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CONVERGENCE RATES IN THE SOBOLEV H^s-NORM OF APPROXIMATIONS BY DISCRETE CONVOLUTIONS

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ABSTRACT – We consider expansion series in terms of scaled translates $\Phi(h^{-1}x-k)$ of a basic function Φ. The coefficients are given by sampled values $R_h f(kh)$ where $R_h f$ are averaging operators obtained by convolutions $f * d\nu_h$. Examples of such expansions have been used in finite element approximations and sampling theory. More recently, they have received considerable attention in wavelet analysis. In this context, the basic functions Φ are named scaling functions. They generate nice decompositions of $L^2(\mathbf{R})$ which are called multiresolution analysis of $L^2(\mathbf{R})$. This article concerns itself with the convergence of these series in the Sobolev H^s-norm. Results already known in the finite element context are extended for basic functions Φ not necessarely having compact support. Besides a regularity condition, Φ is supposed to satisfy the so called m-criterion of convergence. In order to obtain approximations with accuracy $O(h^{m+1-s})$, Φ and ν must also be connected by a moment relation. Emphasis is placed on expansions in a multiresolution analysis of $L^2(\mathbf{R})$. We give special attention to those cases where the coefficients are given by discrete convolutions. Some examples, including sampling series, interpolation and approximate orthogonal projections, are discussed.

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1 Introduction

In this paper we shall deal with expansion series of the form

$$\sum_{k=-\infty}^{\infty} c_{h,k}\Phi(h^{-1}x-k), \tag{1}$$

in terms of the scaled translates of a basic function Φ.

Expansions (1) have been extensively used in the development of approximate methods for the solution of partial differencial equations. These methods consist in searching for the solution in spaces V_h spanned by the basic functions $\Phi(h^{-1}x-k)$. In finite element approximations based on a regular mesh with mesh-width h, the trial spaces V_h are formed by piecewise polynomial functions. A typical choice for the basic function Φ is a B-spline.

The success of such methods depends on the "density" of the spaces V_h, i.e., in the accuracy of the approximations of functions f from V_h. Strang and Fix [21] have isolated those conditions on the finite element spaces V_h which determine the order of approximation in the H^s-norm. For instance, assuming that Φ is compactly supported and is in $H^m(\mathrm{R})$ they stablished that

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smooth functions can be approximated from V_h with error $O(h^{m+1-s})$ in the H^s-norm, $s \leq m$, if and only if the polynomials of degree $\leq m$ can be written as linear combinations of Φ and its translates. Equivalently, the Fourier transform $\widehat{\Phi}$ must have zeros of order $m+1$ at all the points $\xi = 2\pi j, j \neq 0$ and $\widehat{\Phi}(0) \neq 0$. This condition is also known as the *m-criterion of convergence* (cf. [1]).

Assume the m-criterion of convergence. Now the question is: Given an arbitrary approximation of a funtion f in V_h, does this approximation have the same order of accuracy $O(h^{m+1-s})$ in the H^s-norm ? Of course this is not true in general. Additional hypotheses must be satisfied. For example, consider the sampling series

$$S_h f(x) = \sum_{k=-\infty}^{\infty} f(hk)\Phi(h^{-1}x - k). \tag{2}$$

Here the coefficients c_{hk} are the sampled values $f(hk)$ of f at the node points hk (see [5] for an historical overview of this matter). If

$$\widehat{\Phi}(\xi) = 1 + O(\xi^{m+1}) \tag{3}$$

then (see [20] and [4])

$$||f - S_h f||_{H^s} = O(h^{m+1-s}).$$

Condition (3) means that all the moments $\lambda_l = \int y^l \Phi(y)dy$ $l = 1, ..., m$, must be equal to zero.

Consider now the class of expansions

$$A_h f(x) = S_h(R_h f)(x) = \sum_{-\infty}^{\infty} R_h f(hk)\Phi(h^{-1}x - k) \tag{4}$$

where the coefficients are the sampled values $R_h f(hk)$ obtained from f by the "averaging" operation

$$R_h f(x) = \int_{-\infty}^{\infty} f(x-y)h^{-1}\mu(h^{-1}y)dy, \tag{5}$$

with $\mu \in L^1(\mathbf{R})$ and $\int_{-\infty}^{\infty} \mu(y)dy = 1$. That is, $R_h f = f * \mu_h$ is the convolution of f with a kernel of Fejér's type $\mu_h(y) = h^{-1}\mu(h^{-1}y)$. Observe that sampling series correspond to the limit case when μ is the delta distribution. In finite element theory R_h and S_h are known as *restricition* and *prolongation* operators, respectively. For this class of approximation it is required that:

$$\hat{\Phi}(\xi)\hat{\mu}(\xi) = 1 + O(\xi^{m+1}). \tag{6}$$

The m-criterion of convergence together with the *moment relation* (6) implies that (cf. [1] and also [8])

$$||f - A_h f||_{H^s} = O(h^{m+1-s}).$$

In the present paper, we shall extend the results mentioned above for approximations (4) with averaging operations $R_h f$ given by

$$R_h f(x) = (f * d\nu_h)(x) = \int_{-\infty}^{\infty} f(x - hy)d\nu(y), \tag{7}$$

where ν are functions of bounded variation. We shall also replace the regularity hypotheses usually made in finite element theory on the basic functions Φ. Instead of $\Phi \in H^m(\mathrm{R})$ with compact support, we shall assume that Φ is m-regular in the sense that $\Phi(x)$ and its derivatives up to order m must have fast decay as $|x| \to \infty$ (see Hypothesis 2), not necessarely with compact support.

The moment relation in the present case becomes

$$\hat{\Phi}(\xi)\check{\nu}(\xi) = 1 + O(\xi^{m+1}),$$

where $\check{\nu}$ is the Fourier-Stieltjes transform of ν. We shall prove in Section 3 that the same order of accuracy $O(h^{m+1-s})$ also holds for approximations (4)-(7) under the above conditions.

This class includes expansions of type (4)-(5) corresponding to $d\nu(y) = \mu(y)dy$, as well as the sampling series (2) with $d\nu(y) = \delta(y - 0)dy$. Other approximations of interest in the class (4)-(7) are obtained with coefficientes given by discrete convolutions

$$c_{hk} = \sum_l \tau_{l-k} f(lh), \tag{8}$$

where $\tau = (\tau_l)$, $l \in \mathbf{Z}$ are sequences in ℓ^1 with $\sum_l \tau_l = 1$. For these examples, the function ν satisfies $d\nu(y) = \sum_l \tau_l \delta(y - l)dy$. They can also be expressed as sampling series in terms of another basic function Φ^* obtained from Φ as

$$\Phi^*(x) = \sum_l \tau_l \Phi(x + l).$$

Our motiviation for studying this subject comes from the recent interest in the approximations called *multiresolution analysis of* $L^2(\mathbf{R})$ (cf. [17] and [18]). In this context expansions of type (4)-(5) appear naturally. In a multiresolution analysis of $L^2(\mathbf{R})$ a function $f \in L^2(\mathbf{R})$ can be decomposed as follows:

$$f(x) = \sum_{k\in\mathbf{Z}} < f, \Phi_{jk} > \Phi_{jk}(x) + \sum_{l\geq j}\sum_{k\in\mathbf{Z}} < f, \Psi_{lk} > \Psi_{lk}(x). \tag{9}$$

The function $\Phi(x)$ appearing in the first term in the right hand side of the above expression is called a *scaling function* and for $j \in \mathbf{Z}$ the sets

$$\{\Phi_{jk}(x) = 2^{j/2}\Phi(2^j x - k), k \in \mathbf{Z}\}$$

form orthonormal bases of embedded closed subspaces $V_j \subset L^2(\mathbf{R})$ and

$$\Pi_j f(x) = \sum_{k\in\mathbf{Z}} < f, \Phi_{jk} > \Phi_{jk}(x) \tag{10}$$

is the orthogonal projection of f onto V_j. Similarly, the functions $\Psi_{lk}(x)$ in the second term of (9) are defined as

$$\Psi_{lk}(x) = 2^{l/2}\Psi(2^l x - k)$$

in terms of the function $\Psi(x)$, which is usually called a *basic wavelet*, and the set $\{\Psi_{lk}(x), l, k \in \mathbf{R}\}$ constitutes an orthonormal basis for $L^2(\mathbf{R})$. Moreover, for every j, the closed subspace W_j spanned by $\{\Psi_{jk}(x), k \in \mathbf{Z}\}$ is the orthogonal complement of V_j in V_{j+1}. Consequently, in this kind of decomposition the higher resolution approximation $\Pi_{j+1}f$ is obtained by just adding to $\Pi_j f$ a *high frequency* component

$$D_j f = \sum_{k\in\mathbf{Z}} < f, \Psi_{jk} > \Psi_{jk}(x)$$

corresponding to the orthogonal projection of f onto W_j. A multiresolution expansion (9) is a discrete version of the *wavelet transform*

$$Wf(a,b) = a^{-1/2}\int_{-\infty}^{\infty} f(y)\Psi\left(\frac{y-b}{a}\right) dy$$

This technique provides an adequate framework to analise those phenomena that are well localised in time or frequency domains. It has received considerable attention in the last years and had been succesfully applied in several fields of Mathematics, Physics and Signal Analysis (cf. [7], [16] and [19]).

Observe that the approximation $\Pi_j f \in V_j$ has the form (1), where the coefficients are the L^2-scalar products $< f, \Phi_{jk} > 2^{j/2}$. It can also be represented in the convolution form (4)-(5) with $h = 2^{-j}$ and $\mu(y) = \Phi(-y)$. According to Meyer [18], r-regular scaling functions corresponding to multiresolution analysis of $L^2(\mathbf{R})$ satisfy the m-criterion of convergence with m at least equal to r. The moment relation is also verifyied by the operators Π_j. Indeed, orthonormality implies that

$$\sum_k |\hat{\Phi}(\xi + 2k\pi)|^2 = 1$$

for all $\xi \in \mathbf{R}$. Consequently, since $\mu(x) = \Phi(-x)$,

$$\hat{\Phi}(\xi)\hat{\mu}(\xi) = |\hat{\Phi}(\xi)|^2 = 1 - \sum_{k \neq 0} |\hat{\Phi}(\xi + 2k\pi)|^2 = 1 + O(|\xi|^{2m+2}).$$

Note in this case that the moment relation and the m-criterion of convergence have distinct orders, namely, $2m + 1$ and m, respectively. Our results of Section 3 implies that, for smooth f, m-regular muliresolution approximations $\Pi_j f$ have accuracy

$$\| f - \Pi_j f \|_{H^s} = O(2^{-j(m+1-s)}). \tag{11}$$

In fact a sharper result holds in this case (cf. [18]). Let $\epsilon_j = 2^{js} \| D_j f \|_{L^2}$, where D_j is the orthogonal projection onto W_j. Then the H^s-norm is equivalent to the sum of the L^2-norm of $\Pi_0 f$ plus the ℓ^2-norm of the sequence $\epsilon_j, j \geq 0$.

There also exist generalizations of expansions (9) in the form

$$f(x) = \sum_{k \in \mathbf{Z}} < f, \Phi^*_{jk} > \Phi_{jk}(x) + \sum_{l \geq j+1} \sum_{k \in \mathbf{Z}} < f, \Psi^*_{lk} > \Psi_{lk}(x) \tag{12}$$

differing in that the set $\{\Phi_{jk}, \Psi_{jk}\}$ of the *synthesizing functions* is not orthogonal and the *analysing functions* $\{\Phi^*_{jk}, \Psi^*_{jk}\}$ are not necessarily the same as the synthesizing functions. This kind of decompositions includes the *phi-transform* [14] and biorthogonal multiresolution expansions ([6] and [24]), all of them been appropiate for local time-frequency analysis. Note that the approximations

$$\sum_{k \in \mathbf{Z}} < f, \Phi^*_{jk} > \Phi_{jk}(x),$$

obtained by truncation of expansions (12), are also in the class defined by (4)-(5).

Section 4 is dedicated to the study of some usual approximations having the coefficients in the discrete convolution form (8). We shall identify in each case the required condition in order to obtain a moment relation. Even though in the examples presented the emphasis is placed on the context of orthogonal multiresolution analysis, the results are also valid in other contexts.

We shall discuss first the case of sampling series and how to get vanishing moments. For instance, despite the fact that the scaling functions do not always satisfy a moment relation for sampling series, we shall see that it is possible to construct other sampling series aproximations $Q_j f$ in V_j in terms of different basic functions $\Phi^* \in V_0$ with moment relation

$$\widehat{\Phi^*}(\xi) = 1 + O(\xi^{2m+2})$$

Such operators, namely, sampling series with many vanishing moments and based on modified versions of the trial functions, are known in the finite element literature as *quasi-interpolant* operators. Galerkin methods for PDE's based on such schemes of approximation give rise to the so called superconvergence at the node points (cf. [23]).

Whe shall call $I_j f$ the interpolant operator which coincides with f at the node points $k2^{-j}$. This operator can be well defined and will have the coefficients in the discrete convolution form (8), provided that the discrete Fourier transform $\widetilde{\Phi}(\xi)$ never vanishes. Under this condition, the m-criterion of comvergence implies a moment relation of same order.

We shall also study the expansion $D\Pi_j f$ in a multiresolution analysis V_j with coefficients

$$\bar{c}_{hk} = \sum_l f(l2^{-j})\Phi(l-k).$$

They correspond to approximations of the L^2-scalar products $< f, \Phi_{jk} > 2^{j/2}$ using the simple numerical integration by rectangles based on the mesh points $x = k2^{-j}$. We shall prove that both $D\Pi_j$ satisfies a moment relation with

$q = m$. This means that the "discrete orthogonal projection" $D\Pi_j f$ has the same accuracy as $\Pi_j f$.

Before proving our main result in Section 3, let us state some preliminary results.

2 Notations, definitions and preliminary results

The Fourier transform of a function g is defined by

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi y} g(y) dy$$

and the discrete Fourier transform

$$\tilde{\tau}(\xi) = \sum_l \tau_l e^{-il\xi}$$

is associated to sequences $\tau = (\tau_l)$ in ℓ^1 as well as in ℓ^2.

Let us also introduce the Sobolev spaces $H^s(\mathbf{R})$ endowed with the more convenient norm

$$||f||_{H^s} = \left(\int (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{1/2}.$$

The following relation

$$\sum_k g(\xi + 2k\pi) = \frac{1}{2\pi} \sum_k \widehat{g}(k) e^{ik\xi} \tag{13}$$

is known as the Poisson summation formula. It holds almost everywhere (a.e.) if $g \in L^1(\mathbf{R})$ and the sequence $\widehat{g}(k)$ is in ℓ^1. By Riesz-Fischer theorem, this is also true if $g \in L^2(\mathbf{R})$ is such that its periodized version $\sum_k g(\xi + 2k\pi)$ is in $L^2([-\pi, \pi])$.

Through this paper we shall use frequently the following version of (13)

$$h^{-1} \sum_k \widehat{w}(h^{-1}(\xi + 2k\pi)) = \sum_k w(hk) e^{-ik\xi}. \tag{14}$$

In the proof of Theorem 3.1 we shall need the results of the following lemma.

Lemma 2.1 *Let $w \in H^s(\mathrm{R})$, $s > 1/2$. Then the Poisson summation formula (14) holds. Furthermore, for $r \geq 0$*

$$\int_{-\pi}^{\pi} |\xi|^{2r} |\tilde{w}_h(\xi)|^2 d\xi \;\leq\; C\left\{ h^{2r-1} \int_{-\pi/h}^{\pi/h} |\xi|^{2r} |\widehat{w}(\xi)|^2 d\xi \right.$$

$$\left. + \; h^{2s-1} \int_{-\infty}^{\infty} |\xi|^{2s} |\widehat{w}(\xi)|^2 d\xi \right\}, \tag{15}$$

where $C = C(r,s)$ and $\tilde{w}_h(\xi) = \sum_k w(hk) e^{-ik\xi}$.

Proof. First observe that

$$|\sum_k \widehat{w}(h^{-1}(\xi + 2k\pi))|^2 \leq 2\left\{ |\widehat{w}(h^{-1}\xi)|^2 + |\sum_{k\neq 0} \widehat{w}(h^{-1}(\xi + 2k\pi))|^2 \right\}.$$

By Cauchy-Schwartz inequality,

$$|\sum_{k\neq 0} \widehat{w}(h^{-1}(\xi + 2k\pi))|^2 \leq \sum_{k\neq 0} |\xi + 2k\pi|^{-2s} \sum_{k\neq 0} |\xi + 2k\pi|^{2s} |\widehat{w}(h^{-1}(\xi + 2k\pi))|^2$$

$$\leq C \sum_k |\xi + 2k\pi|^{2s} |\widehat{w}(h^{-1}(\xi + 2k\pi))|^2. \tag{16}$$

Consequently

$$\int_{-\pi}^{\pi} |\xi|^{2r} |\sum_k \widehat{w}(h^{-1}(\xi + 2k\pi))|^2 d\xi \leq$$

$$\leq \; C\left\{ \int_{-\pi}^{\pi} |\xi|^{2r} |\widehat{w}(h^{-1}\xi)|^2 d\xi + \sum_k \int_{-\pi}^{\pi} |\xi|^{2r} |\xi + 2k\pi|^{2s} |\widehat{w}(h^{-1}(\xi + 2k\pi))|^2 d\xi \right\}$$

$$\leq \; C\left\{ h^{2r+1} \int_{-\pi/h}^{\pi/h} |u|^{2r} |\widehat{w}(u)|^2 du + \pi^{2r} h^{2s+1} \sum_k \int_{\frac{2(k-1)\pi}{h}}^{\frac{2(k+1)\pi}{h}} |u|^{2s} |\widehat{w}(u)|^2 du \right\}$$

$$= \; C\left\{ h^{2r+1} \int_{-\pi/h}^{\pi/h} |u|^{2r} |\widehat{w}(u)|^2 du + \pi^{2r} h^{2s+1} \int_{-\infty}^{\infty} |u|^{2s} |\widehat{w}(u)|^2 du \right\}.$$

This inequality for $r = 0$ impliy that the function $\sum_k \widehat{w}(h^{-1}(\xi + 2k\pi))$ is square integrable in the interval $[-\pi, \pi]$ and then the Poisson summation formula (14) holds. Therefore,

$$\int_{-\pi}^{\pi} |\xi|^{2r} |\tilde{w}_h(\xi)|^2 d\xi = h^{-2} \int_{-\pi}^{\pi} |\xi|^{2r} |\sum_k \widehat{w}(h^{-1}(\xi + 2k\pi))|^2 d\xi$$

$$\leq C\left\{h^{2r-1}\int_{-\pi/h}^{\pi/h}|u|^{2r}|\hat{w}(u)|^2du+\pi^{2r}h^{2s-1}\int_{-\infty}^{\infty}|u|^{2s}|\hat{w}(u)|^2du\right\},$$

from which the result immediately follows.

As an immediate consequence of equation (15), the discrete ℓ^2-norm

$$||w||_h^2=h\sum_k|w(hk)|^2=\frac{h}{2\pi}\int_{-\pi}^{\pi}|\tilde{w}_h(\xi)|^2d\xi$$

is finite for $w\in H^s(\mathbf{R}), s>1/2$.

Let $BV=BV(\mathbf{R})$ be the set of all functions ν of bounded variation with $||\nu||_{BV}=\int|d\nu(y)|$. For $\nu\in BV$ the Fourier-Stieltjes transform

$$\check{\nu}(\xi)=\int_{-\infty}^{\infty}e^{-i\xi y}d\nu(y)$$

defines a bounded linear transformation from BV into $C(\mathbf{R})$ and

$$|\check{\nu}(\xi)|\leq||\nu||_{BV}. \tag{17}$$

As an example, let $\mu\in L^1(\mathbf{R})$. The function ν defined by $\nu(x)=\int_{-\infty}^{x}\mu(x)dx$ is in BV, $||\nu||_{BV}=||\mu||_{L^1}$ and $\check{\nu}$ coincides with $\hat{\mu}$. Other examples are given by functions ν such that $d\nu(y)=\sum_l\tau_l\delta(y-l)dy$, where $\tau(l)$ are sequences in ℓ^1. In these cases $\check{\nu}(\xi)=\tilde{\tau}(\xi)$.

If $f\in L^2(\mathbf{R})$ and $\nu\in BV$, the convolution $f*d\nu$ as defined by

$$(f*d\nu)(x)=\int_{-\infty}^{\infty}f(x-y)d\nu(y)$$

exists (a.e.) as an absolutely convergent integral, $f*d\nu\in L^2(\mathbf{R})$ and

$$||f*d\nu||_{L^2}\leq||f||_{L^2}||\nu||_{BV}.$$

Furthermore,

$$\widehat{f*d\nu}(\xi)=\hat{f}(\xi)\check{\nu}(\xi),\quad\text{(a.e.)}.$$

Consequently, for $f\in L^2(\mathbf{R})$, the transformation R_hf defined in (7) satisfies

$$\widehat{R_hf}(\xi)=\hat{f}(\xi)\check{\nu}(h\xi). \tag{18}$$

Moreover, if $f\in H^s(\mathbf{R})$ then also $R_hf\in H^s(\mathbf{R})$ and, for $s>1/2$

$$\widetilde{R_hf}(\xi)=\sum_k R_hf(hk)e^{-ik\xi}=h^{-1}\sum_k\widehat{R_hf}(h^{-1}(\xi+2k\pi)). \tag{19}$$

The functions defined in (4) have Fourier transform

$$\widehat{A_h f}(\xi) = h\widehat{R_h f}(h\xi)\hat{\Phi}(h\xi). \tag{20}$$

Therefore,

$$\widehat{A_h f}(\xi) = \hat{f}(\xi)\check{\nu}(h\xi)\hat{\Phi}(h\xi) + \sum_{k\neq 0} \widehat{R_h f}(h^{-1}(h\xi + 2k\pi))\hat{\Phi}(h\xi). \tag{21}$$

3 Convergence in $H^s(\mathbf{R})$

In this section we will study the convergence in the H^s-norm of expansions (4)-(7). We shall assume the following hyphoteses:

Hypothesis 1. $\nu \in BV$ and $\int d\nu(y) = 1$.

Hypothesis 2. Φ is *r-regular*, $r \geq 1$, i.e., for all indices β such that $0 \leq \beta \leq r$

$$|\partial^\beta \Phi/\partial x^\beta| \leq C_n(1 + |x|)^{-n} \tag{22}$$

for all integer $n > 0$.

Hypothesis 3. (The m-criterion of convergence) $\hat{\Phi}(\xi)$ has zeros of order $m+1$ at $\xi = 2k\pi$, $k \neq 0$ and $\hat{\Phi}(0) = 1$.

Hypothesis 4. ν and Φ are related by

$$\hat{\Phi}(\xi)\check{\nu}(\xi) = 1 + O(|\xi|^{q+1}) \tag{23}$$

Theorem 3.1 *Assume that ν and Φ satisfy Hypotheses 1-4 and let $p = \min\{m, q\}$. If $f \in H^{p+1}(\mathbf{R})$ then the approximation $A_h f$ defined by (4)-(7) satisfy the error estimate*

$$\|f - A_h f\|_{H^s} \leq Ch^{p+1-s}\|f\|_{H^{p+1}}, \tag{24}$$

where $0 \leq s \leq \min\{p+1, r\}$ and the constant C is independent of f.

Proof. The proof can be carried out in an analogous way as in [21], Theorem I. In order to estimate

$$\| f - A_h f \|_{H^s}^2 = \int_{-\infty}^{\infty} (1+|\xi|^2)^s |\hat{f}(\xi) - \widehat{A_h f}(\xi)|^2 d\xi$$

we have to show that the integrals

$$I_1 = \int_{|\xi|\le\pi/h} (1+|\xi|^2)^s |\hat{f}(\xi) - \widehat{A_h f}(\xi)|^2 d\xi,$$

and

$$I_2 = \int_{|\xi|\ge\pi/h} (1+|\xi|^2)^s |\hat{f}(\xi) - \widehat{A_h f}(\xi)|^2 d\xi$$

are all bounded by $Ch^{2(p+1-s)} \| f \|_{H^{p+1}}^2$. For I_1 it is a consequence of Hypothesis 4 and Poisson sumation formula. For the estimation of I_2 it is necessary to show that for $0 \le s \le r$

$$\sum_{j\neq 0} |\hat{\Phi}(\xi + 2j\pi)|^2 |\xi + 2j\pi|^{2s} = O(|\xi|)^{2m+2} \tag{25}$$

as $\xi \to 0$. For compactly supported $\Phi \in H^r(\mathrm{R})$ satisfying Hypothesis 3 the proof is based on the Paley-Wiener theorem and the theory of entire functions (cf. [21]). The result in (25) also holds for r-regular functions Φ satisfying Hypothesis 3 and correspond to Proposition 1 in [12].

Replacing expression (21) in I_1 we have

$$I_1 \le 2\left\{\int_{|\xi|\le\pi/h} (1+|\xi|^2)^s |\hat{f}(\xi)[1 - \hat{\mu}(h\xi)\hat{\Phi}(h\xi)]|^2 d\xi\right.$$

$$\left. + \int_{|\xi|\le\pi/h} (1+|\xi|^2)^s |\sum_{j\neq 0} \widehat{R_h f}(\xi + 2\pi j h^{-1})|^2 |\hat{\Phi}(h\xi)|^2 d\xi \right\}$$

$$= 2\{(a) + (b)\}.$$

Hyphotesis 4 implies that

$$\begin{aligned}
(a) \;&\le\; C\int_{|\xi|\le\pi/h} (1+|\xi|^2)^s |h\xi|^{2(q+1)} |\hat{f}(\xi)|^2 d\xi \\
&\le\; C\int_{|\xi|\le\pi/h} (1+|\xi|^2)^s |h\xi|^{2(q+1-s)} |\hat{f}(\xi)|^2 d\xi \\
&\le\; Ch^{2(q+1-s)} \int_{-\infty}^{\infty} (1+|\xi|^2)^{q+1} |\hat{f}(\xi)|^2 d\xi.
\end{aligned}$$

To evaluate (b), consider the estimate (16) together with formulas (17) and(18) to obtain

$$
\begin{aligned}
(b) &\leq Ch^{-2s} \int_{|\xi|\leq\pi/h} \sum_j (h\xi + 2\pi j)^{2(q+1)} |\hat{f}(\xi + 2\pi j h^{-1})|^2 d\xi \\
&= Ch^{2(q-s+1)} \sum_j \int_{\frac{(2j-1)\pi}{h}}^{\frac{(2j+1)\pi}{h}} |\xi|^{2(q+1)} |\hat{f}(\xi)|^2 d\xi \\
&= Ch^{2(q+1-s)} \int_{-\infty}^{\infty} |\xi|^{2(q+1)} |\hat{f}(\xi)|^2 d\xi.
\end{aligned}
$$

I_2 is also splitted in two terms

$$
I_2 \leq 2\left\{ \int_{|\xi|\geq\pi/h} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi + \int_{|\xi|\geq\pi/h} (1+|\xi|^2)^s |\widehat{A_h f}(\xi)|^2 d\xi \right\}.
$$

$$
= 2\{(i) + (ii)\}.
$$

For $|\xi| > \pi/h$ we notice that $|h\xi| > \pi$ and, therefore

$$
(i) \leq C \int_{|\xi|\geq\pi/h} |h\xi|^{2(p+1-s)} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \leq Ch^{2(p+1-s)} \int_{-\infty}^{\infty} (1+|\xi|^2)^{2(p+1)} |\hat{f}(\xi)|^2 d\xi.
$$

Replacing (20) in (ii) we get

$$
\begin{aligned}
(ii) &= h^2 \int_{|\xi|\geq\pi/h} (1+|\xi|^2)^s |\widetilde{R_h f}(h\xi)|^2 |\hat{\Phi}(h\xi)|^2 d\xi \\
&\leq Ch^2 \sum_{j\neq 0} \int_{\frac{(2j-1)\pi}{h}}^{\frac{(2j-1)\pi}{h}} |\xi|^{2s} |\widetilde{R_h f}(h\xi)|^2 |\hat{\Phi}(h\xi)|^2 d\xi \\
&= Ch^2 \sum_{j\neq 0} \int_{-\pi/h}^{\pi/h} |\xi - h^{-1} 2\pi j|^{2s} |\widetilde{R_h f}(h\xi - 2\pi j)|^2 |\hat{\Phi}(h\xi - 2\pi j)|^2 d\xi \\
&= Ch^2 \int_{-\pi/h}^{\pi/h} |\widetilde{R_h f}(h\xi)|^2 \sum_{j\neq 0} |\hat{\Phi}(h\xi + 2\pi j)|^2 |\xi + h^{-1} 2\pi j|^{2s} d\xi \\
&\leq Ch^{1-2s} \int_{-\pi}^{\pi} |\widetilde{R_h f}(\xi)|^2 \sum_{j\neq 0} |\hat{\Phi}(\xi + 2j\pi)|^2 |\xi + 2j\pi|^{2s} d\xi.
\end{aligned}
$$

From (25) and Lemma 2.1

$$
\begin{aligned}
(ii) &\leq Ch^{1-2s} \int_{-\pi}^{\pi} |\widetilde{R_h f}(\xi)|^2 |\xi|^{2p+2} d\xi \\
&\leq Ch^{1-2s+2(p+1-1/2)} \| R_h f \|_{H^{p+1}}^2 \\
&\leq Ch^{2(p+1-s)} \| f \|_{H^{p+1}}^2 .
\end{aligned}
$$

The above estimates for (a), (b), (i) and (ii) led to the result in (24).

3.1 Convergence at the node points

We shall see in this section that the order of convergence $O(h^{p+1})$ also holds in the discrete ℓ^2-norm.

Theorem 3.2 *Let $A_h f$ be defined by (4)-(7). Under the same hypotheses of Theorem 3.1 with $r \geq 1$*

$$||f - A_h f||_{2,h} \leq Ch^{p+1}||f||_{H^{p+1}}. \tag{26}$$

Proof. It is a consequence of Lemma 2.1 and Theorem 3.1. Using equation (15) with $w = f - A_h f$, $r = 0$ and $s = 1$ we have

$$\begin{aligned} ||f - A_h f||^2_{2,h} &= h\sum_k |f(kh) - A_h f(kh)|^2 \\ &\leq C\left\{\int_{-\pi/h}^{\pi/h} |\hat{f}(\xi) - \widehat{A_h f}(\xi)|^2 d\xi + h^{2s}\int_{-\infty}^{\infty} |\xi|^{2s}|\hat{f}(\xi) - \widehat{A_h f}(\xi)|^2 d\xi\right\}. \end{aligned}$$

The first intergral on the right hand side of the above expression can be estimated as I_1 from the proof of Theorem 3.1. Consequently,

$$\begin{aligned} ||f - A_h f||^2_{2,h} &\leq C\left\{h^{2(p+1)}||f||^2_{H^{p+1}} + h^2||f - A_h f||^2_{H^1}\right\} \\ &\leq Ch^{2(p+1)}||f||^2_{H^{p+1}}. \end{aligned}$$

4 Examples

In the examples discussed below the emphasis is placed on expansions in terms of a scaling function Φ associated to a multiresolution analysis of $L^2(\mathbf{R})$. So, let us first present some examples of scaling functions.

4.1 Examples of scaling functions

Example 4.1 The Shannon scaling function. One of the most simple examples of a multiresolution analysis of $L^2(\mathbf{R})$ is given by the spaces V_j of all functions whose Fourier transforms have support in the intervals $[-2^j\pi, 2^j\pi]$.

The correspondig scaling function is the function Φ whose Fourier transform is the indicator function

$$\hat{\Phi}(\xi) = \begin{cases} 1 & \text{if } -\pi \leq |\xi| \leq \pi \\ 0 & \text{otherwise} \end{cases}.$$

This scaling function has compact support in the Fourier domain but has slow decay in the spatial domain: it does not satisfy any regularity condition.

Example 4.2 Meyer [18] showed that one can built a scaling function which is r-regular, for all r, and compact supported in the Fourier domain. This means that it satisfies the m-criterion of convergence for all m. But for many numerical applications, the decay $O(x^{-n})$ for any $n > 0$ is too slow.

Example 4.3 **Splines.** Let V_0 be the subspace of $L^2(\mathbf{R})$ constituted by all C^{r-1}-functions which coincide on each interval $[k, k+1]$ with a polynomial of degree less or equal r. Then, the family of embedded subspaces V_j defined by the relation $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j-1}$ form a multiresolution analysis of $L^2(\mathbf{R})$. Defining $\Phi(x) = \varphi_r(x)$ for odd r and $\Phi(x) = \varphi_r(x + 1/2)$ otherwise, where φ_r is the B-spline of degree r, then the translates $\Phi(x - k), k \in \mathbf{Z}$, constitute an unconditional basis for the subspace V_0. φ_r is defined recursively as $\varphi_r = \varphi_0 * \varphi_{r-1}$, where $\varphi_0(x) = \chi(x)$ is the characteristic function of the interval $[-1/2, 1/2]$. φ_r is an even function in $C^{r-1}(\mathrm{R})$ which coincides with a polynomial of degree r in its support $[-\frac{r+1}{2}, \frac{r+1}{2}]$. Note that the Fourier transform $\hat{\varphi}_r(\xi) = \left(\frac{2sin(\xi/2)}{\xi}\right)^{r+1}$ has zeros of order $r+1$ at all the points $\xi = 2k\pi, k \neq 0$. Consequently Φ is r-regular and satisfies the r-criterion of convergence. However, for each j, the basis $\{\Phi_{j,k}\}, k \in \mathbf{Z}$ is not orthogonal, unless $r = 0$. Through the orthonormalization process defined by

$$\hat{\phi}(\xi) = \hat{\Phi}(\xi) \left(\sum_k |\hat{\Phi}(\xi + 2kpi)|^2 \right)^{-1/2},$$

the resulting new basis $\{\phi_{j,k}, k \in \mathbf{Z}\}$ is orthonormal and inheritates from φ_r all the regularity properties. However, the compact support is lost . But ϕ has exponential decay, which make it good for numerical applications. The construction of such bases is due to Battle [2] and Lemarie [15].

Example 4.4 Daubechies' scaling functions. The scaling functions of Daubechies $\Phi =_N \Phi, N = 2,3,...$ are supported on the intervals $[0, 2N-1]$ and their diadic scaled translates Φ_{jk} form orthonormal basis for multiresolution analysis of $L^2(\mathbf{R})$ (cf. [9]). They satisfy the m-criterion of convergence with $m = N-1$. Consequently the corresponding projection operators Π_j satisfy

$$||f - \Pi_j f||_{H^s} \leq C 2^{-j(N-s)} ||f||_{H^N}$$

for $0 \leq s \leq p(N)$ where $p = p(N)$ is the Sobolev index such that $_N\Phi \in H^p(\mathbf{R})$. However, $p(N) << N-1$ (cf. [10]).

4.2 Discrete convolutions

In the next examples V_j are subspaces of an r-regular multiresolution analysis of $L^2(\mathbf{R})$. Recall that the corresponding sampling series Φ satisfy the m-criterion of convergence with $m \geq r$. Excepting Example 4.8, all the expansions considered here are in the form (4)-(7) with $R_h f$ given in the discrete convolution form

$$R_h f(x) = (f * d\nu)(x) = \sum_l \tau_l f(x - lh), \tag{27}$$

where $\tau \in \ell^1$. It corresponds to $d\nu(y) = \sum_l \tau_l \delta(y - l)dy$. In this case Hypothesis 4 becomes

$$\hat{\Phi}(\xi)\tilde{\tau}(\xi) = 1 + O(\xi^{q+1}).$$

Example 4.5 Sampling series As discussed in the previous section, the accuracy of approximations by sampling series depends on the number of vanishing moments $\lambda_l = \int y^l \Phi(y) dy, l > 0$. Note that for r-regular functions Φ satisfying the m-criterion of convergence, and for $l \leq m$, λ_l can be expressed as

$$\lambda_l = \sum_k k^l \Phi(l).$$

This is also a consequence of Poisson summation formula (cf. [13]).

Scaling functions do not always satisfy a moment relation for sampling series. For instance, calculations with Daubechies' scaling functions for $N = 2,3,...,11$ show that their first moments λ_1 are not equal to zero. However, in an r-regular multiresolution analysis, higher vanishing moments can be obtained with modified basic functions Φ^* constructed from the integer translates of the scaling function Φ. Following [18], let α_k be the Fourier coefficients of $e^{-i\eta(\xi)}$, where $\eta(\xi)$ is a C^∞ and 2π-periodic extension of the argument of $\widehat{\Phi}(\xi)$. Defining

$$\Phi^*(x) = \sum_k \alpha_k \Phi(x+k),$$

then Φ^* is also r-regular and the resulting families $\Phi^*_{jk}(x)$ form orthonormal bases for the same spaces V_j. However, with this procedure, the compact support property of Φ is lost. Note that

$$\begin{aligned}
\widehat{\Phi^*}(x) &= \sum_k \alpha_k e^{ik\xi}\widehat{\Phi}(\xi) \\
&= e^{-i\eta(\xi)}\widehat{\Phi}(\xi) = |\widehat{\Phi}(\xi)| \\
&= 1 + (\xi^{2m+2}).
\end{aligned}$$

As a consequence, the expansions

$$Q_j f(x) = \sum_k f(k2^{-j})\Phi^*(2^j x - k)$$

are in V_j, and for smooth f

$$\|f - Q_j f\|_{H^s} = O(2^{-j(m+1-s)}),$$

where $0 \leq s \leq r$.

Due to their symmetry property, $\lambda_1 = 0$ for all B-splines φ_r. However, since $\varphi_r(x) > 0$ in the interior of their support, $\lambda_2 \neq 0$. In this case, with no orthonormality requirement, it is possible to find a spline $\varphi_r^*(x)$ with vanishing moments λ_l, $1 \leq \lambda \leq 2r+1$, using only finitely many coefficients α_k (see [23]). Therefore, $\varphi_r^*(x)$ also has compact support.

Example 4.6 Interpolation One might wish to take an approximation $I_j f$ from V_j that coincides with f on the mesh points $x = k2^{-j}$, $k \in \mathbb{Z}$. This leads us to consider the possibility of defining an interpolation operator in terms of the translates and dilates of Φ. Let $y_k = f(k2^{-j})$. The problem is to find coeficients c_{jl} such that the function

$$I_j f(x) = \sum_l c_{jl}\Phi(2^j x - l)$$

coincides with y_k at $x = k2^{-j}$. Consequently, the sequence $c = (c_{jl})$ must satisfy

$$\sum_l c_{jl}\Phi(k-l) = y_k \tag{28}$$

for all $k \in \mathbf{Z}$. The above convolution transformation $c \to Tc = y$ is a bounded linear transformation from ℓ^s into itself, $1 \leq s \leq \infty$. Furthermore, assuming that

$$\tilde{\Phi}(\xi) = \sum_k \Phi(k)e^{-ik\xi} \neq 0 \tag{29}$$

for all $0 \leq \xi \leq 2\pi$, then (28) has an inverse in ℓ^s which is explicitly given by

$$c_{jk} = \sum_l \beta_{k-l}y_l, \tag{30}$$

where β_n are the Fourier coefficients of $\frac{1}{\tilde{\Phi}(-\xi)}$. The proofs of the above statements are in [22]. From the relation (30) we deduce that the interpolation operator $I_j f$ has the discrete convolution form (4)-(27) with $d\nu(y) = \sum_k \beta_k \delta(y-k)dy$. Consequently, $\check{\nu}(\xi) = \tilde{\beta}(\xi) = \frac{1}{\tilde{\Phi}(\xi)}$. Recalling that Φ is r-regular an satisfy the m-criterion of convergence and applying the Poisson summation formula we get

$$\begin{aligned} \tilde{\Phi}(\xi) &= \sum_k \hat{\Phi}(\xi + 2k\pi) \\ &= \hat{\Phi}(\xi) + O(\xi^{m+1}). \end{aligned} \tag{31}$$

If in adition Φ satisfies condition (29), then

$$\begin{aligned} \hat{\Phi}(\xi)\check{\nu}(\xi) - 1 &= \frac{\hat{\Phi}(\xi)}{\tilde{\Phi}(\xi)} - 1 \\ &= \frac{\hat{\Phi}(\xi) - \tilde{\Phi}(\xi)}{\tilde{\Phi}(\xi)} = O(\xi^{m+1}). \end{aligned}$$

As a consequence of Theorem 3.1, the interpolation operator satisfies

$$||I_j f - f||_{H^s} \leq C2^{-j(m+1-s)}||f||_{H^{m+1}}$$

for $f \in H^{m+1}(\mathbf{R})$ and $0 \leq s \leq r$.

Now the question is: Do all scaling functions satisfy the interpolation condition (29)? The answer is afirmative for the B-splines (cf. (cf. [22])). As can be deduced from the graphs of $|\widetilde{\Phi}|$ for $\Phi =_N \Phi$, $N = 2, 3, ..., 11$ shown in [12], it is also true for these Daubechies' scaling functions. However, the conjecture for all scaling functions should be analysed carefully. Consider for example the case $N = 5$. Even though condition (29) is satisfied by $_5\Phi$, it is just barely so.

Example 4.7 Discrete orthogonal projections Let $\{V_j\}$ be an r-regular multiresolution analysis of $L^2(\mathbf{R})$ and consider $D\Pi_j f \in V_j$ be the expansion (1) with coefficients

$$\bar{c}_{hk} = \sum_l f(l2^{-j})\Phi(l-k).$$

They correspond to approximations of the L^2-scalar products $< f, \Phi_{jk} > 2^{j/2}$ using the simple numerical integration by rectangles based on the mesh points $x = l2^{-j}$. Here the discrete convolution operator (27) corresponds to $\tau_l = \Phi(-l)$. Consequently, $\tilde{\tau}(\xi) = \widetilde{\Phi}(-\xi)$. Using (31), the moment relation becomes

$$\begin{aligned} \widehat{\Phi}(\xi)\widetilde{\Phi}(-\xi) &= \widehat{\Phi}(\xi)\left(\widehat{\Phi}(-\xi) + O(\xi^{m+1})\right) \\ &= |\widehat{\Phi}(\xi)|^2 + O(\xi^{m+1}) \\ &= 1 + O(\xi^{2m+2}) + O(\xi^{m+1}) \\ &= 1 + O(\xi^{m+1}). \end{aligned}$$

Therefore, $D\Pi_j$ satisfies the hypotheses of Theorem 3.1 with $q = m$. This means that $D\Pi_j f$ has the same accuracy as $\Pi_j f$.

Example 4.8 Sampling series based on different node points We have seen that if the first moment λ_1 of the basic function Φ does not vanish then the sampling series $S_h f$, defined in terms of Φ and based on the node points kh, does not have good accuracy. In such cases another alternative procedure is to consider sampling series based on different node points as

$$A_h f(x) = \sum_k f(h(k+c))\Phi(h^{-1}x - k).$$

This expansion series has the form (4)-(7) with $d\nu(y) = \delta(y+c)dy$. Then the desired moment relation is

$$e^{ic\xi}\widehat{\Phi}(\xi) = 1 + O(\xi^{q+1}).$$

Since $e^{ic\xi}\widehat{\Phi}(\xi) = \widehat{\varphi}(\xi)$ where $\varphi(x) = \Phi(x+c)$, the the possible choice is $c = \lambda_1$ which gives a moment relation with $q = 1$. Consequently, under the m-criterion of convergence, this procedure results in approximations with accuracy of order $O(h^{2-s})$ in the H^s-norm, $0 \leq s \leq 1$. The application of these approtimations to the numerical solution of elliptic problems appeared in [25].

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