systems

Discrete-time T-S systems can be defined similarly to the continuous-time case. However, due to the locality of the model, some particularities related to the presence of the states inside the validity domain in the subsequent instants of time must be defined.

Consider the state-space dynamics

$$\begin{cases} x(k+1) = f(x(k))x(k) + g(x(k))u(k), \\ y(k) = h(x(k))x(k) \end{cases} \tag{3.7}$$

which corresponds to a discrete-time nonlinear system, where the vectors $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^q$, and $y(k) \in \mathbb{R}^s$ respectively represent the state, input, and output. This system can be expressed locally in a compact region of interest containing the origin as a T-S fuzzy model.

For the nonlinear system (3.7), the i-th rule of the T-S fuzzy model takes the form [7, 60]

If $\quad z_1(k)$ is M_{i1} and $\ldots$ and $z_p(k)$ is M_{ip}

Then $\quad \begin{cases} x(k+1) = A_i x(k) + B_i u(k) \\ y(k) = C_i x(k), \quad i \in \mathbb{N}_r, \end{cases}$

where matrices $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times q}$, and $C_i \in \mathbb{R}^{s\times n}$ correspond to the local model components, M_{ij} represents the fuzzy set associated with the j-th premise in the i-th rule and r denotes the number of model rules. The premise variables, denoted as $z(k)^\top = [z_1(k), \ldots, z_p(k)]$, can be designed to encompass the states, external disturbances, and time dependencies. Particularly when dealing with discrete-time systems, it is assumed that these premise variables solely rely on linear combinations of the state variables, i.e.,

$$z(k) = \mathcal{T}x(k) \in \mathbb{R}^p, \quad \mathcal{T}^\top = \begin{bmatrix} \mathcal{T}_1^\top & \cdots & \mathcal{T}_p^\top \end{bmatrix} \in \mathbb{R}^{n\times p}. \tag{3.8}$$

Although this assumption restricts the classes of nonlinear systems that can be investigated (for instance, quadratic systems [144, 145]), stability analysis and especially control design remain challenging problems. Moreover, this assumption has been frequently adopted in the literature on T-S systems. As in the continuous-time case, it is assumed that all the premise variables can be measured or estimated in real time.

Given a certain triple $(x(k), u(k), y(k))$, the fuzzy model represents the nonlinear system as [7, 60]

$$\begin{cases} x(k+1) = A(\alpha(z(k)))x(k) + B(\alpha(z(k)))u(k) \\ y(k) = C(\alpha(z(k)))x(k) \end{cases}, \tag{3.9}$$

valid for all $x(k) \in \mathcal{L} \subseteq \mathbb{R}^n$, a set that includes the origin. Put simply, $\mathcal{L}$ represents the set that delineates the boundaries within which the T-S model remains applicable and it is formally defined later in the text. Matrices $A(\alpha(z(k)))$, $B(\alpha(z(k)))$ and $C(\alpha(z(k)))$, obtained using the center-of-gravity method, can be expressed as

$$(A, B, C)(\alpha(z(k))) = \sum_{i=1}^{r} \alpha_i(z(k))(A, B, C)_i,$$

where $\alpha_i(z(k))$ represents the MF associated with the i-th rule and, for all $k \geq 0$ and $x \in \mathcal{L}$, the vector of MFs $\alpha(z(k))$ belongs to the simplex, i.e.,

$$\alpha\left(z(k)\right) = \begin{bmatrix} \alpha_1(z(k)) & \ldots & \alpha_r(z(k)) \end{bmatrix}^\top \in \Lambda_r.$$

Thus, the nonlinear system described by equation (3.7) can be accurately represented [142] by the T-S fuzzy model provided in (3.9). Consequently, the construction of a fuzzy model becomes pivotal in this approach. Despite the extensive literature on this topic, identification using input-output data for fuzzy modeling is more suitable for systems that cannot be represented by analytical models. In cases where the nonlinear dynamic model is known, the sector nonlinearity approach [7] is more appropriate, ensuring an exact construction of the fuzzy model by considering a local sector. Therefore, considering the representation of the premise variables as linear combinations of the states as in (3.8), it is possible to describe the region of validity of the local sector as a set that includes the origin as

$$\mathcal{P} = \left\{\xi \in \mathbb{R}^n : \mathcal{T}_i\xi \in \left[-p_{\max,i}, p_{\max,i}\right], i \in \mathbb{N}_p\right\}, \tag{3.10}$$

where $p_{\max,i} > 0$, for $i \in \mathbb{N}_p$, are predefined real numbers.

Additionally, in this thesis, it is assumed that all state variables associated to discrete-time systems are confined to a specific interval, defined as

$$\mathcal{X} = \left\{ \xi \in \mathbb{R}^n : e_j \xi \in \left[-x_{\max,j}, x_{\max,j} \right], \ j \in \mathbb{N}_n \right\},$$

where $x_{\max,j} > 0$, for $j \in \mathbb{N}_n$, are predetermined real numbers. This assumption is important for the proposed approach and, moreover, it has a practical justification since variables of physical systems are inherently bounded. Consequently, the domain of validity of the T-S model can be described as the polytope

$$\mathcal{L} = \mathcal{P} \cap \mathcal{X} = \left\{ \xi \in \mathbb{R}^n : L_\ell \xi \in [-z_{\max,\ell}, z_{\max,\ell}], \ \ell \in \mathbb{N}_{p+n} \right\}, \tag{3.11}$$

where

$$L^\top = \begin{bmatrix} \mathcal{T}_1^\top & \cdots & \mathcal{T}_p^\top & e_1^\top & \cdots & e_n^\top \end{bmatrix} = \begin{bmatrix} \mathcal{T}^\top & I_n \end{bmatrix}$$

and

$$z_{\max,\ell} = \begin{cases} p_{\max,\ell}, & \ell \in \{1, \ldots, p\} \\ x_{\max,\ell-p}, & \ell \in \{p+1, \ldots, p+n\} \end{cases}.$$

In the particular case where a premise variable corresponds to a state variable, it is not necessary to constrain such variable in the definition of both $\mathcal{P}$ and $\mathcal{L}$. Therefore, matrix L can be simplified to exclude repeated constraints on the state variables. Alternatively, the domain of validity $\mathcal{L}$ can be described by the vertices that define the polytope, and the computation of the vertices can be performed using the Vertex Enumeration algorithm, a well known linear programming-based tool [143].

For simplicity, dependence on the time index k and premise variables z are explicitly mentioned only when essential, and an abbreviated notation is introduced, where $x(k) = x$ and $x(k+1) = x^+$ denote the state vectors at instants k and $k+1$, respectively, while $\alpha(z(k)) = \alpha$ and $\alpha(z(k+1)) = \alpha^+$ are the MFs at instants k and $k+1$.

Regarding the T-S fuzzy system as delineated in (3.9), the implementation of the PDC methodology consists on using a fuzzy output-feedback controller that mirrors the structure of the associated T-S model. Therefore, the PDC design ensures that each control rule is precisely associated with the corresponding rule of the T-S model [7], i.e.,

If $z_1(k)$ is M_{i1} and $\ldots$ and $z_p(k)$ is M_{ip}
Then $u(k) = K_i y(k), \quad i \in \mathbb{N}^r.$

Consequently, this leads to the derivation of the control law $u(k) = K(\alpha)y(k)$, with

$$K(\alpha) = \sum_{i=1}^{r} \alpha_i K_i, \ K_i \in \mathbb{R}^{q \times s}, \ \alpha \in \Lambda_r$$

and, by extension, the closed-loop T-S fuzzy system can be described as

$$x(k+1) = A_{cl}(\alpha)x(k), \; \forall x(k) \in \mathcal{L} \tag{3.12}$$

where

$$A_{cl}(\alpha) = A(\alpha) + B(\alpha)K(\alpha)C(\alpha). \tag{3.13}$$

Chapter 4

Global Stabilization of Continuous-time T-S Systems

Given the scenario where the variation rates of the MFs are unknown, this chapter proposes using Homogeneous Polynomial Lyapunov Functions (HPLFs) of arbitrary degree on the state for global state- or output-feedback stabilization of continuous-time T-S fuzzy systems. This approach generalizes results based on quadratic stability, aiming to provide more flexible and less conservative conditions.

Consider the candidate Lyapunov function

$$v(x) = x^{\{m\}\top} V x^{\{m\}} \tag{4.1}$$

of degree $2m$ on the states. The global asymptotic stability of the origin of the closed-loop continuous-time T-S given in equation (3.6) can be proved using the next theorem, where $\xi > 0$ is a lower bound for the system decay rate [22].

Theorem 4.1

Let $v : \mathbb{R}^n \to \mathbb{R}$, $v(x) = x^{\{m\}\top} V x^{\{m\}}$, $V \in \mathbb{S}^{\sigma(n,m)}$, *be a function satisfying*

$$\begin{cases} v(x) > 0, \ \forall x \in \mathbb{R}^n, \ x \neq 0 \\ \dot{v}(x) \leq -2\gamma v(x), \ \forall x \in \mathbb{R}^n, \ x \neq 0, \end{cases}$$

along the trajectories of (3.6). *Then* $v(x)$ *is a Homogeneous Polynomial Lyapunov Function (HPLF) of degree* $2m$ *for the T-S system* (3.6) *that proves the global exponential stability of the origin of this system and provides a lower bound for the decay rate, given by* $\xi = \gamma/m$, *when* $\gamma > 0$.

Proof. See [22, 77]. □

The choice $m = 1$ in Theorem 4.1 retrieves the well known result based on the quadratic Lyapunov function $v(x) = x^\top V x$, extensively used in the robust control literature. The main interest of this chapter is to investigate the case $m > 1$, where standard linearization techniques, as change of variables and congruence transformations, cannot be applied. The motivation comes from the fact that, as far as stability is concerned, the conditions of Theorem 4.1 are progressively less conservative with the increase of m, tending to the necessity for a sufficiently large m [76].

The first condition in Theorem 4.1, $v(x) > 0$, can be written as $V > 0$. For the second condition, according to the definition of the extended matrix (2.3), one has

$$\begin{aligned} \dot{v}(x) &= \dot{x}^{\{m\}\top} V x^{\{m\}} + x^{\{m\}\top} V \dot{x}^{\{m\}} \\ &= x^{\{m\}\top} \big(A_{cl}^{\#\top}(\alpha) V + V A_{cl}^{\#}(\alpha)\big) x^{\{m\}}. \end{aligned} \tag{4.2}$$

Taking into account that $A_{cl}^{\#}(\alpha)$ is a cubic fuzzy summation on α (due to the product $B(\alpha)K(\alpha)C(\alpha)$), consider the following matrix

$$E(\alpha) = \sum_{k \in \mathcal{D}_{r,3}} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_r^{k_r} E(\beta^k). \tag{4.3}$$

where $E(\beta^k) \in \mathcal{E}_{n,m}$, $k \in \mathcal{D}_{r,3}$ and $x^{\{m\}\top} E(\alpha) x^{\{m\}} = 0$, $\forall \alpha \in \Lambda_r$. As an illustrative example, the case $n = 2$, $m = 3$, $r = 2$ yields

$$E(\alpha) = \sum_{k \in \mathcal{D}_{2,3}} \alpha_1^{k_1} \alpha_2^{k_2} E(\beta^k), \quad E(\beta^k) = \begin{bmatrix} 0 & 0 & -\frac{1}{2}\beta_1^k & -\frac{1}{2}\beta_2^k \\ 0 & \beta_1^k & \frac{1}{2}\beta_2^k & -\frac{1}{2}\beta_3^k \\ -\frac{1}{2}\beta_1^k & \frac{1}{2}\beta_2^k & \frac{1}{2}\beta_3^k & 0 \\ -\frac{1}{2}\beta_2^k & -\frac{1}{2}\beta_3^k & 0 & 0 \end{bmatrix},$$
$$\mathcal{D}_{2,3} = \{(3,0), (2,1), (1,2), (0,3)\}$$

Thus, including $E(\alpha)$ into (4.2), gives

$$\dot{v}(x) = x^{\{m\}\top} \Big(\mathrm{He}\big(V A_{cl}^{\#}(\alpha)\big) + E(\alpha)\Big) x^{\{m\}} \tag{4.4}$$

and, as a result, the second condition in Theorem 4.1 can be written as

$$x^{\{m\}\top} \Big(\mathrm{He}\big(V A_{cl}^{\#}(\alpha)\big) + E(\alpha)\Big) x^{\{m\}} \le -2\gamma x^{\{m\}\top} V x^{\{m\}}$$

or

$$\mathrm{He}\big(V A_{cl}^{\#}(\alpha)\big) + E(\alpha) + 2\gamma V = \mathrm{He}\big(V(A_{cl}^{\#}(\alpha) + \gamma I)\big) + E(\alpha) \le 0.$$

The next theorem formalizes the above results.

Theorem 4.2

Let $m \geq 1$ be given. If there exist matrices $0 < V \in \mathbb{S}^{\sigma(n,m)}$, $K(\alpha) \in \mathbb{R}^{q\times n}$ and $E(\alpha) \in \mathbb{R}^{\sigma(n,m)}$ as in (4.3), and a scalar $\gamma > 0$ satisfying

$$\mathrm{He}\big(V(A_{cl}^{\#}(\alpha) + \gamma I)\big) + E(\alpha) \leq 0, \quad \forall \alpha \in \Lambda_r, \tag{4.5}$$

then $v(x)$ as in (4.1) is a HPLF of degree $2m$ for the T-S fuzzy system (3.6) that proves the global exponential stability of the origin of the system and $\xi = \gamma/m$ is a lower bound for the decay rate.

Proof. Follows from the previous development. □

In (4.5), one can notice the product of the Lyapunov matrix V by the feedback controller $K(\alpha)$ (inside the closed-loop matrix $A_{cl}^{\#}(\alpha)$), resulting in a Bilinear Matrix Inequality (BMI). Exclusively for the case $m = 1$, it is possible to deal with this problem applying congruence transformations and change of variables ($K(\alpha)V = Z(\alpha)$), as in Tanaka and Wang [7], for the state-feedback problem. Considering output-feedback synthesis, there are some techniques that can derive sufficient LMI conditions, at the price of introducing conservatism or restricting the structure of the matrices of the system or of the variables of the problem. However, for $m \geq 2$, those techniques can no longer be applied. The primary reason is that the entries of the control gains K_i, although appearing affinely, are dispersed among the entries of the extended matrix $A_{cl}^{\#}(\alpha)$. Consequently, the product $K(\alpha)V$ no longer appears, and the change of variables used in the case $m = 1$ is not feasible. Taking a different approach, this work proposes to segregate the factors of the product, constructing LMI conditions that can be iteratively solved. This follows the methodology proposed in Felipe and Oliveira [112], where changes of variables are avoided.

4.1 Main Results

From Finsler's Lemma 2.1, $\mathscr{B}^{\perp\top}(\alpha)\mathscr{Q}(\alpha)\mathscr{B}^{\perp}(\alpha) \leq 0$ if and only if

$$\exists \mathscr{X}(\alpha) \;:\; \mathscr{Q}(\alpha) + \mathrm{He}\left(\mathscr{X}(\alpha)\mathscr{B}(\alpha)\right) \leq 0. \tag{4.6}$$

This equivalence can be used to construct alternative conditions where the extended matrix $A_{cl}^{\#}(\alpha)$ does not multiply any other variable.

First, equation (4.5) in Theorem 4.2 can be written as $\mathscr{B}_1^{\perp\top}(\alpha)\mathscr{Q}_1(\alpha)\mathscr{B}_1^{\perp}(\alpha) \leq 0$ with

$$\mathscr{Q}_1(\alpha) = \begin{bmatrix} E(\alpha) & \star \\ V & 0 \end{bmatrix}, \quad \mathscr{B}_1^{\perp}(\alpha) = \begin{bmatrix} I \\ (A_{cl}^{\#}(\alpha) + \gamma I) \end{bmatrix}$$

and an equivalent condition obtained according to (4.6) is achieved, i.e.,

$$\exists \mathscr{X}(\alpha) \ : \ \mathscr{Q}_1(\alpha) + \mathrm{He}\left(\mathscr{X}(\alpha)\mathscr{B}_1(\alpha)\right) \leq 0 \tag{4.7}$$

with

$$\mathscr{B}_1(\alpha) = \begin{bmatrix} A_{cl}^{\#}(\alpha) + \gamma I & -I \end{bmatrix}, \quad \mathscr{X}(\alpha) = \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \end{bmatrix}.$$

An interesting feature of the structure of inequality (4.6) is that it can be easily rewritten in the form $\mathscr{B}^{\perp\top}(\alpha)\mathscr{Q}(\alpha)\mathscr{B}^{\perp}(\alpha) \leq 0$ because

$$\mathscr{Q}(\alpha) + \mathrm{He}\left(\mathscr{X}(\alpha)\mathscr{B}(\alpha)\right) = \begin{bmatrix} I & \mathscr{X}(\alpha) \end{bmatrix} \begin{bmatrix} \mathscr{Q}(\alpha) & \star \\ \mathscr{B}(\alpha) & 0 \end{bmatrix} \begin{bmatrix} I \\ \mathscr{X}(\alpha)^\top \end{bmatrix}.$$

Thus, rewriting (4.7) again as $\mathscr{B}_2^{\perp\top}(\alpha)\mathscr{Q}_2(\alpha)\mathscr{B}_2^{\perp}(\alpha) \leq 0$ with

$$\mathscr{Q}_2(\alpha) = \begin{bmatrix} E(\alpha) & \star & \star \\ V & 0 & \star \\ \left(A_{cl}^{\#}(\alpha) + \gamma I\right) & -I & 0 \end{bmatrix} \tag{4.8}$$

$$\mathscr{B}_2^{\perp}(\alpha) = \begin{bmatrix} I & 0 \\ 0 & I \\ X_1^\top(\alpha) & X_2^\top(\alpha) \end{bmatrix},$$

a new equivalent condition with slack variables in the form of (4.6) can be derived, i.e., $\exists \mathscr{Y}(\alpha) \ : \ \mathscr{Q}_2(\alpha) + \mathrm{He}\left(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)\right) \leq 0$ where $\mathscr{Y}(\alpha)$ and $\mathscr{B}_2(\alpha)$ are given by

$$\mathscr{Y}(\alpha) = \begin{bmatrix} Y_1(\alpha) \\ Y_2(\alpha) \\ Y_3(\alpha) \end{bmatrix}, \quad \mathscr{B}_2^\top(\alpha) = \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \\ -I \end{bmatrix}.$$

The result is presented in next theorem.

Theorem 4.3

Let the integers $m \geq 1$ and $d \geq 0$ be given. If there exist matrices $0 < V \in \mathbb{S}^{\sigma(n,m)}$, $K(\alpha) \in \mathbb{R}^{q \times s}$, $\mathscr{Y}(\alpha) \in \mathbb{R}^{3\sigma(n,m) \times \sigma(n,m)}$, and $\mathscr{B}_2(\alpha) \in \mathbb{R}^{\sigma(n,m) \times 3\sigma(n,m)}$, and a scalar $\gamma > 0$ such that the robust BMIs

$$\Big(\sum_{i=1}^{r} \alpha_i\Big)^d \Big(\mathscr{Q}_2(\alpha) + \mathrm{He}\big(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)\big)\Big) \leq 0 \tag{4.9}$$

hold for all $\alpha \in \Lambda_r$ with $\mathscr{Q}_2(\alpha)$ given in (4.8)*, then $v(x)$ as in* (4.1) *is a HPLF of degree* $2m$ *for the T-S system* (3.6) *that proves the global exponential stability of the origin of the* *system and* $\xi = \gamma/m$ *is a lower bound for the decay rate.*

Proof. Follows the development (in reverse sense) of Finsler's equivalent conditions that preceded the theorem. □

The parameter d used in inequality (4.9) is related to the application of Pólya's relaxations [136, 137], which are useful when deriving numerically implementable solutions, as discussed in the sequel. Regarding the interesting properties of Theorem 4.3, note that both $A^{\#}_{cl}(\alpha)$ and the Lyapunov matrix V appear affinely in (4.9). Thus, the control gains can be dealt with as decision variables in the optimization problem, facilitating the treatment of structural constraints (e.g., decentralized control) or of magnitude restrictions on the entries of $K(\alpha)$, which can be helpful to avoid actuator saturation. For instance, one can solve (4.9) jointly with the linear constraints (elementwise inequalities)

$$K_{\min} \leq K_i \leq K_{\max}, \; i = 1, \dots, r, \tag{4.10}$$

where $K_{\min}$ and $K_{\max}$ are given matrices. Note that such constraints are hard to be considered in the approaches where the control gains are obtained through change of variables, specially in the case of output-feedback.

Notwithstanding, conditions (4.9) are BMIs due to the product $\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)$. By fixing some variables ($X_1(\alpha)$ and $X_2(\alpha)$) that transform the conditions into LMIs, an iterative algorithm (with local convergence) starting from an initial feasible solution is proposed to search for a solution to Theorem 4.3. The fact that γ appears affinely in (4.9) is explored to construct feasible initial values for $X_1(\alpha)$ and $X_2(\alpha)$. The strategy relies on computing an open-loop decay rate δ/m and, then, to solve (4.6) for the open-loop system, as proposed in next theorem.

Theorem 4.4

Let δ^* *be the optimal solution for the following maximization problem considering the open-loop T-S system:*

$$\begin{aligned} &\underset{\delta,\, E(\alpha),\, V=V^{\top}>0}{\text{maximize}} && \delta \\ &\text{subject to} && \mathrm{He}\big(V(A^{\#}(\alpha)+\delta I)+E(\alpha)\big) \leq 0. \end{aligned}$$

Then, by solving the robust LMIs

$$\begin{bmatrix} E(\alpha) & V \\ V & 0 \end{bmatrix} + \mathrm{He}\Big(\begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \end{bmatrix} \begin{bmatrix} \tilde{A}^{\#}(\alpha) & -I \end{bmatrix} \Big) \leq 0,$$

where $\tilde{A}^{\#}(\alpha) = A^{\#}(\alpha) + \delta^* I$, one obtains $X_1(\alpha)$ and $X_2(\alpha)$ matrices such that, by fixing

$$\mathscr{B}_2(\alpha) = \begin{bmatrix} X_1(\alpha)^\top & X_2(\alpha)^\top & -I \end{bmatrix}$$

in Theorem 4.3 (the conditions in this case become robust LMIs), a feasible solution with $\gamma \leq \delta^*$ is always obtained.

Proof. Immediate from the equivalences of Finsler's Lemma. □

It is interesting to note that $\xi = \gamma/m$, a bound for the decay rate of the system, can also be considered as a relaxation factor in stability. In fact, for sufficiently large negative values of γ, matrix $A^{\#}(\alpha) + \gamma I$ is always stable. As a consequence, it is immediate to construct feasible fixed values for $X_1(\alpha)$ and $X_2(\alpha)$ such that Theorem 4.3 always has a solution. Moreover, since matrix $A^{\#}_{cl}(\alpha) + \gamma I$ appears affinely in (4.9), γ can be conserved as an optimization variable (objective function) to be maximized. Whenever a positive γ is obtained, $\xi = \gamma/m$ is a lower bound for the decay rate of the system. If the outcome is such that $\gamma < 0$, Theorem 4.3 can be tested again. Actually, since $\mathrm{He}\big(\mathscr{Y}(\alpha)\bar{\mathscr{B}}_2(\alpha)\big) = \mathrm{He}\big(\bar{\mathscr{B}}_2^\top(\alpha)\mathscr{Y}^\top(\alpha)\big)$, one has that every $\mathscr{Y}^\top(\alpha)$ is a valid choice for $\bar{\mathscr{B}}_2(\alpha)$ in a new test of conditions (4.9).

Theorem 4.3 gives rise to an iterative algorithm with local convergence (non-decreasing γ). In fact, assuming the feasibility of conditions (4.9) at some iteration it, the particular choices $\mathscr{Q}_{2,it+1}(\alpha) = \mathscr{Q}_{2,it}(\alpha)$ and $\bar{\mathscr{B}}_{2,it+1}(\alpha) = \mathscr{Y}_{it}^\top(\alpha)$ assure that Theorem 4.3 yields $\gamma_{it+1} \geq \gamma_{it}$ as solution at iteration $it + 1$. The stopping criterion is defined in terms of maximum number of iterations (without success), or the value of γ. A stabilizing control gain is obtained if $\gamma = \xi m > 0$ or, if a target decay rate ξ_a is pursued, the algorithm ends when $\gamma \geq \xi_a m$.

Based on the above discussion, Algorithm 2 is proposed. The iterative procedure computes a feedback gain $K(\alpha)$ subject to a magnitude limitation while maximizing a bound to the decay rate. The input parameters are the matrices of the system, $m \geq 1$ (the HPLF degree corresponds to $2m$), the maximum number of Pólya's relaxations $d_{\max}$, the maximum number of iterations $it_{\max}$, the restrictions on the entries of gains K_i, given by $K_{\max}$ and $K_{\min}$ (optional), and the target decay rate ξ_a. A tolerance ε is used to evaluate the evolution of γ along consecutive iterations. When no significant increase is observed on γ, a escape from a local maximum can be tried by increasing the number of Pólya's relaxations.

Whenever available, solutions for $m = 1$ (computed by Algorithm 2 or any other technique) can be used to construct a initial $\bar{\mathscr{B}}_{2,0}(\alpha)$ to search for control gains using HPLFs of order $m \geq 2$, as presented in next theorem.

Algorithm 2 Decay Rate Maximization

Input: A_i, B_i, C_i, m, $d_{\max}$, it_{max}, $K_{\min}$, $K_{\max}$, ξ_a, ε;
Solve the problem in Theorem 4.4 to obtain $\bar{\mathscr{B}}_{2,0}(\alpha)$;
$it \leftarrow 0$; $d \leftarrow 0$;
While $it < it_{\max}$
    $it \leftarrow it + 1$;
    maximize γ **subject to** (4.9) **and** (4.10);
    If $\gamma_{it} \geq \xi_a m$ **Then**
        Return $\xi_{\max} = \gamma_{it}/m$, $K(\alpha)$, V;
    End If
    If $\gamma_{it} - \gamma_{it-1} \leq \varepsilon$ **and** $d < d_{\max}$ **Then**
        $d \leftarrow d + 1$;
    Else
        Return $\xi_{\max} = \gamma_{it}/m$, $K(\alpha)$, V;
    End If
    $\bar{\mathscr{B}}_{2,it}(\alpha) \leftarrow \mathscr{Y}^\top(\alpha)$;
End While
Return $\xi_{\max} = \gamma_{it}/m$, $K(\alpha)$, V;

Theorem 4.5

If feasible for some integer $m \geq 1$, the conditions of Theorem 4.3 are also feasible for ρm, where $\rho > 1$ is any integer.

Proof. First, a HPLF of degree $2\rho m$ is constructed from one of degree $2m$. Considering the HPLF candidate of degree $2m$, one has $v(x) = x^{\{m\}\top} V x^{\{m\}}$ and (4.4) with $E(\alpha) \in \mathcal{E}_{n,m}$. A feasible solution to Theorem 4.3 implies that Theorem 4.2 is also solvable. Thus, one has

$$V > 0, \quad \mathrm{He}\big(VA_{cl}^{\#}(\alpha)\big) + E(\alpha) \leq -2\gamma V. \tag{4.11}$$

For a HPLF of degree $2\rho m$, considering the extended matrix $\tilde{A}_{cl}^{\#}(\alpha)$, such that $\dot{x}^{\{\rho m\}} = \tilde{A}_{cl}^{\#}(\alpha) x^{\{\rho m\}}$, and a linear parametrization $\tilde{E}(\alpha) \in \mathcal{E}_{n,\rho m}$, one can write

$$\tilde{v}(x) = x^{\{\rho m\}\top} \tilde{V} x^{\{\rho m\}}$$

and

$$\dot{\tilde{v}}(x) = x^{\{\rho m\}\top} \big(\mathrm{He}(\tilde{V}\tilde{A}_{cl}^{\#}(\alpha)) + \tilde{E}(\alpha)\big) x^{\{\rho m\}}$$

implying

$$\dot{\tilde{v}}(x) + 2\tilde{\gamma}\tilde{v}(x) = x^{\{\rho m\}\top} \big(\mathrm{He}(\tilde{V}\tilde{A}_{cl}^{\#}(\alpha)) + \tilde{E}(\alpha) + 2\tilde{\gamma}\tilde{V}\big) x^{\{\rho m\}}. \tag{4.12}$$

A HPLF of degree ρm can be constructed as $\tilde{v}(x) = v(x)^\rho$ in terms of Kronecker powers:

$$\begin{aligned}\tilde{v}(x) &= \big((x^{\{m\}})^{[\rho]}\big)^\top V^{[\rho]} \big((x^{\{m\}})^{[\rho]}\big) \\ &= x^{\{\rho m\}\top} T^\top V^{[\rho]} T x^{\{\rho m\}}\end{aligned} \tag{4.13}$$

with $(x^{\{m\}})^{[\rho]} = Tx^{\{\rho m\}}$ where T has full column rank. Imposing that (4.13) describes the same candidate polynomial $\tilde{v}(x)$, and since the Kronecker power of a matrix preserves its positivity, one has $\tilde{V} = T^\top V^{[\rho]} T > 0$, concluding the first part of the proof.

Taking into account the time-derivative of $\tilde{v}(x)$, one has

$$\dot{\tilde{v}}(x) = \frac{dv(x)^\rho}{dv(x)} \dot{v}(x) = \rho v(x)^{\rho-1} \otimes \dot{v}(x)$$

or, using Kronecker powers,

$$\rho v(x)^{\rho-1} = \rho\big((x^{\{m\}})^{[\rho-1]}\big)^\top V^{[\rho-1]} \big((x^{\{m\}})^{[\rho-1]}\big)$$

yielding

$$\dot{\tilde{v}}(x) = x^{\{\rho m\}\top} T^\top \big(\rho V^{[\rho-1]} \otimes \big(\mathrm{He}(VA^{\#}_{cl}(\alpha)) + E(\alpha)\big)\big) T x^{\{\rho m\}}.$$

Then,

$$\dot{\tilde{v}}(x) + 2\tilde{\gamma}\tilde{v}(x) = x^{\{\rho m\}\top} M(\alpha) x^{\{\rho m\}} \tag{4.14}$$

with

$$M(\alpha) = T^\top \Big(\rho V^{[\rho-1]} \otimes \big(\mathrm{He}(VA^{\#}_{cl}(\alpha)) + E(\alpha)\big) + 2\tilde{\gamma} V^{[\rho]}\Big) T.$$

A bound to $x^{\{\rho m\}\top} M(\alpha) x^{\{\rho m\}}$ can be constructed from the feasibility of (4.11), yielding

$$\begin{aligned} x^{\{\rho m\}\top} M(\alpha) x^{\{\rho m\}} &\leq x^{\{\rho m\}\top} T^\top \big(\rho V^{[\rho-1]} \otimes (-2\gamma V) + 2\tilde{\gamma} V^{[\rho]}\big) T x^{\{\rho m\}} \\ &\leq (-2\gamma\rho + 2\tilde{\gamma}) x^{\{\rho m\}\top} T^\top V^{[\rho]} T x^{\{\rho m\}} \\ &= (-2\gamma\rho + 2\tilde{\gamma}) x^{\{\rho m\}\top} \tilde{V} x^{\{\rho m\}}. \end{aligned} \tag{4.15}$$

To have (4.12) and (4.14) as the same polynomial and taking into account (4.15), one has

$$\mathrm{He}(\tilde{V}\tilde{A}^{\#}_{cl}(\alpha)) + \tilde{E}(\alpha) + 2\tilde{\gamma}\tilde{V} = M(\alpha) \leq (-2\gamma\rho + 2\tilde{\gamma})\tilde{V}.$$

Defining $\gamma = \xi m$ and $\tilde{\gamma} = \xi \rho m$, then $-2\gamma\rho + 2\tilde{\gamma} = 0$ and

$$\mathrm{He}\big(\tilde{V}\tilde{A}^{\#}_{cl}(\alpha)\big) + \tilde{E}(\alpha) + 2\tilde{\gamma}\tilde{V} \leq 0.$$

Therefore, when condition (4.11) holds for some $\gamma = \xi m > 0$, the above condition is also guaranteed with $\tilde{V} > 0$ for $\tilde{\gamma} = \xi \rho m > 0$. □

The main utility of Theorem 4.5 is that whenever a controller $K(\alpha)$ is designed using a HPLF of degree m, matrix $(A^{\#}_{cl}(\alpha) - \gamma I)$ can also be certified as robustly stable using a HPLF of degree ρm, $\rho > 1$. In this case, Theorem 4.4 is not necessary and feasible values for $X_1(\alpha)$ and $X_2(\alpha)$ can be computed directly using (4.6) with $(A^{\#}_{cl}(\alpha) - \gamma I)$ constructed with degree ρm. Thus, the new decay rate computed with Algorithm 2 cannot be smaller than the previous one.

4.2 Numerical Examples

Regarding the implementation of the inequalities involving fuzzy summations in Theorems 4.3 and 4.4, the Robust LMI Parser (ROLMIP) is used to extract a finite set of LMIs from the robust LMIs [115, 136]. It is important to emphasize that ROLMIP operates with homogeneous polynomials, where the coefficients associated with monomials of the same degree but different order (such as $\alpha_i\alpha_j$ and $\alpha_j\alpha_i$) are merged into a single coefficient. This can possibly lead to slightly more conservative results compared to a case where the merge is not performed. Basically, a fuzzy summation $X(\alpha)$ is tested to be positive by imposing that each matrix coefficient is positive. The following values are used in Algorithm 2: $it_{\max} = 30$ for Example 1 and $it_{\max} = 50$ for Example 2, $d_{\max} = 5$, $\varepsilon = 10^{-3}$. Whenever Algorithm 2 is tested for consecutive values of m, the gain found for the degree $\bar{m}$ is used in the bisection of Theorem 4.4 when considering $\rho\bar{m}$, that is, matrix $A^{\#}(\alpha)$ is replaced by $A^{\#}_{cl}(\alpha)$. In other words, the bisection is performed for a closed-loop matrix with a previously computed gain, converging to a better (larger) value of δ and, overall, to a better decay rate when testing Algorithm 2. The optimization variables $\mathscr{Y}(\alpha)$, $X_1(\alpha)$ and $X_2(\alpha)$ have been chosen as affine fuzzy summations. The experiments were performed in a PC equipped with Ubuntu 20.04 64 bits, Core i7, 12 GB RAM, Matlab (R2017), Yalmip [114], Mosek 10.0.26 [146].

Regarding comparisons with other techniques from the literature, the following conditions were considered:

- Agulhari, Oliveira, and Peres [147, Theorem 2]: the technique has been adapted to cope with PDC design for T-S systems, basically fixing $P(\alpha) = P$ and considering $K(\alpha)$ affine on α. Concerning the scalar search demanded by the method, the following values were tested: $\xi \in \{10^{-6}, 10^{-5}, \ldots, 1, \ldots, 10^{5}, 10^{6}\}$. Finally, to facilitate the implementation, ROLMIP was used to program the synthesis conditions.

- Jeung and Lee [148, Theorem 4]: the previous values of ξ were also considered for the scalar search.

- Bouarar, Guelton, and Manamanni [149, Theorem 1]: To avoid the necessity of bounds for the time-derivative of the MFs, the choice was made $W_1(h) = W_1$ (as suggested by the authors).

- Montagner, Oliveira, and Peres [71]: particularly in the case of state-feedback design, this method, convergent for a sufficient large number of Pólya's relaxations, is used in the comparisons.

- Lo and Liu [150]: This method was specially developed to cope with polynomial fuzzy systems, but can be adapted to deal with T-S systems. The parser Yalmip [114] was used to implement the sum-of-squares based conditions. In the first stage, the same values of ξ informed above were considered.

Example 1

Consider a T-S system with matrices (randomly generated) of the local models given by

$$A_1 = \begin{bmatrix} 1 & -4 & -1 \\ 2 & -3 & -3 \\ 3 & -1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 7 & -2 \\ 0 & 2 & 5 \\ 5 & -2 & -5 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

The aim is to compare the proposed design technique with some conditions from the literature in the context of state-feedback and output-feedback for two situations:

$$\text{case I} : C_i = \begin{bmatrix} c_i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{case II} : C_i = \begin{bmatrix} c_i & 0 & 1 \end{bmatrix}.$$

with $c_1 = 1,\ c_2 = 1.5$. Table 4.1 presents the maximum values of the decay rate obtained by each condition and the associated computational complexity, given in terms of the number of scalar variables (V_s), LMI rows (L_{LMIs}) and computational times[1] (in seconds).

The results show that all relaxations, including Algorithm 2, are not capable to stabilize the system by state-feedback through a quadratic Lyapunov function, yielding negative values for $\xi_{\max}$. On the other hand, Algorithm 2 with degree four ($m = 2$) and six ($m = 3$) HPLFs can stabilize and provide bounds for the decay rate that become larger as m grows. In the output-feedback case, quadratic Lyapunov functions continue to fail in providing stabilizing controllers, while Algorithm 2 yields feasible results with $m = 2$ and $m = 3$. This experiment shows the importance of employing Lyapunov functions of degrees higher than two to improve performance in the context of control design for continuous-time T-S systems. As an illustration, the PDC gain designed with $m = 3$ in the output-feedback — case II is given by (truncated to 4 decimal digits)

$$K(\alpha(t)) = \alpha_1(t) \begin{bmatrix} 0.7330 & 0.6695 \end{bmatrix} + \alpha_2(t) \begin{bmatrix} -1.6186 & -0.7627 \end{bmatrix}$$

[1] Since the methods have been implemented and executed in the same computer, the computational time of each method provides a reliable relative evaluation of the different numerical complexities. Moreover, the times informed are the average of 10 tests performed.

Table 4.1: Maximum decay rates (ξ_{max}) obtained by the conditions of Agulhari, Oliveira, and Peres [147], Jeung and Lee [148], and Bouarar, Guelton, and Manamanni [149] and Algorithm 2. V_s is the number of scalar variables, L_{LMIs} the number of LMI rows and computational times are given in seconds.

	Condition	$\xi_{\max}$	V_s	L_{LMIs}	time (s)
State-feedback	Jeung and Lee [148]	-1.6673	56	61	1.97
	Bouarar, Guelton, and Manamanni [149]	-4.4808	44	34	0.09
	Montagner, Oliveira, and Peres [71]$_{g=1,d=1}$	-0.0266	48	27	0.09
	Montagner, Oliveira, and Peres [71]$_{g=5,d=5}$	-0.0266	120	75	0.36
	Agulhari, Oliveira, and Peres [147]	-0.0266	49	31	0.50
	Lo and Liu [150]	-0.0269	128	62	0.76
	Algorithm $2_{m=1}$	-0.0294	67	30	0.37
	Algorithm $2_{m=2}$	0.1021	256	60	1.78
	Algorithm $2_{m=3}$	0.2347	716	100	7.83
Output-feedback (case I)	Jeung and Lee [148]	-1.6673	54	61	1.18
	Bouarar, Guelton, and Manamanni [149]	-4.4808	30	30	0.06
	Agulhari, Oliveira, and Peres [147]	-0.0821	47	31	0.88
	Lo and Liu [150]	-0.1111	126	60	1.19
	Algorithm $2_{m=1}$	-0.0549	65	30	0.48
	Algorithm $2_{m=2}$	0.1249	254	60	1.44
	Algorithm $2_{m=3}$	0.1534	714	100	8.48
Output-feedback (case II)	Jeung and Lee [148]	-1.6673	52	61	1.28
	Bouarar, Guelton, and Manamanni [149]	-4.4808	20	26	0.06
	Agulhari, Oliveira, and Peres [147]	-0.2499	45	31	1.39
	Lo and Liu [150]	-0.2197	126	58	1.15
	Algorithm $2_{m=1}$	-0.0582	63	30	0.40
	Algorithm $2_{m=2}$	0.0487	252	60	1.55
	Algorithm $2_{m=3}$	0.0675	712	100	10.70

Considering $\alpha_1(t) = 0.5\sin(x_1(t)) + 0.5$ and $\alpha_2(t) = 1 - \alpha_1(t)$, a time simulation for the closed-loop system is performed considering $\begin{bmatrix}1 & -1 & 2\end{bmatrix}^\top$. Figure 4.1 shows the behavior of the state and control effort, illustrating the effectiveness of the designed static-output feedback control law.

As a final investigation, consider the following constraints on the magnitude of the control gains: $K_i = [k_{i1},\ k_{i2}],\ |k_{ij}| \leq 0.80$, $i = 1,2$ and $j = 1,2$. In this scenario, Algorithm 2 provides a stabilizing solution for $m = 3$, with $\gamma = 0.0549$ and

$$K(\alpha(t)) = \alpha_1(t)\begin{bmatrix}0.2417 & 0.6527\end{bmatrix} + \alpha_2(t)\begin{bmatrix}-0.8000 & -0.8000\end{bmatrix}.$$

The time simulation depicted in Figure 4.2 shows a decrease in performance (a longer interval required to drive the states towards zero), but the system is asymptotically stable. It is noteworthy that handling magnitude bounds is more challenging using the existing methods based on change of variables, which generally require structural constraints on the optimization variables (sources of conservativeness). In the proposed approach, these constraints can be straightforwardly incorporated.

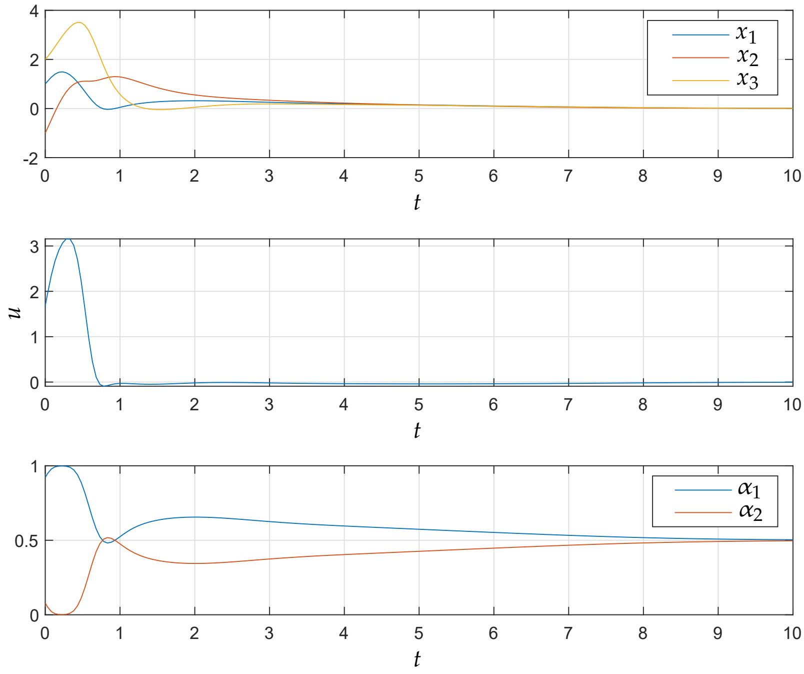


Figure 4.1: Closed-loop response (states, control signal and MFs) with the PDC controller $K(\alpha(t)) = \alpha_1(t)\begin{bmatrix}0.7330 & 0.6695\end{bmatrix} + \alpha_2(t)\begin{bmatrix}-1.6186 & -0.7627\end{bmatrix}$, for the output-feedback — case II.

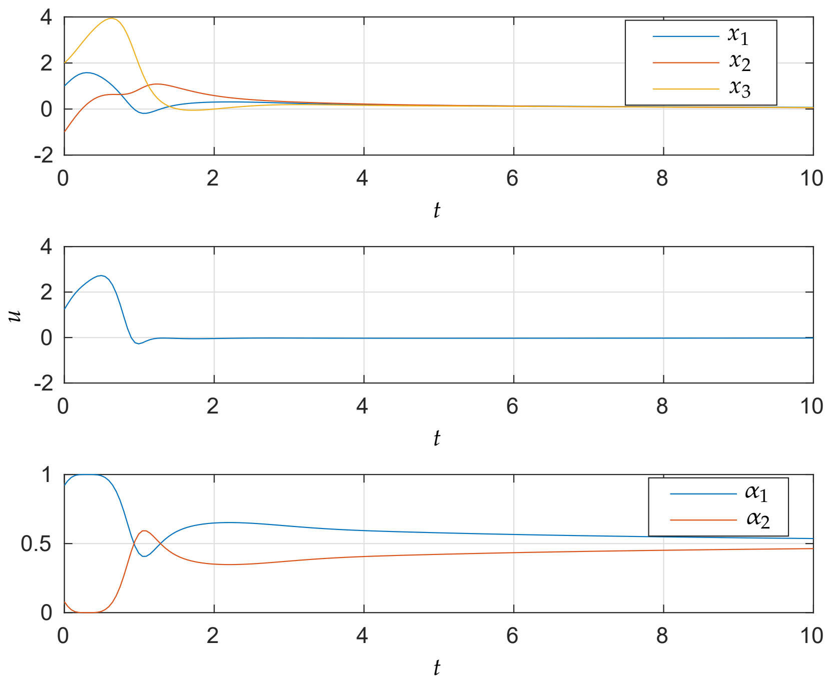


Figure 4.2: Closed-loop response (states, control signal and MFs) of with the PDC controller with magnitude bounds $K(\alpha(t)) = \alpha_1(t)\begin{bmatrix}0.2417 & 0.6527\end{bmatrix} + \alpha_2(t)\begin{bmatrix}-0.8000 & -0.8000\end{bmatrix}$, for the output-feedback — case II.

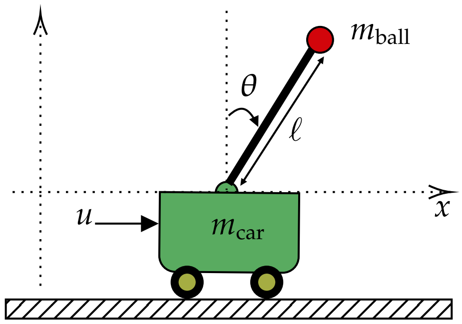


Figure 4.3: Inverted pendulum on a car.

Example 2

To illustrate the applicability of the proposed method in a T-S model arising from a physical system, consider the inverted pendulum depicted in Figure 4.3, where $m_{\text{ball}} = 0.2\ Kg$, $m_{\text{car}} = 0.8\ Kg$ and $\ell = 0.5\ m$. The states x_1 and x_3 are chosen as the pendulum angular position and angular velocity, respectively. The cart position and velocity are chosen as states x_2 and x_4. A T-S fuzzy model (borrowed from Hmidi et al. [151]) as in (3.2) has the following matrices

$$A_1 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 17.3118 & 0 & 0 & 0.0882 \\ 0 & 0 & 0 & 1.0000 \\ -1.7312 & 0 & 0 & -0.0441 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 14.3223 & 0 & 0 & 0.0573 \\ 0 & 0 & 0 & 1.0000 \\ -1.0127 & 0 & 0 & -0.0405 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ -1.7647 \\ 0 \\ 1.1765 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -1.1467 \\ 0 \\ 1.0811 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$\alpha_1(t) = \frac{1 - \dfrac{1}{1+\exp(-14(x_1(t)-\pi/8))}}{1+\exp(-14(x_1(t)-\pi/8))}, \ \alpha_2(t) = 1 - \alpha_1(t)$$

The aim is to design an output-feedback PDC controller maximizing the decay rate of the trajectories but also satisfying magnitude constraints on the entries of gains K_1 and K_2 to assure not excessively large control signals. With this purpose, the following must hold: $|K_{i_j}| \leq 20,\ i = 1,2,\ j = 1,2,3$. The results are presented in Table 4.2, where $it_{\max}$ is set to 50. As can be seen, the conditions in Agulhari, Oliveira, and Peres [147], Lo and Liu [150], and Algorithm 2 using quadratic in the state functions all yielded stabilizing controllers. In the implementation of the methods [147–150], the magnitude limits over the entries of K_i were achieved by constraining the variables involved, in some cases fixing some matrices as diagonal. Note that the proposed method does not suffer from this potential source of conservativeness. Notably, Algorithm 2 demonstrated an advantage in both the decay rate and computational time. When employing a quartic function, Algorithm 2 provided a significantly better decay rate, albeit with a substantially longer computational time, though still within a reasonable range. The PDC gain designed using $m = 2$ is

$$K(\alpha) = \alpha_1 \begin{bmatrix} 15.4246 & 2.0370 & 0.5603 \end{bmatrix} + \alpha_2 \begin{bmatrix} 20.0000 & 2.3371 & 0.5965 \end{bmatrix}$$

Adopting the initial condition $x(0) = [0.3491\ 0\ 0\ 0]^\top$ (corresponding to an initial angle of 20^o on θ), Figure 4.4 shows a time simulation for the closed-loop system.

Table 4.2: Maximum decay rates (ξ_{max}) obtained by the conditions of Agulhari, Oliveira, and Peres [147], Jeung and Lee [148], and Bouarar, Guelton, and Manamanni [149] and Algorithm 2 in Example 2 where the following constraint must hold: $|K_{i_j}| \leq 20,\ i = 1,2,\ j = 1,2,3$. V_s is the number of scalar variables, L_{LMIs} the number of LMI rows and computational times are given in seconds.

Condition	$\xi_{\max}$	V_s	L_{LMIs}	time (s)
Jeung and Lee [148]	-2.5011	90	68	1.33
Bouarar, Guelton, and Manamanni [149]	-4.1567	50	40	0.08
Agulhari, Oliveira, and Peres [147]	0.1260	81	40	1.29
Lo and Liu [150]	0.1052	202	89	3.31
Algorithm $2_{m=1}$	0.1973	113	52	1.18
Algorithm $2_{m=2}$	0.3522	702	112	25.39

As a final note, it is important to highlight that while a HPLF is utilized during the synthesis phase to improve the decay rate, the practical implementation of the resulting PDC controller is identical to the standard PDCs designed through quadratic-on-the-states Lyapunov functions. This fact further emphasizes the advantage of the proposed procedure, where the extra computational effort required to design the controller is spent offline. Moreover, the fact that the closed-loop matrix $A_{cl}(\alpha)$ appears affinely in the synthesis conditions facilitates the immediate treatment of dynamic output-feedback controllers and the handling of discrete-time systems.

4.3 Conclusion

This chapter provided an LMI-based algorithm for output-feedback PDC design for continuous-time T-S systems. The novelty is the possibility of employing homogeneous polynomial Lyapunov functions of arbitrary degree, generalizing the results based on quadratic functions. A numerical experiment showed a situation where quadratic functions may not be enough even to stabilize the system. Conversely, employing higher-degree Lyapunov functions cannot only stabilize the system but also enhance performance by improving the decay rate. It is worth noting that while the

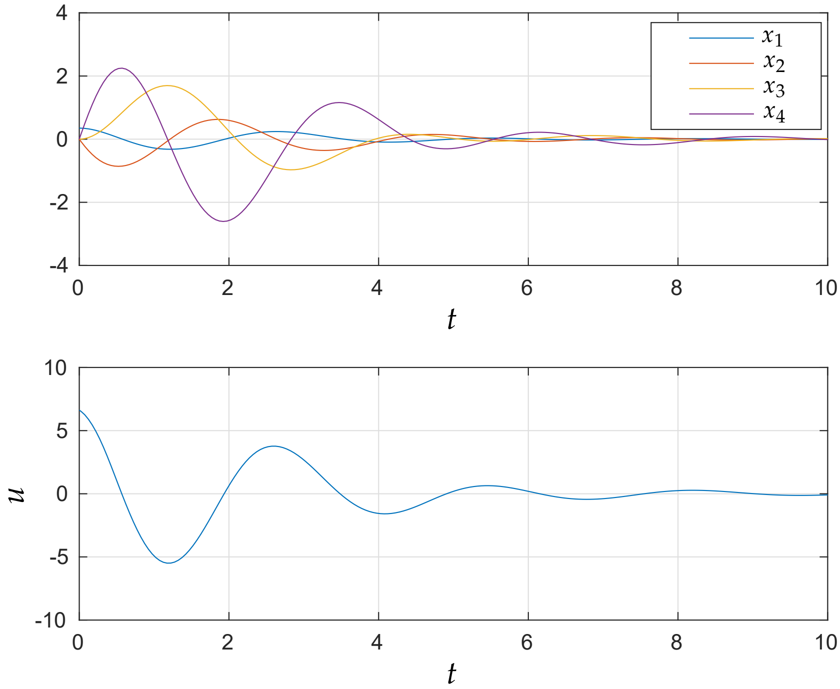


Figure 4.4: Closed-loop response (states and control signal) of the inverted pendulum considering magnitude bounds $K(\alpha) = \alpha_1 \begin{bmatrix} 15.4246 & 2.0370 & 0.5603 \end{bmatrix} + \alpha_2 \begin{bmatrix} 20.0000 & 2.3371 & 0.5965 \end{bmatrix}$ and $x(0) = [0.3491\ 0\ 0\ 0]^\top$.

proposed approach is initially developed to deal with T-S fuzzy systems, the results can also be beneficial in the context of Linear Parameter-Varying (LPV) systems with unknown variation rates of parameters (as switched systems) or other classes of systems where quadratic-on-the-states Lyapunov functions are not enough to characterize stability.

Chapter 5

Regional Stabilization of Continuous-time T-S Systems

Consider the closed-loop continuous-time T-S given in equation (3.6). In this chapter, the locality of the T-S model is further explored, considering Lyapunov candidate functions that depend on the MFs, allowing the incorporation of information regarding the T-S model validity domain, leading to new stability analysis and PDC control synthesis conditions where the knowledge of bounds for the time-derivative of the MFs is not necessary. Those conditions are derived considering two different Lyapunov functions: the first, a Fuzzy Lyapunov Function (FLF) that depends quadratically on the states and polynomially on the MFs; the second, a Homogeneous Polynomial Parameter-Dependent Lyapunov Function (HPPDLF), that depends polynomially on both the MFs and the states.

5.1 Fuzzy Lyapunov Function

Consider a quadratic-on-the-states parameter-dependent (membership function-dependent) candidate Lyapunov function (i.e., a FLF) of the form

$$v(x) = x^\top V(\alpha)x, \tag{5.1}$$

where $V_g(\alpha)$ is a homogeneous polynomial matrix of arbitrary degree g on $\alpha(z(t)) = \alpha$ defined as [136]

$$V_g(\alpha) = \sum_{k\in\mathcal{D}_{r,g}} \alpha^k V_k, \tag{5.2}$$

where α^k are the monomials and $V_k \in \mathbb{S}^n$, $\forall k \in \mathcal{D}_{r,g}$ are matrix-valued coefficients.

The exponential stability of the origin of system (3.6) can be proved using the presented concepts, according to the following theorem.

Theorem 5.1

Let $v : \mathbb{R}^n \to \mathbb{R}$, $v(x) = x^\top V(\alpha)x$, $V(\alpha)$ *as in* (5.2), *be a function satisfying*

$$\begin{cases} v(x) > 0, \ \forall x \in \mathbb{R}^n, \ x \neq 0 \\ \dot{v}(x) \leq -2\xi v(x), \ \forall x \in \mathbb{R}^n, \ x \neq 0, \ \forall \alpha \in \Lambda_r. \end{cases}$$

Then, $v(x)$ *is a FLF for the T-S system* (3.6) *that proves the exponential stability of the origin of this system and, when* $\xi > 0$, ξ *is a lower bound for the decay rate.*

Proof. See, for instance, [3, 22, 77]. □

Taking into account the Lyapunov function candidate of equation (5.1), the first exponential stability condition in Theorem 5.1 can be simply written as $V(\alpha) > 0$. The expression for $\dot{v}(x)$ is given by

$$\dot{v}(x) = \dot{x}^\top V(\alpha)x + x^\top V(\alpha)\dot{x} + x^\top \dot{V}(\alpha)x$$

and, according to the definition of the closed-loop T-S system (3.6):

$$\dot{v}(x) = x^\top \left(A_{cl}^\top(\alpha)V(\alpha) + V(\alpha)A_{cl}(\alpha) + \dot{V}(\alpha)\right)x.$$

Then, the second condition in Theorem 5.1 can be written as

$$x^\top \left(\mathrm{He}\,(V(\alpha)A_{cl}(\alpha)) + \dot{V}(\alpha)\right)x \leq -2\xi x^\top V(\alpha)x$$

or, equivalently,

$$\mathrm{He}\big(V(\alpha)\,(A_{cl}(\alpha) + \xi I_n)\,\big) + \dot{V}(\alpha) \leq 0.$$

The term $\dot{V}(\alpha)$ can be manipulated as follows [96]:

$$\begin{aligned} \dot{V}(\alpha) &= \frac{\partial V(\alpha)}{\partial \alpha_1}\dot{\alpha}_1 + \cdots + \frac{\partial V(\alpha)}{\partial \alpha_r}\dot{\alpha}_r \\ &= \sum_{i=1}^{r} \dot{\alpha}_i \left[\nabla_\alpha V(\alpha)\right]_i = \nabla_\alpha V(\alpha)\left(\dot{\alpha} \otimes I_n\right) \end{aligned} \tag{5.3}$$

where $\dot{\alpha} = \begin{bmatrix} \dot{\alpha}_1 & \cdots & \dot{\alpha}_r \end{bmatrix}^\top$ and

$$\nabla_\alpha V(\alpha) = \begin{bmatrix} \frac{\partial V(\alpha)}{\partial \alpha_1} & \cdots & \frac{\partial V(\alpha)}{\partial \alpha_r} \end{bmatrix} \in \mathbb{R}^{n \times rn}.$$

Since the premise variables z depend on the states, one has

$$\dot{\alpha}(z) = \nabla_x \alpha(z)\dot{x} = J(\theta)A_{cl}(\alpha)x, \tag{5.4}$$

$$[\nabla_x \alpha(z)]_{ij} = \frac{\partial \alpha_i}{\partial x_j}, \quad i = 1, \ldots, r, \quad j = 1, \ldots, n,$$

$$J(\theta) = \nabla_x \alpha(z) = \sum_{k=1}^{v} \theta_k(x) J_k, \quad J(\theta) \in \mathbb{R}^{r \times n},$$

and $\theta_k(x) \in \Lambda_v$. Matrices J_k and the number of vertices v are obtained from the knowledge of $\alpha(z)$ using a sector nonlinearity approach on the nonlinear terms found in $J(\theta)$ inside the set $\mathcal{X}$ where the T-S model is valid [7] (in Subsection 5.1.1 the computation of $J(\theta)$ is illustrated through an example). Therefore, using (5.4), $\dot{V}(\alpha)$ in (5.3) results in

$$\dot{V}(\alpha, \theta, \zeta) = \nabla_\alpha V(\alpha)\big(J(\theta) A_{cl}(\alpha) x(\zeta) \otimes I_n\big) \tag{5.5}$$

where the state vector is replaced by $x(\zeta)$, valid inside $\mathcal{X}$. Note that, using this approach, there is no need to specify bounds on the variation rate of the MFs, considered as a function of the T-S system closed-loop dynamics. The next theorem formalizes the above development.

Theorem 5.2

Given $K(\alpha) \in \mathbb{R}^{q \times s}$, if there exist $0 < V(\alpha) \in \mathbb{S}^n$ as in (5.2) and $\xi > 0$ satisfying

$$\text{He}\big(V(\alpha)\,(A_{cl}(\alpha) + \xi I_n)\,\big) + \dot{V}(\alpha, \theta, \zeta) \leq 0 \tag{5.6}$$

for all $\alpha \in \Lambda_r, \theta \in \Lambda_v, \zeta \in \Lambda_\kappa$ with $\dot{V}(\alpha, \theta, \zeta)$ as in (5.5), then $v(x)$ as in (5.1) is a FLF for the T-S fuzzy system (3.6) that proves the local exponential stability of the origin of the system and ξ is a lower bound for the system decay rate.

In (5.6), one can notice the product of the Lyapunov matrix $V(\alpha)$ by the feedback controller $K(\alpha)$ — inside the closed-loop matrix —, resulting in a Bilinear Matrix Inequality (BMI). An extra complicating factor in this case is the presence of the closed-loop matrix $A_{cl}(\alpha)$ inside $\dot{V}(\alpha, \theta, \zeta)$ in (5.5). Change of variables — at least for state-feedback — could handle this bilinearity, considering for instance the stability of the dual system $A_{cl}(\alpha)^\top$ (as in Gomes et al. [97], that solved the problem in two stages). However, a more general approach is pursued in this thesis, applying Finsler's lemma to separate $\nabla_\alpha V(\alpha)$ from the other matrices in (5.5), dealing directly with $K(\alpha)$ in the optimization problem (no change of variables is required) and facilitating the treatment of output-feedback (a much more challenging control problem). Inspired by Felipe and Oliveira [112], an iterative algorithm based on LMIs similar to the one in Section 2.4 is proposed to solve the problem.

According to Finsler's lemma, condition (5.6) in Theorem 5.2 can be written as $\mathscr{B}_1^{\perp\top}(\alpha, \theta, \zeta)\mathscr{Q}_1(\alpha)\mathscr{B}_1^{\perp}(\alpha, \theta, \zeta) \leq 0$, where

$$\mathscr{Q}_1(\alpha) = \begin{bmatrix} 0 & \star & \star \\ V(\alpha) & 0 & \star \\ 0.5\nabla_\alpha V(\alpha)^\top & 0 & 0 \end{bmatrix} \in \mathbb{S}^{(r+2)n}, \tag{5.7}$$

$$\mathscr{B}_1(\alpha,\theta,\zeta)^{\perp} = \begin{bmatrix} I_n \\ \big(A_{cl}(\alpha) + \xi I_n\big) \\ \big(J(\theta)A_{cl}(\alpha)x(\zeta) \otimes I_n\big) \end{bmatrix} \in \mathbb{R}^{(r+2)n\times n}.$$

Since $\sum_{i=1}^{r} \dot{\alpha}_i = 0$, from (5.4) one has

$$\sum_{i=1}^{r} \dot{\alpha}_i = \mathbb{1}_r^{\top}\dot{\alpha} = \mathbb{1}_r^{\top} J(\theta)A_{cl}(\alpha)x(\zeta) = 0.$$

Moreover, with $\mathscr{B}_1^{\perp}(\alpha,\theta,\zeta)$ given above,

$$\mathscr{B}_1(\alpha,\theta,\zeta) = \begin{bmatrix} A_{cl}(\alpha) + \xi I_n & -I_n & \mathbb{1}_r^{\top} \otimes I_n \\ J(\theta)A_{cl}(\alpha)x(\zeta) \otimes I_n & 0 & -I_{rn} \end{bmatrix} \in \mathbb{R}^{(r+1)n\times(r+2)n} \tag{5.8}$$

and the Finsler's equivalent condition with a slack variable $\mathscr{X}(\alpha)$, statement (iv), reads

$$\exists \mathscr{X}(\alpha) \,:\, \mathscr{Q}_1(\alpha) + \mathrm{He}\big(\mathscr{X}(\alpha)\mathscr{B}_1(\alpha,\theta,\zeta)\big) \leq 0 \tag{5.9}$$

where $\mathscr{X}(\alpha)$ can be partitioned as

$$\mathscr{X}(\alpha) = \begin{bmatrix} X_{11}(\alpha) & X_{12}(\alpha) \\ X_{21}(\alpha) & X_{22}(\alpha) \\ X_{31}(\alpha) & X_{32}(\alpha) \end{bmatrix} \in \mathbb{R}^{(r+2)n\times(r+1)n}. \tag{5.10}$$

Rewriting inequality (5.9) as $\mathscr{B}_2^{\perp\top}(\alpha)\mathscr{Q}_2(\alpha,\theta,\zeta)\mathscr{B}_2^{\perp}(\alpha) \leq 0$, with

$$\mathscr{Q}_2(\alpha,\theta,\zeta) = \begin{bmatrix} \mathscr{Q}_1(\alpha) & \star \\ \mathscr{B}_1(\alpha,\theta,\zeta) & 0 \end{bmatrix} \in \mathbb{S}^{(2r+3)n}, \tag{5.11}$$

where $\mathscr{Q}_1(\alpha)$, $\mathscr{B}_1(\alpha,\theta,\zeta)$ are given respectively in (5.7), (5.8) and

$$\mathscr{B}_2^{\perp}(\alpha) = \begin{bmatrix} I_{(r+2)n} \\ \mathscr{X}(\alpha)^{\top} \end{bmatrix} \in \mathbb{R}^{(2r+3)n\times(r+2)n},$$

the condition with slack variables can be applied once more, providing

$$\exists \mathscr{Y}(\alpha) \,:\, \mathscr{Q}_2(\alpha,\theta,\zeta) + \mathrm{He}\big(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)\big) \leq 0$$

with $\mathscr{Q}_2(\alpha,\theta,\zeta)$ as in (5.11), $\mathscr{Y}(\alpha)$ partitioned as

$$\mathscr{Y}(\alpha) = \begin{bmatrix} Y_{11}(\alpha) & Y_{12}(\alpha) \\ Y_{21}(\alpha) & Y_{22}(\alpha) \\ Y_{31}(\alpha) & Y_{32}(\alpha) \\ Y_{41}(\alpha) & Y_{42}(\alpha) \\ Y_{51}(\alpha) & Y_{52}(\alpha) \end{bmatrix} \in \mathbb{R}^{(2r+3)n\times(r+1)n},$$

and $\mathscr{B}_2(\alpha)$, from the orthogonality condition $\mathscr{B}_2(\alpha)\mathscr{B}_2^{\perp}(\alpha) = 0$, given by

$$\mathscr{B}_2(\alpha) = \begin{bmatrix} \mathscr{X}(\alpha)^{\top} & -I_{(r+1)n} \end{bmatrix} \in \mathbb{R}^{(r+1)n\times(2r+3)n},$$

where $\mathscr{X}(\alpha)$ is presented in (5.10).

From the above discussion, sufficient conditions for local exponential stability by output feedback of the T-S fuzzy system (3.6) with $\xi > 0$ as a lower bound for the decay rate are presented in next theorem.

Theorem 5.3

If there exist parameter-dependent matrices $0 < V(\alpha) \in \mathbb{S}^n$ *as in* (5.2) , $K(\alpha) \in \mathbb{R}^{q\times n}$, $\mathscr{Y}(\alpha) \in \mathbb{R}^{(2r+3)n\times(r+1)n}$, $\mathscr{B}_2(\alpha) \in \mathbb{R}^{(r+1)n\times(2r+3)n}$ *and a scalar* $\xi > 0$ *such that the robust BMIs*

$$\mathscr{Q}_2(\alpha,\theta,\zeta) + \mathrm{He}(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)) \leq 0, \tag{5.12}$$

hold for all $\alpha \in \Lambda_r$, $\theta \in \Lambda_v$ *and* $\zeta \in \Lambda_\kappa$ *with* $\mathscr{Q}_2(\alpha,\theta,\zeta)$ *as in* (5.11), *then* $v(x)$ *as in* (5.1) *is a FLF for the T-S system* (3.6) *that proves the local exponential stability of the origin of the system and* ξ *is a lower bound for the decay rate.*

Since the closed-loop matrix of the T-S system (as well as the control gain $K(\alpha)$) and the Lyapunov matrix appear affinely in the conditions of Theorem 5.3, magnitude bounds can be directly imposed on the entries of the gain matrices through linear constraints. In this case, the conditions in Theorem 5.3 must be solved considering

$$K_m \leq K_i \leq K_M, \quad i = 1, \dots, r \tag{5.13}$$

where K_m e K_M are given matrices. Therefore, there is no additional conservatism, contrary to what is observed in standard techniques that impose structural constraints on the optimization variables involved in the computation of K_i.

Notwithstanding, the conditions in Theorem 5.3 are BMIs due to the product $\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)$. To solve them, an iterative algorithm with local convergence is proposed. Phase 1 of the algorithm starts from an initial feasible solution by fixing some variables — the matrices in $\mathscr{B}_2(\alpha)$, precisely —, that transform the conditions into LMIs. The existence of a solution can be guaranteed by considering ξ as a relaxation parameter on the system stability, which ensures that the conditions in Theorem 5.3 are feasible for a sufficiently large negative value of ξ. Moreover, ξ (that appears affinely in the conditions) can be considered as the objective function to be maximized in the iterative procedure. A stabilizing feedback gain is found when $\xi > 0$.

As discussed in Section 2.4, the main characteristic of the strategy used in this work, that allows the solution of the problem in an iterative way, is the fact that

$$\mathrm{He}\big(\mathscr{Y}(\alpha)\bar{\mathscr{B}}_2(\alpha)\big) = \mathrm{He}\big(\bar{\mathscr{B}}_2(\alpha)^\top\mathscr{Y}(\alpha)^\top\big).$$

Therefore, every $\mathscr{Y}(\alpha)^\top$ is a valid choice for $\bar{\mathscr{B}}_2(\alpha)$ at the next iteration. The local convergence of the algorithm, with non-decreasing ξ, can be proved assuming the feasibility of the conditions in Theorem 5.3 at some iteration it, showing that the conditions in Theorem 5.3 remain feasible at the iteration $it+1$ with the particular choices $\mathscr{Q}_{2,it+1}(\alpha) = \mathscr{Q}_{2,it}(\alpha)$ and $\bar{\mathscr{B}}_{2,it+1}(\alpha) = \mathscr{Y}_{it}(\alpha)^\top$, ensuring that $\xi_{it+1} \geq \xi_{it}$. Thus, the stopping criterion of phase 1 of the algorithm is defined in terms of the evolution of ξ. When a certain target decay rate ξ_a must be attained, the algorithm ends when $\xi \geq \xi_a$ ($\xi_a = 0$ if only stabilization is pursued).

After the required decay rate is ensured, the same iterative procedure can be used to enlarge an estimate of the DOA contained in the domain of validity of the T-S system, that can be represented as the largest invariant set Ω contained in the polytope $\mathcal{X}$ [22], given by $\Omega \triangleq x \in \mathbb{R}^n$ such that $x^\top V(\alpha)x \leq 1$. The constraint $\Omega \in \mathcal{X}$ holds if $a_k^\top V(\alpha)^{-1}a_k \leq c_k^2$, $k = 1, \ldots, m$ [22]. Applying the Schur complement, the condition results in

$$\begin{bmatrix} V(\alpha) & a_k \\ a_k^\top & c_k^2 \end{bmatrix} \geq 0, \quad k = 1, \ldots, m, \quad \forall \alpha \in \Lambda_r. \tag{5.14}$$

One of the possibilities for the enlargement of Ω is to consider the following optimization problem [152]

$$\min_{K(\alpha),V(\alpha),W} \quad \sigma^* = \mathrm{Tr}(W) \quad \text{s.t. (5.12), (5.13), (5.14)},\ V(\alpha) \leq W$$

where $W \in \mathbb{S}^n$, which corresponds to a homogeneous increase of the ellipsoid Ω in all directions [152]. This optimization is performed in phase 2 of the algorithm.

A feasible initial solution for the algorithm can be obtained as follows. Use Theorem 5.2 to compute the open-loop decay rate δ^* of the T-S fuzzy system inside the polytope $\mathcal{X}$. Then, solve equation (5.9) with slack variables considering the modified open-loop T-S system $\tilde{A}(\alpha) = (A(\alpha) + \delta^* I)$ that is guaranteed to be stable. This procedure is described in next theorem.

Theorem 5.4

Let δ^* be the solution of the maximization problem (considering the open-loop T-S system)

$$\begin{aligned} &\underset{\delta,V(\alpha)}{\textit{maximize}} && \delta \\ &\textit{subject to} && (5.14),\ \mathrm{He}\left(V(\alpha)\left(A(\alpha) + \delta I_n\right)\right) + \dot{V}(\alpha,\theta,\zeta) \leq 0. \end{aligned}$$

Then, replace the closed-loop matrix $A_{cl}(\alpha)$ in $\mathscr{B}_1(\alpha,\theta,\zeta)$ defined in (5.8) *by $\tilde{A}(\alpha) = A(\alpha) + \delta^* I$ and solve* (5.9) *on $\mathscr{X}(\alpha)$ to obtain the feasible initialization*

$$\bar{\mathscr{B}}_2(\alpha) = \begin{bmatrix} X_{11}(\alpha)^\top & X_{21}(\alpha)^\top & X_{31}(\alpha)^\top & -I_n & 0 \\ X_{12}(\alpha)^\top & X_{22}(\alpha)^\top & X_{32}(\alpha)^\top & 0 & -I_{rn} \end{bmatrix}.$$

Proof. Immediate considering the equivalent conditions in Finsler's lemma. □

Based on the results presented, Algorithm 3 is proposed, which, at phase one, consists on an iterative procedure to compute the feedback gain $K(\alpha)$ subject to a magnitude limitation and that ensures the performance criterion defined by a certain decay rate. The input parameters are matrices A_i, B_i and C_i, $i \in \mathbb{N}_r$ of the T-S fuzzy system local models, matrices J_j, $j \in \mathbb{N}_v$ of the Jacobian of the MFs, x_k, $k \in \mathbb{N}_\kappa$ that define the polytope $\mathcal{X}$, the maximum number of iterations it_{max}, the restrictions on the gains K_M and K_m (optional), and the target decay rate ξ_a. A tolerance ε is used to evaluate the evolution of the algorithm between consecutive iterations. In phase one ($ph = 1$), the evolution is based on the parameter ξ: once the decay rate reaches the target value, the algorithm starts the second stage ($ph = 2$), where the decay rate is fixed at the value found on the first stage, and the conditions are solved iteratively to maximize the estimate for the region of attraction. The evolution of the algorithm during this stage is monitored based on the difference of $\mathrm{Tr}(W)$ between consecutive iterations.

After fixing a homogeneous polynomial structure for the optimization variables, the complexity of the algorithm can be evaluated based on the number V_s of scalar variables and L_{LMIs} of LMI rows. Considering the definition $\mu(g) = \frac{(r+g-1)!}{(r-1)!g!}$ and assuming polynomial degrees g_1 and g_2 on α for $V(\alpha)$ and $\mathscr{Z}(\alpha)$, respectively, V_s and L_{LMIs} in phase 2 for output-feedback (the most complex case because $A_{cl}(\alpha)$ is cubic on α) are

$$
\begin{aligned}
V_s &= 0.5(\mu(g_1)+1)n(n+1) + rqs + \mu(g_2)(2r^2+5r+3)n^2 \\
L_{LMIs} &= v\kappa(2r+3)n\mu(g_L) + m(n+1)\mu(g_1) + \mu(g_1)n + 2rqs,
\end{aligned}
$$

where $g_L = \max(g_1, 2g_2, 3)$ and $2rqs$ only appears if magnitude bounds are imposed for the gains.

5.1.1 Numerical Example

The Robust LMI Parser (ROLMIP) is used to transform the robust LMIs (that depend on α, θ, ζ) in a finite set of LMIs [115, 136]. The variables $V(\alpha)$ and $K(\alpha)$ are fixed as polynomials of degree one (affine on α), and the slack matrix $\mathscr{Y}(\alpha)$ is fixed with the same degree of $A_{cl}(\alpha)$. Higher degrees could provide better results at the price of a larger computation effort. The algorithm, built in Matlab 9.4 (R2018a), was solved with Mosek 9.3.18 [146], in an 11th Gen Intel(R) Core(TM) i7-1165G7 @ 2.80GHz with 16 GB RAM, Windows 11 Home Single Language. For the computation of areas of ellipsoids in the plane, the script `polyarea` from Matlab was used.

Algorithm 3 Control Design

Input: A_i, B_i, C_i, J_j, x_k, it_{max}, K_m, K_M, ξ_a, ε;
Solve the problem in Theorem 5.4 to obtain $\bar{\mathscr{B}}_{2,0}(\alpha)$;
$it \leftarrow 0$; $ph \leftarrow 1$;
While $it < it_{max}$
$it \leftarrow it + 1$;
If $ph = 1$ **Then**
maximize ξ **subject to** (5.12)**,** (5.13) **and** (5.14);
If $\xi_{it} \geq \xi_a$ **Then**
$ph \leftarrow 2$, $\xi \leftarrow \xi_{it}$;
Else If $\xi_{it} - \xi_{it-1} \leq \varepsilon$ **Then**
break;
End If
Else If $ph = 2$ **Then**
minimize $\mathrm{Tr}(W)$ **subject to** (5.12)**,** (5.13)**,** (5.14) **and** $V(\alpha) \leq W$;
If $\mathrm{Tr}(W)_{it-1} - \mathrm{Tr}(W)_{it} \leq \varepsilon$ **Then**
break;
End If
End If
$\bar{\mathscr{B}}_{2,it}(\alpha) \leftarrow \mathscr{Y}_{it}(\alpha)^\top$;
End While
If $ph = 1$ **Then**
Return $\xi_{max} = \xi_{it}$, $K(\alpha)$, $V(\alpha)$;
Else If $ph = 2$ **Then**
Return $\sigma^* = \mathrm{Tr}(W)_{it}$, $\xi_{max} = \xi$, $K(\alpha)$, $V(\alpha)$;
End If

Consider the nonlinear system [92, 97]:

$$\dot{x} = \begin{bmatrix} -\frac{7}{2} - \frac{3}{2}\sin(x_1) & -4 \\ \frac{19}{2} - \frac{21}{2}\sin(x_1) & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{13}{2} + \frac{7}{2}\sin(x_1) \end{bmatrix} u.$$

Using the sector nonlinearity approach over the nonlinear term $z = \sin(x_1)$, this system can be exactly represented in the compact set $\mathcal{X} = \{x \in \mathbb{R}^2 : |x_i| \leq \pi/2, i = 1, 2\}$ as the T-S fuzzy system (3.2) with

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

$$\alpha_1(z) = (1 + \sin(x_1))/2, \quad \alpha_2(z) = 1 - \alpha_1(z).$$

The Jacobian of the MFs is given by

$$J(\theta) = \nabla_x \alpha(z) = \begin{bmatrix} \frac{\partial \alpha_1(z)}{\partial x_1} & \frac{\partial \alpha_1(z)}{\partial x_2} \\ \frac{\partial \alpha_2(z)}{\partial x_1} & \frac{\partial \alpha_2(z)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \cos(x_1)/2 & 0 \\ -\cos(x_1)/2 & 0 \end{bmatrix}.$$

Applying the sector nonlinearity technique over the nonlinear term $\cos(x_1)$, then $J(\theta) = \theta_1 J_1 + \theta_2 J_2$, $\theta \in \Lambda_2$,

$$J_1 = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The estimate for the DOA obtained by Algorithm 3 (Alg. 3) with $V(\alpha)$ of degree one in α, tolerance $\varepsilon = 10^{-3}$ and target decay rate $\xi_a = 0$ — i.e., stabilizing gains — is shown in Figure 5.1 for both the state- (in black) and output-feedback (in red) — with $C_1 = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 10 & 0 \end{bmatrix}$ —, almost completely overlapping, as well as the estimates provided by other conditions available in the literature for state-feedback: Gomes et al. [97] with Lyapunov matrix of degree $g = 2$ in α and $\xi = 10$ in magenta, Lee and Kim [93] using degree $q = 3$ and $\phi = 10000$ (values also used in Gomes et al. [97] for comparison) in blue, and Pan et al. [92] in green. The corresponding areas are given in Table 5.1, with the number V_s of scalar variables and L_{LMIs} of LMI rows and the computational time demanded by each condition — providing an estimate for their complexity. The results clearly show that, despite the computational cost, the proposed method presents superior results for the estimates of the region of attraction, even when considering the output-feedback, when compared to the other conditions for state-feedback, that are clearly more general. The PDC output-feedback gain provided by the algorithm is given by ($\alpha \in \Lambda_2$)

$$K(\alpha) = \alpha_1 \begin{bmatrix} 0.1049 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.5143 \end{bmatrix}.$$

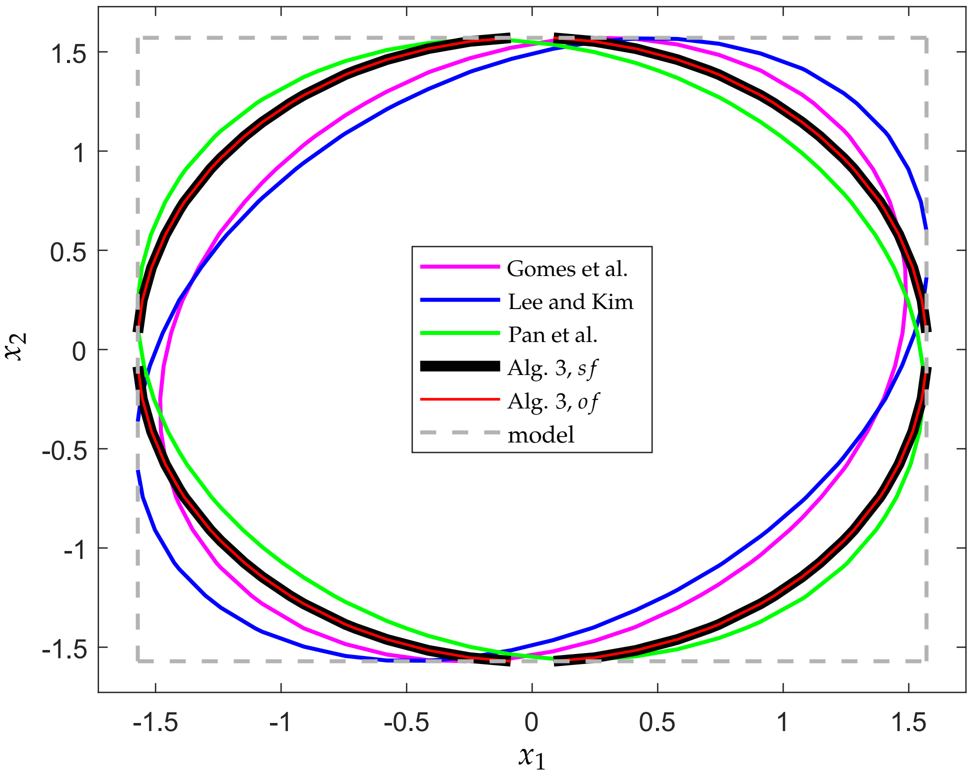


Figure 5.1: Estimates of the DOA obtained with Alg 3 for the state- (sf, black) and output-feedback (of,red), and provided by the state-feedback conditions of Gomes et al. [97] (magenta), Lee and Kim [93] (blue) and Pan et al. [92] (green). The domain of validity of the model is depicted in gray (dashed).

Table 5.1: Areas of the estimates of the DOA for conditions in Pan et al. [92], Lee and Kim [93], and Gomes et al. [97] and Algorithm 3. V_s and L_{LMIs} are respectively the numbers of scalar variables and LMI rows and the computational time is given in seconds.

	Condition	area ($u.a.$)	V_s	L_{LMIs}	time (s)
State-feedback	Gomes et al. [97]$_{g=2,\xi=10}$	7.1824	41	242	1.46
	Lee and Kim [93]$_{q=3,\phi=10000}$	7.3724	129	180	0.94
	Pan et al. [92]	7.6018	22	32	0.41
	Algorithm 3	9.6705	265	588	16.21
Output-feedback	Algorithm 3	9.6776	347	812	43.27

5.2 Homogeneous Polynomial Parameter-Dependent Lyapunov Functions

In this section, the previous results are generalized to consider Lyapunov functions that depend polynomially on the states. Unlike the results in Chapter 4, an additional difficulty arises because the time-derivative of the Lyapunov matrix must be suitably addressed.

Consider the candidate Lyapunov function given by

$$v(x,\alpha) = x^{\{m\}\top} V_g(\alpha) x^{\{m\}} \tag{5.15}$$

where $V_g(\alpha)$ is a homogeneous polynomial matrix of arbitrary degree g on $\alpha(z(t)) = \alpha$ defined as [136]

$$V_g(\alpha) = \sum_{k \in \mathcal{D}_{r,g}} \alpha^k V_k, \tag{5.16}$$

where α^k are the monomials and $V_k \in \mathbb{S}^{\sigma(n,m)}$, $\forall k \in \mathcal{D}_{r,g}$ are matrix-valued coefficients.

Hence, the Lyapunov function $v(x,\alpha)$ given in (5.15) — here nominated Homogeneous Polynomial Parameter-Dependent Lyapunov Function, HPPDLF) — possesses two degrees of freedom. First, matrix $V_g(\alpha)$ exhibits a homogeneous polynomial dependency of arbitrary degree g on the MFs $\alpha_i(z(t))$. Second, and more relevant, $v(x,\alpha)$ is a homogeneous polynomial of degree $2m$ with respect to the state variables x.

In the context of polytopic linear systems characterized by time-varying uncertainties and finite bounds on their variation rate, a similar structure for Lyapunov functions was introduced in Chesi et al. [77] to address robust stability analysis. Referred to as Homogeneous Parameter-Dependent Homogeneous Lyapunov Functions (HPDHLFs), this class of functions also incorporates homogeneous polynomials in both the state variables and the uncertain parameters. Notably, in this formulation, robust stability conditions are derived considering an SMR in terms of an expanded vector encompassing the state variables and the uncertain parameters altogether. In contrast, the strategy pursued in this thesis follows a different path, considering the dependence on x and α separately (more on this in the subsequent sections). Such separation is crucial when designing feedback laws, especially for output-feedback control.

Considering the Lyapunov function (5.15), the exponential stability of the origin of system (3.6) can be assessed through the next theorem [22].

Theorem 5.5

Let $v : \mathbb{R}^n \to \mathbb{R}$, $v(x,\alpha) = x^{\{m\}\top} V_g(\alpha) x^{\{m\}}$, $V_g(\alpha) \in \mathbb{S}^{\sigma(n,m)}$ as in (5.16), *be a function* *satisfying*

$$\begin{cases} v(x,\alpha) > 0, \ \forall x \in \mathbb{R}^n, \ x \neq 0, \ \forall \alpha \in \Lambda_r \\ \dot{v}(x,\alpha) \leq -2\gamma v(x,\alpha), \ \forall x \in \mathbb{R}^n, \ x \neq 0, \ \forall \alpha \in \Lambda_r, \end{cases}$$

along the trajectories of (3.6) *with a scalar* $\gamma > 0$*. Then* $v(x,\alpha)$ *is a HPPDLF of degree* $2m$ *on the states* x *and degree* g *on the MFs* α *for the T-S system* (3.6) *that certifies that the origin of this system is exponentially stable, and provides a lower bound for the decay rate, given by* $\xi = \gamma/m$*.*

Proof. See, for instance, [3, 22, 77]. □

The case $m = 1$ and degree g on the MFs has been explored in previous studies, either by imposing bounds on the time-derivative of the MFs [19, 20, 90–93], or in a stability analysis scenario without such constraints[19, 90, 94–96]. This particular Lyapunov function has also been investigated in the previous section.

Fixing $m = 1$ and $g = 0$ (constant Lyapunov matrix) in Theorem 5.5 leads to the familiar outcome built upon the quadratic Lyapunov function $v(x) = x^\top V x$, extensively employed within the T-S fuzzy literature. However, a synthesis procedure where $m > 1$ certainly induces an algebraic challenge because conventional linearization methods, such as the widely used change of variables ($K(\alpha)V = Z(\alpha)$) and congruence transformations, find limited applicability in such cases, especially for output-feedback control. Moreover, the option $g > 1$ also demands suitable algebraic manipulation for the time-derivative of the Lyapunov matrix, i.e., $\dot{V}_g(\alpha)$. Finsler's lemma and adequate relaxations play important roles in dealing with these technical difficulties, as presented next.

Considering the Lyapunov function (5.15), the first inequality in Theorem 5.5 is equivalent to

$$V_g(\alpha) > 0.$$

Equation (5.15) also allows one to get the expression for the time derivative $\dot{v}(x,\alpha)$ as

$$\dot{v}(x,\alpha) = \dot{x}^{\{m\}\top} V_g(\alpha) x^{\{m\}} + x^{\{m\}\top} \dot{V}_g(\alpha) x^{\{m\}} + x^{\{m\}\top} V_g(\alpha) \dot{x}^{\{m\}}, \tag{5.17}$$

and, from the definition of the closed-loop T-S system (3.6) and the extended matrix (2.3), the term $\dot{x}^{\{m\}}$ is equivalent to

$$\dot{x}^{\{m\}} = A_{cl}^{\#}(\alpha) x^{\{m\}},$$

which can be used in (5.17), resulting in

$$\dot{v}(x,\alpha) = x^{\{m\}\top} \left(A_{cl}^{\#\top}(\alpha) V_g(\alpha) + V_g(\alpha) A_{cl}^{\#}(\alpha) + \dot{V}_g(\alpha) \right) x^{\{m\}}. \tag{5.18}$$

Thus, the second inequality in Theorem 5.5 can be written as

$$x^{\{m\}\top}\left(\mathrm{He}\left(V_g(\alpha)A_{cl}^{\#}(\alpha)\right)+\dot{V}_g(\alpha)\right)x^{\{m\}}\leq -2\gamma x^{\{m\}\top}V_g(\alpha)x^{\{m\}}$$

i.e.,

$$\mathrm{He}\left(V_g(\alpha)A_{cl}^{\#}(\alpha)\right)+\dot{V}_g(\alpha)+2\gamma V_g(\alpha)\leq 0$$

or

$$\mathrm{He}\left(V_g(\alpha)\left(A_{cl}^{\#}(\alpha)+\gamma I_\sigma\right)\right)+\dot{V}_g(\alpha)\leq 0. \tag{5.19}$$

The time-derivative of the parameter-dependent Lyapunov matrix, $\dot{V}_g(\alpha)$, can be written as

$$\begin{aligned}\dot{V}_g(\alpha) &= \frac{\partial V_g(\alpha)}{\partial \alpha_1}\dot{\alpha}_1+\ldots+\frac{\partial V_g(\alpha)}{\partial \alpha_r}\dot{\alpha}_r\\ &= \sum_{i=1}^{r}\dot{\alpha}_i\left[\nabla_\alpha V_g(\alpha)\right]_i\\ &= \nabla_\alpha V_g(\alpha)\left(\dot{\alpha}\otimes I_\sigma\right)\end{aligned}$$

where

$$\nabla_\alpha V_g(\alpha)=\begin{bmatrix}\frac{\partial V_g(\alpha)}{\partial\alpha_1} & \ldots & \frac{\partial V_g(\alpha)}{\partial\alpha_r}\end{bmatrix},\quad \dot{\alpha}^\top=\begin{bmatrix}\dot{\alpha}_1 & \cdots & \dot{\alpha}_r\end{bmatrix}.$$

Since the MFs $\alpha(z(t))$ are functions of the premise variables z, and those are functions of the state variables, as previously defined in Chapter 3, one has

$$\begin{aligned}\dot{\alpha}(z) &= \nabla_x\alpha(z)\dot{x}\\ &= J(\theta)A_{cl}(\alpha)x\end{aligned} \tag{5.20}$$

where

$$\begin{aligned}\left[\nabla_x\alpha(z)\right]_{ij} &= \frac{\partial\alpha_i}{\partial x_j},\\ J(\theta)=\nabla_x\alpha(z) &= \sum_{i=1}^{v}\theta_i(x)J_i,\end{aligned}$$

and $\theta(x)\in\Lambda_v$; matrices J_i are obtained from the knowledge of $\alpha(z)$ and the set $\mathcal{X}$ within the T-S model is valid [7]. Therefore, using (5.20), one has

$$\dot{V}_g(\alpha)=\nabla_\alpha V_g(\alpha)\left(J(\theta)A_{cl}(\alpha)x(\zeta)\otimes I_\sigma\right) \tag{5.21}$$

where the state vector is replaced by $x(\zeta)$ as in (3.5), valid inside $\mathcal{X}$. The advantage of this modeling is that no additional information (as upper bounds) for variation rates of the premise variables is necessary since they can be represented as functions of the closed-loop dynamics of the system.

Considering matrix

$$E_{g_L}(\alpha) = \sum_{k \in \mathcal{D}_{r,g_L}} \alpha^k E(\beta^k).$$

where $E(\beta^k) \in \mathcal{E}_{n,m}$, $k \in \mathcal{D}_{r,g_L}$ and $x^{\{m\}\top} E_{g_L}(\alpha) x^{\{m\}} = 0$, $\forall \alpha \in \Lambda_r$, Theorem 5.6 combines (5.19) and (5.21) to present sufficient conditions to prove that the origin of the T-S system is exponentially stable using an HPPDLF. Aiming for a more compact notation, consider $\vartheta = (\alpha^\top, \theta^\top, \zeta^\top)^\top$.

Theorem 5.6

Given $K(\alpha) \in \mathbb{R}^{q \times n}$, if there exist $0 < V_g(\alpha) \in \mathbb{S}^{\sigma(n,m)}$ as in (5.16) and a scalar $\gamma > 0$ satisfying

$$\mathrm{He}\left(V_g(\alpha)\left(A^{\#}_{cl}(\alpha) + \gamma I_\sigma\right)\right) + \nabla_\alpha V_g(\alpha)\left(J(\theta) A_{cl}(\alpha) x(\zeta) \otimes I_\sigma\right) + E_{g_L}(\alpha) \leq 0, \tag{5.22}$$

for all $\vartheta \in \Lambda_r \times \Lambda_v \times \Lambda_\kappa$, then $v(x, \alpha)$ as in (5.15) is an HPPDLF of degree $2m$ on the states x and degree g on the MFs α for the T-S fuzzy system (3.6) that certifies that the origin of the system is locally exponentially stable. Moreover, $\xi = \gamma/m$ is a lower bound for the system decay rate.

Proof. The proof is omitted since it can be constructed from the development presented at the beginning of the section. □

Although the stability analysis problem of the open-loop system (or the closed-loop system when $K(\alpha)$ is given) is not the main focus of this chapter, it is important to highlight that Theorem 5.6 is a new result for the T-S fuzzy system literature. This theorem can be solved in terms of LMI relaxations, and the results tend to be much less conservative than the design conditions presented subsequently due to the linearization techniques required (which are sources of conservativeness) to handle $K(\alpha)$ as a design variable.

Equation (5.22) comprises a Bilinear Matrix Inequality (BMI) due to the presence of the feedback controller $K(\alpha)$ within the closed-loop matrices. An interesting aspect is that both closed-loop matrices $A^{\#}_{cl}(\alpha)$ and $A_{cl}(\alpha)$ appear in this inequality. The linearity of matrix $A^{\#}_{cl}(\alpha)$ with respect to the entries of $K(\alpha)$ is preserved, but the Kronecker operations (see equation (2.6)) spread the entries of $K(\alpha)$. This separation makes applying the classical change of variables, even in the case of state-feedback [7], unfeasible, necessitating a strategy different from the methodologies frequently employed in the literature.

The distinct approach adopted in this work is, essentially, to separate the matrices $A^{\#}_{cl}(\alpha)$ and $A_{cl}(\alpha)$ from the optimization variables of the problem. This separation enables the derivation of LMI conditions that can be solved iteratively, following the methodology introduced by Felipe and Oliveira [112] and described in Section 2.4. In this strategy, alternative equivalent conditions based on applying Finsler's lemma play a pivotal role.

From Finsler's lemma, equation (5.22) in Theorem 5.6 can be rewritten as

$$\mathscr{B}_1^{\perp\top}(\vartheta)\mathscr{Q}_1(\alpha)\mathscr{B}_1^{\perp}(\vartheta) \leq 0$$

with

$$\mathscr{Q}_1(\alpha) = \begin{bmatrix} E_{g_L}(\alpha) & V_g(\alpha) & 0.5\nabla_\alpha V_g(\alpha) \\ V_g(\alpha) & 0 & 0 \\ 0.5\nabla_\alpha V_g(\alpha)^\top & 0 & 0 \end{bmatrix}, \tag{5.23}$$

$$\mathscr{B}_1^{\perp}(\vartheta) = \begin{bmatrix} I_\sigma \\ \left(A^{\#}_{cl}(\alpha) + \gamma I_\sigma\right) \\ \left(J(\theta)A_{cl}(\alpha)x(\zeta) \otimes I_\sigma\right) \end{bmatrix}.$$

Considering that the sum of the time-derivative of the premise variables is null, i.e.,

$$\sum_{i=1}^{r} \dot{\alpha}_i = 0,$$

then, according to (5.20),

$$\sum_{i=1}^{r} \dot{\alpha}_i = \mathbb{1}_r^\top \dot{\alpha} = \mathbb{1}_r^\top J(\theta)A_{cl}(\alpha)x(\zeta) = 0.$$

Therefore, considering $\mathscr{B}_1(\vartheta) \in \mathbb{R}^{(r+1)\sigma\times(r+2)\sigma}$, the condition $\mathscr{B}_1(\vartheta)\mathscr{B}_1^{\perp}(\vartheta) = 0$ can be attended choosing

$$\mathscr{B}_1(\vartheta) = \begin{bmatrix} A^{\#}_{cl}(\alpha) + \gamma I_\sigma & -I_\sigma & \mathbb{1}_r^\top \otimes I_\sigma \\ J(\theta)A_{cl}(\alpha)x(\zeta) \otimes I_\sigma & 0 & -I_{r\sigma} \end{bmatrix}. \tag{5.24}$$

Thus, it is possible to write the Finsler's equivalent condition with slack variables

$$\exists \mathscr{X}(\vartheta) \,:\, \mathscr{Q}_1(\alpha) + \mathrm{He}\left(\mathscr{X}(\vartheta)\mathscr{B}_1(\vartheta)\right) \leq 0 \tag{5.25}$$

with $\mathscr{Q}_1(\alpha)$ and $\mathscr{B}_1(\vartheta)$ as in (5.23) and (5.24), respectively, and the slack variable

$$\mathscr{X}(\vartheta) = \begin{bmatrix} X_{11}(\vartheta) & X_{12}(\vartheta) \\ X_{21}(\vartheta) & X_{22}(\vartheta) \\ X_{31}(\vartheta) & X_{32}(\vartheta) \end{bmatrix} \in \mathbb{R}^{(r+2)\sigma\times(r+1)\sigma}. \tag{5.26}$$

Re-writing inequality (5.25) in the form $\mathscr{B}_2^{\perp\top}(\vartheta)\mathscr{Q}_2(\vartheta)\mathscr{B}_2^{\perp}(\vartheta) \leq 0$, statement (ii) of Lemma 2.1, with

$$\mathscr{Q}_2(\vartheta) = \begin{bmatrix} \mathscr{Q}_1(\alpha) & \star \\ \mathscr{B}_1(\vartheta) & 0 \end{bmatrix}, \tag{5.27}$$

where $\mathscr{Q}_1(\alpha)$ and $\mathscr{B}_1(\vartheta)$ are given respectively in (5.23) and (5.24), and

$$\mathscr{B}_2^{\perp}(\vartheta) = \begin{bmatrix} I_{(r+2)\sigma} \\ \mathscr{X}(\vartheta)^\top \end{bmatrix},$$

with $\mathscr{X}(\vartheta)$ as in (5.26), Finsler's lemma can be applied once more, resulting in the equivalent condition $\exists \mathscr{Y}(\vartheta) \, : \, \mathscr{Q}_2(\vartheta) + \mathrm{He}\left(\mathscr{Y}(\vartheta)\mathscr{B}_2(\vartheta)\right) \leq 0$ with $\mathscr{Q}_2(\vartheta)$ as in (5.27), the slack variables

$$\mathscr{Y}(\vartheta) = \begin{bmatrix} Y_{11}(\vartheta) & Y_{12}(\vartheta) \\ Y_{21}(\vartheta) & Y_{22}(\vartheta) \\ Y_{31}(\vartheta) & Y_{32}(\vartheta) \\ Y_{41}(\vartheta) & Y_{42}(\vartheta) \\ Y_{51}(\vartheta) & Y_{52}(\vartheta) \end{bmatrix} \in \mathbb{R}^{(2r+3)\sigma \times (r+1)\sigma},$$

and $\mathscr{B}_2(\vartheta)$, from the orthogonality conditions $\mathscr{B}_2(\vartheta)\mathscr{B}_2(\vartheta)^\perp = 0$, given by

$$\mathscr{B}_2(\vartheta) = \begin{bmatrix} \mathscr{X}(\vartheta)^\top & -I_{(r+1)\sigma} \end{bmatrix}. \tag{5.28}$$

The main result of this section is the next theorem, built upon the above algebraic manipulation.

Theorem 5.7

If there exist matrices $0 < V_g(\alpha) \in \mathbb{S}^{\sigma(n,m)}$ *as in* (5.16), $K(\alpha) \in \mathbb{R}^{q \times n}$, $\mathscr{Y}(\vartheta) \in \mathbb{R}^{(2r+3)\sigma \times (r+1)\sigma}$, $\mathscr{B}_2(\vartheta) \in \mathbb{R}^{(r+1)\sigma \times (2r+3)\sigma}$ *and a scalar* $\gamma > 0$ *such that the robust BMIs*

$$\mathscr{Q}_2(\vartheta) + \mathrm{He}\left(\mathscr{Y}(\vartheta)\mathscr{B}_2(\vartheta)\right) \leq 0, \tag{5.29}$$

hold for all $\vartheta \in \Lambda_r \times \Lambda_v \times \Lambda_\kappa$ *with* $\mathscr{Q}_2(\vartheta)$ *as in* (5.27), *then* $v(x, \alpha)$ *as in* (5.15) *is a HPPDLF of degree* $2m$ *on the states* x *and degree* g *on the MFs* α *for the T-S system* (3.6) *that certifies that the origin of the system is locally exponentially stable, and* $\xi = \gamma/m$ *is a lower bound for the decay rate.*

Proof. Immediate from the presented development. □

The main advantage of Theorem 5.7 over Theorem 5.6 is that both $A_{cl}(\alpha)$ and $A_{cl}^{\#}(\alpha)$ appear affinely in inequality (5.29). Consequently, the PDC gain $K(\alpha)$ also appears affinely and can be computed without resorting to a change of variables, which, as previously explained, is not possible in this case. Moreover, dealing with $K(\alpha)$ directly as a variable of the problem makes it straightforward to establish magnitude bounds on its entries. This is in stark contrast to approaches based on change of variables, where structural constraints come at the expense of constraining other variables in the problem (a source of conservatism). Consequently, the conditions presented in Theorem 5.7 can be tackled while considering the constraint formulation:

$$K_{\min} \leq K_i \leq K_{\max}, \quad i = 1, \ldots, r \tag{5.30}$$

where $K_{\min}$ and $K_{\max}$ are given matrices.

However, it is worth noting that the conditions established in Theorem 5.7 also take the form of BMIs due to the presence of the product $\mathscr{Y}(\vartheta)\mathscr{B}_2(\vartheta)$. Unlike the BMIs in (5.22), this bilinearity can be suitably addressed using the parameter γ as a relaxation factor in an iterative algorithm where local convergence can be assured. The first step in devising the iterative algorithm is to linearize inequality (5.29) by converting it into LMIs. This can be achieved by treating the matrix $\mathscr{B}_2(\vartheta)$ as constant (possibly ϑ-dependent). To ensure feasibility with a pre-specified matrix $\mathscr{B}_2(\vartheta)$, the parameter γ plays an important role. Besides serving as a lower bound for the decay rate of the system, it can also act as a relaxation parameter. This is because the conditions of Theorem 5.7 can always be made feasible by selecting a sufficiently large negative value for γ. At this stage, the affine dependency of the left-hand side of (5.29) with respect to $A^{\#}_{cl}(\alpha)$ proves to be useful since γ can be maximized as an objective function throughout the iterative process. If γ exceeds 0, it indicates that a stabilizing PDC gain has been found.

In order to compute the first feasible solution, Theorem 5.6 can be employed to derive a bound for the decay rate δ^*/m associated with the open-loop T-S system (i.e., with $K(\alpha) = 0$). Subsequently, equation (5.25) can be solved by considering the modified open-loop matrix $\tilde{A}(\alpha) = (A^{\#}(\alpha) + \delta^* I)$, which is guaranteed to be stable. This process yields the slack variables $\mathscr{X}(\vartheta)$ required to produce the initial value of $\mathscr{B}_2(\vartheta)$ as in (5.28). The theorem presented in the sequence provides a comprehensive description of this procedure.

Theorem 5.8

Consider the following maximization problem associated to the open-loop T-S system

$$\begin{aligned} \underset{\delta,\, E_{g_L}(\alpha),\, V_g(\alpha)>0}{\text{maximize}} \quad & \delta \\ \text{s.t.} \quad & \text{He}\left(V_g(\alpha)\left(A^{\#}(\alpha) + \delta I_\sigma\right)\right) \\ & + \nabla_\alpha V_g(\alpha)\left(J(\theta)A(\alpha)x(\zeta) \otimes I_\sigma\right) + E_{g_L}(\alpha) \leq 0. \end{aligned}$$

Let δ^* be the optimal solution. Then, substitute the extended closed-loop matrix $A^{\#}_{cl}(\alpha)$ in $\mathscr{B}_1(\vartheta)$, given in expression (5.24)*, by $\tilde{A}(\alpha) = A^{\#}(\alpha) + \delta^* I$, yielding*

$$\mathscr{B}_1(\vartheta) = \begin{bmatrix} \tilde{A}(\alpha) & -I_\sigma & \mathbb{1}_r^\top \otimes I_\sigma \\ J(\theta)A(\alpha)x(\zeta) \otimes I_\sigma & 0 & -I_{r\sigma} \end{bmatrix},$$

and solve (5.25) *for $\mathscr{X}(\vartheta)$ to obtain the feasible initial condition*

$$\bar{\mathscr{B}}_2(\vartheta) = \begin{bmatrix} \mathscr{X}(\vartheta)^\top & -I_{(r+1)\sigma} \end{bmatrix}.$$

Proof. Straightforward from the conditions of Finsler's lemma. □

Suppose that a stabilizing PDC gain is available, computed using any other technique from the literature. In such cases, the procedure suggested in Theorem 5.8 could be applied considering the closed-loop matrix $A_{cl}(\alpha)$, which is already stable, certainly leading to a positive δ^*. Consequently, Algorithm 4 can be utilized to obtain improved decay rates or expand the DOA (as explained in the sequence).

The primary feature of the approach employed in this study, facilitating an iterative problem-solving strategy, lies in the property that

$$\mathrm{He}\big(\mathscr{Y}(\vartheta)\bar{\mathscr{B}}_2(\vartheta)\big) = \mathrm{He}\big(\bar{\mathscr{B}}_2(\vartheta)^\top \mathscr{Y}(\vartheta)^\top\big).$$

As a result, any choice of $\mathscr{Y}(\vartheta)^\top$ can serve as a valid selection for $\bar{\mathscr{B}}_2(\vartheta)$ in the subsequent iteration. The algorithm demonstrates local convergence characterized by a γ that cannot decrease along the iterations, which can be verified by assuming that the conditions outlined in Theorem 5.7 are feasible at a given iteration it. It is then demonstrated that the same conditions remain feasible at the $(it+1)$-th iteration through specific choices, such as $\mathscr{Q}_{2,it+1}(\vartheta) = \mathscr{Q}_{2,it}(\vartheta)$ and $\bar{\mathscr{B}}_{2,it+1}(\vartheta) = \mathscr{Y}_{it}(\vartheta)^\top$, thereby ensuring $\gamma_{it+1} \geq \gamma_{it}$. As a result, the evolution of γ along the iterations is used as the stop criterion of the algorithm. If a decay rate ξ_a is given as a target to be achieved, the algorithm concludes when $\gamma \geq \xi_a m$ ($\xi_a = 0$ if solely aiming for stabilization).

Once the desired decay rate is achieved, the iterative process can be slightly modified to produce an estimate for the DOA of the closed-loop trajectories. This domain can be represented as the largest invariant set Ω confined within the polytope $\mathcal{X}$ [22]. Since in this section the Lyapunov function depends polynomially on the states, Ω is formally defined as

$$\Omega = \{x \in \mathbb{R}^n \,:\, x^{\{m\}\top} V_g(\alpha) x^{\{m\}} \leq 1\}. \tag{5.31}$$

As discussed in the previous section, the condition to assure that Ω is contained within the polytope $\mathcal{X}$ (domain of validity of the T-S system) can be easily established in the case $m = 1$ because in this case the inequality in (5.31) can be made affinely in x by a

Schur complement, and thus tested on the vertices h^k defined in (3.5). However, for $m > 1$, a different strategy is necessary. Assuming that the linear constraints in (3.4) are symmetric, that is, the valid region can always be defined through $|a_k^\top x| \leq 1$, it is possible to equivalently redefine the set $\mathcal{X}$ as

$$\mathcal{X} = \{x \in \mathbb{R}^n \ : \ x^{\{m\}\top} M_k x^{\{m\}} \leq 1,\ k = 1, \dots, \mu\},$$

where

$$x^{\{m\}\top} M_k x^{\{m\}} = (a_k^\top x)^{2m},$$
$$M_k = G_m^\top Q_k^{[m]} G_m,\ Q_k = a_k a_k^\top,$$

and G_m is defined in (2.5). Therefore, with this new representation, the constraint $\Omega \subset \mathcal{X}$ can be accomplished through

$$V_g(\alpha) \geq M_k, \quad k = 1, \dots, \mu. \tag{5.32}$$

It is important to emphasize that the restriction given by (5.32) must be considered when obtaining the initial condition for the iterative procedure, i.e., when solving the conditions presented in Theorem 5.8.

The following optimization problem can be used as a heuristic for the enlargement of Ω [153]

$$\min_{K(\alpha), V_g(\alpha), W} \quad \mathrm{tr}(W) \quad \text{s.t. (5.29), (5.30), (5.32)},\ V_g(\alpha) \leq W,$$

Since two performance criteria are pursued, i.e., the optimization of the decay rate and the enlargement of Ω, an algorithm with two phases, is proposed, as detailed in the sequence.

As input parameters, Algorithm 4 has the data associated with the T-S model A_i, B_i, C_i, $i \in \mathbb{N}_r$, J_j, $j \in \mathbb{N}_v$, x_k, $k \in \mathbb{N}_\kappa$, the degrees m and g related to the Lyapunov function, the constraints on the PDC gains (optional) $K_{\min}$ and $K_{\max}$, the target decay rate ξ_a, the maximum number of iterations $it_{\max}$, and tolerance ε to end up the procedure if the progress (in terms of the values of γ_k or $\mathrm{tr}(W)$, in phase 1 or 2, respectively) between two consecutive iterations is not significant. In phase 1, the procedure tries to achieve the desired decay rate, and in the case of success, the procedure goes to phase 2, where the enlargement of the estimate of the region of attraction is performed.

The flexibility of the proposed technique regarding the choices of m and g raises a question about the conservativeness of the results when the values of m and g grow. The next theorem provides an important outcome addressing this issue.

Algorithm 4 PDC Design

```
Input: g, m, A_i, B_i, C_i, J_j, x_k, it_max, K_min, K_max, ξ_a, ε;
Solve the problem in Theorem 5.8 to obtain B̄_{2,0}(ϑ);
it ← 0; phase ← 1;
While it < it_max
    it ← it + 1;
    If phase = 1 Then
        maximize γ subject to (5.29), (5.30) and (5.32);
        If γ_it ≥ ξ_a m Then
            phase ← 2, γ ← γ_it;
        Else If γ_it − γ_{it−1} ≤ ε Then
            break;
        End If
    Else If phase = 2 Then
        minimize tr(W) subject to (5.29), (5.30), (5.32) and V_g(α) ≤ W;
        If |tr(W)_it − tr(W)_{it−1}| ≤ ε Then
            break;
        End If
    End If
    B̄_{2,it}(ϑ) ← Y_it(ϑ)^⊤;
End While
If phase = 1 Then
    Return γ_max = γ_it, K(α), V_g(α);
Else If phase = 2 Then
    Return σ* = σ_it, γ_max = γ, K(α), V_g(α);
End If
```

Theorem 5.9

If the conditions of Theorem 5.7 are feasible for some integer $m \geq 1$ and $g \geq 1$, then they are also feasible for ρm and ρg, for any given integer $\rho > 1$.

Proof. Considering the Lyapunov function

$$v(x,\alpha) = x^{\{m\}\top} V_g(\alpha) x^{\{m\}},$$

and that the exponential stability conditions ($v(x,\alpha) > 0$ and $\dot{v}(x,\alpha) \leq -2\gamma v(x,\alpha)$ with $\gamma = \xi m > 0$) hold, it is possible to write another Lyapunov function

$$\tilde{v}(x,\alpha) = x^{\{\rho m\}\top} \tilde{V}_{\rho g}(\alpha) x^{\{\rho m\}}, \tag{5.33}$$

as $\tilde{v}(x,\alpha) = v(x,\alpha)^{\rho}$. Using Kronecker powers, one has

$$\begin{aligned} \tilde{v}(x,\alpha) &= \left(\left(x^{\{m\}}\right)^{[\rho]}\right)^{\top} V_g(\alpha)^{[\rho]} \left(\left(x^{\{m\}}\right)^{[\rho]}\right) \\ &= x^{\{\rho m\}\top} T^{\top} V_g(\alpha)^{[\rho]} T x^{\{\rho m\}}, \end{aligned} \tag{5.34}$$

where

$$\left(x^{\{m\}}\right)^{[\rho]} = T x^{\{\rho m\}} \tag{5.35}$$

and since the polynomials in (5.33) and (5.34) are the same, one has

$$\tilde{V}_{\rho g}(\alpha) = T^\top V_g(\alpha)^{[\rho]} T.$$

Noticing that T has full-column rank, considering that the condition $v(x,\alpha) > 0$ is valid, then $V_g(\alpha) > 0$ and, according to the previous expression, $\tilde{V}_{\rho g}(\alpha) > 0$.

Computing the time-derivative,

$$\begin{aligned}\dot{\tilde{v}}(x,\alpha) &= \frac{dv(x,\alpha)^\rho}{dv(x,\alpha)}\dot{v}(x,\alpha)\\ &= \rho v(x,\alpha)^{\rho-1} \otimes \dot{v}(x,\alpha)\end{aligned}$$

and writing $v(x,\alpha)^{\rho-1}$ in terms of Kronecker powers and $\dot{v}(x,\alpha)$ as in (5.18) yields

$$\begin{aligned}\dot{\tilde{v}}(x,\alpha) &= \left(\rho\left(\left(x^{\{m\}}\right)^{[\rho-1]}\right)^\top V_g(\alpha)^{[\rho-1]}\left(\left(x^{\{m\}}\right)^{[\rho-1]}\right)\right)\\ &\quad \otimes\left(x^{\{m\}\top}\left(A_{cl}^{\#\top}V_g(\alpha) + V_g(\alpha)A_{cl}^{\#}\right.\right.\\ &\quad \left.\left. +\ \dot{V}_g(\alpha)\right)x^{\{m\}}\right)\\ &= \left(\left(x^{\{m\}}\right)^{[\rho]}\right)^\top \left(\rho V_g(\alpha)^{[\rho-1]} \otimes \left(A_{cl}^{\#\top}V_g(\alpha)\right.\right.\\ &\quad \left.\left. +V_g(\alpha)A_{cl}^{\#} + \dot{V}_g(\alpha)\right)\right)\left(\left(x^{\{m\}}\right)^{[\rho]}\right)\\ &= x^{\{\rho m\}\top}T^\top\left(\rho V_g(\alpha)^{[\rho-1]}\otimes\left(A_{cl}^{\#\top}V_g(\alpha)\right.\right.\\ &\quad \left.\left. +V_g(\alpha)A_{cl}^{\#} + \dot{V}_g(\alpha)\right)\right)Tx^{\{\rho m\}}. \end{aligned} \tag{5.36}$$

Taking into account Equations (5.34) and (5.36), one has

$$\begin{aligned}\dot{\tilde{v}}(x,\alpha) + 2\tilde{\gamma}\tilde{v}(x,\alpha) = x^{\{\rho m\}\top}T^\top &\left(\left(\rho V_g(\alpha)^{[\rho-1]}\otimes\right.\right.\\ &\left.\left(A_{cl}^{\#\top}V_g(\alpha) + V_g(\alpha)A_{cl}^{\#} + \dot{V}_g(\alpha)\right)\right)\\ &\left. +2\tilde{\gamma}V_g(\alpha)^{[\rho]}\right)Tx^{\{\rho m\}}.\end{aligned}$$

Since the condition $\dot{v}(x,\alpha) \le -2\gamma v(x,\alpha)$ is valid with $\gamma = \xi m > 0$, then

$$\rho V_g(\alpha)^{[\rho-1]} \otimes \left(A_{cl}^{\#\top}V_g(\alpha) + V_g(\alpha)A_{cl}^{\#} + \dot{V}_g(\alpha)\right) \le -2\gamma V_g(\alpha)$$

and considering that the decay rate is constant (i.e., $\tilde{\gamma} = \rho\gamma$), one has

$$\begin{aligned}\dot{\tilde{v}}(x,\alpha) + 2\tilde{\gamma}\tilde{v}(x,\alpha) \le x^{\{\rho m\}\top}T^\top &\left(\left(\rho V_g(\alpha)^{[\rho-1]}\right.\right.\\ &\left.\otimes\left(-2\gamma V_g(\alpha)\right)\right) + 2\rho\gamma V_g(\alpha)^{[\rho]}\Big)Tx^{\{\rho m\}}\end{aligned}$$

$$\leq x^{\{\rho m\}\top} T^\top \left(-2\rho\gamma V_g(\alpha)^{[\rho]} + 2\rho\gamma V_g(\alpha)^{[\rho]}\right) T x^{\{\rho m\}} \leq 0,$$

i.e., $\dot{\tilde{v}}(x,\alpha) \leq -2\tilde{\gamma}\tilde{v}(x,\alpha)$.

Therefore, if the exponential stability conditions $v(x,\alpha) > 0$ and $\dot{v}(x,\alpha) \leq -2\gamma v(x,\alpha)$ with $\gamma = \xi m > 0$ hold, then $\tilde{v}(x,\alpha) > 0$ and $\dot{\tilde{v}}(x,\alpha) \leq -2\tilde{\gamma}\tilde{v}(x,\alpha)$, with $\tilde{\gamma} = \rho\xi m > 0$, are also feasible for $\rho \geq 1$.

□

A final issue worth highlighting is the effect of increasing m in the enlargement of Ω. As presented, there is no guarantee that the volume produced by a certain degree ρm will be larger than the one obtained with degree m since there is no apparent connection between the trace of the matrix W and the volume of Ω. In this context, the representation given in (5.31) can be useful again. Suppose that Ω in (5.31) is obtained for certain degrees m and g. Now consider the estimates of the domain using the degrees ρg and ρm, for an integer $\rho > 1$, are given by

$$\Omega_\rho = \left\{x \in \mathbb{R}^n \,:\, x^{\{\rho m\}\top} V_{\rho g}(\alpha) x^{\{\rho m\}} \leq 1\right\}.$$

To assure that $\Omega \subset \Omega_\rho$, it is enough to produce an equivalent representation of $x^{\{m\}\top} V_g(\alpha) x^{\{m\}} \leq 1$ using a left-hand side with degree ρm. This can be straightforwardly done by applying the constraint

$$T^\top V_g(\alpha)^{[\rho]} T \geq V_{\rho g}(\alpha), \tag{5.37}$$

with T satisfying (5.35). Thus, whenever a Lyapunov function of degrees m and g is available, with a certain associated volume for Ω, it is possible to search for a new Lyapunov function of degrees ρm and ρg with the guarantee that the new volume cannot be smaller, simply taking into account the constraint (5.37) in the parameter-dependent LMI conditions that are considered in Algorithm 4.

5.2.1 Numerical Examples

Algorithm 4 and all associated conditions are presented in terms of inequalities depending polynomially (also known as fuzzy summations) on the parameters $\vartheta = (\alpha, \theta, \zeta)$. In this form, no numerically tractable procedure is directly applicable. However, relaxations can be applied (exploring the non-negativity of the vector of parameters ϑ), and a finite set of LMIs can be used to test the inequalities. Moreover, this procedure can nowadays be performed with software support, for instance, using the Robust LMI Parser (ROLMIP) [115], which makes the implementation of the parameter-dependent LMIs user-friendly. In the implementation, it was chosen to keep $E_{g_L}(\alpha)$

with fixed degree $g_L = 1$, and the slack variables $\mathscr{Y}(\vartheta)$ depending only on the MFs α, i.e., $\mathscr{Y}(\alpha)$, also with degree one. In this section, the PDC controller $K(\alpha)$ depends affinely on α, but higher polynomial degrees could be straightforwardly considered, especially using ROLMIP. The programming of all codes was built in Matlab 9.4 (R2018a), and the SDP solver Mosek 9.3.18 [146] was employed. The PC used is equipped with the following setup: Core(TM) i7-1165G7 @ 2.80GHz 11th Gen Intel(R) with 16 GB RAM, Windows 11 Home Single Language. The numerical complexity of the design techniques presented used in the experiments are evaluated in terms of the numbers V_s of scalar variables and L_{LMIs} of LMI rows.

Example 1

Consider the nonlinear system from Gomes et al. [97],

$$\dot{x} = \begin{bmatrix} -\frac{2+a}{2} + \frac{2-a}{2}\sin(x_1) & -4 \\ \frac{19}{2} - \frac{21}{2}\sin(x_1) & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \frac{10+b}{2} + \frac{10-b}{2}\sin(x_1) \end{bmatrix} u,$$

where $-4 \leq a \leq 4$ and $-2 \leq b \leq 2$. In this example, the aim is to evaluate the existence of state-feedback ($C(\alpha) = I$) PDC controllers considering different values of (a, b) lying in the defined intervals and comparing the proposed technique with the methods in Gomes et al. [97] and Lee and Kim [93].

Considering the premise variable $z = x_1$ and using the section nonlinearity approach over the nonlinear term $\sin(x_1)$, the dynamics can be precisely represented inside the compact set $\mathcal{X} = \{x \in \mathbb{R}^2 : |x_i| \leq \pi/2, i = 1, 2\}$ as the T-S system (3.2) with

$$A_1 = \begin{bmatrix} -a & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}, \quad -4 \leq a \leq 4,$$

$$B_1 = \begin{bmatrix} 1 \\ 10 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ b \end{bmatrix}, \quad -2 \leq b \leq 2$$

and the MFs

$$\alpha_1 = \frac{1 + \sin(x_1)}{2}, \quad \alpha_2 = \frac{1 - \sin(x_1)}{2}.$$

The Jacobian of the MFs yields

$$J(\theta) = \Delta_x \alpha(z) = \begin{bmatrix} \frac{\partial \alpha_1}{\partial x_1} & \frac{\partial \alpha_1}{\partial x_2} \\ \frac{\partial \alpha_2}{\partial x_1} & \frac{\partial \alpha_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \cos(x_1) & 0 \\ -\cos(x_1) & 0 \end{bmatrix}$$

and the sector nonlinearity approach applied on the nonlinear term $\cos(x_1)$ leads to the representation $J(\theta) = \theta_1 J_1 + \theta_2 J_2$, $\theta \in \Lambda_2$ with

$$J_1 = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In Algorithm 4, no restriction on the magnitude of the gains is imposed, and the target decay rate is established as $\xi_a = 0$, i.e., only assuring stability. The maximum number of iterations is set at $it_{max} = 25$ and the tolerance $\varepsilon = 10^{-4}$. Tests are carried out for the Lyapunov function of degrees $m = 1$ and $g = 1$, where the initial condition $\bar{\mathscr{C}}_1(\vartheta)$ is obtained through Theorem 5.8. The proposed technique is compared with LMIs conditions from Gomes et al. [97] with Lyapunov matrix of degree $g = 1$ and $\xi = 5$, and Lee and Kim [93] with degree $q = 2$ and $\phi = 10000$ (the same value used in Gomes et al. [97]). The feasible pairs depicted in Figure 5.2 clearly show the less conservatism of the iterative algorithm proposed in this section in stabilizing the system.

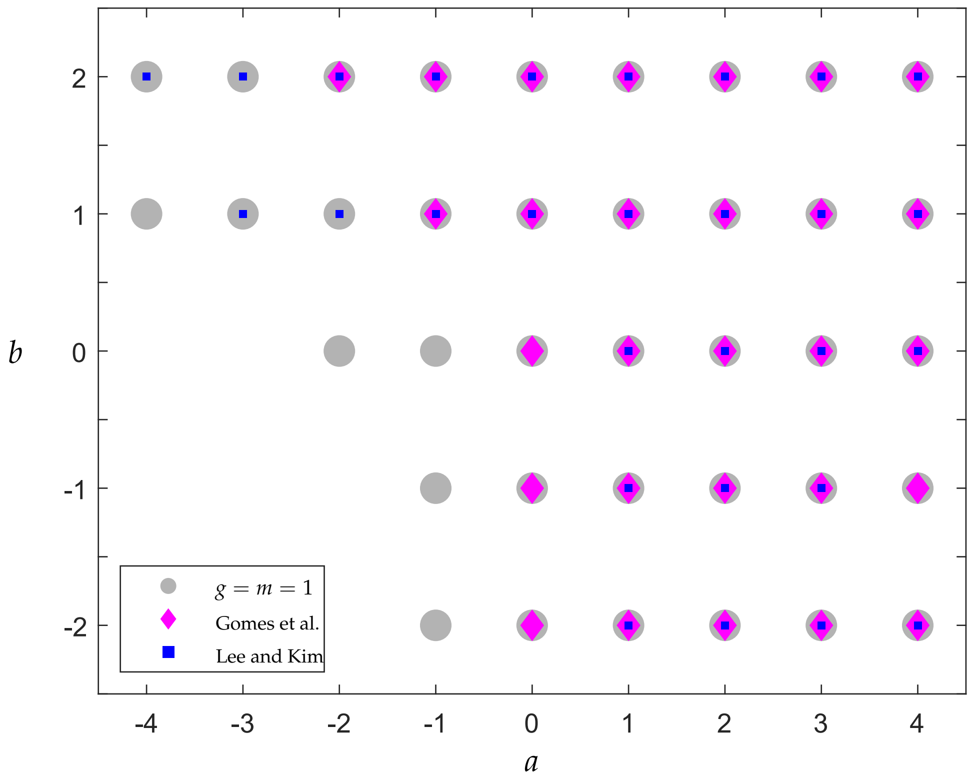


Figure 5.2: Feasible pairs $(a, b) \in [-4, 4] \times [-2, 2]$ for Example 1 estimated using Algorithm 4 (circles), Gomes et al. [97] (diamonds), and Lee and Kim [93] (squares).

Example 2

This example shows the effectiveness of the proposed technique dealing with output-feedback, particularly with the purpose of providing a DOA with the largest possible area inside the validity of the model. Consider the T-S system (3.2) adapted from Pan et al. [16] valid in

$$\mathcal{X} = \left\{ x \in \mathbb{R}^2 : |x_i| \leq 1.5, i = 1, 2 \right\}$$

with

$$A_1 = \begin{bmatrix} -0.1125 & -0.02 \\ 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.1125 & -1.527 \\ 1 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and the MFs defined as

$$\alpha_1 = \frac{2.25 - x_2^2}{2.25}, \; \alpha_2 = \frac{x_2^2}{2.25}$$

with the corresponding Jacobian

$$J(\theta) = \nabla_x \alpha(z) = \begin{bmatrix} \frac{\partial \alpha_1}{\partial x_1} & \frac{\partial \alpha_1}{\partial x_2} \\ \frac{\partial \alpha_2}{\partial x_1} & \frac{\partial \alpha_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-2x_2}{2.25} \\ 0 & \frac{2x_2}{2.25} \end{bmatrix}.$$

The sector nonlinearity approach applied on the nonlinear term x_2 leads to the representation $J(\theta) = \theta_1 J_1 + \theta_2 J_2$, $\theta \in \Lambda_2$ with

$$J_1 = \frac{1}{2.25} \begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix}, \quad J_2 = \frac{1}{2.25} \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}.$$

The estimation of the DOA obtained by Algorithm 4 for state- and output-feedback with different degrees g and m for the Lyapunov function, and a target decay rate $\xi_a = 0$ (i.e., only stabilization), are depicted in figures 5.3 and 5.4, respectively. For the Lyapunov function of degrees $m = 1$ and $g = 1$, the initial condition $\bar{\mathscr{C}}_1(\vartheta)$ is obtained through Theorem 5.8. For the other degrees, Theorem 5.9 is used to find an initial condition starting from the results found for $m = 1$ and $g = 1$ — therefore, it is worthy to emphasize that the increase of the area is expected in relation to the area found with those degrees only. Additionally, to further illustrate the advantages of the proposed technique, comparisons with state-feedback conditions (clearly more general to provide larger DOAs) from the literature are included for comparison.

The corresponding areas are provided in Table 5.2. The numbers V_s and L_{LMIs} and the computational time (in seconds) are also presented, offering an estimate of the numerical complexity of the conditions. The outcomes show that, in exchange for an increase of the computational load, the iterative algorithm consistently yields superior results in terms of DOA estimates when compared to the other state-feedback techniques from the literature, even using output-feedback with degrees $g = m = 2$.

5.3 Conclusion

This chapter introduces an LMI-based algorithm for local output-feedback PDC control of continuous-time T-S fuzzy systems using two different types of Lyapunov

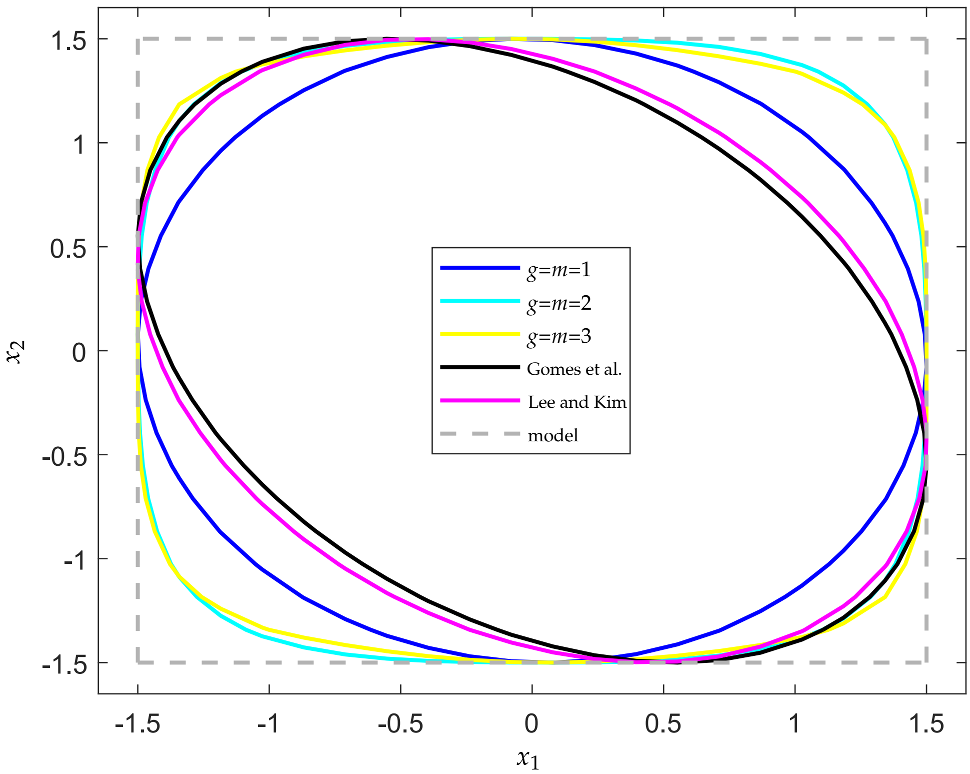


Figure 5.3: Estimates of the DOA for Example 2 considering state-feedback control law obtained with Algorithm 4 for different degrees g and m, and with the conditions of Gomes et al. [97] (black) and Lee and Kim [93] (magenta). The gray box (dashed) indicates the domain of validity of the model.

Table 5.2: Areas of the estimated DOA obtained by the conditions of Lee and Kim [93] and Gomes et al. [97] and Algorithm 4 and the corresponding number of scalar variables (V_s), LMI rows (L_{LMIs}) and computational time (in seconds) for Example 2.

	Condition	area ($u.a.$)	V_s	L_{LMIs}	time (s)
State-feedback	Gomes et al. [97]$_{g=2,\xi=10}$	6.5615	41	242	1.00
	Lee and Kim [93]$_{q=3,\phi=10000}$	6.7247	129	180	1.06
	Algorithm 4$_{g=m=1}$	7.0387	181	356	12.41
	Algorithm 4$_{g=m=2}$	8.1644	409	558	25.60
	Algorithm 4$_{g=m=3}$	8.1281	733	992	55.56
Output-feedback	Algorithm 4$_{g=m=1}$	4.8443	179	356	19.15
	Algorithm 4$_{g=m=2}$	7.2307	407	558	37.17
	Algorithm 4$_{g=m=3}$	6.1460	731	992	83.21

functions: Fuzzy Lyapunov Functions (FLFs) and Homogeneous Polynomial Parameter-Dependent Lyapunov Functions (HPPDLFs). Both methods eliminate the need for bounds on the time-derivative of the MFs. The main advantage of HPPDLFs is their ability to extend the results of FLFs and the existing literature, which predominantly rely on quadratic-on-the-states Lyapunov functions.

The proposed algorithms first guarantee a specified decay rate, then search for the largest estimate of the DOA within the model's valid space. The algorithms have

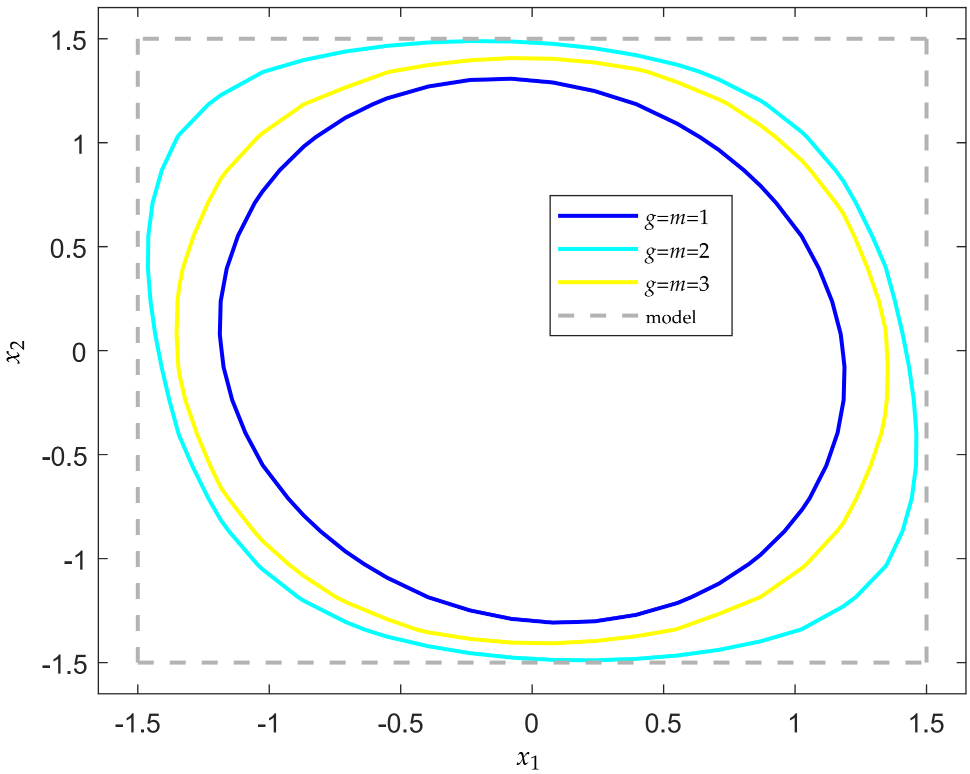


Figure 5.4: Estimates of the DOA for Example 2 considering output-feedback control law obtained with Algorithm 4 for different degrees g and m. The gray box (dashed) indicates the domain of validity of the model.

local convergence and optimize the decay rate of closed-loop trajectories, thereby providing extended DOA estimates. Numerical examples demonstrate the effectiveness of the proposed technique in stabilizing T-S fuzzy systems and improving DOA estimates. Moreover, the results indicate that the proposed method can provide less conservative outcomes than existing state-feedback control approaches. Finally, although not explored in this chapter, designing PDC gains that also depend on θ and ζ presents a promising strategy for further improving closed-loop performance.

Chapter 6

Regional Stabilization of Discrete-time T-S Systems

Consider the discrete-time T-S system described as

$$x(k+1) = A_d(\alpha)x(k),\ \forall x(k) \in \mathcal{L}, \tag{6.1}$$

where $A_d(\alpha) = A(\alpha)$ — i.e., the open-loop system — when considering the stability analysis and $A_d(\alpha) = A_{cl}(\alpha)$, according to equation (3.13), for the feedback stabilization problem, according to the definitions presented in Chapter 3. The goal of regional stability analysis is to determine whether the zero equilibrium point of the open-loop T-S system is locally asymptotically stable. When considering the feedback stabilization of this system, the objective is to ensure the asymptotic stability of the origin. Additionally, in both scenarios, an inner estimate of the Domain of Attraction (DOA) is computed. This involves identifying a set that is entirely contained within the actual DOA, providing a conservative but reliable estimate of the region where the system will converge to the equilibrium point. To achieve this, one can consider a Fuzzy Lyapunov Function (FLF) given by

$$v(x) = x^\top V(\alpha)x \tag{6.2}$$

where $V(\alpha)$ is a parameter-dependent matrix defined as

$$V(\alpha) = \sum_{i=1}^{r} \alpha_i V_i,\ V_i \in \mathbb{S}^n. \tag{6.3}$$

Therefore, the candidate Lyapunov function $v(x)$ incorporates the parameter dependence through matrix $V(\alpha)$, which is a linear combination of V_i matrices weighted by the values of the MFs α_i, $\alpha \in \Lambda_r$.

It is important to highlight that the T-S system and the MFs are defined only for $x \in \mathcal{L}$. Therefore, to ensure that x^+ and α^+ can be consistently defined, it is necessary to define the set [110]

$$\mathcal{R} = \left\{\xi \in \mathcal{L} : A_d(\alpha(\omega))\xi \in \mathcal{L},\ \omega = \mathcal{T}\xi\right\}, \tag{6.4}$$

guaranteeing that, if $x \in \mathcal{R}$, then $x^+ \in \mathcal{L}$ (i.e., the subsequent states remain in $\mathcal{L}$) and α^+ is well defined (more details later).

By considering the Lyapunov function (6.2), the asymptotic stability of the origin of the discrete-time T-S system can be evaluated using the following theorem.

Theorem 6.1

Let $v : \mathbb{R}^n \to \mathbb{R}$, $v(x) = x^\top V(\alpha)x$, $V(\alpha) \in \mathbb{S}^n$ as in (6.3), *be a function satisfying*

$$\begin{cases} v(x) > 0, \ \forall x \in \mathcal{L}, \ x \neq 0, \ \forall \alpha \in \Lambda_r \\ \Delta v(x) = v(x^+) - v(x) < 0, \ \forall x \in \mathcal{R}, \ x \neq 0, \ \forall \alpha \in \Lambda_r, \end{cases}$$

along the trajectories of the discrete-time T-S system. Then $v(x)$ is a FLF for the T-S system that certifies that the origin of the system is asymptotically stable.

Proof. See, for instance, [3]. □

Besides the asymptotic stability of the origin of the T-S system, the regional stability conditions of this thesis also aid in determining the DOA — in other words, the maximum distance that a trajectory can reach while still converging to the origin as $k \to \infty$. Despite the complexity of analytically determining the DOA, Lyapunov functions can be utilized to estimate the DOA by employing sets contained within it [3], as presented in Section 2.2.

Consider

$$\Omega(v, c) = \{x \in \mathbb{R}^n \ : \ v(x) \leq c\}$$

the c-level invariant set of $v(x)$, where c is a positive real number. Based on LaSalle's invariance theorem [3], it becomes evident that if a FLF satisfying the conditions of Theorem 6.1 exists and if $\Omega(v, c) \subset \mathcal{R}$ is bounded, then every trajectory originating from $\Omega(v, c)$ remains within $\Omega(v, c)$ and converges to the origin as $k \to \infty$. Consequently, $\Omega(v, c)$ serves as a positively invariant set and an estimate of the DOA [3, 22]. Overall, the use of the one-sublevel set $\Omega(v, 1)$ for estimating the DOA via Lyapunov functions strikes a balance between conservatism, practicality, and guarantee of stability, being widely adopted in control theory. Moreover, using the one-sublevel set allows for a uniform analysis across different Lyapunov functions and systems, providing a standardized approach for estimating the DOA, facilitating comparisons between different methods and systems.

As thoroughly discussed in Lee and Joo [110], the invariant subset $\Omega(v, 1)$ of the DOA satisfies the following conditions:

1. For all $x \in \partial\mathcal{L}$, the boundary of $\mathcal{L}$, $v(x) > 1$;

2. The set $\Omega(v, 1)$ must be a subset of $\mathcal{R}$;

3. The set $\Omega(v, 1)$ is contained within $\{\xi \in \mathcal{R} : \Delta v(\xi) < 0\}$, where $\Delta v(x) = v(x^+) - v(x)$.

Condition 1) assures that $\Omega(v, 1)$, which is the level-one subset of $v(x)$, is strictly contained within $\mathcal{L}$. Condition 3), based on Theorem 6.1 for asymptotic stability, ensures that $\Omega(v, 1)$ is an invariant subset of the DOA. Therefore, as stated in condition 2), trajectories originating within $\Omega(v, 1)$ remain inside the regions defined by $\mathcal{L}$, $\mathcal{R}$, and $\Omega(v, 1)$.

6.1 Main Results

To obtain stability analysis conditions that allow the maximization of the estimated DOA using a polytopic representation for the MFs, the proposed strategy starts employing the Kronecker product between α and the identity matrix I_n to form a block matrix, writing $V(\alpha)$ in (6.3) as

$$V(\alpha) = \mathscr{V}(\alpha \otimes I_n), \tag{6.5}$$

where $\mathscr{V} = \begin{bmatrix} V_1 & \dots & V_r \end{bmatrix} \in \mathbb{R}^{n \times rn}$ is a matrix consisting of the concatenated vertices of $V(\alpha)$.

It is also important to define the candidate Lyapunov functions at instant $k + 1$. Using (6.2) and considering the closed-loop T-S system (6.1), $v(x(k+1)) = v(x^+)$ can be expressed as

$$v(x^+) = x^\top A_d(\alpha)^\top V(\alpha(z(k+1)))A_d(\alpha)x$$

and, using equation (6.5), one has

$$V(\alpha(z(k+1))) = V(\alpha^+) = \mathscr{V}(\alpha^+ \otimes I_n). \tag{6.6}$$

Now, a central point of this study involves the polytopic representation of the MFs. To achieve this, one must consider that the premise variables are linear combinations of the states, as presented in Chapter 3 and depicted in equation (3.8), enabling the expression of the MFs as

$$\alpha(z(k)) = E(z(k))\mathcal{T}x + e(z(k)), \tag{6.7}$$

where $E = E(z(k)) \in \mathbb{R}^{r \times p}$ comprises terms of the MFs that are a linear combination of the state variables, and $e = e(z(k)) \in \mathbb{R}^{r \times 1}$ may encompass constant or nonlinear terms of the MFs. Consequently, the MFs at time $k+1$ can be expressed in terms of $E^+ = E(z(k+1))$ and $e^+ = e(z(k+1))$, the transformation matrix $\mathcal{T}$, and the preceding state x (as per the dynamics of the T-S system (6.1)), that is,

$$\alpha^+ = E^+\mathcal{T}x^+ + e^+ = E^+\mathcal{T}A_d(\alpha)x + e^+. \tag{6.8}$$

This particular formulation holds paramount significance for the outcomes of this thesis. It facilitates the polytopic representation of the MFs at the instant $k+1$ through a linear combination of the state variables using the transformation matrix $\mathcal{T}$. Moreover, it allows for the application of the sector nonlinearity approach to either E^+ or e^+, if deemed necessary.

In this expression the state x can be replaced by a polytopic representation, valid within $\mathcal{R}$, enabling the mathematical definition of the MFs at time $k+1$. Therefore, using (6.8) on (6.6), one gets

$$V(\alpha^+) = \mathscr{V}\left((E^+\mathcal{T}A_d(\alpha)x(\zeta) + e^+) \otimes I_n\right). \tag{6.9}$$

The examples presented in the sequence illustrate the proposed representation.

Example 1

Consider the nonlinear discrete-time system

$$x^+ = 0.5\begin{bmatrix} -1 & 1.5x_1(k) \\ 1 & -0.5 \end{bmatrix} x, \tag{6.10}$$

where $-z_{\text{max}} \leq x_1(k) \leq z_{\text{max}}$, $\forall k \geq 0$. Considering the premise variable $z = x_1$, then $\mathcal{T} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, according to (3.8). Applying the sector nonlinearity approach on the nonlinear term $1.5z$, the nonlinear system (6.10) can be exactly represented by the discrete-time T-S fuzzy system with $A(\alpha) = \alpha_1 A_1 + \alpha_2 A_2$,

$$A_1 = 0.5\begin{bmatrix} -1 & 1.5z_{\text{max}} \\ 1 & -0.5 \end{bmatrix}, \; A_2 = 0.5\begin{bmatrix} -1 & -1.5z_{\text{max}} \\ 1 & -0.5 \end{bmatrix},$$

defined in $\mathcal{P} = \left\{\xi \in \mathbb{R}^2 : \mathcal{T}\xi \in [-z_{\text{max}}, z_{\text{max}}]\right\}$ according to (3.10), with the MFs given by $\alpha_1 = (z + z_{\text{max}})/(2z_{\text{max}})$ and $\alpha_2 = (-z + z_{\text{max}})/(2z_{\text{max}})$. As previously stated, this work assumes that all the state variables are limited. Therefore, for x_2, the limits $-5z_{\text{max}} \leq x_2(k) \leq 5z_{\text{max}}$ are imposed. Thus, from (3.11), the domain $\mathcal{L}$ can be defined with

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad z_{\text{max},1} = z_{\text{max}}, \quad z_{\text{max},2} = 5z_{\text{max}}.$$

Writing the vector of MFs according to the representation (6.7), one gets

$$\begin{aligned}\alpha =& \frac{0.5}{z_{\max}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} z + 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ =& \frac{0.5}{z_{\max}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} x(\zeta) + 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = E\mathcal{T}x(\zeta) + e\end{aligned}$$

where

$$E = \frac{0.5}{z_{\max}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e = 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{6.11}$$

At instant $k+1$, according to (6.9), the Lyapunov function can be obtained considering E^+ and e^+, along with the region $\mathcal{R}$ to define $x(\zeta)$, where the MFs are mathematically defined. From (6.11), one gets

$$E^+ = \frac{0.5}{z_{\max}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^+ = 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To define the set $\mathcal{R}$, condition (6.4) can be used, which requires that $A(\alpha(\omega))\xi \in \mathcal{L}$, i.e., $\mathcal{T}A(\alpha(\omega))\xi \in [-z_{\max}, z_{\max}]$. Considering $\omega = \mathcal{T}\xi = \xi_1$, one can obtain $A(\alpha(\omega))$ based on the previously derived T-S system

$$A(\alpha(\omega)) = 0.5 \begin{bmatrix} -1 & 1.5\xi_1 \\ 1 & -0.5 \end{bmatrix}.$$

Therefore,

$$\begin{aligned}\mathcal{T}A(\alpha(\omega))\xi =& \begin{bmatrix} 1 & 0 \end{bmatrix} 0.5 \begin{bmatrix} -1 & 1.5\xi_1 \\ 1 & -0.5 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \\ =& 0.5(-\xi_1 + 1.5\xi_1\xi_2) \in [-z_{\max}, z_{\max}],\end{aligned}$$

and the set $\mathcal{R}$ can be written as

$$\mathcal{R} = \left\{ \begin{matrix} \xi \in \mathbb{R}^2 \, : \, \xi_1 \in [-z_{\max}, z_{\max}], \\ 0.5\xi_1 (1.5\xi_2 - 1) \in [-z_{\max}, z_{\max}] \end{matrix} \right\} \tag{6.12}$$

Even though the set defined in equation (6.12) is non-convex, it is possible to use the convex hull that contains this set to polytopically represent $x(\zeta)$. Therefore, an outer approximation of the set $\mathcal{R}$ can be obtained, ensuring that $x(\zeta)$ remains within $\mathrm{co}(\mathcal{R})$. Thus, convex optimization techniques can be used to provide a numerically tractable analysis of the dynamics of the system, as illustrated in Figure 6.1. The figure displays the sets $\mathcal{L}$ and $\mathcal{R}$, represented by the gray dot-dashed and blue solid lines, respectively. The blue dashed line represents the convex hull containing $\mathcal{R}$. The lines in the figure correspond to the mathematical definition of these sets. The colored areas, namely red and blue, indicate unstable and stable trajectories, respectively, while the green area contains trajectories that escape the region of validity $\mathcal{L}$ before converging to the origin. These areas, along with the arrows indicating the state at the next instant, are obtained through numerical simulations on the T-S system.

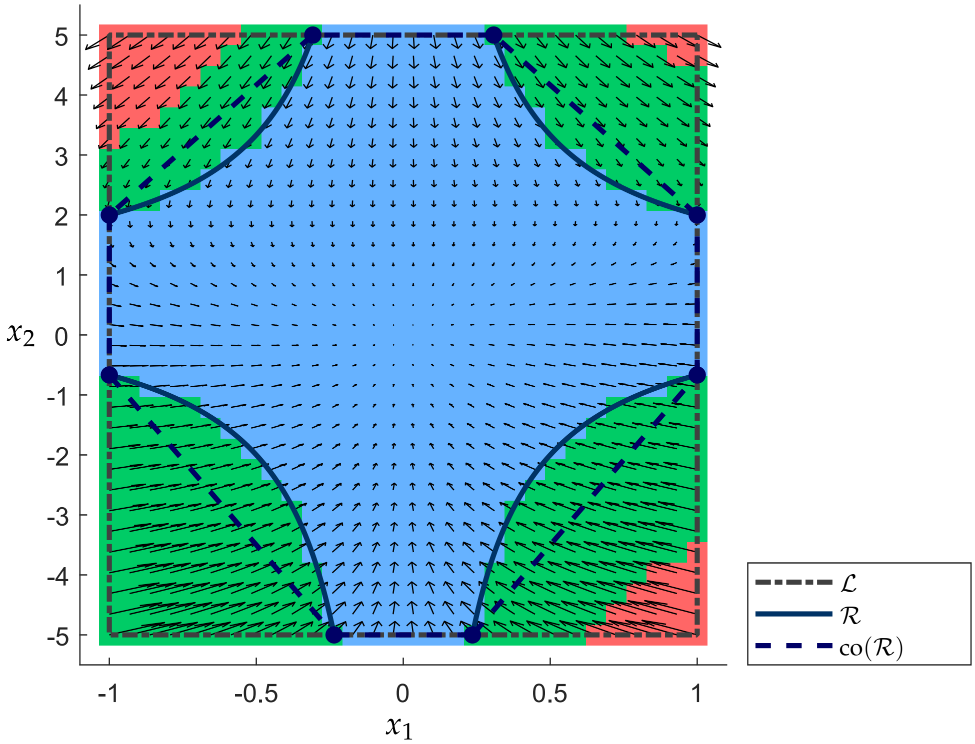


Figure 6.1: Distinct regions of the state space for Example 1. The validity domain $\mathcal{L}$ of the model is depicted in gray as a dot-dashed line. The solid blue line represents the limits of the set $\mathcal{R}$, while the dashed blue line corresponds to the convex hull co($\mathcal{R}$). Trajectories originating from the red region are unstable, whereas the ones starting from the blue are stable; the green region contains trajectories that escape $\mathcal{L}$ before converging to the origin.

Example 2

Consider the T-S system from Lee and Joo [110] where $A(\alpha) = \alpha_1 A_1 + \alpha_2 A_2$,

$$A_1 = \begin{bmatrix} a_1 & -0.7 \\ 0 & -0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.9 & 0 \\ -0.7 & a_2 \end{bmatrix},$$

with the premise variable $z(k) = x_1(k)$ and the MFs

$$\alpha^\top = 1/2 \begin{bmatrix} 1 + \sin(z(k)) & 1 - \sin(z(k)) \end{bmatrix}$$

defined in $\mathcal{P} = \left\{ \xi \in \mathbb{R}^2 : \mathcal{T}\xi \in [-\pi/2, \pi/2], \mathcal{T} = \begin{bmatrix} 1 & 0 \end{bmatrix} \right\}$. Imposing the limits $-\pi \leq x_2(k) \leq \pi$, from (3.11), the domain $\mathcal{L}$ can be defined with

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad z_{\max,1} = \pi/2, \quad z_{\max,2} = \pi.$$

According to (6.7), it is possible to write the MFs as

$$\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} x + 0.5 \begin{bmatrix} 1 + \sin(z(k)) \\ 1 - \sin(z(k)) \end{bmatrix} = E\mathcal{T}x + e$$

and, at instant $k+1$,

$$E^{+} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad e^{+} = 0.5 \begin{bmatrix} 1 + \sin(z(k+1)) \\ 1 - \sin(z(k+1)) \end{bmatrix}.$$

The presence of the term proportional to the sine of the premise variable at instant $k+1$ may initially appear problematic. However, this issue can be addressed by considering that the T-S system is defined within $\mathcal{L}$, where $-\pi/2 \leq x_1(k) \leq \pi/2$ holds for all $k \geq 0$. Therefore, since $z(k+1) = x_1(k+1)$, then $-\pi/2 \leq z(k+1) \leq \pi/2$ and applying the sector nonlinearity approach in the nonlinear term $\sin(z(k+1))$ yields

$$e^{+} = \beta_1(z(k))e_1 + \beta_2(z(k))e_2$$

where $e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$, $e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ and $\beta \in \Lambda_2$.

Then, by incorporating the revised representation of e^{+} and the polytopic representation of the MFs, the candidate Lyapunov function at instant $k+1$ can be expressed in a polytopic form. Similarly to Example 1, it is possible to mathematically define the set $\mathcal{R}$ — whose expression is omitted for simplicity. Again, $\text{co}(\mathcal{R})$ can be used to polytopically represent $x(\zeta)$. This representation, although more conservative, enables the proposed method to be implemented. Note that if the MFs are only polynomial expressions of the states — as in Example **1** —, the term e^{+} is constant, and no further application of the sector nonlinearity approach is required, leading to a simpler representation of $V(\alpha^{+})$.

Theorem 6.2, in the sequel, uses $V(\alpha^{+})$ given in (6.9) to propose sufficient conditions for the local asymptotic stability of the origin of the T-S system (6.1).

Theorem 6.2

Let $v : \mathbb{R}^n \to \mathbb{R}$, $v(x) = x^\top V(\alpha)x$, $V(\alpha) \in \mathbb{S}^n$ *as in* (6.3), *be a function satisfying*

$$z_{max,\ell}^{-2} L_\ell^\top L_\ell - V(\alpha) < 0 \quad \forall x \in \mathcal{L}, \ell \in \mathbb{N}_{p+n}, \tag{6.13}$$

$$\begin{bmatrix} -V(\alpha) & \star \\ z_{max,\ell}^{-1} L_\ell A_d(\alpha) & -1 \end{bmatrix} < 0 \quad \forall x \in \mathcal{L}, \ell \in \mathbb{N}_{p+n}, \tag{6.14}$$

$$A_d(\alpha)^\top V(\alpha^{+}) A_d(\alpha) - V(\alpha) < 0 \quad \forall x \in \mathcal{R} \tag{6.15}$$

where $V(\alpha^+)$ is defined in (6.9). *Then $v(x)$ as in* (6.2) *is a FLF for the T-S fuzzy system* (6.1) *that proves the local asymptotic stability of the origin. Moreover, the set $\Omega(v,1)$ is an invariant subset of the DOA of the system.*

Proof. Given that $L_\ell x = z_{\max,\ell}$ for $x \in \partial\mathcal{L}$, it is straightforward to notice that (6.13), when multiplied by $x^\top$ on the left and x on the right, i.e.,

$$z_{\max,\ell}^{-2} x^\top L_\ell^\top L_\ell x < v(x),$$

implies that $1 < v(x)$ for all $x \in \partial\mathcal{L}$. Hence, $v(x) = 1$ for all $x \in \partial\Omega(v,1)$, indicating that $\Omega(v,1)$ is strictly contained in $\mathcal{L}$ and $\Omega(v,1)$ serves as a sublevel set of $v(x)$.

The proof that $\Omega(v,1) \subset \mathcal{R}$ necessitates the left-multiplication of (6.14) by $\xi^\top$ and the right-multiplication by ξ, where

$$\xi = \begin{bmatrix} I_n \\ z_{\max,\ell}^{-1} L_\ell A_d(\alpha) \end{bmatrix} x,$$

yielding

$$z_{\max,\ell}^{-2} x^\top A_d(\alpha)^\top L_\ell^\top L_\ell A_d(\alpha) x - v(x) < 0 \quad \forall x \in \mathcal{L} \setminus \{0_n\}, \, \ell \in \mathbb{N}_{p+n}. \tag{6.16}$$

Considering the definition of the validity domain (3.11), it is possible to write $x_\ell = L_\ell x$ for the instant k and, for the instant $k+1$, considering the system dynamics, $x_\ell(k+1) = L_\ell A_d(\alpha) x$. Therefore, expression (6.16) implies that

$$z_{\max,\ell}^{-2} x_\ell(k+1)^2 < v(x), \quad \forall x \in \mathcal{L} \setminus \{0_n\}, \ell \in \mathbb{N}_{p+n}$$

which can be further manipulated to

$$x_\ell(k+1)^2 - z_{\max,\ell}^2 < z_{\max,\ell}^2 (v(x) - 1) \quad \forall x \in \mathcal{L} \setminus \{0_n\}, \ell \in \mathbb{N}_{p+n}.$$

From the previous expression, if $x_\ell \in \Omega(v,1)$, then $z_{\max,\ell}^2(v(x)-1) < 0$, implying $x_\ell(k+1)^2 < z_{\max,\ell}^2$. Thus, $-z_{\max} < x_\ell(k+1) < z_{\max}$, indicating $x_\ell(k+1) \in \mathcal{L}$, i.e., $x_\ell \in \mathcal{R}$. Hence, one can conclude that

$$\Omega(v,1) \subset \bigcap_{\ell=1}^{p+n} \left\{ \xi \in \mathcal{L} \,:\, \eta_\ell \in [-z_{\max,\ell}, z_{\max,\ell}], \, \eta = L A_d(\alpha(\omega))\xi, \, \omega = \mathcal{T}\xi \right\}$$

which implies $\Omega(v,1) \subset \mathcal{R}$.

By pre- and post-multiplying equation (6.15) by $x^\top$ and x, respectively, and considering the system dynamics (3.9), one obtains

$$x^{+\top} V(\alpha^+) x^+ - x^\top V(\alpha) x < 0 \quad \forall x \in \mathcal{R} \setminus \{0_n\},$$

and, therefore, $v(x^+) - v(x) < 0$, $\forall x \in \mathcal{R} \setminus \{0_n\}$. Hence, the Lyapunov function $v(x)$ is strictly decreasing along the trajectories within $\mathcal{R} \setminus \{0_n\}$. Since $\Omega(v,1) \subset \mathcal{R}$ is guaranteed, one can conclude that $\Omega(v,1) \setminus \{0_n\} \subseteq \{\xi \in \mathcal{R} : \Delta v(\xi) < 0\}$. □

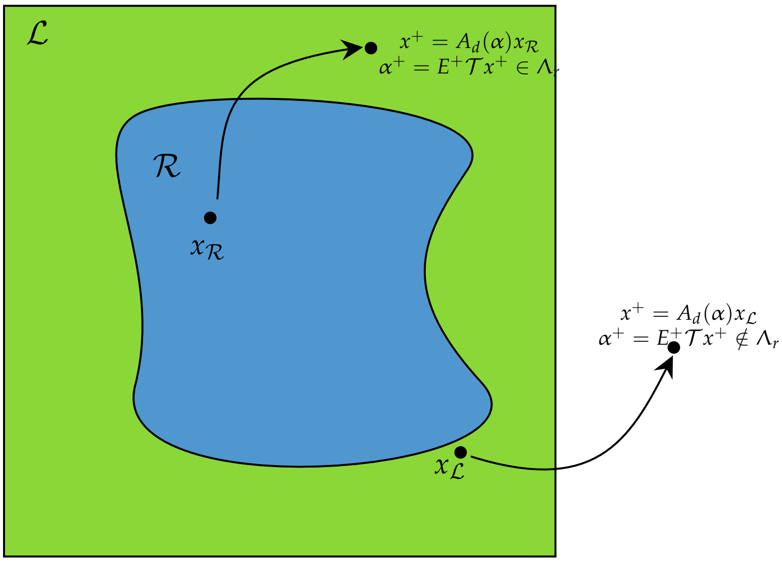


Figure 6.2: Illustrative trajectories starting inside $\mathcal{R}$ and $\mathcal{L}$.

Special attention should be given when addressing inequality (6.15), which must be fulfilled within the region $\mathcal{R}$.

Consider the illustration depicted in Figure 6.2, where the states $x_{\mathcal{R}}$ and $x_{\mathcal{L}}$ are such that $x_{\mathcal{R}} \in \mathcal{R}$ and $x_{\mathcal{L}} \notin \mathcal{R}$ but $x_{\mathcal{L}} \in \mathcal{L}$. In both cases, the T-S model is capable of producing a valid x^+ even if this value is outside $\mathcal{L}$, because the equality $x^+ = A_d(\alpha)x$ holds for any state x belonging to $\mathcal{L}$, where the T-S model is valid. However, the same conclusion cannot be drawn for the MFs. Specifically, if $x^+ \notin \mathcal{L}$, the expression $\alpha^+ = E^+\mathcal{T}x^+$ does not hold anymore because $E^+\mathcal{T}x^+$ can lead to values not covered by the T-S model. In other words, one can obtain values of α^+ outside the unit simplex, i.e., $\alpha \notin \Lambda_r$ (where α_i can assume values greater than one, or negative).

One strategy to address this issue is to impose bounds on the variation rates of the MFs, for instance, $\Delta\alpha = \alpha^+ - \alpha$, with $|\Delta\alpha| \leq b$, $0 \leq b \leq 1$, as proposed in Lee and Joo [110]. Thus, the evaluation of (6.15) subject to these constraints assures that only feasible pairs (according to b) $(\alpha, \alpha^+) \in \Lambda_r \times \Lambda_r$ are taken into account. Note that in this case the considered region is equal or is strictly contained in $\mathcal{R}$. However, the use of bounds has the drawback of loosing the precise relation between α^+ and the current values of α and x, established in (6.8).

In this work, a different approach is proposed, maintaining the connection between α^+ and the dynamics of the system (which turns to be a key feature of this work). Thus, condition (6.15) is tested in: *i)* the convex hull of $\mathcal{R}$ whenever this region can be computed; *ii)* $\mathcal{L}$, the polyhedral region where the T-S model is valid. In both cases, some conservatism may be added since $E^+\mathcal{T}x^+$ can assume values not reachable by α^+, depending on how large is the difference between $\mathcal{R}$ compared to $co(\mathcal{R})$ or $\mathcal{L}$. The second option (i.e., test (6.15) in region $\mathcal{L}$) is mandatory for the synthesis problem, since the shape of $\mathcal{R}$ depends on the gain $K(\alpha)$ to be computed.

6.2 Stability analysis

If the open-loop T-S system is considered — or if the PDC gain $K(\alpha)$ is given —, the conditions from Theorem 6.2 can be used as an LMI-based stability analysis problem.

In this case, the conditions from Theorem 6.2 can be considered as restrictions to an optimization problem given by

$$\begin{aligned} \underset{\gamma, V(\alpha)}{\text{minimize}} \quad & \gamma \\ \text{subject to} \quad & \text{equation (6.13)}, \\ & \text{equation (6.14)}, \\ & \text{equation (6.15)}, \\ & V(\alpha) - \gamma I_n \leq 0 \quad \forall x \in \mathcal{L}. \end{aligned} \tag{6.17}$$

Pre- and post multiplying the last restriction of the previous minimization problem by $x^\top$ and x, respectively, yields

$$x^\top V(\alpha) x - \gamma x^\top x \leq 0 \Rightarrow v(x) \leq \gamma x^\top x \Rightarrow v(x) - 1 \leq \gamma x^\top x - 1 \, \forall x \in \mathcal{L} \setminus \{0_n\},$$

which implies $\{\xi \in \mathcal{L} : \xi^\top \xi \leq 1/\gamma\} \subseteq \Omega(v, 1)$. Therefore, minimizing γ while considering the constraint $\{\xi \in \mathcal{L} : \xi^\top \xi \leq 1/\gamma\} \subseteq \Omega(v, 1)$ results on the enlargement of $\Omega(v, 1)$.

6.2.1 Numerical Example

The Robust LMI Parser (ROLMIP) is used to transform the robust LMIs of the optimization problem (6.17) in a finite set of LMIs [115, 136]. The code, built in Matlab 9.4 (R2018a), is solved with Mosek 9.3.18 [146], in an Intel(R) Core(TM) i7-5500U @ 2.40GHz with 16 GB RAM, Windows 10 Home Single Language.

With the given values of $a_1 = -0.35$ and $a_2 = -0.10$, the T-S system of **Example 2** is simulated, and the results are depicted in Figure 6.3, following the same color schemes as the previous example. For a comparison in terms of conservatism with another method, the stability conditions of Lee and Joo [110, Th. 1] and the conditions proposed in equation (6.17) are applied. For this particular system, the conditions of Lee and Joo [110, Th. 1] remained feasible[1] even with $b = 0.99$, resulting in the estimation of

[1]Parameter b is a bound for the variation rates $\Delta\alpha_1(k)$ and $\Delta\alpha_2(k)$.

the DOA depicted in cyan in Figure 6.3. However, it can be observed that those conditions are more conservative compared to the conditions proposed in equation (6.17), as evidenced by the larger DOA estimated (in magenta) shown in Figure 6.3. The results presented in Table 6.1 further validate the superiority of the proposed method in terms of estimating a larger area of stability.

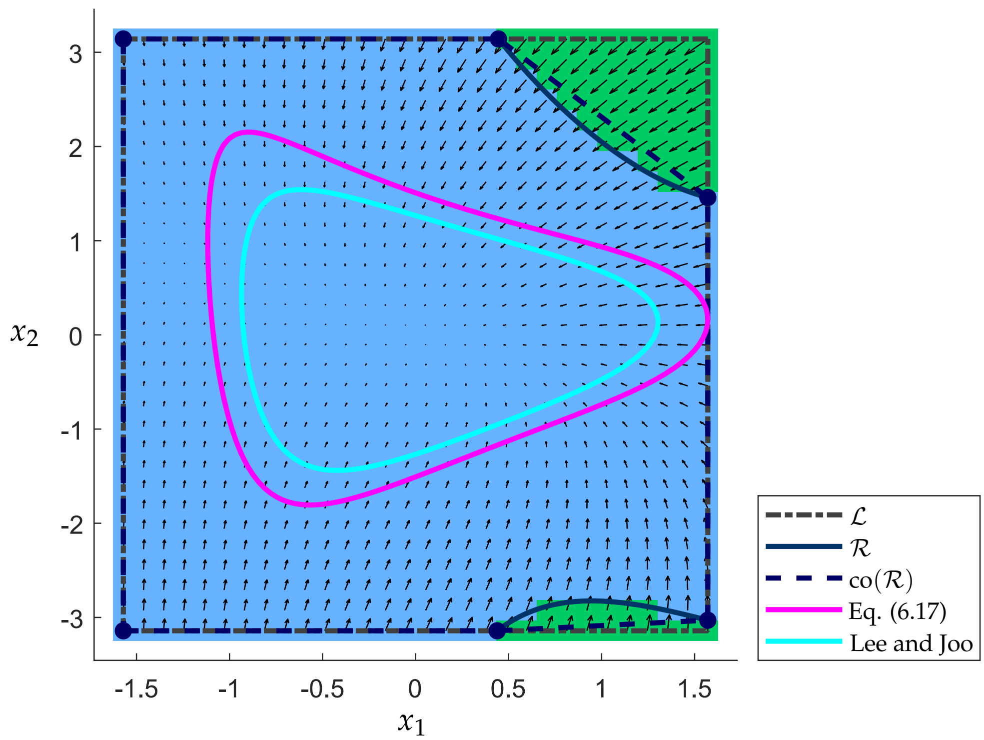


Figure 6.3: Estimated DOA for Example 2 obtained using Lee and Joo [110, Th. 1] with $(g, b) = (1, 0.99)$ (cyan), and using the optimization problem of (6.17) (magenta). The validity domain $\mathcal{L}$ of the model is depicted in gray as a dot-dashed line. The solid blue line represents the limits of the set $\mathcal{R}$, while the dashed blue line corresponds to co($\mathcal{R}$). Trajectories originating from the red region are unstable, whereas the ones starting from the blue are stable; the green region contains trajectories that escape $\mathcal{L}$ before converging to the origin.

Table 6.1: Areas of the estimated DOA obtained by the conditions of Lee and Joo [110] and the optimization problem in Equation (6.17) and the corresponding number of scalar variables (V_s), LMI rows (L_{LMIs}) and computational time (in seconds) for Example 2.

Condition	area (*u.a.*)	V_s	L_{LMIs}	time (*s*)
Lee and Joo [110, Th. 1]$_{(g,b)=(1,0.99)}$	4.5739	55	126	0.54
Equation (6.17)	6.7560	7	110	1.93

Regarding the regional stabilization of T-S systems employing a PDC gain $K(\alpha)$, the complexity of the synthesis problem notably increases compared to the stability analysis, as discussed in the following section.

6.3 PDC design

Notably, condition (6.15) in Theorem 6.2 involves the product between the Lyapunov matrix at the next instant $V(\alpha^+)$ and the feedback controller $K(\alpha)$, appearing inside the closed-loop matrix $A_{cl}(\alpha)$. To address this challenge, this thesis proposes employing Finsler's equivalent conditions to separate the products between the closed-loop matrix — that replaces $A_d(\alpha)$ in this feedback stabilization problem — and the Lyapunov matrix in (6.15), yielding LMI conditions that can be solved iteratively, following the methodology introduced by Felipe and Oliveira [112], as presented in Chapter 2. Another noteworthy characteristic is that condition (6.9) also involves the product between the feedback controller $K(\alpha)$ and the matrix $\mathcal{V}$ representing the concatenated vertices of the Lyapunov matrix $V(\alpha)$, which provides an extra complicating factor when fully separating the variables $K(\alpha)$ and $V(\alpha)$ using Finsler's lemma.

It is possible to write condition (6.15) in Theorem 6.2 in the form of statement (ii) of Finsler's lemma, i.e., $\mathscr{B}_1(\alpha)^{\perp\top}\mathscr{Q}_1(\alpha)\mathscr{B}_1(\alpha)^{\perp} < 0$, where

$$\mathscr{Q}_1(\alpha) = \begin{bmatrix} -V(\alpha) & \star \\ 0 & V(\alpha^+) \end{bmatrix}, \quad \mathscr{B}_1(\alpha)^{\perp} = \begin{bmatrix} I_n \\ A_{cl}(\alpha) \end{bmatrix},$$

and from the orthogonal projection property, $\mathscr{B}_1(\alpha)\mathscr{B}_1(\alpha)^{\perp} = 0$, resulting in

$$\mathscr{B}_1(\alpha) = \begin{bmatrix} A_{cl}(\alpha) & -I_n \end{bmatrix}. \tag{6.18}$$

Therefore, from statement (iv) of Lemma 2.1, an equivalent condition with slack variables can be obtained

$$\exists \mathscr{X}(\alpha) \,:\, \mathscr{Q}_1(\alpha) + \mathrm{He}(\mathscr{X}(\alpha)\mathscr{B}_1(\alpha)) < 0, \quad \mathscr{X}(\alpha) = \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \end{bmatrix} \in \mathbb{R}^{2n\times n}. \tag{6.19}$$

Condition (6.19) can also be rewritten as $\mathscr{B}_2(\alpha)^{\perp\top}\mathscr{Q}_2(\alpha)\mathscr{B}_2(\alpha)^{\perp} < 0$, according to statement (ii) of Lemma 2.1, where

$$\mathscr{Q}_2(\alpha) = \begin{bmatrix} -V(\alpha) & \star & \star \\ 0 & V(\alpha^+) & \star \\ A_{cl}(\alpha) & -I_n & 0 \end{bmatrix}, \quad \mathscr{B}_2(\alpha)^{\perp} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \\ X_1(\alpha)^\top & X_2(\alpha)^\top \end{bmatrix},$$

and using the orthogonal property to obtain

$$\mathscr{B}_2(\alpha) = \begin{bmatrix} X_1(\alpha)^\top & X_2(\alpha)^\top & -I_n \end{bmatrix}, \tag{6.20}$$

it is possible to employ Finsler's lemma again and introduce more slack variables to obtain another equivalent condition, i.e.,

$$\exists \mathscr{Y}(\alpha) \,:\, \mathscr{Q}_2(\alpha) + \mathrm{He}(\mathscr{Y}(\alpha)\mathscr{B}_2(\alpha)) < 0, \quad \mathscr{Y}(\alpha) \in \mathbb{R}^{3n\times n}. \tag{6.21}$$

Even though the product between the closed-loop matrix $A_{cl}(\alpha)$ and the Lyapunov matrix at instant $k+1$, $V(\alpha^+)$, is separated in (6.21), the expression for the latter one, according to (6.9), indicates that there is another product between the variables that must be separated. Considering expression (6.6) for $V(\alpha^+)$ — to simplify the notation — and exploring Lemma 2.1 once more, one can manipulate (6.21) to get the form (ii), $\mathscr{B}_3(\alpha)^{\perp\top}\mathscr{Q}_3(\alpha)\mathscr{B}_3(\alpha)^{\perp} < 0$, with

$$\mathscr{Q}_3(\alpha) = \begin{bmatrix} -V(\alpha) & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ A_{cl}(\alpha) & -I_n & 0 & \star & \star \\ 0 & 0.5(\alpha^+ \otimes I_n) & 0 & 0 & \star \\ X_1(\alpha)^\top & X_2(\alpha)^\top & -I_n & 0 & 0 \end{bmatrix},$$

$$\mathscr{B}_3(\alpha)^{\perp} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \\ 0 & \mathcal{V}^\top & 0 \\ Y_1(\alpha)^\top & Y_2(\alpha)^\top & Y_3(\alpha)^\top \end{bmatrix},$$

where α^+ is defined in (6.8).

Similarly to the previous developments, it is possible to obtain

$$\mathscr{B}_3(\alpha) = \begin{bmatrix} Y_1(\alpha)^\top & Y_2(\alpha)^\top & Y_3(\alpha)^\top & 0 & -I_n \\ 0 & \mathcal{V}^\top & 0 & -I_{rn} & 0 \end{bmatrix}$$

and apply Finsler's Lemma to introduce more slack variables, resulting in

$$\exists \mathscr{X}(\alpha) \,:\, \mathscr{Q}_3(\alpha) + \mathrm{He}(\mathscr{X}(\alpha)\mathscr{B}_3(\alpha)) < 0, \quad \mathscr{X}(\alpha) \in \mathbb{R}^{(r+4)n\times(r+1)n}. \tag{6.22}$$

Finally, following again the same steps, writing $\mathscr{B}_4(\alpha)^{\perp\top}\mathscr{Q}_4(\alpha)\mathscr{B}_4(\alpha)^{\perp} < 0$ with

$$\mathscr{Q}_4(\alpha) = \begin{bmatrix} -V(\alpha) & \star & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star & \star \\ A_{cl}(\alpha) & -I_n & 0 & \star & \star & \star & \star \\ 0 & 0.5(\alpha^+ \otimes I_n) & 0 & 0 & \star & \star & \star \\ X_1(\alpha)^\top & X_2(\alpha)^\top & -I_n & 0 & 0 & \star & \star \\ Y_1(\alpha)^\top & Y_2(\alpha)^\top & Y_3(\alpha)^\top & 0 & -I_n & 0 & \star \\ 0 & \mathcal{V}^\top & 0 & -I_{rn} & 0 & 0 & 0 \end{bmatrix}, \tag{6.23}$$

$$\mathscr{B}_4(\alpha)^\perp = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_{rn} & 0 \\ 0 & 0 & 0 & 0 & I_n \\ Z_{11}(\alpha)^\top & Z_{21}(\alpha)^\top & Z_{31}(\alpha)^\top & Z_{41}(\alpha)^\top & Z_{51}(\alpha)^\top \\ Z_{12}(\alpha)^\top & Z_{22}(\alpha)^\top & Z_{32}(\alpha)^\top & Z_{42}(\alpha)^\top & Z_{52}(\alpha)^\top \end{bmatrix},$$

it is possible to get

$$\mathscr{B}_4(\alpha) = \begin{bmatrix} Z_{11}(\alpha)^\top & Z_{21}(\alpha)^\top & Z_{31}(\alpha)^\top & Z_{41}(\alpha)^\top & Z_{51}(\alpha)^\top & -I_n & 0 \\ Z_{12}(\alpha)^\top & Z_{22}(\alpha)^\top & Z_{32}(\alpha)^\top & Z_{42}(\alpha)^\top & Z_{52}(\alpha)^\top & 0 & -I_{rn} \end{bmatrix} \tag{6.24}$$

and write a final equivalent condition

$$\exists \mathscr{W}(\alpha) \, : \, \mathscr{Q}_4(\alpha) + \mathrm{He}(\mathscr{W}(\alpha)\mathscr{B}_4(\alpha)) < 0, \quad \mathscr{W}(\alpha) \in \mathbb{R}^{(2r+5)n\times(r+1)n}.$$

Building upon the previous algebraic manipulation, next theorem presents the main result of this subsection.

Theorem 6.3

If there exist matrices $V(\alpha) \in \mathbb{S}^n$ *as in* (6.3), $\mathscr{W}(\alpha) \in \mathbb{R}^{(2r+5)n\times(r+1)n}$ *and* $\mathscr{B}_4(\alpha) \in \mathbb{R}^{(r+1)n\times(2r+5)n}$ *satisfying*

$$z_{max,\ell}^{-2} L_\ell^\top L_\ell - V(\alpha) < 0 \quad \forall x \in \mathcal{L}, \ell \in \mathbb{N}_{p+n}, \tag{6.25}$$

$$\begin{bmatrix} -V(\alpha) & \star \\ z_{max,\ell}^{-1} L_\ell A_{cl}(\alpha) & -1 \end{bmatrix} < 0 \quad \forall x \in \mathcal{L}, \ell \in \mathbb{N}_{p+n}, \tag{6.26}$$

$$\mathscr{Q}_4(\alpha) + \mathrm{He}(\mathscr{W}(\alpha)\mathscr{B}_4(\alpha)) < 0 \tag{6.27}$$

where $\mathscr{Q}_4(\alpha)$ *is defined in* (6.23), *then* $v(x)$ *as in* (6.2) *is a FLF for the T-S fuzzy system* (3.12) *that proves the regional asymptotic stability of the origin. Moreover, the set* $\Omega(v, 1)$ *is an invariant subset of the DOA of the system.*

Proof. Immediate from the development previously presented. □

The primary advantage of Theorem 6.3 lies in the affine appearance of the closed-loop matrix in inequality (6.27) and, consequently, of the controller gain $K(\alpha)$, facilitating direct treatment of this gain as a variable in the problem. Hence, it becomes straightforward to impose bounds on its entries as

$$K_{\min_{\ell,j}} \le K_{i_{\ell,j}} \le K_{\max_{\ell,j}}, \quad i \in \mathbb{N}_r, \tag{6.28}$$

where $K_{\min}$ and $K_{\max}$ are given matrices. These constraints can be useful in preventing control signals from having excessive magnitudes, for instance.

However, it is important to note that conditions in (6.27) of Theorem 6.3 take the form of Bilinear Matrix Inequalities (BMIs) due to the presence of the product $\mathscr{W}(\alpha)\mathscr{B}_4(\alpha)$. To address this challenge, an iterative algorithm with local convergence is proposed, starting from an initial feasible solution through an appropriate choice of variables $\bar{\mathscr{B}}_4(\alpha)$. The existence of a feasible solution can be ensured by considering a relaxed stability condition. This involves examining the stability of the modified system

$$x^+ = \rho A_{cl}(\alpha)x = (\rho A(\alpha) + B(\alpha)\tilde{K}(\alpha)C(\alpha))x, \tag{6.29}$$

where $\tilde{K}(\alpha) = \rho K(\alpha)$. The introduced real positive scalar ρ can be viewed as a *relaxation* parameter because for ρ close to zero, a stable dynamic matrix can be easily obtained with any gain (for instance, $\tilde{K}(\alpha) = 0$). If $\rho \geq 1$ is obtained, the stabilizing gain for the original system can be recovered. Therefore, ensuring the existence of a feasible initial solution requires selecting a sufficiently small positive value for ρ. The procedure to determine the first initial feasible values for $\bar{\mathscr{B}}_4(\alpha)$ and ρ is be described in detail in what follows.

The key feature of condition (6.27), enabling the iterative solution, is that

$$\text{He}(\mathscr{W}(\alpha)\bar{\mathscr{B}}_4(\alpha)) = \text{He}(\bar{\mathscr{B}}_4(\alpha)^\top \mathscr{W}(\alpha)^\top).$$

Therefore, any $\mathscr{W}(\alpha)^\top$ serves as a valid choice for $\bar{\mathscr{B}}_4(\alpha)$ in the subsequent iteration. The local convergence of the algorithm, with non-decreasing ρ, can be demonstrated by assuming the feasibility of the conditions in Theorem 6.3 at iteration *it* and showing that they remain feasible at the subsequent iteration $it + 1$. This is achieved with specific selections $\mathscr{Q}_{4,it+1}(\alpha) = \mathscr{Q}_{4,it}(\alpha)$ and $\bar{\mathscr{B}}_{4,it+1}(\alpha) = \mathscr{W}_{it}(\alpha)^\top$, ensuring $\rho_{it+1} \geq \rho_{it}$. Moreover, as the relaxation parameter ρ appears affinely in the conditions of Theorem 6.3, it can be considered as an objective function to be maximized in the iterative procedure; a stabilizing PDC gain $K(\alpha) = \tilde{K}(\alpha)/\rho$ is attained when $\rho \geq 1$.

After achieving stability, the iterative process can be slightly adjusted to maximize the estimate of the DOA. Various approaches can be employed to tackle this maximization problem. One method involves homogeneously enlarging the DOA by expanding an ellipsoid contained within $\Omega(v, 1)$, as demonstrated in Lee and Joo [110], or employing the convex function $-\log(\det(\cdot))$ method. Alternatively, a heuristic approach imposes the constraint $V(\alpha) \leq W$, with the optimization problem aimed at minimizing the trace of matrix W. This strategy also results in a homogeneous increase of $\Omega(v, 1)$ in all directions [152].

To obtain an initial feasible solution for the iterative algorithm, one can first consider the relaxed open-loop system $x^+ = \delta A(\alpha)x$ and utilize Theorem 6.2 to determine the upper bound for the relaxation δ^* performing, for instance, a bisection procedure. Subsequently, LMI (6.19) can be solved to find the slack variables $\mathscr{X}(\alpha)$; then, these results can be utilized to solve LMI (6.21) for $\mathscr{Y}(\alpha)$. Following these steps, the relaxed closed-loop system (6.29) can be considered to solve LMI (6.22) and determine $\mathscr{Z}(\alpha)$, which is then employed to initialize $\bar{\mathscr{B}}_4(\alpha)$ as outlined in (6.24). The subsequent theorem formalizes this procedure in details.

Theorem 6.4

Consider the relaxed open-loop T-S system

$$x^+ = \tilde{A}x, \quad \tilde{A} = \delta A(\alpha).$$

1. *Solve the optimization problem*

$$\underset{\delta, V(\alpha)}{\text{maximize}} \quad \delta \quad \text{subject to}$$

$$z_{max,\ell}^{-2} L_\ell^\top L_\ell - V(\alpha) < 0 \quad \forall x \in \mathcal{L}, \ell \in \mathbb{N}_{p+n},$$

$$\begin{bmatrix} -V(\alpha) & \star \\ z_{max,\ell}^{-1} L_\ell \tilde{A}(\alpha) & -1 \end{bmatrix} < 0 \quad \forall x \in \mathcal{L}, \ell \in \mathbb{N}_{p+n},$$

$$\tilde{A}(\alpha)^\top V(\alpha^+) \tilde{A}(\alpha) - V(\alpha) < 0 \quad \forall x \in \mathcal{R}$$

where, according to (6.9),

$$V(\alpha^+) = \mathscr{V}\big((E^+ \mathcal{T} \tilde{A}(\alpha) x(\zeta) + e^+) \otimes I_n\big).$$

2. *Let δ^* be the optimal solution and consider, according to* (6.18),

$$\mathscr{B}_1(\alpha) = \begin{bmatrix} \delta^* A(\alpha) & -I_n \end{bmatrix}$$

to solve (6.19) *on $V(\alpha)$ to obtain the slack variables $\mathscr{X}(\alpha)$.*

3. *Proceed to solve* (6.21), *still considering the relaxed open-loop system and, according to* (6.20), $\mathscr{B}_2(\alpha) = \begin{bmatrix} X_1(\alpha)^\top & X_2(\alpha)^\top & -I_n \end{bmatrix}$, *to obtain $V(\alpha)$ and $\mathscr{Y}(\alpha)$.*

4. *With the computed values of $V(\alpha)$ and $\mathscr{Y}(\alpha)$, consider*

$$\mathscr{B}_3(\alpha) = \begin{bmatrix} Y_1(\alpha)^\top & Y_2(\alpha)^\top & Y_3(\alpha)^\top & 0 & -I_n \\ 0 & \mathcal{V}^\top & 0 & -I_{rn} & 0 \end{bmatrix}$$

and the relaxed closed-loop system (6.29) *to solve* (6.22) *on* $\mathscr{X}(\alpha)$*, finally resulting in the feasible initial condition*

$$\bar{\mathscr{B}}_4(\alpha) = \begin{bmatrix} Z_{11}(\alpha)^\top & Z_{21}(\alpha)^\top & Z_{31}(\alpha)^\top & Z_{41}(\alpha)^\top & Z_{51}(\alpha)^\top & -I_n & 0 \\ Z_{12}(\alpha)^\top & Z_{22}(\alpha)^\top & Z_{32}(\alpha)^\top & Z_{42}(\alpha)^\top & Z_{52}(\alpha)^\top & 0 & -I_{rn} \end{bmatrix},$$

according to (6.24).

Proof. It follows directly from the equivalent conditions of Finsler's lemma and the preceding developments. □

Based on the presented results, Algorithm 5 is proposed. This algorithm consists of two phases: phase one involves an iterative procedure to compute a stabilizing PDC gain, while phase two focuses on maximizing the estimation of the DOA using the same iterative approach. As input parameters, Algorithm 5 takes the local linear matrices of the T-S model, A_i, B_i and C_i, $i \in \mathbb{N}_r$, along with x_j, $j \in \mathbb{N}_{p+n}$ — a polytopic representation of the states for the validity domain $\mathcal{L}$. Additionally, it requires the validity domain $\mathcal{L}$ itself, the polytopic representation of the MFs at instant $k+1$ (E^+ and e^+), and, optionally, the constraints on the entries of the PDC gains $K_{\min}$ and $K_{\max}$. Other inputs include the maximum number of iterations $it_{\max}$ and a tolerance ε, which determines the end of the iterative procedure if the progress between two consecutive iterations is not significant. During phase one, the algorithm attempts to stabilize the closed-loop T-S system, and the evolution of the algorithm is assessed in terms of the relaxation parameter ρ_{it}. Upon success (i.e., $\rho_{it} \geq 1$), the algorithm proceeds to phase two, where it focuses on enlarging the estimated DOA, fixing $\rho = 1$ and evaluating the progress in terms of $\mathrm{tr}(W)$. Note that, starting from any feasible initial condition (as the one provided by Theorem 6.4), the algorithm always converges in phase 1 to a local maximum. Whenever a solution with $\rho_{it} \geq 1$ is obtained, the algorithm enters phase 2 where an estimate for the DOA is provided.

6.3.1 Numerical Examples

Despite the conditions of theorems 6.3 and 6.4 being represented in terms of inequalities that depend polynomially on the parameters, α and ζ, it is possible to apply relaxations and exploit the non-negativity of the parameters to obtain a finite set of LMIs to test the inequalities while solving Algorithm 5. The Robust LMI Parser (ROLMIP) is employed to obtain the finite set of LMIs [115, 136]. The code, developed in Matlab 9.4 (R2018a), is solved with Mosek 10.0.26 [146], on a PC with an Intel(R) Core(TM) i7-5500U @ 2.40GHz with 16 GB RAM, running Windows 10 Home Single Language. In the implementation, the dependence of both the PDC controller $K(\alpha)$

Algorithm 5 PDC Design

Input: A_i, B_i, C_i, x_j, E^+, e^+, $\mathcal{L}$, $it_{\max}$, $K_{\min}$, $K_{\max}$, ε;
Solve the problem in Theorem 6.4 to obtain $\bar{\mathscr{B}}_{4,0}(\alpha)$;
$it \leftarrow 0$; $phase \leftarrow 1$;
While $it < it_{\max}$
  $it \leftarrow it + 1$;
  If $phase = 1$ **Then**
    maximize ρ **subject to** (6.25), (6.26), (6.27), **and** (6.28);
    If $\rho_{it} \geq 1$ **Then**
      $phase \leftarrow 2$, $\rho \leftarrow 1$;
    Else If $\rho_{it} - \rho_{it-1} \leq \varepsilon$ **Then**
      break;
    End If
  Else If $phase = 2$ **Then**
    minimize $\mathrm{tr}(W)$ **subject to** (6.25), (6.26), (6.27), (6.28), **and** $V(\alpha) \leq W$;
    If $|\mathrm{tr}(W)_{it} - \mathrm{tr}(W)_{it-1}| \leq \varepsilon$ **Then**
      break;
    End If
  End If
  $\bar{\mathscr{B}}_{4,it}(\alpha) \leftarrow \mathscr{W}_{it}(\alpha)^\top$;
End While
If $phase = 2$ **Then**
  Return $\mathrm{tr}(W)^* = \mathrm{tr}(W)_{it}$, $K(\alpha) = \tilde{K}(\alpha)/\rho$, $V(\alpha)$;
End If

and the slack variables is kept affine on α, but higher polynomial degrees could be considered straightforwardly, potentially leading to less conservative results at the price of a higher computational effort.

Two numerical examples are presented next. The first example involves a sinusoidal nonlinear dynamic system, where the premise variable is a linear combination of the states of the system. The second example, borrowed from [99], features a polynomial nonlinear dynamic system, with the premise variable corresponding to the first state. Both examples are concerned with state-feedback stabilization, allowing for comparison with another approach available in the literature, specifically the one from [110]. Additionally, the second example also includes output-feedback stabilization, providing a broader perspective on the effectiveness of the proposed approach.

Example 3

Consider the nonlinear discrete-time system

$$x^+ = \begin{bmatrix} \sin(x_1 + x_2) & 5 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{6.30}$$

where the state variables are both limited, i.e., $-\pi/2 \leq x_1(k) \leq \pi/2$, $-\pi/2 \leq x_2(k) \leq \pi/2$, as well as their sum, $-\pi/2 \leq x_1(k) + x_2(k) \leq \pi/2$, for all $k \geq 0$.

Considering the premise variable $z = x_1 + x_2$, i.e., the linear combination of the states, then $\mathcal{T} = \begin{bmatrix} 1 & 1 \end{bmatrix}$, according to (3.8). Applying the sector nonlinearity approach on the nonlinear term $\sin(x_1 + x_2)$, the nonlinear system (6.30) can be exactly represented by the discrete-time T-S system (3.9) where

$$(A, B)(\alpha) = \alpha_1 (A, B)_1 + \alpha_2 (A, B)_2, \quad \alpha \in \Lambda_2,$$

$$A_1 = \begin{bmatrix} 1 & 5 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 5 \\ 1 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

defined in $\mathcal{L}$, according to (3.11), with

$$L = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad z_{\max,\ell} = \pi/2, \ \ell \in \mathbb{N}_3,$$

which can be used to find the vertices of the polytope that defines $x(\zeta)$, for instance, through the Multi-Parametric Toolbox (MPT) [143].

The MFs, given by $\alpha_1 = 0.5(\sin(z) + 1)$ and $\alpha_2 = 0.5(-\sin(z) + 1)$, can be represented according to (6.7), i.e.,

$$\alpha = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0.5 \begin{bmatrix} \sin(z) + 1 \\ -\sin(z) + 1 \end{bmatrix} = E\mathcal{T}x + e,$$

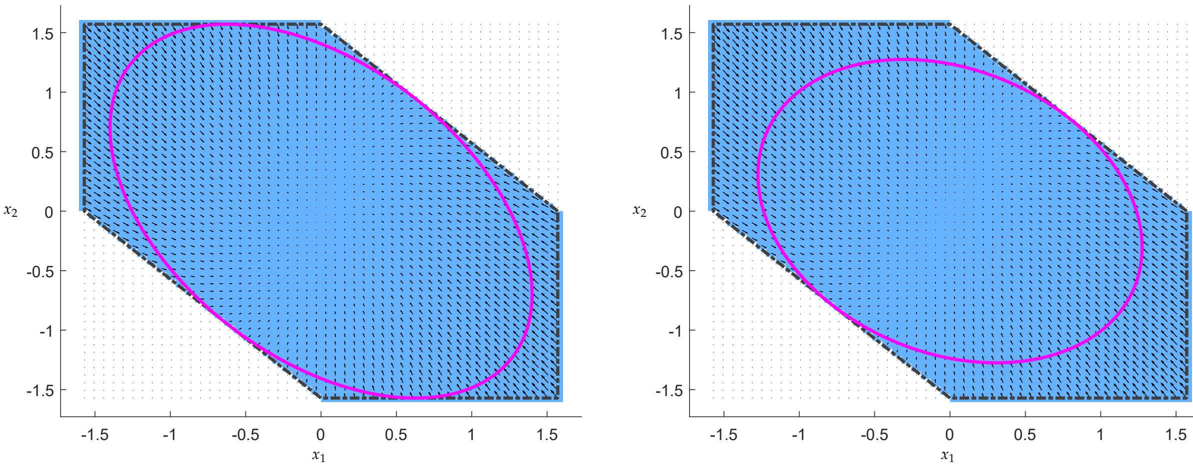
and, at instant $k + 1$,

$$E^+ = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad e^+ = 0.5 \begin{bmatrix} \sin(z^+) + 1 \\ -\sin(z^+) + 1 \end{bmatrix}.$$

As previously discussed, the presence of the sine of the premise variable at instant $k + 1$ in the expression for e^+ may seem problematic. However, exploring the locality of the T-S model and the definition of region $\mathcal{R}$ according to (6.4), it is ensured that x^+ remains within $\mathcal{L}$. Therefore, $-\pi/2 \leq z^+ \leq \pi/2$ and the sector nonlinearity approach can be applied to the nonlinear term $\sin(z^+)$, yielding

$$e^+ = \beta_1 e_1 + \beta_2 e_2, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \beta \in \Lambda_2.$$

With the polytopic representation, Algorithm 5 is applied, setting the maximum number of iterations as $it_{\max} = 100$ and the tolerance $\varepsilon = 10^{-4}$. The results are illustrated in Figure 6.4a. In this figure, the blue area represents points x such that x^+ remain within the validity domain, i.e., the region $\mathcal{R}$ (in this case $\mathcal{R}$ coincides with $\mathcal{L}$). The limit of the validity domain $\mathcal{L}$ is depicted by the gray dot-dashed line and the solid magenta line indicates the boundary of the estimated DOA. For comparison reasons,



(a) Algorithm 5. **(b)** Lee and Joo [110, Th. 2].

Figure 6.4: Estimated DOA for Example 3 is depicted in magenta. The boundary of the validity domain $\mathcal{L}$ of the model is shown in gray as a dot-dashed line, while the blue shaded region represents the domain $\mathcal{R}$ (which coincides with $\mathcal{L}$ in this case) under consideration.

the conditions of Theorem 2 in Lee and Joo [110] are also tested, considering the MFs variation bounded by $b = 0.9$, resulting in the DOA shown in Figure 6.4b.

The numerical complexities of the techniques are shown in Table 6.2, where V_s corresponds to the number of scalar variables and L_{LMIs} to the number of LMI rows. Algorithm 5, which required 37 iterations to converge, exhibits a significantly higher computational cost compared to the synthesis conditions of [110]. However, this cost directly correlates with the size of the estimated DOA, indicating the superiority of the conditions proposed in this work.

Table 6.2: Areas of the estimated DOA obtained by the conditions of Lee and Joo [110] and Algorithm 5 and the corresponding number of scalar variables (V_s), LMI rows (L_{LMIs}) and computational time (in seconds) for Example 3.

Condition	area ($u.a.$)	V_s	L_{LMIs}	time (s)
Lee and Joo [110, Th. 2]$_{b=0.9}$	4.9526	19	143	0.76
Algorithm 5	6.1843	269	682	24.76

Example 4

Consider the nonlinear model

$$x^+ = \begin{bmatrix} 1 & -x_1 \\ -1 & -0.5 \end{bmatrix} x + \begin{bmatrix} 5 + x_1 \\ 2x_1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \tag{6.31}$$

where only the state variable x_1 is limited in Guerra and Vermeiren [99], i.e., $-z_m \leq x_1(k) \leq z_m$ (z_m to be defined).

Considering the premise variable $z = x_1$, then $\mathcal{T} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, according to (3.8). Applying the sector nonlinearity approach on the nonlinear term x_1, the nonlinear system (6.31) can be exactly represented by the discrete-time T-S system (3.9) where

$$(A, B, C)(\alpha) = \alpha_1 (A, B, C)_1 + \alpha_2 (A, B, C)_2, \quad \alpha \in \Lambda_2$$

$$A_1 = \begin{bmatrix} 1 & -z_{\max} \\ -1 & -0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & z_{\max} \\ -1 & -0.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 5 + z_{\max} \\ 2z_{\max} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5 - z_{\max} \\ -2z_{\max} \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

defined in $\mathcal{P}$, according to (3.10). However, as previously stated, this work assumes that all the state variables are limited. Therefore, for x_2, the limits $-10z_m \le x_2(k) \le 10z_m$ are imposed. Thus, from (3.11), the domain $\mathcal{L}$ can be defined with

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad z_{\max,1} = z_m, \quad z_{\max,2} = 10z_m,$$

which can be used to determine the vertices of the polytope that defines $x(\zeta)$. In this case, since the premise variable corresponds to the state x_1, matrix L that defines the domain $\mathcal{L}$ is simplified, as previously explained.

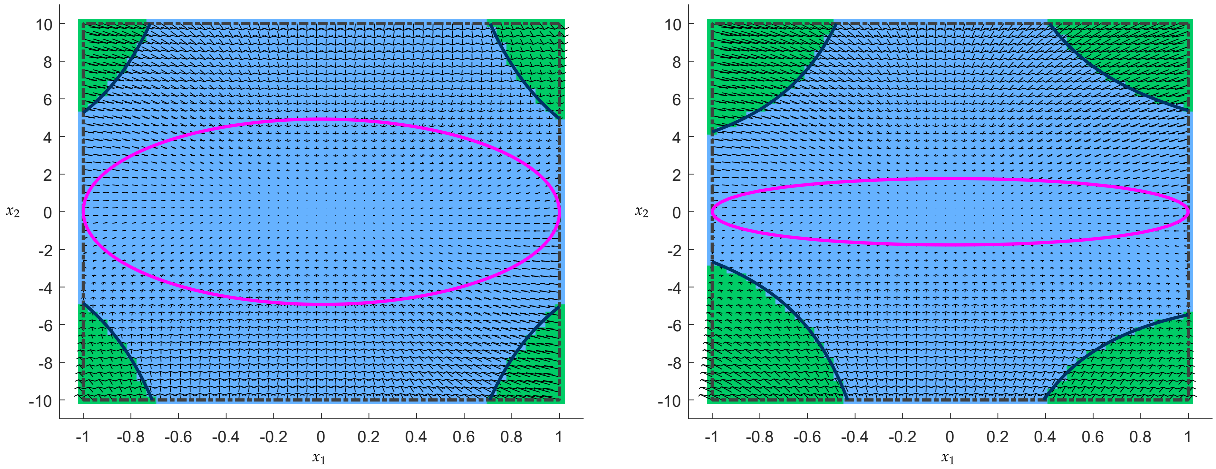
The MFs can be represented according to (6.7), i.e.,

$$\alpha = \frac{1}{2z_m} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} x + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = E\mathcal{T}x + e,$$

and, at instant $k+1$,

$$E^+ = \frac{1}{2z_m} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^+ = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Using the polytopic representation and considering first the state-feedback stabilization, Algorithm 5 is applied using the same values for the maximum number of iterations and tolerance as in the previous example. The results are illustrated in Figure 6.5a. The limit of the validity domain $\mathcal{L}$ is depicted by the gray dot-dashed line, while the solid magenta line indicates the boundary of the estimated domain of attraction. In this case, in addition to the blue area representing the region $\mathcal{R}$, a green area is also observed. Within this green area, although $x \in \mathcal{L}$, at the next instant, $x^+ \notin \mathcal{L}$, meaning that the state vector escapes the validity domain of the T-S system. However, it is important to note that the estimated DOA remains inside $\mathcal{R}$ and therefore, the conditions of Theorem 6.3 remain valid.



(a) Algorithm 5.   **(b)** Lee and Joo [110, Th. 2].

Figure 6.5: Estimated DOA for Example 4 considering state-feedback PDC design is depicted in magenta. The boundary of the validity domain $\mathcal{L}$ of the model is shown in gray as a dot-dashed line, while the blue shaded region represents the domain $\mathcal{R}$, bounded by the solid blue line. Trajectories originating from the green region escape the domain $\mathcal{L}$ in the subsequent instant $k+1$.

Comparing the iterative conditions of this work with the synthesis conditions of Theorem 2 in Lee and Joo [110] with $b = 0.9$, the superiority of the iterative algorithm is even more evident in this example, with the area of the estimated DOA being around 2.75 times the area of the DOA estimated with Theorem 2 in Lee and Joo [110]. The cost, again, is the numerical complexity, shown in Table 6.3, which nevertheless remains reasonable. In this example, the iterative algorithm converges in 27 iterations and the control gain is given by (truncated with four decimal digits)

$$K(\alpha) = \alpha_1 \begin{bmatrix} -0.0825 & 0.1000 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.1200 & -0.1003 \end{bmatrix}.$$

Table 6.3: Areas of the estimated DOA obtained by the conditions of Algorithm 5 and the corresponding number of scalar variables (V_s), LMI rows (L_{LMIs}) and computational time (in seconds) for Example 4 considering state-feedback.

Condition	area ($u.a.$)	V_s	L_{LMIs}	time (s)
Lee and Joo [110, Th. 2]$_{b=0.9}$	5.6331	19	125	0.81
Algorithm 5	15.4679	269	246	13.32

Finally, Algorithm 5 is also employed to compute an output-feedback PDC controller. The resulting DOA is depicted in Figure 6.6, following the same color scheme as the previous figures. It is interesting to note how the shape of the region $\mathcal{R}$ differs from that in Figure 6.5, as it depends on the dynamics of the final closed-loop T-S system, demonstrating that it is difficult to obtain a sketch of region $\mathcal{R}$ when performing control design. Additionally, the estimated DOA, as claimed, lies inside region $\mathcal{R}$, illustrating the validity of the assumptions made in this work. Note that the output-feedback provides a good estimate of the DOA, only 5.87% smaller than the one provided by Lee and Joo [110] in the state-feedback case. It is worthy to emphasize that the synthesis conditions of Lee and Joo [110] cannot be applied in the output-feedback case. The numerical complexity of the technique considering output-feedback PDC design is shown in Table 6.4. In this case, the iterative algorithm is terminated with 61 iterations and the control gain is given by

$$K(\alpha) = \alpha_1 \begin{bmatrix} 0.1183 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.0709 \end{bmatrix}.$$

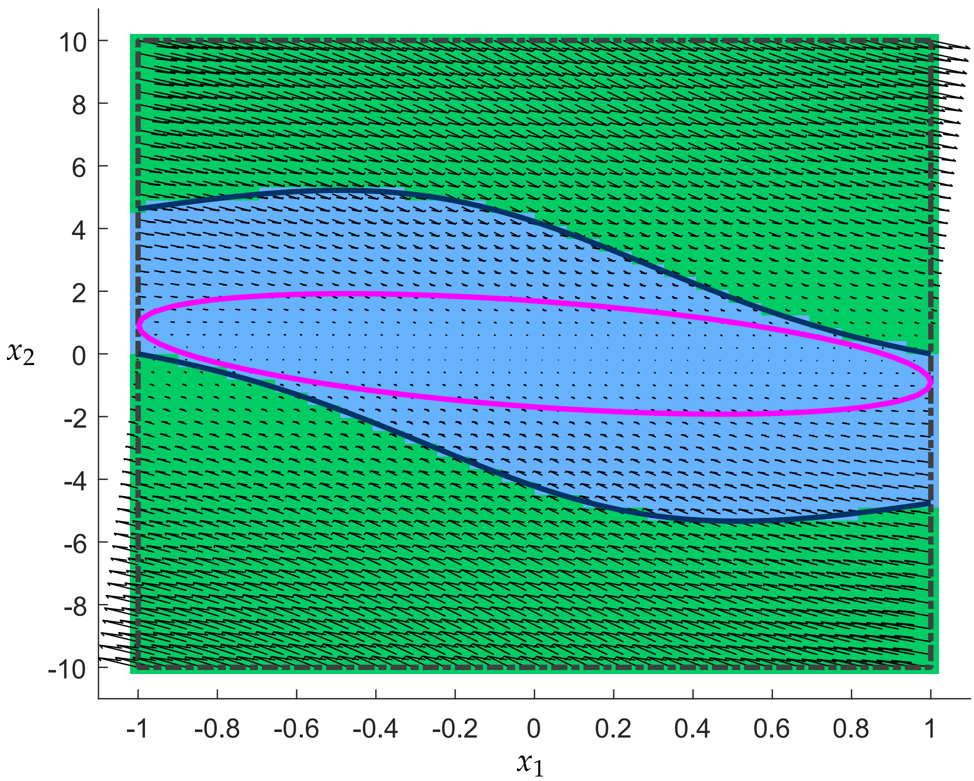


Figure 6.6: Estimated DOA for Example 4 considering output-feedback PDC design, obtained through Algorithm 5, is depicted in magenta. The boundary of the validity domain $\mathcal{L}$ of the model is shown in gray as a dot-dashed line, while the blue shaded region represents the domain $\mathcal{R}$, bounded by the solid blue line. Trajectories originating from the green region escape the domain $\mathcal{L}$ in the subsequent instant $k+1$.

Table 6.4: Areas of the estimated DOA obtained by the conditions of Algorithm 5 and the corresponding number of scalar variables (V_s), LMI rows (L_{LMIs}) and computational time (in seconds) for Example 4 considering output-feedback.

Condition	area (*u.a.*)	V_s	L_{LMIs}	time (*s*)
Algorithm 5	5.3023	267	246	22.67

6.4 Conclusion

This chapter addressed the problem of regional stability, in the context of stability analysis and synthesis, as well as the maximization of the estimated DOA for discrete-time T-S fuzzy systems. The proposed approach utilizes a fuzzy-modeling technique to represent the MFs in a polytopic form based on the premise variables. Unlike existing methods in the literature, this representation eliminates the need for explicit information about the bounds of variation of the MFs. In the context of stability analysis, the estimation of the DOA is formulated as a single-parameter minimization problem subject to LMI constraints, providing an efficient solution demonstrated through numerical examples and comparisons with other methods. Furthermore, in the context of synthesis, the chapter tackles regional stabilization and DOA maximization for discrete-time T-S fuzzy systems using state- or output-feedback PDC control. The proposed fuzzy-modeling technique, representing MFs in a polytopic form, bypasses the necessity for explicit MF bounds and by conveniently exploring Finsler's lemma, the chapter presents an iterative algorithm for the PDC design problem. Numerical examples highlight the effectiveness and superiority of the proposed method in providing the largest estimated DOA and flexibility in imposing gain limits, despite increased computational effort.

Chapter 7

Conclusions and Future Steps

This thesis has advanced the field of control for T-S fuzzy systems, focusing on both continuous-time and discrete-time domains. By addressing the limitations of traditional quadratic Lyapunov functions, novel synthesis conditions were proposed utilizing Homogeneous Polynomial Lyapunov Functions (HPLFs), Fuzzy Lyapunov Functions (FLFs) and Homogeneous Polynomial Parameter-Dependent Lyapunov Functions (HPPDLFs) — that depend polynomially on system states and MFs. These approaches significantly reduce conservatism and improve performance in control design.

For continuous-time T-S fuzzy systems, FLF, HPLF and HPPDLF stabilization methods were developed. Those methods generalize beyond quadratic stability, the latter two introducing an innovative strategy that incorporates extended matrix representations without the need for the standard change of variables. Those methods were shown to effectively reduce conservatism and enhance system performance through an iterative LMI-based algorithm. Furthermore, regional stability analysis was extended by employing polytopic representations of the gradient of MFs, enabling the design of more efficient output-feedback controllers without requiring known bounds on the time-derivatives of the MFs.

In the discrete-time domain, FLFs were used to handle the regional stability and estimation of the domain of attraction (DOA). The proposed polytopic representation approach eliminates the need for prior bounds on the variation rates of the MFs, leading to less conservative stability conditions and better DOA estimates. The effectiveness of the method was validated through numerical examples, demonstrating superiority over existing techniques in terms of reduced conservatism and improved performance.

The synthesis procedures developed in this thesis were given in terms of iterative LMI-based algorithms that can be applied to both state- and output-feedback stabilization, with constrained structure or magnitude bounds on the entries, showing flexibility and practicality. These procedures not only ensure closed-loop stability but also optimize the decay rate — in the continuous-time domain — and maximize the DOA estimate. It is worth to emphasize that, since the synthesis procedures involve the design of a PDC gain, the practical implementation of the designed gains has the same complexity of other PDC techniques available in the literature. Moreover, since the controller synthesis is performed offline, the typically higher computational times required by the proposed method do not pose an issue.

Future work will aim to extend these methods to accommodate Lyapunov functions with rational dependence on system states and to address performance criteria such as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Additionally, exploring the potential of these advanced Lyapunov functions in other complex nonlinear systems remains an exciting avenue for further research. Immediate extensions can be considered to treat uncertain time-invariant linear systems and switched systems. Finally, the investigation of line integral Lyapunov functions [154] depending polynomially on the states is also a topic for further research.

In conclusion, this thesis has made significant contributions to the robust control of T-S fuzzy systems, offering less conservative and more effective solutions for both continuous- and discrete-time applications. These advancements pave the way for future research and development in nonlinear control systems, promising enhanced stability, performance, and robustness in practical applications.

Publications

International Journal

Submitted

1. QUILLES-MARINHO, Y.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. Output-feedback Control Design for Takagi-Sugeno Fuzzy Systems Through Lyapunov Functions Depending Polynomially on the States. **Fuzzy Sets and Systems**, 2024. Under Review (second round).

2. QUILLES-MARINHO, Y. et al. Local Output-feedback PDC control for T-S Fuzzy Systems Through Polynomially Membership and State Dependent Lyapunov Functions. **IEEE Transactions on Systems, Man, and Cybernetics**, 2024. Awaiting Decision.

3. QUILLES-MARINHO, Y. et al. Regional Output-feedback PDC control for Discrete-time T-S Fuzzy Systems via Fuzzy-modeled Membership Functions. **IEEE Trans-**

actions on Fuzzy Systems, 2024. Awaiting Decision.

National Conference

Published

1. MARINHO, Y. Q.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. Estabilização Paralela Distribuída de Sistemas Nebulosos de Takagi-Sugeno Utilizando Funções de Lyapunov Polinomiais Homogêneas no Estado. In: ANAIS do XXII Congresso Brasileiro de Automática. Fortaleza, CE, Brasil: [s.n.], Oct. 2022

International Conference

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1. MARINHO, Y. Q.; OLIVEIRA, R. C. L. F.; PERES, P. L. D. An LMI-based approach for local output-feedback stabilization of continuous-time Takagi-Sugeno fuzzy systems. In: PROCEEDINGS of the 2023 IEEE International Conference on Fuzzy Systems. Songdo Incheon, South Korea: [s.n.], Aug. 2023
2. QUILLES-MARINHO, Y. et al. Local Stability Analysis and Estimation of Domains of Attraction for Discrete-Time Takagi-Sugeno Fuzzy Systems via Fuzzy-Modeled Membership Functions. In: 2024 American Control Conference (ACC). [s.l.: s.n.], 2024. P. 2630–2635. DOI: 10.23919/ACC60939.2024.10644856

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