GLOBAL STRONG SOLUTIONS OF THE EQUATIONS FOR THE MOTION OF NONHOMOGENEOUS INCOMPRESSIBLE FLUIDS

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ABSTRACT – By using the spectral semi-Galerkin method, we prove a result on global existence in time of strong solutions of the Navier-Stokes equations for the motion of nonhomogeneous incompressible fluids. This was obtained without assuming that the external force field decay with time. We reach in this way basically the same level of knowledge as in the case of the classical Navier-Stokes equations. We also derive estimates that are useful for obtaining error bounds for the approximate solutions. Stronger forms of these estimates, including an uniform in time estimate for the gradient of the density, are obtained when the external force field decays exponentially.

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Abstract

By using the spectral semi-Galerkin method, we prove a result on global existence in time of strong solutions of the Navier-Stokes equations for the motion of nonhomogeneous incompressible fluids. This was obtained without assuming that the external force field decay with time. We reach in this way basically the same level of knowledge as in the case of the classical Navier-Stokes equations. We also derive estimates that are useful for obtaining error bounds for the approximate solutions. Stronger forms of these estimates, including an uniform in time estimate for the gradient of the density, are obtained when the external force field decays exponentially.

1 Introduction

In this work we will be concerned with global existence in time of strong solutions of the three dimensional stratified (or nonhomogeneous) Navier-Stokes equations, that is, the equations for the motion of a nonhomogeneous incompressible fluid (obtained as a mixture of miscible incompressible fluids, for instance). Being $\Omega \subset I\!R^n$, $n = 2$ or 3, a $C^{1,1}$-regular bounded open set, these equations are

$$
(1.1)\qquad \begin{cases} \rho \dfrac{\partial u}{\partial t} + \rho u.\nabla u - \Delta u - \text{grad} p = \rho f \ , \\ \\ \text{div } u = 0 \ , \\ \\ \dfrac{\partial \rho}{\partial t} + u \nabla \rho = 0 \quad \text{in} \quad \Omega \, , \\ \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \ , \\ \\ \rho|_{t=0}(x) = \rho_0 \quad \text{in} \quad \Omega \ , \\ \\ u|_{t=0}(x) = u_0 \quad \text{in} \quad \Omega \ . \end{cases}
$$

Here Ω is the container where the fluid is inside; $u(x,t) \in I\!R^n$ denotes the velocity of the fluid at a point $x \in \Omega$ and time $t \in [0,\infty)$; $\rho(x,t) \in I\!R$ and $p(x,t) \in I\!R$ denote the density and the hydrostatic pressure of the fluid respectively; $u_0(x)$ and $\rho_0(x)$ are initial velocity and density, respectively; $f(x,t)$ is the density by unit of mass of the external force acting on the fluid. Here, without loosing generality, we have scaled the variables in order to the viscosity to be one; the fluid adheres to the wall $\partial\Omega$ of the container which is assumed to be at rest. The expressions grad, Δ and div denote the gradient, Laplacian and divergence operators, respectively (we also denote the gradient operator by ∇ and $\dfrac{\partial u}{\partial t}$ by u_t); the $i^{\underline{\text{th}}}$ cartesian component of $u.\nabla u$ is given by $(u.\nabla u)_i = \sum_{j=1}^{n} u_j \dfrac{\partial u_i}{\partial x_j}$; $u.\nabla \rho = \sum_{j=1}^{n} u_j \dfrac{\partial \rho}{\partial x_j}$. The first equation in (1.1) corresponds to the balance of linear momentum; the third equation to the balance of mass, and the second one states that the fluid is incompressible. The unknowns in the problem are u, ρ and p.

The classical Navier-Stokes equations correspond to the special case where $\rho(x,t) = \rho_0$ is a positive constant; in this case the third equation in (1.1) is automatically satisfied. This case has been much studied (see Ladyszhenskaya [12] and Teman [19] and the references there in).

Equations (1.1) have been less studied, maybe due to their mixed parabolic-hyperbolic character. Antonzev and Kazhikov [2], Kazhikov [9], Lions [14], Simon [18] and Kim [11] have studied local and global existence for weak solutions of (1.1). Stronger local and global solution were obtained by Ladyszhenkaya and Solonnikov [13] by linearization and fixed point arguments, and also by Okamoto [15] by using evolution operators techniques and also fixed point arguments; they assume exponential decay of the external forces. The more constructive spectral semi-Galerkin method was used by Salvi [17] to obtain local in time strong solutions and to study conditions for regularity at $t = 0$. That technique was also used by Boldrini and Rojas-Medar [3] to obtain global strong solutions corresponding to external forces with a mild form of decay in time.

We observe that all these known results on global existence of strong solutions require some sort of decay in time of the associated external force. Ladyszhenkaya and Solonnikov worked in the L^q-space $(q > n)$ and required exponential decay in time of the (small) $L^q(\Omega)$-norm of the external force. Okamoto worked in the L^2-space and f identically zero, and, in order to obtain the same result for nonzero force field, at least an exponential decay in time of the $L^2(\Omega)$-norm of f would be required. Boldrini and Rojas-Medar worked in the L^2-space by requiring the milder form of decay $f \in L^2([0,\infty);(L^2(\Omega))^n)$.

However, in the case of the classical Navier-Stokes equations, this kind of requirement is not necessary (see for instance, Heywood and Rannacher [9]), and, therefore, one excepts to prove global existence without it in the case of equations (1.1).

This is indeed true, and we prove it in this paper by assuming f belonging $L^\infty([0,\infty);(L^2(\Omega))^n)$ (with small enough norms, as usual, in the three dimensional case; any norm in the two dimensional case) and certain other regularity assumptions that will he detailed later on in the paper.

Thus we reach basically the same level of knowledge as the one in the case of the classical Navier-Stokes equations.

Also, we present a sequence of estimates for the (strong) solutions of (1.1) and their spectral approximations. These estimates are important because they are used in an essencial way in a forthcoming paper by Boldrini and Rojas-Medar [4] to obtain uniform in time error bounds for the spectral approximations of (1.1). These estimates are similar to the ones in Heywood [8] in the case of the classical Navier-Stokes, and they are derived under a certain assumption on the stability of the solution being approximated. Also, thanks to the estimates present here, these uniform in time error bounds are obtained without non realistic assumptions like the ones in Salvi [16], which require a global compatibility condition on the initial data.

2 Global Existence in the Case of External Forces without Decay

We start by recalling certain definition and fact that will be used in the rest of the paper.

In what follows we will assume Ω of class $C^{1,1}$. We will consider the usual Sobolev spaces

$$W^{m,q}(D) = \{f \in L^q(D);\ \|\partial^\alpha f\|_{L^q(D)} < +\infty,\ (|a| \leq m)\}\ ,$$

$m = 0, 1, 2, \ldots, 1 \leq q \leq \infty$, $D = \Omega$ or $\Omega \times (0,T)$, $0 < T \leq +\infty$, with the usual norm. When $q = 2$, we denote $H^m(D) = W^{m,2}(D)$ and $H_0^m(D) =$ closure of $C_0^\infty(\Omega)$ in $H^m(D)$. If B is a Banach-space, we denote by $L^q([0,T); B)$ the Banach space of the B-valued functions defined in the interval $[0,T)$ that are L^q-integrable in the sense of Bochner.

Let $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega)^n;\ \text{div } v = 0 \text{ in } \Omega\}$; $V =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $H_0^1(\Omega)^n$, and $H =$ closure of $C_{0,\sigma}^\infty(\Omega)$ in $L_0^2(\Omega)^n$.

Let P be the orthogonal projection from $L^2(\Omega)^n$ onto H obtained by the usual Helmholtz decomposition. Then the operator $A : H \to H$ given by $A = -P\Delta$ with domain $D(A) = H^2(\Omega)^n \cap V$ is called the Stokes operator. It is well known that A is a positive definite self-adjoint operator and is characterized by the relation

$$(Aw, v) = (\nabla w, \nabla v) \quad \text{for all} \quad w \in D(A),\quad v \in V\ .$$

From now on, we denote the inner product in H (i.e., the L^2-inner product) by (,). The general L^p-norm will be denoted by $\|\ \|_{L^p(\Omega)}$; to easy the notation, in the case $p = 2$ we simply denote the L^2-norm by $\|\ \|$.

We observe that for the regularity properties of the Stokes operator, it is usually assumed that Ω is of class C^3; this being in order to use Cattabriga's results [5]. We use instead the stronger results of Amrouche and Girault [1] which implies, in particular, that when $Au \in (L^2(\Omega))^n$ then $u \in H^2(\Omega)$ and $\|u\|_{H^2}$ and $\|Au\|$ are equivalent norms when Ω is of class $C^{1,1}$. This will be enough for all of the results in this paper except for the last one (Proposition 4.3) which will require that Ω is of class $C^{2,1}$ because in that proposition we want to work with H^3-regularity.

We will denote respectively by φ_k and λ_k $(k \in N)$ the eigenfunctions and eigenvalues the Stokes operator defined on $V \cap (H^2(\Omega))^n$. It is well know that $\{\varphi^k\}_{k=1}^{\infty}$ form an orthogonal complete system in the spaces H, V and $V \cap (H^2(\Omega))^n$ with their usual inner products (u,v), $(\nabla u, \nabla v)$ and (Au, Av), respectively.

We denote by $V_k = \text{span}\ [\varphi_1, \ldots, \varphi_k]$.

Still concerning the matter of notation, as it is usual we will denote $c, c_1, \ldots, C, C_1, \ldots$ generic constants depending only on the fixed data of the problem.

The following assumptions on the initial data will hold throughout this paper:

(A.1) the initial value for the density ρ_0 belongs to $C^1(\overline{\Omega})$ and satisfies $0 < \alpha \leq \rho_0(x) \leq \beta < +\infty$ ae in Ω.

(A.2) the initial value u_0 belongs to $V \cap (H^2(\Omega))^n$.

Now, we rewrite problem (1.1) as follows: find $\rho \in C^1(\Omega \times (0,T))$ and $u \in C([0,T); D(A))$, $u_t \in L^\infty((0,T); H) \cap L^2((0,T); V)$ $(0 < T \leq +\infty)$ such that

$$
(2.1) \qquad \begin{cases} (\rho u_t, v) + (\rho u.\nabla u, v) + (Au, v) = (\rho f, v)\ , \ \forall v \in H\ , \\ \rho_t + u.\nabla \rho = 0\ , \ \text{ae}\ (x,t) \in \Omega \times (0,T)\ , \\ u|_{t=0} = u_0\ , \quad \rho|_{t=0} = \rho_0\ , \ \text{ae}\ x \in \Omega\ . \end{cases}
$$

The espectral semi-Galerkin approximations for (u, ρ) are defined for each $k \in N$ as the solution $(u^k, \rho^k) \in C^2([0,T]; V_k) \cap C^1(\overline{\Omega} \times [0,T))$ of

$$
(2.2) \qquad \begin{cases} (\rho^k u_t^k, v) + (\rho^k u^k.\nabla u^k, v) + (Au^k, v) = (\rho^k f, v)\ , \ \forall v \in V_k\ , \\ \rho_t^k + u^k.\nabla \rho^k = 0\ , \quad \text{for}\ \ \forall (x,t) \in \Omega \times (0,T)\ , \\ u(x,0) = u_0^k(x) \quad \rho(0,x) = \rho_0(x)\ , \quad \forall x \in \Omega\ . \end{cases}
$$

Here, u_0^k are the projections of u_0 on V_k.

By using these approximations, Salvi [17] proved a local in time existence theorem for (2.1) under assumptions (A.1) and (A.2). His result is the following. (Theorem 1 and 2, in Salvi [17])

Proposition 2.1 Suppose that (A.1) and (A.2) are true and that $f \in L^2((0,T);\ (H^1(\Omega))^n,\ f_t \in L^2((0,T);\ (L^2(\Omega))^n);\ 0 < T < +\infty$. Then there is $0 < T' \leq T$ such that the solution (u, ρ) of (2.1) satisfies

$$u \in L^\infty((0,T'); V \cap (H^2(\Omega))^n) \cap L^2((0,T'); (H^3(\Omega))^n \cap V) \quad , \quad \rho \in C^1(\Omega \times (0,T'))$$

$\alpha \leq \rho(x,t) \leq \beta$, for almost all (x,t) in $\Omega \times (0,T')$. This solution is unique.

Remarks.

i) The result stated by Salvi requires only $f \in (L^2(\Omega \times (0,T)))^n$ and $f_t \in (L^2(\Omega \times (0,T)))^n$. In fact, his proof is done with $f \equiv 0$, and, in order for it to be true for $f \not\equiv 0$, it is necessary to assume what we stated above.

ii) During the proof of Proposition 2.1, it is easily seen that the semi-Galerkin approximations exist globally in time. (See also Kim [9])

We observe that the result stated in Proposition 2.1 can be improved.

Proposition 2.2. Under the conditions Propisition 2.1, we have

$$u \in C([0,T'], D(A))$$

Proof. We will prove the continuity at $t = 0$; for other points $t_0 > 0$ the argument is quite similar.

We observe that $u \in C([0,T'], V)$ as it was proved by Simon [18] (Theorem 14, p. 1110-1111); so it is enough to prove the continuity in the H^2-norm.

For this we observe that it is enough to show that $\lim\limits_{t \to 0^+} ||P\Delta u(\cdot\ ,\ t)|| \leq ||P\Delta u_0||$, since we already know $u(\cdot\ ,\ t) \to u_0$ in H^1. To this end, setting $v = P\Delta u_t^k$ in (2.2), we obtain

$$||(\rho^k)^{1/2}\nabla u_t^k||^2 + \frac{1}{2}\frac{d}{dt}||P\Delta u^k||^2$$

$$= (\rho^k u^k \nabla u^k - \rho^k f, P\Delta u_t^k) - \int_\Omega u_t^k \nabla \rho^k \nabla u_t^k$$

$$= \frac{d}{dt}(\rho^k u^k \nabla u^k - \rho^k f, P\Delta u^k) - (\rho_t^k u^k \nabla u^k + \rho^k u_t^k \nabla u^k +$$

$$+\rho^k u^k \nabla u_t^k - \rho_t^k f - \rho^k f_t, P\Delta u^k) - \int_\Omega u_t^k \nabla \rho^k \Delta u_t^k \,.$$

By integrating this expression with respect to time, and by using the estimates (4.3), p. 6, (4.17), p. 10 and (4.24), p. 12 in Salvi [17], we have

$$\|P\Delta u^k(t)\|^2 \le \|P\Delta u_0^k\|^2 + 2\{(\rho^k u^k \nabla u^k - \rho^k f, P\Delta u^k)$$
$$-(\rho_0 u_0^k \nabla u_0^k - \rho_0 f(0), P\Delta u_0^k)\} + Nt$$

uniformly in k. From this, we conclude

$$\|P\Delta u^k(t)\|^2 \le \|P\Delta u_0\|^2 + 2\{(\rho u \nabla u - \rho f, P\Delta u)$$
$$-(\rho_0 u_0 \nabla u_0 - \rho_0 f(0), P\Delta u_0)\} + Nt \,.$$

Since $\rho u \nabla u - \rho_0 u_0 \nabla u_0$ in $L^2(\Omega)$ and $P\Delta u \to P\Delta u_0$ weakly in L^2 as $t \to 0^+$, we obtain the desired result. □

Now, we state our first result.

Theorem 2.3. Suppose that $n = 3$, that (A.1) and (A.2) are true and that $f \in L^\infty([0,\infty);(H^1(\Omega))^n)$, $f_t \in L^\infty([0,\infty);(L^2(\Omega))^n)$. If $\|u_0\|_{H^1(\Omega)}$ and $\|f\|_{L^\infty([0,\infty);(L^2(\Omega))^n)}$ are sufficiently small, then the solution (ρ, u) of problem (2.1) exists globally in time and satisfies $u \in C([0,\infty);(H^2(\Omega))^n \cap V)$, $\rho \in C^1(\overline{\Omega} \times [0,T])$ for any finite $T > 0$. Moreover, for any $\gamma > 0$ there exists some finite positive constants M and C such that

$$\sup_{t\ge 0} \|\nabla u(t)\| = M$$
$$\sup_{t\ge 0} \|u_t\| \le C$$
$$\sup_{t\ge 0} \|Au(t)\| \le C$$
$$\sup_{t\ge 0} e^{-\gamma t} \int_0^t e^{\gamma s} \|\nabla u_t(s)\|^2 ds \le C$$
$$\sup_{t\ge 0} e^{-\gamma t} \int_0^t e^{\gamma s} \|u(s)\|^2_{W^{2,6}} ds \le C$$
$$\sup_{t\ge 0} e^{-\gamma t} \int_0^t e^{\gamma s} \|\nabla u(s)\|^2_{C(\Omega)} ds \le C$$

Also, the same kind of estimates hold uniformly in k for the semi-Galerkin approximations.

Remark. We have notationally distinguished the constant in the first of the above estimates because it will play a special role in many points in the arguments that follow.

Proof. We start by proving the boundness in time of $||\nabla u(t)||$.

From Kim [11], p. 94, we have the following differential inequality

$$\frac{d}{dt}||\nabla u||^2 + c||Au||^2 \leq c_1||\nabla u||^{10} + c_3 \tag{2.3}$$

where c, c_1 and $c_3 = c \sup_{t\geq 0}||f(t)||$ are positive constants. Now, we observe that there exist $c_2 > 0$ such that

$$c_2||\nabla u||^2 \leq c||Au||^2,$$

so, in the above inequality we obtain

$$\frac{d}{dt}||\nabla u||^2 \leq c_1||\nabla u||^{10} - c_2||\nabla u||^2 + c_3\,. \tag{2.4}$$

Let $\psi(t) = ||\nabla u(t)||^2$; then we have the following differential inequality

$$\frac{d\psi}{dt} \leq c_1\psi^5 - c_2\psi + c_3$$

$$\psi(0) = ||\nabla u_0||^2\,.$$

We consider the corresponding differential equation

$$\frac{d\phi}{dt} = c_1\phi^5 - c_2\phi + c_3 = F(c_3, \phi)$$

$$\phi(0) = \psi(0)\,.$$

Results from the differential inequalities implies $\psi(t) \leq \phi(t)$ for all t in the interval of existence. Now, we observe that, when $c_3 = 0$, $F(0,\phi) = c_1\phi^5 - c_2\phi$.

Consequently, $F(0,\phi)$ has are simple root given by $r(0) = (c_2/c_1)^{1/4}$; this root is unstable, and so, for c_3 small, $F(c_3,\phi)$ also has one unstable simple root $r(c_3)$ close to

$r(0)$. Thus, if $0 < \psi(0) = \phi(0) < r(c_3)$, we have $0 \leq \psi(t) \leq \phi(t) \leq r(c_3) < +\infty$ for all t in the interval of existence. Thus, there is a constant $M > 0$ such that

(2.5)
$$\sup_{t \geq 0} ||\nabla u|| = M < +\infty .$$

Now, we proceed to prove the other stated estimates. They should be proved first for the approximations (ρ^k, u^k) and then carried to (ρ, u) in the limit. Since that is a standard procedure and the computations are exactly the same, to easy the notation, we will work directly with (ρ, u) in the rest of the paper.

Also, we would like to mention that the technique of using exponentials as weightning functions in time was inspired on Heywood and Rannacher [9].

Again from an inequality in Kim [11], we have

$$\frac{d}{dt}||\nabla u||^2 + c||Au||^2 + c||\rho^{1/2}u_t||^2 \leq c_1||\nabla u||^{10} + c_3 ,$$

Multiplying the above inequality by $e^{\gamma t}, \gamma > 0$, and integrating in time from 0 to t, we have

$$e^{\gamma t}||\nabla u||^2 + c\int_0^t e^{\gamma s}||Au(s)||^2 ds + c\int_0^t e^{\gamma s}||\rho^{1/2}u_t||^2 ds$$

$$\leq c\int_0^t e^{\gamma s}||\nabla u||^{10} ds + c_3\int_0^t e^{\gamma s} ds + \gamma\int_0^t e^{\gamma s}||\nabla u||^2 ds .$$

Multiplying by $e^{-\gamma t}$ and recalling that $||\nabla u(t)||$ is uniformly bounded, we get that

$$e^{-\gamma t}\int_0^t e^{\gamma s}||Au(s)||^2 \quad \text{and} \quad e^{-\gamma}\int_0^t e^{\gamma s}||\rho^{1/2}u_t||^2 ds$$

are also uniformly bounded. Now, by differentiating (2.1) (actually (2.2)) with respect to t and setting $v = u_t$, (actually $v = u_t^k$), after of some computations we have

(2.6)
$$\begin{aligned}&\frac{d}{dt}||\rho^{1/2}u_t||^2 + 2||\nabla u_t||^2 \\ &= 2(\rho_t f, u_t) + 2(\rho f_t, u_t) - 4(\rho u\nabla u_t, u_t) \\ &-2(\rho u_t\nabla u, u_t) - 2(\rho_t u\nabla u, u_t) .\end{aligned}$$

By using the Sobolev type inequality (See Duff [6], p. 464)

(2.7)
$$||\varphi||_{L^4} \leq ||\varphi||^{1/4}||\nabla\varphi||^{3/4} .$$

We have

$$\begin{aligned}
|4(\rho u \nabla u_t, u_t)| &\leq 4||\rho||_{L^\infty}||u||_{L^4}||\nabla u_t|| \; ||u_t||_{L^4} \\
&\leq 4||\rho||_{L^\infty}||\nabla u|| \; ||\nabla u_t||\{||u_t||^{1/4}||\nabla u_t||^{3/4}\} \\
&\leq c_\varepsilon||\rho||^2_{L^\infty}||\nabla u||^8 \; ||u_t||^2 + \varepsilon||\nabla u_t||^2 ,
\end{aligned}$$

thanks to the embedding $H^1 \hookrightarrow L^4$ and Young's inequality. Anagously we can prove that

$$2|(\rho u_t \nabla u, u_t)| \leq c_\varepsilon||\rho||^2_{L^\infty}||\nabla u||^4 \; ||u_t||^2 + \varepsilon||\nabla u_t||^2 .$$

Also, we have

$$|(\rho f_t, u_t)| \leq c_\varepsilon||\rho||^2_{L^\infty}||f_t||^2 + \varepsilon||\nabla u_t||^2 .$$

Using the relation $\rho_t = -\text{div}(\rho u)$, we have

$$\begin{aligned}
2|(\rho_t f, u_t)| &\leq 2|(\rho u \nabla f, u_t)| + 2|(\rho u \nabla u_t, f)| \\
&\leq c_\varepsilon||\rho||^2_{L^\infty}||\nabla u||^2||\nabla f||^2 + c_\varepsilon||\rho||^2_{L^\infty}||\nabla u||^2 \; ||f||^2_{L^4} + 2\varepsilon||\nabla u_t||^2
\end{aligned}$$

and

$$\begin{aligned}
-2\int_\Omega \rho_t u \nabla u \; u_t &= 2\int_\Omega \text{div}(\rho u) u \nabla u \; u_t \\
&= 2\sum_{i,j,k}^n \int_\Omega \frac{\partial}{\partial x_k}(\rho u_k) u_i \frac{\partial}{\partial x_i} u_j \; u_{j,t} \\
&= -2\sum_{i,j,k}^n \int_\Omega \rho u_k \frac{\partial}{\partial x_k} u_i \frac{\partial}{\partial x_i} u_j \; u_{j,t} \\
&\quad -2\sum_{i,j,k}^n \int_\Omega \rho u_k u_i \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i} u_j \; u_{j,t} \\
&\quad -2\sum_{i,j,k}^n \int_\Omega \rho u_k u_i \frac{\partial}{\partial x_k} u_j \frac{\partial}{\partial x_i} \; u_{j,t}
\end{aligned}$$

$$\leq c\|\rho\|_{L^\infty}\|u\|_{L^6}\|\nabla u\|_{L^6}\|\nabla u\|\ \|u_t\|_{L^6}$$

$$+c\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|Au\|\ \|u_t\|_{L^6}$$

$$+c\|\rho\|_{L^\infty}\|u\|_{L^6}^2\|\nabla u\|_{L^6}\ \|\nabla u_t\|$$

$$\leq c\|\rho\|_{L^\infty}^2\|\nabla u\|^4\|Au\|^2 + \varepsilon\|\nabla u_t\|^2 .$$

thanks to the Hölder inequality, the Sobolev embedding $H^1 \hookrightarrow L^6$ and Young's inequality.

By choosing $\varepsilon = 1/8$, from (2.5) and (2.6) we obtain

$$\frac{d}{dt}\|\rho^{1/2}u_t\|^2 + \|\nabla u_t\|^2$$

$$\leq c\beta^2\|\nabla u\|^4\|Au\|^2 + c\beta^2\|\nabla u\|^2\|\nabla f\|^2$$

$$+c\beta^2\|\nabla u\|^2\|f\|_{L^4}^2 + c\beta^2\|f_t\|^2$$

$$+c\beta^2\|\nabla u\|^8\|u_t\|^2 + c\beta^2\|\nabla u\|^4\|u_t\|^2$$

$$\leq c\beta^2(M^2 + M^4 + M^8)(\|u_t\|^2 + \|Au\|^2 + \|f_t\|^2 + \|\nabla f\|^2 + \|f\|_{L^4}^2)$$

$$\leq c(\beta, M)(\|u_t\|^2\|Au\|^2 + \|f_t\|^2 + \|\nabla f\|^2) .$$

Now, we multiply the above inequality by $e^{\gamma t}$, $\gamma > 0$, and integrate in time from 0 to t, we have

$$e^{\gamma t}\|\rho^{1/2}u_t\|^2 + \int_0^t e^{\gamma s}\|\nabla u_t(s)\|^2 ds \tag{2.8}$$

$$\leq \|\rho_0^{1/2}u_t(0)\|^2 + \gamma\int_0^t e^{\gamma s}\|\rho^{1/2}u_t(s)\|^2 ds$$

$$+c(\beta, M)\int_0^t e^{\gamma s}(\|u_t(s)\|^2 + \|Au(s)\|^2 + \|f_t\|^2 + \|\nabla f\|^2)ds$$

$$\leq \beta||u_t(0)||^2 + \beta\gamma \int_0^t e^{\gamma s}||u_t(s)||^2 ds$$

$$+c(\beta, M) \int_0^t e^{\gamma s}(||u_t(s)||^2 + ||Au(s)||^2 + c)ds \ ,$$

where we have used the hypothesis on f.

By multiplying the above inequality by $e^{-\gamma t}$, we get

$$||\rho^{1/2}u_t||^2 + e^{-\gamma t} \int_0^t e^{\gamma s}||\nabla u_t(s)||^2 ds$$

$$\leq \beta e^{-\gamma t}||u_t(0)||^2 + \beta\gamma e^{-\gamma t} \int_0^t e^{\gamma s}||u_t(s)||^2 ds$$

$$+c(\beta, M)e^{-\gamma t} \int_0^t e^{\gamma s}(||u_t(s)||^2 + ||Au(s)||^2 + c)ds$$

$$\leq \beta e^{-\gamma t}||u_t(0)||^2 + K(\beta, M, \gamma)$$

thanks to the previous estimates. So, it is enough to find estimates for $||u_t(0)||^2$ (actually $||u_t^k(0)||$).

For this, recall that u_0 $(u_0^k) \in V \cap H^2(\Omega)$; consequently, by setting $v = u_t$ in (2.1) (actually $v = u_t^k$ in (2.2)), we get

$$||\rho^{1/2}u_t||^2 = (\rho f, u_t) - (Au, u_t) - (\rho u \nabla u, u_t)$$

So, since $||\rho^{1/2}u_t||^2 \geq \alpha||u_t||^2$, the above implies

$$||u_t(0)||^2 \leq \frac{1}{\alpha}\{||\rho_0||_{L^\infty}||f(0)|| + ||Au_0|| + c||\rho_0||\ ||Au_0||\ ||\nabla u_0||\} \leq c\ .$$

Thus, we conclude that

(2.9)
$$\sup_{t\geq 0} ||u_t(t)|| \leq c$$

(2.10)
$$\sup_{t\geq 0} e^{-\gamma t} \int_0^t e^{\gamma s}||\nabla u_t(s)||^2 ds \leq c$$

Setting $v = Au$ in (2.1) (actually $v = Au^k$ in (2.2)), we get

$$||Au||^2 = (\rho f, Au) - (\rho u_t, Au) - (\rho u \nabla u, Au)\ ,$$

which implies

$$\begin{aligned}||Au|| &\leq ||\rho||_{L^\infty}||f|| + ||\rho||_{L^\infty}||u_t|| + ||\rho||_{L^\infty}||u||_{L^4}||\nabla u||_{L^4} \\ &\leq \beta||f|| + \beta||u_t|| + c\beta||\nabla u||\ ||\nabla u||^{1/4}||Au||^{3/4} \\ &\leq \beta||f|| + \beta||u_t|| + c\beta||\nabla u||^5 + \frac{1}{2}||Au||\end{aligned}$$

thanks to (2.6) applied to ∇u, the fact that $||u||_{H^2} \leq c||Au||$ by Amrouche and Girault's result [1] and Young's inequality.

Finally, we conclude that

$$||Au|| \leq c\beta||f|| + c\beta||u_t|| + c\beta||\nabla u||^5 \leq c\ ,$$

and so, from this, (2.8) and the hypothesis on f, we get

$$\sup_{t\geq 0} ||Au(t)|| \leq c \tag{2.11}$$

Now, from the equivalent form of (2.1)

$$Au = P(\rho(u_t + u\nabla u - f) = F$$

(actually, the equivalent form of (2.2): $Au^k = P_k(\rho^k(u^k_t + u^k\nabla u^k - f)) = F^k$, which holds because we are using a spectral basis), and the fact that

$$e^{-\gamma t}\int_0^t e^{\gamma s}||F(s)||^2_{L^6} ds \leq c$$

from our previous estimates (2.10), (2.11) (the same holds for F^k), we conclude that

$$e^{-\gamma t}\int_0^t e^{\gamma s}||u(s)||^2_{W^{2,6}} ds \leq c\ ,$$

from Amrouche and Girault's results.

Thus, the usual Sobolev embeddings imply that

$$e^{-\gamma t}\int_0^t e^{\gamma s}||\nabla u(s)||^2_{C(\Omega)} ds \leq c\ .$$

Now, we observe that the previous estimates hold true for $\gamma \geq 0$ if we are considering finite time intervals $[0,T]$, $0 < T < +\infty$ (with the suprema obviously depending on T). This comes from the way that the proofs were done.

Thus, in a finite interval $[0,T]$, we can take the last estimate with $\gamma = 0$. This estimate together with the ones by Ladyzhenskaya and Solonikov ([13], p. 705) for $\nabla\rho$ and $\nabla\rho_t$ imply that $\rho \in C^1(\Omega \times [0,T])$ for any finite $T > 0$. □

Remark. As in the end of the previous proof, we observe that all the estimates (except the one on the derivatives of the density) hold true for $\gamma \geq 0$ on the time interval $[0,\infty)$ if we also include in the hypothesis $f \in L^2([0,\infty); H^1(\Omega))^n)$, $f_t \in L^2([0,\infty); (L^2(\Omega))^n)$.

In what follows we prove that for nonhomogeneous fluids it is possible to recover the classical result for the usual Navier-Stokes equations that in the two dimensional case it is not necessary to assume the smallness of the initial data and external force. We have the following result.

Theorem 2.4. Suppose $n = 2$, that (A.1) and (A.2) are true,

$$f \in L^\infty([0,\infty); (H^1(\Omega))^n),\; f_t \in L^\infty([0,\infty); (L^2(\Omega))^n)$$

and satisfies $u \in C([0,\infty); (H^2(\Omega))^n \cap V)$, $\rho \in C^1(\overline{\Omega} \times [0,T])$ for any finite $T > 0$. Moreover, the estimates given in Theorem 2.1 are true for any $\gamma \geq 0$.

Proof. The same remarks made at the beginning of the proof of Theorem 2.1 hold here.

Working as in Lions [14], p. 65, we have

$$\frac{d}{dt}||\rho^{1/2}u||^2 + ||\nabla u||^2 = (\rho f, u) .$$

Consequently, by multiplying the above equation by $e^{\overline{\gamma}t}$ and recalling that $||u||^2 \leq C_\Omega ||\nabla u||^2$ for $u \in (H_0^1(\Omega))^n$, we conclude

$$\frac{d}{dt}(e^{\overline{\gamma}t}||\rho^{1/2}u||^2) + \frac{1}{2}e^{\overline{\gamma}t}||\nabla u||^2 \leq \frac{1}{2}C_\Omega^2\beta^2 e^{\overline{\gamma}t}||f||^2$$

for $0 < \overline{\gamma} \leq \dfrac{1}{4\beta C_\Omega}$.

The above inequality implies,

$$(2.12) \qquad 2||\rho^{1/2}u||^2 + e^{-\overline{\gamma}t}\int_0^t e^{\overline{\gamma}s}||\nabla u(s)||^2 ds$$

$$\leq 2e^{-\overline{\gamma}t}||\rho_0^{1/2}u_0||^2 + \beta^2 C_\Omega^2 e^{-\overline{\gamma}t}\int_0^t e^{\overline{\gamma}s}||f(s)||^2 ds$$

$$\leq 2\beta||u_0||^2 + \frac{C_\Omega^2\beta^2}{\overline{\gamma}}||f||_{L^\infty}^2 .$$

Also, working as in Simon [18], p. 1115, we have

$$\frac{d}{dt}||\nabla u||^2 \leq c||\nabla u||^4 + c||f||^2$$

where c is a positive constant that depends only on β and Γ (the boundary of Ω). Setting $\psi(t) = ||\nabla u(t)||^2$ in the last inequality, we have

$$\frac{d}{dt}\psi \leq c\psi^2 + c||f||^2 . \tag{2.13}$$

Now, we observe that $c\psi^2 + c_1 \leq 2c\psi^2$ for $\psi \geq \left(\frac{c_1}{c}\right)^{1/2}$, where $c_1 = c \sup_{t\in[0,T)} ||f(t)||$. If we call $\ell^* = \max\left\{\left(\frac{c_1}{c}\right)^{1/2}, 1, ||\nabla u_0||^2\right\}$, then either we have $0 \leq \psi(t) \leq \ell^*$ for all $t \geq 0$ or there exists some interval $[t_1, t_2]$, $t_2 > t_1$ for which $||\nabla u(t_1)||^2 = \ell^*$ and for $t \in [t_1, t_2]$ it is true that $||\nabla u(t)||^2 \geq \ell^*$. Then, due to our choice of ℓ^*, in this interval $[t_1, t_2]$ there holds $\frac{d}{dt}\psi \leq c\psi^2$ or, equivalently,

$$\frac{d}{dt}\ell n\psi \leq c\psi .$$

Multiplying the above inequality by $e^{\overline{\gamma}t}$ we have

$$\frac{d}{dt}e^{\overline{\gamma}t}\ell n\psi \leq ce^{\overline{\gamma}t}\psi + \overline{\gamma}e^{\overline{\gamma}t}\ell n\psi .$$

Now, we observe that there exists a positive constant k such that $\ell n\psi \leq k + k\psi$, consequently, using this and integrating the last inequality from t_1 to $t \in [t_1, t_2]$, we get

$$e^{\overline{\gamma}t}\ell n\psi(t_2) - e^{\overline{\gamma}t_1}\ell n\psi(t_1)$$
$$\leq (c + \overline{\gamma}k)\int_{t_1}^t e^{\overline{\gamma}\tau}\psi(\tau)d\tau + \overline{\gamma}k\int_{t_1}^t e^{\overline{\gamma}\tau}d\tau ,$$

so,

$$\ell n\psi(t_2) - e^{\overline{\gamma}(t_1-t)}\ell n(t_1)$$

$$\leq (c+\overline{\gamma}k)e^{-\overline{\gamma}t}\int_{t_1}^{t} e^{\overline{\gamma}t}\psi(\tau)d\tau + \overline{\gamma}ke^{-\overline{\gamma}t}\int_{t_1}^{t} e^{\gamma\tau}d\tau$$

$$\leq (c+\overline{\gamma}k)[2\beta||u_0||^2 + \frac{C_\Omega^2\beta^2}{\overline{\gamma}}||f||^2_{L^\infty}] + k[1-e^{-\overline{\gamma}t_2}]$$

$$\leq (c+\overline{\gamma}k)[2\beta||u_0||^2 + \frac{C_\Omega^2\beta^2}{\overline{\gamma}}||f||^2_{L^\infty}] + k \equiv \overline{M} .$$

Consequently, since $-e^{\gamma(t_1-t)}\ell n\psi(t_1) \geq -\ell n\psi(t_1)$, we have $\ell n\dfrac{\psi(t)}{\psi(t_1)} \leq \overline{M}$, which implies, for all $t \in [t_1, t_2]$

$$||\nabla u(t)||^2 \leq ||\nabla u(t_1)||^2 e^{\overline{M}} = \ell^* e^{\overline{M}} .$$

Since this is independent of t_1 and t_2, we conclude that for all $t \geq 0$ we have

$$||\nabla u(t)||^2 \leq \max\{\ell^*, \ell^* e^{\overline{M}}\} = \ell^* e^{\overline{M}} .$$

The rest of analysis is now done exactly as in the tridimensional case. □

3 Global Existence in the Case of Exponentially Decaying External Forces

In this Section we assume that the external force field decays exponentially in time. We will show that the solutions of (2.1) have better regularity than in the previous case, and we will be even able to prove an uniform in time estimate for the L^∞-norm of the gradient of the density.

In fact, we have the following result

Theorem 3.1. Suppose $n = 3$, that (A.1) and (A.2) are true and that for some constant $\overline{\gamma} > 0$, $e^{\overline{\gamma}t}f \in L^\infty([0,\infty); (H^1(\Omega))^n)$, $e^{\overline{\gamma}t}f_t \in L^\infty([0,\infty); (L^2(\Omega))^n)$. Then, if $||u_0||_{H^1(\Omega)}$ and $||e^{\overline{\gamma}t}f||_{L^\infty([0,\infty);(L^2(\Omega))^n)}$ sufficiently small, the solution (ρ, u) of problem (2.1) exists globally in time. Moreover, there is a positive constant $\gamma^* \leq \overline{\gamma}$ such that for any $0 \leq \theta < \gamma^*$, there hold the following estimates

$$\sup_{t\geq 0} e^{\gamma^* t}||\nabla u(t)||^2 < +\infty$$

$$\sup_{t\geq 0} e^{\theta t}(||u_t(t)||^2 + ||Au(t)||^2) < +\infty$$

$$\sup_{t\geq 0}\int_0^t e^{\theta s}(||\nabla u_t(s)||^2 + ||u(s)||^2_{W^{2,6}} + ||\nabla u(s)||^2_{L^\infty} ds < +\infty$$

$$\sup_{t\geq 0}(||\nabla\rho(t)||_{L^\infty} + ||\rho_t(t)||_{L^\infty}) < +\infty$$

$$\sup_{t\geq 0}\sigma(t)(||\nabla u_t(t)||^2) < +\infty$$

$$\sup_{t\geq 0}\int_0^t \sigma(s)(||u_{tt}(s)||^2 + ||Au_t(s)||^2)ds < +\infty\ .$$

In the last two estimates $\sigma(t) = \min\{1,t\}e^{\theta t}$. The same kind of estimates hold for the semi-Galerkin approximations.

Proof. There hold the same remarks as the ones made in the proof of Theorem 2.3 (just after (2.5)) concerning the fact that the estimates should first be derived for the approximations and then carried out to the limit. Multiplying the differential inequality (2.4) by $e^{\gamma t}$, with $0 \leq \gamma \leq \overline{\gamma}$, we have for certain positive constants c_1, c_2, c_3

$$\frac{d}{dt}(e^{\gamma t}||\nabla u||^2) \leq c_1 e^{\gamma t}||\nabla u||^{10} - c_2 e^{\gamma t}||\nabla u||^2 + \gamma e^{\gamma t}||\nabla u||^2 + c_3.$$

Here c_1 and c_2 are independent of γ and $c_3 = c\sup_{t\geq 0} e^{\gamma t}||f(t)|| \leq c\sup_{t\geq 0} e^{\overline{\gamma} t}||f(t)|| < +\infty$. Now, let $\gamma^* = \min\left\{\overline{\gamma}, \frac{c_2}{\beta C_\Omega}\right\}$ and $\psi(t) = e^{\gamma^* t}||\nabla u||^2$; then

$$\frac{d}{dt}\psi \leq c_1\psi^5 - \overline{c}_2\psi + c_3$$

$$\psi(0) = ||\nabla u_0||^2$$

with $\overline{c}_2 = \frac{c_2}{2}$ by our choice of γ^*.

Now, analogously as in Theorem (2.1), we prove that

$$\sup_{t\geq 0} e^{\gamma^* t}||\nabla u||^2 = \sup_{t\geq 0}\psi(t) \leq c < +\infty \tag{3.1}$$

for all $t \geq 0$ if $||\nabla u_0||$ and $||e^{\overline{\gamma} t}f(t)||_{L^\infty([0,\infty);(L^2(\Omega))^n)}$ are small enough.

Now, we multiply the differential inequality (2.3) by $e^{\theta t}$, with $0 \leq \theta < \gamma^*$, to obtain

$$\frac{d}{dt}(e^{\theta t}||\nabla u||^2) + ce^{\theta t}||Au||^2 + ce^{\theta t}||\rho^{1/2}u_t||^2$$

$$\leq c_1 e^{\theta t}||\nabla u||^{10} + \theta e^{\theta t}||\nabla u||^2 + ce^{\theta t}||f(t)||^2 .$$

By integrating the above inequality, we get for all $t \geq 0$,

$$e^{\theta t}||\nabla u(t)||^2 + c\int_0^t e^{\theta s}||Au(s)||^2 ds + c\int_0^t e^{\theta s}||\rho^{1/2}u_t||^2 ds$$

$$\leq ||\nabla u_0|| + c_1(\sup_{t\geq 0}(||\nabla u||)^8(\sup_{t\geq 0} e^{\gamma^* t}||\nabla u||^2)\int_0^t e^{(\theta-\gamma^*)s}ds$$

$$+\theta(\sup_{t\geq 0} e^{\gamma^* t}||\nabla u||^2)\int_0^t e^{(\theta-\gamma^*)s}ds$$

$$+c(\sup_{t\geq 0}||f(t)||)(\sup_{t\geq 0} e^{\gamma^* t}||f||)\int_0^t e^{(\theta-\gamma^*)s}ds$$

Since $\theta - \gamma^* < 0$ and (3.1) and the hypothesis of this Theorem.

Since $\theta < \gamma^*$, (A.1) and the hypothesis of the theorem of f together with estimate (3.1) imply the following results

$$\sup_{t\geq 0} e^{\theta t}||\nabla u(t)||^2 \leq C < +\infty \tag{3.2}$$

$$\sup_{t\geq 0}\int_0^t e^{\theta s}(||Au(s)||^2 ds \leq C < +\infty \tag{3.3}$$

$$\sup_{t\geq 0}\int_0^t e^{\theta s}||u_t(s)||^2 ds \leq C < +\infty \tag{3.4}$$

To obtain the other estimates, we consider the inequality (2.8) with $\gamma = \theta$, together with (3.2)-(3.4):

$$e^{\theta t}||\rho^{1/2}u_t||^2 + \int_0^t e^{\theta s}||\nabla u_t(s)||^2 ds$$

$$\leq ||\rho^{1/2}u_t(0)||^2 + \theta\int_0^t e^{\theta s}||\rho^{1/2}u_t(s)||^2 ds$$

$$+c\int_0^t e^{\theta s}(||u_t(s)||^2 + ||Au(s)||^2)ds + c\int_0^t e^{\theta s}(||f_t(s)||^2 + ||\nabla f(s)||^2)ds$$

$$\leq \beta||u_t(0)||^2 + c + \int_0^t e^{\overline{\gamma} s}(||f_t(s)||^2 + ||\nabla f(s)||^2)e^{-(\overline{\gamma}-\theta)s}ds .$$

We can estimate $||u_t(0)||$ as we did previously and observe that

$$\int_0^t e^{\overline{\gamma}s}(||f_t(s)||^2 + ||\nabla f(s)||^2)e^{-(\overline{\gamma}-\theta)s}ds$$

$$\leq (\sup_{s\geq 0}(||f_t(s)|| + ||\nabla f(s)||))(\sup_{s\geq 0}(e^{\overline{\gamma}s}(||f_t(s)|| + ||\nabla f(s)||)) \int_0^t e^{-(\overline{\gamma}-\theta)s}ds$$

$$\leq \frac{C^2}{\overline{\gamma}-\theta} < +\infty ,$$

since $\theta < \overline{\gamma}$, and we have exponential decaying of f.

Thus, we have

$$\sup_{t\geq 0} e^{\theta t}||u_t(t)||^2 < C < +\infty , \tag{3.5}$$

$$\sup_{t\geq 0} \int_0^t e^{\theta s}||\nabla u_t(s)||^2 ds < C < +\infty . \tag{3.6}$$

Now, analogously as in Theorem 2.3, we can prove

$$\sup_{t\geq 0} e^{\theta s}(||Au(t)||^2 \leq C < +\infty . \tag{3.7}$$

To obtain estimates for the density, we observe that (2.1) is equivalent to $Au = P(\rho(-u_t - u\nabla u + f)$. Thus, $A(e^{\theta t}u(t)) = P(e^{\theta t}\rho(-u_t - u\nabla u + f))$, with $0 < \theta < \gamma^*$, which implies, by Amrouch Girault's results [1], estimates (3.1)-(3.7) and our hypothesis on f, that $e^{\frac{\theta}{2}t}u \in L^2((0,\infty); W^{2,6}(\Omega))$.

Consequently, by Sobolev embedding

$$e^{\frac{\theta}{2}t}u \in L^2([0,\infty); W^{1,\infty}(\Omega)) . \tag{3.8}$$

Thus, the formula of Ladyzhenskaya and Solonnikov [13], p. 705,

$$||\nabla\rho(t)||_{L^\infty} \leq C||\nabla\rho_0||_{L^\infty}\, e^{\int_0^t ||\nabla u(s)||_{L^\infty} ds} ,$$

furnishes that

$$\sup_{t\geq 0} ||\nabla\rho(t)||_{L^\infty} \leq C < +\infty , \tag{3.9}$$

since estimate (3.8) implies that for all $t \geq 0$

$$\begin{aligned}
\int_0^t ||\nabla u(s)||_{L^\infty} ds &= \int_0^\infty e^{\frac{\theta}{2}t} ||\nabla u(t)||_{L^\infty} e^{-\frac{\theta}{2}t} dt \\
&\leq \left(\int_0^\infty e^{\theta t} ||\nabla u(t)||^2_{L^\infty} dt\right)^{1/2} \left(\int_0^\infty e^{-\theta t} dt\right)^{1/2} \\
&\leq C < +\infty \,.
\end{aligned}$$

Now, the continuity equation (the equation for the density) and the above estimates imply

(3.10) $$\sup_{t \geq 0} ||\rho_t(t)||_{L^\infty} \leq C < +\infty \,.$$

To obtain higher order estimates, we proceed as follows:
We differentiate the equation (2.1) with respect to t and set $v = u_{tt}$,

(3.11) $$\begin{aligned}
&||\rho^{1/2} u_{tt}||^2 + \frac{1}{2}\frac{d}{dt} ||\nabla u_t||^2 \\
&(\rho_t f, u_{tt}) + (\rho f_t, u_{tt}) - (\rho_t u \nabla u, u_{tt}) \\
&-(\rho u_t \nabla u, u_{tt}) + (\rho u \nabla u_t, u_{tt}) - (\rho_t u_t, u_{tt}) \,.
\end{aligned}$$

Now, we observe that for all $\delta > 0$

$$|(\rho_t f, u_{tt})| \leq C_\delta ||\rho_t||^2_{L^\infty} ||f||^2 + \delta ||u_{tt}||^2 \,;$$

$$|(\rho f_t, u_{tt})| \leq C_\delta ||\rho||^2_{L^\infty} ||f_t||^2 + \delta ||u_{tt}||^2 \,;$$

$$|(\rho_t u \nabla u, u_{tt})| \leq C_\delta ||\rho_t||^2_{L^\infty} ||u||^2_{L^\infty} ||\nabla u||^2 + \delta ||u_{tt}||^2 \,;$$

$$|(\rho u_t \nabla u, u_{tt})| \leq C_\delta ||\rho||^2_{L^\infty} ||u_t||^2_{L^4} ||\nabla u||^2_{L^4} + \delta ||u_{tt}||^2 \,;$$

$$|(\rho u \nabla u_t, u_{tt})| \leq C_\delta ||\rho||^2_{L^\infty} ||u||^2_{L^\infty} ||\nabla u_t||^2 + \delta ||u_{tt}||^2 \,;$$

$$|(\rho_t u_t u_{tt})| \leq C_\delta ||\rho_t||^2_{L^\infty} ||u_t||^2 + \delta ||u_{tt}||^2 \; .$$

Consequently, by using these estimates with $\delta = \dfrac{\alpha}{10}$ in (3..11) we are left with

$$\alpha ||u_{tt}||^2 + \frac{d}{dt} ||\nabla u_t||^2$$

$$\leq C(||f||^2 + ||f_t||^2 + ||\nabla u||^2 + ||\nabla u_t||^2 + ||u_t||^2)$$

Multiplying the above inequality by $\sigma(t) = e^{\theta t} \min\{1, t\}$ and integrating in time from $\varepsilon > 0$ to t, we get

(3.12) $$\alpha \int_\varepsilon^t \sigma(s) ||u_{tt}(s)||^2 ds + \sigma(t) ||\nabla u_t(t)||^2$$

$$\leq \sigma(\varepsilon) ||\nabla u_t(\varepsilon)||^2 + C \int_\varepsilon^t \sigma(t) \{||f||^2 + ||f_t||^2 + ||\nabla u||^2 + ||\nabla u_t||^2 + ||u_t||^2\} ds \; .$$

Now, we observe that $\sigma(t) \leq e^{\theta t}$, and, therefore,

$$\int_\varepsilon^t \sigma(t) (||f||^2 + ||f_t||^2 + ||\nabla u||^2 + ||\nabla u_t||^2 + ||u_t||^2) ds$$

$$\leq \int_0^t e^{\theta t} \{||f||^2 + ||f_t||^2 + ||\nabla u||^2 + ||\nabla u_t||^2 + ||u_t||^2\} ds$$

$$\leq \left(\int_0^t e^{(\theta - \gamma) s} ds \right) \left(\sup_{t \geq 0} e^{\gamma t} (||f||^2 + ||f_t||^2 + ||\nabla u||^2) \right)$$

$$+ \int_0^t e^{\theta s} ||\nabla u_t||^2 ds + \int_0^t e^{\gamma s} ||u_t||^2 ds \leq C < +\infty$$

in virtue of our hypothesis and estimates.

So, by taking $\varepsilon_n \to 0$, along a convenient sequence in (3.12)

$$\alpha \int_0^t \sigma(s) ||u_{tt}(s)||^2 ds + \sigma(t) ||\nabla u_t(t)||^2$$

$$\leq C + \lim_{\varepsilon_n \to 0} \sigma(\varepsilon_n) ||\nabla u_t(\varepsilon_n)||^2 = C < +\infty \; .$$

Here, the subsequence was chosen according to the following easily proved lemma:

Lemma. If $h(t) \geq 0$, $g(t) \geq 0$ and

$$\int_a^t h(s)ds = g(t)$$

where $g(t)$ is finite. Then there exist are subsequence $\varepsilon_n \to a^+$ such that

$$\varepsilon_n h(\varepsilon_n) \to 0 \quad \text{as} \quad n \to \infty \ .$$

The lemma was applied with $h(t) = \sigma(t)||\nabla u_t(t)||^2$, which satisfies the conditions in it due to (3.6) and the fact that $0 \leq \sigma(t) \leq e^{\theta t}$.

Now, the estimate for $\int_0^t \sigma(s)||Au_t||^2 ds$ follows easily from the previous one by using $v = Au_t$ in equation (2.1). □

In the two dimensional case we have a stronger result

Theorem 3.2. Suppose $n = 2$, that (A.1) and (A.2) are true and that for some constant $\bar{\gamma} > 0$, $e^{\bar{\gamma}t} f \in L^\infty([0,\infty);(H^1(\Omega))^n)$, $e^{\bar{\gamma}t} f_t \in L^\infty([0,\infty);(L^2(\Omega))^n)$. Then, the solution (ρ, u) of problem (2.1) exists globally in time. Moreover, the estimates in Theorem 3.1 hold true for any $0 \leq \theta < \gamma^*$ with $\gamma^* = \bar{\gamma}$.

Proof. Again working as in Simon [18], op. 1115, we have

$$\frac{d}{dt}||\nabla u||^2 \leq C||\nabla u||^4 + C||f||^2 \ ,$$

with C a positive constant. Multiplying this inequality by $e^{\bar{\gamma}t}$ we can rewrite it as

$$\begin{aligned} \frac{d}{dt}(e^{\bar{\gamma}t}||\nabla u||^2) &\leq ce^{\bar{\gamma}t}||\nabla u||^4 + \bar{\gamma}e^{\bar{\gamma}t}||\nabla u||^2 + ce^{\bar{\gamma}t}||f||^2 \\ &\leq c_1 e^{2\bar{\gamma}t}||\nabla u||^4 + c_2 + ce^{\bar{\gamma}t}||f||^2 \ . \end{aligned}$$

Calling $\psi(t) = e^{\bar{\gamma}t}||\nabla u||^2, c_3 = c_2 + \sup_{t\geq 0} e^{\bar{\gamma}t}||f||^2$, the above inequality becomes

$$\frac{d\psi}{dt} \leq c_1\psi^2 + c_3$$

which can be analysed exactly as in Theorem 2.3, furnishing that $\sup_{t\geq 0} e^{\overline{\gamma}t}||\nabla u(t)||^2 = \sup_{t\geq 0} \psi(t) < +\infty$.

Thus, we can take $\gamma^* = \overline{\gamma}$, and the rest of the analysis is completely analogous to the one in the previous theorem. □

Remark. We observe that the given estimate for u_{tt} in the previous two theorems hold true in the case of the local existence if we take the supremum in the interval of time $[0,T]$, where $0 < T < +\infty$ is less than the time for which the solution is known to exist. This estimate implies that the solution of the local existence theorem actually satisfies $u_t \in C((0,T],H)$ and, therefore, $u \in C^1((0,T),H)$.

4 Results on the Pressure and Further Higher Order Estimates

In a standard way we can obtain information on the pressure. In fact, we have:

Proposition 4.1. Under the hypothesis of Theorem 2.2 or 2.3, if (u,p) is a solution of (2.1), there exists $p \in L^\infty([0,\infty); H^1(\Omega)/\mathbb{R})$ such (u,ρ,p) is solution of (1.1).

Proof. We observe that (2.1) is equivalent to $Au = P(g)$ where $g = \rho(f - u\nabla u - u_t)$. Now, our estimates for ρ and u imply that $g \in L^\infty([0,\infty), L^2(\Omega))$, and, therefore, Amrouch and Girault's results [1] imply that there is a unique $p \in L^\infty([0,\infty); H^1(\Omega)/\mathbb{R})$ such that

$$-\Delta u + \nabla p = g\ .$$

This and the equation for ρ imply that (u,ρ,p) satisfies (1.1). □

In the case in which the external force field decay exponentially in time, we have better regularity for the pressure p.

Proposition 4.2. Under the hypothesis of Theorem 3.1 or 3.2, if (u,ρ) is a solution of (2.1), there exists p such that (u,ρ,p) is solution of (1.1). Moreover, for any $\varepsilon > 0$,

$$p \in L^\infty([0,\infty); H^1(\Omega)/\mathbb{R}) \cap C([\varepsilon,\infty); H^1(\Omega)/\mathbb{R}),$$

$$p_t \in L^\infty([\varepsilon,\infty); H^1(\Omega)/\mathbb{R}).$$

Proof. Differentiating $Au = P(g)$ with respect to t, which is possible by the regularity results of Theorem 3.1 on 3.2, we have $Au_t = P(g_t)$. Now these same regularity results imply $g_t \in L^\infty([\varepsilon,\infty), L^2(\Omega))$ for any $\varepsilon > 0$.
Now, Amrouche and Girault's results imply that the corresponding pressure p satisfies $p_t \in L^\infty([\varepsilon,\infty); H^1(\Omega)/\mathbb{R})$ for any $\varepsilon > 0$. Therefore, $p \in C([\varepsilon,\infty); H^1(\Omega)/\mathbb{R})$. □

By using the above estimates on the pressure we can derive heigher order estimates in the case of exponentially decaying force fields.

Proposition 4.3. Under the conditions of Theorem 3.1, if Ω is of class $C^{2,1}$, there holds

$$\sup_{t\geq 0} \sigma(t)(||u||^2_{H^3} + ||\nabla u||^2_{L^\infty}) < +\infty$$

Proof. By the previous propositions we have regularity on the pressure, and we can work directy with equations (1.1). Also, since Ω is of class $C^{2,1}$, Amrouch and Girault's [1] imply that the eigenfunctions of the Stokes operator have H^3-regularity. Thus, the spectral approximations also have this H^3-regularity in the space variable and we can to try to estimate their H^3-norm in an uniform way with respect to the approximationns. This will imply in a standard way that the same estimate will hold for the solution.

Therefore, exactly as we did in all our previous results, since the computations are the same, to easy the notation we will work directly with the solution.

By differentiating the equation (1.1) with respect to x_i and by taking the inner product in $(L^2(\Omega))^n$ with $-A\frac{\partial u}{\partial x_i}$, we get

$$\begin{aligned} ||A\frac{\partial u}{\partial x_i}|| \;\leq\; & ||\frac{\partial \rho}{\partial x_i}||_{L^\infty}(||u_t|| + ||u||_{L^\infty}||\nabla u|| + ||f||) \\ & + \;||\rho||_{L^\infty}(||\frac{\partial u_t}{\partial x_i}|| + ||\frac{\partial u}{\partial x_i}||_{L^4}||\nabla u||_{L^4} \\ & + \;||u||_{L^\infty}||\frac{\partial u}{\partial x_i}|| + ||\frac{\partial f}{\partial x_i}||) \end{aligned}$$

since

$$\left(\nabla\frac{\partial p}{\partial x_i},\, A\frac{\partial u}{\partial x_i}\right) = \left(P\nabla\frac{\partial p}{\partial x_i},\, \Delta\frac{\partial u}{\partial x_i}\right) = 0\,.$$

Consequently,

$$\sigma(t)||A\frac{\partial u}{\partial x_i}||^2 \leq c\sigma(t)\{||u_t||^2 + M^2||Au||^2 + ||f||^2||\}$$

$$+ \quad c\sigma(t)\{||\nabla u_t||^2 + ||Au||^4 + M^2||Au||^2$$

$$+ \quad ||\nabla f||^2\} \leq C < +\infty$$

thanks to the previous estimates.

So, we conclude that

$$\sigma(t)||u||^2_{H^3} \leq \sigma(t)\sum_i ||A\frac{\partial u}{\partial x_i}||^2 < +\infty$$

for all $t \geq 0$. Now, we utilize standard Sobolev embedding results to obtain

$$\sigma(t)||\nabla u||^2_{L^\infty} \leq C < +\infty$$

for all $t \geq 0$. □

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