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Complexity of Sliding Shilnikov Connections in Filippov Systems & Hyperbolicity for Infinite Delayed Difference Equations

Complexidade de Conexões Deslizantes de Shilnikov em Sistemas de Filippov & Hiperbolicidade para Equações de Diferenças com Atraso Infinito

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Supervisor: Douglas Duarte Novaes

Co-supervisor: Gabriel Ponce

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"Soy porque somos."
("I am because we are.")
Francia Márquez, Vice President of Colombia

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"A pior coisa que você pode fazer com um problema é resolvê-lo completamente."
Daniel Kleitman, matemático americano

"The worst thing you can do to a problem is solve it completely."
Daniel Kleitman, American mathematician

"Mas, ou muito me engano, ou acabo de escrever um capítulo inútil."
Machado de Assis, em **Memórias Póstumas de Brás Cubas**
(Capítulo CXXXVI / Inutilidade)

"But, unless I am much mistaken, I have just written an utterly pointless chapter."
Machado de Assis, in **The Posthumous Memoirs of Brás Cubas**
(Chapter CXXXVI / Pointlessness)

Resumo

O presente trabalho é desenvolvido em duas partes distintas.

Na primeira parte tratamos da questão de analisar conexões deslizantes de Shilnikov, um tipo de conexão homoclínica no contexto de campos vetoriais suaves por partes, seguindo a convenção de Filippov. A partir da caracterização pelo objeto abstrato de Sistemas Conformes de Funções Iteradas (ou *Conformal Iterated Function Systems*, geralmente abreviado como *CIFS*), utilizamos as ferramentas da teoria para investigar propriedades de caráter dinâmico, topológico e geométrico do conjunto invariante local do sistema dinâmico gerado pela conexão deslizante de Shilnikov, principalmente a dimensão de Hausdorff e a existência de medidas conformes relacionadas à dinâmica em questão.

Na segunda parte, estudamos o conceito de equações de diferenças lineares, em particular a noção de dicotomia exponencial, que caracteriza um comportamento de crescimento com taxas exponenciais de contração e expansão. Em particular, focamos em equações com atraso infinito, onde a imagem de um ponto possivelmente depende de todos os tempos anteriores. A partir daí, utilizamos o conceito de cociclos, onde definimos sua hiperbolicidade, e associamos a uma equação um cociclo. Deduzimos então o resultado principal, onde se caracteriza a equivalência entre a dicotomia exponencial de uma equação e a hiperbolicidade do cociclo associado a ela. Também são apresentados exemplos e aplicações do resultado.

Palavras-chave: Conexões deslizantes de Shilnikov. Conjunto invariante. Sistemas de funções iteradas. Dimensão de Hausdorff. Medidas conformes. Equações de Diferenças Lineares. Atraso Infinito. Dicotomia exponencial. Cociclos. Hiperbolicidade.

Abstract

This work is developed in two distinct parts.

In the first part, we analyze sliding Shilnikov connections, a type of homoclinic connection in the context of piecewise smooth vector fields, following Filippov's convention. From the characterization by the abstract object of Conformal Iterated Function Systems, usually abbreviated as CIFS, we utilize theoretical tools to investigate properties of a dynamic, topological, and geometric nature of the local invariant set of the dynamical system generated by the Shilnikov sliding connection, mainly the Hausdorff dimension and the existence of conformal measures related to the dynamics in question.

In the second part, we study the concept of linear difference equations, particularly the notion of exponential dichotomy, which characterizes a growth behavior with exponential rates of contraction and expansion. Specifically, we focus on equations with infinite delay, where the image of a point may depend on all previous times. From there, we utilize the concept of cocycles, where we define their hyperbolicity and associate a cocycle with an equation. We then deduce the main result, which characterizes the equivalence between the exponential dichotomy of an equation and the hyperbolicity of the associated cocycle. Examples and applications of the result are also presented.

Keywords: Sliding Shilnikov connections. Invariant set. Conformal Iterated Function Systems. Hausdorff dimension. Conformal measures. Linear difference equations. Infinite delay. Exponential dichotomy. Cocycles. Hyperbolicity.

Contents

Introduction

This thesis deals with two main results, which are quite unrelated with each other. This explains the choice to separate the text in two parts.

In Part I, we study an object that arises in the theory of piecewise smooth vector fields, more specifically we investigate some geometric, topological, and dynamical properties of sliding Shilnikov connections on Filippov fields.

Meanwhile, Part II tackles a question about linear difference equations with infinite delay, and the relationship about exponential dichotomy of the equation and hyperbolicity of a related cocycle.

Now, we provide a short introduction of the content and main results of each part.

Part I: Complexity of Sliding Shilnikov Connections in Filippov Systems

Some of the main objects in the study of continuous dynamical systems are vector fields on a manifold and their flows. This concept is firstly introduced by presenting continuous, in fact smooth, vector fields, which is a natural approach. With them, we can model many kinds of phenomena, for example some physical and biological systems.

However, there are some instances where the transition between states is not smooth, but vary in a discontinuous fashion. The theory of piecewise smooth vector fields is a way to understand how we could deal with those kind of problems, and the *Filippov's convention* is one method to determine the behavior of systems in this context.

In this new case, there are some questions about the objects that arise in this study. "When do they act like their continuous counterparts?" and "Are there new objects that arises in this generalized notion?" are some examples that we might ask, among many others. One such object, in particular, is the so-called *sliding Shilnikov connection*, which is a trajectory Γ that connects a hyperbolic (pseudo-)saddle-point q to itself by the flow of both the sliding vector field given by Filippov's convention and the continuous vector fields themselves.

When we study what happens in a neighborhood of the sliding Shilnikov connection Γ, we notice the appearance of points that remain forever in the neighborhood (from which we can define a first-return map π), so we have an invariant set Λ. Thus, we have a dynamical system (Λ, π). We then take a local approach in a neighborhood U of q, defining appropriately the dynamical system (Λ_U, π_U).

This dynamical system and its invariant set possess a lot of fascinating behavior, for instance properties related to chaos and infinite (topological) entropy, to name a few.

In this thesis, we further investigate (Λ_U, π_U), relating them to the theory of (conformal) iterated function systems, which roughly speaking is a family of contractions of a space into itself that contains a lot of information about its invariant set. Then we state and prove that the Hausdorff dimension of Λ_U satisfies $0 < \dim_{\mathrm{H}}(\Lambda_U) < 1$, its Lebesgue measure is zero, and that $\overline{\Lambda_U}$ retain both properties and additionaly is a Cantor set, while also giving an explicit construction of it.

Further, we conjugate the dynamics of (Λ_U, π_U) to a Bernoulli system with infinite symbols, and show that a probability measure that is invariant and ergodic relatively to π_U, with some additional interesting properties, can be defined on (Λ_U, π_U).

This part of the thesis is derived from the work produced in (CUNHA; NOVAES; PONCE, 2024).

Structure of Part I

This part of the thesis is structured as follows.

We begin with Chapter 1, giving a quick overview about the importance and applications of Filippov systems and sliding Shilnikov connections, our objects of interest.

In Chapter 2 we will introduce Filippov systems (Section 2.1) and sliding Shilnikov connections (Subsection 2.1.1), which is where live our notions. We then state our main result in Section 2.2.

Then, in Chapter 3, we will show the relationship between the abstract notion of Iterated Function Systems (IFS) and Conformal IFS (CIFS) and the Hausdorff dimension of a special set linked to them. For this, we define the Hausdorff dimension and give some important properties in Section 3.1; then we introduce IFS in general (Section 3.2) and we focus our attention in the Conformal IFS (CIFS), explained in Section 3.3. We deal with the attractor set of an IFS/CIFS, an object inherent to the theory that we will relate with the invariant set of our dynamical system. After that, the notion of the pressure function of a CIFS is studied (Subsection 3.3.1). At last, (Section 3.4) is going to present the relationship between IFS, CIFS, the pressure function and the Hausdorff dimension of the attractor set.

Chapter 4 is dedicated to the steps to prove our main results. In Section 4.1 it is defined a first-return map in a neighborhood of a important point of a sliding Shilnikov connection, while in Section 4.2 we will see how this map can be related to an CIFS; Sections 4.3 and 4.4 analyzes the closure of the set Λ_U, offering two approaches. Finally, Section 4.5 is where we put all information together for the proof.

After that, in Chapter 5 we will provide some additional comments about the structure of the invariant set of the dynamical system, conjugating it to a Bernoulli system (Section 5.1), and the existence of a conformal measure with support in Λ_U and that is invariant and ergodic relatively to π_U (Section 5.2).

Part II: Hyperbolicity for Infinite Delayed Difference Equations

Given a Banach space $(X, |\cdot|)$, for a given sequence $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ of operators, we could consider the difference equation

$$x(m+1) = L_m x(m), \quad \text{for } m \in \mathbb{Z}.$$

For any element v (the initial condition) and integer $n \in \mathbb{Z}$, there is only one solution $x : \mathbb{Z}_{\geq n} \to X$ that satisfies $x(n) = v$, for $j \in \mathbb{Z}_{\leq 0}$, and the difference equation.

More generally, for an integer $r \leq 0$ (the finite delay) and denoting $x_m(j) := x(m+j)$, for $m \in \mathbb{Z}$, $j \in [r, 0] \cap \mathbb{Z}$, we could consider the equation

$$x(m+1) = L_m x_m, \quad \text{for } m \in \mathbb{Z},$$

with $\{L_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{L}(B, X)$, where $B = B_r := \{\phi : [r, 0] \cap \mathbb{Z} \to X \text{ such that } \|\phi\|_B < +\infty\}$ is a Banach space. Similarly, for a given $v \in B$ and $n \in \mathbb{Z}$, there exists only one solution $x = x(\cdot, n, v) : \mathbb{Z}_{n+r} \to X$ satisfying $x_n = v$ and the finitely delayed difference equation.

Of course, when $r = 0$, we simply have the previous case.

We then define, for each $m, n \in \mathbb{Z}$ with $m \geq n$, the operator $T(m, n) \in \mathcal{L}(B)$ by $T(m, n)v = x_m(\cdot, n, v)$. When the system has no delay, this amounts to the composition of linear maps.

The concept of exponential dichotomy in these equations captures the notion of the solutions converging with exponential rate in a stable and unstable spaces measured by the norms $\|T(m, n)\|_{\mathcal{L}(B)}$ of the operators. As just noted, since $T(m, n)$ is the generalization of the notion of composing linear maps, we are investigating a behavior relating the dynamics to contractions and expansions with exponential rate of compositions of linear operators.

Meanwhile, we may define a cocycle that tries to capture these behaviors. The idea is to get a space where we can get the information of all shifts of $\{L_m\}_{m\in\mathbb{Z}}$, which is a bigger space, but we need only one argument to decide about the exponential rates of decay. We say then that the cocycle is *hyperbolic*.

Then, we have the following equivalence: the equation with finite delay has exponential dichotomy if and only if the associated cocycle is hyperbolic.

In this thesis, we establish the same kind of equivalence, valid in a large class of spaces, for the case with infinite delay, that is, for the equation of the form

$$x(m+1) = L_m x_m, \quad \text{for } m \in \mathbb{Z},$$

where now $setL_{kk\in\mathbb{Z}} \subseteq \mathcal{L}(B, X)$, where $B := \{\phi : \mathbb{Z}_{\leq 0} \to X \text{ such that } \|\phi\|_B < +\infty\}$.

Besides, we analyze some examples, showing properties like the dependence of exponential dichotomy on the norm, and we make some applications, for example in the theory of perturbed equations.

This part of the thesis is derived from the work produced in (BARREIRA; CUNHA; VALLS, 2025).

Structure of Part II

First, in Chapter 6 we will make a brief exposition about the necessary concepts needed to contextualize and state our main result. More specifically, Section 6.1 is where we are going to define linear difference equations with no delay, with finite delay and with infinite delay, as well as introduce the notion of exponential dichotomy for each kind. Then, Section 6.2 will approach cocyles, making an introduction on the subject and showing how to define a cocycle associated with a linear difference equation. Later, in Section 6.5 we will state our main result, after stating the correspondent versions for equations with no delay and with finite delay for context.

Chapter 7 will be devoted to some preliminaries that we will need to prove our result, where in Section 7.1 will be devoted to analyze the spaces that we will be working on, and Section 7.2 will be where we will discuss some bound conditions for our work.

In Chapter 8 we are going to prove or main result.

An important part of the theory involves finding examples and counterexamples, so we dedicate Chapter 9 for that. Section 9.1 will show that the concept of exponential dichotomy depends heavily on the norm and may change behavior accordingly to changes on the underlying Banach space. Then, Section 9.2 will provide two examples of non-autonomous equations with exponential dichotomy, each one with a particular characteristic, and Section 9.3 will discuss how a robustness property may be used to find another examples.

At last, Chapter 10 will show how to use the main result of this part of the thesis to infere some other properties. In Section 10.1 we will extend some properties obtained for one equation to a set of equations, meanwhile Section 10.2 will show the robustness of exponential dichotomy for linear perturbations. Then, in Section 10.3, we will relate our result to a generalized notion of the spectra of a sequence of bounded linear operators.

Part I

Complexity of Sliding Shilnikov Connections in Filippov Systems

1 Introduction

When modeling various phenomena, it is often observed that the rules governing their evolution change abruptly at specific thresholds, introducing discontinuities in their models (see (JEFFREY, 2018; JEFFREY, 2020) for a general discussion on discontinuities in applied models). These sudden variations are commonly associated with processes involving decisions, or switches, such as those in neurons or electronic systems, light refraction, body collisions, changes in dry friction regimes, or any other scenarios that exhibit abrupt shifts in behavior. In ecology, for example, the phenomenon of prey-switching describes a predator's adaptive diet in response to the availability of different prey species. This behavior, observed in many predator species, creates discontinuities in prey-predator models (see, for example, (PILTZ; PORTER; MAINI, 2014; LEEUWEN et al., 2013)). Similar discontinuities are found in other applied models, such as prey-predator models with prey refuge (KŘIVAN, 2011), mechanical systems (SZALAI; JEFFREY, 2014), electromagnetic processes (BACHAR et al., 2010), and others. The mathematical framework used to model and understand these phenomena includes the concept of piecewise smooth differential systems. Therefore, obtaining a better understanding of the geometric and dynamical properties of these systems is highly valuable. For surveys on piecewise smooth dynamical systems and their applications, see (BERNARDO et al., 2008; MAKARENKOV; LAMB, 2012).

These types of differential systems, however, raise a fundamental question: what constitutes a solution? In (FILIPPOV, 1988), Filippov used the theory of differential inclusions to address this issue. He formulated what is now known as *Filippov's convention* for the trajectories of piecewise smooth differential systems. Systems that follow this convention are referred to as *Filippov systems* (for discussions on other conventions, see (JEFFREY, 2014; JEFFREY et al., 2022; NOVAES; JEFFREY, 2015)). The set of discontinuities of a Filippov system is called *switching set*, which, for our purposes, will always be assumed to be a smooth manifold.

Filippov's convention and its related concepts will be formally defined in the following subsections but, before that, let us briefly discuss in an informal manner this convention and the main object of our study. First, for points on the switching manifold where the vector fields in both sides cannot be concatenated to create a trajectory that crosses it, the Filippov's convention induces a dynamics on the switching manifold, allowing trajectories to slide along it, a phenomenon known as *sliding dynamics*. Also, the Filippov's convention extends the classical concept of singularities for smooth vector fields by inducing new types of singularities on the switching manifold. In these scenarios, the trajectories of a Filippov system may asymptotically approach these singularities or reach them in finite time, potentially leading to a loss of uniqueness of trajectories. The combination of these new singularities and the sliding dynamics gives rise to unique global phenomena in Filippov systems, such as the *sliding Shilnikov connection* (see Definition 1) that has been recently introduced and was discussed in, for example, (NOVAES; TEIXEIRA, 2019).

The sliding Shilnikov connection is an important notion in Filippov systems, as their existence implies chaotic behavior within an invariant subset of the system, as demonstrated in (NOVAES; PONCE; VARÃO, 2017). This phenomenon has practical applications in applied science. For instance, in (PILTZ; PORTER; MAINI, 2014), numerical evidences of chaotic behavior were observed in a prey-switching Filippov-type prey-predator model. This observation was analytically explained in (CARVALHO; NOVAES; GONÇALVES, 2020) by proving that such a model exhibits a sliding Shilnikov connection. The study of the properties and applicability of this connection is in its early stages, and understanding the topology and complexity of its associated invariant sets is of interest.

2 Filippov Systems and Sliding Shilnikov Connections

Here we introduce our objects of study. Let $V \subseteq \mathbb{R}^n$ be an open subset, and then we recall that a function $f : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class $\mathcal{C}^{l,\varepsilon}(V, \mathbb{R}^m)$ if $f \in \mathcal{C}^l(V, \mathbb{R}^m)$ and its l-th derivative, $D^l f$, satisfies the ε-Hölder condition, that is, there exists a constant L such that $\left\|D^l f(x) - D^l f(y)\right\|_{\text{op}} \leq L\,|x-y|^{\varepsilon}$ for all $x, y \in V$, where $\|\cdot\|_{\text{op}}$ is the standard norm among linear operators and $|\cdot|$ is the usual euclidian norm on $\mathbb{R}^n$. When the context is clear, we may abbreviate this as $\mathcal{C}^{l,\varepsilon}(V)$ or $\mathcal{C}^{l,\varepsilon}$.

2.1 Filippov Systems

For an open subset $V \subseteq \mathbb{R}^n$, let us consider the following piecewise smooth vector field:

$$Z(u) = \begin{cases} X(u), & \text{if } g(u) > 0, \\ Y(u), & \text{if } g(u) < 0, \end{cases} \qquad u \in V, \tag{2.1}$$

where $X, Y \in \mathcal{C}^{l,\varepsilon}(V, \mathbb{R}^n)$, with $l \geq 1$ an integer, $0 < \varepsilon \leq 1$, and $g \in \mathcal{C}^1(V, \mathbb{R})$ is a function with 0 as a regular value (that is, $Dg(u) : \mathbb{R}^n \to \mathbb{R}$ is surjective for all $u \in g^{-1}(0)$). Its switching manifold is given by $M = g^{-1}(0)$.

The piecewise smooth vector field (2.1) is concisely denoted as $Z = (X, Y)_g$ (or simply $Z = (X, Y)$). The space of all piecewise smooth systems of the form (2.1) is denoted by $\Omega_g^{l,\varepsilon}(V, \mathbb{R}^n) \cong \mathcal{C}^{l,\varepsilon}(V, \mathbb{R}^n) \times \mathcal{C}^{l,\varepsilon}(V, \mathbb{R}^n)$, allowing us to endow it with the product topology. When the context is clear, we may abbreviate this as $\Omega_g^{l,\varepsilon}(V)$, $\Omega_g^{l,\varepsilon}$, $\Omega^{l,\varepsilon}(V)$, or simply $\Omega^{l,\varepsilon}$.

The Filippov's convention establishes that the local trajectories of Z (i.e. local solutions of the differential system $\dot{u} = Z(u)$) correspond to solutions of the differential inclusion

$$\dot{u} \in \mathfrak{F}_Z(u), \;\; u \in V, \tag{2.2}$$

where $\mathfrak{F}_Z : V \rightsquigarrow \mathbb{R}^n$ is the following set-valued function

$$\mathfrak{F}_Z(u) := \begin{cases} \{X(u)\}, & \text{if } g(u) > 0 \\ \{(1-s)X(u) + sY(u) : s \in [0,1]\}, & \text{if } g(u) = 0, \\ \{Y(u)\}, & \text{if } g(u) < 0. \end{cases}$$

We recall that $\varphi : I \to V$, defined on an open interval $I \subseteq \mathbb{R}$, is said to be a solution of the differential inclusion (2.2), if it is an absolutely continuous function satisfying

$\dot{\varphi}(t) \in \mathfrak{F}_Z(\varphi(t))$ for almost every $t \in I$. For an introduction on differential inclusions, see (AUBIN; CELLINA, 1984).

The local trajectories of (2.1) have an intuitive geometric interpretation. To explore this, let $\varphi_F^t(u)$ denote the flow of a vector field $F : V \subseteq \mathbb{R}^n \to \mathbb{R}^n$ at time t starting from u, and also define

$$Fg(u) := \langle F(u), \nabla g(u) \rangle, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^n$. For points in V where $g(u) \neq 0$, the local trajectory corresponds to the local trajectories of either X or Y, depending on whether $g(u) > 0$ or $g(u) < 0$, respectively. To describe the local solutions for points on the switching manifold M, we first distinguish between some open regions on M (see Figure 1):

- The points on M satisfying $Xg(u)Yg(u) > 0$ define the *crossing region* M^c. This implies that there exists $t_1 < 0 < t_2$ such that $g(\varphi_X^t(u)) > 0$ for $t \in (t_1, 0)$ and $g(\varphi_Y^t(u)) < 0$ for $t \in (0, t_2)$, or $g(\varphi_X^t(u)) > 0$ for $t \in (0, t_2)$ and $g(\varphi_Y^t(u)) < 0$ for $t \in (t_1, 0)$, so that both trajectories can be concatenated at u to form a local trajectory of Z at u.

- The points on M satisfying $Xg(u) > 0$ and $Yg(u) < 0$ define the *escaping region* M^e. This implies that there exists $t_2 > 0$ such that $g(\varphi_X^t(u)) > 0$ for $t \in (0, t_2)$ and $g(\varphi_Y^t(u)) < 0$ for $t \in (0, t_2)$, so that both trajectories also cannot be concatenated at u to form a trajectory of Z.

- The points on M satisfying $Xg(u) < 0$ and $Yg(u) > 0$ define the *sliding region* M^s. This implies that there exists $t_1 < 0$ such that $g(\varphi_X^t(u)) > 0$ for $t \in (t_1, 0)$ and $g(\varphi_Y^t(u)) < 0$ for $t \in (t_1, 0)$, so that both trajectories cannot be concatenated at u to form a trajectory of Z.

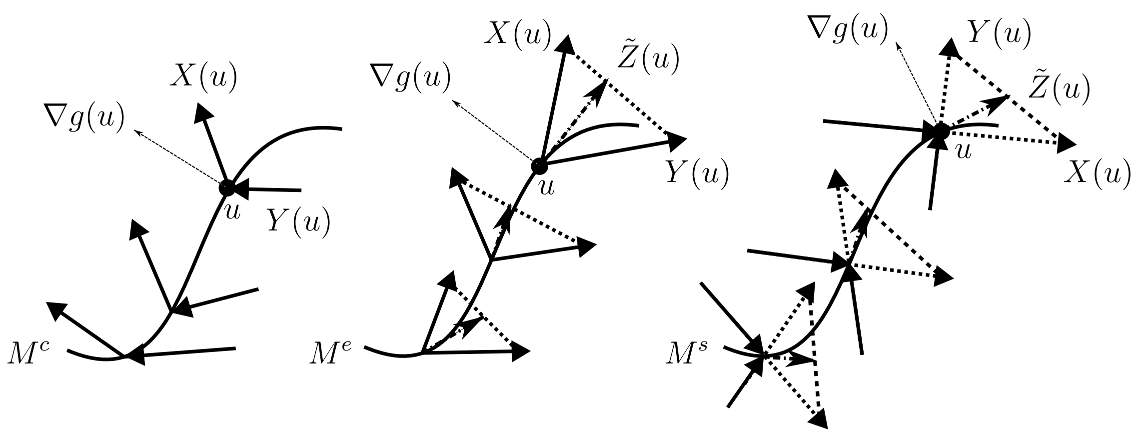

Figure 1 – A graphical representation of the crossing, escaping and sliding regions on the switching manifold.

For a point $u \in M^c$, the local trajectory of (2.1) at u is uniquely determined as a suitable concatenation of the local trajectories of X and Y at u, as previously described.

For a point $u \in M^{s,e} := M^s \cup M^e$, the local trajectories of X and Y at u cannot be concatenated, as they are either both approaching or both departing from M at u. However, for each $u \in M^{s,e}$, there exists a unique vector within the convex combination $\mathfrak{F}_Z(u)$ that is tangent to M at u. This vector is given by:

$$\tilde{Z}(u) := \frac{Yg(u)X(u) - Xg(u)Y(u)}{Yg(u) - Xg(u)}, \quad u \in M^{s,e}.$$

Since $\tilde{Z}(u) \in T_u M = T_u M^{s,e}$ for every $u \in M^{s,e}$, $\tilde{Z}(u)$ defines a vector field on $M^{s,e}$, referred to as the *sliding vector field*. Additionally, because $\tilde{Z}(u) \in \mathfrak{F}_Z(u)$ for all $u \in M^{s,e}$, any local trajectory of $\tilde{Z}(u)$ satisfies the differential inclusion (2.2) and, therefore, corresponds to a local trajectory of the Filippov vector field (2.1). It should be noted that the local trajectories of X or Y at u can eventually be concatenated with the local trajectory of $\tilde{Z}$ at u to form additional local trajectories of Z at u. Consequently, the uniqueness of local trajectories is not guaranteed for points in $M^{s,e}$.

The sliding dynamics, described above, naturally introduces a first new type of singularity of the Filippov system Z, corresponding to the singularities of the sliding vector field $\tilde{Z}$. These are points $u^* \in M^{s,e}$ where $\tilde{Z}(u^*) = 0$, known as *pseudo-equilibria of* Z. Notably, trajectories of $\tilde{Z}$ can asymptotically approach a pseudo-equilibrium, while trajectories of X and Y can either reach or depart from it in finite time. This type of singularity is crucial to the definition of sliding Shilnikov connections. A pseudo-equilibrium is considered hyperbolic if it is a hyperbolic singularity of $\tilde{Z}$. Additionally, if $u^* \in M^s$ is an unstable hyperbolic focus of $\tilde{Z}$, or if $u^* \in M^e$ is a stable hyperbolic focus of $\tilde{Z}$, then u^* is referred to as a *hyperbolic pseudo-saddle-focus* (this corresponds to the point p in Figure 2).

Finally, we consider the set of *tangency points* M^t, which consists of the points $u \in M$ where $Xg(u)Yg(u) = 0$. A point $u \in M^t$ is called a tangency point of X if $Xg(u) = 0$, or a tangency point of Y if $Yg(u) = 0$. There are many possible configurations of points in M^t, leading to different definitions of their local trajectories. As a result, the uniqueness of local trajectories is also not guaranteed at points in M^t. In the following, we will introduce the concept of a *visible fold-regular point*, a specific type of tangency point that occurs in sliding Shilnikov connections.

A tangency point $u \in M^t$ is referred to as a visible fold of X (resp. Y) if $X^2g(u) := X(Xg)(u) > 0$ (resp. $Y^2g(u) := Y(Yg)(u) < 0$). Conversely, if the inequalities are reversed, the point u is called an invisible fold of X (resp. Y). A visible/invisible fold $u \in M$ of X (resp. Y) is called a visible/invisible fold-regular point if $Yg(u) \neq 0$ (resp. $Xg(u) \neq 0$). If $Yg(u) > 0$ (resp. $Xg(u) < 0$), the point lies on the boundary of the sliding region, ∂M^s. Conversely, if $Yg(u) < 0$ (resp. $Xg(u) > 0$), it lies on the boundary of the escaping region, ∂M^e. For instance, the point q in Figure 2 corresponds to a visible fold-regular point lying on ∂M^s.

Remark 1. *Visible fold-regular points have several important properties (see (TEIXEIRA, 1990)), of which we will highlight two.*

(R1) Firstly, the sliding vector field $\tilde{Z}$ is always transverse to these points, which means that trajectories of $\tilde{Z}$ either reach or depart from them transversely within a finite time.

(R2) Secondly, when the dimension of the space is greater than or equal to 3 ($n \geq 3$), a visible fold-regular point q is never isolated, which means that there exists a neighborhood U around q such that $U \cap \partial M^{s,e}$ is a set consisting entirely of visible fold-regular points.

As a consequence of the above Remark (R1), the local trajectory of Z at a visible fold-regular point u is determined by an appropriate combination of local trajectories of X, Y, and $\tilde{Z}$ at u.

2.1.1 Sliding Shilnikov Connections

The concept of sliding Shilnikov connection was recently introduced and was discussed more profoundly in, for example, (NOVAES; TEIXEIRA, 2019). Roughly speaking, it consists of a trajectory Γ of Z, passing though a visible fold-regular $q \in \partial M^{s,e}$ and connecting a hyperbolic pseudo-saddle-focus p to itself asymptotically on one side and in finite time on the other side (see Figure 2).

Some of its dynamical properties, such as chaotic behavior, was further explored in (NOVAES; PONCE; VARÃO, 2017). It has also demonstrated significant applied importance, as shown in (CARVALHO; NOVAES; GONÇALVES, 2020), by proving that a family of prey-switching Filippov-type prey-predator models exhibits chaotic behavior, which had previously been supported only by numerical evidence in (PILTZ; PORTER; MAINI, 2014).

In what follows, we introduce the definition of sliding Shilnikov connection.

Definition 1 (Sliding Shilnikov Connection)**.** *Let $Z = (X, Y) \in \Omega^{1,\varepsilon}$ be a Filippov system with a hyperbolic pseudo-saddle-focus $p \in M^s$ (resp. $p \in M^e$) and a visible fold-regular point $q \in \partial M^s$ (resp. $q \in \partial M^e$), which is a visible fold point of X. Assume that:*

1. *The trajectory of $\tilde{Z}$ passing through q converges to p backward in time (resp. forward in time), that is, $\lim_{t\to-\infty} \varphi^t_{\tilde{Z}}(q) = p$ (resp. $\lim_{t\to+\infty} \varphi^t_{\tilde{Z}}(q) = p$).*

2. *The trajectory of X passing through q reaches M^s (resp. M^e) in a finite time $t_q > 0$ (resp. $t_q < 0$) at p, that is, $\varphi_X^{t_q}(q) = p$ and $g(\varphi_X^t(q)) \neq 0$, for any $t \in (0, t_q)$ (resp. $t \in (t_q, 0)$).*

Then, the sliding loop Γ passing through q and connecting p to itself is called a sliding Shilnikov connection *(see Figure 2).*

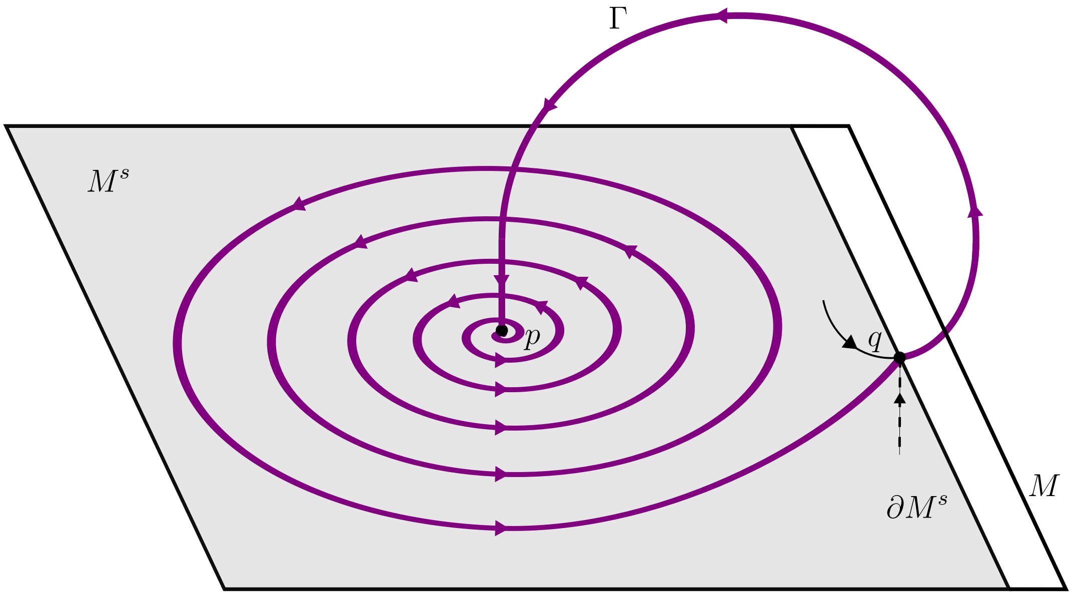

Figure 2 – Representation of a sliding Shilnikov connection. The point $p \in M^s$ is a hyperbolic pseudo-saddle-focus and the point $q \in \partial M^s$ is a visible fold-regular point for X. The forward trajectory of Z at q follows the flow of the vector field X until it reaches, in finite time, the sliding region M^s at the hyperbolic pseudo-saddle-focus p. The backward trajectory of Z at q follows the backward flow of the sliding vector field $\tilde{Z}$ which approaches asymptotically to the hyperbolic pseudo-saddle-focus p.

In (NOVAES; PONCE; VARÃO, 2017), it was demonstrated the existence of a neighborhood $B \subseteq \mathbb{R}^3$ of q such that, for $\gamma := B \cap \partial M^{s,e}$, which is a curve of visible fold-regular points (see (R2) from Remark 1), a first-return map $\pi : \mathrm{Dom}(\pi) \to \gamma$ is well-defined on a subset $\mathrm{Dom}(\pi) \subseteq \gamma$. This map captures the complete dynamics of the Filippov system Z in a neighborhood of the sliding Shilnikov connection Γ. The dynamics of π exhibits a rich structure with many interesting properties. For instance, in (NOVAES; PONCE; VARÃO, 2017), it was shown that the restriction of π to a local invariant set is topologically conjugate to a Bernoulli shift with infinite topological entropy. In Section 4.1, we will provide details on the construction of the first-return map π.

The main goal of this study is to examine, from a local perspective, certain geometric properties of the invariant set of the first-return map restricted to U, $\pi_U :=$

$\pi|_{\mathrm{Dom}(\pi)\cap U}$, where $U \subseteq \gamma$ is a neighborhood of q. Such invariant set is given by

$$\Lambda_U := \left\{ w \in U : \pi^k(w) \in \mathrm{Dom}(\pi) \cap U, \text{ for all } k \geq 0 \right\}. \tag{2.4}$$

More specifically, our goal is to estimate the Hausdorff dimension, denoted by $\dim_{\mathrm{H}}(\cdot)$, and the one-dimensional Lebesgue measure, denoted by $\mathrm{m}_1(\cdot)$, as well as to explore the Cantor set topological structure for both the invariant set and its closure.

As we will show, the first-return map can be decomposed into maps exhibiting expanding behavior, such that the functions corresponding to their inverses are contractions (see Figure 5). This brings us into the realm of *Iterated Function Systems (IFS)*, or more specifically, *Conformal Iterated Function Systems (CIFS)* when the functions meet suitable criteria. We will use tools from IFS theory to analyze the geometric aspects of Λ_U mentioned before (see Section 4.2).

2.2 Statement of Main Result

Our main findings can be summarized as follows.

Theorem A1. *Let $V \subseteq \mathbb{R}^3$ be an open subset and consider a Filippov system $Z = (X, Y) \in \Omega_g^{1,1}(V, \mathbb{R}^3)$ possessing a sliding Shilnikov connection Γ passing through a visible fold-regular point $q \in \partial M^{s,e}$. Consider the first-return map $\pi : \mathrm{Dom}(\pi) \subseteq \gamma \to \gamma$ associated to Γ. Then, there exists a neighborhood $U \subseteq \gamma$ of q such that:*

(a) The invariant set Λ_U satisfies $0 < \dim_{\mathrm{H}}(\Lambda_U) < 1$ and $\mathrm{m}_1(\Lambda_U) = 0$.

(b) The closure of the invariant set, $\overline{\Lambda_U}$, is a Cantor set, $\dim_{\mathrm{H}}\left(\overline{\Lambda_U}\right) = \dim_{\mathrm{H}}(\Lambda_U)$, and $\mathrm{m}_1(\overline{\Lambda_U}) = \mathrm{m}_1(\Lambda_U)$. Furthermore, $\overline{\Lambda_U} = \Lambda_U \,\dot{\cup}\, Q_U$, where $Q_U = \left(\bigcup_{k \geq 0} \pi^{-k}(q) \right) \cap U$ is a countable set.

The notion of the Hausdorff dimension of a set will be properly defined in Section 3.1. Besides, we recall that a Cantor set is a non-empty, compact, totally disconnected (the singletons are the only subsets that are connected), perfect (all of its points are accumulation points) and metrizable set. It is known that every set with these topological characteristics are homeomorphic (see, for instance, (PUGH, 2015, Chapter 2, Theorem 67 and Theorem 73)).

Additionally, in Chapter 5 we provide other results: a conjugation of the first-return map in the invariant set Λ_U with a Bernoulli system with countably infinite symbols, and we also discuss the existence of a conformal measure with invariant and ergodic properties in the invariant set Λ_U.

3 Hausdorff Dimension, Iterated Function Systems (IFS) and Conformal IFS (CIFS)

3.1 Hausdorff Dimension

The Hausdorff dimension is a notion of dimension related to the "size", in a measure sense, of a set. More precisely, let us consider the diameter of an arbitrary subset $V \subseteq \mathbb{R}^d$,

$$\operatorname{diam}(V) := \sup\{|x-y| : x, y \in V\}.$$

Given a set S and a real number $s \geq 0$, for each $\delta > 0$ we look at all countable (possibly finite) covers of S with subsets of diameter at most δ (also called a *δ-cover*) and calculate the quantity

$$\mathcal{H}^s_\delta(S) := \inf\left\{\sum_{k\geq 1}(\operatorname{diam} V_k)^s : \ \{V_k\}_{k\geq 1} \text{ is a } \delta\text{-cover of } S\right\}.$$

We define the *s-dimensional Hausdorff measure* of S as the limit

$$\mathcal{H}^s(S) := \lim_{\delta\to 0}\mathcal{H}^s_\delta(S).$$

The *Hausdorff dimension* of S, denoted by $\dim_{\mathrm{H}}(S)$, is then determined by

$$\dim_{\mathrm{H}}(S) := \inf\{s \geq 0 : \mathcal{H}^s(S) = 0\},$$

which is well-defined.

Notice that if $\mathcal{H}^{\tilde{s}}(S) \in \mathbb{R}$, for some $\tilde{s} > 0$, then $\mathcal{H}^t(S) = +\infty$, for all $t < s$. For this reason, usually we have $\inf\{s \geq 0 : \mathcal{H}^s(S) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(S) = +\infty\}$, except when $\dim_{\mathrm{H}}(S) = 0$, when $\{s \geq 0 : \mathcal{H}^s(S) = +\infty\}$ may be empty.

For more details in this discussion and some important properties of the Hausdorff dimension, we refer to (SCHLEICHER, 2007) and (FALCONER, 2014, Chapter 2). The properties that will be relevant to our discussion are listed below:

(H1) If $S_i \subseteq S_j$, then $\dim_{\mathrm{H}}(S_i) \leq \dim_{\mathrm{H}}(S_j)$ ((SCHLEICHER, 2007, Theorem 2(1))).

(H2) If $\{S_k\}_{k\geq 1}$ is a countable collection of sets, then $\dim_{\mathrm{H}}\left(\bigcup_{k\geq 1} S_k\right) = \sup_{k\geq 1}\{\dim_{\mathrm{H}}(S_k)\}$ ((SCHLEICHER, 2007, Theorem 2(2))).

(H3) If f is a bi-Lipschitz map, then $\dim_{\mathrm{H}}(S) = \dim_{\mathrm{H}}(f(S))$, which implies that diffeomorphic compact sets have the same Hausdorff dimension ((SCHLEICHER, 2007, Theorem 2(5))).

(H4) If $\dim_{\mathrm{H}}(S) < 1$, then the one-dimensional Lebesgue measure of S is 0 ((SCHLEICHER, 2007, Theorem 2(8))).

(H5) If $\dim_{\mathrm{H}}(S) < 1$, then S is totally disconnected ((FALCONER, 2014, Proposition 2.5)).

3.2 Iterated Function Systems (IFS)

For this section, we assume that $K \subseteq \mathbb{R}^d$ is a compact subset satisfying $K = \overline{K^\circ}$, and that I is a countable (finite or infinite) index set.

An *iterated function system (IFS) on* K is a family of contractions on K, $\mathcal{F} = \{f_i : K \to K\}_{i \in I}$, that is, a set of functions such that, for each $i \in I$, there exists $c_i < 1$ for which

$$|f_i(x) - f_i(y)| \leq c_i |x - y|, \quad \text{for all } x, y \in K.$$

Given an IFS $\mathcal{F} := \{f_i\}_{i \in I}$, we consider finite words of I of the form $\eta = (\eta_1, \ldots, \eta_k) \in \bigcup_{j \geq 1} I^j$, $k \geq 1$. For such a word η, we define the corresponding function f_η as

$$f_\eta = f_{(\eta_1, \ldots, \eta_k)} := f_{\eta_1} \circ \cdots \circ f_{\eta_k}.$$

A key concept concerning an IFS is its *attractor set*. To define it, let $\mathrm{proj} : I^{\mathbb{N}} \to 2^K$ be the projection map from the symbol space $I^{\mathbb{N}}$ into 2^K, that is, for each $\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in I^{\mathbb{N}}$, $\mathrm{proj}(\omega)$ is defined satisfying

$$\{\mathrm{proj}(\omega)\} = \bigcap_{k \geq 1} f_{\omega|_k}(K), \tag{3.1}$$

where $\omega|_k := (\omega_1, \ldots, \omega_k)$, for $k \geq 1$ (notice that, in general, it is a multi-valued function; however, as well shall see, in our applications it will always be a bijection). Then, the attractor set of the IFS is defined as

$$\Delta := \mathrm{proj}\left(I^{\mathbb{N}}\right) = \bigcup_{\omega \in I^{\mathbb{N}}} \bigcap_{k \geq 1} f_{\omega|_k}(K). \tag{3.2}$$

This set satisfies the self-similarity property, that is, $\Delta = \bigcup_{i \in I} f_i(\Delta)$. When the index set I is finite, there is only one non-empty set that satisfies this property, which may not be the case when I is infinite. However, the attractor set is the largest set that satisfies the self-similarity property (see (MAULDIN, 1995, Remark of Section 3)).

Remark 2. *It is worth noting that if there exists $0 < s < 1$ for which the contraction constants satisfy $c_i \le s$, the set in (3.1) becomes a singleton for each $\omega \in I^{\mathbb{N}}$. In this case,* proj *can also be regarded as a function from $I^{\mathbb{N}}$ into K, in a minor abuse of notation.*

Also, we should note that, if $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ are IFS (we say that $\tilde{\mathcal{F}}$ is a sub-system of $\mathcal{F}$), then their respective attractor sets, namely $\tilde{\Delta}$ and Δ, satisfy $\tilde{\Delta} \subseteq \Delta$.

We now present a classical example of an IFS:

Example 1 (The ternary Cantor set)**.** *Consider the set of functions $\mathcal{F} = \{f_i\}_{i\in\{0,2\}} = \{f_0, f_2\}$, where $f_0, f_2 : [0,1] \to [0,1]$ defined by $f_0(x) = \dfrac{x}{3}$ and $f_2(x) = \dfrac{x}{3} + \dfrac{2}{3}$. Since each function is a contraction, it is an IFS.*

Given $\omega = (\omega_1, \omega_2, \omega_3, \ldots) \in \{0,2\}^{\mathbb{N}}$, we have that $f_{\omega|_k}(K) = f_{\omega_1} \circ \ldots \circ f_{\omega_k}(K)$. This is the set of real numbers in $[0,1]$ with ternary expansion, up to the k-th ternary digit, equal to $0.\omega_1\omega_2\ldots\omega_k$.

Clearly, the union for all $\omega \in \{0,2\}^{\mathbb{N}}$ gives all the numbers in $[0,1]$ with the k initial ternary digits equal to 0 and 2. This, of course, implies that the attractor set Δ of this IFS is the ternary Cantor set.

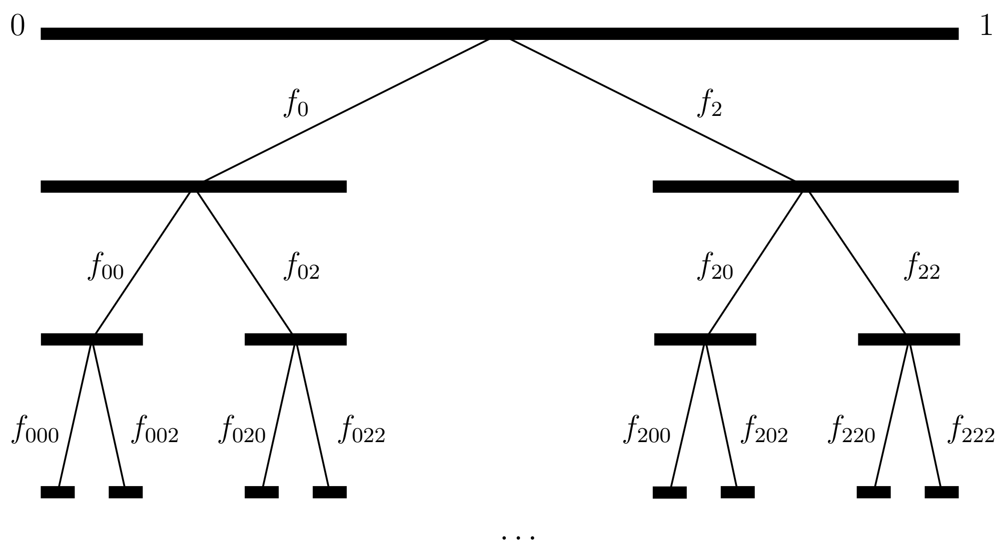

Figure 3 – Representation of the initial steps in the construction of the Cantor set, expliciting its relationship with an IFS.

3.3 Conformal Iterated Function Systems (CIFS)

As we shall see, the Hausdorff dimension of the attractor set of an IFS is given by a simple result (Proposition 3), but it cannot be estimated as quite as simple when the index set I is infinite. Nevertheless, if an (infinite) IFS satisfies the so-called *conformal conditions*, we have some results that allow us to consider approximations by finite IFS.

Let $\mathcal{F} := \left\{f_i : K \subseteq \mathbb{R}^d \to K\right\}_{i\in I}$ be an IFS. In what follows, we list the conformal conditions:

(C1) For each $i \in I$, the function f_i is an injection of K into itself;

(C2) The system is uniformly contractive on K, that is, there exists some $s < 1$ such that

$$|f_i(x) - f_i(y)| \leq s\,|x - y|\,, \quad \text{for all } i \in I \text{ and for all } x, y \in K;$$

(C3) (Open Set Condition) The set K is connected, and each function satisfies $f_i(K^\circ) \subseteq K^\circ$ and $f_i(K^\circ) \cap f_j(K^\circ) = \emptyset$, for all $i, j \in I, i \neq j$;

(C4) There is an open set $V \subseteq \mathbb{R}^d$, with $K \subseteq V$, such that each f_i extends to f_i^V, a $\mathcal{C}^{1,\varepsilon}$ diffeomorphism on V, and the extensions are conformal functions, that is, the derivatives $D_x f_i^V$ satisfy $D_x f_i^V = \kappa_{x,i}\,\mathrm{Isom}_{x,i}$, where $\kappa_{x,i} \in \mathbb{R}$ and $\mathrm{Isom}_{x,i} : \mathbb{R}^d \to \mathbb{R}^d$ is an isometry, for any $x \in V$ and $i \in I$ (for more details on conformal functions, see (PESIN, 2008, Chapter 7));

(C5) The following inequality holds:

$$\inf_{x\in\partial K}\inf_{0<r<1}\frac{m_d(B_r^\circ(x)\cap K^\circ)}{m_d(B_r^\circ(x))} > 0,$$

where m_d is the d-dimensional Lebesgue measure;

(C6) There are constants $L \geq 1$ and $\alpha > 0$ such that, for every $i \in I$,

$$\left|\left\|D_x f_i^V\right\|_{\mathrm{op}} - \left\|D_y f_i^V\right\|_{\mathrm{op}}\right| \leq L \cdot \left\|(Df_i^V)^{-1}\right\|_{\mathrm{unif}}^{-1} \cdot |x - y|^\alpha\,,$$

where $x, y \in V$ (the open set in Condition (C4)), f_i^V, $i \in I$, are the extensions also established in Condition (C4), $\|\cdot\|_{\mathrm{op}}$ is the operator norm, and $\left\|(Df_i^V)^{-1}\right\|_{\mathrm{unif}} := \sup_{z\in V}\left\{\left\|(D_z f_i^V)^{-1}\right\|_{\mathrm{op}}\right\}$.

In (MAULDIN; URBAŃSKI, 1996, Lemma 2.2), we see a proof that Condition (C6) implies the following result:

(C6a) (Bounded Distortion Property) There exists $M > 1$ such that

$$\frac{1}{M} \leq \frac{\|D_x(f_\eta)\|_{\mathrm{op}}}{\|D_y(f_\eta)\|_{\mathrm{op}}} \leq M,$$

for every $\eta \in \bigcup_{j\geq 1} I^j$ and every $x, y \in V$.

An IFS that satisfies all these conditions (C1)-(C6) is said to be a *conformal IFS* or just a *CIFS*.

It is important to note that (C5) and (C6) are alternative formulations to those traditionally stated, such as in (MAULDIN, 1995). These alternative conditions are more suitable for our purposes and are discussed, for instance, in (MAULDIN, 1995, Theorem 3.2), (MAULDIN; URBAŃSKI, 1996, Lemma 2.2), and (MAULDIN; GRAF; WILLIAMS, 1987).

Now we present an important result concerning CIFS, which states that in the definition of the attractor set (3.2), the union and intersection can be interchanged, in a certain sense.

Proposition 1 ((MAULDIN, 1995, Theorem 3.1))**.** *Let $\mathcal{F} = \{f_i : i \in I\}$ be a CIFS. Then, the following relationship holds:*

$$\Delta = \bigcup_{\omega \in I^{\mathbb{N}}} \bigcap_{k\geq 1} f_{\omega|_k}(K) = \bigcap_{k\geq 1} \bigcup_{\eta \in I^k} f_\eta(K).$$

3.3.1 Pressure of a CIFS

When dealing with conformal systems, the notion of the *pressure* of the system is important (see, for instance, (MAULDIN; URBAŃSKI, 1996, Chapter 3) and (MAULDIN, 1995, Chapter 6)). We give the definition:

Definition 2 (Pressure of a CIFS)**.** *Let us define, for each $k \geq 0$, the functions $P_k : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{\infty\}$ defined by*

$$P_k(t) := \sum_{\eta \in I^k} \|D(f_\eta)\|_{\text{unif}}^t .$$

We then define the pressure of the system*, a function $P : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{\infty\}$ given by*

$$P(t) := \lim_{k\to\infty} \frac{1}{k} \log P_k(t) = \lim_{k\to\infty} \frac{1}{k} \log \left(\sum_{\eta \in I^k} \|D(f_\eta)\|_{\text{unif}}^t \right).$$

Some properties of the pressure function will help us. First of all, let us note that $P_1(t)$ is a non-increasing function. Let us define $\mathcal{P} := \{t \geq 0 \text{ such that } P_1(t) < +\infty\}$ and $\zeta := \inf \mathcal{P}$, so $\mathcal{P} = [\zeta, +\infty)$ or $\mathcal{P} = (\zeta, +\infty)$.

Now we present some properties of the pressure function:

Proposition 2 ((MAULDIN, 1995, Theorem 6.1 and 6.2))**.** *The pressure function satisfies the following properties:*

2*(a)* $P_k(t)$ *is non-increasing on* $[0, \infty)$*, strictly decreasing on* $[\zeta, \infty)$*, and continuous on* $\mathcal{P}$*, for all* $k \geq 1$*.*

2*(b)* $P(t)$ *is non-increasing on* $[0, \infty)$*, strictly decreasing on* $[\zeta, \infty)$*, and continuous on* $\mathcal{P}$*.*

2*(c)* $-t \log M + \log P_1(t) \leq P(t) \leq \log P_1(t)$*, where* $M > 1$ *is the constant from Condition (C6a).*

When the pressure function of a CIFS has a zero, we can deduce additional properties about our system; so we make the following definition:

Definition 3. *A CIFS is* regular *if there exists a zero for its pressure function, that is, there exists* $\hat{t} \geq 0$ *such that* $P(\hat{t}) = 0$*. In this case, we say that the CIFS is* $\hat{t}$*-regular.*

3.4 IFS, CIFS, and the Hausdorff Dimension of Δ

Certain results on the Hausdorff dimension of the attractor set of an IFS provide bounds in terms of the contraction constants. For a finite IFS, we have the following result:

Proposition 3 ((FALCONER, 2014, Propositions 9.6 and 9.7))**.** *Let* $\{f_1, \ldots, f_k\}$ *be an IFS on* K*, with attractor set* Δ *and satisfying the following condition: given any* $1 \leq i \leq k$*, there exists* $0 < b_i \leq c_i < 1$ *such that*

$$b_i|x - y| \leq |f_i(x) - f_i(y)| \leq c_i|x - y|, \quad \text{for all } x, y \in K.$$

Besides, let us assume that $f_i(\Delta) \cap f_j(\Delta) = \emptyset$*, for every* $i \neq j$*. Then,*

$$s \leq \dim_{\mathrm{H}}(\Delta) \leq t,$$

where s *and* t *are the unique real numbers satisfying*

$$\sum_{1 \leq i \leq k} b_i^s = 1 \quad \text{and} \quad \sum_{1 \leq i \leq k} c_i^t = 1.$$

As a consequence, we obtain the following lower bound for the Hausdorff dimension of the attractor set of a non-trivial IFS.

Corollary 1. *Let* $\mathcal{F} = \{f_i\}_{i \in I}$ *be an IFS having at least two different functions,* f_{i_1} *and* f_{i_2}*, satisfying* $b\,|x - y| \leq \left|f_{i_j}(x) - f_{i_j}(y)\right|$*, for some* $b > 0$ *and* $j = 1, 2$*. Then* $\dim_{\mathrm{H}}(\Delta) > 0$*.*

Proof. Let us consider the sub-system $\tilde{\mathcal{F}} = \{f_{i_1}, f_{i_2}\} \subseteq \mathcal{F}$. We can then apply Proposition 3 and Property (H1) on the attractor set $\tilde{\Delta}$ and note that $s \leq \dim_{\mathrm{H}}\left(\tilde{\Delta}\right) \leq \dim_{\mathrm{H}}(\Delta)$, where s is the non-negative number that satisfies $b^s + b^s = 1$. This number has to be greater than 0, since $b^0 + b^0 = 2 \neq 1$. Therefore, we have $0 < s \leq \dim_{\mathrm{H}}\left(\tilde{\Delta}\right) \leq \dim_{\mathrm{H}}(\Delta)$. □

Now, as an example, we are going to use Proposition 3 to calculate the Hausdorff dimension of the ternary Cantor set, using its description detailed in Example 1:

Example 2 (The Hausdorff dimension of the ternary Cantor set)**.** *Consider the IFS in Example 1. For both* f_0 *and* f_2*, we have that* $|f_i(x) - f_i(y)| = \frac{1}{3}|x - y|$*. By Proposition 3, we must have that the Hausdorff dimension of the Cantor set is the number* s *satisfying*

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1 \iff \frac{2}{3^s} = 1 \iff 3^s = 2 \iff s = \log_3 2 = \frac{\log 2}{\log 3}.$$

That is, the Hausdorff dimension of the ternary Cantor set is $\log 2/\log 3$*.*

The following proposition establishes a relationship between the Hausdorff dimension of the attractor set of a CIFS and its finite approximations.

Proposition 4 ((MAULDIN; URBAŃSKI, 1996, Theorem 3.15))**.** *Let* $\mathcal{F} = \{f_i\}_{i \in I}$ *be a CIFS with attractor set* Δ*. Then, the identity*

$$\dim_{\mathrm{H}}(\Delta) = \sup_{J \in \mathrm{Fin}(I)} \left\{ \dim_{\mathrm{H}}(\Delta_J) : \Delta_J \textit{ is the attractor set of the sub-system } \mathcal{F}_J = \{f_j\}_{j \in J} \right\}$$

holds, where $\mathrm{Fin}(I) \subseteq 2^I$ *is the set of finite subsets of* I*.*

The main result that concerns us is the connection between the Hausdorff dimension of the invariant set of a CIFS and its pressure function:

Proposition 5 ((MAULDIN, 1995, Theorem 7.4))**.** *Let* $\mathcal{F} = \{f_i\}_{i \in I}$ *be a CIFS with invariant set* Δ*. If* $P(\hat{t}) = 0$*, then* $\hat{t}$ *is the only zero of* P *and* $\dim_{\mathrm{H}}(\Delta) = \hat{t}$*.*

4 Proof of the Main Theorem (of Part I)

The local dynamics of a sliding Shilnikov connection is captured by the first-return map π, defined on a subset of $\partial M^{s,e}$ within a small neighborhood $U \subseteq \partial M^{s,e}$ of the visible fold-regular point q. Thus, our initial task is to construct this map. Subsequently, based on its construction, we will relate this map to the theory of CIFS by showing that it can be branched into countably many smooth maps such that the functions corresponding to their inverses constitute a CIFS.

For clarity, we assume the hypotheses of Theorem A1 in this section. Without loss of generality, we will focus exclusively on the sliding case, where $p \in M^s$ and $q \in \partial M^s$, as depicted in Figure 2.

4.1 Construction of the First-Return Map

Let us consider a sliding Shilnikov connection Γ (see Definition 1) with a hyperbolic pseudo-saddle-focus $p \in M^s$ and passing through a visible fold-regular point $q \in \partial M^s$, which is a visible fold point of X. Recall that, for $u \in V \subseteq \mathbb{R}^3$ and $w \in \overline{M^s}$, the flows of X and $\tilde{Z}$ are denoted, respectively, by $\varphi_X^t(u)$ and $\varphi_{\tilde{Z}}^t(w)$.

First, for $r > 0$, let us define the set

$$\gamma_r := \overline{(B_r^\circ(q) \cap \partial M^s)}.$$

Remark (R2) implies that, for sufficiently small $r > 0$, γ_r is a smooth curve of visible fold-regular points of Z, which are also visible fold points for X. In addition, Remark (R1) states that γ_r is a transversal section of the sliding vector field $\tilde{Z}$.

Second, from Definition 1, there exists $t_q > 0$ such that $\varphi_X^{t_q}(q) = p$, in particular, $g(\varphi_X^{t_q}(q)) = 0$, and $g(\varphi_X^t(q)) \neq 0$, for any $t \in (0, t_q)$. Since

$$\frac{d}{dt} g(\varphi_X^t(q))\Big|_{t=t_q} = Xg(p) \neq 0,$$

the Implicit Function Theorem implies the existence of $r > 0$, and a function $t_X(w)$, defined in γ_r, satisfying

$$t_X(q) = t_q,\ g\Big(\varphi_X^{t_X(w)}(w)\Big) = 0,\ \text{and } g\left(\varphi_X^t(w)\right) \neq 0 \text{ for all } t \in (0, t_X(w)).$$

Thus, define

$$\mu_r := \left\{\varphi_X^{t_X(w)}(w) : w \in \gamma_r\right\}.$$

Notice that the flow of the vector field X maps γ_r onto μ_r by means of the following diffeomorphism

$$\begin{aligned} \theta_X : \gamma_r &\to \mu_r \\ w &\mapsto \theta_X(w) := \varphi_X^{t_X(w)}(w), \end{aligned} \tag{4.1}$$

implying that μ_r is a smooth curve containing p. Furthermore, since $p \in \mu_r \subseteq M^s$ is a hyperbolic focus of $\tilde{Z}$, we can see that $\mu_r \setminus \{p\}$ is transversal to $\tilde{Z}$. Indeed, if this were not the case, the vector $0 \neq v \in T_pM$ tangent to μ_r at p would be a real eigenvector of $D\tilde{Z}(p)$ restricted to T_pM. However, this is impossible because the restriction of $D\tilde{Z}(p)$ to T_pM has a pair of complex conjugate eigenvalues.

Third, consider the backward saturation, S_r, of γ_r induced by $\varphi_{\tilde{Z}}$, that is,

$$S_r := \bigcup_{t \geq 0} \varphi_{\tilde{Z}}^{-t}(\gamma_r).$$

From Definition 1, we have $\lim\limits_{t\to-\infty} \varphi_{\tilde{Z}}^t(q) = p$. Therefore, taking into account that γ_r is a transversal section of $\tilde{Z}$ and that $p \in \mu_r \subseteq M^s$ is a hyperbolic focus of $\tilde{Z}$, the Implicit Function Theorem can be used to ensure that, for sufficiently small $r > 0$, $\varphi_{\tilde{Z}}^t(w)$ converges to p as $t \to -\infty$ for all $w \in \gamma_r$. Now, since $\mu_r \setminus \{p\}$ is transversal to $\tilde{Z}$ and $p \in \mu_r \subseteq M^s$ is a hyperbolic focus of $\tilde{Z}$, we deduce that

$$S_r \cap \mu_r = \bigcup_{i \geq 0} \left(L_i^{\mu_r} \cup R_i^{\mu_r} \right),$$

where the collections $\{L_i^{\mu_r}\}_{i\geq 0}, \{R_i^{\mu_r}\}_{i\geq 0} \subseteq \mu_r$, of the connect components of $S_r \cap \mu_r$, are located in opposite sides of $p \in \mu_r$, $L_i^{\mu_r} \cap L_j^{\mu_r} = \emptyset = R_i^{\mu_r} \cap R_j^{\mu_r}$, if $i \neq j$, and $L_i^{\mu_r}, R_i^{\mu_r}$ converge to $\{p\}$ in the Hausdorff distance as i increases (see Figure 4). In addition, for each $\xi \in S_r \cap \mu_r$, there exists $t_{\tilde{Z}}(\xi) > 0$ satisfying $\varphi_{\tilde{Z}}^{t_{\tilde{Z}}(\xi)}(\xi) \in \gamma_r$ and $\varphi_{\tilde{Z}}^t(\xi) \in M^s$, for all $t \in (0, t_{\tilde{Z}}(\xi))$. This induces the following maps from each $J \in \{L_i^{\mu_r}\}_{i\geq 0} \cup \{R_i^{\mu_r}\}_{i\geq 0}$ into γ_r,

$$\begin{aligned} \theta_{\tilde{Z}}^J : J &\to \gamma_r \\ \xi &\mapsto \pi(w) := \varphi_{\tilde{Z}}^{t_{\tilde{Z}}(\xi)}(\xi). \end{aligned} \tag{4.2}$$

For $J \in \{L_i^{\mu_r}\}_{i\geq 1} \cup \{R_i^{\mu_r}\}_{i\geq 1}$, such maps are diffeomorphisms. However, for $J \in \{L_0^{\mu_r}, R_0^{\mu_r}\}$, the induced map $\theta_{\tilde{Z}}^J$ may not be surjective (see Figure 4).

Finally, for each $i \geq 0$, we define

$$L_i^{\gamma_r} := \theta_X^{-1}(L_i^{\mu_r}) \subseteq \gamma_r \quad \text{and} \quad R_i^{\gamma_r} := \theta_X^{-1}(R_i^{\mu_r}) \subseteq \gamma_r,$$

where θ_X is the diffeomorphism given by (4.1). This ensures that the properties $L_i^{\gamma_r} \cap L_j^{\gamma_r} = \emptyset$ and $R_i^{\gamma_r} \cap R_j^{\gamma_r} = \emptyset$ hold for $i \neq j$, and that $L_i^{\gamma_r}, R_i^{\gamma_r}$ converge to $\{q\}$ in the Hausdorff distance as i increases. Thus, we define

$$\mathcal{W}_r := \bigcup_{i \geq 0} \left(L_i^{\gamma_r} \cup R_i^{\gamma_r} \right) \subseteq \gamma_r.$$

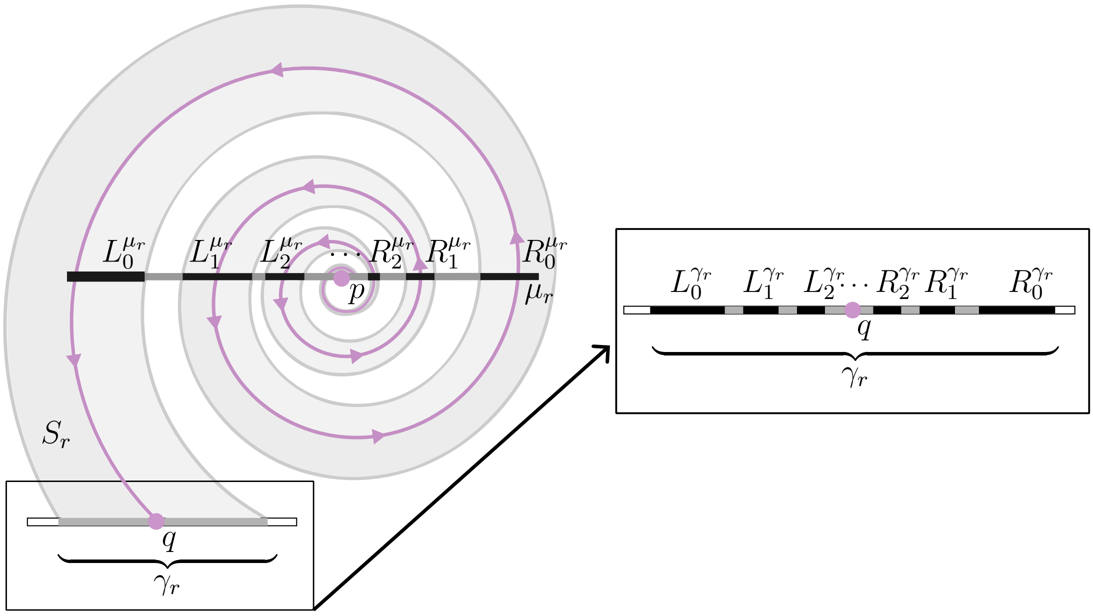

Figure 4 – On the left, we have a representation of the sliding dynamics, illustrating the intersection between the backward saturation S_r and μ_r, which generates the collections $\{L_i^{\mu_r}\}_{i\geq 0}$ and $\{R_i^{\mu_r}\}_{i\geq 0}$. On the right, a zoomed-in view of γ_r shows the collections $\{L_i^{\gamma_r}\}_{i\geq 0}$ and $\{R_i^{\gamma_r}\}_{i\geq 0}$.

Therefore, for $r > 0$ sufficiently small, the first-return map

$$\begin{aligned} \pi : \mathcal{W}_r &\to \gamma_r \\ w &\mapsto \pi(w) := \varphi_{\tilde{Z}}^{t_{\tilde{Z}}(\theta_X(w))}(\theta_X(w)) \end{aligned} \tag{4.3}$$

is well-defined.

4.2 The First-Return Map as a CIFS

We will now demonstrate that π can be branched into countably many smooth maps exhibiting expanding behavior, such that the functions corresponding to their inverses are contractions (see Figure 5) and constitute a CIFS.

Consider the index set $\mathcal{J} = \mathcal{J}_r := \{L_i^{\gamma_r}\}_{i\geq 0} \cup \{R_i^{\gamma_r}\}_{i\geq 0}$. Note that the function π maps each $J \in \mathcal{J}$ onto γ_r, with the possible exceptions of $L_0^{\gamma_r}$ and $R_0^{\gamma_r}$, where π may not be surjective. Again, this is not an issue, as we are interested in the local behavior of the map π. Specifically, we will focus on the restriction of π to $U \cap \mathcal{W}_r$, where $U \subseteq \gamma_r$ is a neighborhood of q satisfying the following condition:

(T) If $J \in \mathcal{J}$ and $J \cap U \neq \emptyset$, then $J \subseteq U$ and $\pi(J) = \gamma_r$.

For a given neighborhood $U \subseteq \gamma_r$ of q satisfying (T), we denote $\mathcal{J}_U = \{J \in \mathcal{J} : J \subseteq U\}$.

For each $J \in \mathcal{J}_U$, the restriction $\pi_J := \pi|_J : J \to \gamma_r$ is a diffeomorphims, thus its inverse $\psi_J := \pi_J^{-1} : \gamma_r \to J$ is well-defined. In what follows we will investigate the behavior of π_J and ψ_J.

We start by analyzing more deeply the sliding dynamics close to the unstable hyperbolic focus of the sliding vector field $\tilde{Z}$. Since $\mu_R \to \{p\}$ as $R \to 0$, p is and unstable hyperbolic focus of $\tilde{Z}$, and $\mu_R \setminus \{p\}$ is transverse to $\tilde{Z}$, we can establish the existence of $\hat{R} > 0$ such that for any $R \in (0, \hat{R})$, there exists $\overline{R} > 0$ for which a $\mathcal{C}^{1,1}$ first-return map of $\tilde{Z}$ on μ_R, denoted by $\rho : \mu_R \to \mu_{\overline{R}}$, is well-defined. The map ρ is given by $\rho(\xi) := \varphi_{\tilde{Z}}^{t_\rho(\xi)}(\xi)$, where $t_\rho(\xi)$ is the time taken for the trajectory of $\tilde{Z}$, starting at ξ, to turn around the focus p and return to $\mu_{\overline{R}}$ on the same side as ξ relative to p.

Now, we are going to need the following result, for which a proof can be found in (BARREIRA; VALLS, 2007):

Proposition 6 (Hartman-Grobman Theorem, (HARTMAN, 1960), (RODRIGUES; SOLÀ–MORALES, 2012, Theorem 1))**.** *Let $V \subseteq \mathbb{R}^n$ be a neighborhood of the origin. Also, consider $M \in GL_n(\mathbb{R})$ such that $\|M\| < 1$, and $F \in \mathcal{C}^{1,1}(V, \mathbb{R}^n)$ such that $F(0) = 0$ and $D_0F = 0$.*

Then, for the (possibly non-linear) map $f(x) = Mx + F(x)$, there exists a conjugacy map $g(x) = x + h(x)$ in a neighborhood of the origin, with $h \in \mathcal{C}^{1,\beta}$ (for some $0 < \beta < 1$), with $h(0)$, $D_0h = 0$, and such that $gfg^{-1} = M$.

As similarly pointed out by (NOVAES; PONCE; VARÃO, 2017, Proposition 2), since p is a hyperbolic unstable fixed point of ρ, we can apply Proposition 6 with the convenient changes (for example, reversing the direction of time, we have the same result changing the hypothesis of the matrix norm to $\|M\| > 1$) in order to get a local $\mathcal{C}^{1,\beta}$-linearization of ρ, for some $\beta > 0$, in some neighborhood $\mathcal{O} \subseteq \mu_{\hat{R}}$. That is, there exists a diffeomorphism $H : \mathcal{O} \to H(\mathcal{O})$ of class $\mathcal{C}^{1,\beta}$ with $H(p) = 0$, such that $\rho(\xi) = H^{-1}(\lambda H(\xi))$, with $\lambda > 1$, for every $\xi \in \mu_R \cap \mathcal{O}$. Notice that, if $\rho^{k-1}(\xi) \in \mathcal{O}$, then $\rho^k(\xi) = H^{-1}(\lambda^k H(\xi))$.

Now, in the definition of the first-return map (4.3), take $r \in (0, \hat{R})$ such that $\mu_r \subseteq \mathcal{O}$. Denote by ρ_X the restriction of θ_X, given by (4.1), to $\mathcal{W}_r$, which is a diffeomorphim onto its image $S_r \cap \mu_r$. In addition, given $w \in \mathcal{W}_r$, since $\rho_X(w) \in S_r \cap \mu_r \subseteq \mathcal{O}$, we can define an integer $c(w)$ corresponding to the number of times that ρ must be applied to $\rho_X(w)$ before it enters $L_1^{\mu_r} \cup R_1^{\mu_r}$, that is,

$$c(w) = i - 1 \text{ for } w \in L_i^{\gamma_r} \cup R_i^{\gamma_r}.$$

Note that $c(w) = -1$ if $w \in L_0^{\gamma_r} \cup R_0^{\gamma_r}$. However, this is not an issue, as we will consider w in a small neighborhood of q. Since the function c is constant in each $J \in \mathcal{J}$, we denote $c_J := c(w)$, for $w \in J$, that is,

$$c_J = i - 1, \text{ if } J \in \{L_i^{\gamma_r}, R_i^{\gamma_r}\}, \quad i \geq 0. \tag{4.4}$$

Finally, define the auxiliary map

$$\rho_{\tilde{Z}}^{J} := \begin{cases} \theta_{\tilde{Z}}^{L_1^{\mu_r}}(\xi) & \text{if } J \in \{L_i^{\gamma_r}\}_{i\geq 0}, \\ \theta_{\tilde{Z}}^{R_1^{\mu_r}}(\xi) & \text{if } J \in \{R_i^{\gamma_r}\}_{i\geq 0}. \end{cases}$$

Notice that, for $J \in \{L_i^{\gamma_r}\}_{i\geq 0}$, we have $\rho_{\tilde{Z}}^{J}|_{L_1^{\mu_r}} = \theta_{\tilde{Z}}^{L_1^{\mu_r}}$, and for $J \in \{R_i^{\gamma_r}\}_{i\geq 0}$, we have $\rho_{\tilde{Z}}^{J}|_{R_1^{\mu_r}} = \theta_{\tilde{Z}}^{R_1^{\mu_r}}$. These maps are diffeomorphisms, respectively, from $L_1^{\mu_r}$ and $R_1^{\mu_r}$ onto γ_r, induced by $\tilde{Z}$ as defined in (4.2). It is important to mention that the guaranteed diffeomorphic nature of these maps is the reason we chose $L_1^{\mu_r} \cup R_1^{\mu_r}$ to define $c(w)$, rather than $L_0^{\mu_r} \cup R_0^{\mu_r}$.

With the notation introduced above, for $J \in \mathcal{J}_U$, the restricted first-return map π_J and its inverse ψ_J are given, respectively, by

$$\begin{aligned} \pi_J(w) &:= \rho_{\tilde{Z}}^{J} \circ \rho^{c_J} \circ \rho_X(w) = \rho_{\tilde{Z}}^{J} \circ H^{-1}(\lambda^{c_J} H \circ \rho_X(w)), \\ \psi_J(x) &:= \rho_X^{-1} \circ \rho^{-c_J} \circ (\rho_{\tilde{Z}}^{J})^{-1}(x) = \rho_X^{-1} \circ H^{-1}\Big(\lambda^{-c_J} H \circ (\rho_{\tilde{Z}}^{J})^{-1}(x)\Big). \end{aligned} \tag{4.5}$$

Proposition 7. *There exists a neighborhood $U \subseteq \gamma_r$ of q satisfying (T) for which $0 < \epsilon_J \leq |\psi_J'(x)| \leq s < 1$, for every $J \in \mathcal{J}_U$ and for every $x \in \gamma_r$, and, consequently, $|\pi'(w)| \geq 1/s > 1$, for every $w \in U \cap \mathcal{W}_r$.*

Proof. Let $J \in \mathcal{J}_U$. First, since ψ_J is a diffeomorphism on a compact set γ_r, then

$$\epsilon_J := \inf_{x\in\gamma_r} \psi_J'(x) > 0.$$

Now, since $\rho_{\tilde{Z}}^{J}$, ρ_X and $H|_{\mu_r}$ are diffeomorphisms on compact sets, we can set

$$A_{\min} := \min_{\xi\in J} \left|(\rho_{\tilde{Z}}^{J})'(\xi)\right| \cdot \min_{z\in H(\mu_r)} \left|(H^{-1})'(z)\right| \cdot \min_{\xi\in\mu_r} |H'(\xi)| \cdot \min_{x\in\gamma_r} |\rho_X'(x)| > 0. \tag{4.6}$$

Thus, taking the expression (4.5) into account, we get

$$s := \left|\sup_{x\in\gamma_r} \psi_J'(x)\right| \leq \frac{1}{A_{\min}\lambda^{c_J}}. \tag{4.7}$$

Hence, we can take a neighborhood $U \subseteq \gamma_r$ of q satisfying (T) for which $c_J > -\log_\lambda A_{\min}$ for every $J \in \mathcal{J}_U$. This implies that $s < 1$.

Finally, since

$$\pi_J'(w) = \frac{1}{\psi_J'(\pi_J(w))},$$

we conclude that $\pi_J'(w) \geq s$ for every $w \in J$ and $J \in \mathcal{J}_U$. Therefore, $|\pi'(w)| \geq \dfrac{1}{s} > 1$, for every $w \in U \cap \mathcal{W}_r$. □

Remark 3. *The combination of Proposition 7 and the compactness of each set $J \in \mathcal{J}$ ensures that the collection of functions*

$$\Psi_U := \{\psi_J : J \in \mathcal{J}_U\}$$

forms a countable IFS, thereby admitting an attractor set.

We need to clarify an important detail. Formally, the functions of the IFS Ψ_U are defined on the curve γ_r, but working with them in this context can be cumbersome. Instead, we can consider an orientation-preserving diffeomorphism $h : \gamma_r \to [-1,1]$ such that $h(q) = 0$. By defining $\tilde{\psi}_J := h \circ \psi_J \circ h^{-1} : [-1,1] \to h(J)$, we form the set $\tilde{\Psi}_U := \left\{\tilde{\psi}_J : J \in \mathcal{J}, J \subseteq U\right\}$, which is an IFS. Additionally, we have $h(L_i^{\gamma_r}) \subseteq [-1,0)$ and $h(R_i^{\gamma_r}) \subseteq (0,1]$. This provides a nice graphical representation of the first-return map π in Figure 5. Since Property (H3) implies that the Hausdorff dimension is invariant under bi-Lipschitz maps, which is the case for h, this transformation does not change any of the properties we are investigating. Therefore, from now on, we will assume, by an abuse of notation, that the functions in Ψ_U are defined in the interval $[-1,1]$ and that the sets $J \in \mathcal{J}_U$ are subintervals of $[-1,1]$.

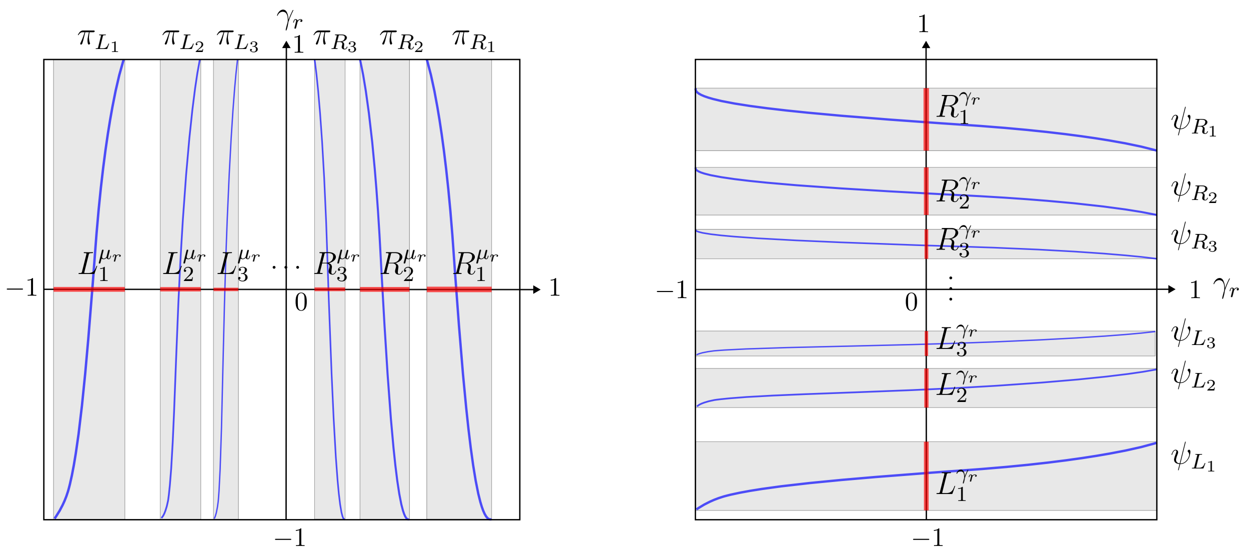

Figure 5 – Representation of the branches of the first-return map π_U, starting with $i = 1$, and the IFS Ψ_U, respectively.

In the following, we will prove that Ψ_U is indeed a CIFS. This is important because it allows the use of Proposition 1 to investigate the attractor set Δ_U of Ψ_U, defined in (3.2).

Proposition 8. *There exists a neighborhood $U \subseteq \gamma_r$ of q satisfying Condition (T) for which the Ψ_U is conformal.*

Proof. In what follows, we will examine the conformal conditions individually.

Let $U \subseteq \gamma_r$ be the neighborhood of q given by Proposition 7.

We start by the most immediate conditions. First, for $J \in \mathcal{J}_U$, ψ_J is a diffeomorphism, in particular injective, so that Condition (C1) holds. Also, Proposition 7 provides $|\psi_J'(x)| \leq s < 1$, for every $J \in \mathcal{J}_U$ and for every $x \in \gamma_r$, which directly implies Condition (C2). Moreover, γ_r is connected and the images of each function ψ_J, $J \in \mathcal{J}_U$, are mutually disjoint and, therefore, Condition (C3) is satisfied. Finally, Condition (C5) holds because $K = \gamma_r$ is diffeomorphic to the closed interval $[-1, 1]$.

It remains to show that Conditions (C4) and (C6) hold.

Consider the set $\gamma_{r+\delta} = \overline{(B^\circ_{r+\delta}(q) \cap \partial M^s)}$ for $\delta > 0$, and for each $J \in \mathcal{J}_U$, define the extension $\psi^\delta_J : \gamma_{r+\delta} \to \gamma_{r+\delta}$ of ψ_J by joining affine functions with slopes equal to the lateral derivatives of ψ_J at the endpoints of γ_r on each side. This extension is a diffeomorphism, and since $\gamma_{r+\delta}$ is compact, the derivative $(\psi^\delta_J)'$ is Lipschitz continuous and, consequently, ψ^δ_J is also $\mathcal{C}^{1,\varepsilon}$. Furthermore, by Proposition 7, $(\psi^\delta_J)'(x) \neq 0$ for every $x \in \gamma_r$, and therefore, for every $x \in \gamma_{r+\delta}$. Finally, since we are dealing with functions defined on a one-dimensional set, the extension $(\psi^\delta_J)|_{\gamma^\circ_{r+\delta}}$ is trivially conformal, satisfying the requirements for Condition (C4) to hold.

At last, we will show that Condition (C6) holds. Notice that, for $x, y \in \gamma_r$, we have

$$\left\| (\psi'_J)^{-1} \right\|^{-1}_{\text{unif}} = \left(\sup_{x \in \gamma_r} \left| (\psi'_J(x))^{-1} \right| \right)^{-1} = \left(\frac{1}{\inf_{x \in \gamma_r} |\psi'_J(x)|} \right)^{-1} = \inf_{x \in \gamma_r} |\psi'_J(x)| \,.$$

By applying the Chain Rule in (4.5), we deduce that

$$\psi'_J(x) = \left[\left(\rho^{-1}_X\right)' (f_{H^{-1}}(x)) \right] \cdot \left[\left(H^{-1}\right)' (f_\lambda(x)) \right] \cdot \lambda^{-c_J} \cdot \left[H' \left(\left(\rho^J_{\tilde{Z}}\right)^{-1} (x) \right) \right] \cdot \left[\left(\left(\rho^J_{\tilde{Z}}\right)^{-1} \right)' (x) \right],$$

where $f_{H^{-1}} := H^{-1} \left(\lambda^{-c_J} H \circ \left(\rho^J_{\tilde{Z}}\right)^{-1} \right)$ and $f_\lambda := \lambda^{-c_J} H \circ \left(\left(\rho^J_{\tilde{Z}}\right)^{-1} \right)$.

Similarly to (4.6), given that each function in the composition is a diffeomorphism, we can set

$$C := \left(\min_{\zeta \in \mu_r} \left| \left(\rho^{-1}_X\right)' (\zeta) \right| \right) \cdot \left(\min_{z \in H(\mu_r)} \left| (H^{-1})'(z) \right| \right) \cdot \left(\min_{\zeta \in \mu_r} |H'(\zeta)| \right) \cdot \left(\min_{x \in \gamma_r} \left| \left(\left(\rho^J_{\tilde{Z}}\right)^{-1} \right)' (x) \right| \right),$$

which is greater than 0.

This implies that

$$\inf_{x \in \gamma_r} |\psi'_J(x)| \geq C \, \lambda^{-c_J}. \tag{4.8}$$

Let us denote $T_J(x) := \lambda^{c_J} \cdot \psi_J(x)$. Notice that $T_J(x)$ is $\mathcal{C}^{1,\alpha}$, for some $\alpha > 0$, since each of the functions $\rho^{-1}_{\tilde{Z}}$ and ρ^{-1}_X is a $\mathcal{C}^{1,1}$ diffeomorphism (the same class of differentiability of the fields X and Y that induce them), and H, H^{-1} are $\mathcal{C}^{1,\beta}$ maps, and in compact domains, products and compositions preserve the Hölder condition, with some adjustments in the multiplicative and exponential constants (this can be proven using many times the Lipschitz and Hölder conditions). Keeping this in mind and applying in (4.8), we deduce the following inequality, for all $x, y \in \gamma_r$:

$$\Big| |\psi'_J(x)| - |\psi'_J(y)| \Big| \leq |\psi'_J(x) - \psi'_J(y)| = \left| \lambda^{-c_J} \cdot T'_J(x) - \lambda^{-c_J} \cdot T'_J(y) \right| =$$

$$= \lambda^{-c_J} \cdot |T'_J(x) - T'_J(y)| \leq \frac{\inf_{v \in \gamma_r} |\psi'_J(v)|}{C} \cdot D \, |x - y|^\alpha = C^{-1} D \cdot \left\| (\psi'_J)^{-1} \right\|^{-1}_{\text{unif}} \cdot |x - y|^\alpha \,,$$

for some $D > 1, \alpha > 0$. We can, then, extend the inequality for ψ_J^δ and $x, y \in \gamma_{r+\delta}^\circ$, since the derivatives $(\psi_J^\delta)'$ in $\gamma_{r+\delta}$ attain values already obtained by ψ_J' in γ_r (by construction), so we finally deduce (C6).

Therefore, our IFS satisfies all the necessary conditions, making it CIFS. □

Now, we will demonstrate that the attractor set of CIFS Ψ_U, given by (3.2), coincides with the invariant set of the restricted first-return map $\pi_U := \pi|_{\mathcal{W}_r \cap U}$ given by (2.4).

Proposition 9. *The invariant set Λ_U of π_U coincides with the attractor set Δ_U of the CIFS Ψ_U.*

Proof. First, consider the sets

$$\Lambda_U^k := \bigcap_{0 \le i \le k} \pi^{-i}\left(\mathcal{W}_r \cap U\right), \text{ for } k \ge 0, \quad \text{and} \quad \Delta_U^k := \bigcup_{\eta \in \mathcal{J}_U^k} \psi_\eta\left(\gamma_r\right), \text{ for } k \ge 1. \tag{4.9}$$

Therefore, the invariant set Λ_U of π_U, as defined by (2.4), can be expressed as

$$\Lambda_U = \bigcap_{k \ge 0} \pi^{-k}\left(\mathcal{W}_r \cap U\right) = \bigcap_{k \ge 0} \Lambda_U^k. \tag{4.10}$$

Furthermore, by Proposition 1, the attractor set of the CIFS Ψ_U is given by

$$\Delta_U = \bigcap_{k \ge 1} \bigcup_{\eta \in \mathcal{J}_U^k} \psi_\eta\left(\gamma_r\right) = \bigcap_{k \ge 1} \Delta_U^k. \tag{4.11}$$

Let us see that the following relationship holds $\Lambda_U^k = \Delta_U^{k+1}$, for every $k \ge 0$. Indeed,

$$\begin{aligned}
x \in \Lambda_U^k &\iff \pi^i(x) \in J_i \subseteq \mathcal{W}_r \cap U \text{ where } J_i \in \mathcal{J}_U \text{ for } 0 \le i \le k, \text{ and } \pi^{k+1}(x) \in \gamma_r \\
&\iff \pi_{J_k} \circ \pi_{J_{k-1}} \circ \ldots \circ \pi_{J_0}(x) = y, \text{ for some } y \in \gamma_r \text{ and } J_i \in \mathcal{J}_U \text{ for } 0 \le i \le k \\
&\iff \psi_{J_k}^{-1} \circ \psi_{J_{k-1}}^{-1} \circ \ldots \circ \psi_{J_0}^{-1}(x) = y, \text{ for some } y \in \gamma_r \text{ and } J_i \in \mathcal{J}_U \text{ for } 0 \le i \le k \\
&\iff x = \psi_{J_0} \circ \ldots \circ \psi_{J_{k-1}} \circ \psi_{J_k}(y), \text{ for some } y \in \gamma_r \text{ and } J_i \in \mathcal{J}_U \text{ for } 0 \le i \le k \\
&\iff x \in \psi_\eta\left(\gamma_r\right), \text{ for some } \eta \in \mathcal{J}_U^{k+1} \\
&\iff x \in \Delta_U^{k+1}.
\end{aligned}$$

Thus, taking (4.10) and (4.11) into account, we conclude that $\Lambda_U = \Delta_U$. □

4.3 The Closure of Λ_U

Now, we turn our attention to $\overline{\Lambda_U}$, the closure of Λ_U. The characterization $\Lambda_U = \Delta_U = \bigcap_{k \ge 1} \Delta_U^k$ provided by Proposition 9 allows to explore its structure and properties in detail. Notice that the sets Δ_U^k, $k \geq 1$, as given by (4.9), satisfy the following recursive property

$$\Delta_U^{k+1} = \bigcup_{\eta \in \mathcal{J}_U^{k+1}} \psi_\eta(\gamma_r) = \bigcup_{J \in \mathcal{J}_U} \psi_J(\Delta_U^k), \quad \text{for any } k \geq 1. \tag{4.12}$$

In order to provide a characterization for the closure $\overline{\Delta_U}$, define

$$Q_U^k := \left(\bigcup_{j=1}^{k-1} \bigcup_{\eta \in \mathcal{J}_U^j} \psi_\eta(\{q\}) \right) \cup \{q\}, \text{ for } k \geq 1. \tag{4.13}$$

Notice that $Q_U^1 = \{q\}$, and that the sets Q_U^k, for $k \geq 1$, also satisfy a recursive property

$$Q_U^{k+1} = \left(\bigcup_{J \in \mathcal{J}_U} \psi_J(Q_U^k) \right) \cup \{q\}. \tag{4.14}$$

Finally, define

$$\Delta_U^q := \bigcap_{k \geq 1} \left(\Delta_U^k \cup Q_U^{k-1} \right).$$

Clearly, since $\Delta_U^k \subseteq \Delta_U^k \cup Q_U^{k-1}$, we must have $\Delta_U \subseteq \Delta_U^q$. In the next two propositions, we will show that Δ_U^q is compact and $\overline{\Delta_U} = \Delta_U^q$.

Proposition 10. *The sets $\Delta_U^k \cup Q_U^k$, for $k \geq 1$, and Δ_U^q are compact.*

Proof. The proof will be done by induction on k.

For $k = 1$, we have that $\Delta_U^1 \cup Q_U^1 = (\mathcal{W}_r \cap U) \cup \{q\}$, which is compact.

Now, assume as inductive hypothesis that the set $\Delta_U^k \cup Q_U^k$ is compact, for some $k > 1$. In what follows, we will show that $\Delta_U^{k+1} \cup Q_U^{k+1}$ is also compact.

First, notice that the relationships (4.12) and (4.14) imply

$$\Delta_U^{k+1} \cup Q_U^{k+1} = \left(\bigcup_{J \in \mathcal{J}_U} \psi_J(\Delta_U^k) \right) \cup \left(\bigcup_{J \in \mathcal{J}_U} \psi_J(Q_U^k) \right) \cup \{q\} = \left(\bigcup_{J \in \mathcal{J}_U} \psi_J(\Delta_U^k \cup Q_U^k) \right) \cup \{q\}.$$

Let $\mathcal{V}$ be an open cover of $\Delta_U^{k+1} \cup Q_U^{k+1}$, so in particular it has an open set $V_q \in \mathcal{V}$ that covers q. Denote by $\tilde{\mathcal{J}}_U = \{J \in \mathcal{J}_U : J \subseteq V_q\}$. Since $L_i^{\gamma_r}, R_i^{\gamma_r}$ are converging to $\{q\}$ as i increases, we conclude that $\mathcal{J}_U \setminus \tilde{\mathcal{J}}_U$ is a finite set. Furthermore, since for $J \in \mathcal{J}_U$, we have $\psi_J(\Delta_U^k \cup Q_U^k) \subseteq J$, it follows that $\psi_J(\Delta_U^k \cup Q_U^k) \subseteq V_q$ for every $J \in \tilde{\mathcal{J}}_U$. Finally, from the inductive hypothesis, we have that

$$\bigcup_{J \in \mathcal{J}_U \setminus \tilde{\mathcal{J}}_U} \psi_J(\Delta_U^k \cup Q_U^k)$$

is compact since it is a finite union of compact sets. Therefore, it has finite subcover $\mathcal{V}' \subseteq \mathcal{V}$. Hence, $\mathcal{V}' \cup \{V_q\} \subseteq \mathcal{V}$ is a finite subcover of $\Delta_U^{k+1} \cup Q_U^{k+1}$, which implies that it is compact.

In addition, Δ_U^q is compact since it is an intersection of compact sets. $\square$

Proposition 11. *We have that* $\Delta_U^q = \overline{\Delta_U}$.

Proof. First of all, since $\Delta_U \subseteq \Delta_U^q$, Proposition 10 implies that $\overline{\Delta_U} \subseteq \Delta_U^q$. Thus, it remains to show that $\Delta_U^q \subseteq \overline{\Delta_U}$ to conclude that $\Delta_U^q = \overline{\Delta_U}$.

Let $x \in \Delta_U^q = \bigcap_{k\geq 1} \left(\Delta_U^k \cup Q_U^k\right)$, that is, $x \in \Delta_U^k \cup Q_U^k$ for all $k \geq 1$. First, if $x \in \Delta_U^k$, for all $k \geq 1$, then $x \in \Delta_U \subseteq \overline{\Delta_U}$ and there is nothing to prove. Thus, suppose that $x \in Q_U^\kappa$, for some $1 \leq \kappa \leq k$. If $\kappa = 1$, then $x = q \in \overline{\Delta_U}$. Now, assume that $\kappa \geq 2$. This means that there exists $\overline{\eta} = (J_1, \ldots, J_\ell) \in \mathcal{J}_U^\ell$, for some $\ell \leq \kappa$, such that $x = \psi_{\overline{\eta}}(q) = \psi_{(J_1,\ldots,J_\ell)}(q)$. Consider a sequence $(x_i)_{i\geq 0}$ in Δ_U converging to q. Since $\psi_{\overline{\eta}}$ is continuous, we have that

$$x = \psi_{\overline{\eta}}(q) = \psi_{\overline{\eta}}\left(\lim_{i\to\infty} x_i\right) = \lim_{i\to\infty} \psi_{\overline{\eta}}(x_i).$$

Since $\psi_{\overline{\eta}}(x_i) \in \Delta_U$, because Δ_U is π-invariant, we conclude that x the limit of a sequence of elements of Δ_U, therefore $x \in \overline{\Delta_U}$. $\square$

Now, recall that

$$Q_U = \left(\bigcup_{k\geq 0} \pi^{-k}(q)\right) \cap U = \bigcup_{k\geq 0} \pi_U^{-k}(q). \tag{4.15}$$

as defined in the statement of Theorem A1. The next proposition details the structure of $\overline{\Lambda_U}$.

Proposition 12. *The set* Q_U *is countable and* $\overline{\Delta_U} = \Delta_U \mathbin{\dot\cup} Q_U$ *or, equivalently,* $\overline{\Lambda_U} = \Lambda_U \mathbin{\dot\cup} Q_U$.

Proof. First of all, we can easily see that

$$Q_U^k = \bigcup_{j=0}^{k-1} \pi_U^{-j}(\{q\}), \text{ for } k \geq 1. \tag{4.16}$$

This implies that the set Q_U, as defined in (4.15), can be written as

$$Q_U = \bigcup_{k\geq 1} Q_U^k. \tag{4.17}$$

Notice that, since Ψ_U is a countable set of functions and taking the relationships (4.17) and (4.13) into account, we conclude that Q_U is a countable union of countable sets, therefore it is countable.

In what follows, we proceed with of proof of the equality $\overline{\Delta_U} = \Delta_U \mathbin{\dot\cup} Q_U$.

First, we will prove that $\overline{\Delta_U} \subseteq \Delta_U \cup Q_U$. Let $x \in \overline{\Delta_U} = \bigcap_{k\geq 1} \left(\Delta_U^k \cup Q_U^k\right)$, that is, $x \in \Delta_U^k \cup Q_U^k$, for all $k \geq 1$. If $x \in \Delta_U^k$, for all $k \geq 1$, taking (4.11) into account, we have that $x \in \Delta_U \subseteq \Delta_U \cup Q_U$. Otherwise, for some $\ell \geq 1$ and taking (4.17) into account, $x \in Q_U^\ell \subseteq Q_U \subseteq \Delta_U \cup Q_U$. Therefore, $\overline{\Delta_U} \subseteq \Delta_U \cup Q_U$.

Now, for proving the opposite inclusion, $\Delta_U \cup Q_U \subseteq \overline{\Delta_U}$, let $x \in \Delta_U \cup Q_U$. Since $\Delta_U \subseteq \overline{\Delta_U}$, it only remains to consider the case $x \in Q_U$. Taking (4.17) into account, let $\kappa \geq 1$ be the first non-negative integer satisfying $x \in Q_U^\kappa$. If $\kappa = 1$, then $x = q \in Q_U^k$, for all $k \geq 1$, so $x \in \bigcap\limits_{k \geq 1} \left(\Delta_U^k \cup Q_U^{k-1}\right) = \overline{\Delta_U}$. On the other hand, assume $\kappa > 1$. Notice that, from (4.16), $x \in Q_U^\kappa \subseteq Q_U^k$ for every $k \geq \kappa$. Thus, let us prove that $x \in \Delta_U^k$ for every $1 \leq k \leq \kappa$. Indeed, from (4.13), $x \in Q_U^\kappa$ implies that $x \in \bigcup\limits_{\eta \in \mathcal{J}_U^\ell} \psi_\eta(\{q\})$, for some $1 \leq \ell \leq \kappa$. However, if $\ell < \kappa$, then $x \in Q_U^\ell$ contradicting the fact that $\kappa \geq 1$ is the first non-negative integer satisfying $x \in Q_U^\kappa$. Thus, $x \in \bigcup\limits_{\eta \in \mathcal{J}_U^\kappa} \psi_\eta(\{q\})$, that is, $x = \psi_{\overline{\eta}}(q)$, for some $\overline{\eta} = (J_1, \ldots, J_\kappa) \in \mathcal{J}_U^\kappa$. This means that $x = \psi_{J_1} \circ \cdots \circ \psi_{J_\kappa}(q) \in \Delta_U^k$, for $1 \leq k \leq \kappa$. Therefore, $x \in \Delta_U^k \cup Q_U^k$ for every $k \geq 1$, which implies that $x \in \bigcap\limits_{k \geq 1} \left(\Delta_U^k \cup Q_U^k\right) = \overline{\Delta_U}$.

Furthermore, the union is disjoin, given that Δ_U is π_U-invariant and $q \notin \Delta_U$. $\square$

4.4 The Closure of Λ_U (Alternative Version)

Now we present another discussion on the closure of Λ_U, offering an alternative view, more closely related by the theory of IFS.

Let us define the constant function $\psi_q : I \to \{q\}$, and the index set $\mathcal{J}_q := \mathcal{J}_U \cup \{q\}$. Then, we consider the family $\Psi_q := \{\psi_J : J \in \mathcal{J}_q\}$. Notice that, even if it strongly resembles one, this is not a CIFS, since the function ψ_q is not injective.

Remember that $\Delta_U = \cap_{k \geq 1} \Delta_U^k$, where $\Delta_U^k := \cup_{\eta \in \mathcal{J}_U^k} \psi_\eta(\gamma_r)$. Now consider the set $\Delta_q := \cap_{k \geq 1} \Delta_q^k$, where $\Delta_q^k := \cup_{\eta \in \mathcal{J}_q^k} \psi_\eta(\gamma_r)$, for $k \geq 1$. Clearly, since $\Delta_q^k \supseteq \Delta_U^k$, we have $\Delta_q \supseteq \Delta_U$. Here we may see that the theory of CIFS is an inspiration for our arguments, although they are purely set-theoretical.

We aim to prove the following results: Δ_q is, in fact, the closure of Δ_U; and Δ_q is the disjoint union of Δ_U with the set of the pre-images of q by π_U, which is a countable set.

Our first step is to prove that Δ_q is compact (in particular, it contains $\overline{\Delta_U}$). For that, we need the following lemma:

Lemma 1. *For all $k \geq 1$, we have $\Delta_q^{k+1} = \bigcup\limits_{J \in \mathcal{J}_q} \psi_J(\Delta_q^k)$.*

Proof. We have the following identities:

$$\Delta_q^{k+1} = \bigcup_{\eta \in \mathcal{J}_q^{k+1}} \psi_\eta(\gamma_r) = \bigcup_{J \in \mathcal{J}_q} \bigcup_{\eta \in \mathcal{J}_q^k} (\psi_J \circ \psi_\eta)(\gamma_r) = \bigcup_{J \in \mathcal{J}_q} \psi_J \left(\bigcup_{\eta \in \mathcal{J}_q^k} \psi_\eta(\gamma_r) \right) = \bigcup_{J \in \mathcal{J}_q} \psi_J \left(\Delta_q^k \right),$$

as desired. □

We can then proceed with the following result:

Proposition 13. *For all $k \geq 1$, we have that Δ_q^k is compact. Additionally, the set Δ_q is compact as well.*

Proof. The proof is by induction on k.

For the base, $\Delta_q^1 = \mathrm{Dom}(\pi_U) \cup \{q\} = \overline{\mathrm{Dom}(\pi_U)}$, therefore it is compact.

Now suppose the result valid for k, and let us analyze Δ_q^{k+1}, using Lemma 1:

$$\Delta_q^{k+1} = \bigcup_{J \in \mathcal{J}_q} \psi_J(\Delta_q^k) = \left(\bigcup_{J \in \mathcal{J}_U} \psi_J(\Delta_q^k) \right) \cup \psi_q(\Delta_q^k) = \left(\bigcup_{J \in \mathcal{J}_U} \psi_J(\Delta_q^k) \right) \cup \{q\}.$$

Let $\mathcal{U}$ be an open cover of Δ_q^{k+1}, so in particular there exists $U_q \in \mathcal{U}$ such that $q \in U_q$, therefore $U_q \supseteq \cup_{J \in \tilde{\mathcal{J}}_U} \psi_J(\Delta_q^k)$, where $\tilde{\mathcal{J}}_U \subseteq \mathcal{J}_U$ and $\mathcal{J}_U \setminus \tilde{\mathcal{J}}_U$ is finite.

But $\cup_{\mathcal{J}_U \setminus \tilde{\mathcal{J}}_U} \psi_J(\Delta_q^k)$ is the finite union of the image of a compact set (by the induction hypothesis) under continuous functions, therefore it is compact and then there exists a finite subcover $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ satisfying $\cup_{J \in \mathcal{J}_U \setminus \tilde{\mathcal{J}}_U} \psi_J(\Delta_q^k) \subseteq \tilde{\mathcal{U}}$.

Then, $\tilde{\mathcal{U}} \cup \{U_q\}$ is a finite subcover of Δ_q^{k+1}. Additionally, since $\Delta_q = \cap_{k \geq 1} \Delta_q^k$ is the intersection of compact sets, it is compact as well. □

This proposition implies that $\Delta_q \supseteq \overline{\Delta_U}$, therefore we now need to prove that $\overline{\Delta_U} \supseteq \Delta_q$:

Proposition 14. *We have that $\Delta_q \subseteq \overline{\Delta_U}$.*

Proof. Let $x \in \Delta_q = \cap_{k \geq 1} \cup_{\eta \in \mathcal{J}_q^k} \psi_\eta(\gamma_r)$, therefore $x \in \cup_{\eta \in \mathcal{J}_q^k} \psi_\eta(\gamma_r)$, for all $k \geq 1$.

If $x \in \cup_{\eta \in \mathcal{J}_U^k} \psi_\eta(\gamma_r)$, for all $k \geq 1$, then in fact $x \in \Delta_U \subseteq \overline{\Delta_U}$, so there is nothing to prove.

Otherwise, there exists $n_q > 0$ with $x \in \psi_{\overline{\eta}}(\gamma_r)$, where $\overline{\eta} = (J_1, \ldots, J_{n_q - 1}, q)$ and $\left\{J_1, \ldots, J_{n_q-1}\right\} \subseteq \mathcal{J}_U$. Therefore, we can deduce that $x \in \left(\psi_{J_1} \circ \ldots \circ \psi_{J_{n_q-1}} \circ \psi_q\right)(\gamma_r) = \left(\psi_{J_1} \circ \ldots \circ \psi_{J_{n_q-1}} \circ\right)(\{q\})$, which amounts to $x = \psi_{(J_1, \ldots, J_{n_q-1})}(q)$.

Now, let $(x_j)_{j \geq 0} \subseteq \Delta_U$ be a sequence of elements of Δ_U, with $x_j \to q$. Since $\psi_{\overline{\eta}}$ is continuous, we have $x = \psi_{\overline{\eta}}(q) = \psi_{\overline{\eta}}\left(\lim_{j \to \infty} x_j\right) = \lim_{j \to \infty} \psi_{\overline{\eta}}(x_j)$. Each $\psi_{\overline{\eta}}(x_j) \in \Delta_U$, because Δ_U is π-invariant, then we conclude that x is the limit of a sequence of elements of Δ_U, therefore $x \in \overline{\Delta_U}$.

Of course, this proves that $\Delta_U^q \subseteq \overline{\Delta}$. □

At last we have the following proposition, making explicit the structure of $\overline{\Delta}$:

Proposition 15. *We have that $\Delta_q = \overline{\Delta_U} = \Delta_U \,\dot\cup\, Q_U$, where $Q_U = \cup_{k\geq 0}\pi_U^{-k}(\{q\})$ is a countable set.*

Proof. Clearly, we have $\cup_{k\geq 0}\pi_U^{-k}(\{q\}) = \cup_{k\geq 0}\cup_{\eta\in\mathcal{J}_U^k}\psi_\eta(\{q\})$, so we just need to prove that the union is disjoint. This comes from the fact that Δ_U is π_U-invariant and $q \notin \Delta_U$, therefore $\Delta_U \cap Q_U = \emptyset$.

The countability of Q_U stems from the fact that it is a countable union of countable sets. □

4.5 Proof of Theorem A1

Proof. For the proof of Theorem A1, we take U to be a neighborhood where Ψ_U is a CIFS, whose existence is guaranteed by Proposition 8. Additionally, we can choose U such that

$$\sum_{J\in\mathcal{J}_U}\frac{1}{A_{\min}\lambda^{c_J}} < 1. \tag{4.18}$$

This choice is possible since, from (4.4), $c_J = i-1$, for $J \in \{L_i^{\gamma_r}, R_i^{\gamma_r}\}$ and $i \geq 0$.

Proof of Theorem A1(a). Consider

$$P_1^U(t) := \sum_{J\in\mathcal{J}_U}\left|\sup_{x\in\gamma_r}\psi_J'(z)\right|^t.$$

This function is related to the discussion on the pressure function of an IFS detailed in Subsection 3.3.1. Notice that $P_1^U(t)$ is positive and unbounded as $t \to 0^+$. In addition, from Proposition 7, $\left|\sup_{x\in\gamma_r}\psi_J'(z)\right| \leq s < 1$, thus $P_1^U(t)$ is strictly decreasing. Moreover, the relationship (4.7), from the proof of Proposition 7, says that

$$P_1^U(t) \leq \sum_{J\in\mathcal{J}_U}\left(\frac{1}{A_{\min}\lambda^{c_J}}\right)^t,$$

which, taking (4.18) into account, implies $P_1^U(1) < 1$. Thus, there exists $0 < \hat{t} < 1$ such that $P_1^U(\hat{t}) = 1$.

Since $|\psi_J(x) - \psi_J(y)| \leq \sup_{z\in\gamma_r}|\psi_J'(z)|\cdot|x-y|$ for each $J \in \mathcal{J}_U$, Proposition 3 and Proposition 4 imply the upper bound $\dim_{\mathrm{H}}(\Lambda_U) \leq \hat{t} < 1$. Consequently, Property (H4) implies $m_1(\Lambda_U) = 0$. Besides, by applying Corollary 1 and the lower bounds for $|\psi_J'|$ provided by Proposition 7, we obtain the lower bound $\dim_{\mathrm{H}}(\Lambda_U) > 0$. □

We now offer an alternative proof for this part of the main result. While for now it does not offer any advantage, this proof in particular is used in Section 5.2 for some additional results.

Alternative Proof of Theorem A1(a). To prove that, by Proposition 5 we only need to prove that Ψ_U is a t_U-regular CIFS, for some $0 < t_U < 1$.

First, similarly to (4.6), we define

$$A_{\max} := \max_{\xi \in J} \left|(\rho_{\tilde{Z}}^J)'(\xi)\right| \cdot \max_{z \in H(\mu_r)} \left|(H^{-1})'(z)\right| \cdot \max_{\xi \in \mu_r} |H'(\xi)| \cdot \max_{x \in \gamma_r} |\rho_X'(x)| > 0,$$

from where we can deduce $\sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\max}\lambda^{c_J}}\right)^t \leq \sum_{J \in \mathcal{J}_U} \left|\inf_{x \in \gamma_r} \psi_J'(z)\right|^t \leq \sum_{J \in \mathcal{J}_U} \left|\sup_{x \in \gamma_r} \psi_J'(z)\right|^t = P_1^U(t)$.

We then have the following inequalities:

$$\sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\max}\lambda^{c_J}}\right)^t \leq P_1^U(t) \leq \sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\min}\lambda^{c_J}}\right)^t,$$

which by applying Proposition 2(c) tells us that

$$\log\left[\frac{1}{M^t} \sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\max}\lambda^{c_J}}\right)^t\right] \leq P^U(t) \leq \log\left[\sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\min}\lambda^{c_J}}\right)^t\right], \tag{4.19}$$

where $M > 1$ and $P^U(t)$ is the pressure function of Ψ_U.

By the construction in Proposition 7, we have that $A_{\max}\lambda^{c_J} \geq A_{\min}\lambda^{c_J} > 1$, for all $J \in \mathcal{J}_U$, with $c_J = i - 1$ when $J \in \{L_i^{\gamma_r}, R_i^{\gamma_r}\}$. This implies that by taking the limits in (4.19), we have

$$\lim_{t \to \infty} P^U(t) \leq \lim_{t \to \infty} \log\left[\sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\min}\lambda^{c_J}}\right)^t\right] = -\infty \tag{4.20}$$

and

$$\lim_{t \to 0^+} P^U(t) \geq \lim_{t \to 0^+} \log\left[\frac{1}{M^t} \sum_{J \in \mathcal{J}_U} \left(\frac{1}{A_{\max}\lambda^{c_J}}\right)^t\right] = \infty. \tag{4.21}$$

Therefore, the continuity given by Proposition 2(a) and Proposition 2(b) applied to the bounds in (4.20) and (4.21) guarantees that there exists $t_U \geq 0$ such that $P^U(t_U) = 0$.

Therefore, Ψ_U is a t_U-regular CIFS. Since $P^U(0)$ is bounded below by a divergent series, we may deduce that $t_U > 0$. Since $P^U(1) \leq \log\left(\sum_{j \in \mathcal{J}_U} \frac{1}{A_{\min}\lambda^{c_J}}\right) < \log 1 = 0$ (by (4.18)) and the function $P^U(t)$ is decreasing, we must have that $t_U < 1$.

Therefore we conclude that $0 < \dim_{\mathrm{H}}(\Lambda_U) < 1$, and consequently $m_1(\Lambda_U) = 0$, by applying Property (H4). $\square$

Finally, we give a proof for the final item of our main result:

Proof of Theorem A1(b). Proposition 12 provides that $\overline{\Lambda_U} = \Lambda_U \,\dot{\cup}\, Q_U$, where Q_U is a countable set and, therefore, $\dim_{\mathrm{H}}(Q_U) = 0$. Thus, Property (H2) and Theorem A1(a) imply that $\dim_{\mathrm{H}}\left(\overline{\Lambda_U}\right) = \dim_{\mathrm{H}}(\Lambda_U) < 1$. Again, from Property (H4), we have $\mathrm{m}_1(\overline{\Lambda_U}) = 0$.

Now we are going to prove that $\overline{\Lambda_U}$ is a Cantor set, that is, a non-empty metric space which is compact, totally disconnected and perfect.

Clearly, $\overline{\Lambda_U} \neq \emptyset$, it inherits the metric structure from γ_r, and is compact. Besides, given that $\dim_{\mathrm{H}}\left(\overline{\Lambda_U}\right) < 1$, Property (H5) says that $\overline{\Lambda_U}$ is a totally disconnected set. In order to see that it is a perfect set, let $x \in \Lambda_U$; from (3.2), we know that x is represented by some $\overline{\eta} = (J_1, J_2, J_3, \dots) \in \mathcal{J}_U^{\mathbb{N}}$, that is, $x = \mathrm{proj}(\overline{\eta})$, where proj denotes the projection defined in (3.1), which is well-defined due to the uniformly contractive property (C2) satisfied by the CIFS Ψ_U (see Remark 2). Taking a sequence $\{x_j\}_{j\geq 0} \subseteq \Lambda_U$ satisfying $x_k \in \psi_{(J_1,\dots,J_k)}(\gamma_r)$, but $x_k \notin \psi_{(J_1,\dots,J_k,J_{k+1})}(\gamma_r)$, for $k \geq 1$, we have that $\{x_j\}_{j\geq 0} \subseteq \Lambda_U \setminus \{x\}$ and $x_j \to x$. This implies that every point of Λ_U is an accumulation point. Since the closure of a set does not add any isolated points, $\overline{\Lambda_U}$ is a perfect set. □

This concludes the proof of Theorem A1. □

5 Further Comments

5.1 Topological Dynamics

Let I be a countable (finite or infinite) set. We consider the set $I^{\mathbb{N}}$ and the function $\sigma : I^{\mathbb{N}} \to I^{\mathbb{N}}$ given by $\sigma(i_1, i_2, i_3, \ldots) := (i_2, i_3, i_4, \ldots)$. We also consider the function $\operatorname{dist} : I^{\mathbb{N}} \times I^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ given by

$$\operatorname{dist}(a, b) = \operatorname{dist}((a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}) := \begin{cases} 0, & \text{if } a = b; \\ \\ \dfrac{1}{\kappa}, & \text{if } a \neq b, \end{cases}$$

where $\kappa = \min\{j \in \mathbb{N} \text{ such that } a(j) \neq b(j)\}$.

This constitutes a metric on $I^{\mathbb{N}}$, which turns σ in a continuous function. The dynamical systems given by this setting are known as *Bernoulli systems*.

These systems have been studied for a long time and are well understood. Therefore, if a dynamical system is topologically conjugated to a Bernoulli system, we gain a lot of information. So, in the next propositions, we now focus our attention to prove that our dynamical system is, in fact, topologically conjugated to a Bernoulli system.

Proposition 16. *Consider the Bernoulli system $(\mathcal{J}_U^{\mathbb{N}}, \sigma)$. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{J}_U^{\mathbb{N}} \arrow[r, "\sigma"] \arrow[d, "\operatorname{proj}"'] & \mathcal{J}_U^{\mathbb{N}} \arrow[d, "\operatorname{proj}"] \\ \Lambda_U \arrow[r, "\pi_U"'] & \Lambda_U \end{tikzcd}$$

Proof. Take $\omega = (J_1, J_2, J_3, \ldots) \in \mathcal{J}_U^{\mathbb{N}}$.

We have $\operatorname{proj} \circ\, \sigma(\omega) = \operatorname{proj} \circ\, \sigma(J_1, J_2, J_3, \ldots)(\gamma_r) = \operatorname{proj}(J_2, J_3, J_4 \ldots) = \psi_{(J_2,J_3,J_4\ldots)}(\gamma_r)$.

On the other hand, we also have $\pi_U \circ \operatorname{proj}(\omega) = \pi_U \circ \operatorname{proj}(J_1, J_2, J_3, \ldots) = \pi_U \circ \psi_{(J_1,J_2,J_3,\ldots)}(\gamma_r) = \pi_U \circ \underbrace{\psi_{J_1}}_{\pi_{J_1}^{-1}} \circ\, \psi_{(J_2,J_3,J_4,\ldots)}(\gamma_r) = \psi_{(J_2,J_3,J_4,\ldots)}(\gamma_r)$.

Therefore, $\operatorname{proj} \circ\, \sigma = \pi_U \circ \operatorname{proj}$, so the diagram commutes. $\square$

Next, we prove that this particular conjugation maintains the topological properties of the system.

Proposition 17. *The function* proj : $\mathcal{J}_U^{\mathbb{N}} \to \Lambda_U$ *is an homeomorphism.*

Proof. We are going to prove the continuity of proj and proj^{-1} separately:

- proj is continuous:

  Let $a, b \in \mathcal{J}_U^{\mathbb{N}}$ with $a \neq b$, satisfying $\text{dist}(a, b) < \epsilon$. Then, there exists $\kappa = \kappa(\epsilon) \geq 1$, such that $a(i) = b(i)$, for all $1 \leq i \leq \kappa$.

  Now, consider $\text{proj}(a) = \psi_a(\gamma_r) = \psi_{a(1)} \circ \ldots \psi_{a(\kappa)} \left(\psi_{\sigma^\kappa(a)}(\gamma_r)\right)$ and $\text{proj}(b) = \psi_b(\gamma_r) = \psi_{b(1)} \circ \ldots \psi_{b(\kappa)} \left(\psi_{\sigma^\kappa(b)}(\gamma_r)\right) = \psi_{a(1)} \circ \ldots \psi_{a(\kappa)} \left(\psi_{\sigma^\kappa(b)}(\gamma_r)\right)$.

  Therefore, we have that

  $$\begin{aligned} |\text{proj}(a) - \text{proj}(b)| &\leq \sup_{x,y \in \gamma_r} \left|\psi_{(a(1),\ldots,a(\kappa))}(x) - \psi_{(a(1),\ldots,a(\kappa))}(y)\right| \\ &\leq s^\kappa \sup_{x,y \in \gamma_r} |x - y| \leq cs^\kappa, \end{aligned}$$

  for $c = \text{diam}(\gamma_r) > 0$. This, of course, proves that proj is continuous.

- proj^{-1} is continuous:

  Let $x, y \in \Lambda_U$, satisfying $|x - y| < \epsilon$. Then, there exists $\kappa = \kappa(\epsilon) \geq 1$ such that $x, y \in \psi_{J_1} \circ \ldots \circ \psi_{J_\kappa}(\gamma_r)$.

  Therefore, we have that $\text{proj}^{-1}(x) = (J_1, \ldots, J_\kappa, J^x_{\kappa+1}, J^x_{\kappa+2}, \ldots) \in \mathcal{J}_U^{\mathbb{N}}$ and $\text{proj}^{-1}(x) = (J_1, \ldots, J_\kappa, J^y_{\kappa+1}, J^y_{\kappa+2}, \ldots) \in \mathcal{J}_U^{\mathbb{N}}$, for some $J_i \in \mathcal{J}_U$ for $1 \leq i \leq \kappa$, and $J^x_i, J^y_i \in \mathcal{J}_U$ for $i \geq \kappa + 1$.

  This, of course, implies that $\text{dist}(\text{proj}(x), \text{proj}(y)) \leq \dfrac{1}{\kappa}$, which proves that proj^{-1} is continuous as well.

$\square$

This shows that, at least topologically speaking, the dynamics of (Λ_U, π_U) and $(\mathcal{J}_U^{\mathbb{N}}, \sigma)$ are the same. Therefore, all information regarding the topological dynamics of the former is exactly the same of the latter. For instance, the topological entropy of the system is infinite (as already proved in (NOVAES; PONCE; VARÃO, 2017) by another conjugation in a proper subset of the invariant set Λ_U) and the set of periodic orbits is dense.

5.2 Conformal Measures

As said in Section 3.3.1, the regularity of a CIFS (see Definition 3) gives us more information about the invariant set of the CIFS. One such information is the existence of an interesting probability measure on Δ closely related to aspects of the functions of the CIFS $\mathcal{F}$. Let us introduce some concepts and propositions.

Definition 4 (t-Conformal Measures)**.** *Let $\mathcal{F}$ be a CIFS and let $t \geq 0$ be a positive real number. We say that a measure m_t is a t-*conformal measure *for the CIFS $\mathcal{F}$ if*

1. $m_t(\Delta) = 1$*;*
2. $m_t(f_i(B)) = \int_B |D(f_i)|^t \, dm_t$*, for all $i \in I$ and for all borelian sets B;*
3. $m_t(f_i(K) \cap f_j(K)) = 0$*, for all $i \in I$, $i \neq j$.*

The next proposition relates the regularity of a CIFS with the existence of a t-conformal measure on the invariant set:

Proposition 18 ((MAULDIN, 1995, Theorem 7.2))**.** *A CIFS $\mathcal{F}$ is a regular system, with $P(\hat{t}) = 0$, if and only if there exists a $\hat{t}$-conformal measure $m_{\hat{t}}$ for $\mathcal{F}$.*

Now, let us define a measure in the symbol space, applying the $\hat{t}$-conformal measure $m_{\hat{t}}$ obtained from the last proposition. For each $\eta = (\eta_1 \ldots, \eta_k) \in \bigcup_{j \geq 1} I^j$, we define $|\eta| := k$ (the *length of* η), and then consider $[\eta] := \left\{\omega \in I^{\mathbb{N}} \text{ such that } \omega|_{|\eta|} = \eta\right\}$, the cilinder given by η. We then define the measure ν on the σ-algebra generated by the cilinders by

$$\nu[\eta] := \int_K |D(f_\eta)|^{\hat{t}} \, dm_{\hat{t}}.$$

We can extend the measure ν defined on $\bigcup_{j \geq 1} I^j$ to a measure ν^* on $I^{\mathbb{N}}$, following the details on (MAULDIN, 1995, Section 8). This extended measure satisfies the following proposition:

Proposition 19 ((MAULDIN, 1995, Theorem 8.1))**.** *Let $\hat{t}$ be such that $P(\hat{t}) = 0$. Then there is a unique measure ν^* on $I^{\mathbb{N}}$ which is invariant under the shift σ and is equivalent to ν. Moreover, $\dfrac{1}{M^{\hat{t}}} \leq \dfrac{d\nu^*}{d\nu} \leq M^{\hat{t}}$, where M is the constant from Condition (C6a).*

If each point of the invariant set Δ is given by a unique $\omega \in I^{\mathbb{N}}$ by the projection, the measure ν^* on the coding space gives rise to a measure on the invariant set itself, in such a way that it interacts with the dynamics induced by the CIFS. For that, consider $x_\omega = f_\omega(X)$, with $\omega = (\omega_1, \omega_2, \omega_3, \ldots)$, and let us define $T : \Delta \to \Delta$ such that $T(x_\omega) = f_{\omega_1}^{-1}(x) = \text{proj}(\sigma(\omega))$.

Now we state the following result:

Proposition 20 ((MAULDIN, 1995, Theorem 8.2)). *Suppose $P(\hat{t}) = 0$, and define the measures $m := \nu \circ \mathrm{proj}^{-1}$ on $\mathrm{proj}(\cup_{j\geq 1} I^j)$ and $m^* := \nu^* \circ \mathrm{proj}^{-1}$ on Δ. Then m^* is the unique invariant and ergodic measure with respect to T and satisfying $m^* \sim m$ on Δ.*

Applying this to our context, we deduce following additional result:

Theorem A2. *There exists a t_U-conformal probability measure m_U on Λ_U, where $t_U = \dim_{\mathrm{H}}(\Lambda_U)$. Besides, m_U is the unique measure satisfying $m_U \sim m_U^*$ which is invariant and ergodic with respect to π_U.*

Proof. Notice that in the alternative proof for Theorem A1(a), given in Section 4.5, we actually showed that Ψ_U is a t_U-regular CIFS. By applying Proposition 18, Proposition 19, and Proposition 20, we prove the existence of a t_U-conformal measure m_U that is invariant and ergodic relative to π_U, with $t_U = \dim_{\mathrm{H}}(\Lambda_U)$. $\square$

Part II

Hyperbolicity for Infinite Delayed Difference Equations

6 Introduction

In this part of the thesis, we introduce dynamical systems defined by linear difference equations with various types of delay; more specifically, we consider linear difference equations with no delay, with finite delay, and with infinite delay. We then examine the concept of exponential dichotomy for each case, which essentially means that our phase space is composed of subspaces exhibiting expanding and contracting behaviors at an exponential rate.

After that, we discuss cocycles, an object that gives us information about how the elements of our space interact with each other accordingly to the dynamics of the system, and later on we define the hyperbolicity property for a cocycle.

Then, the two subjects are related by making a cocycle induced by a linear difference equation. This background will allow us to state the known results about the equivalence between exponential dichotomy of a difference equation and the hyperbolicity of its induced cocycle, for the cases with no or finite delay. This put in context the main statement of this part of the thesis, that is the same kind of equivalence for infinite delay in a large class of spaces.

The infinite delay is the case where the difference equations have solutions possibly depending on their values at all former times. This creates the situation in which we have to deal with a dynamical system that is infinite-dimensional, due to the phase space of the sequence of previous steps being infinite. This creates some particular characteristics for these equations, as we shall see, among other examples.

We also give several applications of our main result. These essentially show that several properties related to the hyperbolicity of a given delay-difference equation with infinite delay can be extended to all equations in its invariant hull (Section 6.2 contains the definitions and detailed descriptions). The properties that we consider in these applications are:

1. Extension of hyperbolicity from a specific equation to all equations within the invariant hull, maintaining identical constants in the notion of exponential dichotomy;

2. Robustness of hyperbolicity under sufficiently small linear perturbations, applicable to all equations within the invariant hull of the original equation;

3. Uniformity of spectra across the invariant hull, employing a generalized spectrum concept compatible with non-autonomous systems.

Theorem B1 happens to clarify the somewhat peculiar situation that the two notions of hyperbolicity in the theorem have been used, in different situations, to study hyperbolicity. The cocycle is already present in (SACKER; SELL, 1978) in problems regarding linear differential equations. Some versions of Theorem B1 were established in (LATUSHKIN; SCHNAUBELT, 1999) and (CHICONE; LATUSHKIN, 1999). For continuous time, there is the work in (PLISS; SELL, 1999). More recently, the suitable version of the statement in Theorem B1 was established for equations without delay both for discrete and continuous time in (BARREIRA; VALLS, 2020) and for delay-differential equations with finite delay in (BARREIRA; HOLANDA; VALLS, 2021).

So, now we make an exposition to present these subject. Since we will be working plenty with sequences (on an arbitrary set A), we already define the bilateral shift function $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ given by

$$(\sigma\phi)(k) := \phi(k+1), \quad \text{for any } k \in \mathbb{Z}, \text{ for any } \phi \in A^{\mathbb{Z}}, \tag{6.1}$$

and the unilateral shift function $\tau : A^{\mathbb{Z}_{\leq 0}} \to A^{\mathbb{Z}_{\leq 0}}$ given by

$$(\tau\phi)(0) := 0 \quad \text{and} \quad (\tau\phi)(k) := \phi(k+1), \text{ for } k < 0, \text{ for any } \phi \in A^{\mathbb{Z}_{\leq 0}}. \tag{6.2}$$

6.1 Linear Delay Difference Equations and Exponential Dichotomy

In this section, we present linear difference equations with different kinds of delay, and explain the notion of exponential dichotomy (the idea behind it being that the totality of our space either contracts or expands with exponential rate, according to the dynamics).

6.1.1 Linear Delay Equations with No Delay

Let $(X, |\cdot|)$ be a Banach space, and let $L := \{L_m\}_{m\in\mathbb{Z}} \subseteq \mathcal{L}(X)$ be a sequence of bounded linear operators. Then the linear difference equation with no delay defined by L is the following:

$$x(m+1) = L_m x(m), \quad \text{for } m \in \mathbb{Z}. \tag{6.3}$$

For a given $n \in \mathbb{Z}$ and $v \in X$, there is a unique solution $x = x^L(\cdot, n, v) : \mathbb{Z}_{\geq n} \to X$ satisfying $x(n) = v$ and (6.3), for all $m \geq n$.

6.1.2 Exponential Dichotomy for Linear Difference Equations with No Delay

The behavior of successive operator compositions is crucial when dealing with the dynamics of the system created by (6.3). Trying to have a better understand of them, we define the following object:

Definition 5 (Evolution Family for Linear Equations with No Delay)**.** *For each $m, n \in \mathbb{Z}$ with $m \geq n$, we define the operator*

$$T_L(m,n) = T(m,n) := \begin{cases} L_{m-1} \cdots L_n, & \text{if } m > n, \\ \mathrm{Id}, & \text{if } m = n. \end{cases}$$

Note that $T(n,n) = \mathrm{Id}$ and $T(m,n) = T(m,k)T(k,n)$, for all $n \leq k \leq m$, justifying the use of the nomenclature *evolution family*.

A useful approach when dealing with dynamics is try to decompose our space in stable and unstable subspaces, which amounts to, respectively, exponential rates of contraction and expansion under the action of our maps. This motivates the following definition:

Definition 6 (Exponential Dichotomy for Linear Equations with No Delay)**.** *We say that* (6.3)*, or the sequence of operators $L = \{L_m\}_{m\in\mathbb{Z}}$, or yet the evolution family $(T(m,n))_{m\geq n}$ has* exponential dichotomy *if it satisfies the following criteria:*

1. *There exist projections $P_n \in \mathcal{L}(X)$, for $n \in \mathbb{Z}$, such that*
$$P_m T(m,n) = T(m,n)P_n, \quad \text{for all } m \geq n,$$
2. *The linear operator*
$$\hat{T}(m,n) := T(m,n)|_{\ker P_n} : \ker P_n \to \ker P_m$$
is onto and invertible for all $m \geq n$;
3. *There exist $\lambda, D > 0$ such that, for every $m \geq n$, we have*
$$\|T(m,n)P_n\|_{\mathcal{L}(X)} \leq De^{-\lambda(m-n)} \quad \text{and} \quad \left\|\hat{T}(n,m)Q_m\right\|_{\mathcal{L}(X)} \leq De^{-\lambda(m-n)},$$
where $\hat{T}(n,m) := \hat{T}(m,n)^{-1}$ and $Q_m := \mathrm{Id} - P_m$.

6.1.3 Linear Delay Equations with Finite Delay

Now, we are going to investigate the case of finite delay. For that, we fix a non-positive integer $r \leq 0$ (the delay of the system), and we consider the Banach spaces $(X, |\cdot|)$ and $B_r = B := \{\phi : [r,0] \cap \mathbb{Z} \to X\}$, with some norm $\|\cdot\|_B$. We are also given a sequence of bounded operators $\{L_m\}_{m\in\mathbb{Z}} \subseteq \mathcal{L}(B,X)$. Also, for any sequence $x : (-\infty, m] \cap \mathbb{Z} \to X$ and any $n \leq m$, we introduce the function $x_n \in B$ given by $x_n(j) := x(j+n)$, for $r \leq j \leq 0$.

The linear difference equation with finite delay is then written as:

$$x(m+1) = L_m x_m, \quad \text{for } m \in \mathbb{Z}. \tag{6.4}$$

Again, for a given $n \in \mathbb{Z}$ and $v \in B$, there is a unique solution $x = x^L(\cdot, n, v) : \mathbb{Z}_{\leq n+r} \to X$ satisfying $x_n = v$ and (6.4), for all $m \geq n$.

6.1.4 Exponential Dichotomy for Linear Difference Equations with Finite Delay

Definition 7 (Evolution Family for Linear Equations with Finite Delay)**.** *Consider the system described by* (6.4)*. For any $m, n \in \mathbb{Z}$ with $n \leq m$, we define the* evolution family (associated to (6.4)) *as $T(m,n) : B \to B$ given by $T(m,n)\phi := x_m(\cdot, n, \phi)$.*

Again, this family satisfies $T(n,n) = \mathrm{Id}$ and $T(m,n) = T(m,k)T(k,n)$, for all $n \leq k \leq m$.

Now, we have the necessary background to define exponential dichotomy for linear difference equations with finite delay:

Definition 8 (Exponential Dichotomy for Linear Difference Equations with Finite Delay)**.** *We say that that* (6.4)*, or the sequence of operators $L = \{L_m\}_{m\in\mathbb{Z}}$, or yet the associated evolution family $(T(m,n))_{m\geq n}$ has an* exponential dichotomy *if the following properties hold:*

1. *There exist projections $P_n \in \mathcal{L}(B)$ for $n \in \mathbb{Z}$ such that*
$$P_m T(m,n) = T(m,n)P_n \quad \text{for } m \geq n;$$
2. *The linear operator*
$$\hat{T}(m,n) = T(m,n)|_{\ker P_n} : \ker P_n \to \ker P_m$$
is onto and invertible for each $m \geq n$;
3. *There exist $\lambda, D > 0$ such that for each $m \geq n$ we have*
$$\|T(m,n)P_n\|_{\mathcal{L}(B)} \leq De^{-\lambda(m-n)} \quad \text{and} \quad \left\|\hat{T}(n,m)Q_m\right\|_{\mathcal{L}(B)} \leq De^{-\lambda(m-n)},$$
where $\hat{T}(n,m) := \hat{T}(m,n)^{-1}$ and $Q_m := \mathrm{Id} - P_m$.

Naturally, when $r = 0$, with the convenient identifications we get the same notion for linear difference equations with no delay, so it is a generalization of the previous objects.

6.1.5 Linear Delay Equations with Infinite Delay

Finally, we may introduce the concept of linear difference equations with infinite delay. The generalization is straightforward: we consider the Banach space $(X, |\cdot|)$, but now we have to take care with some convergence details, since our phase space is infinite-dimensional, so we define our sequence space as the subspace of sequences given by $B := \{\phi : \mathbb{Z}_{\leq 0} \to X \text{ such that } \|\phi\|_B < +\infty\}$. A particular specificity of this case is that we restrict our spaces of interest to accommodate the fact that the relationship between the growth of the sequence and the norm may interfere with our desired outcomes.

Thinking about this, we follow (MATSUNAGA; MURAKAMI, 2004) and shall always assume that there exist $\alpha > 0$ and sequences $K_1, K_2 : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ such that given $x : \mathbb{Z} \to X$ with $x_0 \in B$, for each $n \in \mathbb{N}_0$ we have $x_n \in B$ and

$$\alpha \left|x(n)\right| \leq \left\|x_n\right\|_B \leq K_1(n) \max_{0\leq l\leq n} \left|x(l)\right| + K_2(n) \left\|x_0\right\|_B . \tag{6.5}$$

We emphasize that this is not an overly restrictive requirement, since the usual Banach spaces $B = \ell^p := \left\{\phi : \mathbb{Z}_{\leq 0} \to X \text{ such that } \|\phi\|_p < +\infty\right\}$ satisfy it, for $1 \leq p \leq +\infty$, and also a much larger class of spaces too (some of them will be presented in Section 7.1). We again reinforce the fact that each Banach space B may be infinite-dimensional even if X is finite-dimensional.

Also, let $\{L_m\}_{m\in\mathbb{Z}} \subseteq \mathcal{L}(B, X)$ be a sequence of bounded operators, and given any sequence $x : (-\infty, m] \to x$ and any $n \leq m$, we introduce the sequence $x_n \in B$ given by $x_n(j) := x(j + n)$, for $j \leq 0$.

We write the linear difference equation with infinite delay as:

$$x(m + 1) = L_m x_m, \quad \text{for } m \in \mathbb{Z}. \tag{6.6}$$

As expected, for a given $n \in \mathbb{Z}$ and $v \in B$, there is a unique solution $x = x^L(\cdot, n, v) : \mathbb{Z} \to X$ satisfying $x_n = v$ and (6.6), for all $m \geq n$.

The definitions of the evolution family $T(m, n)$ and of exponential dichotomy for this case are the same as in Definition 8, with the adjustments of B being a set of sequences indexed by $\mathbb{Z}_{\leq 0}$ now. We give the definitions for the sake of completeness:

Definition 9 (Evolution Family for Linear Equations with Infinite Delay)**.** *Consider the system described by* (6.6)*. For any $m, n \in \mathbb{Z}$ with $n \leq m$, we define the* evolution family (associated to (6.6)) *as $T(m, n) : B \to B$ given by $T(m, n)\phi := x_m(\cdot, n, \phi)$.*

As expected, they also satisfy $T(n, n) = \mathrm{Id}$ and $T(m, n) = T(m, k)T(k, n)$, for all $n \leq k \leq m$.

Definition 10 (Exponential Dichotomy for Linear Difference Equations with Infinite Delay)**.** *We say that that* (6.6)*, or the sequence of operators $L = \{L_m\}_{m\in\mathbb{Z}}$, or yet the associated evolution family $(T(m, n))_{m\geq n}$ has an* exponential dichotomy *if the following properties hold:*

1. *There exist projections $P_n \in \mathcal{L}(B)$ for $n \in \mathbb{Z}$ such that*

$$P_m T(m, n) = T(m, n)P_n \quad \textit{for } m \geq n;$$

2. *The linear operator*

$$\hat{T}(m,n) = T(m,n)|_{\ker P_n} : \ker P_n \to \ker P_m$$

is onto and invertible for each $m \geq n$*;*

3. *There exist* $\lambda, D > 0$ *such that for each* $m \geq n$ *we have*

$$\|T(m,n)P_n\|_{\mathcal{L}(B)} \leq De^{-\lambda(m-n)} \quad \text{and} \quad \left\|\hat{T}(n,m)Q_m\right\|_{\mathcal{L}(B)} \leq De^{-\lambda(m-n)}, \tag{6.7}$$

where $\hat{T}(n,m) := \hat{T}(m,n)^{-1}$ *and* $Q_m := \mathrm{Id} - P_m$*.*

Again, for an integer $r \leq 0$, when we deal with the (finite dimensional) subspace $B_r := \{\phi : \mathbb{Z}_{\leq 0} \to X \text{ such that } \phi(j) = 0, \forall j \leq r-1\} \subseteq B$, we have the same notion for linear difference equations with finite delay.

6.2 Cocycles and Hyperbolicity

We now introduce the concept of cocycles. Those are an association between the states of the phase space of our dynamical system (a point and a time) and a linear transformation, satisfying some compatibility conditions.

Here is the formal definition:

Definition 11 ((Linear) Cocycles)**.** *Let* X *be a Banach space, and* $\mathcal{Y} := \mathcal{L}(Y)$ *be the set of bounded linear operators acting on a Banach space* Y *(which does not need to be* X *in general). A* (linear) cocycle on X over an invertible map $f : X \to X$ *is a function* $S : X \times \mathbb{N}_0 \to \mathcal{Y}$ *such that*

$$S(x, m+n) = S(f^m(x), n)\ S(x,m),$$

for every $m \geq n \geq 0$ *and* $x \in X$*.*

A common example of a cocycle is the behavior of the derivative in relation to iterates of a function. Writing the derivative of f at x as $D_x(f)$, the chain rule gives us that $D_x(f^{m+n}) = D_{f^m(x)}(f^n)\ D_x(f^m)$, which is the cocycle equation.

Now, we introduce the definition of a hyperbolic cocycle:

Definition 12 (Hyperbolic Cocycle)**.** *We say that a cocycle* S *over an invertible map* $f : X \to X$ *is* hyperbolic *if the following properties hold:*

1. *There exist projections* $P(M) \in \mathcal{L}(Y)$*, for* $M \in X$ *such that*

$$P(f^n(M))S(M,n) = S(M,n)P(M) \quad \text{for } (M,n) \in X \times \mathbb{N}_0;$$

2. *The linear operator*

$$\hat{S}(M,n) = S(M,n)|_{\ker P(M)} : \ker P(M) \to \ker P(f^n(M))$$

is onto and invertible for each $(M,n) \in X \times \mathbb{N}_0$*;*

3. *There exist* $\lambda, D > 0$ *such that for each* $(M,n) \in X \times \mathbb{N}_0$ *we have*

$$\|S(M,n)P(M)\| \leq De^{-\lambda n} \quad \text{and} \quad \left\|\hat{S}(M,-n)Q(M)\right\| \leq De^{-\lambda n},$$

where $\hat{S}(M,-n) := \hat{S}(f^{-n}(M),n)^{-1}$ *and* $Q(M) = \mathrm{Id} - P(M)$*.*

6.3 The Cocycle Associated to a Linear Delay Difference Equation

Now we consider a specific cocycle. Let $\mathcal{Z}$ be the set of all pointwise limits of the set $\{\sigma^n L : n \in \mathbb{Z}\}$ (see (6.1) for the definition of $\sigma^n L$), which we call the *invariant hull* of (6.6). More precisely, the elements of $\mathcal{Z}$ are the sequences $M = \{M_m\}_{m\in\mathbb{Z}}$ of linear operators in $\mathcal{L}(B,X)$ for which there exists a sequence $\{r_k\}_{k\in\mathbb{N}}$ in $\mathbb{Z}$ such that

$$L_{m+r_k} \to M_m \quad \text{when } k \to \infty, \tag{6.8}$$

for all $m \in \mathbb{Z}$. We define a cocycle $S_L : \mathcal{Z} \times \mathbb{N}_0 \to \mathcal{L}(B)$ over the shift map $\sigma : \mathcal{Z} \to \mathcal{Z}$ by

$$S_L(M,n) = T_M(n,0) \quad \text{for } (M,n) \in \mathcal{Z} \times \mathbb{N}_0.$$

6.4 Previous Results

The relationship between the exponential dichotomy of linear difference equations and the hyperbolicity of their associated cocycles is already established, with the results presenting the equivalence of both properties. We provide these results:

Proposition 21 ((BARREIRA; VALLS, 2020, Theorem 5 and Theorem 6))**.** *Consider the dynamical system given by the linear delay difference equation with no delay described by* (6.3)*. Assume that* $L = \{L_m\}_{m\in\mathbb{Z}}$ *is a bounded sequence. Then the following statements are equivalent:*

1. (6.3) *has exponential dichotomy;*

2. *The cocycle* S_L *is hyperbolic.*

This proposition can be, in a certain sense, generalized. Now we focus in this generalization:

For a given linear difference equation with *finite* delay $r \leq 0$, we may associate a linear difference equation with *no* delay in the space $X^{|r|+1}$ as follows: consider the context of (6.4), and for a given $v = (v(r), v(r+1), \ldots, v(1), v(0)) \in B_r$ there is the isomorphism taking $v \in B = B_r$ to $\tilde{v} = (v(r), v(r+1), \ldots, v(1), v(0)) \in X^{|r|+1}$. Besides, since $|r|+1$ is finite, the norms of the spaces B_r, $X^{|r|+1}$ and X are equivalent.

Now, considering a sequence $\tilde{x} : \mathbb{Z} \to X^{|r|+1}$, and the sequence of operators $\left\{\tilde{L}_m\right\}_{m\in\mathbb{Z}} \subseteq \mathcal{L}(X^{|r|+1})$ given by

$$\tilde{L}_m(\tilde{v}) = \tilde{L}_m(v(r), \ldots, v(0)) := (v(r+1), \ldots, v(0), L_m v),$$

and the linear difference equation with no delay defined by $\tilde{x}(m+1) = \tilde{L}_m \tilde{x}(m)$, it is not hard to notice that this association defines the same structure of the dynamical system, with the same exponential behaviour.

Therefore the theories of hyperbolicity for linear equations with no delay and with finite delay are essentially the same. With this point of view, we can apply Proposition 21 to the context of linear difference equations with finite delay as well.

Continuous versions

There is also similar definitions and constructions in a continuous setting, providing a version of the previous theorem in the case of differential equations (for example, in (BARREIRA; VALLS, 2020)). For the sake of completeness, we provide a quick overview, without getting into the finer details.

Given a continuous function $L : \mathbb{R} \to \mathcal{L}(X)$, we consider the linear equation

$$x' = L(t)x. \tag{6.9}$$

This equation determines an evolution family $(T(t,s))_{t\geq s} \subseteq \mathcal{L}(X)$, defined by $T(t,s)x(t) = x(s)$, for any solution of the equation (6.9). Notice that $T(t,t) = \mathrm{Id}$ and $T(t,\tau)T(\tau,s) = T(t,s)$. The definition for exponential dichotomy is analogous, with the correspondent changes. For instance, now the index set is $\mathbb{R}$. The definition is:

1. There exists projections $P(t) \in \mathcal{L}(X)$, for $t \in \mathbb{R}$, satisfying

   $$P(t)T(t,s) = T(t,s)P(s),$$

   for $t \geq s$;

2. The linear operator

   $$\hat{T}(t,s) := T(t,s)|_{\ker P(s)} : \ker P(s) \to P(t)$$

   is onto and invertible, for each $t \geq s$;

3. There exist $\lambda, D > 0$ such that, for all $t \geq s$, we have

$$\|T(t,s)P(s)\| \leq De^{\lambda(t-s)} \text{ and } \left\|\hat{T}(s,t)Q(t)\right\| \leq De^{\lambda(t-s)},$$

where $\hat{T}(s,t) := \hat{T}(t,s)^{-1}$ and $Q(t) = \mathrm{Id} - P(t)$.

Furthermore, we also determine the cocycle, only now using the index set as $\mathbb{R}_{\geq 0}$: a (linear) cocycle over a flow φ_t on X is a function $\mathcal{C} : X \times \mathbb{R}_{\geq 0}$ satisfying

1. $\mathcal{C}(x, 0) = \mathrm{Id}$, for all $x \in X$;
2. $\mathcal{C}(x, t+s) = \mathcal{C}(\varphi_s(x), t)\mathcal{C}(x, s)$, for all $x \in X$ and $t, s \geq 0$.

Similar notions for this kind of cocycles are used to define its *hyperbolicity*, and with them the following result holds:

Proposition 22 ((BARREIRA; VALLS, 2020, Theorem 12 and Theorem 13))**.** *Consider the dynamical system given by the linear differential equation described by* (6.9)*. Then the following statements are equivalent:*

1. (6.9) *has exponential dichotomy;*
2. *The cocycle* $\mathcal{C}_L$ *is hyperbolic.*

In the same spirit, for differential equations there is a version of the theorem for finite delay too, with the convenient definitions (see (BARREIRA; HOLANDA; VALLS, 2021)).

6.5 Statement of the Main Result

Now, we have the context to properly state our main result, which is the following generalization of Proposition 21:

Theorem B1. *Consider the dynamical system given by the linear delay difference equation with infinite delay described by* (6.6)*. Assume that* $L = \{L_m\}_{m\in\mathbb{Z}}$ *is bounded, and also that* K_1 *and* K_2 *of* (6.5) *are bounded sequences as well. Then the following statements are equivalent:*

1. (6.6) *has exponential dichotomy;*
2. *The cocycle* S_L *is hyperbolic.*

7 Preliminaries

In this section we introduce the background concepts used in this thesis. In particular, we specify the class of Banach spaces with the necessary properties for the phase space of a linear difference equation with infinite delay, as well as some notions related to hyperbolicity.

7.1 Spaces of Interest

In our study, we are going to restrict our attention to Banach spaces $(B, \|\cdot\|_B)$, where $B \subseteq X^{\mathbb{Z}_{\leq 0}}$. Following (MATSUNAGA; MURAKAMI, 2004) we shall always assume that there exist $\alpha > 0$ and sequences $K_1, K_2 : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ such that given $x : \mathbb{Z} \to X$ with $x_0 \in B$, for each $n \in \mathbb{N}_0$ we have $x_n \in B$ and

$$\alpha \, |x(n)| \leq \|x_n\|_B \leq K_1(n) \max_{0 \leq l \leq n} \|x(l)\| + K_2(n) \, \|x_0\|_B \, . \tag{7.1}$$

We emphasize that this is not a very restrictive requirement, since the usual Banach spaces $B = \ell^p := \left\{ \phi : \mathbb{Z}_{\leq 0} \to X \text{ such that } \|\phi\|_p < +\infty \right\}$ satisfy it, for $1 \leq p \leq +\infty$. Indeed, as we shall see next, a generalization of those spaces satisfy this condition.

For all $p \in [1, +\infty]$ and any sequence $w : \mathbb{Z}_{\leq 0} \to \mathbb{R}_{>0}$, consider the spaces

$$\ell^p_w := \left\{ v : \mathbb{Z}_{\leq 0} \to X \text{ such that } \|v\|_{p,w} < +\infty \right\},$$

where

$$\|v\|_{p,w} := \left(\sum_{k \leq 0} w(k) \, \|v(k)\|^p \right)^{1/p} \quad \text{for } p < +\infty, \tag{7.2}$$

and

$$\|v\|_{\infty,w} = \sup_{k \leq 0} (w(k) \, \|v(k)\|). \tag{7.3}$$

Proposition 23 ((BARREIRA; RIJO; VALLS, 2020, Proposition 1))**.** *The spaces $\ell^p_w = (\ell^p_w, \|\cdot\|_{p,w})$ are Banach spaces, for all $p \in [1, +\infty]$. Besides, if the function w satisfies*

$$L(n) := \sup_{k \leq -n} \frac{w(k)}{w(k+n)} < +\infty \quad \text{for } n \geq 0, \tag{7.4}$$

then the space ℓ^p_w satisfies property (6.5), with

$$\alpha := w(0)^{1/p}, \quad K_1(n) := \left(\sum_{k=-n}^{0} w(k) \right)^{1/p}, \quad K_2(n) := L(n)^{1/p},$$

for $p \in [1, +\infty)$*, and*

$$\alpha := w(0), \quad K_1(n) := \sup_{-n \leq k \leq 0} w(k), \quad K_2(n) := L(n),$$

for $p = +\infty$*.*

Proposition 23 provides us with a family of examples of spaces satysfying (6.5), for each nondecreasing functions w (since they satisfy the condition in (7.4)).

7.2 Bounded Growth

(7.1) is not the only place where we put some constraints in the growth of our objects. As usual, we ask for $\{L_m\}_{m\in\mathbb{Z}}$ in (6.6) to be a bounded sequence of bounded operators, that is, we will assume that there exists $C_L > 0$ such that

$$\|L_m\| \leq C_L \quad \text{for } m \in \mathbb{Z}. \tag{7.5}$$

Moreover, we are going consider Banach spaces B for which the sequences K_1 and K_2 in (7.1) are bounded. For the spaces ℓ^p_w, this is the case if the sequence w in (7.2) is nondecreasing and $\sum_{k\leq 0} w(k)$ converges, for $p \in [1, +\infty)$, and if the sequence w in (7.3) is nondecreasing, for $p = +\infty$ (this follows readily from Proposition 23).

If those boundedness requirements are met, then the linear operators in $(T_L(m, n))_{m\geq n}$ also satisfy a bounded growth property. This is a simple consequence of the following proposition:

Proposition 24. *Assume that there exists* $C_K > 0$ *such that*

$$K_1(n), K_2(n) \leq C_K \quad \textit{for } n \geq 0. \tag{7.6}$$

Then for each bounded sequence L *in* $\mathcal{L}(B, X)$*, the sequence of linear operators* $T_L(m+1, m)$ *for* $m \in \mathbb{Z}$ *is bounded.*

Proof. By (6.5), for each $v \in B$ we have

$$\begin{aligned}
\|T_L(m+1, m)v\|_B &= \|T_{\sigma^m L}(1, 0)v\|_B = \left\|x_1^{\sigma^m L}(\cdot, 0, v)\right\|_B \\
&\leq C_K \max_{0\leq l\leq 1} \left\|x^{\sigma^m L}(l)\right\| + C_K \left\|x_0^{\sigma^m L}(\cdot, 0, v)\right\|_B \\
&= C_K \max\left\{\left\|x^{\sigma^m L}(0, 0, v)\right\|, \left|x^{\sigma^m L}(1, 0, v)\right|\right\} + C_K \|v\|_B \\
&= C_K \max\{|v(0)|, |L_m v|\} + C_K \|v\|_B \\
&\leq C_K \left(\max\left\{\alpha^{-1}, C_L\right\} + 1\right) \|v\|_B .
\end{aligned}$$

Therefore, the sequence $(T_L(m + 1, m))_{m\in\mathbb{Z}}$ is bounded. $\square$

The proposition readily implies that

$$\|T_L(m,n)\| \leq \beta^{m-n} \quad \text{for } m \geq n \tag{7.7}$$

and some constant $\beta > 1$.

8 Proof of the Main Theorem (of Part II)

As stated before (see Theorem B1), we are going to show that the hyperbolicity of the cocycle S_L obtained from some linear difference equation with infinite delay as described in (6.6) is equivalent to the existence of an exponential dichotomy for the same equation.

Proof. We separate the proof in two statements.

- **The cocycle S_L is hyperbolic $\implies$ Equation (6.6) has exponential dichotomy:**

  First we assume that the cocycle S_L is hyperbolic. Replacing (M, n) by $(\sigma^n L, m - n)$ in Definition 12 and noting that

  $$S_L(\sigma^n L, m-n) = T_{\sigma^n L}(m-n, 0) = T_L(m, n), \tag{8.1}$$

  we readily obtain the various properties in Definition 10 taking the projections $P_n = P(\sigma^n L)$. For the sake of completeness, we explain them here:

  The first property is given by the identities

  $$\begin{aligned} P_m T_L(m,n) &= P(\sigma^m L) S_L(\sigma^n L, m-n) \\ &= S_L(\sigma^n L, m-n) P(\sigma^n L) \\ &= T_L(m,n) P_n. \end{aligned}$$

  The second property comes from the fact that, since by hypothesis

  $$\hat{S}_L(\sigma^n L, m-n) = S_L(\sigma^n L, m-n)|_{\ker P(\sigma^n L)} : \ker P(\sigma^n L) \to \ker P(\sigma^{m-n}(\sigma^n L))$$

  is onto and invertible, and by taking the identification in (8.1) and the projections as described, this is the same as

  $$\hat{T}_L(m,n) = T_L(m,n)|_{\ker P_n} : \ker P_n \to \ker P_m$$

  also being onto and invertible, thus satisfying the second property.

  At last, since S_L is hyperbolic, taking $M = \sigma^n L$, we have that there exist $\lambda, D > 0$ satisfying

  $$\|T_L(m,n)P_n\| = \|S_L(\sigma^n L, m-n)P(\sigma^n L)\| = \|S_L(M, m-n)P(M)\| \le De^{-\lambda(m-n)}.$$

Similarly, we can also deduce

$$\left\|\hat{T}_L(n,m)Q_m\right\| = \left\|\hat{T}_L(m,n)^{-1}(\mathrm{Id}-P_m)\right\| = \left\|\hat{S}_L(\sigma^n L, m-n)^{-1}(\mathrm{Id}-P_m)\right\| = \left\|\hat{S}_L(\sigma^{n-m}(\sigma^m L), m-n)^{-1}(\mathrm{Id}-P(\sigma^m L))\right\|,$$

and by taking $M=\sigma^m L$ this time, we have that the last expression is equal to

$$\left\|\hat{S}_L(\sigma^{n-m}(M), m-n)^{-1}(\mathrm{Id}-P(M))\right\| = \left\|\hat{S}_L(M, n-m)Q(M)\right\| = \left\|\hat{S}_L(M, -(m-n))Q(M)\right\| \leq De^{-\lambda(m-n)};$$

both inequalities show that the third property is satisfied.

Therefore, (6.6) has an exponential dichotomy. □

- **Equation (6.6) has exponential dichotomy $\Longrightarrow$ The cocycle S_L is hyperbolic:**

Now assume that (6.6) has an exponential dichotomy. We first show that the dynamics defined by the delay-difference equation

$$y(m+1) = M_m y_m \quad \text{for } m \in \mathbb{Z}, \tag{8.2}$$

with M_m as in (6.8) for each $m \in \mathbb{Z}$, induces the same operators $T_M(m,n)$ as the limit of the operators $T_L(m+r_k, n+r_k)$ when $k \to \infty$ (this includes the statement that the limit exists).

Lemma 2. *For each $m,n \in \mathbb{Z}$ with $m \geq n$ we have*

$$T_L(m+r_k, n+r_k) \to T_M(m,n) \quad \textit{when } k \to \infty.$$

Proof of Lemma 2. Without loss of generality we take $n=0$ (the general case can be brought to this one using the second identity in (8.1)). We denote the solution of the equation

$$x(m+1) = L_{m+r_k} x_m \quad \text{for } m \in \mathbb{Z}$$

by x^k and the solution of (8.2) by y. By (7.7) we have $\|x_m\| \leq \beta^m \|v\|$ for all $m \geq 0$ and some $\beta > 1$, both for $x = x^k$ and $x = y$, taking $x_0 = v$.

We first show that there exists $\bar{\beta} \geq \beta$ such that

$$\max_{0 \leq l \leq m} \left\|x^k(l) - y(l)\right\| \leq \bar{\beta}^m \|v\| \max_{0 \leq s \leq m-1} \|L_{s+r_k} - M_s\| \tag{8.3}$$

for all $m \geq 1$. We proceed by induction on m. Since $x^k(0) = y(0) = v(0)$, for $m=1$ we have

$$\begin{aligned}\max_{0 \leq l \leq 1} \left\|x^k(l) - y(l)\right\| &= \left\|x^k(1) - y(1)\right\| = \|(L_{r_k} - M_0)v\| \\ &\leq \|v\| \cdot \|L_{r_k} - M_0\| \leq \bar{\beta} \|v\| \cdot \|L_{s+r_k} - M_s\|_{|s=0}\end{aligned}$$

for any $\bar{\beta} \geq \beta$. Now we assume that the result holds for some $m-1$ with $m \geq 2$. Then

$$\begin{aligned}
&\max_{0\leq l\leq m}\left\|x^k(l)-y(l)\right\| \\
&= \max\left\{\max_{0\leq l\leq m-1}\left\{\left\|x^k(l)-y(l)\right\|\right\}, \left\|L_{m-1+r_k}x^k_{m-1}-M_{m-1}y_{m-1}\right\|\right\} \\
&\leq \max\Bigg\{\bar{\beta}^{m-1}\|v\|\max_{0\leq s\leq m-2}\{\|L_{s+r_k}-M_s\|\}, \|L_{m-1+r_k}-M_{m-1}\|\cdot\left\|x^k_{m-1}\right\| \\
&\quad + \|M_{m-1}\|\cdot\left\|x^k_{m-1}-y_{m-1}\right\|\Bigg\},
\end{aligned}$$

using the induction hypothesis. The last expression can be bound by

$$\begin{aligned}
&\max\Bigg\{\bar{\beta}^{m-1}\|v\|\max_{0\leq s\leq m-2}\{\|L_{s+r_k}-M_s\|\}, \beta^{m-1}\|v\|\cdot\|L_{m-1+r_k}-M_{m-1}\| \\
&\quad + C_LC_K\max_{0\leq l\leq m-1}\left\{\left\|x^k(l)-y(l)\right\|\right\}\Bigg\} \\
&\leq \max\Bigg\{\bar{\beta}^{m-1}\|v\|\max_{0\leq s\leq m-2}\{\|L_{s+r_k}-M_s\|\}, \bar{\beta}^{m-1}\|v\|\cdot\|L_{m-1+r_k}-M_{m-1}\| \\
&\quad + C_LC_K\bar{\beta}^{m-1}\|v\|\max_{0\leq s\leq m-2}\{\|L_{s+r_k}-M_s\|\}\Bigg\} \\
&\leq \bar{\beta}^{m}\|v\|\max_{0\leq s\leq m-1}\{\|L_{s+r_k}-M_s\|\},
\end{aligned}$$

provided that $\bar{\beta} \geq \max\{\beta, 2+C_LC_K\}$. This establishes property (8.3). Therefore,

$$\left\|x^k_m-y_m\right\| \leq C_K\max_{0\leq l\leq m}\left\{\left\|x^k(l)-y(l)\right\|\right\} \leq \bar{\beta}^m\|v\|\max_{0\leq s\leq m-1}\{\|L_{s+r_k}-M_s\|\},$$

and so $x^k_m \to y_m$ when $k\to\infty$, for all $m \geq 1$. We conclude that

$$T_M(m,0)v = y_m = \lim_{k\to\infty}x^k_m = \lim_{k\to\infty}T_L(m+r_k,r_k)v,$$

which completes the proof of the lemma. □

Now we consider the sequences of projections $P_{k,n} = P_{n+r_k}$ and $Q_{k,n} = Q_{n+r_k}$ when $k\to\infty$. For simplicity of the notation, for each $m \geq n$ we write

$$U_k(m,n) = T_L(m+r_k, n+r_k)$$

and

$$\hat{U}_k(n,m) = T_L(m+r_k,n+r_k)|^{-1}_{\ker P_{n+r_k}}.$$

Lemma 3. *$(P_{k,n})_{k\in\mathbb{N}}$ is a Cauchy sequence for each $n\in\mathbb{Z}$.*

Proof of Lemma 3. We make the remark that this proof is inspired on related arguments in (BARREIRA; VALLS, 2020) for difference equations without delay. Given $i,k,p,n\in\mathbb{Z}$ with $p\geq n$, we define

$$A^{i,k}_p = \hat{U}_i(n,p)Q_{i,p}U_k(p,n)P_{k,n}.$$

Then

$$\begin{aligned}\sum_{p=n}^{+\infty}(A_p^{i,k}-A_{p+1}^{i,k}) &= \lim_{q\to+\infty}\sum_{p=n}^{q}(A_p^{i,k}-A_{p+1}^{i,k})\\ &= Q_{i,n}P_{k,n}-\lim_{q\to+\infty}\hat{U}_i(n,q)Q_{i,q}U_k(q,n)P_{k,n}\\ &= Q_{i,n}P_{k,n}=P_{k,n}-P_{i,n}P_{k,n},\end{aligned}$$

since it is a telescopic sum and by the vanishing of the limit term, in view of the exponential bounds in the notion of exponential dichotomy. Therefore,

$$\begin{aligned}P_{k,n} &= P_{i,n}P_{k,n}+\sum_{p=n}^{+\infty}(A_p^{i,k}-A_{p+1}^{i,k})\\ &= P_{i,n}P_{k,n}\\ &\quad+\sum_{p=n}^{+\infty}\hat{U}_i(n,p+1)Q_{i,p+1}(U_i(p+1,p)-U_k(p+1,p))U_k(p,n)P_{k,n}.\end{aligned}\tag{8.4}$$

In a similar fashion, by writing

$$B_p^{i,k}=U_i(n,p)P_{i,p}\hat{U}_k(p,n)Q_{k,n},$$

we also have

$$\begin{aligned}\sum_{p=-\infty}^{n-1}(B_p^{i,k}-B_{p+1}^{i,k}) &= \lim_{q\to-\infty}\sum_{p=q}^{n-1}(B_p^{i,k}-B_{p+1}^{i,k})\\ &= \lim_{q\to-\infty}U_i(n,q)P_{i,q}\hat{U}_k(q,n)Q_{k,n}-P_{i,n}Q_{k,n}\\ &= -P_{i,n}Q_{k,n}=-Q_{k,n}+Q_{i,n}Q_{k,n},\end{aligned}$$

again in view of the exponential bounds. Therefore,

$$\begin{aligned}Q_{k,n} &= Q_{i,n}Q_{k,n}-\sum_{p=-\infty}^{n-1}(B_p^{i,k}-B_{p+1}^{i,k})\\ &= Q_{i,n}Q_{k,n}\\ &\quad-\sum_{p=-\infty}^{n-1}U_i(n,p+1)P_{i,p+1}(U_i(p+1,p)-U_k(p+1,p))\hat{U}_k(p,n)Q_{k,n}.\end{aligned}\tag{8.5}$$

Note that

$$\begin{aligned}P_{i,n}P_{k,n}-Q_{i,n}Q_{k,n} &= P_{i,n}P_{k,n}-(\mathrm{Id}-P_{i,n})(\mathrm{Id}-P_{k,n})\\ &= -\mathrm{Id}+P_{k,n}+P_{i,n}=P_{i,n}-Q_{k,n}.\end{aligned}\tag{8.6}$$

Letting

$$E_p^{i,k}=U_i(p+1,p)-U_k(p+1,p)\tag{8.7}$$

and subtracting (8.4) from (8.5), it follows from (8.6) that

$$\begin{aligned}P_{k,n}-Q_{k,n} &= P_{i,n}-Q_{k,n}+\sum_{p=n}^{+\infty}\hat{U}_i(n,p+1)Q_{i,p+1}E_p^{i,k}U_k(p,n)P_{k,n}\\ &\quad+\sum_{p=-\infty}^{n-1}U_i(n,p+1)P_{i,p+1}E_p^{i,k}\hat{U}_k(p,n)Q_{k,n}\end{aligned}$$

and so

$$P_{k,n} - P_{i,n} = \sum_{p=n}^{+\infty} \hat{U}_i(n, p+1) Q_{i,p+1} E_p^{i,k} U_k(p,n) P_{k,n} + \sum_{p=-\infty}^{n-1} U_i(n, p+1) P_{i,p+1} E_p^{i,k} \hat{U}_k(p,n) Q_{k,n}.$$

By Proposition 24 there exists $d > 0$ such that $\left\|E_p^{i,k}\right\| \leq d$ for all i, k and p. Thus, by the bounds in (6.7) we obtain

$$\begin{aligned}
\|P_{k,n} - P_{i,n}\| &\leq \sum_{p=n}^{+\infty} \left\|\hat{U}_i(n, p+1) Q_{i,p+1} E_p^{i,k} U_k(p,n) P_{k,n}\right\| \\
&\quad + \sum_{p=-\infty}^{n-1} \left\|U_i(n, p+1) P_{i,p+1} E_p^{i,k} \hat{U}_k(p,n) Q_{k,n}\right\| \\
&\leq \sum_{p=n}^{+\infty} ce^{-\lambda(p+1-n)} \left\|E_p^{i,k}\right\| ce^{-\lambda(p-n)} \\
&\quad + \sum_{p=-\infty}^{n-1} ce^{-\lambda(n-p-1)} \left\|E_p^{i,k}\right\| ce^{-\lambda(n-p)} \\
&\leq c^2 e^{\lambda} (\sum_{p \notin I_q} de^{-2\lambda|n-p|} + (2q-1) \left\|E_p^{i,k}\right\|)
\end{aligned}$$

for each $q \in \mathbb{N}$, where $I_q = (n-q, n+q)$. Given $\varepsilon > 0$, take q such that

$$\frac{e^{-2\lambda q}}{1 - e^{-2\lambda}} < \varepsilon.$$

On the other hand, by Lemma 2 there exists $l \in \mathbb{N}$ such that

$$\left\|E_p^{i,k}\right\| = \|U_k(p+1, p) - U_i(p+1, p)\| < \varepsilon$$

for all $i, k \geq l$ and $p \in I_q$ (since p belongs to a finite set). Therefore,

$$\|P_{k,n} - P_{i,n}\| \leq c^2 e^{\lambda} (2d + 2q - 1)\varepsilon,$$

which shows that $(P_{k,n})_{k \in \mathbb{N}}$ is a Cauchy sequence. □

By Lemma 3, one can define

$$P(\sigma^n M) = \lim_{k\to\infty} P_{k,n} = \lim_{k\to\infty} P_{n+r_k}$$

for each $n \in \mathbb{Z}$. These operators are projections. Indeed,

$$P(\sigma^n M)^2 = \lim_{k\to\infty} P_{n+r_k} P_{n+r_k} = \lim_{k\to\infty} P_{n+r_k}^2 = P(\sigma^n M).$$

We also establish an invertibility property along what later on will be the unstable direction.

Lemma 4. *For each $n \in \mathbb{N}_0$ the linear operator $T_M(n,0)|_{\ker P(M)}$ is onto and invertible.*

Proof of Lemma 4. We proceed in a similar manner to that in the proof of Lemma 3 to show that $(\hat{U}_k(\bar{n},n)Q_{k,n})_{k\in\mathbb{N}}$ is a Cauchy sequence for any integers $\bar{n} \le n$. Given $i,k,p,n,\bar{n} \in \mathbb{Z}$ with $p \ge n \ge \bar{n}$, let

$$C_p^{i,k} = \hat{U}_i(\bar{n},p)Q_{i,p}U_k(p,n)P_{k,n}.$$

It follows as in the proof of Lemma 3 that

$$\begin{aligned}\hat{U}_i(\bar{n},n)Q_{i,n}P_{k,n} &= \sum_{p=n}^{+\infty}(C_p^{i,k} - C_{p+1}^{i,k}) \\ &= \sum_{p=n}^{+\infty}\hat{U}_i(\bar{n},p+1)Q_{i,p+1}E_p^{i,k}U_k(p,n)P_{k,n},\end{aligned}$$

with $E_p^{i,k}$ as in (8.7). Similarly, writing

$$D_p^{i,k} = U_i(\bar{n},p)P_{i,p}\hat{U}_k(p,n)Q_{k,n}$$

for $n \ge \bar{n} \ge p$, we obtain

$$\begin{aligned}P_{i,\bar{n}}\hat{U}_k(\bar{n},n)Q_{k,n} &= -\sum_{p=-\infty}^{\bar{n}-1}(D_p^{i,k} - D_{p+1}^{i,k}) \\ &= -\sum_{p=-\infty}^{\bar{n}-1}U_i(\bar{n},p+1)P_{i,p+1}E_p^{i,k}\hat{U}_k(p,n)Q_{k,n}.\end{aligned}$$

Now observe that

$$\begin{aligned}\hat{U}_i(\bar{n},n)Q_{i,n}P_{k,n} - P_{i,\bar{n}}\hat{U}_k(\bar{n},n)Q_{k,n} &= \hat{U}_i(\bar{n},n)Q_{i,n} - \hat{U}_i(\bar{n},n)Q_{i,n}Q_{k,n} \\ &\quad - \hat{U}_k(\bar{n},n)Q_{k,n} + Q_{i,\bar{n}}\hat{U}_k(\bar{n},n)Q_{k,n}\end{aligned}$$

and

$$\hat{U}_i(\bar{n},n)Q_{i,n}Q_{k,n} - Q_{i,\bar{n}}\hat{U}_k(\bar{n},n)Q_{k,n} = \sum_{p=\bar{n}}^{n-1}\hat{U}_i(\bar{n},p+1)Q_{i,p+1}E_p^{i,k}\hat{U}_k(p,n)Q_{k,n}.$$

Therefore,

$$\begin{aligned}\hat{U}_i(\bar{n},n)Q_{i,n} - \hat{U}_k(\bar{n},n)Q_{k,n} &= \sum_{p=n}^{+\infty}\hat{U}_i(\bar{n},p+1)Q_{i,p+1}E_p^{i,k}U_k(p,n)P_{k,n} \\ &\quad + \sum_{p=-\infty}^{\bar{n}-1}U_i(\bar{n},p+1)P_{i,p+1}E_p^{i,k}\hat{U}_k(p,n)Q_{k,n} \\ &\quad - \sum_{p=\bar{n}}^{n-1}\hat{U}_i(\bar{n},p+1)Q_{i,p+1}E_p^{i,k}\hat{U}_k(p,n)Q_{k,n}.\end{aligned}$$

One could now proceed as in the proof of Lemma 3 to show that the sequence $(\hat{U}_k(\bar{n},n)Q_{k,n})_{k\in\mathbb{N}}$ converges. Since

$$\hat{U}_k(0,n)Q_{k,n}U_k(n,0)Q_{k,0} = Q_{k,0}$$

and

$$U_k(n,0)Q_{k,0}\hat{U}_k(0,n)Q_{k,n} = Q_{k,n},$$

taking limits when $k \to \infty$ we find that

$$R_nT_M(n,0)|_{\ker P(M)} = Q(M)$$

and

$$T_M(n,0)|_{\ker P(M)}R_n = Q(\sigma^nM),$$

where R_n is the limit of the sequence $(\hat{U}_k(0,n)Q_{k,n})_{k\in\mathbb{N}}$ when $k \to \infty$. This yields the desired statement. □

Finally, we use the former properties to establish the hyperbolicity of the cocycle S. First note that

$$\begin{aligned} S(M,n)P(M) = T_M(n,0)P(M) &= \lim_{k\to\infty} T_L(n+r_k,r_k)P_{r_k} \\ &= \lim_{k\to\infty} P_{n+r_k}T_L(n+r_k,r_k) = P(\sigma^nM)S(M,n). \end{aligned}$$

Moreover, for any integer $n \geq 0$ we have

$$\|T_L(n+r_k,r_k)P_{r_k}\| \leq ce^{-\lambda n} \quad \text{and} \quad \left\|\hat{T}_L(r_k-n,r_k)Q_{r_k}\right\| \leq ce^{-\lambda n}. \tag{8.8}$$

On the other hand, by Lemma 4 the operator $T_M(n,0)|_{\ker P(M)}$ is invertible and so one can introduce

$$\hat{S}(M,-n) = S(\sigma^{-n}M,n)|^{-1}_{\ker P(\sigma^{-n}M)} = \hat{T}_{\sigma^{-n}M}(n,0)^{-1}$$

as in Definition 12. Hence, taking limits in (8.8) when $k \to \infty$ gives

$$\|T_M(n,0)P(M)\| \leq ce^{-\lambda n} \quad \text{and} \quad \left\|\hat{T}_{\sigma^{-n}M}(n,0)^{-1}Q(M)\right\| \leq ce^{-\lambda n}$$

or, equivalently,

$$\|S(M,n)P(M)\| \leq ce^{-\lambda n} \quad \text{and} \quad \left\|\hat{S}(M,-n)Q(M)\right\| \leq ce^{-\lambda n},$$

where $Q(M) = \mathrm{Id} - P(M)$. Therefore, the cocycle S is hyperbolic. □

Therefore, since we proved the implications in both directions, we have the equivalence between exponential dichotomy for linear difference equations with infinite delay and the hyperbolicity of its induced cocycle, as desired. □

9 Examples

In this chapter, we are going to introduce interesting examples of linear difference equations with infinite delay, and we will see some aspects of the theory presented so far.

9.1 Exponential Dichotomy Depends on the Norm

Here we will show a case illustrating that even an *autonomous* linear dynamics may have very different exponential behaviors with respect to different norms.

Example 3 (see (BARREIRA; RIJO; VALLS, 2020))**.** *Given $\gamma > 0$, we consider the equation*

$$x(m+1) = \sum_{k\leq 0} e^{\gamma k} x(k+m) \quad \text{for } m \in \mathbb{Z}. \tag{9.1}$$

We also consider the Banach space

$$\ell_\gamma = \left\{ v : \mathbb{Z}_{\leq 0} \to \mathbb{R} \text{ such that } \|v\|_\gamma < +\infty \right\}, \text{ where } \|v\|_\gamma = \sum_{k\leq 0} e^{\gamma k} \|v(k)\|.$$

Note that $\ell_\gamma = \ell^1_w$ with $w : \mathbb{Z}_{\leq 0} \to \mathbb{R}_{>0}$ given by $w(k) = e^{\gamma k}$ for $k \in \mathbb{Z}_{\leq 0}$ (see Section 7.1). Moreover, (7.1) holds for some bounded sequences K_1 and K_2.

Define an operator L_γ on the space ℓ_γ by

$$L_\gamma v = \sum_{k\leq 0} e^{\gamma k} v(k). \tag{9.2}$$

Then (9.1) can be written in the form

$$x(m+1) = L_\gamma x_m \quad \text{for } m \in \mathbb{Z}. \tag{9.3}$$

Note that $L_\gamma \in \mathcal{L}(\ell_\gamma, \mathbb{R})$ since

$$\|L_\gamma v\| = \left\| \sum_{k\leq 0} e^{\gamma k} v(k) \right\| \leq \sum_{k\leq 0} e^{\gamma k} \|v(k)\| = \|v\|_\gamma$$

for each $v \in \ell_\gamma$. In (BARREIRA; RIJO; VALLS, 2020, Section 3) it is shown that (9.3) has an exponential dichotomy on each space ℓ_γ.

We will show that (9.3) does not have an exponential dichotomy on the space ℓ^∞, seeing now L_γ in (9.2) as an element of $\mathcal{L}(\ell^\infty, \mathbb{R})$.

Example 4. *We consider the Banach space*

$$\ell^\infty = \{ v : \mathbb{Z}_{\leq 0} \to \mathbb{R} \text{ such that } \|v\|_\infty < +\infty \}, \text{ where } \|v\|_\infty = \sup_{k\leq 0} \|v(k)\|.$$

Note that $\ell^\infty = \ell^\infty_w$ *with* $w : \mathbb{Z}_{\le 0} \to \mathbb{R}_{>0}$ *given by* $w(k) = 1$ *for* $k \in \mathbb{Z}_{\le 0}$*. Again property* (7.1) *holds for some bounded sequences* K_1 *and* K_2*. Note also that* $L_\gamma \in \mathcal{L}(\ell^\infty, \mathbb{R})$ *since whenever* $\|v\|_\infty = 1$ *we have*

$$\|L_\gamma v\| = \left\|\sum_{k\le 0} e^{\gamma k} v(k)\right\| \le \sum_{k\le 0} e^{\gamma k} \|v(k)\| \le \sum_{k\le 0} e^{\gamma k}.$$

To show that (9.3) *does not have an exponential dichotomy on the space* ℓ^∞*, we proceed by contradiction. Namely, assume that there exists such an exponential dichotomy with nonzero projection* $P_n \in \mathcal{L}(\ell^\infty, \ell^\infty)$ *for some* $n \in \mathbb{Z}$*. For* $v \in \ell^\infty$ *with* $\|v\|_\infty = 1$ *such that* $P_n v \neq 0$ *we have*

$$\begin{aligned}\left\|T_{L_\gamma}(m,n)P_n v\right\|_\infty &= \|x_m(\cdot, n, P_n v)\|_\infty = \sup_{j\le 0} |x_m(j, n, P_n v)| \\ &\ge \sup_{j\le 0} |x_n(j, n, P_n v)| = \|P_n v\|_\infty > 0\end{aligned}$$

for every $m \ge n$*, and so one cannot have an exponential dichotomy (since the first bound in* (6.7) *cannot hold). Finally, assume that there exists such an exponential dichotomy with* $P_n = 0$ *for all* $n \in \mathbb{Z}$*. Then* $Q_n = \mathrm{Id}$ *for all* $n \in \mathbb{Z}$ *and so the operator* $T_{L_\gamma}(m,n) : \ell^\infty \to \ell^\infty$ *is onto and invertible for each* $m \ge n$*. On the other hand, given* $v \in \ell^\infty$*, we have* $T_{L_\gamma}(n+1,n)v \in \ell^\infty$ *satisfying*

$$\left(T_{L_\gamma}(n+1,n)v\right)(j) = \begin{cases} v(j+1) & \text{for } j \le -1, \\ Lv & \text{for } j = 0. \end{cases}$$

This shows that the operator $T_{L_\gamma}(n, n+1)$ *cannot be onto, and so again one cannot have an exponential dichotomy.*

9.2 Non-Autonomous Equations with Infinite Delay Possessing Exponential Dichotomy

We also give examples of *nonautonomous* linear delay-difference equations with infinite delay that has an exponential dichotomy. Each example has a particular characteristic, showing that these kind of equations may have many properties.

In these examples, we are going to use the following lemma:

Lemma 5. *Suppose that* $f, g : \mathbb{N} \to \mathbb{R}$ *are such that* $f(n) \le Me^{-\alpha n}$ *and* $g(n) \le Ne^{-\beta n}$*, for all* $n \in \mathbb{N}$ *and for some* $M, N, \alpha, \beta > 0$*.*

Then there exists $D, \lambda > 0$ *such that* $f(n) + g(n) \le De^{-\lambda n}$*, for all* $n \in \mathbb{N}$*.*

Proof of Lemma 5. We have the following chain of inequalities:

$$f(n) + g(n) \leq Me^{-\alpha n} + Ne^{-\beta n} \leq Me^{-\lambda n} + Ne^{-\lambda n},$$

where $\lambda := \min\{\alpha, \beta\}$.

Then, we have:

$$Me^{-\lambda n} + Ne^{-\lambda n} \leq Ke^{-\lambda n} + Ke^{-\lambda n} \leq 2Ke^{-\lambda n},$$

where $K := \max\{M, N\}$.

This is the desired result, taking $D = 2K$. □

Now, we present the examples:

Example 5. *Consider the Banach space $B = \ell^1_w$ with $w : \mathbb{Z}_{\leq 0} \to \mathbb{R}_{>0}$ given by $w(k) = 2^k$ for $k \in \mathbb{Z}_{\leq 0}$. The norm of $v \in B$ is given by $\|v\| = \sum_{k\leq 0} 2^k |v(k)|$ and property (7.1) holds for some bounded sequences K_1 and K_2.*

Now we consider (6.6) for the operators $L_m : B \to \mathbb{R}$ given by

$$L_m v = \begin{cases} v(0)/|m|! & \text{if } m < 0, \\ \sum\limits_{k\leq 0} 2^k v(k) & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

Note that property (7.5) holds. We will show that (6.6) has an exponential dichotomy with projections $P_n = \mathrm{Id}$ for all $n \in \mathbb{Z}$.

For each $m, n \in \mathbb{Z}$ with $m > n$ we have

$$T(m,n)v = (\cdots, v(-1), v(0), x(n+1), x(n+2), \cdots, x(m-1), x(m)), \tag{9.4}$$

where $x = x^L(\cdot, n, v)$. Assume first that $n \geq 1$. Then $m > n \geq 1$ and

$$T(m,n)v = (\cdots, v(-1), v(0), 0^{m-n}) = \tau^{m-n}v,$$

where τ is defined in (6.2). Therefore,

$$\|T(m,n)v\| = \left\|\tau^{m-n}v\right\| = 2^{-(m-n)}\|v\|.$$

Now assume that $m \leq 0$. Then $0 \geq m > n$ and

$$\begin{aligned} \|T(m,n)v\| &= \left\|\left(\cdots, v(-1), v(0), \frac{v(0)}{a_0}, \frac{v(0)}{a_1}, \cdots, \frac{v(0)}{a_{m-n-1}}\right)\right\| \\ &\leq \left\|\tau^{m-n}v\right\| + \left\|\left(\cdots, 0, 0, \frac{v(0)}{a_0}, \frac{v(0)}{a_1}, \cdots, \frac{v(0)}{a_{m-n-1}}\right)\right\|, \end{aligned} \tag{9.5}$$

where $a_k = |n|! \cdots |n+k|!$ *for each* $k \geq 0$. *Each of the terms* $a_0, \ldots, a_{m-n-1}$ *is greater than or equal to* $(m-n)!$ *since* $|n| > m-n \Leftrightarrow m < 0$. *Therefore,*

$$\begin{aligned} \|T(m,n)v\| &\leq 2^{-(m-n)} \|v\| + \sum_{k=n-m+1}^{0} 2^k \frac{|v(0)|}{(m-n)!} \\ &\leq 2^{-(m-n)} \|v\| + \frac{m-n}{(m-n)!} \|v\| \\ &\leq 2^{-(m-n)} \|v\| + 4 \cdot 2^{-(m-n)} \|v\|, \end{aligned} \tag{9.6}$$

using in the last inequality that

$$k/k! \leq 2^{-k+2} \quad \text{for any integer } k \geq 0. \tag{9.7}$$

Finally, assume that $n \leq 0$ *and* $m \geq 1$. *Then* $m \geq 1 > 0 \geq n$ *and*

$$\begin{aligned} \|T(m,n)v\| &= \left\|(\cdots, v(-1), v(0), x(n+1), \cdots, x(-1), x(0), x(1), 0^{m-1})\right\| \\ &\leq \left\|\tau^{m-n} v\right\| + \left\|\tau^{m-1}(\cdots, 0, 0, x(n+1), \cdots, x(-1), x(0), x(1))\right\| \\ &= 2^{-(m-n)} \|v\| + 2^{-(m-1)} C, \end{aligned}$$

where

$$\begin{aligned} C &= \left\|\left(\cdots, 0, 0, \frac{v(0)}{a_0}, \frac{v(0)}{a_1}, \cdots, \frac{v(0)}{a_{|n|-1}}, x(1)\right)\right\| \\ &\leq \sum_{k=n}^{-1} 2^k \frac{|v(0)|}{|n|!} + |x(1)| \\ &\leq \frac{|n|}{|n|!} \|v\| + |x(1)| \leq 2^{-|n|+2} \|v\| + |x(1)|, \end{aligned}$$

using again (9.7). *Moreover,*

$$\begin{aligned} |x(1)| &\leq \sum_{k \leq 0} 2^{k-|n|} |v(k)| + \sum_{k=n+1}^{0} \frac{2^k |v(0)|}{|n|!} \\ &\leq 2^{-|n|+1} \|v\| + \frac{|n|}{|n|!} \|v\| \leq 2^{-|n|+1} \|v\| + 2^{-\|n\|+2} \|v\|. \end{aligned}$$

Putting everything together we obtain

$$\|T(m,n)v\| \leq 21 \cdot 2^{-(m-n)} \|v\|.$$

This readily implies that there exist constants $D, \lambda > 0$ *such that*

$$\|T(m,n)v\| \leq De^{-\lambda(m-n)} \|v\|$$

for any $m > n$, *which shows that* (6.6) *has an exponential dichotomy with projections* $P_n = \mathrm{Id}$ *for* $n \in \mathbb{Z}$.

In the former example, the image of one of the operators L_m depends on infinitely many components. The following example shows that exponential dichotomies can also occur when all operators L_m depend on finitely many components (in fact, even in one component), although the delay in each step become arbitrarily large, so the infinite-dimensional space is indeed required. We continue to consider the same Banach space $B = \ell^1_w$, with $w(k) = 2^k$ for $k \in \mathbb{Z}_{\leq 0}$.

Example 6. *Consider equation (6.6) for the operators $L_m \colon B \to \mathbb{R}$ given by*

$$L_m v = \begin{cases} v(0)/|m|! & \text{if } m < 0, \\ v(-m)/c_m & \text{if } m \geq 0, \end{cases}$$

where $c_k = \prod_{i=0}^{k} i!$. For $m \geq 0$ we have $|L_m v| \leq 2^m \|v\|/c_m$ and so property (7.6) holds. Assume first that $n < m \leq 0$. Then (9.5) and (9.6) hold. Now assume that $0 \leq n < m$. By (9.4) we have

$$\begin{aligned} \|T(m,n)v\| &= \left\| \left(\cdots, v(-1), v(0), \frac{v(-n)}{c_n}, \cdots, \frac{v(-n)}{,} c_{m-1} \right) \right\| \\ &\leq 2^{-(m-n)}\|v\| + \sum_{k=0}^{m-n-1} \frac{|v(-n)|}{c_{n+k} 2^{m-n-1-k}} \\ &\leq 2^{-(m-n)}\|v\| + \frac{2^n \|v\|}{c_n} \cdot \frac{1}{2^{m-n-1}} \sum_{k=0}^{\infty} \frac{2^k}{k!} \leq \kappa 2^{-(m-n)} \|v\| \end{aligned}$$

for some $\kappa > 0$, since $|v(-n)| \leq 2^n \|v\|$ and $2^n/c_n$ is bounded for $n \geq 0$. Finally, assume that $n \leq 0 < 1 \leq m$ and write $z = v(0)/c_{|n|}$. Then

$$\begin{aligned} \|T(m,n)v\| &= \left\| \left(\cdots, v(-1), v(0), \frac{v(0)}{a_0}, \frac{v(0)}{a_1}, \cdots, z, \frac{z}{c_0}, \frac{z}{c_1}, \cdots, \frac{z}{c_{m-1}} \right) \right\| \\ &\leq 2^{-(m-n)}\|v\| + 2^{-m} \sum_{k=n}^{-1} \frac{|v(0)|}{a_{k-n}} + \sum_{k=0}^{m-1} \frac{|z|}{k! 2^{m-1-k}} \\ &\leq 2^{-(m-n)}\|v\| + 2^{-m} \frac{|n|}{|n|!}\|v\| + \frac{\|v\|}{2^{m-1}|n|!} \sum_{k=0}^{\infty} \frac{2^k}{k!} \end{aligned}$$

that again can be bound by $\kappa 2^{-(m-n)}\|v\|$ for some $\kappa > 0$. Thus, equation (6.6) has an exponential dichotomy with $P_n = \mathrm{Id}$ for $n \in \mathbb{Z}$.

At last, we have an equation that has the same characteristic of having each step a larger delay, but it also grows the number of coordinates in which it depends. Again, the underlying space is $B = \ell^1_w$, where $w(k) = 2^k$ for $k \in \mathbb{Z}_{\leq 0}$.

Example 7. *We consider the equation*

$$x(m+1) = L_m x_m \quad \text{for } m \in \mathbb{Z}, \tag{9.8}$$

where

$$L_m v = \begin{cases} \dfrac{v(0)}{|m|!}, & \text{for } m < 0 \\ \\ \displaystyle\sum_{k=m}^{2m} \frac{2^{-k} v(-k)}{F_m}, & \text{for } m \geq 0, \end{cases}$$

where $F_k := \prod_{j=0}^{k} j!$.

Note that $\|L_m\| \leq C_L$, *for some* $C_L > 0$ *and for all* $m \in \mathbb{Z}$. *For each* $m, n \in \mathbb{Z}$ *with* $m > n$ *we have*

$$T(m,n)v = (\cdots, v(-1), v(0), x(n+1), x(n+2), \cdots, x(m-1), x(m)),$$

where $x = x^L(\cdot, n, v)$.

Assume first that $n < m \leq 0$, *therefore*

$$\begin{aligned} \|T(m,n)v\| &= \left\| \left(\cdots, v(-1), v(0), \frac{v(0)}{|n|!}, \frac{v(0)}{|n|! \cdot |n+1|!}, \cdots, \frac{v(0)}{|n|! \cdots |m-1|!} \right) \right\| \\ &\leq 2^{-(m-n)} \|v\| + \frac{(m-n)}{(m-n)!} \|v\| \\ &\leq 2^{-(m-n)} \|v\| + K \cdot 2^{-(m-n)} \|v\|, \end{aligned}$$

for some $K > 0$. *Using Lemma 5, we obtain the exponential dichotomy for this case.*

For the next case, we are going to use a lemma:

Lemma 6. *Let* $0 \leq n$. *Then, for all* $k \geq n+1$, *we have that*

$$|x(k,n)| \leq \frac{\|v\|}{F_{k-1}}.$$

Proof of Lemma 6. Proof by induction.

The base case $k = n+1$ follows from

$$|x(n+1)| = |L_n v| \leq \sum_{j=0}^{\infty} \frac{\left|2^{-k} v(-k)\right|}{F_n} = \frac{\|v\|}{F_n}.$$

Now, suppose the statement valid for $k \geq n+1$. Then

$$\begin{aligned}
|x(k+1)| &= |L_k x_k| = |L_k(\dots v(-1)v(0)x(n+1)\dots x(k))| \\
&\leq \sum_{j=k}^{2k} \frac{|2^{-j}v(-n+k-j)|}{F_k} \\
&= \sum_{j=0}^{k} \frac{\left|2^{-k-j}v(-n-j)\right|}{F_k} \\
&\leq \frac{1}{2^k F_k} \sum_{j=0}^{\infty} \frac{|2^{-n-j}v(-n-j)|}{2^{-n}} \\
&\leq \frac{2^n}{2^k F_k} \sum_{j=0}^{\infty} |2^{-n-j}v(-n-j)| \leq \underbrace{\frac{2^n}{2^k}}_{\leq 1} \cdot \frac{1}{F_k} \|v\| \leq \frac{\|v\|}{F_k}.
\end{aligned}$$

$\square$

Now assume that $0 \leq n < m$, thus

$$\begin{aligned}
\|T(m,n)v\| &= \left\| \Big(\cdots, v(-1), v(0), x(n+1), \cdots, x(m) \Big) \right\| \\
&\leq 2^{-(m-n)} \|v\| + \sum_{k=n+1}^{m} \frac{1}{2^{m-k}} \cdot \frac{\|v\|}{F_{k-1}} \\
&\leq 2^{-(m-n)} \|v\| + \|v\| \cdot \sum_{k=0}^{m-n-1} \frac{1}{2^{m-n-1-k} F_{n+k}} \\
&\leq 2^{-(m-n)} \|v\| + \frac{\|v\|}{2^{m-n-1} F_n} \cdot \sum_{k=0}^{\infty} \frac{2^k}{k!} \\
&\leq 2^{-(m-n)} \|v\| + D \cdot 2^{-(m-n)} \|v\|,
\end{aligned}$$

since $\sum_{k=0}^{\infty} \frac{2^k}{k!}$ *converges. Again, using Lemma 5, we conclude that this case also has exponential dichotomy.*

For the last case, we also are going to need another lemma:

Lemma 7. *Let $n \leq 0$. Then, for all $k \geq 1$, we have that*

$$|x(k)| \leq \frac{1}{(k-1)!} \cdot \left\{ 2^n \|v\| + \frac{|n|}{|n|!} \|v\| \right\}.$$

Proof of Lemma 7. Proof by induction.

For the base case $k = 1$, we have that

$$|x(1)| = |L_0 x_0| = \left| L_0 \left(\cdots, v(-1), v(0), \frac{v(0)}{|n|!}, \frac{v(0)}{|n|! \cdot |n+1|!}, \cdots, \frac{v(0)}{F_{|n|}} \right) \right| = \frac{|v(0)|}{F_{|n|}} \leq \|v\|,$$

which proves the base case.

Now, suppose the statement valid for $k \geq 1$. Then

$$
\begin{aligned}
&|x(k+1)| = |L_k x_k| \\
=&\left| L_k \left(\cdots, v(-1), v(0), \frac{v(0)}{|n|!}, \frac{v(0)}{|n|! \cdot |n+1|!}, \cdots, \frac{v(0)}{F_{|n|}}, x(1), \cdots, x(k) \right) \right| \\
\leq& \frac{1}{k!} \cdot \left\{ \sum_{j=n+1}^{0} \frac{|v(0)|}{|n|!} + \sum_{j=-k}^{n} 2^j \left| v(j-n) \right| \right\} \\
\leq& \frac{1}{k!} \cdot \left\{ \frac{|n|}{|n|!} \|v\| + \sum_{j=-k-n}^{0} 2^{j+n} |v(j)| \right\} \\
\leq& \frac{1}{k!} \cdot \left\{ \frac{|n|}{|n|!} \|v\| + 2^n \sum_{j=-\infty}^{0} 2^j \left| v(j) \right| \right\} \\
\leq& \frac{1}{k!} \cdot \left\{ \frac{|n|}{|n|!} \|v\| + 2^n \|v\| \right\},
\end{aligned}
$$

as desired. □

Note that this implies that $|x(k)| \leq \dfrac{c \cdot 2^n}{(k-1)!} \|v\|$*, for some* $c > 0$*, for all* $k \geq 1$*.*

At last, we must check the case $n \leq 0 < 1 \leq m$*:*

$$
\begin{aligned}
\|T(m,n)v\| &= \left\| \left(\underbrace{\cdots, v(-1), v(0),}_{A} \underbrace{\frac{v(0)}{|n|!}, \frac{v(0)}{|n|! \cdot |n+1|!}, \cdots, \frac{v(0)}{F_{|n|}},}_{B} \underbrace{x(1), \cdots, x(m)}_{C} \right) \right\| \\
&\leq \underbrace{2^{-(m-n)} \|v\|}_{A} + \underbrace{2^{-m} \cdot \sum_{k=n+1}^{0} \frac{|v(0)|}{|n|!}}_{B} + \underbrace{\sum_{k=1}^{m} \frac{1}{2^{m-k}} \cdot |x(k)|}_{C} \\
&\leq \underbrace{2^{-(m-n)} \|v\|}_{A} + \underbrace{2^{-m} \cdot \frac{|n|}{|n|!} \|v\|}_{B} + \underbrace{2^{-m} \cdot \sum_{k=1}^{m} 2^k \cdot \frac{D \cdot 2^n}{(k-1)!} \|v\|}_{C} \\
&\leq \underbrace{2^{-(m-n)} \|v\|}_{A} + \underbrace{2^{-m} \cdot \frac{|n|}{|n|!} \|v\|}_{B} + \underbrace{D \cdot 2^{-(m-n)} \cdot \sum_{k=1}^{\infty} \frac{k 2^k}{k!} \|v\|}_{C} \\
&\leq \underbrace{2^{-(m-n)} \|v\|}_{A} + \underbrace{D_1 \cdot 2^{-(m-n)} \|v\|}_{B} + \underbrace{D_2 \cdot 2^{-(m-n)} \|v\|}_{C},
\end{aligned}
$$

given that $\sum_{k=0}^{\infty} k 2^k / k!$ *converges. Lemma 5 allows us to conclude that this case also has exponential dichotomy.*

Therefore, since all cases have exponential dichotomy, taking the appropriate constants, we conclude that the equation (9.8) has exponential dichotomy, with $P_n = \mathrm{Id}$*, for all* $n \in \mathbb{Z}$*.*

9.3 Examples Obtained From Perturbations

Example 8. *As we shall see in Lemma 9, the property of having exponential behavior is robust under small linear perturbations. This allows us to create further examples of exponential dichotomies from known examples. This property may even assures that desired properties are obtained.*

More precisely, assume that the property in (7.6) holds and let L be a bounded sequence such that (6.6) has an exponential dichotomy. If a given sequence of bounded linear operators $N = (N_m)_{m\in\mathbb{Z}} \subseteq \mathcal{L}(B, X)$ is such that there exists $\delta > 0$ satisfying $\sup_{m\in\mathbb{Z}} \|N_m\| < \delta$, then the equation

$$y(m+1) = (L_m + N_m)y_m \quad \text{for } m \in \mathbb{Z}$$

also has an exponential dichotomy (for possibly different multiplicative and exponential constants).

10 Applications

This section is devoted to present some applications of the main result. They follow from the extension of the exponential dichotomy property of an equation to any other equation in its invariant hull. This amounts to the persistence of hyperbolicity under sufficiently small perturbations and for the spectra of delay-difference equations.

10.1 Extension of the Hyperbolicity

In view of giving some applications of our main result in Theorem B1 we first show, as consequence of this theorem, that if the delay-difference (6.6) has an exponential dichotomy, then any (8.2) with $M \in \mathcal{Z}$ also has an exponential dichotomy. Here $\mathcal{Z}$ denotes the invariant hull of (6.6) (see Section 7.2).

Lemma 8. *Assume that property* (7.6) *holds and let L be a bounded sequence such that* (6.6) *has an exponential dichotomy with constants λ and c. Then for each sequence $M \in \mathcal{Z}$* (8.2) *also has an exponential dichotomy, with the same constants λ and c.*

Proof. It follows from Theorem B1 that the cocycle S_L is hyperbolic. Given $M \in \mathcal{Z}$, we define the operators

$$\bar{T}(m,n) = S_L(\sigma^n M, m-n) \quad \text{and} \quad \bar{P}_n = P(\sigma^n M)$$

for each $m, n \in \mathbb{Z}$ with $m \geq n$. It follows readily from the definition of S_L that

$$\bar{T}(m,n) = T_{\sigma^n M}(m-n,0) = T_M(m,n).$$

Moreover, since the cocycle S_L is hyperbolic:

1. For each $m \geq n$ we have

$$\begin{aligned} S_L(\sigma^n M, m-n)P(\sigma^n M) &= P(\sigma^{m-n}\sigma^n M)S_L(\sigma^n M, m-n) \\ &= P(\sigma^m M)S_L(\sigma^n M, m-n) \end{aligned}$$

   and so

$$\bar{T}(m,n)\bar{P}_n = \bar{P}_m \bar{T}(m,n);$$

2. The linear operator

$$\bar{T}(m,n)|_{\ker \bar{P}_n} = S_L(\sigma^n M, m-n)|_{\ker P(\sigma^n M)} : \ker \bar{P}_n \to \ker \bar{P}_m$$

   is onto an invertible for each $m \geq n$;

3. For each $m \geq n$ we have

$$\left\| \bar{T}(m,n)\bar{P}_n \right\| \leq ce^{-\lambda(m-n)}$$

and

$$\left\| \bar{T}(m,n)|_{\ker \bar{P}_n}^{-1}(\mathrm{Id} - \bar{P}_n) \right\| \leq ce^{-\lambda(m-n)}.$$

We emphasize that the constants λ and c are independent of M. This shows that (8.2) has an exponential dichotomy with projections $\bar{P}_n$ and with the desired constants λ and c. $\square$

Finally, as a consequence of Theorem B1 and Lemma 8 we obtain the following result, which considers both (6.6) and (8.2).

Theorem B2. *Assume that property* (7.6) *holds and let L be a bounded sequence. Then the following properties are equivalent:*

1. *The cocycle S_L is hyperbolic;*
2. (6.6) *has an exponential dichotomy;*
3. (8.2) *has an exponential dichotomy for each $M \in \mathcal{Z}$.*

10.2 Robustness Property

In this section we show that the existence of an exponential dichotomy not only persists under sufficiently small perturbations, but also that the same happens to any (8.2) with $M \in \mathcal{Z}$. More precisely, this amounts to consider the perturbed equations

$$y(m+1) = (M_m + N_m)y_m \quad \text{for } m \in \mathbb{Z} \tag{10.1}$$

with $M \in \mathcal{Z}$, where N_m for $m \in \mathbb{Z}$ are linear operators in $\mathcal{L}(B, X)$, and show that all of them have an exponential dichotomy provided that all N_m are sufficiently small.

To prove that, we need the following lemma, which is a particular case of (BARREIRA; RIJO; VALLS, 2020, Theorem 2), which deals with more general cases than small linear perturbations:

Lemma 9. *Assume that property* (7.6) *holds and let L be a bounded sequence such that* (6.6) *has an exponential dichotomy. Then there exists $\delta > 0$ (depending only on the constants λ and c of the exponential dichotomy) such that if $\sup_{m\in\mathbb{Z}} \|N_m\| < \delta$, then the equation*

$$y(m+1) = (L_m + N_m)y_m \quad \textit{for } m \in \mathbb{Z}$$

has an exponential dichotomy.

Theorem B3. *Assume that property (7.6) holds and let L be a bounded sequence such that (6.6) has an exponential dichotomy. Then there exists $\delta > 0$ such that (10.1) has an exponential dichotomy whenever $M \in \mathcal{Z}$ and $\sup_{m\in\mathbb{Z}} \|N_m\| < \delta$.*

Proof. Since the number δ in the lemma depends only on the constants λ and c of the exponential dichotomy, it follows readily from Lemma 8 that one can use the same constant δ for each (8.2) with $M \in \mathcal{Z}$. Hence, the desired statement follows readily from Lemma 9. □

10.3 Spectra

As another consequence of Lemma 8, we show in this section that the spectra of all delay-difference equations in (8.2) with $M \in \mathcal{Z}$ coincide.

We start by recalling the notion of spectrum, which is a natural generalization of the notion of spectrum for a single linear operator.

Definition 13. *The spectrum Σ_L of (6.6) is the set of all numbers $a \in \mathbb{R}$ such that the evolution family $(T_L^a(m,n))_{m\geq n}$ defined by*

$$T_L^a(m,n) = e^{-a(m-n)}T_L(m,n)$$

has an exponential dichotomy.

Similarly, given $a \in \mathbb{R}$, we define a cocycle $S_L^a : \mathcal{Z} \times \mathbb{N}_0 \to \mathcal{L}(B)$ by

$$S_L^a(M,n) = e^{-an}S_L(M,n) = e^{-an}T_M(n,0).$$

Essentially repeating the arguments in the proof of Theorem B1, one can easily obtain the following result.

Theorem B4. *Assume that property (7.6) holds and let L be a bounded sequence. Then for each $a \in \mathbb{R}$ the following properties are equivalent:*

1. *The cocycle S_L^a is hyperbolic;*
2. *The evolution family $(T_L^a(m,n))_{m\geq n}$ has an exponential dichotomy.*

As a consequence of Theorem B4, one can be conveniently rephrase Lemma 8 for each constant $a \in \mathbb{R}$ as follows.

Corollary 2. *Assume that property (7.6) holds and let L be a bounded sequence. Then for each $a \in \mathbb{R}$ the evolution family $(T_L^a(m,n))_{m\geq n}$ has an exponential dichotomy if and only if $(T_M^a(m,n))_{m\geq n}$ has an exponential dichotomy for all $M \in \mathcal{Z}$, in which case the constants λ and c are independent of the sequence M.*

The following result is now a simple consequence of this corollary.

Corollary 3. *Assume that property (7.6) holds and let L be a bounded sequence. Then $\Sigma_L = \Sigma_M$ for all $M \in \mathcal{Z}$.*

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