ON FRAISSE'S PROOF OF COMPACTNESS

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Janeiro RP 02/93

Relatório de Pesquisa

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ABSTRACT – Appart from four trivial cases, a universal Horn class of graphs generated by finitely many finite graphs can not be finitely axiomatizable.

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Janeiro – 1993

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Campinas, November 1992

"Viajando por caminhos que não sei onde vão dar"

(Brazilian popular song)

Fraisse's proof of countable compactness in [F], Th. 1.5.3, ammounts to the construction of a sheaf of structures over the one point compactification $\omega \cup \{\infty\}$ of the discrete space ω of natural numbers, so that the fiber in ∞ is generic (forcing and truth coincide). This gives perhaps the simplest non-boolean sheaves with all their fibers being generic.

If τ is a countable type of structures, then the space E_τ of structures of type τ topologized by the elementary classes as basic open classes is pseudometric. In fact the following is a totally bounded metric generating the elementary topology. Choose first a sequence $\tau_1 \subseteq \tau_2 \subseteq \ldots$ of finite subtypes of τ such that $\tau = \cup_n \tau_n$ and define a ***modified rank*** for any sentence of type τ by

$$mr(\varphi) = \min\{n \ : \ \varphi \in L_{\omega\omega}(\tau_n), \ \ n \geq qr(\varphi)\}$$

where qr is the ordinary quantifier rank. In this context, $\mathcal{A} \equiv^n_* \mathcal{C}$ will denote elementary equivalence with respect to sentences of modified rank less than n (hence, $\mathcal{A} \equiv^0_* \mathcal{C}$ always holds). Finally define the metric by

$$d(\mathcal{A}, \ \mathcal{C}) = \inf \ \{\frac{1}{n+1} \ : \ \mathcal{A} \equiv^n_* \mathcal{C}\}.$$

It is easy to see that this pseudo-metric is totally bounded (see [W]); hence, to show compactness it is enough to show that every Cauchy sequence converges.

Fraisse's proof may be construed as extracting first a Cauchy sequence from an arbitrary sequence (this may be done because the pseudometric is totally bounded) and then showing the Cauchy sequence converges. He assumes τ is finite, but with a slight modification

Supported by FAPESP, Fundação de Amparo à Pesquisa do Estado de São Paulo.

of the notion of n-partial isomorphism his construction works for countable τ.

Following Fraissé, construct a subsequence $\{\mathcal{A}_n\}_n$ together with finite subsets $E_n \subseteq A_n$ and injective functions $f_n : E_n \to E_{n+1}$ such that

i) $f_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ is an n-partial isomorphism for each n.
ii) For any $a \in A_n$ there is $g \supseteq f_n$ such that g is a $(n-1)$-partial isomorphism, $a \in dom(g)$, and $g(a) \in E_{n+1}$.

It may be shown that if $f_{nk} = f_k \circ \ldots \circ f_m$ for $k > n$ and $\mathcal{A}_\infty =_{\underrightarrow{\lim}} \{f_{nk} : \mathcal{A}_n|E_n \to \mathcal{A}_k|E_k\}_{n \leq k}$ then the limit morphisms $f_{m\infty} : \mathcal{A}_n \to \mathcal{A}_\infty$ are n-partial isomorphisms for any n. Here, a n-partial isomorphism $f : \mathcal{A} \to \mathcal{C}$ is defined as in [F], except that it will be assumed to be isomorphism only with respect to the relation symbols in τ_n. In particular, we have

$$(\mathcal{A}_n, a_1, \ldots, a_m) \equiv^n_* (\mathcal{A}_k, f_{nk}(a_1), \ldots, f_{nk}(a_m)) \tag{1}$$

for any $a_1, \ldots, a_m \in E_n$ and $k = n, n+1, \ldots, \infty$.

Let $\omega \cup \{\infty\}$ be the one point (or Alexandroff) compactification of discrete ω. Hence, the topology of $\omega \cup \{\infty\}$ is discrete in the open subspace ω and a neighborhood basis for ∞ consists of the sets:

$$[n, \infty] = \{j \in \omega : j \geq n\} \cup \{\infty\}$$

we will write also:

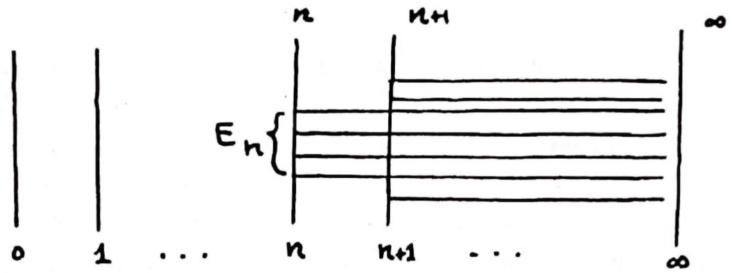
$$[n, \infty) = \{j \in \omega : j \geq n\}.$$

Define a sheaf $\mathcal{A}$ over $\omega \cup \{\infty\}$ as follows:

$$\mathcal{A}(S) = \Pi_{i \in S} A_i \quad \text{if} \quad S \subseteq \omega$$

$$\mathcal{A}([n, \infty]) = \{(f_{nk}(a))_{k \in [n, \infty]} \mid a \in E_n\}, \quad n \in \omega;$$

the projections $\mathcal{P}_{ST} : \mathcal{A}(S) \to \mathcal{A}(T)$, $S \supseteq T$ are simply the restrictions. It is readily seen that this is a sheaf where the fiber over $n \in \omega$ is $\mathcal{A}_n$ and the fiber over ∞ is $\mathcal{A}_\infty$:



The sections over $[n, \infty]$ are in one to one correspondence with the elements of E_n, so that there are finitely many only. If σ is such a section then

$$\begin{aligned}\sigma(k) &= f_{nk}(a)\,, \quad k \in [n, \infty) \\ \sigma(\infty) &= f_{n\infty}(a) = \hat{a}.\end{aligned}$$

For the semantics of forcing in sheaves we refer to [E]. A sheaf of structures wilt be said to be *generic* at a point x of the base space if forcing at that point coincides with classical truth in the fiber. More, precissely, if one has

$$\mathcal{A} \Vdash_x \varphi(\sigma_1, \ldots, \sigma_k) \leftrightarrow \mathcal{A}_x \models \varphi(\sigma,\ (x), \ldots, \sigma_n(x)) \tag{2}$$

for each formula $\varphi(x_1, \ldots, x_n)$ and (partial) sections $\sigma_1, \ldots, \sigma_n$ of $\mathcal{A}$ defined in x. Here, $\mathcal{A}_x$ is *fiber of* $\mathcal{A}$ *at* x.

Clearly, Fraisse's sheaf is generic at any point $n \in \omega$, because $\{n\}$ is open. Less obviously, it is generic at ∞. First we need:

LEMMA. *Let* $n \in \omega$, $n \geq mr(\varphi)$, *then the following are equivalent for* $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \mathcal{A}([n, \infty])$:

i) $\mathcal{A} \Vdash_n \varphi(\sigma)$

ii) $\mathcal{A}_j \models \varphi(\overline{\sigma}(j))$ *for some* $j \in [n, \infty]$.

Proof. We have already noticed that $\mathcal{A} \Vdash_n \varphi(\overline{\sigma})$ iff $\mathcal{A}_n \models \varphi(\overline{\sigma})$. Now, if $j \in [n, \infty]$ let $\overline{a} = \overline{\sigma}(n)$, then $\overline{\sigma}(j) = f_{nj}(\overline{a})$ and by (1), $\mathcal{A}_n \models \varphi(\overline{\sigma}(n))$ iff $\mathcal{A}_j \models \varphi(\overline{\sigma}(j))$, including $j = \infty$, because $n \geq mr(\varphi)$. □

PROPOSITION. $\mathcal{A}$ *is generic at* ∞.

Proof. We show (2) by induction in formulas

φ *atomic.* Follows by definition of forcing.

$\varphi = \psi \wedge \theta,\ \psi \vee \theta$. Trivial inductive step.

$\varphi = \psi \rightarrow \theta$. If $\mathcal{A} \Vdash_{\infty} (\psi \rightarrow \theta)(\overline{\sigma})$ and $\mathcal{A}_{\infty} \models \psi(\overline{\sigma}(\infty))$ then $\mathcal{A} \Vdash_{\infty} \psi(\overline{\sigma})$ by induction hypothesis and so $\mathcal{A} \Vdash_{\infty} \theta(\overline{\sigma})$. By induction again, $\mathcal{A}_{\infty} \models \theta(\overline{\sigma}(\infty))$. Conversely, assume $\mathcal{A}_{\infty} \models (\psi \rightarrow \theta)(\overline{\sigma}(\infty))$ and let $n \in \omega$ be large enough so that $n \geq qr(\varphi)$ and $\overline{\sigma}$ is defined in $[n, \infty]$. If $\phi \neq S \subseteq [n, \infty]$ is such that $\mathcal{A} \Vdash_S \psi(\overline{\sigma})$ then by genericity in $j \in \omega$ and by induction hypothesis in ∞, we have $\mathcal{A}_j \models \psi(\overline{\sigma}(j))$ for any $j \in S$. By the Lemma, $\mathcal{A}_{\infty} \models \psi(\overline{\sigma}(\infty))$ and so $\mathcal{A}_{\infty} \models \theta(\overline{\sigma}(\infty))$. By the lemma again $\mathcal{A}_j \models \theta(\overline{\sigma}(\sigma))$ for any $j \in S$. By induction hypothesis, $\mathcal{A} \Vdash_S \theta(\overline{\sigma})$. This shows $\mathcal{A} \Vdash_{\infty} (\psi \rightarrow \theta)(\overline{\sigma}))$.

$\varphi = \exists \mu \theta(\mu, \overline{x})$. Trivial induction.

$\varphi = \forall \mu \theta(\mu, \overline{x})$. If $\mathcal{A}_{\infty} \Vdash \forall \mu \theta(\mu, \overline{\sigma})$ then for any τ defined in ∞, $\mathcal{A} \Vdash_{\infty} \theta(\tau, \overline{\sigma})$ and by induction hypothesis $\mathcal{A}_{\infty} \models \theta(\tau(\infty), \overline{\sigma}(\infty))$. As any element of $\mathcal{A}_{\infty}$ is to the form $\tau(\infty)$, we have $\mathcal{A}_{\infty} \models \forall \mu \theta(\mu, \overline{\sigma}(\infty))$.

Conversely, if $\mathcal{A}_{\infty} \models \forall \mu \theta(\mu, \overline{\sigma})$, let $n = mr(\theta) + 1$ We must show that for any τ defined in $j \in [n, \infty]$,

$$\mathcal{A} \Vdash_j \theta(\tau, \overline{\sigma}). \tag{3}$$

For $j = \infty$ this follows by induction since by hypothesis $\mathcal{A}_{\infty} \models \theta(\tau(\infty), \overline{\sigma}(\infty))$. For $j \in [n, \infty)$, let $a = \tau(j) \in \mathcal{A}_j$. Since $f_{j\infty}$ is an n-partial isomorphism and $f_{j\infty}(\overline{\sigma}(j)) = \overline{\sigma}(\infty)$, then there is $\hat{b} \in \mathcal{A}_{\infty}$ such that

$$(\mathcal{A}_j, a, \overline{\sigma}(\sigma)) \equiv^{n-1}_{\cdot} (\mathcal{A}_{\infty}, \hat{b}, \overline{\sigma}(\infty)).$$

As $\mathcal{A}_{\infty} \models \theta(\hat{b}, \overline{\sigma}(\infty))$ by hypothesis and $mr(\theta) = n - 1$, then $\mathcal{A}_j \models \theta(a, \overline{\sigma}(j))$ and so $\mathcal{A}_j \Vdash \theta(\tau, \overline{\sigma})$. With this we have shown (3). □

Although Fraisse's performs his limit construction with a subsequence of the given Cauchy sequence, it is clear that we may from a sheaf as above with the original sequence.

In this way, the Cauchy sequences of E_τ are extendible to sheaves of structures of type τ over $\omega \cup \{\infty\}$. If we identify a structure $\mathcal{A}$ with the constant sheaf $\overline{\mathcal{A}}$ over $\omega \cup \{\infty\}$, we have an embedding

$$E_\tau \overset{J}{\hookrightarrow} Sh_\tau(\omega \cup \{\infty\})$$

where $Sh_\tau(\omega \cup \{\infty\})$ is the category of sheaves of structures of type τ over $\omega \cup \{\infty\}$. Now, $Sh_\tau(\omega \cup \{\infty\})$ has the natural topology induced by intuitionist logic where the basic closed classes are of the form

$$\mathcal{O}_\varphi = \{S \in Sh_\tau(\omega \cup \{\infty\})\ :\ S \Vdash \varphi\}.$$

Under this topology, the embedding is continuous, because for the constant sheaf: $\overline{\mathcal{A}} \Vdash \varphi \Leftrightarrow \mathcal{A} \models \varphi$.

Questions: 1) Is $Sh_\tau(\omega \cup \{\infty\})$ compact?
2) Is $\{S \in Sh_\tau(\omega \cup \{\infty\}) : S \text{ generic at } \infty\}$ a closed compact subspace of Sh_τ?
3) May every sheaf over $\omega \cup \{\infty\}$ with generic limit be construed as a Fraisse's limit?

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