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Parallel and semi-parallel immersions into space forms (**)

Introduction

The geometry of a submanifold of a space form is described by the second fundamental form. Therefore it is natural to study isometric immersions whose second fundamental form are *simple* in some sense. In this paper we will describe some (by now classical) results on *parallel immersions*, i.e. immersions whose second fundamental form is covariantly constant, and some more recent results on *semi-parallel immersions*, i.e. immersions whose second fundamental form verifies the corresponding integrability condition.

It is probably worthwhile to observe that the above condition are the analog, in submanfolds theory, of what symmetric and semi-symmetric spaces are in intrinsic riemannian geometry.

1 - Notations and basic facts

Let $Q = Q^N(c)$ be a *complete simply connected N-dimensional riemannian manifold of costant curvature* c, i.e. a sphere ($c > 0$), an euclidean space ($c = 0$) or a hyperbolic space ($c < 0$). Superscripts will, usually, denote dimensions and will be dropped when clear from the context.

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Let M be a *riemannian manifold* and $f\colon M \to Q(c)$ be an *isometric immersion*. We will use the following (standard) notations: ∇ and $\overline{\nabla}$ will denote the *Levi-Civita connections* of M and $Q(c)$ respectively, $\nu(M)$ will denote the *normal bundle of the immersion*, $\alpha = \overline{\nabla} - \nabla\colon TM \oplus TN \to \nu(M)$ *the second fundamental form*, $\nabla^{\perp}$ the *connection on* $\nu(M)$, and, if $\xi \in \nu(M)$, $A_\xi\colon TM \to TM$ will denote the *shape operator* in the ξ direction, $A_\xi X = \nabla^{\perp}_X \xi - \overline{\nabla}_X \xi$. A_ξ and α are symmetric and related by $\langle A_\xi X, Y\rangle = \langle \alpha(X, Y), \xi\rangle$.

The local geometry of the immersion is described by the above data and the basic equations

$$\begin{aligned}\langle R(X, Y)Z, W\rangle &= c(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle)\\ &\quad + \langle \alpha(Y, W), \alpha(Y, Z)\rangle - \langle \alpha(X, Z), \alpha(Y, W)\rangle\end{aligned} \tag{1.1 (Gauss)}$$

$$(\nabla_X \alpha)(Y, Z) = (\nabla_Y \alpha)(X, Z) \tag{1.2 (Codazzi-Mainardi)}$$

$$\langle R^{\perp}(X, Y)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{1.3 (Ricci)}$$

where ∇ is extended to act on a bilinear form $\beta\colon TM \oplus TM \to \nu(M)$ by $\nabla_X \beta(Y, Z) = \nabla^{\perp}_X(\beta(Y, Z)) - \beta(\nabla_X Y, Z) - \beta(Y, \nabla_X Z)$ and $R^{\perp}$ denotes the curvature of $\nabla^{\perp}$, i.e. $R^{\perp}(X, Y)\xi = \nabla^{\perp}_X \nabla^{\perp}_Y \xi - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X, Y]} \xi$.

The basic equations (1.1), (1.2), (1.3), are the *integrability conditions* for an isometric immersion in the sense of the *fundamental theorem of submanifolds theory*:

Theorem 1. *Given a n-dimensional riemannian manifold M, a p-dimensional riemannian vector bundle* ν *over M with a metric connection* ∇^{ν} *and a symmetric bilinear bundle map* $\alpha\colon TM \oplus TM \to \nu$, *such that the above data verify the basic equations* (1.1), (1.2), *and* (1.3), *then there exist a local isometric immersion of M in* $Q^{n+p}(c)$ *such that* ν *is isometric to the normal bundle,* ∇^{ν} *corresponds to the normal connection and* α *to the second fundamental form. Moreover two such isometric immersions* (*locally*) *coincide up to an isometry of the ambient space and, if M is simply connected, the immersion is globally defined.*

It is natural to try to classify isometric immersions whose second fundamental form is somehow *simple*. We will start discussing the case in which α is constant in the following sense

Definition 1. The immersion $f\colon M \to Q^N(c)$ is said to be *parallel* if and only if $\nabla\alpha = 0$.

Parallel immersions are geometrically characterized by the following property: Given a point $p \in M$, consider the unique isometry of the ambient space which fixes p and whose differential fixes the normal space of M at p and reflects the tangent space. The immersion is *parallel* if and only if the above isometry maps $f(M)$ into itself (see [**10**], [**15**]). In particular the geodesic symmetries of M are isometries and M is a *symmetric space* (for the above reason, such immersions are also called *extrinsically symmetric*).

Example 1 (*Umbilical submanifolds*). An isometric immersion is *umbilical* if $A_\xi = \lambda_\xi I$ for all $\xi \in \nu(M)$. In this case $\alpha(X, Y) = \langle X, Y \rangle H$ where $H = \frac{1}{n}(\operatorname{trace} \alpha)$ is the *mean curvature vector*. It is a standard result that in this case $\nabla^\perp_X H = 0$ for all tangent vectors X. It follows that those immersions are parallel. Umbilical submanifolds have constant curvature $k = c + \|H\|^2$, and, besides the totally geodesics ones, they are sphere if $c \geqslant 0$ or, if $c < 0$, geodesic spheres ($k > 0$), equidistant hypersurfaces ($k < 0$) or horospheres ($k = 0$). It is important to observe the (also standard) fact that *given $p \in Q^N(c)$, $\boldsymbol{E} \subseteq T_p Q^N(c)$, and $H \in \boldsymbol{E}^\perp$, there exists a unique umbilical submanifold S containing p, with $T_p S = \boldsymbol{E}$ and mean curvature vector H*. An other simple fact, we want to observe explicitly, is that *the composition of a parallel immersion and an umbilical one is still parallel*, but, in general, composition of parallel immersions is not parallel even if the first one is totally geodesic (for example an helix on a circular cylinder in $\boldsymbol{R}^3$).

Example 2 (*Extrinsic products of umbilical submanifolds*). Let M be a riemannian manifold which decomposes as riemannian product $M_1 \times \ldots \times M_m$ of riemannian manifolds and $f: M \to Q^N(c)$ an isometric immersion. Then f is said to be an *extrinsic product* if for all X tangent to M_i and Y tangent to M_j, $i \neq j$, $\alpha(X, Y) = 0$. If $c = 0$, by a result of Moore, there exist an orthogonal decomposition $Q^N(0) = \boldsymbol{R}^N = \boldsymbol{R}^{N_1} \oplus \ldots \oplus \boldsymbol{R}^{N_m}$ and isometric immersions $f_i: M_i \to \boldsymbol{R}^{N_i}$, $i = 1, \ldots, m$, such that $f = f_1 \times \ldots \times f_m$. A similar result for $c > 0$ is a rather obvious consequence of the above result and for $c < 0$ a bit more involved (see [**18**]).

If f is an extrinsic product and $f|_{M_i}$ is totally umbilical, we will say that f is an *extrinsic product of umbilical submanifolds*. It is clear that an extrinsic product of umbilical submanifolds is a parallel immersion.

Observation. As in Example 1 we can ask if, given a subspace $\boldsymbol{E}$ of $T_p Q^N(c)$, an orthogonal decomposition $\boldsymbol{E} = \oplus \boldsymbol{E}_i$, $i = 1, \ldots, m$ and distinct vectors $H_1, \ldots, H_m$ in $\boldsymbol{E}^\perp$, there exists an extrinsic product of umbilical submanifolds $f: M = M_1 \times \ldots \times M_m \to Q^N(c)$, such that the tangent space at M_i in a given fixed point is $\boldsymbol{E}_i$ and the mean curvature vector of $f|_{M_i}$ is H_i. There is an obvious necessary condition: Given X tangent to M_i and Y tangent to M_j, $i \neq j$, $\|X\| = \|Y\| = 1$, we have

$$0 = k(X, Y) = c + \langle \alpha(X, X), \alpha(Y, Y) \rangle - \|\alpha(X, Y)\|^2 .$$

Therefore $\langle \alpha(X, X), \alpha(Y, Y) \rangle = -c$ and, taking traces, $\langle H_i, H_j \rangle = -c$. It is rather clear that the condition is also sufficient if $c = 0$, and by a standard extension, if $c > 0$. It turns out that the condition is sufficient also if $c < 0$ (see [3]). An interesting observation is the following: Let k_i be the (constant) sectional curvature of M_i and suppose $k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m$. Then

$$0 < \|H_2 - H_1\| = k_1 + k_2 \leqslant 2k_2 .$$

Therefore $k_i > 0$ if $i > 1$. In particular we can embed in a hyperbolic space, as extrinsic product of umbilical submanifolds, a product of an hyperbolic space and a sphere, but we can not embed a product of an hyperbolic space and an Euclidean space.

As in Example 1, we observe explicitly that *the composition of a parallel immersion with an extrinsic product of umbilical immersions is parallel.*

Example 3 (*Standard embeddings of some symmetric R-spaces*). Let $\boldsymbol{K}$ denote either the reals, the complex or the quaternions and $M(p, q; \boldsymbol{K})$ the space of $p \times q$ matrices with coefficients in $\boldsymbol{K}$. If $A \in M(p, q; \boldsymbol{K})$, we will denote by A^* the transpose conjugate. Consider the projective n-space over $\boldsymbol{K}$, $\boldsymbol{K}P^n$, as quotient of the unit sphere in $\boldsymbol{K}^{n+1}$ (i.e. the set $\{x \in M(n+1, 1; \boldsymbol{K}) : x^* x = 1\}$), by the equivalence relation $x \sim y$ if and only if $x = \lambda y$, $\lambda \in \boldsymbol{K}$. Define $f: \boldsymbol{K}P^n \to M(n+1, n+1; \boldsymbol{K})$ by $f(x) = xx^*$. Then f is a parallel immersion, whose image lies in the affine subspace

$$\{A \in M(n+1, n+1; \boldsymbol{K}) \mid A = A^* \text{ and trace } A = 1\} .$$

Those embeddings, up to a congruence, are the so called *Veronese embeddings* of the projective spaces and are parallel. More in general, if we consider the Grasmann manifold of the n-dimensional subspaces of $\boldsymbol{K}^N$, $G_n(\boldsymbol{K}^N)$, we have a natural embedding into $M(N, N; \boldsymbol{K})$, given by sending an n-space π into the matrix that represents the projection onto π. Those embeddings are also parallel. Also compact matrix groups are naturally parallel embedded in the corresponding matrix spaces. All those embeddings are particular cases of a general construction that we will discuss in the next section.

The importance of the above examples is that they are the basic blocks which allow to construct all parallel immersions.

2 - Jordan triple systems and the classification of parallel immersions

The starting point of the classification of parallel immersions is the following uniqueness theorem due to Reckziegel (see [**14**])

Theorem 2. *If two parallel immersions coincide at some point with the tangent space and second fundamental form, then they (locally) coincide.*

The problem is then reduced to study which *second fundamental form* are induced by parallel immersions. We will treat this problem in terms of triple systems.

Let $f\colon M \to Q(c)$ be an isometric immersion. For $x \in M$ we have an *associated triple system* for $T_x M$, $L\colon T_x M \oplus T_x M \oplus T_x M \to T_x M$

$$L(X, Y)(Z) = c\langle X, Y\rangle Z + A_{\alpha(X,Y)} Z + R(X, Y) Z. \tag{2.1}$$

The symmetric part $S(X, Y) = c\,\langle X, Y\rangle I + A_{\alpha(X,Y)}$ describes the extrinsic geometry, while the anti symmetric part $R(X, Y)$ depends only on the intrinsic geometry of M. When convenient, we will consider a triple system as the induced bilinear map $L\colon T_x M \oplus T_x M \to \mathrm{End}\,(T_x M)$.

We want to abstract the situation and study it algebraically. Let $\boldsymbol{V}$ be an inner product real vector space, $c \in \boldsymbol{R}$, $\boldsymbol{E}$ a subspace of $\boldsymbol{V}$ and $\alpha\colon \boldsymbol{E} \times \boldsymbol{E} \to \boldsymbol{E}^{\perp}$ a symmetric bilinear map. The *triple system of* $(\boldsymbol{V}, c)$ *with initial data* $(\boldsymbol{E}, \alpha)$ is the triple system $L = S + R\colon \boldsymbol{E} \oplus \boldsymbol{E} \to \mathrm{End}\,(\boldsymbol{E})$ whose symmetric part S and anti symmetric part R are given by

$$\begin{aligned} \langle S(X, Y)V, W\rangle &= c\langle X, Y\rangle\langle V, W\rangle + \langle \alpha(X, Y), \alpha(V, W)\rangle \\ \langle R(X, Y)V, W\rangle &= \langle S(Y, V)X, W\rangle - \langle S(X, V)Y, W\rangle. \end{aligned} \tag{2.2}$$

Direct calculations show

$$\begin{aligned} \langle R(X, Y)V, W\rangle = c(\langle X, W\rangle\langle Y, V\rangle - \langle X, V\rangle\langle Y, W\rangle) \\ + \langle \alpha(X, W), \alpha(Y, V)\rangle - \langle \alpha(X, V), \alpha(Y, W)\rangle \end{aligned} \tag{2.3}$$

(algebraic Gauss equation),

$$L(X, Y) = L^{*}(Y, X) \tag{2.4}$$

(adjoint with respect to the fixed inner product in V),

$$L(X, Y)Z = L(Z, Y)X. \tag{2.5}$$

The triple system L is called a *Jordan triple system*, JTS for short, if, in addition, it verifies the following condition:

$$[L(X, Y), L(V, W)] = L(L(X, Y)V, W) - L(V, L(Y, X)W) \tag{2.6}$$

where [,] denote, as usual, the commutator of endomorphisms.

For $\xi \in \boldsymbol{E}^{\perp}$, we define $A_{\xi}: \boldsymbol{E} \to \boldsymbol{E}$ by $\langle A_{\xi}X, Y\rangle = \langle\alpha(X, Y), \xi\rangle$ and the *normal curvature* $R^{\perp}(X, Y): \boldsymbol{E}^{\perp} \to \boldsymbol{E}^{\perp}$ by

$$R^{\perp}(X, Y)\xi = \alpha(X, A_{\xi}Y) - \alpha(Y, A_{\xi}X).$$

Let $\boldsymbol{E}_1^{\perp}$ be the span of the image of α and $\boldsymbol{E}_2^{\perp}$ it's orthogonal complement in $\boldsymbol{E}^{\perp}$. Then $\boldsymbol{E}_2^{\perp} = \{\xi \in \boldsymbol{E}^{\perp} \mid A_{\xi} = 0\}$ and $R^{\perp}(X, Y)$ acts trivially in $\boldsymbol{E}_2^{\perp}$. The triple system is said to be *full*, if $\boldsymbol{E}_2^{\perp} = \{0\}$. Taking inner product of $R^{\perp}$ with $\eta \in \boldsymbol{E}^{\perp}$, we get $\langle R^{\perp}(X, Y)\xi, \eta\rangle = [A_{\xi}, A_{\eta}]X, Y\rangle$ and therefore $R^{\perp}(X, Y) = 0$ $\forall X, Y \in \boldsymbol{E}$, if and only if all the A_{ξ}'s commutes, i.e. if and only if they are all simultaneously diagonalizable.

Theorem 3. *The triple system L is a* JTS *if and only if*

$$R^{\perp}(X, Y)[\alpha(Z, W)] = \alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W). \tag{2.7}$$

Proof. The fact that (2.7) implies that L is a JTS is given in [**10**]. For the converse we could trace back the above mentioned proof or simply wait for the theorem, which guarantees that any JTS is the triple system of a parallel immersion (and (2.7) is just the integrability condition of $\nabla\alpha = 0$).

Corollary 1. *The triple system associated to a parallel immersion is a* JTS

A basic algebraic result is the following (see [2] for a proof)

Theorem 4. *For a* JTS *L as above, $\forall X, Y \in \boldsymbol{E}$ we have*

i) $\quad \operatorname{trace} S(S(X, Y)X, Y) \geqslant 0$

ii) $\quad \operatorname{trace} S(R(X, Y)X, Y) \leqslant 0$

and equality holds in i), *if and only if* $S(X, Y) = 0$ *and in* ii), *if and only if* $\boldsymbol{R}(X, Y) = 0$.

We want to classify JTS for $(\boldsymbol{V}, c)$ with initial data $(\boldsymbol{E}, \alpha)$, which are *minimal* in the sense that $H = \frac{1}{n}\operatorname{trace}\alpha = 0$. For $c \leqslant 0$ Theorem 4 gives an easy answer. In fact if $n = \dim \boldsymbol{E} = 1$, then clearly $\alpha = 0$. Suppose $n \geqslant 2$. Given independent vectors X and Y and computing traces we get

$$\operatorname{trace} L(X, Y) = \operatorname{trace} S(X, Y) = nc\langle X, Y\rangle \quad \text{(by minimality).}$$

Therefore

$$\operatorname{trace} S(S(X, Y)X, Y) = nc(c\langle X, Y\rangle^2 + \|\alpha(X, Y)\|^2)\,. \tag{2.8}$$

If $c = 0$, Theorem 4 and (2.8) imply $\boldsymbol{S}(X, Y) = 0$ and therefore $\alpha = 0$, i.e. the JTS is *totally geodesic.*

If $c < 0$, Theorem 4 and (2.8) give $c\langle X, Y\rangle^2 + \|\alpha(X, Y)\|^2 \leqslant 0$ and therefore $\|\alpha(X, Y)\|^2 \leqslant -c\langle X, Y\rangle^2$. Hence $\alpha(X, Y) = 0$ for all X, Y orthogonal. In particular $\alpha(X, X) = \alpha(Y, Y)$ for all unit vectors X and Y and, since trace $\alpha = 0$, again the JTS is totally geodesic. Therefore we have

Theorem 5. *A minimal* JTS *with* $c \leqslant 0$ *is totally geodesic* (*and therefore is the triple system associated to a parallel immersion into* $Q(c)$).

If $c > 0$ the classification is strictly related to the theory of symmetric R-spaces. We will give a short outline following [**10**] closely.

We start with a JTS for $(\boldsymbol{W}, c)$ with initial data $(\boldsymbol{E}, \alpha)$, which is minimal and full, and define a new JTS for $(\boldsymbol{V}, 0)$ where $\boldsymbol{V} = \boldsymbol{W} \oplus \boldsymbol{R}$, with initial data $(\boldsymbol{E}, \tilde{\alpha})$ where $\tilde{\alpha}(X, Y) = \alpha(X, Y) + c\eta$ where $\eta = (0, 1) \in \boldsymbol{W} \oplus \boldsymbol{R}$. Geometrically this means that, instead of looking for the second jet of an immersion in a N-sphere of curvature c, we look at the composition with thestandard embedding of this sphere in $\boldsymbol{R}^{N+1}$.

Let $\Lambda = \operatorname{span}\{L(X, Y) \mid X, Y \in \boldsymbol{E}\}$ and consider the vector space $\mathfrak{G} = \boldsymbol{E} \oplus \Lambda \oplus \boldsymbol{E}$, and in $\mathfrak{G}$, define a Lie product

$$[(X, F, Y), (\tilde{X}, \tilde{F}, \tilde{Y})] = (F\tilde{X} - \tilde{F}X, [F, \tilde{F}] - \frac{1}{2}(L(X, \tilde{Y}) - L(\tilde{X}, Y), \tilde{F}^t Y - F^t \tilde{Y})\,.$$

With the above product $\mathfrak{G}$ is a semisimple Lie algebra with a Cartan involution $(X, F, Y) \to (Y, -F^t, X)$ and a Cartan decomposition $\mathfrak{G} = \mathfrak{m} \oplus \mathfrak{p}$ where

$$\mathfrak{m} = \operatorname{span}(X, R(U, V), X)\,, \qquad \mathfrak{p} = \operatorname{span}(X, A_{\alpha(U, V)}, -X)\,.$$

Moreover the element $\tilde{\eta} = (0, A_{-r/c}, 0) = (0, -\mathrm{Id}, 0)$ is such that $(\mathrm{ad}\,\tilde{\eta})^3 = \mathrm{ad}\,\tilde{\eta}$ and the eigenspaces of $\mathrm{ad}\,\tilde{\eta}$ relative to the eigenvalues -1, 0, and 1 are $\boldsymbol{E} \oplus \{0\} \oplus \{0\}$, $\{0\} \oplus \Lambda \oplus \{0\}$ and $\{0\} \oplus \{0\} \oplus \boldsymbol{E}$ respectively.

Conversely, let G be a (semisimple non compact) Lie group with Lie algebra $\mathfrak{G}$, with Cartan decomposition $\mathfrak{G} = \mathfrak{m} \oplus \mathfrak{p}$, and K a maximal compact subgroup, associated to the Cartan decomposition. Let $\tilde{\eta}$ be an element in $\mathfrak{p}$ and let $K_0 = \{k \in K \mid \mathrm{Ad}(k)\,\tilde{\eta} = \tilde{\eta}\}$ be the isotropy subgroup of the adjoint representation. Then the map $f\colon M = K/K_0 \to \mathfrak{p}$, $f([k]) = \mathrm{Ad}(k)\,\tilde{\eta}$ is an embedding of $M = K/K_0$ into $\mathfrak{p}$ (which is an euclidean space with respect to the scalar product given by the Killing form). If $(\mathrm{ad}\,\tilde{\eta})^3 = \mathrm{ad}\,\tilde{\eta}$, the induced metric is riemannian symmetric, and M is a *symmetric R-space*, and f is called the *standard embedding of the symmetric R-space M*. Those standard embedding are parallel and minimal in the sphere of radius $\|\tilde{\eta}\|$.

Remark 1. Symmetric R-space are symmetric spaces of the type $M = K/K_0$, obtained starting from a semisimple Lie group of non compact type G, with Lie algebra $\mathfrak{G}$ with a Cartan decomposition as above and a non zero element $\tilde{\eta} \in \mathfrak{p}$ such that $(\mathrm{ad}\,\tilde{\eta})^3 = \mathrm{ad}\,\tilde{\eta}$. This is equivalent to the fact that M is the quotient of G by a parabolic subgroup. Symmetric R-spaces are classified. The classification includes all classical symmetric spaces of compact type and some of the exceptional ones. By the above, a symmetric R-space admits a parallel immersion into some euclidean space, whose image lies in some sphere where it is minimal.

We observe, finally, that the manifold of Example 2 are particular cases of this general construction.

The above construction gives a bijection between isomorphism classes of minimal JTS for $c > 0$, and symmetric R-spaces. In conclusion, we get

Theorem 6. *A minimal* JTS *for* $c > 0$ *is the triple system associated to the standard embedding of a symmetric R-space (and conversely the triple system associated to a standard embedding of a symmetric R-spaces is a minimal* JTS *in a sphere).*

We want to decompose a JTS into *irreducible factors*, and show that each irreducible factor is the *composition* of a minimal JTS and an umbilical one. So first we define *composition* (this will be the algebraic analog of composition of isometric immersions): Let L_1 be a JTS for $(\tilde{\boldsymbol{E}}, \tilde{c})$ with initial data $(\boldsymbol{E}, \bar{\alpha})$ and L_2 a

JTS for (V, c) with initial data $(\tilde{E}, \tilde{\alpha})$. The composition of L_1 and L_2 is the JTS for (V, c) with initial data (E, α) where $\alpha = \tilde{\alpha} + \bar{\alpha}$.

Let L be a JTS for (V, c) with initial data (E, α). The *trace operator* $A: E \to E$, $\langle AX, Y \rangle = \operatorname{trace} S(X, Y)$, is a symmetric operator and determines an orthogonal decomposition in eigenspaces, $E = \oplus E_i$. We set $n = \dim E$ and $n_i = \dim E_i$. Let $H = \frac{1}{n}\operatorname{trace}\alpha$ be the mean curvature vector, so that E_i is also an eigenspace of A_H.

Lemma 1. *$L(X, Y)(E_i) \subseteq E_i$ (in particular L induces a* JTS *for (V, c) with initial data $(E_i, \alpha_i E_i)$).*

Proof. We have

$$\langle [A, L(X, Y)]V, W\rangle = \langle AL(X, Y)V, W\rangle - \langle AV, L(Y, X)W\rangle$$

$$= \operatorname{trace}(L(L(X, Y)V, W) - L(V, L(Y, X)W) = \operatorname{trace}([L(X, Y), L(V, W)]) = 0 .$$

Since A commutes with $L(X, Y)$, the latter leaves the eigenspaces of A invariant.

Set now $$H_i = \frac{1}{n_i}\operatorname{trace}\alpha_{|E_i} \qquad c_i = c + \|H_i\|^2$$

and $$\tilde{E}_i = E_i \oplus \operatorname{span}\{\alpha(X, X) - \|X\|^2 H_i \colon X \in E_i\}.$$

If $X \in E_i$, $Y \in E_j$, $i \neq j$, the above lemma and (2.5) imply

$$\begin{aligned} &\langle \alpha(X, X), \alpha(Y, Y)\rangle = -c\|X\|^2\|Y\|^2, \quad \langle \alpha(X, Y), H_j\rangle = -c\|X\|^2, \\ &\alpha(X, Y) = 0, \quad \langle H_i, H_j\rangle = -c, \quad \langle H_i, H\rangle = \lambda_i = \langle A_H X, X\rangle. \end{aligned} \tag{2.9}$$

In particular the H_i's are distinct, the $\tilde{E}_i$'s are orthogonal and the H_i's are orthogonal to $\oplus\tilde{E}_i$.

Resuming, each $\tilde{E}_i$ is an inner product vector space and

$$\tilde{\alpha}_i \colon E_i \oplus E_i \to E_i^{\perp} \cap \tilde{E}_i, \qquad \tilde{\alpha}_i(X, Y) = \alpha(X, Y) - \langle X, Y\rangle H_i,$$

determines a minimal JTS for $(\tilde{E}_i, c_i)$ with initial data $(E_i, \tilde{\alpha}_i)$. Moreover we have an umbilical JTS for (V, c) with initial data $(\tilde{E}_i, \bar{\alpha}_i)$, $\bar{\alpha}_i(X, Y) = \langle X, Y\rangle H_i$, and the triple system for (V, c) with initial data $(E_i, \alpha|_{E_i})$ is the composition of the two.

We have now all the algebra ready for the *classification theorem*

Theorem 7. *Let $f: M \to Q(c)$ be a parallel immersion, of a connected manifold. Then there exist an extrinsic product of umbilical manifolds $g: Q_1(c_1) \times \ldots \times Q_k(c_k) \to Q(c)$, a riemannian decomposition of $M = M_1 \times \ldots \times M_k$ and minimal parallel immersions $f_i: M_i \to Q_i(c_i)$ such that $f = g \circ (f_1 \times \ldots \times f_k)$.*

Proof. Let $x \in M$, $p = f(x) \in Q(c)$, $\boldsymbol{E} = (\mathrm{d}f)_x(T_x M) \subseteq T_p Q(c) = \boldsymbol{V}$, and α the second fundamental form of f at x. The triple system of the immersion is a JTS for $(\boldsymbol{V}, c)$ with initial data $(\boldsymbol{E}, \alpha)$. Keeping the notation of the above discussion, $\boldsymbol{E}$ decomposes orthogonally, $\boldsymbol{E} = \oplus \boldsymbol{E}_i$, $i = 1, \ldots, k$. Suppose to have ordered the subspaces such that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_k$. Since the H_i's are distinct we have $0 < \|H_1 - H_i\|^2 = c_1 + c_i \leqslant 2c_i$ and therefore $c_i > 0$ if $i > 1$. Moreover the $\tilde{\boldsymbol{E}}_i$'s are orthogonal and the H_i's are orthogonal to $\oplus \tilde{\boldsymbol{E}}_i$. By (2.9) and the Observation in Example 2, there exist an extrinsic product of umbilical immersions g: $Q_1(c_1) \times Q_2(c_2) \times \ldots \times Q_k(c_k) \to Q(c)$ with tangent space at some point $(x_1, \ldots, x_k)$ being $\oplus \tilde{\boldsymbol{E}}_i$, and mean curvature vector of $g|_{Q_i(c_i)}$ at x_i being H_i. Now we have a JTS for $(\tilde{\boldsymbol{E}}_i, c_i)$ with initial data $(\boldsymbol{E}_i, \tilde{\alpha}_i)$, which is minimal and therefore we have a parallel immersion $\tilde{f}_i$: $M_i \to Q_i(c_i)$ with that triple system (by (2.8) and Remark 1). The immersion $\tilde{f} = g \circ (\tilde{f}_1 \times \ldots \times \tilde{f}_k)$: $M_1 \times \ldots \times M_k \to Q(c)$ is parallel and has the same JTS of f, and therefore, by Theorem 2, coincides with f.

Corollary 2. *If M is irreducible, then either f is umbilical* ($c \leqslant 0$) *or a standard embedding of a symmetric R-space.*

3 - Semi-parallel immersions

In section 3 we described a scheme of classification of parallel immersions based on the following two facts

1. *The triple system associated to a parallel immersion is a Jordan triple system.*

2. *Two parallel immersions with the same triple system at some point, coincide.*

While condition 1 is *punctual*, condition 2 is *local*. This fact leads naturally to the following concept:

Definition 2. An isometric immersion is *semi-parallel* if the triple system at each point is a JTS

Remark 2. The condition of semi-parallelism is equivalent to

$$R(X, Y)\alpha = \nabla_X \nabla_Y \alpha - \nabla_Y \nabla_X \alpha - \nabla_{[X, Y]}\alpha = 0$$

for all tangent vectors X, Y (this follows from the basic equations (1.1), (1.2) and (1.3) and from Theorem 3). The above, and again the basic equations, imply

$$R(X, Y)R = \nabla_X \nabla_Y R - \nabla_Y \nabla_X R - \nabla_{[X, Y]}R = 0$$

i.e. the metric is *semi-symmetric*. Semi-symmetric manifolds are classified, at least generically, in **[16]** and **[17]**. In analogy with the intrinsic case, where the condition $R(X, Y)R = 0$ is the integrability condition of $\nabla R = 0$, the condition $R(X, Y)\alpha = 0$ is the *integrability condition* of $\nabla\alpha = 0$.

Semi-parallel immersions may be thought as *2nd order envelopes of parallel immersions* in the sense that, at each point, there exists a parallel submanifold with the same fundamental triple system. An easy but good example to keep in mind is a right cylinder over a plane curve. At each point of this surface the right cylinder over the circle osculating the curve is a parallel submanifold with the same triple system.

A classification of semi-parallel immersions is not known yet (depending up to what we want to classify!). In this section we will discuss some of the known results on the problem.

We will start with the case of *hypersurfaces*. Let $f\colon M^n \to Q^{n+1}(c)$ be an isometric immersion which is semi-parallel, ξ a unit normal vector and $\{e_1, \dots, e_n\}$ an orthonormal basis for the tangent space of M, which diagonilizes the Weigarten operator A_ξ. Let $\lambda_i = \langle A_\xi e_i, e_i \rangle$ be the principal curvatures in the ξ direction. In this situation the condition of being semi-parallel is equivalent to

$$(\lambda_i \lambda_j + c)(\lambda_i - \lambda_j) = 0\,. \tag{3.1}$$

It follows from the above that there are at most two distinct principal curvatures, say λ and μ. If $\lambda \neq \mu$ then $\lambda\mu = -c$ and the two eigenspaces T_λ and T_μ determine two distributions. It is a standard consequence of the equations of Codazzi-Mainardi that those distributions are involutive and, if the multiplicity of one of the principal curvatures, say λ, is at least two, then λ is constant along the leaves of T_λ and those leaves are totally umbilical. From those observations we get that if $c \neq 0$ and at one point there are two distinct principal curvatures, both of multiplicity bigger than one, then the hypersurface is isoparametric (i.e. has constant principal curvatures). An isoparametric, connected hypersurface with only two distinct principal curvatures is a tube around a totally geodesic submanifold, if it is connected. If one of the principal curvatures is simple, then M is a 1-parameter envelope of umbilical submanifolds and those may be described quite explicitly (see [4]). If $c = 0$, besides such tubes we can have cylindrical immersions (i.e. rank $A_\xi \leq 1$), cones over spheres and products of such cones with Euclidean spaces (see [8]).

Strictly related to the case of semi-parallel hypersurfaces is the case of *semi-parallel immersions with flat normal connection.* If the normal curvature vanishes at a point $x \in M$, then there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_xM such that $\alpha(e_i, e_j) = 0$ if $i \neq j$. In this case semi-parallelism is equivalent to

$$k(e_i, e_j)(\alpha(e_i, e_i) - \alpha(e_j, e_j)) = 0 \tag{3.2}$$

where $k(e_i, e_j)$ is the sectional curvature of the plane spanned by e_i and e_j. If $c \geq 0$, the sectional curvature is non negative by the Gauss equation, and, at least if M is complete, the topology is well understood: Its universal covering space is the riemannian product of manifolds homeomorphic to spheres and a manifold diffeomorphic to an Euclidean space (see [9]). An interesting geometric result, which we will use later in a weaker form, is the following

Theorem 8. *Let f: $M \to Q(c)$ be a semi-parallel immersion with flat normal connection. If $c \geq 0$, M is connected and the Ricci curvature is positive at some point, then f is parallel.*

Proof. We will consider the case $c = 0$. The case $c > 0$ is an obvious consequence. We will start proving

Proposition 1. *The immersion f has parallel mean curvature and constant scalar curvature.*

Proof. Let $p \in M$ be a point where the Ricci curvature is positive and $\{e_1, \ldots, e_n\}$ be a basis of T_pM such that $\alpha(e_i, e_j) = 0$ if $i \neq j$. We will write α_{kl} for $\alpha(e_k, e_l)$. From (3.2) and the positivity of the Ricci curvature at p, we get

a) $\alpha_{ii} = \alpha_{jj}$ or $k(e_i, e_j) = \langle \alpha_{ii}, \alpha_{jj} \rangle = 0$

b) $\alpha_{ii} \neq 0$ for all i and there exists $j \neq i$ such that $\alpha_{ii} = \alpha_{jj}$.

Since the condition $\mathrm{Ricci}_p > 0$ is open, the above conditions hold in a neighborhood of p and it is not difficult to see that we can choose a smooth orthonormal frame $\{e_1, \ldots, e_n\}$ in a possibly smaller open set, such that the two conditions above hold true in this open set. From the Codazzi-Mainardi equations we get, if $i \neq j$

1. $$\nabla^{\perp}_{e_j}\alpha_{ii} = (\nabla_{e_j}\alpha)(e_i, e_i) + 2\alpha(\nabla_{e_j}e_i, e_i)$$
$$= (\nabla_{e_i}\alpha)(e_j, e_i) + 2\langle\nabla_{e_j}e_i, e_i\rangle\alpha_{ii} = \langle\nabla_{e_i}e_i, e_j\rangle(\alpha_{ii} - \alpha_{jj}).$$

Therefore

2. If $i \neq j$ and $\alpha_{ii} = \alpha_{jj}$, then $\nabla^{\perp}_{e_i}\alpha_{jj} = 0$.
3. For all i, $\nabla^{\perp}_{e_i}\alpha_{ii} = 0$. (Take $j \neq i$ with $\alpha_{jj} = \alpha_{ii}$).
4. If $i \neq j$ and $\alpha_{ii} \neq \alpha_{jj}$ then $\nabla^{\perp}_{e_i}\alpha_{jj} = 0$. In fact $\langle\alpha_{ii}, \alpha_{jj}\rangle = 0$ by a) and therefore $0 = e_j\langle\alpha_{ii}, \alpha_{jj}\rangle = -\langle\nabla_{e_i}e_i, e_j\rangle\|\alpha_{jj}\|^2$. Since $\alpha_{jj} \neq 0$ the conclusion follows from 1.

From the above we conclude that the α_{ii}'s are parallel on the open set we are working in and therefore the mean curvature vector is parallel and the scalar curvature is constant on that open set. A simple connectness argument gives the desired conclusion.

We go back now to the proof of the theorem. We start observing that $H \neq 0$. We can choose, therefore, a local orthonormal frame in the normal bundle, $\{\xi_1, \ldots, \xi_{N-n}\}$ such that $\xi_1 = H/\|H\|$. Set $\lambda_i^k = \langle A_{\xi_k}e_i, e_i\rangle$. The Laplacian of $\|\alpha\|^2$ is given by

$$\frac{1}{2}\Delta(\|\alpha\|^2) = \sum_{i<j}\Big(\sum_{k=1}^{M-n}(\lambda_i^k - \lambda_j^k)^2 k(e_i, e_j)\Big) + \|\nabla\alpha\|^2$$

(see [**6**], p. 133). Since $\|\alpha\|^2$ is constant and the sectional curvatures are non negative, it follows that α is parallel.

It follows easily from the classification of parallel immersions, that the only ones with flat normal connections are the extrinsic products of umbilical immersions. Therefore

Corollary 3. *In the above hypothesis, f is an extrinsic product of umbilical immersions.*

Remark 3. A scheme of classification of semi-parallel immersions into $\boldsymbol{R}^N$ with flat normal connection may be found in [**13**], where a version of the above result is attributed to Riives (whose paper however we were not able to find). The interest in semi-parallel immersions with flat normal connection comes from the fact that, at least if $c \leqslant 0$, all semi-parallel immersions in codimension 2 have flat normal connection. In fact, it follows easily by (2.7) that the mean curvature vector is in the kernel of $R^{\perp}(X, Y)$ for all tangent vectors X, Y. If $H \neq 0$ the antisymmetry of $R^{\perp}(X, Y)$ (and the condition on the codimension) implies the vanishing of $R^{\perp}$. If $H = 0$, (2.8) implies that the point is totally geodesic ($c \leqslant 0$) and therefore, again, $R^{\perp} = 0$.

4 - Semi-parallel surfaces

In this section we will study semi-parallel immersions of a 2-dimensional manifold M^2 into an N-dimensional space form $Q = Q^N(c)$.

Let $\{e_1, e_2\}$ be a local orthonormal tangent frame and we will set, as before, $\alpha_{ij} = \alpha(e_i, e_j)$. Also k will denote the Gaussian curvature and $R^{\perp} = R(e_1, e_2)$ the normal curvature operator. With these notations the semi-parallelism condition becomes

$$R^{\perp}\alpha_{11} = R^{\perp}\alpha_{22} = 2k\alpha_{12} \qquad R^{\perp}\alpha_{12} = k(\alpha_{11} - \alpha_{22}). \tag{4.1}$$

An immediate consequence of (4.1) is that if $R^{\perp} \equiv 0$, then either f is umbilical or M^2 is flat. Moreover an immersion of a flat surface is semi-parallel if and only if $R^{\perp} \equiv 0$.

The above observation is of some interest also because it allows us to classify semi-parallel immersions into $Q^4(c)$. In fact, in this case, if $c \leqslant 0$ then $R^{\perp} \equiv 0$ (see (2.8)) and, if $c > 0$, either $R^{\perp} \equiv 0$ or $f(M^2)$ is a piece of a Veronese surface (see Proposition 2 below).

Let ξ be a unit normal field. The Ricci equation and (4.1) give

$$\begin{aligned} &\langle \alpha_{ii}, \alpha_{12}\rangle(\alpha_{11} - \alpha_{22}) + (\langle \alpha_{ii}, \alpha_{22} - \alpha_{11}\rangle - (-1)^i 2k)\alpha_{12} = 0 \qquad i = 1, 2 \\ &(\|\alpha_{12}\|^2 - k)(\alpha_{11} - \alpha_{22}) + \langle \alpha_{12}, \alpha_{22} - \alpha_{11}\rangle\alpha_{12} = 0 . \end{aligned} \tag{4.2}$$

If $R^{\perp} \neq 0$, α_{12} and $\alpha_{11} - \alpha_{22}$ are linearly independent and in this case (4.2) and the Gauss equation give

$$\begin{aligned} &\|\alpha_{12}\|^2 = k \qquad \|\alpha_{ii}\|^2 = 4k - c \qquad \langle \alpha_{ii}, \alpha_{12}\rangle = 0 \\ &\|H\|^2 = 3k - c \qquad \|\alpha_{11} - \alpha_{22}\|^2 = 4k \qquad \langle \alpha_{11}, \alpha_{22}\rangle = 2k - c . \end{aligned} \tag{4.3}$$

From the above we deduce the following result of Deprez ([7]):

Theorem 9. *Let $f: M^2 \to Q^N(c)$ be a semi-parallel immersion. Then there exists an open and dense set $U \subseteq M^2$ such that the connected components of U are of the following types*

i) *Open parts of umbilical $Q^2(k)$ in $Q^N(c)$, $k \geq c$*

ii) *Flat surfaces with $R^\perp = 0$*

iii) *Isotropic immersions with $R^\perp \neq 0$ and $\|H\|^2 = 3k - c$.*

For further use we will give a look at the case $c = 1$. By composition with the inclusion $Q^N(1) = S^N \hookrightarrow \boldsymbol{R}^{N+1}$, we get a semi-parallel immersion into a euclidean space. We will denote by $\tilde{\alpha}$, $\tilde{H}$ the second fundamental form and the mean curvature vector of the new immersion. Since $\|f(x)\| = 1$, differentiating twice we get $\langle \tilde{\alpha}_{ii}, f(x) \rangle = -1$ and hence $\langle \tilde{H}, f(x) \rangle = -1$. If $R^\perp \neq 0$, by (4.3) $k = \frac{1}{3}\|\tilde{H}\|^2$ and, since $\|\tilde{H}\| \geq 1$, $k \geq \frac{1}{3}$.

Proposition 2. *$f: M^2 \to S^N$ be a semi-parallel immersion of a connected surface with $R^\perp \neq 0$ somewhere. Then the following are equivalent:*

i) $k = \frac{1}{3}$

ii) *f is minimal*

iii) *$f(M^2)$ is a piece of a Veronese surface*

iv) *$f(M^2)$ is contained in a totally geodesic S^4 in S^N.*

Proof. Since $\langle \tilde{H}, f(x) \rangle = -1$ and $k = \frac{1}{3}\|\tilde{H}\|^2 \geq \frac{1}{3}$ we have that $k = \frac{1}{3}$ if and only if $\tilde{H}$ is parallel to $f(x)$ and therefore if and only if f is minimal in S^N. Moreover a minimal surface in S^N with constant curvature $\frac{1}{3}$ is a piece of a Veronese surface, by a theorem of Bryant (see [5]). So the first three conditions are equivalent and clearly implies the fourth. Suppose now $N = 4$. Differentiating $\|f(x)\| = 1$ we get $\langle \tilde{\alpha}_{12}, f(x) \rangle = 0 = \langle \tilde{\alpha}_{11} - \tilde{\alpha}_{22}, f(x) \rangle$ and therefore $\tilde{H}$ is parallel to $f(x)$ since $e_1, e_2, \tilde{\alpha}_{12}, \tilde{\alpha}_{11} - \tilde{\alpha}_{22}$ and $\tilde{H}$ are orthogonal (by (4.3)). Therefore f is minimal is S^4.

We will discuss now two of the main results known for this problem. The first one is due to Lumiste and Asperti-Mercuri (see [12] and [1]):

Theorem 10. *Let $f: M^2 \to Q^5(c)$ be a semi-parallel immersion, M^2 con-*

nected. Suppose $R^{\perp} \neq 0$ *somewhere. Then* k *is constant and* H *is parallel in the normal connection.*

Remark 4. The proof of Theorem 10 is based in the moving frame method and involves horrible calculations. Our aspectation is that the theorem holds true for all codimensions, but we where not able to handle the extra terms which appears in codimension bigger than three. Just for fun, and to have an idea of the calculations, the above results, for $c < 0$, is obtained by showing that the function $\phi = 3k - c$, is solution of the polynomial equation

$$42336\phi^3 + 13361712\phi^2 + 41446782\phi + 67855359 = 0 \qquad (!)$$

In the hypothesis of Theorem 10 the normal curvature is everywhere non zero, if M^2 is connected. By theorems of Chen and Yau (see [**6**], p. 106) the parallelism of H implies that f is a minimal immersion in a totally umbilical hypersurface $Q^4(\tilde{c})$ of $Q^5(c)$. Since the immersion is minimal $\tilde{c} \geq k > 0$. In conclusion

Corollary 4. *In the hypothesis of Theorem* 10 f *is an immersion of* M^2 *as a piece of a Veronese surface in some* 4*-sphere totally umbilical in* $Q^5(c)$.

We want to discuss, briefly, the case of semi-parallel immersion of *compact* surfaces into $\boldsymbol{R}^N$. Since $k \geq 0$, such a surface is either flat or diffeomorphic to a 2-sphere or a projective plane. An important concept in the study of such immersion is the *total absolute curvature.*

We recall that the total absolute curvature is defined by

$$\tau(f) = (c_{N-1})^{-1} \int_M \Big(\int_{\|\xi\|=1} |\det(A_\xi)| \, d\sigma_{N-3} \Big) \, dM \tag{4.4}$$

where $d\sigma_n$ is the volume density of the n-sphere, $c_n = \int_{S^n} d\sigma_n$, and dM is the volume density of M.

We want to estimate the total absolute curvature of a semi-parallel surface. If the mean curvature vector at x, $H(x)$, is zero, then x is a totally geodesic point and so $A_\xi = 0$ for all ξ's normal to M^2 at x. Suppose $H(x) \neq 0$ and choose an adapted frame for $\boldsymbol{R}^N$, $\{e_1, \ldots, e_N\}$ with $e_3 = H\|H\|^{-1}$ and, if $R^{\perp} \neq 0$,

$e_4 = (\alpha_{11} - \alpha_{22})\|\alpha_{11} - \alpha_{22}\|^{-1}$ and $e_5 = \alpha_{12}\|\alpha_{12}\|^{-1}$. Set $A_\lambda = A_{e_\lambda}$. Now (4.3) implies

1. If $R^\perp = 0$, $A_3 = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \sqrt{k} \end{pmatrix}$ $A_\lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if $\lambda \geqslant 4$.

2. If $R^\perp \neq 0$, $A_3 = \begin{pmatrix} \sqrt{3k} & 0 \\ 0 & \sqrt{3k} \end{pmatrix}$ $A_4 = \begin{pmatrix} \sqrt{k} & 0 \\ 0 & -\sqrt{k} \end{pmatrix}$

$$A_5 = \begin{pmatrix} 0 & \sqrt{k} \\ \sqrt{k} & 0 \end{pmatrix} \qquad A_\lambda = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ if } \lambda \geqslant 6.$$

Let ξ be a unit normal vector at x. Write $\xi = \sum_{\lambda \geqslant 3} \xi_\lambda e_\lambda$. Then $|\det(A_\xi)|$ is given by $k\varphi(\xi_3, \ldots, \xi_N)$, where

$$\varphi(\xi_3, \ldots, \xi_N) = \begin{cases} (\xi_3)^2 & \text{if } R^\perp = 0 \\ |3(\xi_3)^2 - (\xi_4)^2 - (\xi_5)^2| & \text{if } R^\perp \neq 0. \end{cases}$$

We observe now that φ is the integrand that appears in the expression of the total absolute curvature of the immersion of the unit 2-sphere in $\boldsymbol{R}^N$ as totally umbilical surface, in the first case, and as a Veronese surface in the second. Therefore the integral of φ on the unit $(N-3)$-sphere is $\frac{c_{N-1}}{2\pi}$ in the first case and $\frac{3c_{N-1}}{2\pi}$ in the second case. From the above and the Gauss-Bonnet theorem we get:

$$\chi(M) = (c_{N-1}) \int_M k \, dM \int_{S^{N-3}} (\xi_3)^2 d\sigma_{N-3} \leqslant \tau(f)$$

$$\leqslant (c_{N-1}) \int_M k \, dM \int_{S^{N-3}} |3(\xi_3)^2 - (\xi_4)^2 - (\xi_5)^2| d\sigma_{N-3} = 3\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

Theorem 11. *In the above hypothesis, if M^2 is not flat and non orientable, then f is a Veronese surface in some 4-sphere in $\boldsymbol{R}^N$.*

Proof. The hypothesis imply that M^2 is a projective plane and so $\chi(M^2) = 1$. By the Morse theory interpretation of τ, $\tau \geqslant 3$ and therefore $\tau = 3$, by the above discussion. By a result of Kuiper and Pohl (see [**11**]) such immersion is projectively equivalent to a Veronese surface and therefore contained in a 5-dimensional affine subspace. The conclusion then follows from Corollary 4.

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