**DOI: https://doi.org/10.5540/tcam.2023.024.03.00427**

Trends in Computational and Applied Mathematics

Trends in Computational and Applied Mathematics, **24**, N. 3 (2023), 427-436
Sociedade Brasileira de Matemática Aplicada e Computacional
Online version ISSN 2676-0029
www.scielo.br/tcam
doi: 10.5540/tcam.2023.024.03.00427

# A Combinatorial Bijection between Ordered Trees and Lattice Paths

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Received on December 16, 2021 / Accepted on November 16, 2022

**ABSTRACT.** This work presents a combinatorial bijection between the set of lattice paths and the set of ordered trees, both counted by the central coefficients of the expansion of the trinomial $(1+x+x^2)^n$. Moreover, using a combinatorial interpretation of Catalan numbers, we establish a new set of ordered trees counted by a new sequence.

**Keywords:** lattice paths, ordered trees, combinatorial identity, central trinomial coefficients.

## 1 INTRODUCTION

The combinatorial relationships between different objects are an interesting area, which has attracted much attention recently. Some finite sets have the same cardinality, and then, there is combinatorial bijection between these sets. In some cases, we can observe that such combinatorial relations are trivial, but in other cases the relation between the sets is not easy to establish.

Considering this type e of combinatorial enumeration problem, we can associate different mathematical objects by making bijective mappings, where each element corresponds to another, with combinatorial operations. In this way, we can prove identities extending results from one object to another, such as the generating functions of sets.

Several methods and techniques have been developed for studying combinatorial enumerations problems, specially involving lattices and graphs (see, for instance, [3,4,7]). In this context, a particular interest has been brought to the enumerative relationship between the lattice paths, ordered trees and the central trinomial coefficients.

The most important fact between the lattice theory and combinatorics is that each lattice can be represented as an inclusion-order on a set system. Then the lattice structure another viewpoint associated with sets of combinatorial objects.

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On the other hand, trinomial coefficients appear in relation to many generating functions, as Legendre polynomial, and are solution for some combinatorial problems. For example, the central trinomial coefficients are the number of permutations of $n$ simbols, each $-1,0$ or 1, wich sum to 0, or are the solution of a problem with restrictions of number of ordered ballots from $n$ voters, [see more in [8]].

The $n-$th central trinomial coefficients, $T_n$, is defined as the coefficients of $x^n$ in the expansion of $(a+bx+cx^2)^n$, for integers $a,b$ and $c$ ( for more details see in [ [8], Table 1] and [6]). The sequence $\{T_n\}_{n\geq 0}$ admits various combinatorial interpretations as lattice paths and ordered trees with restrictions. Especially, the expression

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{n}{2k} b^{n-2k} (ac)^k, \tag{1.1}$$

give us the explicit combinatorial formula for $T_n$ in terms of $a,b$ and $c$.

It is well-know that the set of lattice paths going from $(0,0)$ to $(n,0)$ with steps $U=(1,1)$, $D=(1,-1)$ and $H=(1,0)$ and the set of ordered trees with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most two are counted by the $n-$th central trinomial coefficients, $T_n$, with $a=b=c=1$. Therefore, a natural question is whether there is a closed relationship between these sets, more precisely, what is the combinatorial bijection between them?

First of all, it is important observe that the central trinomial coefficients of $(1+x+x^2)^n$ is the sequence with first terms given by $1,1,3,7,19,51,\cdots$, [A002426, [8]]. This case was studied by Euler [5] and can be seen currently with results in congruence [10].

On the side, Deutsch investigated in [2] the relationship between Schröder's small numbers and Schröder's lattice paths and formulates a bijection among them, [A001003, [8]]. The Schröder's small numbers, also known as super-Catalan numbers, are connected with paths by considering lattices paths from $(0,0)$ to $(n,n)$ with steps $(1,1),(2,0)$, and $(1,-1)$. Another similar paths are Dyck paths and Motzkin paths, according to [A000108 [8]].

The bijection presented by Deutsch used ordered trees with some restriction. The relationship of lattice paths going from $(0,0)$ to $(n,0)$ with steps $U=(1,1)$, $D=(1,-1)$ and $H=(1,0)$ and the set of ordered trees with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most two are cited in [8], both counted by the sequence of central trinomial coefficients.

This paper introduces a bijective combinatorial proof of these sets. More precisely, we give a combinatorial bijection between the set of lattice paths going from $(0,0)$ to $(n,0)$, with steps $U=(1,1), D=(1,-1)$ and $H=(1,0)$, and the set of ordered trees with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most two. Both sets are counted by the sequence of the central trinomial coefficients of $(1+x+x^2)^n$. Moreover, using a combinatorial interpretation of Catalan numbers and the sequence of the central trinomial coefficients of $(1+x+x^2)^n$, we can build a new sequence that count ordered trees in complementary type. To reach our goal, we give some useful elements about the graphs, lattices and trees (Section 2). After formulating some basic lemmas we give the proof of our main result (Section 3). Finally, we consider a new interesting sequence, related to the sequence $T_n$ and the Catalan numbers $C_n$, that count ordered trees in complementary type.

## 2 ABOUT GRAPHS, TREES AND LATTICES

A graph $G$ consists of a non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and a function that associates with each $G$ edge an unordered pair of (not necessarily distinct) vertices of $G$. A connected graph is a graph that there is a path from any vertex to any other vertex in the graph. And acyclic graph is a graph having no graph cycles. A tree is a connected and acyclic graph. We say that a tree is rooted when we differentiate one vertex (node) from the others, which we call the root.

The definition of ordered trees according to Stanley [9] is given by:

1. one specially designated vertex is called the root of $T$, and
2. the remaining vertex (excluding the root) are put into an ordered partition $(T_1,...,T_m)$ of $m \geq 0$ pairwise disjointed, nonempty sets $T_1,...,T_m$ where each of them is a plane tree.

The plane trees or subtrees are said to be children of the root tree and are ordered trees.

Also, according to Dershowitz [1], the set of ordered trees with $n$ edges is counted by the well-known sequence of Catalan numbers $C_n$, with first terms 1, 1, 2, 5, 14, 42, 132, ..., sequence given in [A000108, [8]], and the Expression

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{2.1}$$

For reasons of clarity, we are considering the following definition for lattice path, given by Stanley, [9]. Let $S$ be a subset of $\mathbb{Z}^d$, a lattice path $L$ in $\mathbb{Z}^d$ of length $k$ with steps in $S$ is a sequence $v_0, v_1, ..., v_k \in \mathbb{Z}^d$ such that each consecutive difference $v_i - v_{i-1}$ lies in $S$. We say that $L$ starts at $v_0$ and ends at $v_k$, or more simply that $L$ goes from $v_0$ to $v_k$.

By considering the expansion coefficients of $(1+x+x^2)^n$, it is possible to build an arithmetic triangle, similar to Pascal's, namely we have,

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & 1 & 1 & & & & & \\
 & & & & 1 & 2 & 3 & 2 & 1 & & & & \\
 & & & 1 & 3 & 6 & 7 & 6 & 3 & 1 & & & \\
 & & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 & & \\
 & 1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 & 1 & \\
1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1
\end{array}.$$

Analyzing the numerical sequence [ [8], A002426] determined by the central coefficients of the trinomial expansion $(1+x+x^2)^n$ we can described the arithmetic triangle above. And by interpretations made using the information available in [8], we show that the central coefficients of trinomial expansion $(1+x+x^2)^n$ is nothing else but the number of lattice paths from $(0,0)$ to $(n,0)$ with steps $U=(1,1)$, $D=(1,-1)$, and $H=(1,0)$.

On the other side, also this number count ordered trees with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most 2. Thus, fixed $n$, these sets have the same cardinal. In the literature there is no bijective proof connecting lattices and trees with these restrictions directly. Our goal is to show a combinatorial bijection between them.

## 3 A COMBINATORIAL BIJECTIVE PROOF

Let $\mathbb{G}_n$ be the set of ordered trees with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most 2. And let $\mathbb{P}_n$ be the set of lattice paths from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,-1)$ and $(1,0)$. For a fixed $n \in \mathbb{N}$, as the sets have the same cardinality seen in [8], there is a bijection between the two sets $\mathbb{G}_n$ and $\mathbb{P}_n$. Let $\phi : \mathbb{G}_n \to \mathbb{P}_n$ be the combinatorial bijective function of $\mathbb{G}_n$ and $\mathbb{P}_n$.

For the construction of the preceding bijection $\phi$, we need to consider the following result, which characterizes the number of steps in $\mathbb{P}_n$.

**Lemma 3.1.** *The number of steps* $(1,1)$ *in* $\mathbb{P}_n$ *is the same as steps* $(1,-1)$.

**Proof.** Considering $c$ the set of lattice paths from $(0,0)$ to $(n,0)$ with $k$ steps $(1,1)$, $j$ steps $(1,-1)$ and $q$ steps $(1,0)$. We obtain $k(1,1)+j(1,-1)+q(1,0) = (k+j+q, k-j)$, where $k+j+q=n$ and $k-j=0$. Therefore , we derive that $k=j$. □

The Lemma 3.1 shows us that there is symmetry with respect to the $x-$axis for the elements of the set $\mathbb{P}_n$. Then, let $C_1, C_2, C_3, C_4$ and $C_5$ in $\mathbb{P}_n$, such that $C_1$ is the set of paths that are above the $x$-axis, $C_2$ the set of which are below the $x$-axis, $C_3$ for those with positive and then negative, $C_4$ for those with negative and then positive, and $C_5$ for those with only ordinate zero.

For every fixed $n$, we consider the set $C = C_1 \cup C_3 \cup C_5$ where the sets $C_1, C_3$ and $C_5$ are in $\mathbb{P}_n$.

Note that the number of edges in $\mathbb{G}_n$ is $n+1$, and the number of edges in $\mathbb{P}_n$ is $n$, so let's remove an edge, conveniently, such that this number becomes the same.

To succeed in this process, let call $A$ the set of *clean trees* with $n$ edges, that is, exactly one edge has been removed, and let $f : \mathbb{G}_n \to A$, be the function that removes it. The function $f$ is defined by contracting the rightmost edge emanating from the root, the excess edge, such that we get a tree at $A = Im(f)$; an exception occurs when we have a sequence of nodes with degree one nodes after the excess edge. In this case, the sequence of consecutive adjacent nodes with degree one nodes, if any, will go above the root after the excess edge has contracted. An example for this exception is given in Figure 1, on the right side.

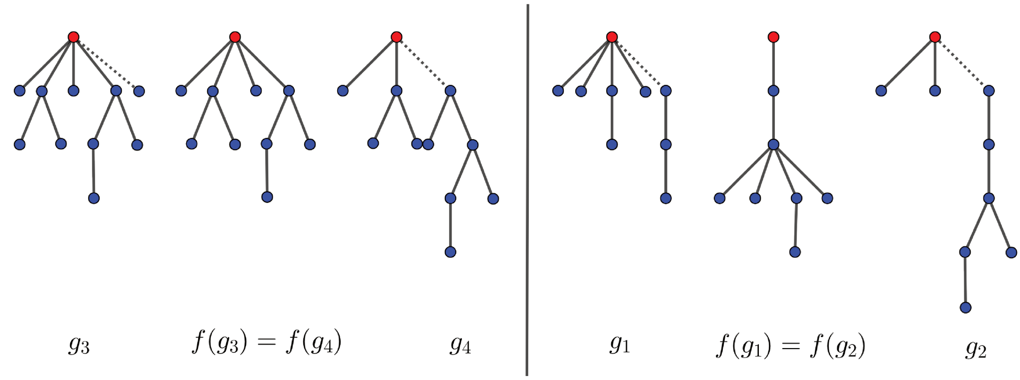


Figure 1: $f$ applied in case of exception.

In this way we see that two trees can determine the same clean tree and as we will see it is unique. Now, note that, given the clean tree, we can reverse the procedure, which in this case generates exactly two trees, except for the tree composed only of nodes and root of degree one. In the inverse procedure, consider the two cases:

I - the degree of clean tree root is greater than one, for this end to obtain the first tree, we expand one edge from the root by pushing the two rightmost edges of the clean root tree. To get the second one, we expand it by creating another right edge emanating from the root.

II - the degree of clean tree root is equal to one, for this end take $v$, if any, with the first node of degree greater than one after the root. To get the first tree, we expand an edge by $v$, pushing the two rightmost edges emanating from this node. Next, we contract the sequence of grade one nodes preceding $v$ and expand that sequence to the adjacent vertex to the rightmost $v$. For the second tree, we expand a rightmost edge of $v$, and again we contract the sequence of grade one nodes preceding $v$ and we expand this sequence at the vertex adjacent to $v$ to the right. If $v$ does not exist, this clean tree derives only from the tree composed only with nodes and root of degree one.

Note that $f(g_1) = a = f(g_2)$, for two trees $g_1$ and $g_2$ in $\mathbb{G}_n$. Then, we can define the inverse procedure $f^{-1}(a) = \{g_1, g_2\}$, $a \in A$. Since $g_1$ and $g_2$ have different degrees for their roots, taking without loss of generality $g_1$ as the tree with root of the highest degree. Let $h : A \to \mathbb{G}_n$, such that $h$ associates each element of $A$ with the $g_1$ tree, that is, always associates with the highest root tree. Taking $B = Im(h)$, thus $f\big|_B$ is bijective.

Moreover, we obtain the cardinality $|A| = \frac{K_n-1}{2} + 1 = \frac{K_n+1}{2}$, that implies that $|A| = |C|$.

**Lemma 3.2.**

*Under the preceding data and notations, consider the two sets $C = C_1 \cup C_3 \cup C_5$ and $B = Im(h)$. Then, there is a bijection between sets $C$ and $B$.*

**Proof.** The main goal is to find a bijective application $\psi : B \to C$. To reach our goal, we consider an element $g \in B$. Let $a \in A$ such that $f(g) = a$. Then $\psi(g)$ is obtained by traversing the edges of the $a$ tree from left to right, bottom to top, and out. The remaining internal edges will be traverse from right to left and top to bottom so that:

I - Does step $(1,0)$ when the vertex has out degree one.

II - Make steps $(1,1)$ and $(1,-1)$ when the vertex has an exit degree greater than or equal to two. Alternately, when the vertex has an out degree greater than two make step $(1,1)$ corresponds to the leftmost edge and $(1,-1)$ to the rightmost edge, (see Figure 2). And if an edge emanating from the root with step $(1,1)$ generates subtrees, makes step $(1,-1)$ and $(1,1)$ when the vertex has an output degree equal to two. Where $(1,-1)$ corresponds to the left edge and $(1,1)$ to the right edge.

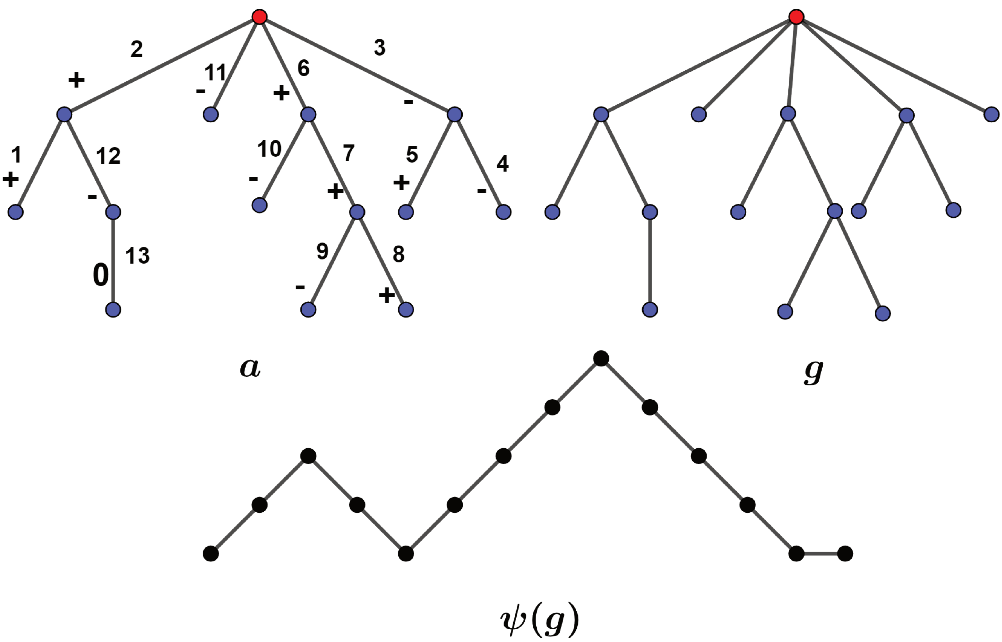


Figure 2: Path $\psi(g)$ $(+ = (1,1), - = (1,-1)$ e $0 = (1,0))$.

Enumerating the trees from left to right and bottom to top, we will always start with steps $(1,0)$ or $(1,1)$, and may later, at some point along the way, have a number of steps $(1,-1)$ higher than that of steps $(1,1)$. So the built path is in $C$. This way, each tree in $A$ will only associate with a single path in $C$. In fact, by switching one edge, we have also changed the enumeration.

Conversely, given a lattice path $c \in C$, we can build a tree $g$ , such that $\psi^{-1}(c) = g$.

That is, let $c$ going from $(0,0)$ to $(n,0)$ with $k$ steps $(1,1)$, and consequently $k$ steps $(1,-1)$ by Lemma 3.1. Consider the first increasing path in $c$ with only steps $(1,1)$ up to the point $(j,s)$, and $t$ the number of steps $(1,1)$ and $(1,-1)$ with upper ends in $(x,y)$ in $c$, with $0 \leq x,y \leq n$, and $y \neq s$, (see Figure 3).

Thus we define the degree of the root by $2k - t + 1$. Observe that $t$ is even, since the number of steps $(1,1)$ and $(1,-1)$ it is the same.

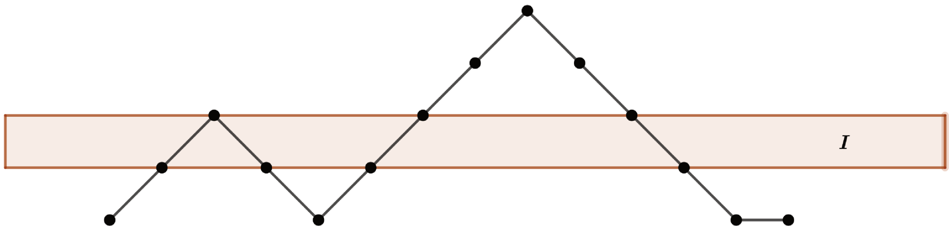


Figure 3: Root degree range.

By construction, the consecutive step requires a node of degree two, decreasing the degree of the root by two. Since the maximum possible degree for the root is $2k+1$ and we have removed as few edges as possible when we have $t > 0$, this root will be the highest degree and therefore it is in $B$. Moreover, we form the degree of the root and the other nodes have degree two or one according to the number of steps $(1,0)$ being further left and emanating from an appropriate node according to the sequence of steps of the lattice path. $\square$

**Theorem 3.1.**

*The number of lattice paths from $(0,0)$ to $(n,0)$ with steps $U = (1,1), D = (1,-1)$ and $H = (1,0)$ is the same as that of ordered trees with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most 2.*

**Proof.** Let $\phi : \mathbb{G}_n \to \mathbb{P}_n$ be the bijective function between $\mathbb{G}_n$ and $\mathbb{P}_n$.

Given a tree $g \in \mathbb{G}_n$, we find a lattice path such that if $g \in B \subset \mathbb{G}_n$, then $\phi(g) = \psi(g) = c \in C$. Otherwise $g \in B^{\complement} \subset \mathbb{G}_n$ then $\phi(g) = c' \in C^{\complement} \subset \mathbb{P}_n$ such that $c'$ is the symmetrical of $c$ relative to the $x$-axis.

Conversely, given a path $c \in \mathbb{P}_n$, we have to $c \in C \subset \mathbb{P}_n$, then $\phi^{-1}(c) = \psi^{-1}(c) = g \in B$. Otherwise $c \in C^{\complement} \subset \mathbb{P}_n$ so we take $c' \in C$ such that $c'$ is the symmetrical of $c$ relative to the $x$-axis. So by applying $\psi^{-1}$ to $c'$ we get $g \in B$. Moreover there is $a \in A$ such that $f^{-1}(a) = \{g, \bar{g}\}$, where $\bar{g}$ is the lowest rooted tree. Thus we define $\phi^{-1}(c) = \bar{g} \in B^{\complement}$. $\square$

## 4 A NEW SEQUENCE

Let $T_n$ be the sequence formed by the central coefficients of the trinomial expansion $(1+x+x^2)^n$.

To find a closed expression for $T_n$ we can take the element with the highest coefficient in the expansion of $[-x+(1+x)^2]^n$. Therefore, we have

$$[-x+(1+x)^2]^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{i=0}^{2k} \binom{2k}{i} x^{n+k-i}.$$

Note that by the symmetry of the arithmetic triangle, the coefficient of greatest value follows $x^n$ in the development of $(1+x+x^2)^n$, that is, the largest value occurs when $x^{n+k-i} = x^n$, so $k = i$.

Thus, the closed expression for $T_n$ is $\displaystyle\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{k}$.

We denote $C_n$ as the sequence formed by Catalan numbers, that is, whose generating function is furnished by considering the sequence $\dfrac{1}{n+1}\dbinom{2n}{n}$. Since $T_n$ counts the number of trees ordered with $n+1$ edges, having root of odd degree and nonroot nodes of out degree at most 2, and $C_n$ counts the number of ordered trees with $n$ edges, we can conclude that $C_{n+1} - T_n$ counts the ordered trees with $n+1$ edges such that either the root has an even degree, or the ordered tree has an odd degree root such that the nodes of out degree are greater than or equal to 3.

Also, since we have $T_0 = 1$, $T_1 = 1$, $T_2 = 3$, $T_3 = 7$, $T_4 = 19$, $\ldots$, and $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, we can compute the first terms of the sequence $C_{n-1} - T_n$ : $0, 1, 2, 7, 23, 81, 288$, not cataloged in the literature, as verified at [8].

Let $L_n = C_{n+1} - T_n$, for $n = 0$, we cannot form a tree with an edge, under the conditions listed above, therefore, we set $L_0 = 0$.

Varying $n$ in $\frac{1}{n+2}\dbinom{2n+2}{n+1} - \displaystyle\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{k}$, we succeed in forming a new sequence, whose first terms given by $, 1, 2, 7, 23, 81, 288, \ldots$, and taking $n = 3$ we have 7 possible ordered trees as we can see in Figure 4. Our new sequence $L_n$, counts the ordered trees with $n+1$ edges such that either the root has an even degree.

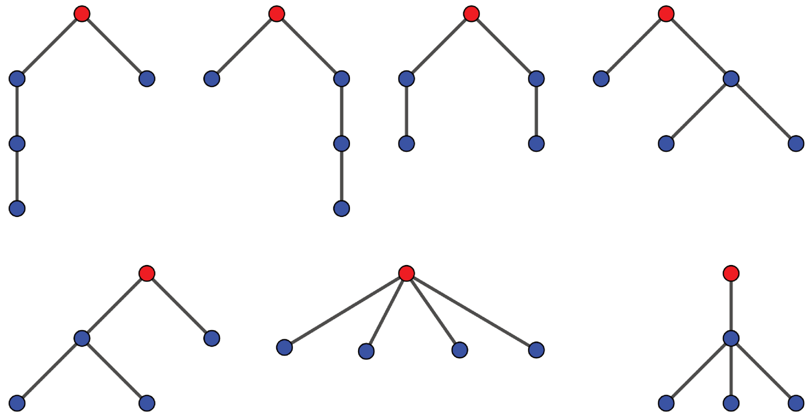

Figure 4: $n = 3$.

## CONCLUSION

Throughout this study we had established some properties involving the set formed by the central coefficients of the trinomial expansion $(1+x+x^2)^n$, lattice paths and ordered trees. Moreover, a theoretical basis for studying bijection between these sets.

This research allowed to determine a bijective function between the set of ordered trees and lattice paths. The combinatorial operation proposed here can be applied to prove relationships between the same types of objects.

Finally, these results have contributed to another approach in the generalized interpretation of closed relation between the trinomial of the form $(a+bx+cx^2)^n$, ordered trees and lattice paths. In the best of our knowledge, our approach is not current in the literature.

### Acknowledgments

The authors thank the Referee for his/her valuable comments and suggestions. The second author thanks Mustapha Rachidi for his valuable suggestions and comments. The authors express their sincere thanks to the INMA and Universidade Federal de Mato Grosso do Sul – UFMS/MEC – Brazil for their valuable support.

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