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LUCAS DE CUNTO COSTANZO

Control of θ-periodic switched systems with application in electrical engineering

Controle de sistemas com comutação periódicos em θ com aplicação em engenharia elétrica

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DISSERTAÇÃO DE MESTRADO

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RESUMO

Neste trabalho, duas regras de comutação são propostas a fim de lidar com o problema de rastreamento de trajetória de sistemas com comutação, cujas matrizes dinâmicas variam periodicamente no tempo de acordo com um parâmetro θ, que pode ser tanto uma variável externa, quanto dependente do estado. Neste último caso, o sistema torna-se altamente não-linear e o projeto de controle mais desafiador. A primeira regra minimiza um custo garantido de desempenho enquanto a segunda leva em consideração o controle com ação integral para lidar com incertezas de modelagem. O grande desafio do controle com ação integral vem do fato de sua matriz dinâmica possuir posto incompleto. Em ambas, as condições são expressas em termos de desigualdades matriciais lineares, que podem ser resolvidas facilmente através de softwares disponíveis. Dois sistemas elétricos trifásicos são usados para validar as soluções propostas: um conversor de potência AC-DC e uma máquina síncrona de imã permanente. Uma montagem experimental foi elaborada para a implementação prática da regra e seus resultados foram comparados com técnicas recentes disponíveis na literatura.

Palavras–chave: Sistemas afins com comutação periódicos em θ; Rastreamento assintótico; Controle com ação integral; Desigualdades matriciais lineares

ABSTRACT

In this work, two different switching rules are proposed to deal with the trajectory tracking problem of switched systems, whose dynamic matrices vary periodically in time with respect to a parameter θ, which can be an external or state-dependent. In the last case, the system becomes highly nonlinear making the control design more challenging. The first rule minimizes a guaranteed cost of performance, while the second takes into account the control with integral action to deal with model uncertainties. The great challenge in the control with integral action arises from the fact that its dynamical matrix is rank deficient. In both, the conditions are expressed in terms of linear matrix inequalities, which are simple to solve using ready available software. Two three-phase electrical systems are used to validate the proposed switching rules: an AC-DC power converter and a permanent magnet synchronous machine (PMSM). An experimental setup has been elaborated to implement the switching strategy in the PMSM and to compare it with recent techniques available in the literature.

Keywords: θ-periodic switched affine systems; asymptotic tracking; control with integral action; linear matrix inequalities

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Abbreviations

AC	Alternating Current
DC	Direct Current
PMSM	Permanent Magnet Synchronous Machine
LTI	Linear Time Invariant
LPV	Linear Parameter Varying
LMI	Linear Matrix Inequality

Symbols

(’)	Transpose of a matrix
$(\bullet)$	Symmetric block of a matrix
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_+$	Set of non-negative real numbers
$\mathbb{R}^n$	Set of real vectors with dimension n
$\mathbb{R}^{n\times m}$	Set of real matrices with dimension $n \times m$
$\mathbb{N}$	Set of natural numbers
$\mathbb{K}$	Set composed by the first N positive natural numbers $\mathbb{K} := \{1, \ldots, N\}$
Λ	Unit simplex $\Lambda := \left\{\lambda \in \mathbb{R}^N \ : \ \lambda_j \geq 0, \sum_{j\in\mathbb{K}} \lambda_j = 1\right\}$
X_λ	Convex combination of matrices $X_\lambda = \sum_{i\in\mathbb{K}} \lambda_i X_i, \ \lambda \in \Lambda$
$\gamma_i(X)$	i-th eigenvalue of matrix X
$\mathrm{diag}(X, Y)$	Block diagonal of matrices X and Y
$\mathrm{Tr}(X)$	Trace of a matrix X

$\mathrm{Re}(x)$	Real part of the complex number x
$\mathrm{He}\{X\}$	Operator defined as $\mathrm{He}\{X\} := X + X'$
$\arg\min_{i \in \mathbb{K}} f_i$	Argument of the minimum of f_i
$\|\cdot\|$	Euclidean norm
$\|\cdot\|_2$	$\mathcal{L}_2$ norm
$\mathrm{Int}(C)$	Interior of the set C
$\mathrm{co}(C)$	Convex hull of the set C
$X < (\leq) 0$	X is (semi-)negative definite
$X > (\geq) 0$	X is (semi-)positive definite

CONTENTS

1 INTRODUCTION

Under the light of technological developments over the past few decades in electrical and computer engineering, specially after digital controllers being widespread in industrial applications, the study of dynamical systems that present an interaction between continuous and discrete behaviors, also called hybrid systems, have become a topic of great interest. In this class of systems, that are usually characterized by presenting nonlinear phenomena, another subclass arises, formed by the switched systems. This subclass is defined by presenting several subsystems and a switching rule that orchestrates the commutation among them, which are called switching events. These events are known to be source of remarkable behaviors that can either provide strong stability and performance properties or act as vicious disturbances, see the seminal survey paper (DeCarlo *et al.*, 2000) and the books (Liberzon, 2003; Sun; Ge, 2011) on this topic for examples. Proper control design, specially within safety-critical and performance-sensible contexts, must not only carefully allow for these phenomena but also be able to exploit them to enhance the closed-loop response.

Many researchers devoted their work to thoroughly study the switching stabilization problem of many classes of switched systems. Naturally, switched linear systems have received most of the attention. By exploring their linearity, necessary and sufficient conditions for stabilizability were developed for continuous-time systems (Lin; Antsaklis, 2009) (under the assumption that sliding modes do not occur) and for discrete-time systems (Fiacchini; Jungers, 2014). Numerically dealing with these conditions, though, is not an easy task. Indeed, the NP-hardness of some instances of switched stabilization problems (e.g, quadratic stabilizability (Lin; Antsaklis, 2009), polytopic-uncertain systems (Vlassis; Jungers, 2014)) is a fair motivation to search for more conservative but computationally treatable design conditions, such as those in (Geromel; Colaneri, 2006a; Fiacchini *et al.*, 2015). Not surprisingly, for more general classes of switched systems, most questions regarding switched stabilizability remain open.

For instance, by merely introducing affine switching terms to the switched linear models, the homogeneity of the system is lost and asymptotic stability of the origin may no longer be achievable. Indeed, this class of switched affine systems has a more intricate stabilizability problem where, besides the design of a stabilizing switching rule, the control expert must determine a desired attractor in the state space. Examples of attractors to which switched affine systems can be stabilized are points (Bolzern; Spinelli, 2004; Deaecto *et al.*, 2010), forward-invariant compact sets (Hetel; Fridman, 2013; Egidio; Deaecto, 2019) and limit cycles (Patino *et al.*, 2009; Egidio *et al.*, 2020; Serieye *et al.*, 2020). Due to the substantial complexity of this class, several problems that were already solved for switched linear systems remain open, for example the stability analysis for polytopic systems.

As one may notice from the aforementioned references, one of the main motivations for the study of switched affine systems is their ability to characterize the behavior of several DC-DC power converters, see (Deaecto *et al.*, 2010; Baldi *et al.*, 2018; Beneux *et al.*, 2019; Garcia; Santos, 2021) for some examples. In fact, switching devices that allow the efficient rerouting of energy in electrical systems are paramount for the development of power electronics and electrical drives, specially considering the increasing concern in finding more effective ways of producing, storing and using energy. However, the time-invariant nature of each subsystem in switched affine models precludes the use of this class of systems to study AC devices directly, requiring case-based analysis (e.g., (Trofino *et al.*, 2009; Scharlau *et al.*, 2013; Hadjeras *et al.*, 2019; Guo; Ren, 2020)) or formulations based on uncertain systems, such as linear parameter-varying systems (e.g., (Delpoux *et al.*, 2014)). The control design problem for AC devices is generally formulated as a tracking problem, where the objective is to enforce the system output to follow a time-varying trajectory profile of interest, which is more complicated than to ensure simply stabilization.

Of course, the control of AC systems have been extensively studied by the power electronics, electric machines and control communities under several different perspectives such as classical frequency domain methods (Bacha *et al.*, 2014), sliding mode control (Repecho *et al.*, 2017), model-predictive control (Rodriguez; Cortes, 2012), and other ad-hoc analyses (Krause *et al.*, 2013; Krishnan, 2017; Wu; Narimani, 2017). Nevertheless, a general methodology for designing stabilizing switching laws to the subclass of switched systems that models AC devices is a point of great importance to be considered in the literature. Recently, a unified way to solve stabilization and tracking problems for power electronics converters have been proposed in (Garcia; Santos, 2021) and (Garcia *et al.*, 2021), respectively, based on a quadratic Lyapunov function. In the current work our main goal is to go further by exploring the time-varying nature of the θ-parameter through the adoption of a parameter-dependent Lyapunov function in order to obtain less conservative design conditions. In this context, the contributions can be summarized as follows:

- we develop a general theorem to deal with the tracking problem of a class of switched systems that are affine on the state and vary periodically on a parameter θ. This parameter can characterize electrical and mechanical angles on AC systems, which provides a general framework for many devices, while avoiding ad-hoc modulation strategies, auxiliary reference frames and pulse-width modulation drivers.

- particular corollaries demonstrate how this general theory can be used to formulate systematic control-design procedures based on convex optimization, where the stability and performance conditions are cast as linear matrix inequalities (LMIs), which are easily handled by off-the-shelf softwares.

- the conceived theory is shown to generalize the preceding (and preliminary) results presented in (Egidio *et al.*, 2019; Egidio *et al.*, 2020; Egidio *et al.*, 2022a) in many directions. A more general Lyapunov-function dependent on the parameter θ is conceived, novel conditions for the trajectory-tracking and integral control of these systems are provided and new simulation and experimental results illustrate the theory.

1.1 Publications

As a result of the developed work, the following papers were produced:

- Costanzo, L. C.; Deaecto, G. S.; Egidio, L. N.; Barros, T. A. Nova metodologia de controle para conversores de potência trifásicos cc-ca. *In: Simpósio Brasileiro de Automação Inteligente-SBAI*. [S.l.: s.n.], 2021., p. 558–563.

- Deaecto, G. S.; Costanzo, L. C.; Egidio, L. N. Trajectory tracking for a class of θ-periodic switched affine systems. Submitted, 2022.

1.2 Chapters Outline

This dissertation is divided in 6 chapters, whose contents are presented below:

- **Chapter 1: Introduction**

  It presents the motivation, the state-of-the-art, the challenges and the objective of this work.

- **Chapter 2: Preliminary Concepts**

  It outlines concepts that are already known in the literature, going from Lyapunov stability theory to an introduction on switched systems, which are the main subject of this work. Those concepts will be important for the next chapters.

- **Chapter 3: θ-periodic Switched Systems**

  It encompasses the main theoretical results obtained in this work. In this chapter, first a general methodology for the control of switched affine systems that are dependent on a periodic parameter is presented, which ensures a guaranteed cost. After that, a switching rule with integral action is developed in order give robustness to the controlled system.

- **Chapter 4: Power Electronics Applications**

  It provides a corollary based on the first theorem presented in Chapter 3 to deal specifically with the control of a three-phase AC-DC power converter. The results obtained were validated and compared to others available in the literature. The effectiveness of the solution is evidenced through computational simulations.

- **Chapter 5: Electrical Machines Applications**

  It proposes a corollary based on the first theorem presented in Chapter 3 to deal specifically with the control of a three-phase Permanent Magnet Synchronous Machine. The results are also compared with other works from the literature. Its efficacy is put in evidence through experimental essays through a switching rule that is equivalent to the one proposed in the corollary, but more efficient in terms of computational effort.

- **Chapter 6: Conclusions**

  It presents a summary about the obtained results and provides a perspective for future work.

2 PRELIMINARY CONCEPTS

In this chapter, some important concepts for the study of dynamical systems will be presented. These concepts, widely studied in the literature, will be important to understand the results obtained later in this work. Firstly, a general idea about equilibrium points and stability of dynamical systems is presented. After that, an introduction to Lyapunov theory together with an analysis about linear time-invariant systems are provided. Lastly, some introductory results about switched systems are presented to put in evidence the intrinsic characteristics of this important class of systems. The switched linear and affine systems are the focus of this introductory part and several examples will be used to illustrate the essential aspects of this theory.

To begin with our discussion, let us consider the general dynamical system with representation given by

$$\dot{x} = f(x) \tag{2.1}$$

where $f \in \mathbb{R}^n$ is a nonlinear or linear function and $x \in \mathbb{R}^n$ is the state vector.

2.1 Equilibrium Point

The vector $x_e \in \mathbb{R}^n$ is said to be an equilibrium point of the dynamical system (2.1), if once the system reaches this point at an instant $t = t_0$, it remains in it for all future time, that is, once $x(t) = x_e$ for $t = t_0$ then $x(t) = x_e \; \forall \; t \geq t_0$, see (Luenberger, 1979). For simplicity, to the analysis of nonlinear systems which present multiple equilibrium points, it is often common to adopt the auxiliary state vector $\xi = x - x_e$ in order to obtain an equivalent system

$$\dot{\xi} = f_e(\xi) \tag{2.2}$$

whose origin is the unique equilibrium point. This change of variables makes the analysis easier and will be used in the subsequent work.

2.2 Stability

The stability of a dynamical system is always defined with respect to an equilibrium point. This point is said to be stable if, for any $R > 0$, there exists $r > 0$ such that for $\|x(0)\| < r$, then $\|x(t)\| < R$ for all $t \geq 0$, otherwise this equilibrium point is said to be unstable, see

(Slotine *et al.*, 1991). In other words, according to (Luenberger, 1979) the equilibrium point will be stable if the state vector tends to return to x_e once moved slightly away from it, or at least does not keep moving further away.

Given an initial condition in the state space, if $x(t) \to x_e$ as $t \to \infty$, the equilibrium point is referred to as asymptotically stable. If this occurs only for a set of initial conditions, the equilibrium point is said to be locally stable. Contrarily, it is globally stable. In short, if for a given system, we have $x(t) \to x_e$ as $t \to \infty$ for any initial condition, the equilibrium point x_e is said to be globally asymptotically stable. In this work, we are interested in obtaining design conditions to ensure asymptotic stability of the switched systems under study.

2.3 Lyapunov Theory

One of the most useful approaches to determine the stability of dynamical systems is the Lyapunov theory, first introduced in the doctoral dissertation "The general problem of the stability of motion" by Aleksandr Mikhailovich Lyapunov, in 1892. This work presents two methods for stability analysis, the linearization method and the direct method. The first one determines the local stability of a nonlinear system equilibrium point analyzing its linear approximation around that point. The second and more general one, which will be extensively used in this work, determines stability by constructing an "energy-like" function for the system and by examining its time variation.

2.3.1 Lyaponov's Direct Method

The main idea of the Lyapunov's direct method is based on the fact that if the total energy of a system is continuously decreasing, the dynamical system, linear or not, will eventually settle down to an equilibrium point. Thus, it is possible to conclude the stability of a system by analyzing the time variation of a scalar function, called Lyapunov function, which must present some specific characteristics.

Definition 2.1. The function $v(x) : \mathbb{D} \to \mathbb{R}$ with domain $\mathbb{D} \subset \mathbb{R}^n$ is a Lyapunov function if it presents the following set of conditions (Khalil, 2002):

- $v(x)$ is continuously differentiable
- $v(x) = 0$ for $x = x_e$

- $v(x) > 0, \forall x \neq x_e$
- $\dot{v}(x) \leq 0, \forall x \neq x_e$

If, for a given system, these requirements are verified, then it is possible to conclude that the system is stable. Besides that, if the last condition is strictly negative and the domain $\mathbb{D} \equiv \mathbb{R}^n$, then the system is globally asymptotically stable.

2.3.2 Linear Time-Invariant Systems

An n-order linear time-invariant (LTI) system can be represented by a series of n linear differential equations of first order, whose unforced system is given by the following state space representation

$$\dot{x}(t) = Ax(t), \;\; x(0) = x_0 \tag{2.3}$$

$$z(t) = Ex(t) \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the state vector, $z \in \mathbb{R}^p$ is the controlled output and x_0 is the initial condition. The first equation (2.3) is called dynamic equation , while the second (2.3) is called output equation.

For an LTI system, the set of equilibrium points can be determined by the null space of matrix A, defined by $Ax = 0$. In the case where the matrix A is non-singular, the system presents a single equilibrium point at the origin $x_e = 0$, otherwise, it can present an infinite number of equilibrium points.

For this class of dynamical systems, a necessary and sufficient condition for stability comes from the analysis of the eigenvalues of A. In this case, the origin is a globally asymptotically stable equilibrium point if and only if $\mathrm{R}e\left(\gamma_i(A)\right) < 0$ for all $i = 1, \ldots, n$, where $\gamma_i(A)$ is the i-th eigenvalue of matrix A. This can be verified from the general solution to (2.3) given by

$$x(t) = e^{At}x_0 \tag{2.5}$$

where using $V^{-1}AV = J = \mathrm{diag}\{J_1, \ldots, J_r\}$, with J_i being a Jordan block associated with the eigenvalue $\gamma_i(A)$, we have

$$e^{At} = Ve^{Jt}V^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}e^{\gamma_i(A)t}R_{ik} \tag{2.6}$$

where m_i is the order of the Jordan block J_i, see (Khalil, 2002) for details. It is clear from (2.6) that the state $x(t)$ of (2.5) always evolves towards the origin whenever $\mathrm{Re}\,(\gamma_i(A)) < 0,\ \forall i = 1, \ldots, n$.

Besides the analysis of the eigenvalues, another possibility to conclude about the stability of an LTI system is to apply the Lyapunov theory. In this case, a Lyapunov function candidate is the quadratic one, given by:

$$v(x) = x'Px, P = P' > 0 \tag{2.7}$$

The time-derivative of this function along an arbitrary trajectory of the system (2.3) results in another quadratic form

$$\begin{aligned}\dot{v}(x) &= \frac{d}{dt}(x'Px) \\ &= \dot{x}'Px + x'P\dot{x} \\ &= x'(A'P + PA)x = -x'Qx \end{aligned} \tag{2.8}$$

where

$$A'P + PA = -Q \tag{2.9}$$

is called Lyapunov equation. Notice that, whenever $Q > 0$, the equation (2.9) provides a solution $P > 0$, we have $\dot{v}(x) < 0$, indicating that the origin $x_e = 0$ is a globally asymptotically equilibrium point, see Definition 2.1. Actually, as it will be formalized in the next lemma, this condition is not only sufficient, but also necessary for the stability of LTI systems.

Lemma 2.1. Given a positive definite matrix $Q > 0$, there exists a unique solution $P > 0$ to the Lyapunov equation (2.9) if and only if matrix A is Hurwitz stable.

Proof. The sufficiency follows directly from the before mentioned reasoning. Indeed, considering the Lyapunov function (2.7) as being the square of the distance between a point $x(t)$ in the state space to the origin $x_e = 0$, if there exists a matrix $P > 0$ satisfying the Lyapunov equation for $Q > 0$, then this distance will always be decreasing, as a consequence from the fact that $\dot{v}(x) < 0,\ \forall x \neq 0$, indicating that the system (2.3) is globally asymptotically stable.

To prove the necessity, we need to suppose that matrix A is Hurwitz stable and to find a unique solution $P > 0$ for the Lyapunov equation (2.9) with $Q > 0$ given. Let us consider a candidate P as being

$$P = \int_0^{\infty} e^{A't} Q e^{At} dt \tag{2.10}$$

Using (2.5) as the analytical solution of (2.3) with an arbitrary $x(0) = x_0 \neq 0 \in \mathbb{R}^n$, we have

$$\begin{aligned} x_0' P x_0 &= \int_0^\infty x_0' e^{A't} Q e^{At} x_0 dt \\ &= \int_0^\infty x(t)' Q x(t) dt > 0 \end{aligned} \tag{2.11}$$

indicating that the candidate P is indeed definite positive, since $Q > 0$. Now replacing (2.10) into (2.9), we obtain

$$\begin{aligned} A'P + PA &= A' \left(\int_0^\infty e^{A't} Q e^{At} dt \right) + \left(\int_0^\infty e^{A't} Q e^{At} dt \right) A \\ &= \int_0^\infty (A' e^{A't} Q e^{At} + e^{A't} Q e^{At} A) dt \\ &= \int_0^\infty \frac{d}{dt} (e^{A't} Q e^{At}) dt \\ &= \lim_{t \to \infty} e^{A't} Q e^{At} - Q \end{aligned} \tag{2.12}$$

Notice that $\lim_{t \to \infty} e^{A't} Q e^{At} = 0$ because the matrix A is Hurwitz stable, resulting back in the equation (2.9).

Now, to prove that (2.10) is a unique solution, let us consider another matrix $\hat{P} \neq P$, which provides

$$A'(P - \hat{P}) + (P - \hat{P})A = 0 \tag{2.13}$$

Multiplying to the left by $e^{A't}$ and to the right by its transpose, we have

$$e^{A't} \left(A'(P - \hat{P}) + (P - \hat{P})A \right) e^{At} = \frac{d}{dt} (e^{A't} (P - \hat{P}) e^{At}) = 0 \tag{2.14}$$

Hence

$$e^{A't} (P - \hat{P}) e^{At} = K, \; \forall t \geq 0 \tag{2.15}$$

where K is a constant matrix for all $t \geq 0$. Evaluating then the equality for $t = 0$ and $t \to \infty$ it is possible to conclude that $P = \hat{P}$ is the unique solution. Thus, whenever the matrix A is Hurwitz stable, the Lyapunov equation has a unique solution $P > 0$, concluding the proof. □

Lastly, whenever the system (2.3)-(2.4) is asymptotically stable, the quality of its transient response can be evaluated by the index

$$J = \int_0^\infty z(t)' z(t) dt \tag{2.16}$$

which can be exactly calculated from the solution of the Lyapunov equation (2.9) for $Q = E'E$. Indeed, in this case $\dot{v}(x(t)) = -x(t)'E'Ex(t) = -z(t)'z(t)$, and we have

$$J = -\int_0^\infty \dot{v} dt = v(x(0)) - \lim_{t \to \infty} v(x(t)) = x_0' P x_0 \tag{2.17}$$

since $\lim_{t \to \infty} v(x(t)) = 0$ due to the asymptotic stability of the system.

2.4 Switched Systems

Hybrid systems are characterized by systems that present an interaction between continuous and discrete dynamics. When details of its discrete behavior are ignored and instead all possible switching combinations from a certain class are considered, a subclass, called switched systems, is obtained, see (Liberzon, 2003). They present a finite set of subsystems and a switching function (or rule) responsible for selecting one of them at each instant of time. This rule can be a disturbance in the system or a control variable to be designed. In the first situation, the purpose is to ensure stability of the overall system for any arbitrary switching rule. In the second, when all subsystems are unstable, the interest lies in designing a switching rule able to ensure stability and suitable performance. Otherwise, if at least one of the subsystems is stable, the rule can still be designed to improve the system overall performance. In this work, the goal is to treat the case where the switching rule is the unique control variable of the system. In both situations, it is common the presence of sliding modes in the system. This behavior is intrinsic of the switched systems and is characterized by a particular dynamic, different from that of each isolated subsystem. Although, in several cases, sliding modes are important to ensure asymptotic stability, they imply in a arbitrarily high switching frequency, also known as chattering, which can damage physical components that are not designed to work under such conditions, (Liberzon, 2003; Sun; Ge, 2011).

Besides the theoretical challenges, the great interest in the study of this class of systems arises from the wide number of practical applications, as this class is able to represent several real systems, going from power electronics and electric machines to automotive control, network control, aircraft and air traffic control, and computer disks.

2.4.1 Switched Linear Systems

A switched linear system is composed of a finite number of subsystems that are linear and time-invariant. They can be described by the following state space representation

$$\dot{x}(t) = A_\sigma x(t), \;\; x(0) = x_0 \tag{2.18}$$

$$z(t) = E_\sigma x(t) \tag{2.19}$$

where $x \in \mathbb{R}^n$ is the state, $z \in \mathbb{R}^p$ is the controlled output and $\sigma(t) = u(x(t)) : \mathbb{R}^n \to \mathbb{K} := \{1, \ldots, N\}$ is the state-dependent switching rule that selects one among N available subsystems at each instant of time. Notice that, if the matrices A_i, $\forall i \in \mathbb{K}$ are non singular, the origin $x_e = 0$ is the unique equilibrium point shared by all the subsystems. The following definition defines the convex combination of matrices with the same dimension and will be important afterward.

Definition 2.2. The subscript X_λ defines the linear combination of matrices $\{X_1, \ldots, X_N\}$ as

$$X_\lambda = \sum_{i \in \mathbb{K}} \lambda_i X_i \tag{2.20}$$

where

$$\Lambda := \left\{ \lambda \in \mathbb{R}^N \ : \ \lambda_j \geq 0, \sum_{j \in \mathbb{K}} \lambda_j = 1 \right\} \tag{2.21}$$

is the unit simplex.

For this class of switched systems, several results have been proposed in the literature, going from stability analysis (Geromel; Colaneri, 2006b; Lin; Antsaklis, 2009; Shorten *et al.*, 2007; Scharlau *et al.*, 2014), state and output feedback control (Geromel *et al.*, 2008; Deaecto *et al.*, 2011a; Duan; Wu, 2014) to robust and linear parameter varying (LPV) control (Geromel; Deaecto, 2009; Deaecto *et al.*, 2011b; Zhai, 2001). They differ among them from the adoption of different Lyapunov functions, as the quadratic in (Zhai, 2001), min-type in (Geromel; Colaneri, 2006b; Deaecto *et al.*, 2011a; Duan; Wu, 2014) and max-type in (Scharlau *et al.*, 2014), leading to sufficient conditions with different degrees of conservatism. The simplest one is the quadratic Lyapunov function $v(x) = x'Px$, whose stability conditions are given in the next theorem which is also available in (Deaecto *et al.*, 2010).

Theorem 2.1. *Consider the switched linear system* (2.18)-(2.19) *and define $Q_i = E_i' E_i$. If there exist a vector $\lambda \in \Lambda$ and a symmetric positive definite matrix $P > 0$ satisfying*

$$A_\lambda' P + P A_\lambda + Q_\lambda < 0 \tag{2.22}$$

then, the switching rule $\sigma(t) = u(x(t))$ with

$$u(x) = \arg\min_{i \in \mathbb{K}} x'(A_i' P + P A_i + Q_i)x \tag{2.23}$$

makes the equilibrium point $x_e = 0$ globally asymptotically stable and ensures that the inequality

$$\|z\|_2^2 < x_0' P x_0 \tag{2.24}$$

is valid.

Proof. The proof is available in (Deaecto *et al.*, 2010) but will be repeated due to its importance for our context of study. It begins by evaluating the time derivative of the quadratic Lyapunov function $v(x) = x'Px$ along an arbitrary trajectory of the system, resulting in

$$\begin{aligned}\dot{v}(x) &= x'(A'_\sigma P + PA_\sigma + Q_\sigma)x - z'z \\ &= \min_{i\in\mathbb{K}} x'(A'_i P + PA_i + Q_i)x - z'z \\ &= \min_{\lambda\in\Lambda} x'(A'_\lambda P + PA_\lambda + Q_\lambda)x - z'z \\ &\leq x'(A'_\lambda P + PA_\lambda + Q_\lambda)x - z'z \\ &< -z'z \end{aligned} \tag{2.25}$$

where the second equality comes from the switching function (2.23), the third equality and the first inequality are due to the minimum operator, and the last inequality is a consequence from the validity of (2.22). Now, integrating both sides of

$$\dot{v}(x) < -z'z \tag{2.26}$$

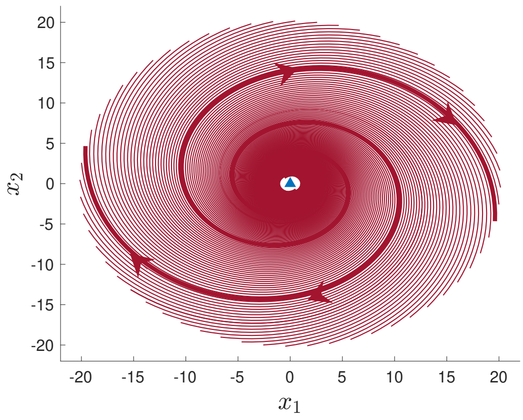
from $t = 0$ to $t \to \infty$, we obtain (2.24) since $\lim_{t\to\infty} v(x(t)) = 0$ as a consequence of the system asymptotic stability. □

Notice that, in the general case, with $\lambda \in \Lambda$ and $P > 0$ as variables, the conditions of this theorem are extremely difficult to solve. Indeed, finding $\lambda \in \Lambda$ such that A_λ is Hurwitz stable is known to be an NP-hard problem, see (Blondel; Tsitsiklis, 1997). However, for a $\lambda \in \Lambda$ given, the conditions become linear matrix inequalities (LMIs) being very simple to solve by off-the-shelf algorithms, see (Boyd *et al.*, 1994). Hence, we can solve Theorem 2.1 by searching the vector $\lambda \in \Lambda$ that provides the better guaranteed performance (2.24). The next numerical example illustrates the use of Theorem 2.1 in a switched linear system composed of two unstable subsystems.

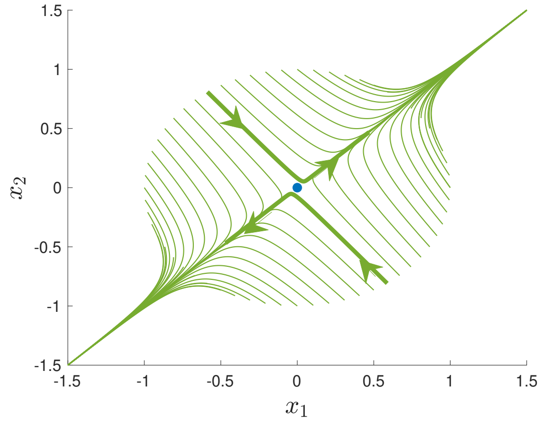
Example 2.4.1. Consider the switched linear system (2.18)-(2.19) defined by the matrices

$$A_1 = \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2.5 & 3.5 \\ 4.5 & -3.5 \end{bmatrix}, \; E_1 = E_2 = I \tag{2.27}$$

with the respective eigenvalues $\gamma_i(A_1) \in \{1 + 5\sqrt{-1},\ 1 - 5\sqrt{-1}\}$ and $\gamma_i(A_2) \in \{1,\ -7\}$ for $i = 1, 2$. Notice that both subsystems are unstable. The first subsystem has an unstable focus at the origin, while the second has a saddle at the origin, as illustrated in the phase portraits of Figure 2.1.



(a) Subsystem 1



(b) Subsystem 2

Figure 2.1 – Phase plane of individual subsystems

For a given $\lambda \in \Lambda$, we have solved the optimization problem

$$\inf_{P>0,\ \rho>0} \rho \tag{2.28}$$

subject to the LMI (2.22) and

$$P - \rho I < 0 \tag{2.29}$$

We have made a search on the vector $\lambda \in \Lambda$, choosing the one that provides the smallest guaranteed cost (2.28). Notice that, this cost is more conservative than (2.24) since the inequality

$$\|z\|_2^2 < x_0' P x_0 < \rho x_0' x_0 \tag{2.30}$$

is valid. However, the correspondent optimization problem is independent of initial conditions chosen around the same circumference, that is, $x_0 \in X_0$ where

$$X_0 = \{x_0 \in \mathbb{R}^n \ : \ \|x_0\| = r\} \tag{2.31}$$

for an arbitrary $r > 0$. The solution to this problem, associated to $\rho = 1.8854$, has provided $\lambda = [0.583\ 0.417]'$ and the positive definite matrix

$$P = \begin{bmatrix} 0.5140 & 0.2540 \\ 0.2540 & 1.8383 \end{bmatrix} \tag{2.32}$$

important to the implementation of the switching rule (2.23). Figure 2.2 provides the resulting phase portrait of the switched linear system for initial conditions taken in a circumference of radius $r = 8$, where the origin is pointed by the blue $\times$. We can observe that the state trajectories

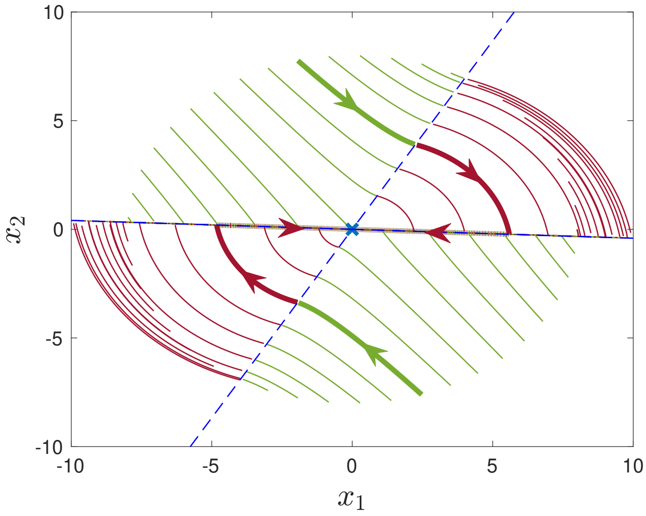


Figure 2.2 – Phase plane of controlled system

converge to the origin as expected. In this figure, the switching surface obtained as the locus of the equation

$$x'P(A_1 - A_2)x = 0 \tag{2.33}$$

is highlighted in dashed line, where it is possible to identify the sliding surface. Figure 2.3 shows the state trajectories for a initial condition $x_0 = [2.4721\ 7.6085]'$ with its associated switching function. We can observe that after 0.3 seconds the system evolves towards the origin in a sliding mode.

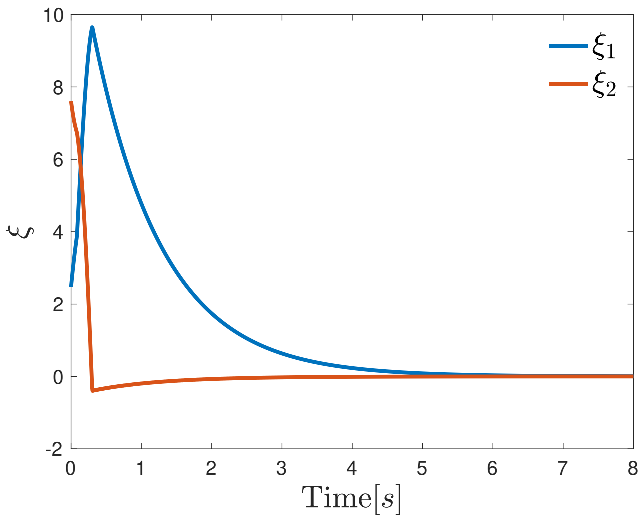


Figure 2.3 – Phase portrait of controlled system

2.4.2 Switched Affine Systems

Switched affine systems can be seen as an generalization of the switched linear systems, which are characterized by the presence of an affine term in its dynamical equation. This class is represented by the following state space realization

$$\dot{x}(t) = A_\sigma x(t) + b_\sigma, \;\; x(0) = x_0 \tag{2.34}$$

$$z(t) = E_\sigma x(t) \tag{2.35}$$

where $x \in \mathbb{R}^n$ is the state vector, $z \in \mathbb{R}^p$ is the controlled output and $\sigma(t) = u(x(t))$ is the switching function to be designed to guarantee global asymptotic stability of an equilibrium point x_e of interest. The matrices A_i and b_i represent the i-th available subsystem for all $i \in \mathbb{K} := \{1, \ldots, N\}$. The presence of the affine term results in a set of equilibrium points forming a region of great interest in the state space, given by

$$X_e = \{-A_\lambda^{-1} b_\lambda \; : A_\lambda \in \mathcal{H}, \; \lambda \in \Lambda\} \tag{2.36}$$

where $\mathcal{H}$ is the set of Hurwitz matrices. This characteristic makes the control problem substantially more complicated than the linear case, because besides the design of a global asymptotic stabilizing switching rule, it is also necessary to determine the set of attainable equilibrium points.

Usually, the equilibrium point x_e of interest does not match with any of the individual subsystems, therefore, an arbitrarily high switching frequency is required to ensure asymptotic stability. Given an equilibrium point of interest $x_e \in X_e$ and considering the auxiliary state vector $\xi = x - x_e$, we can rewrite the system (2.34)-(2.35) as

$$\dot{\xi}(t) = A_\sigma \xi(t) + \ell_\sigma, \;\; \xi(0) = \xi_0 \tag{2.37}$$

$$z_e(t) = E_\sigma \xi(t) \tag{2.38}$$

where $\ell_i = A_i x_e + b_i$ and $z_e = z - E_\sigma x_e$. This change of variable allows us to deal with the stabilization problem to the origin as the unique equilibrium point since $x(t) \to x_e$ whenever $\xi \to 0$. The literature presents some results dealing with this stabilization problem, as for instance (Deaecto *et al.*, 2010; Bolzern; Spinelli, 2004; Patino *et al.*, 2009). All of them adopting the quadratic Lyapunov function

$$v(\xi) = \xi' P \xi, \;\; P > 0 \tag{2.39}$$

The great difficulty in adopting a more general Lyapunov function, as the non-convex ones, is to deal with the reachability of the equilibrium point of interest. The reference (Baldi *et al.*, 2018) has proposed stability conditions based on a convex, but time-varying Lyapunov function, which is naturally less conservative than the ones based on (2.39) but more difficult to implement. The next theorem provides the stabilization conditions also available in (Deaecto *et al.*, 2010).

Theorem 2.2. *Consider the switched affine system* (2.34)-(2.35)*, the equilibrium point of interest $x_e \in X_e$ and its associated vector $\lambda \in \Lambda$ and define $Q_i = E_i' E_i$. If there exists a symmetric positive definite matrix $P > 0$ satisfying the LMI*

$$A_\lambda' P + P A_\lambda + Q_\lambda < 0 \tag{2.40}$$

then, the switching rule $\sigma(t) = u(\xi(t))$ with

$$u(\xi) = \arg\min_{i \in \mathbb{K}} \xi'(A_i' P + P A_i + Q_i)\xi + 2\xi' P \ell_i \tag{2.41}$$

and $\xi = x - x_e$, makes the equilibrium point $x_e \in X_e$ globally asymptotically stable and ensures that the inequality

$$\|z_e\|_2^2 < \xi_0' P \xi_0 \tag{2.42}$$

with $z_e = z - E_\sigma x_e$ is valid.

Proof. The proof is also available in (Deaecto *et al.*, 2010). Consider the equivalent switched affine system (2.37)-(2.38). We start the proof by evaluating the time derivative of (2.39) along an arbitrary trajectory of this system together with the proposed switching rule (2.41) obtaining

$$\begin{aligned}
\dot{v}(\xi) &= \dot{\xi}' P \xi + \xi' P \dot{\xi} \\
&= \xi'(A_\sigma' P + P A_\sigma + Q_\sigma)\xi + 2\xi' P \ell_\sigma - z_e' z_e \\
&= \min_{i \in \mathbb{K}} \xi'(A_i' P + P A_i + Q_i)\xi + 2\xi' P \ell_i - z_e' z_e \\
&= \min_{\lambda \in \Lambda} \xi'(A_\lambda' P + P A_\lambda + Q_\lambda)\xi + 2\xi' P \ell_\lambda - z_e' z_e \\
&\leq \xi'(A_\lambda' P + P A_\lambda + Q_\lambda)\xi + 2\xi' P \ell_\lambda - z_e' z_e \\
&< -z_e' z_e
\end{aligned} \tag{2.43}$$

where the third equality is due to the switching rule, the fourth equality and the first inequality come from the minimum operator and the last inequality is a consequence of (2.40) and of the fact that $\ell_\lambda = 0$ because $x_e \in X_e$. The guaranteed cost is obtained by the same procedure adopted in Theorem 2.1. □

Notice that the proposed conditions do not require any stability property of the isolated subsystems. Only the convex combination A_λ must be Hurwitz stable as a sufficient condition for stabilization. Moreover, different from Theorem 2.1 the conditions are linear matrix inequalities, since $\lambda \in \Lambda$ is not a variable but a given vector associated to the equilibrium point $x_e \in X_e$. The next example illustrates the main features of this control technique.

Example 2.4.2. Consider the switched affine system (2.34)-(2.35) given by the following matrices

$$A_1 = \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} -2.5 & 3.5 \\ 4.5 & -3.5 \end{bmatrix},\ b_1 = \begin{bmatrix} 10 \\ -24 \end{bmatrix},\ b_2 = \begin{bmatrix} -25 \\ 31 \end{bmatrix} \tag{2.44}$$

with $E_1 = E_2 = I$ and the respective equilibrium points

$$x_{e_\blacktriangle} = \begin{bmatrix} -5 \\ -1 \end{bmatrix},\ x_{e_\bullet} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \tag{2.45}$$

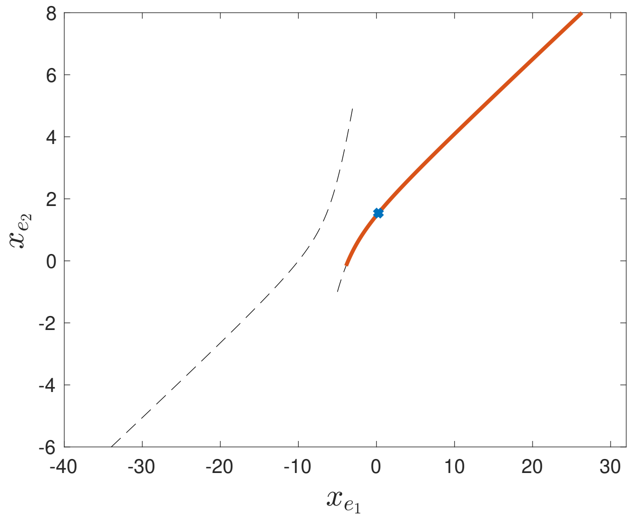
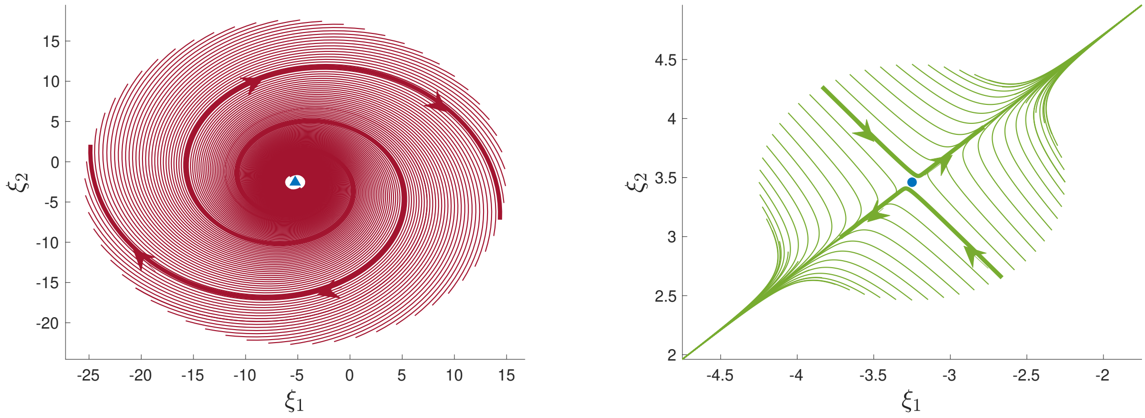


Figure 2.4 – Equilibrium points

The dynamical matrices are the same already adopted in the previous example, being the first subsystem characterized by an unstable focus at $x_{e_\blacktriangle}$ and the second one characterized by a saddle at $x_{e_\bullet}$. The set of attainable equilibrium points was calculated from (2.36) and illustrated in Figure 2.4, where the dashed line represents all the equilibrium points, while the red continuous line only those for which $A_\lambda \in \mathcal{H}$. We have chosen the equilibrium point $x_e = [0.2499 \ \ 1.5393]'$ associated with the vector $\lambda = [0.53 \ \ 0.47]'$. Figure 2.5 provides the phase portraits of the individual subsystems represented by the shifted variable $\xi = x - x_e$.



(a) Subsystem 1　　　(b) Subsystem 2

Figure 2.5 – Phase portraits of individual subsystems

The initial conditions adopted for the first and second subsystems were $\|x_0 - x_{e_\blacktriangle}\| < 1$ and $\|x_0 - x_{e_\bullet}\| < 1$, respectively. We have solved the optimization problem

$$\inf_{P>0,\rho>0} \rho \tag{2.46}$$

subject to the LMI (2.40) and (2.29), which is more conservative than (2.42) but, as discussed in (2.30), is independent of initial conditions chosen around the same circumference. We have obtained the guaranteed cost $\rho = 2.1613$ and the positive definite matrix

$$P = \begin{bmatrix} 0.4281 & 0.4184 \\ 0.4184 & 2.0603 \end{bmatrix} \tag{2.47}$$

important for the switching function implementation (2.41).

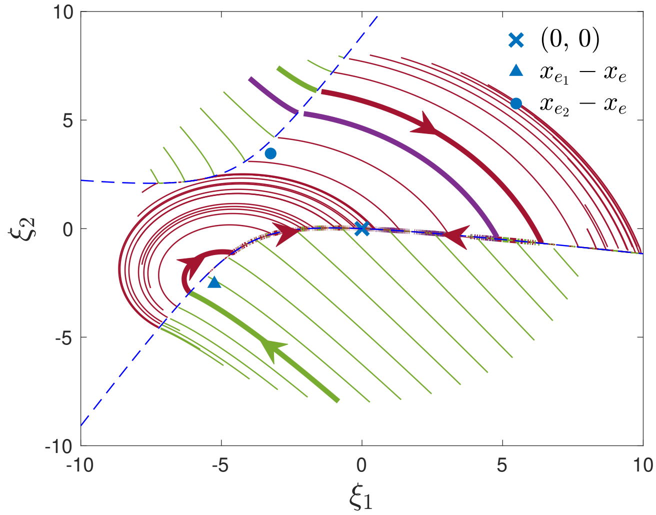


Figure 2.6 – Phase portrait of the controlled system

Choosing initial conditions such that $\|\xi_0\| < 8$ we have obtained the phase portrait of Figure 2.6. In this figure the colors red and green represent the actuation of the first and second subsystems, respectively, and the blue $\times$ indicates the origin. The dashed line corresponds to the switching surface, where the rule can select a different subsystem or makes the system to evolve on a sliding mode. Figures 2.7 and 2.8 shows, respectively, the state trajectories over time and its corresponding switching rule for $x_0 = [-4.00\ 6.9282]'$, highlighted in Figure 2.6 by the color purple. It is possible to see that around 0.425s, the switching signal assumes a arbitrarily high frequency, which characterizes the presence of sliding modes.

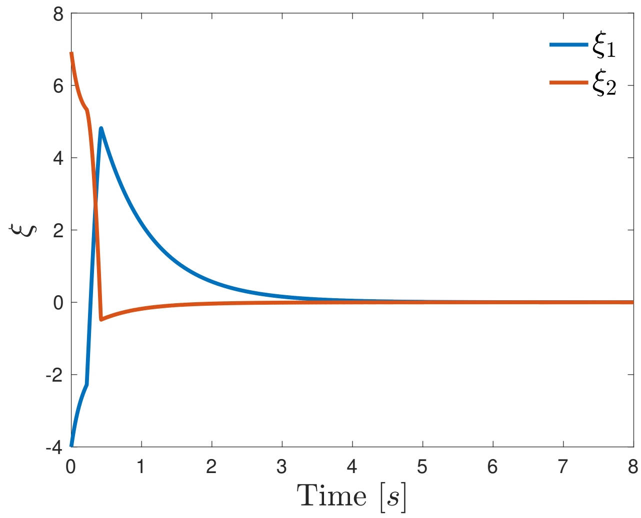


Figure 2.7 – State trajectories

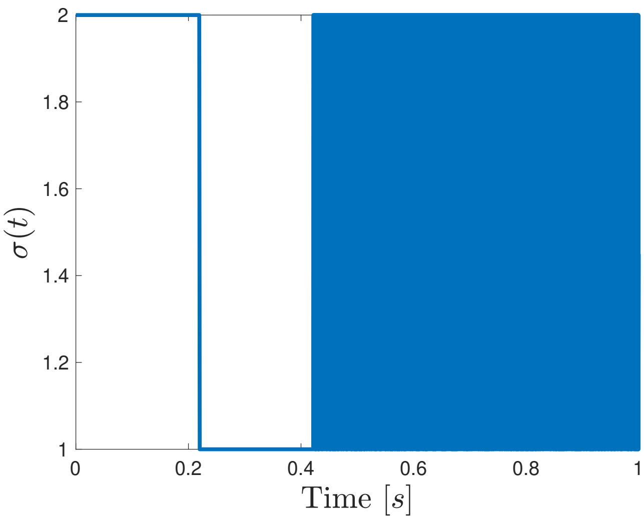


Figure 2.8 – Switching rule over time

The previous example was important to illustrate the main characteristics of the switching rule provided in Theorem 2.2. Notice that in (2.41), matrix $P > 0$ is dependent on the equilibrium point $x_e \in X_e$ through the vector $\lambda \in \Lambda$ and needs to be recalculated whenever a new point is chosen. This can be an inconvenient in practical implementations, as for instance, in the control of DC-DC power converters where it is very common changes in the operating point. The next theorem, also available in (Deaecto *et al.*, 2010), deals with the design of a switching rule that is more conservative than that of Theorem 2.2, but is robust with respect to change in the equilibrium points and can be a good alternative for practical implementations.

Theorem 2.3. *Consider the switched affine system* (2.34)-(2.35)*, an equilibrium point $x_e \in X_e$ and define $Q_i = E_i' E_i$. If there exists a symmetric positive definite matrix $P > 0$ satisfying the LMIs*

$$A_i' P + P A_i + Q_i < 0, \quad i \in \mathbb{K} \tag{2.48}$$

then, the switching rule $\sigma(t) = u(\xi(t))$ with

$$u(\xi) = \arg \min_{i \in \mathbb{K}} \xi' P \ell_i \tag{2.49}$$

and $\xi = x - x_e$, makes the equilibrium point $x_e \in X_e$ globally asymptotically stable and ensures that the inequality (2.42) *is satisfied with $z_e = z - E_\sigma x_e$.*

Proof. The proof, also available in (Deaecto *et al.*, 2010), follows the same steps of the one in Theorem 2.2. The time derivative of the Lyapunov function (2.39) along an arbitrary trajectory of system (2.37)-(2.38) provides

$$\begin{aligned} \dot{v}(\xi) &= \xi'(A_\sigma' P + P A_\sigma + Q_\sigma)\xi + 2\xi' P \ell_\sigma - z_e' z_e \\ &< \min_{i \in \mathbb{K}} 2\xi' P \ell_i - z_e' z_e \\ &\leq 2\xi' P \ell_\lambda - z_e' z_e \\ &= -z_e' z_e \end{aligned} \tag{2.50}$$

where the first inequality comes from the LMI (2.48) and from the switching function (2.49). The last inequality is due to the minimum operator and the last equality is a consequence of the fact that $\ell_\lambda = 0$ because $x_e \in X_e$. The guaranteed cost is obtained as in Theorem 2.2 concluding the proof. □

Notice that in this case, matrix P is robust with respect to changes in the equilibrium points, as a result, any equilibrium point $x_e \in X_e$ can be selected using the same matrix P. Besides, the switching rule is linear and simpler to implement in practical applications. Nevertheless, since the system needs to be quadratically stable, that is, all the subsystems must be

Table 2.1 – Buck-Boost converter parameters

V_o	100 V
R	2 Ω
L	500 μH
C_o	470 μF
R_o	50 Ω

stable and to admit the same solution $P > 0$, the problem becomes much more stringent than the one of Theorem 2.2. An alternative strategy which is also robust with respect to changes in the equilibrium points, but less stringent than (2.49), has been proposed in (Silva *et al.*, 2022). In this reference, several min-type switching rules were experimentally validated and compared, from the theoretical and practical viewpoints. The next example of practical appealing illustrates the main characteristics of the switching rule (2.49).

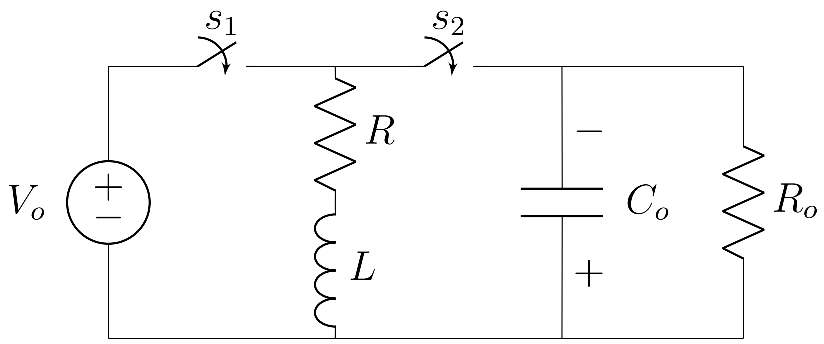


Figure 2.9 – Buck-Booster Inverter schematic

Example 2.4.3. Consider a buck-boost converter modelled as the switched affine system (2.34)-(2.35) with the state vector defined as $x = [i_L \ \ v_C]'$ and the matrices given by

$$A_1 = \begin{bmatrix} -R/L & 0 \\ 0 & -1/(R_oC_o) \end{bmatrix}, \ A_2 = \begin{bmatrix} -R/L & -1/L \\ 1/C_o & -1/(R_oC_o) \end{bmatrix}$$
$$b_1 = \begin{bmatrix} V_o/L \\ 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.51}$$

The electric circuit is presented in Figure 2.9 and the numerical parameters are provided in Table 2.1. The controlled output is defined with

$$E_1 = E_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.52}$$

The equilibrium points of each isolated subsystem is given respectivelly by

$$x_{e_\blacktriangle} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}, \ x_{e_\bullet} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.53}$$

Solving the optimization problem (2.46) subject to (2.48) and (2.29), we have obtained $\rho = 0.012$ associated to the positive definite matrix

$$P = \begin{bmatrix} 0.0109 & 0.0003 \\ 0.0003 & 0.0119 \end{bmatrix} \tag{2.54}$$

used for the implementation of the switching rule (2.49). Figure 2.10 presents the phase portrait of the switched affine system for several equilibrium points.

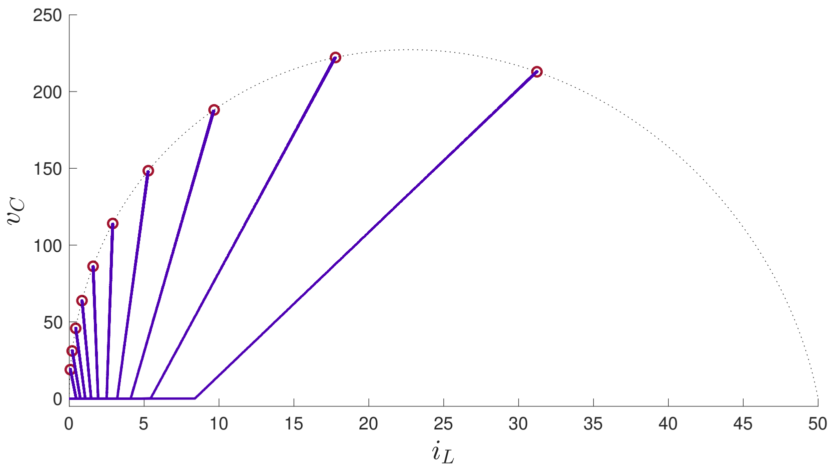


Figure 2.10 – Phase portrait of the controlled system

In this figure the dotted line represents the set of attainable equilibrium points X_e, and the purple continuous lines are the state trajectories evolving from the same initial condition $x_0 = [0 \;\; 0]'$. Notice that all the equilibrium points were attained with success. Figure 2.11 and 2.12 shows the time evolution of the current and voltage for the equilibrium point $x_e = [31.2258 \;\; 212.9032]'$. Observe that the occurrence of sliding modes from $t = 0.045ms$ onward is essential to make the trajectories fixed on the equilibrium point.

An interesting discussion is about robustness of the min-type switching rules with respect to parameter uncertainties, as load variations. In this case, the equilibrium point is not known, since it depends directly on the dynamical matrices of the system, which are subject to uncertainties. It is not simple to design a switching rule without the exact knowledge of this point, which make this problem very challenging from theoretical and practical viewpoints. Due to its complexity, there are only few works dealing with this topic, although it has attracted the attention of the scientific community in the last years, see the recent references (Baldi *et al.*, 2018), (Beneux *et al.*, 2019).

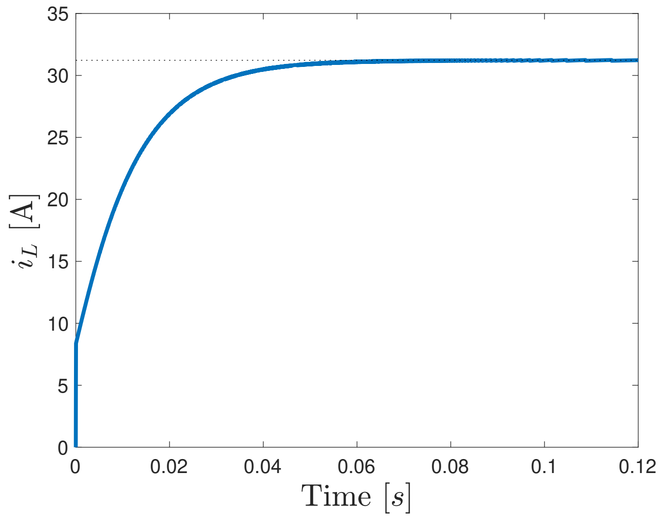


Figure 2.11 – i_L trajectory

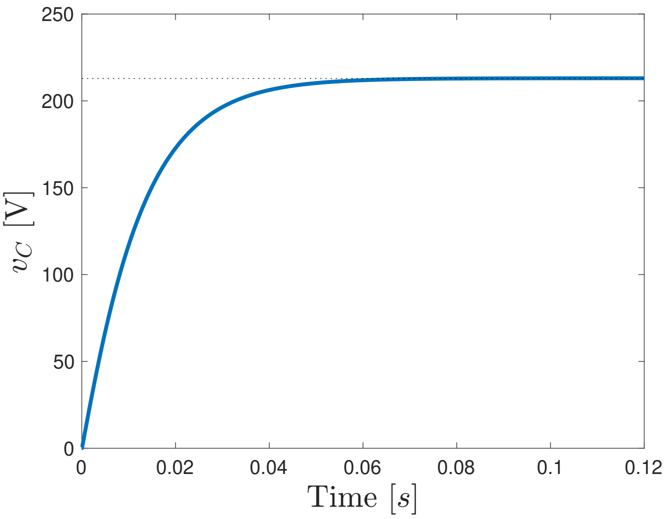


Figure 2.12 – v_C trajectory

2.5 Final Considerations

In this chapter some important concepts for the comprehension of this work were presented. First, we introduced the idea of equilibrium points and stability of dynamical systems. After that, the Lyapunov theory was presented, which is the basis for all theory developed in the subsequent chapters. For more details about those, see (Khalil, 2002),(Slotine *et al.*, 1991), and (Luenberger, 1979). Lastly, we made an introduction to the main features of switched systems, which can be further studied in references (Liberzon, 2003) and (Sun, 2006). We have focused on the switched linear and affine systems and presented several academic and practical examples to illustrate the most important concepts.

3 θ-PERIODIC SWITCHED SYSTEMS

In this chapter, the trajectory tracking problem of a class of switched systems is treated, which is the most important theoretical result of this dissertation. This class is more general than the previously studied switched affine systems and is characterized by presenting matrices in the dynamical equation that are periodic with respect to a parameter θ, describing two important subclass of systems: the parameter-dependent switched affine systems, when θ is an exogenous parameter and, the more challenging one, the switched nonlinear systems, when it is state-dependent. The obtained results are expressed in terms of linear matrix inequalities and take into account a guaranteed cost of performance. In order to consider robustness with respect to model uncertainties, making the problem more realistic to practical implementations, we have proposed a control with integral action. From the theoretical viewpoint, the difficulty in the integral control was to ensure the negative definiteness of the time derivative of the Lyapunov function when all the convex combinations of the dynamical matrices are singular. The results presented in this chapter are available in (Deaecto *et al.*, 2022).

As it will be clear in the next chapters, the theoretical results here presented have a relevant practical appeal and will be applied in two different areas of the electrical engineering: power electronics and electric machines.

3.1 Problem Formulation

Consider the θ-periodic switched affine system given by

$$\dot{x} = A_\sigma(\theta)x + b_\sigma(\theta), \quad x(0) = x_0 \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state and $\sigma \in \mathbb{K} := \{1, ..., N\}$ is the switching signal that selects, at each instant of time, one of the N available subsystems. The matrix-valued functions $A_i(\theta)$ and $b_i(\theta)$ are periodic and continuous with respect to the time-varying parameter $\theta(t) \ : \ \mathbb{R}_+ \to \mathbb{R}$, with period θ_P defined as $\theta_P = \min\{\hat{\theta} > 0 \ : \ X_i(\theta + \hat{\theta}) = X_i(\theta) \ \forall i \in \mathbb{K}, \ X_i = (A_i, b_i), \theta \in \mathbb{R}_+\}$. It is supposed that θ is an almost everywhere differentiable function and belongs to a bounded set $\dot{\theta} \in \Theta_D$ for all $t \geq 0$ with

$$\Theta_D = \{\dot{\theta} \in \mathbb{R} \ : \ \underline{\omega} \leq \dot{\theta} \leq \overline{\omega}, \ \underline{\omega}, \ \overline{\omega} \in \mathbb{R}\} \tag{3.2}$$

In our context, θ can be an exogenous parameter and, therefore, the switched affine system (3.1) is parameter-dependent or it can be state-dependent, making the switched system (3.1) highly nonlinear. The control methodology here proposed allows us to treat both cases with almost the same approach. However, when θ is an exogenous parameter, the asymptotic tracking guarantee is global and becomes local when it is state-dependent. Define $\Theta_P = [0, \theta_P]$ and let $x_e(t, \theta) \in \mathbb{R}^n$ be a desired trajectory to be tracked by the action of the switching function $\sigma(t)$. The trajectory profile $x_e(t, \theta) \in \mathbb{R}^n$ is chosen by the designer according to the system characteristics and must satisfy certain conditions to be presented afterward. With the auxiliary variable $\xi(t) = x(t) - x_e(t, \theta)$, the system (3.1) can be rewritten as

$$\dot{\xi} = A_\sigma(\theta)\xi + \ell_\sigma(t, \theta, \dot{\theta}), \; \xi(0) = \xi_0 \tag{3.3}$$

where $\ell_i(t, \theta, \dot{\theta}) = A_i(\theta)x_e(t, \theta) + b_i(\theta) - \dot{x}_e(t, \theta, \dot{\theta})$ and $\xi_0 = x_0 - x_{e0}$, with $x_{e0} = x_e(0)$.

The control goal is to design a state-dependent switching function $u(x, t, \theta, \dot{\theta}) : \mathbb{R}^n \times \mathbb{R}_+ \times \Theta_P \times \Theta_D \to \mathbb{K}$ such that the switching control $\sigma(t) = u(x(t), t, \theta(t), \dot{\theta}(t))$ guarantees the asymptotic tracking of the desired trajectory $x_e(t, \theta)$, that is

$$\lim_{t\to\infty} x(t) - x_e(t, \theta) = 0 \tag{3.4}$$

This is accomplished by means of design conditions that ensure the asymptotic stability of the origin $\xi = 0$ of (3.3) and the minimization of a suitable upper bound on the quadratic cost

$$\mathcal{J} = \int_0^\infty \xi(t)'Q\xi(t)dt \tag{3.5}$$

where $Q \geq 0$ is a given matrix.

Before continuing let us define the matrix $R(\theta)$ that will be important to obtain the design conditions.

Definition 3.1. Define the periodic matrix-valued function $R(\theta) : \Theta_P \to \mathbb{R}^{n\times n}$ with the following properties:

$$R(\theta)'R(\theta) = I, \;\; \dot{R}(\theta) = R(\theta)\Omega(\dot{\theta}) \tag{3.6}$$

where $\Omega(\dot{\theta}) \in \mathbb{R}^{n\times n}$ is linear with respect to $\dot{\theta}$ and skew-symmetric.

There are several matrices satisfying these properties, as for instance, the rotation matrices in the special orthogonal group $SO(n)$ (Gallier; Xu, 2003), widely used to model rotational dynamic systems, as the robotic ones. Another very useful class of matrices is composed of constant matrices $M \in \mathcal{M} \subset \mathbb{R}^{n\times n}$ for which the commutative property

$$MR(\theta) = R(\theta)M \tag{3.7}$$

is valid. Notice that the class $\mathcal{M}$ must be defined for each $R(\theta)$, since this property is dependent on the structure of this matrix that naturally satisfies the properties (3.6).

We are particularly interested in studying switched affine systems described by (3.1) and satisfying the following assumptions.

Assumptions: There must exist $\lambda(t, \theta, \dot{\theta}) \in \Lambda$ such that:

a) The trajectory $x_e(t, \theta)$ satisfies the equality

$$\ell_\lambda(t, \theta, \dot{\theta}) = 0 \tag{3.8}$$

for all $t \in \mathbb{R}_+$, $\theta \in \Theta_P$ and $\dot{\theta} \in \Theta_D$.

b) The vector λ that satisfies (3.8) must yield a convex combination of θ-dependent matrices $A_i(\theta),\ \forall i \in \mathbb{K}$, satisfying the identity

$$A_\lambda(\theta) = R(\theta)A_R R(\theta)' \tag{3.9}$$

with a suitable constant matrix $A_R \in \mathbb{R}^{n\times n}$.

These assumptions can be satisfied by various real world systems, as for instance, the AC-DC power converters and the permanent magnet synchronous machines (PMSM), which will be studied in the next two chapters, respectively. Under a similar perspective, using less general design conditions, the control of these systems have also been treated in (Egidio *et al.*, 2020) and (Egidio *et al.*, 2022a) . In the next two chapters, we present a comparison with the control techniques proposed in these references with some discussions to put in evidence the main features of the proposed control theory. The assumption (3.8) is not only important to define the class of admissible trajectories but also states a necessary condition for the differential equation (3.1) to have a stabilizable equilibrium at the origin $\xi = 0$, considering Filippov solutions, see (Liberzon, 2003). The assumption (3.9) is useful to obtain conditions expressed in terms of linear matrix inequalities (LMIs), being solved without difficulty by readily available algorithms in the literature, see the book (Boyd *et al.*, 1994).

In the next sections, our main results are presented. First, the trajectory tracking problem is treated taking into account the minimization of an upper bound of (3.5). In a second step, due to its importance in practical applications, the control with integral action is taken into account only for the case when θ is an exogenous parameter. It is more intricate than the previous one, since every convex combination of the dynamic matrices is rank deficient. Nevertheless, we provide design conditions that ensure that the time derivative of the Lyapunov function is always strictly negative. Both results will be particularized, in the next two chapters, to cope with the control of two electrical devices of distinct nature: a three-phase AC-DC power converter and a permanent magnet synchronous machine.

3.2 Tracking With Guaranteed Cost

Let us adopt the following Lyapunov function candidate

$$v(\xi,\theta) = \xi' P(\theta)\xi \tag{3.10}$$

with positive definite matrix

$$P(\theta) = R(\theta) P_R R(\theta)' \tag{3.11}$$

where matrix $0 < P_R \in \mathbb{R}^{n\times n}$ must satisfy some conditions to be presented in the next theorem. The time-derivative of (3.10) with respect to an arbitrary trajectory $(\xi(t), \theta(t))$ is given by

$$\begin{aligned} \dot{v}(\xi,\theta) &= \frac{\partial v(\xi,\theta)}{\partial \xi}' (A_\sigma(\theta)\xi + \ell_\sigma(t,\theta,\dot{\theta})) + \frac{\partial v(\xi,\theta)}{\partial \theta}\dot{\theta} \\ &= \xi' W_\sigma(\theta,\dot{\theta})\xi + 2\xi' P(\theta)\ell_\sigma(t,\theta,\dot{\theta}) \end{aligned} \tag{3.12}$$

with

$$W_i(\theta,\dot{\theta}) = A_i(\theta)' P(\theta) + P(\theta) A_i(\theta) + \dot{P}(\theta) \tag{3.13}$$

and $\dot{P}(\theta) = R(\theta)\mathrm{He}\{P_R \Omega(\dot{\theta})'\}R(\theta)'$. The associated state-dependent switching function $\sigma(t) = u(\xi(t), t, \theta(t), \dot{\theta}(t))$ is the following

$$u(\xi,t,\theta,\dot{\theta}) = \arg\min_{i\in\mathbb{K}} \xi' W_i(\theta,\dot{\theta})\xi + 2\xi' P(\theta)\ell_i(t,\theta,\dot{\theta}) \tag{3.14}$$

Notice that this switching function guarantees the asymptotic tracking to the equilibrium trajectory $x_e(t,\theta)$ whenever $\dot{v}(\xi,\theta) < 0$ for all $\xi \neq 0$ indicating that the origin $\xi = 0$ of (3.3) is an asymptotically stable equilibrium point. This can be ensured by design conditions expressed as the solution to a set of LMIs. The next theorem presents this important result.

Theorem 3.1. *Consider the matrix-valued function* $R(\theta)$ *defined in* (3.6)*, the set* $\dot{\theta} \in \Theta_D$ *provided in* (3.2) *and let* $0 \leq Q \in \mathcal{M}$ *be given. Assuming that system* (3.3) *and the equilibrium* *trajectory* $x_e(t,\theta)$ *satisfy the assumptions* (3.8) *and* (3.9)*, if there exists a matrix* $P_R > 0$ *solution to the linear matrix inequalities*

$$\text{He}\{P_R \mathcal{A}_R(\underline{\omega})\} + Q < 0 \tag{3.15}$$

$$\text{He}\{P_R \mathcal{A}_R(\overline{\omega})\} + Q < 0 \tag{3.16}$$

with $\mathcal{A}_R(\omega) = A_R + \Omega(\omega)'$ *then, the switching function* $\sigma(t) = u(\xi(t), t, \theta(t), \dot{\theta}(t))$ *given in* (3.14) *ensures the asymptotic stability of the origin* $\xi = 0$ *and that the following inequality*

$$\mathcal{J} < \text{Tr}(P_R Y) \tag{3.17}$$

is satisfied with $Y = R(\theta_0)'\xi_0\xi_0' R(\theta_0)$.

Proof. The time-derivative of the Lyapunov function defined in (3.10) along an arbitrary trajectory $(\xi(t), \theta(t))$ and under the action of the switching rule $\sigma(t) = u(\xi(t), t, \theta(t), \dot{\theta}(t))$ defined in (3.14) provides

$$\begin{aligned} \dot{v}(\xi, t, \theta, \dot{\theta}) &= \min_{i \in \mathbb{K}} \xi' W_i(\theta, \dot{\theta})\xi + 2\xi' P(\theta)\ell_i(t, \theta, \dot{\theta}) \\ &= \min_{\lambda \in \Lambda} \xi' W_\lambda(\theta, \dot{\theta})\xi + 2\xi' P(\theta)\ell_\lambda(t, \theta, \dot{\theta}) \\ &\leq \xi' W_\lambda(\theta, \dot{\theta})\xi + 2\xi' P(\theta)\ell_\lambda(t, \theta, \dot{\theta}) \\ &< -\xi' Q \xi < 0 \end{aligned} \tag{3.18}$$

where the second equality and the first inequality are due to the min operator and the last inequality follows from the assumptions that $\ell_\lambda(t, \theta, \dot{\theta}) = 0$ and that $A_\lambda(\theta)$ can be written as in (3.9). Indeed, considering $A_\lambda(\theta) = R(\theta) A_R R(\theta)'$ and using $P(\theta)$ given in (3.11), we obtain

$$\begin{aligned} W_\lambda(\cdot) &= R(\theta)\Big(\text{He}\{P_R \mathcal{A}_R(\dot{\theta})\} + Q\Big)R(\theta)' - R(\theta) Q R(\theta)' \\ &< -R(\theta) Q R(\theta)' = -Q \end{aligned} \tag{3.19}$$

where the inequality comes from the fact that $Q \in \mathcal{M}$ and, as a consequence, the equality $R(\theta) Q R(\theta)' = Q R(\theta) R(\theta)' = Q$ holds and from the validity of (3.15) and (3.16), which is equivalent to ensure that $\text{He}\{P_R \mathcal{A}_R(\dot{\theta})\} + Q < 0$ for some $\dot{\theta} \in \Theta_D$. Indeed, the linear dependence of $\Omega(\dot{\theta})$ with respect to $\dot{\theta}$ allows us to write $\Omega(\dot{\theta}) = \alpha\Omega(\underline{\omega}) + (1-\alpha)\Omega(\overline{\omega})$ for all $0 \leq \alpha \leq 1$ and, therefore, checking if the inequality is valid at these extrema is equivalent to check the same for all $\dot{\theta} \in \Theta_D$. Finally, integrating both sides of (5.1) from $t = 0$ to $t \to \infty$ we have

$$
\begin{aligned}
\mathcal{J} &< v(\xi_0, \theta_0) \\
&= \xi_0' R(\theta_0) P_R R(\theta_0)' \xi_0 \\
&= \mathrm{Tr}(P_R Y)
\end{aligned} \tag{3.20}
$$

where the last equality comes from the cyclic property of the trace function. The proof is concluded. □

The most important point about this theorem is that it provides a switching function $\sigma(\xi(t), t, \theta(t), \dot{\theta}(t))$ able to asymptotically track a trajectory $x_e(t, \theta)$ of interest by means of LMI conditions that do not depend explicitly on θ. Regardless of $\theta(t)$ being an exogenous or a state-dependent parameter, the design conditions in Theorem 3.1 guarantee asymptotic stability of the origin $\xi = 0$ under the assumption that $\dot{\theta} \in \Theta_D$. However, when θ is exogenous, the stability holds globally if the assumption is not violated whereas, when θ is state dependent, the boundedness of $\dot{\theta}$ may only be ensured locally. These aspects will be fully illustrated in the next two chapters through the control of the two already mentioned devices: the AC-DC power converter, where θ is an external parameter, and the permanent magnet synchronous machine where θ is state-dependent.

Another remark is that this result can be analyzed in the light of the Floquet theory for continuous-time periodic linear systems, see (Bittanti; Colaneri, 2009). Based on this theory, matrix $R(\theta(t))$ can be seen as a generalized Floquet transformation for an averaged system defined by $(A_\lambda(\theta), \ell_\lambda(t, \theta, \dot{\theta}))$ and given by

$$
\dot{\xi} = R(\theta) A_R R(\theta)' \xi \tag{3.21}
$$

whenever the assumptions (3.8)-(3.9) are satisfied. Indeed, with the similarity transformation $\xi(t) = R(\theta(t))\hat{\xi}(t)$, we obtain the equivalent system

$$
\dot{\hat{\xi}} = (A_R + \Omega(\dot{\theta})')\hat{\xi} \tag{3.22}
$$

whose stability is ensured by the validity of the LMIs (3.15)-(3.16), which, in turn, imply the stability of the system (3.21). Notice that our context is, however, more general than that of periodic systems, because the periodicity of the system (3.21) is not necessarily with respect to time itself but rather with the time-varying scalar $\theta(t)$. For the particular case where $\dot{\theta}(t) = \omega \in \mathbb{R}$ is constant for all $t \in \mathbb{R}$, matrix $A_R + \Omega(\omega)'$ is a Floquet factor and the inequalities of Theorem

3.1 are equivalent to the well-known Floquet-Lyapunov necessary and sufficient conditions for the stability of the system $\dot{\xi} = R(\omega t)A_R R(\omega t)'\xi$. Moreover, by means of the fundamental matrix $X(t) = R(\theta(t))e^{(A+\Omega(\omega)')t}$ the analytic solution of this system can be directly obtained as being

$$\xi(t) = R(\theta(t))e^{(A_R+\Omega(\omega)')t}R(\theta(0))'\xi(0) \tag{3.23}$$

These points are illustrated in the control of the AC-DC converter that will be treated in the next chapter where the case $\dot{\theta}(t) = \omega \in \mathbb{R}$ occurs. The next subsection provides conditions for the control with integral action of this class of systems.

3.3 Control With Integral Action

Consider the θ-periodic augmented system

$$\dot{\xi}_a = \begin{bmatrix} A_\sigma(\theta) & 0 \\ C_\perp R(\theta)' & 0 \end{bmatrix} \xi_a + \begin{bmatrix} \ell_\sigma(t,\theta,\dot{\theta}) \\ 0 \end{bmatrix} \tag{3.24}$$

with θ being an exogenous parameter such that $\dot{\theta}(t) = \omega$ is constant for all $t \in \mathbb{R}_+$. The augmented state vector is $\xi_a = [\xi' \;\; \xi_\perp']'$ with $\xi_\perp \in \mathbb{R}^m$. This system was obtained by adding the state variable

$$\xi_\perp = \int_0^t C_\perp R(\theta(\tau))'\xi(\tau)d\tau \tag{3.25}$$

to (3.3). Suppose that the assumptions (3.8)-(3.9) remain valid and that matrix $A_R + \Omega(\omega)$ is nonsingular for $\omega \in \Theta_D$. Notice that the augmented dynamic matrix is rank deficient and does not admit a Hurwitz stable convex combination of the matrices of the subsystems. Hence, the conditions provided in the previous subsection cannot be applied to the augmented system (3.24) since, due to the integrator (3.25), there will always exist persistent null eigenvalues. In this case, let us adopt the augmented Lyapunov function candidate

$$V(\xi,\theta) = \xi_a' \begin{bmatrix} P(\theta) & \bullet \\ P_x' R(\theta)' & P_\perp \end{bmatrix} \xi_a \tag{3.26}$$

with P_x and $P_\perp$ of appropriate dimensions. Let us also define the function

$$\begin{aligned} f_i(\cdot) &= \frac{\partial V}{\partial \xi_a}' \dot{\xi}_a \Big|_{\sigma=i} + \frac{\partial V}{\partial \theta}\dot{\theta} \\ &= \begin{bmatrix} \xi \\ \xi_\perp \\ 1 \end{bmatrix}' \begin{bmatrix} \mathrm{He}\{K_i(\theta)\} + \dot{P}(\theta) & \bullet & \bullet \\ U_i(\theta,\omega) & 0 & \bullet \\ \ell_i' P(\theta) & \ell_i' R(\theta) P_x & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \xi_\perp \\ 1 \end{bmatrix} \end{aligned} \tag{3.27}$$

with matrices $U_i(\theta,\omega) = P_{\mathrm{x}}'R(\theta)'A_i(\theta)+P_\perp C_\perp R(\theta)'+P_{\mathrm{x}}'\Omega(\omega)'R(\theta)'$ and $K_i(\theta) = P(\theta)A_i(\theta)+R(\theta)C_\perp' P_{\mathrm{x}}' R(\theta)'$. In this function, the dependency of ℓ_i with respect to (t,θ,ω) was omitted by simplicity. The next theorem provides conditions for the asymptotic stability of the augmented system (3.24) by means of the switching function $\sigma(t) = u(\xi_a,t,\theta,\omega)$ with

$$u(\xi_a,t,\theta,\omega) = \arg\min_{i\in\mathbb{K}} f_i(\xi_a,t,\theta,\omega) \tag{3.28}$$

and f_i given in (3.27). As it will be clear afterward, besides the requirements on the dynamical matrices, establishing conditions on the affine terms is essential to guarantee that the time-derivative of the Lyapunov function be strictly negative definite.

Theorem 3.2. *Consider the augmented system* (3.24) *with θ being an exogenous parameter, where $\dot\theta(t) = \omega \in \Theta_D$ is constant and let the matrix-valued function $R(\theta)$ be defined in* (3.6). *Assume that system* (3.3) *and the trajectory $x_e(t,\theta)$ satisfy the assumptions* (3.8)-(3.9) *and define the vector-valued function $g_i(t,\theta,\omega) = C_\perp(A_R+\Omega(\omega)')^{-1}R(\theta)'\ell_i(t,\theta,\omega)$. If the condition*

$$0 \in \mathrm{Int}(\mathrm{co}(\{g_i \,:\, i\in\mathbb{K}\})) \tag{3.29}$$

is verified for all $t\in\mathbb{R}$, $\theta\in\Theta_P$ and $\omega\in\Theta_D$ and there exist symmetric matrices P_R and $P_\perp$ solution to the inequalities

$$\begin{bmatrix} \mathcal{A}_R' P_R \mathcal{A}_R & \bullet \\ -P_\perp C_\perp & P_\perp \end{bmatrix} > 0 \tag{3.30}$$

$$\mathrm{He}\left\{\mathcal{A}_R'\left(\mathcal{A}_R' P_R \mathcal{A}_R - C_\perp' P_\perp C_\perp\right)\right\} < 0 \tag{3.31}$$

with $\mathcal{A}_R = A_R + \Omega(\omega)'$, then, for the augmented system (3.24)*, the switching function $\sigma(t) = u(\xi_a(t),t,\theta(t),\omega)$ given in* (3.28) *with*

$$P_{\mathrm{x}}' = -P_\perp C_\perp \mathcal{A}_R^{-1} \tag{3.32}$$

ensures asymptotic stability of the origin $\xi_a = 0$.

Proof. Adopting the Lyapunov function candidate (3.26), let us demonstrate that under an arbitrary trajectory of (3.24) and the switching function (3.28), we have $\dot V(\xi_a,t,\theta,\omega) < 0$ for all $t\geq 0$ and $\xi_a\neq 0$. From this point on, we omit the dependence of $\dot V$ with respect to (ξ_a,t,θ,ω) to ease the notation. From (3.27) and the switching function (3.28) we obtain

$$\begin{aligned} \dot V &= \min_{i\in\mathbb{K}} f_i(\xi_a,t,\theta,\omega) \\ &= \min_{\mu_*\in\Lambda} f_{\mu_*}(\xi_a,t,\theta,\omega) \\ &\leq f_\mu(\xi_a,t,\theta,\omega), \qquad \forall\mu\in\Lambda \end{aligned} \tag{3.33}$$

Let us consider two cases: $\xi \neq 0$ and $\xi = 0$. When $\xi \neq 0$, we have

$$\begin{aligned}\dot{V} &\leq f_\lambda(\xi_a, t, \theta, \omega) \\ &= \xi' R(\theta) \left(\mathrm{He}\{P_R \mathcal{A}_R - C_\perp' P_\perp C_\perp \mathcal{A}_R^{-1}\}\right) R(\theta)'\xi \\ &< 0 \end{aligned} \tag{3.34}$$

where it has been used $\lambda \in \Lambda$ that satisfies the assumptions (3.8)-(3.9) leading to $A_\lambda(\theta) = R(\theta) A_R R(\theta)'$ and $\ell_\lambda(t, \theta, \omega) = 0$. Also, it has been chosen P_{x} given in (3.32) which makes $U_\lambda(\theta, \omega)$ null in (3.27). Moreover, the inequality in (3.34) follows from the validity of (3.31) after multiplying to the right by $\mathcal{A}_R^{-1}$ and to the left by its transpose.

Now, for the case where $\xi = 0$ and using P_{x} given in (3.32), the function $f_i(\cdot)$ becomes

$$f_i(t, \theta, \omega) = -2\xi_\perp' P_\perp C_\perp \mathcal{A}_R^{-1} R(\theta)' \ell_i(t, \theta, \omega) \tag{3.35}$$

leading to the following time-derivative of the Lyapunov function

$$\begin{aligned}\dot{V} &= \min_{i \in \mathbb{K}} -2\xi_\perp' P_\perp g_i(t, \theta, \omega) \\ &= \min_{\mu_* \in \Lambda} -2\xi_\perp' P_\perp g_{\mu_*}(t, \theta, \omega) \\ &\leq -2\epsilon \xi_\perp' P_\perp \xi_\perp \\ &< 0 \end{aligned} \tag{3.36}$$

where the last inequality is due to the fact that $\epsilon \xi_\perp \in \mathrm{Int}(\mathrm{co}(\{g_i \ : \ i \in \mathbb{K}\}))$ for $\epsilon > 0$ arbitrarily small, which is implied by the condition $0 \in \mathrm{Int}(\mathrm{co}(\{g_i \ : \ i \in \mathbb{K}\}))$. From the two cases, we have $\dot{V} < 0$ for all $t \geq 0$ and $\xi_a \neq 0$. It is also important to observe that $V(\xi_a, \theta) > 0$ whenever the inequality

$$\begin{bmatrix} P_R & \bullet \\ P_{\mathrm{x}}' & P_\perp \end{bmatrix} > 0 \tag{3.37}$$

holds, since it becomes the Lyapunov matrix (3.26) when multiplied to the left by $\mathrm{diag}(R(\theta), I)$ and to the right by its transpose. This inequality with P_{x} given in (3.32) is ensured by (3.30) after multiplying it to the right by $\mathrm{diag}(\mathcal{A}_R^{-1}, I)$ and to the left by the transpose. The proof is concluded. □

This result is one of the first to provide conditions that ensure that $\dot{V} < 0$ for all $t \geq 0$ and $\xi_a \neq 0$ to θ-periodic switched affine systems, when the dynamic matrix is rank deficient. The stabilization problem for a simpler class of rank-deficient switched systems that does not take into account the periodicity with respect to the θ-parameter has been treated in (Egidio *et al.*, 2022b). Notice that the condition (3.29) concerning the affine terms is essential to accomplish this goal, in the sense that, otherwise, it would only be possible to ensure that $\dot{V} \leq 0$. The condition (3.29) can be checked by verifying if there exists $\mu(t, \theta, \omega) \in \text{Int}(\Lambda)$ such that $g_\mu = 0$ and $\text{rank}([g_i]_{i \in \mathbb{K}}) = m$. Fortunately, for several electrical devices, as the three phase AC-DC converter treated in the next chapter, the same vector λ that satisfies the assumptions (3.8)-(3.9) can be chosen such that $\lambda \in \text{Int}(\Lambda)$ and, therefore, the general condition is simplified by checking if $\text{rank}([g_i]_{i \in \mathbb{K}}) = m$ for all $t \in \mathbb{R}_+$, $\theta \in \Theta_P$ and $\dot{\theta} \in \Theta_D$. Notice that the dependency of g_i with respect to time is inherited by the equilibrium trajectory $x_e(t, \theta)$, which can be chosen by the designer to contain relevant information for a limited time interval $t \in [0, T_F)$. Thus, the rank condition can be verified by making a fine grid on the limited sets $[0, T_F) \times \Theta_p \times \Theta_D$. Unfortunately, at this moment, the integral action was studied only for the case where θ is an exogenous parameter and $\dot{\theta} = \omega$ is constant. The case where θ is state-dependent is more complicated and will be treated in future work. Although, this constraint exists, the developed theory still covers a great variety of systems as the three-phase AC-DC converters to be treated in the next chapter.

3.4 Final Considerations

In this chapter, was presented a general methodology for the control design of a switching rule capable of ensuring asymptotic tracking of a desired trajectory for switched systems whose dynamic matrices depend on a periodic parameter. When this parameter is exogenous, the design conditions ensure global asymptotic tracking and can be generalized to cope with integral action, making the technique robust with respect to model uncertainties and, therefore, more realistic. When this parameter is state-dependent, the system is highly nonlinear and the asymptotic tracking is only local. In this case, the control with integral action is more challenging and an important subject to future research. All the obtained conditions were expressed in terms of linear matrix inequalities and are simple to implement in several real systems. The next two chapters are dedicated to illustrate these results in the control of two electric circuits of distinct nature: a three-phase AC-DC converter considered in Chapter 4 and a permanent magnet synchronous machine in Chapter 5.

4 POWER ELECTRONICS APPLICATIONS

In this chapter, the theoretical results proposed in the previous one will be specialized to cope with the control of three-phase AC-DC power converters that can be modeled as a parameter dependent switched affine system. In this sense, a corollary is proposed to show that the solution of a simply Lyapunov equation is sufficient to regulate the output voltage of the converter to a desired DC value, ensuring unitary power factor. The obtained design conditions are also available in (Deaecto *et al.*, 2022) and are a generalization of the results proposed in (Egidio *et al.*, 2020), used here for the sake of comparison. This last reference takes into account a less general Lyapunov function and leads to a more conservative results. The same theory has been adopted to the control of DC-AC power converters and the results are available in (Costanzo *et al.*, 2021) indicating its applicability in the control of different three-phase power converters. An example shows the efficiency of the proposed theory and illustrates the validity of the integral action in case of load variations.

4.1 Three-Phase AC-DC Power Converter

At this moment, let us first define the vector functions

$$f(\theta) = \begin{bmatrix} \sin(\theta) \\ \sin(\theta - 2\pi/3) \\ \sin(\theta - 4\pi/3) \end{bmatrix}, \; g(\theta) = \begin{bmatrix} \cos(\theta) \\ \cos(\theta - 2\pi/3) \\ \cos(\theta - 4\pi/3) \end{bmatrix} \tag{4.1}$$

Notice that $h'f(\theta) = 0, \; \forall \theta \in \mathbb{R}$, with $h = [1 \; 1 \; 1]'$, allowing us to conclude that the vector function $f(\theta)$ belongs to a plane Π in $\mathbb{R}^3$ perpendicular to the vector h, formally defined as

$$\Pi = \{v \in \mathbb{R}^3 \; : \; h'v = 0\} \tag{4.2}$$

Consequently, along a trajectory of θ, the-time derivative of $f(\theta)$ belongs to the same plane Π, since $\dot{f}(\theta) = \dot{\theta} g(\theta)$ and $h'g(\theta) = 0$. It also follows that

$$f(\theta)'f(\theta) = 3/2, \;\; f(\theta)'g(\theta) = 0, \;\; g(\theta)'g(\theta) = 3/2 \tag{4.3}$$

which will be largely used afterwards.

This system has been borrowed from (Egidio *et al.*, 2020) and (Bouafia *et al.*, 2009). The former is a preliminary version of the present results, but based on a simpler Lyapunov

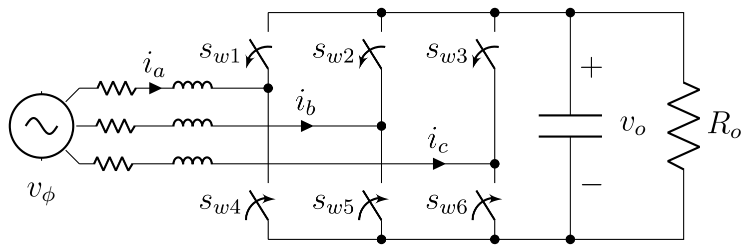


Figure 4.1 – Three-phase AC-DC power converter

Table 4.1 – Switching function σ, switches state and vector S_i

σ	s_{w1}	s_{w2}	s_{w3}	S'_σ
1	0	0	1	[-1/3 -1/3 2/3]
2	0	1	0	[-1/3 2/3 -1/3]
3	0	1	1	[-2/3 1/3 1/3]
4	1	0	0	[2/3 -1/3 -1/3]
5	1	0	1	[1/3 -2/3 1/3]
6	1	1	0	[1/3 1/3 -2/3]
7	1 0	1 0	1 0	[0 0 0]

function and without considering the integral action. The electric circuit is given in Figure 4.1 and its dynamic model is described by the following switched system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b(\theta(t)), \;\; x(0) = x_0 \tag{4.4}$$

where $x(t) = [i_\phi(t)' \;\; v_o(t)]' \in \mathbb{R}^4$ is the state, with $i_\phi(t) = [i_a(t)\; i_b(t)\; i_c(t)]' \in \mathbb{R}^3$ formed by the phase currents and $v_o(t) \in \mathbb{R}$ being the output voltage. The time-varying parameter $\theta(t)$ is supposed to be known and represents the electrical angle given by $\theta(t) = \omega t + \theta_0$ with constant angular frequency $\dot{\theta}(t) = \omega, \; \forall t \in \mathbb{R}_+$. The system matrices are given by

$$A_i = \begin{bmatrix} -(R_L/L)I_3 & -(1/L)S_i \\ (1/C)S_i' & -1/(R_oC) \end{bmatrix}, \; b(\theta) = \begin{bmatrix} (1/L)v_\phi \\ 0 \end{bmatrix} \tag{4.5}$$

where R_L and L are the resistance and inductance of each coupling inductor, C is the dc-link capacitance, R_o is the load resistance, $v_\phi = v_m f(\theta)$ is the input voltage and v_m is the peak phase-to-neutral voltage. The switching signal $\sigma(t) \in \mathbb{K} = \{1, \ldots, 7\}$ and the vectors $S_i \in \mathbb{R}^3, \; i \in \mathbb{K}$, take values according to Table 4.1. In this electric circuit, each pair (s_{w1}, s_{w4}), (s_{w2}, s_{w5}) and (s_{w3}, s_{w6}) is alternately commanded and $s_{wi}, \; \forall i = \{1, 2, 3\}$, is 1 when the switch is closed and 0 when it is open.

Notice that the electrical angle $\theta(t)$ is an external time-varying parameter. Moreover, the switching signal appears only in the dynamic matrix A_σ, whereas the parameter θ

appears only in the affine term $b(\theta)$. Due to the three-phase nature of the system we will adopt the matrix

$$R(\theta) = \begin{bmatrix} \sqrt{2/3}f(\theta) & \sqrt{2/3}g(\theta) & \sqrt{1/3}h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.6}$$

which satisfies the properties (3.6) with

$$\varOmega(\omega) = \begin{bmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.7}$$

The equilibrium trajectory of interest is of the form

$$x_e(\theta(t)) = \begin{bmatrix} i_* f(\theta(t)) \\ v_{o*} \end{bmatrix} \tag{4.8}$$

where the pair (i_*, v_{o*}) has to belong to a suitable set X_e, which fulfills the assumptions (3.8)-(3.9). This equilibrium trajectory is responsible to regulate the output voltage to a desired value v_{o*}, ensuring unitary power factor, where the phase currents must track a sinusoidal reference $i_{\phi*}(t) = i_* f(\theta(t))$ synchronized with the source phase voltages.

Adopting the auxiliary variable $\xi(t) = x(t) - x_e(\theta)$ we obtain the equivalent system

$$\dot{\xi} = A_\sigma \xi + \ell_\sigma(\theta, \omega), \ \xi(0) = \xi_0 \tag{4.9}$$

where $\ell_i(\theta, \omega) = A_i x_e(\theta) + b(\theta) - \dot{x}_e(\theta, \omega)$ and $\xi_0 = x_0 - x_{e0}$, with $x_{e0} = x_e(0)$. The next corollary particularizes Theorem 3.1 to cope with the control of the AC-DC power converter.

Corollary 4.1. Consider the switched affine system (4.4) defined by matrices (4.5), with electrical angle $\theta(t) = \omega t + \theta_0$, the matrix-valued function $R(\theta)$ given in (4.6) with the associated matrix $\varOmega(\omega)$ given in (4.7) and let $0 \leq Q \in \mathcal{M}$ be given. The equilibrium trajectory $x_e(\theta(t))$ given in (4.8), with (i_*, v_{o*}) belonging to the set

$$X_e = \mathcal{C}_1 \bigcap \mathcal{C}_2 \tag{4.10}$$

where

$$\mathcal{C}_1 = \{(i_*, v_{o*}) : (v_m - R_L i_*)^2 + (L\omega i_*)^2 \leq v_{o*}^2/3\} \tag{4.11}$$

$$\mathcal{C}_2 = \{(i_*, v_{o*}) : R_L i_*^2 - v_m i_* + 2v_{o*}^2/(3R_o) = 0\} \tag{4.12}$$

allows the system (4.4) to satisfy the assumptions (3.8)-(3.9) with

$$A_R = \begin{bmatrix} -R_L/L & 0 & 0 & \beta v_d/L \\ 0 & -R_L/L & 0 & \beta\omega i_* \\ 0 & 0 & -R_L/L & 0 \\ -\beta v_d/C & -\beta L\omega i_*/C & 0 & -1/(R_oC) \end{bmatrix} \tag{4.13}$$

where $\beta = \sqrt{6}/(2v_{o*})$ and $v_d = R_L i_* - v_m$. If there exists a matrix $P_R > 0$ solution to the Lyapunov equation

$$\mathrm{He}\{P_R(A_R + \Omega(\omega)')\} + Q = -\varepsilon I \tag{4.14}$$

with $\epsilon > 0$ arbitrarily small, then the switching function $\sigma(t) = u(\xi(t), \theta(t), \omega)$ given in (3.14) ensures the global asymptotic stability of the origin $\xi = 0$ of (3.3) and satisfies the inequality

$$\mathcal{J} < \mathrm{Tr}(P_R Y) \tag{4.15}$$

with $Y = R(\theta_0)'\xi_0\xi_0' R(\theta_0)$.

Proof. First of all, let us show that the assumptions (3.8)-(3.9) are satisfied. Observe that vectors $S_i, \forall i \in \mathbb{K}$, lie in the same plane Π of $f(\theta)$ and $g(\theta)$ defined in (4.2), since $h'S_i = 0,\ \forall i \in \mathbb{K}$. Moreover, they form the vertices of a regular hexagon defined as

$$\mathbb{P} = \{v \in \mathbb{R}^3 \ :\ v = \sum_{i\in\mathbb{K}} \lambda_i S_i,\ \forall \lambda \in \Lambda\} \tag{4.16}$$

From $\ell_i(\theta, \dot\theta) = A_i x_e(\theta) + b(\theta) - \dot x_e(\theta)$ we have

$$\ell_\lambda(\theta, \dot\theta) = \begin{bmatrix} -(1/L)\left(v_d f(\theta) + L\omega i_* g(\theta) + v_{o*}S_\lambda\right) \\ (1/C)\left(i_* S'_{\lambda(\theta)} f(\theta) - v_{o*}/R_o\right) \end{bmatrix} \tag{4.17}$$

and the first row is null whenever we choose $\lambda(\theta, \omega) \in \Lambda$ such that

$$S_\lambda = \frac{v_m - R_L i_*}{v_{o*}} f(\theta) - \frac{L\omega i_*}{v_{o*}} g(\theta) \tag{4.18}$$

Now, it is important to analyze under which conditions this choice is valid. Notice that S_λ is a linear combination of vectors $\{f(\theta), g(\theta)\}$ and, therefore, also belongs to the same plane Π. Hence, for each θ and ω it is always possible to find a convex combination $\lambda(\theta, \omega) \in \Lambda$ satisfying (4.18) if $S_\lambda \in \mathbb{F}$ where

$$\mathbb{F} = \{v \in \mathbb{R}^3 \ :\ h'v = 0,\ v'v \le r^2\} \tag{4.19}$$

Table 4.2 – AC-DC converter parameters

R_o	175 Ω
R_L	0.56 Ω
L	19.5 mH
ω	$2\pi \times 50$ rad/s
C	2.35 mF
v_M	40.825 V

with $r = 1/\sqrt{2}$ being the radius of the greatest circumference with center at the origin inside the hexagon $\mathbb{P}$. Hence, the set $\mathcal{C}_1$ provides the pairs (i_*, v_{o*}) for which $\|S_\lambda\|^2 \leq 1/2$. On the other hand, the pairs (i_*, v_{o*}) in $\mathcal{C}_2$ make null the second row of (4.17) after adopting S_λ given in (4.18). Thus, the equilibrium trajectory $x_e(\theta)$ defined in (4.8), with (i_*, v_{o*}) belonging to X_e, ensures that the assumption (3.8) is satisfied. Replacing S_λ of (4.18) in the matrix $A_\lambda(\theta)$ and rearranging the terms, we verify that the assumption (3.9) is satisfied with A_R given in (4.13). Now, as $\dot{\theta} = \omega$ is constant, the inequalities (3.15)-(3.16) of Theorem 3.1 are replaced by the Lyapunov equation (4.14). The switching function and the guaranteed cost follow from Theorem 3.1 concluding thus the proof. □

It is important to highlight that the control methodology is based on the solution of a simple Lyapunov equation and ensures global asymptotic tracking to $x_e(\theta)$ whenever $\omega \in \Theta_D$. Moreover, this corollary provides a design condition that is less conservative than the LMIs proposed in (Egidio *et al.*, 2020) due to the adoption of a Lyapunov function that is dependent on a more comprehensive matrix $R(\theta)$.

To take into account robustness with respect to model uncertainties, the control with integral action can be obtained directly by the conditions of Theorem 3.2, which are LMIs since $\dot{\theta} = \omega$ is constant for all $t \geq 0$. It is important to recall that the equilibrium trajectory $x_e(\theta)$ must belong to the set (4.11) and A_R is given in (4.13) in order to satisfy the assumptions (3.8)-(3.10).

4.2 Simulation Results

Consider the AC-DC converter with its electric circuit presented in Figure 4.1, whose mathematical model is described by the system (4.4) with matrices (4.5). The numerical parameters were borrowed from (Bouafia *et al.*, 2009) and given in Table 4.2. As already mentioned, the same converter has been studied in the reference (Egidio *et al.*, 2020), which is a preliminary version of the present results, but adopting a less general Lyapunov function and without taking into account the integral action. The main control goal is to regulate the output

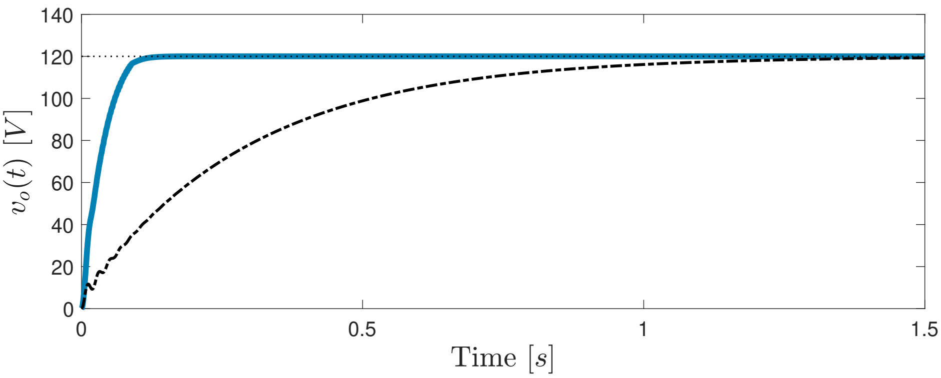


Figure 4.2 – Output voltages from (3.14) and from the averaged system

voltage to a steady-state value of $v_{o*} = 120$ V ensuring unitary power factor for the system. Notice that the pair (i_*, v_{o*}) with $i_* = 1.3694$ A belongs to the set X_e defined in (4.11) and, therefore, the trajectory (4.8) with these values is attainable.

Solving the conditions of Corollary 4.1 with $Q = \text{diag}(0, 0, 0, 1)$, $x(0) = 0$ and $\theta(0) = 0$, we have obtained the same results provided in the preliminary paper (Egidio *et al.*, 2020). However, solving the optimization problem

$$\mathcal{J} < \min_{P_R > 0} \text{Tr}(P(0)) = \text{Tr}(P_R) \tag{4.20}$$

with $P_R > 0$ being the solution of the Lyapunov equation (4.14), we obtained a guaranteed cost of $\mathcal{J} < 0.1851$ while the results of reference (Egidio *et al.*, 2020) have provided a guaranteed cost of $\mathcal{J} < 0.2309$, which is 25% greater. This puts in evidence that the proposed conditions in Corollary 4.1 are more general than the ones of (Egidio *et al.*, 2020). An interpretation for the upper bound (4.20) is that it is valid for initial conditions such that $||\xi_0|| \leq 1$. Indeed, we have

$$\xi_0' P(0) \xi_0 \leq \max_{||\xi_0|| \leq 1} \xi_0' P(0) \xi_0 = \gamma_{max}(P(0)) \leq \text{Tr}(P(0)) \tag{4.21}$$

where $\gamma_{max}(P(0))$ is the maximum eigenvalue of $P(0)$. Figure 4.2 shows in solid line the output voltage obtained through the solution of (4.20) and in dot-dashed line the one obtained by the averaged system (3.21). This puts in evidence the importance of the closed-loop switching function to performance enhancement.

Let us now change our focus to the control with integral action, which has not been treated in (Egidio *et al.*, 2020), and not fully explored in the literature to date. Consider the augmented system (3.24) with

$$C_\perp = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.22}$$

for which the state (3.25) becomes

$$\xi_\perp = \int_0^t v_o(\tau) - v_{o*}\, d\tau \tag{4.23}$$

enforcing $v_o(t) \to v_{o*}$ robustly, whenever the system is stable.

Adopting A_R given in (4.13), and taking into account that for the AC-DC converter, the term $g_i(\theta, \omega)$ is a function of $\theta \in \Theta_P$ and $\dot{\theta} = \omega,\ \forall t \geq 0$, we verified the condition (3.29) of Theorem 3.2 by confirming that $\mathrm{rank}([g_i]_{i\in\mathbb{K}}) = m = 1$ for all $\theta \in \tilde{\Theta}_P \subset \Theta_P$, where $\tilde{\Theta}_P$ is obtained by a sufficiently refined grid of Θ_P. As discussed just after Theorem 3.2, this is possible, because the same vector λ that satisfies the assumptions (3.8)-(3.9) is such that $\lambda \in \mathrm{int}(\Lambda)$. Indeed, notice that the chosen pair (i_*, v_{o*}) is associated to $\lambda(\theta, \omega) \in \Lambda$ such that $\|S_\lambda\|^2 < 1/2$, which corresponds to a circle that lies in the interior of the regular hexagon (4.16), implying that $\lambda \in \mathrm{int}(\Lambda)$. Moreover, matrices

$$P_R = \begin{bmatrix} 0.1840 & 0.1384 & 0.0000 & 0.2580 \\ 0.1384 & 0.5960 & 0.0000 & 0.8520 \\ 0.0000 & 0.0000 & 1.3943 & 0.0000 \\ 0.2580 & 0.8520 & 0.0000 & 1.6105 \end{bmatrix}, P_\perp = 10.0082 \tag{4.24}$$

represent a feasible solution to the LMIs (3.30)-(3.31) and will be used to implement the switching function (3.28) with $f_i(\cdot)$ defined in (3.27). As it is already known in the literature, the integral controller is very useful to ensure robustness w.r.t. the steady-state performance, but can deteriorate considerably the transient response, since it generally causes overshoot and a great settling time as a consequence of the depletion and refilling process necessary to make $\xi_\perp \to 0$ in (4.23), see (Beerens *et al.*, 2019) and (Prieur *et al.*, 2018) for details. An important result to circumvent this problem has been proposed in (Clegg, 1958) and consists in resetting the state $\xi_\perp(t)$ whenever $v_o(t)$ reaches the set point v_{o*}. Inspired on this result we have added the following reset condition to the augmented system (3.24)

$$\xi_\perp(t_+) = 0, \;\; \text{f} \;\; (\mathrm{sign}(\xi_\perp(t))\delta + \Delta v_o(t))\, \xi_\perp(t) < 0 \tag{4.25}$$

where $\Delta v_o = v_o - v_{o*}$ and $\delta > 0$ is adopted to avoid unnecessary resets caused by numerical oscillations or small perturbations enforcing the system to respect an interval $v_{o*} \pm \delta$ where the reset does not occur.

We have implemented the switching function (3.28) with matrices (4.24) considering that during the time interval $t \in [4, 8)$ [s] the load was abruptly reduced to $0.7R_o$ and augmented in the sequel to $1.3R_o$ during the time interval $t \in [8, 12)$. Figure 4.3 presents the output voltages for the pure integral control in dashed-dot lines, and for the reset integral control in continuous lines, with the associated state variables $\xi_\perp$.

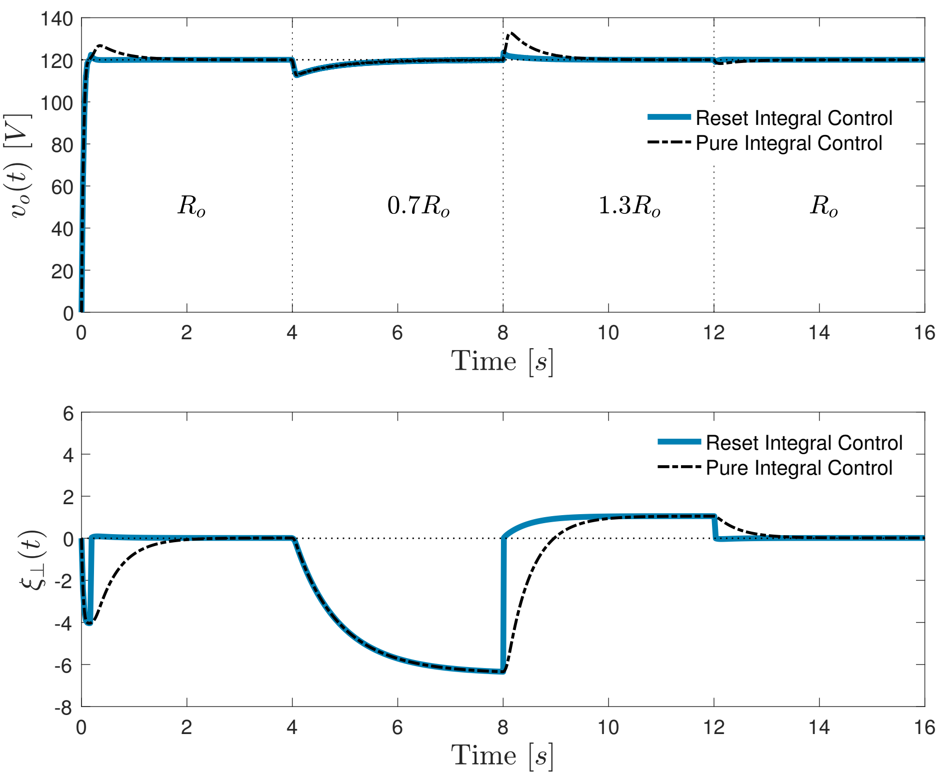


Figure 4.3 – Output voltages and the corresponding $\xi_\perp$

Notice that in both cases the switching strategy successfully brought the output voltage $v_o(t)$ to 120 [V] as desired, even during the intervals where the system suffered perturbations. Also, it is noticeable that the simple reset action proposed in (Clegg, 1958) reduced considerably the overshoot and the settling time. The literature provides other reset strategies, which could be more efficient for the present case, see (Beerens *et al.*, 2019; Zhao; Wang, 2016) as some examples. However, this point is not our focus at this moment and a deep study about reset techniques will be left for future work. Figure 4.4 presents the a and b phase currents, only for the reset integral control, with a highlight showing that they converge to the reference current presented in dashed line. The associated switching function is also presented in the same figure with a highlight in the transient response.

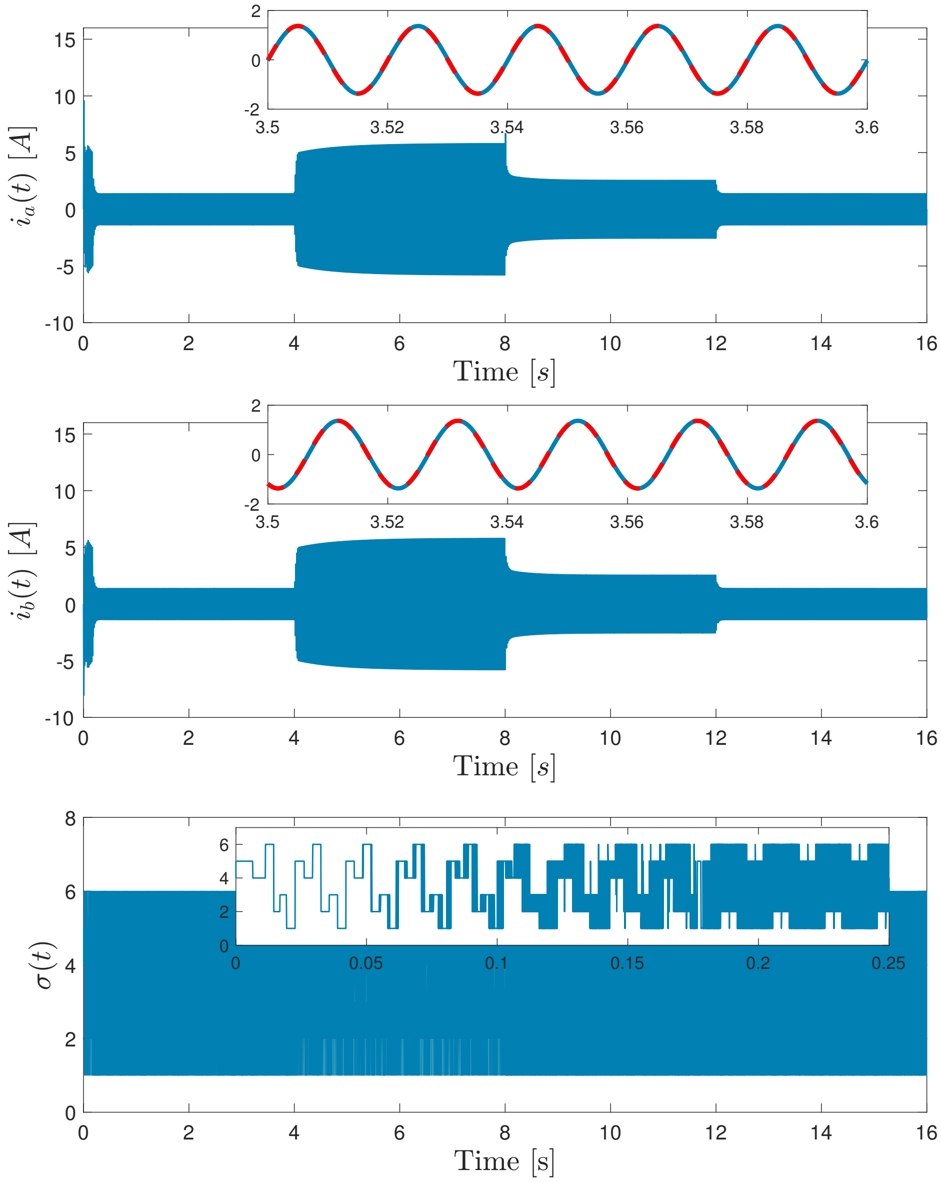


Figure 4.4 – Phase currents (a and b) and the corresponding switching function

4.3 Final Considerations

In this chapter, the conditions obtained in Chapter 3 were applied in an AC-DC power converter, whose dynamic model was borrowed from (Egidio *et al.*, 2020). The design conditions have been described as the solution of a simple Lyapunov equation and were efficient to ensure DC output voltage regulation and unitary power factor. Also, the guaranteed cost obtained was lower than the one provided by (Egidio *et al.*, 2020), which puts in evidence that the new conditions are in fact less conservative. Furthermore, we illustrated the efficacy of the switching rule with integral action by changing the nominal value of R_0 and showing that the switching rule was still capable of ensure asymptotic stability to same trajectory of interest.

5 ELECTRIC MACHINES APPLICATIONS

In this chapter, the theoretical results proposed in Chapter 3 will be specialized to cope with the control of a permanent magnet synchronous machine (PMSM). Differently from the previously studied AC-DC power converter, where θ was an exogenous parameter, here it is state-dependent making the switched system highly nonlinear and more complicated to obtain the design conditions. A corollary based on the general conditions of Theorem 3.1 provides design conditions that ensure local asymptotic stability of a trajectory of interest for the PMSM. The obtained results are also available in (Deaecto *et al.*, 2022) and are compared to the ones of the recent reference (Egidio *et al.*, 2022a), that treats the same problem but using a simpler Lyapunov condition. An experimental arrangement was elaborated to validate the control methodology and to show its efficiency with respect to others available in the literature.

5.1 Three-Phase PMSM

Consider the three-phase permanent magnet synchronous machine with n_p pairs of poles depicted in Figure 5.1 and borrowed from (Egidio *et al.*, 2019) and (Egidio *et al.*, 2022a). Its dynamic model is described by the following switched system

$$\dot{x}(t) = A(\theta(t))x(t) + b_{\sigma(t)},\ x(0) = x_0 \tag{5.1}$$

$$\dot{\theta}(t) = n_p\omega_M(t),\ \theta(0) = \theta_0 \tag{5.2}$$

where $x(t) = [i_\phi(t)'\ \omega_M(t)]' \in \mathbb{R}^4$ is the state with $i_\phi(t) = [i_a(t)\ i_b(t)\ i_c(t)]' \in \mathbb{R}^3$ and the pair $(A(\theta), b_i)$ is given by

$$A(\theta)=\begin{bmatrix} -(R_L/L)I_3 & -\lambda_M f(\theta)/L \\ \lambda_M f(\theta)'/J_M & -c_M/J_M \end{bmatrix},\ b_i = \begin{bmatrix} V_{dc}S_i/L \\ -\tau_M/J_M \end{bmatrix} \tag{5.3}$$

where R_L and L are the resistance and the equivalent inductance per phase, respectively, J_M is the rotor moment of inertia, λ_M is the peak value of the mutual flux linkage, c_M is the viscous friction coefficient and τ_M is the external constant torque. In this case, $\theta_M(t)$ is the shaft angular displacement of the machine and $\omega_M(t)$ is its correspondent angular velocity. According to (5.2), its relationship with the electrical angle θ is $\theta_M = \theta/n_p$. Let us define $\dot{\theta}(t) = \omega(t)$ which naturally is not constant and must belong to the set Θ_D defined in (3.2).

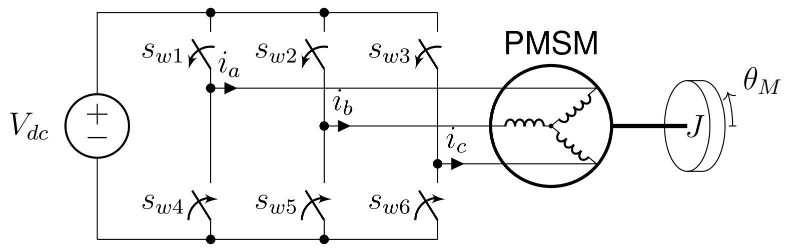


Figure 5.1 – PMSM and inverter schematic

Table 5.1 – Switching function σ, switches state and vector S_i

σ	s_{w1}	s_{w2}	s_{w3}	S'_σ
1	0	0	1	[-1/3 -1/3 2/3]
2	0	1	0	[-1/3 2/3 -1/3]
3	0	1	1	[-2/3 1/3 1/3]
4	1	0	0	[2/3 -1/3 -1/3]
5	1	0	1	[1/3 -2/3 1/3]
6	1	1	0	[1/3 1/3 -2/3]
7	1 0	1 0	1 0	[0 0 0]

The functions $f(\theta)$ and $g(\theta)$ are the same adopted in the previous chapter and will be repeated for convenience

$$f(\theta) = \begin{bmatrix} \sin(\theta) \\ \sin(\theta - 2\pi/3) \\ \sin(\theta - 4\pi/3) \end{bmatrix}, \; g(\theta) = \begin{bmatrix} \cos(\theta) \\ \cos(\theta - 2\pi/3) \\ \cos(\theta - 4\pi/3) \end{bmatrix} \tag{5.4}$$

Notice that they satisfy the properties (4.2) and (4.3) that will be also useful in this chapter. The switching signal $\sigma(t) \in \mathbb{K} = \{1, \ldots, 7\}$ and the vectors $S_i \in \mathbb{R}^3,\ i \in \mathbb{K}$, take values according to Table 5.1.

Notice that, the parameter θ is state-dependent because it is obtained from rotational velocity $\omega_M(t)$ in (5.2), which is one of the state variables. This case is more intricate than the one of the AC-DC converter, since the dependence of θ with respect to the state ω_M makes the PMSM model highly nonlinear. Moreover, unlike the converter, the parameter θ appears only in the dynamic matrix $A(\theta)$ while the switching signal appears only in the affine term b_σ. Again, let us adopt the matrix

$$R(\theta) = \begin{bmatrix} \sqrt{2/3}f(\theta) & \sqrt{2/3}g(\theta) & \sqrt{1/3}h & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.5}$$

which satisfies the properties (3.6) with

$$\Omega(\omega) = \begin{bmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.6}$$

The equilibrium trajectory is of the form

$$x_e(t,\theta) = \begin{bmatrix} i_*(t) f(\theta(t)) \\ \omega_{M*}(t) \end{bmatrix} \tag{5.7}$$

where the main goal is to track a pre-defined rotational velocity profile $\omega_{M*}(t)$ with a compatible current reference $i_{\phi*}(t) = i_*(t)f(\theta(t))$. The next corollary particularizes the result of Theorem 3.1 to cope with this more intricate problem, where the switched system (3.3) is nonlinear and the reference trajectory (5.7) is time-dependent and more general than the one considered in (4.8) for the AC-DC converter.

Corollary 5.1. Consider the switched affine system (5.1)-(5.2) defined by matrices (5.3), the matrix-valued function $R(\theta)$ given in (5.5) with the associated matrix $\Omega(\omega)$ given in (5.6) and let $0 \leq Q \in \mathcal{M}$ be given. The current trajectory $i_{\phi*}(t) = i_*(t)f(\theta(t))$ where

$$i_*(t) = \frac{2(c_M\omega_{M*}(t) + J_M\dot{\omega}_{M*}(t) + \tau_M)}{3\lambda_M} \tag{5.8}$$

with $\omega_{M*}(t)$ being a desired rotational velocity profile belonging, pointwise in time, to the set

$$\mathcal{C}_* = \left\{\omega_M : \Delta(\omega_M)'(\psi\psi' + \kappa^2\varphi\varphi')\Delta(\omega_M) \leq \frac{3(\lambda_M V_{dc})^2}{4}\right\} \tag{5.9}$$

where $\kappa = \max\{|\underline{\omega}|, |\overline{\omega}|\}$ and

$$\psi = \begin{bmatrix} R_Lc_M + 3\lambda_M^2/2 & J_MR_L + Lc_M & J_ML & R_L \end{bmatrix}' \tag{5.10}$$

$$\varphi = \begin{bmatrix} Lc_M & J_ML & 0 & L \end{bmatrix}' \tag{5.11}$$

$$\Delta(\omega_M) = \begin{bmatrix} \omega_M & \dot{\omega}_M & \ddot{\omega}_M & \tau_M \end{bmatrix}' \tag{5.12}$$

allow the system (5.1)-(5.2) to satisfy the assumptions (3.8)-(3.9) with matrix

$$A_R = \begin{bmatrix} -R_L/L & 0 & 0 & -\gamma/L \\ 0 & -R_L/L & 0 & 0 \\ 0 & 0 & -R_L/L & 0 \\ \gamma/J_M & 0 & 0 & -c_M/J_M \end{bmatrix} \tag{5.13}$$

with $\gamma = \sqrt{6}\lambda_M/2$. If there exists a matrix $P_R > 0$ solution to the linear matrix inequalities (3.15)-(3.16) then the switching function $\sigma(t) = u(\xi(t), t, \theta(t), \omega(t))$ with

$$u(\xi, t, \theta, \omega) = \arg\min_{i\in\mathbb{K}} \xi' P(\theta) b_i \tag{5.14}$$

ensures the asymptotic stability of the origin $\xi = 0$ whenever $\omega(t) \in \Theta_D$ and satisfies the inequality (3.17) with $Y = R(\theta_0)'\xi_0\xi_0' R(\theta_0)$.

Proof. The proof follows the same reasoning of Corollary 4.1 and, therefore, will be presented in general lines. It consists in showing that the equilibrium trajectory $x_e(t,\theta)$, with $i_*(t)$ given in (5.8) and $\omega_*(t)$ satisfying (5.9), together with $A_\lambda(\theta)$, with A_R given in (5.13), ensure that the assumptions (3.8)-(3.9) hold. The other conditions follow directly from the validity of Theorem 3.1. Notice that, as the dynamic matrix is independent of σ, the assumption (3.9) is directly verified by using A_R given in (5.13). To check the assumption (3.8) recall that the vectors $S_i, \forall i \in \mathbb{K}$, form the vertices of the regular hexagon $\mathbb{P}$ defined in (4.16), which is in the same plane Π of $f(\theta)$ and $g(\theta)$. Using the equilibrium trajectory $x_e(t,\theta)$ given in (5.7), we have

$$\ell_\lambda = \begin{bmatrix} -\vartheta(t,\theta) f(\theta)/L - \omega i_* g(\theta) + (V_{dc}/L) S_{\lambda(\theta)} \\ 3\lambda_M i_*/(2J_M) - c_M \omega_{M*}/J_M - \dot{\omega}_{M*} - \tau_M/J_M \end{bmatrix} \tag{5.15}$$

where $\vartheta(t) = R_L i_* + \lambda_M \omega_{M*} + L di_*/dt$ and it was omitted the dependence of ℓ_λ with respect to (t,θ,ω). Notice that $i_*(t)$ given in (5.8) makes null the second row of (5.15), while the first row is null whenever we choose $\lambda(t,\theta,\omega) \in \Lambda$ such that

$$S_\lambda = \frac{\vartheta(t,\theta)}{V_{dc}} f(\theta) + \frac{L\omega i_*}{V_{dc}} g(\theta) \tag{5.16}$$

This vector belongs to the same plane Π of the hexagon $\mathbb{P}$. Hence, for all $t \geq 0$, $\theta \in \Theta_P$ and $\omega \in \Theta_D$, it is always possible to find a convex combination $\lambda(t,\theta,\omega) \in \Lambda$ for which the identity (5.16) holds, if S_λ belongs to $\mathbb{F}$ defined in (4.19), with $r = 1/\sqrt{2}$ being the radius of the greatest circumference inside the hexagon $\mathbb{P}$ and centered at the origin. Notice that $\|S_\lambda\|^2 = 3(a_1^2 + a_2\omega^2)/2$ for all $\omega \in \Theta_D$ with $a_1 = \vartheta(t,\omega)/V_{dc}$ and $a_2 = Li_*/V_{dc}$. Thus, after replacing (5.8) in (5.16), we have that $\|S_\lambda\|^2_{\omega=\kappa} \leq 1/2$ for all $\omega \in \Theta_D$ whenever $\omega_{M*}(t)$ belongs to the set $\mathcal{C}_*$ given in (5.9). The switching function follows from (3.14) after taking into account that $A(\theta)$ is index independent and the guaranteed cost is the same provided in Theorem 3.1 concluding thus the proof. □

Notice that differently from the AC-DC converter where the asymptotic tracking is global, in this case it is valid only in the region where $\omega(t) = n_p\omega_M(t)$, dependent on fourth state

Table 5.2 – PMSM parameters

R_L	2.19Ω
L	8.1 mH
λ_M	6.02×10^{-2} V.s/rad
c_M	4.16×10^{-4} N.m.s/rad
τ_M	7.90×10^{-3} V
J_M	3.71×10^{-4}kg.m^2
V_{dc}	100 V

component, is inside the set Θ_D. A theoretical guarantee that $\omega(t)$ does not leave Θ_D is provided in the recent reference (Egidio *et al.*, 2022a), which treats the same problem but by means of conditions based on a less general Lyapunov function. These two control methodologies will be compared afterward by means of experimental results.

5.2 Experimental Results

In this section, the theoretical results are illustrated through the control of a permanent magnet synchronous machine with four pair of poles $n_p = 4$ fed by a voltage source inverter as presented in Figure 5.1. The numerical parameters were identified and are shown in Table 5.2. The designed switching function was embedded in a Texas Instruments TMS320F28069 microcontroller (*MCU*) and a dead time of $1\mu s$ was considered for operating each pair of switches. The motor used in the experiment was the Estun EMJ-04APB24 and a propeller with 50.8 cm of diameter was coupled to its shaft. The phase currents were measured through shunt resistors and data were acquired by analog-to-digital converters and quadrature encoder pulse modules. To measure rotational velocity and displacement, an attached incremental encoder with $2{,}500$ pulses per rotation was used. Its signal was filtered by a first order Butterworth filter with cutoff frequency $\omega_c = 4{,}000$ rad/s and discretized through the bilinear transformation. A photo of the experimental arrangement is presented in Figure 5.2.

The experiment will illustrate the asymptotic tracking of two pre-defined trajectory profiles ensuring a suitable guaranteed cost. It was considered that the set Θ_D in (3.2) is defined with $|\dot{\theta}| < \kappa = 800$ rad/s which is equivalent to have a angular velocity constrained to the set $|\omega_M| < 200$ rad/s. We have used a control frequency of 40 kHz, which is enough to make all the calculations needed for the switching rule implementation in the time interval between two control interruptions. Moreover, this frequency is sufficiently high to allow that the switching function be implemented under a continuous-time perspective. We have solved the conditions

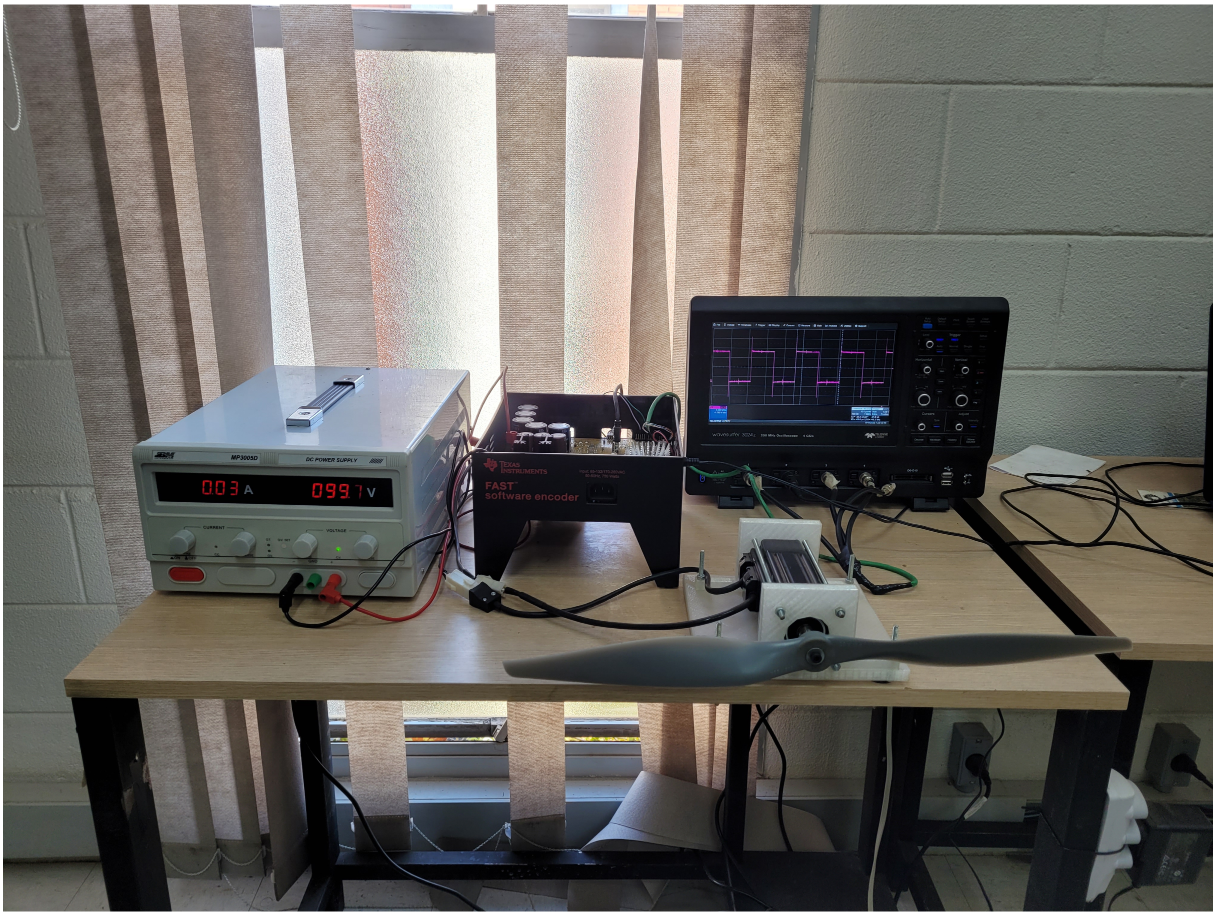


Figure 5.2 – Experimental arrangement

of Corollary 5.1 with the objective function (4.20), adopted previously in the control of the AC-DC converter, but subject to (3.15)-(3.16). As mentioned just after (4.20), the reason for adopting this function is to make the control design independent of the system initial condition $x(0)$. Adopting $Q = \text{diag}(I_3, 1)$, we have obtained a guaranteed cost of $\mathcal{J} < 7.3846$ associated to

$$P_R = \begin{bmatrix} 2.4032 & 0.0000 & 0 & 0.0532 \\ 0.0000 & 4.6439 & 0 & -0.0000 \\ 0 & 0 & 0.0020 & 0 \\ 0.0532 & -0.0000 & 0 & 0.3356 \end{bmatrix} \tag{5.17}$$

important to implement the switching function (5.14). Solving the same problem but with the design conditions proposed in Theorem 3 of reference (Egidio *et al.*, 2022a), we have obtained a guaranteed cost of $\mathcal{J} < 14.2657$, which is 93% greater than the one resulting from the Corollary 5.1, which indicates that the results here proposed are not more conservative. We have implemented the switching function (5.14) for two different rotational velocity profiles that satisfy, pointwise in time, the condition (5.9). The second profile present some discontinuities that

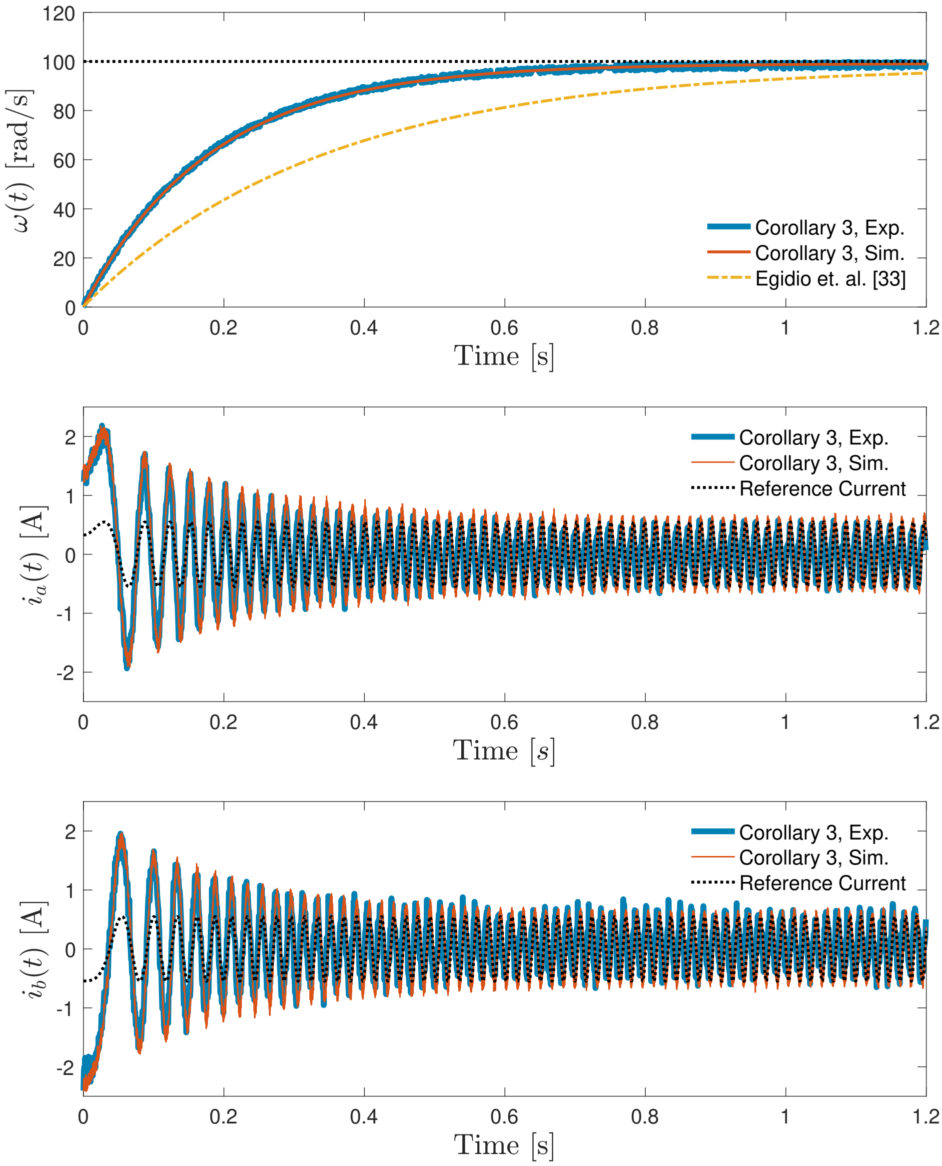


Figure 5.3 – Experimental and simulated angular velocity and a and b phase currents for $\omega_* =$ 100 rad/s

can be viewed as new initial conditions for the remaining reference trajectory, see (Egidio *et al.*, 2022a) for a discussion about this point.

The first profile consists in a step reference of $\omega_*(t) = 100$ rad/s used to validate the identified model and for the sake of comparison. The first plot of Figure 5.3 shows in solid lines the measurement of the angular velocity $\omega(t)$ obtained through the experimental set with its correspondent simulation, both resulting from the switching function of Corollary 5.1 with P_R given in (5.17). It is possible to observe that the responses are very close and the angular velocity $\omega(t)$ has reached $\omega_*(t)$ as expected, validating the model and the proposed control technique. In the same figure, the angular velocity obtained by simulation from (Egidio *et*

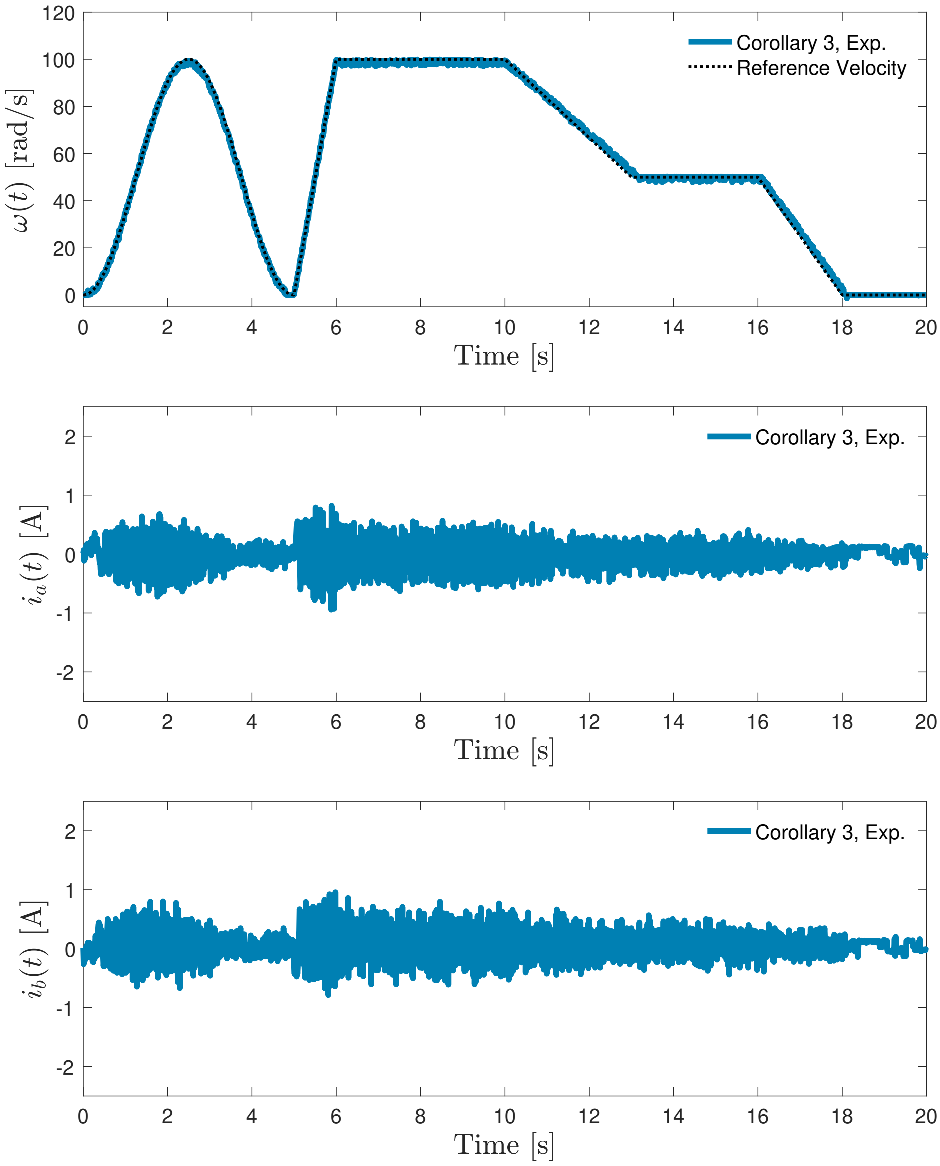


Figure 5.4 – Experimental velocity for a time-varying reference profile

al., 2022a) is presented in dot-dashed line. It is noticeable that with the switching rule (5.14) of Corollary 5.1, the convergence to $\omega_*(t)$ is faster. The second and third plots of Figure 5.3 present the associated a and b phase currents for the technique of Corollary 5.1 together with the associated reference current $i_{\phi*}(t)$, showing that it is indeed reached as expected.

In the second profile, a time-varying velocity trajectory composed of a sinusoid and successive ramps has been considered. Figure 5.4 shows in the first plot the measurement of the angular velocity $\omega(t)$ and the adopted reference $\omega_*(t)$ in solid and dotted lines, respectively, and in the second and third plots the correspondent measurements of the a and b phase currents. Notice that the desired profile was tracked as expected and, compared to the step reference, the peak currents were attenuated leading to a smoother transient response.

5.3 Computational Analysis

The proposed switching rule was embedded in a Texas Instrument (TI) microcontroller through the software Code Composer Studio 9.0 that is an integrated development environment that supports TI's microcontrollers. As this rule was designed to operate in an arbitrarily high switching frequency, which is not possible in practical implementations, an effort had to be made to reduce the number of calculations allowing a control frequency as close as possible to the ideal one. The control frequency of 40 kHz was the one adopted for this practical application and was considered high enough to obtain results near the theoretical ones. For this purpose, it is imperative to embed a switching function less computational demanding than (5.14). In fact, we have that

$$\begin{aligned} \sigma(\xi) &= \arg\min_{i\in\mathbb{K}} \xi' R_\theta P_R R_\theta' b_i \\ &= \arg\min_{i\in\mathbb{K}} \xi' \begin{bmatrix} \hat{R}_\theta \hat{P}_R \hat{R}_\theta' \\ \hat{d}_R' \hat{R}_\theta' \end{bmatrix} S_i \end{aligned} \tag{5.18}$$

where $\hat{R}_\theta \in \mathbb{R}^{3\times 3}$ and $\hat{P}_R \in \mathbb{R}^{3\times 3}$ are given by

$$\hat{R}_\theta = \begin{bmatrix} \sqrt{\frac{2}{3}} f(\theta) & \sqrt{\frac{2}{3}} g(\theta) & \sqrt{\frac{1}{3}} h \end{bmatrix}, \; P_R = \begin{bmatrix} \hat{P}_R & \hat{d}_R \\ \hat{d}_R' & \hat{q}_R \end{bmatrix} \tag{5.19}$$

with $\hat{R}_\theta$ obtained from (5.6). The second equality in (5.18) can be still simplified to the equivalent form

$$\sigma(\xi) = \arg\min_{i\in\mathbb{K}} \xi_r' \mathcal{M}_i \nu(\theta) \tag{5.20}$$

where vector $\nu(\theta) = [\sin(\theta) \;\; \cos(\theta) \;\; 2\sin(\theta)\cos(\theta) \;\; (1-2\sin^2(\theta)) \;\; 1]'$ and ξ_r is obtained from ξ without its third component, because it was used the fact that $i_c = -i_a - i_b$. After some algebraic manipulations and using the trigonometric identities, we have obtained that the constant matrix $\mathcal{M}_i \in \mathbb{R}^{3\times 5}$ is given by

$$\mathcal{M}_i = T_L \begin{bmatrix} \sum_{j=1}^{3} \boldsymbol{p_{1j}} S_{i_j} & \sum_{j=1}^{3} (\boldsymbol{m_{1j}} + \boldsymbol{n_{1j}}) S_{i_j} \\ \sum_{j=1}^{3} \boldsymbol{p_{2j}} S_{i_j} & \sum_{j=1}^{3} (\boldsymbol{m_{2j}} + \boldsymbol{n_{2j}}) S_{i_j} \\ \sum_{j=1}^{3} \boldsymbol{p_{3j}} S_{i_j} & \sum_{j=1}^{3} (\boldsymbol{m_{3j}} + \boldsymbol{n_{3j}}) S_{i_j} \\ \sum_{j=1}^{3} (\hat{d}_{R_1} \boldsymbol{r_j} + \hat{d}_{R_2} \boldsymbol{u_j}) S_{i_j} & 0 \end{bmatrix} T_R \tag{5.21}$$

with matrices T_L and T_R defined as

$$T_L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; T_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix} \tag{5.22}$$

At this point, it is important to make some remarks about the notation. The symbols in boldface represents matrices whose elements are vectors, as for instance, $\boldsymbol{m_{ij}} \in \mathbb{R}^{1\times 3}$ denotes the vector of dimension $\mathbb{R}^{1\times 3}$ placed in position (i, j) of matrix $\boldsymbol{m} \in \mathbb{R}^{3\times 9}$. The same reasoning is valid for the other matrices and the dimension of each element-vector will be clear afterward together with its definition. The notation $G_{g_{ij}} = G_g(i, j)$ corresponds to the scalar placed in the position (i, j) of matrix G_g, and for the vectors S_{i_j} represents the j-th component of the vector S_i. The same reasoning is adopted for the other matrices and vectors.

Thus, the vectors $\boldsymbol{m_{ij}} \in \mathbb{R}^{1\times 3}$ and $\boldsymbol{n_{ij}} \in \mathbb{R}^{1\times 3}$ are defined by

$$\boldsymbol{m_{ij}} = \begin{bmatrix} G_{f_{j1}}G_{a_{i1}} & G_{f_{j2}}G_{a_{i2}} & G_{f_{j2}}G_{a_{i1}} + G_{f_{j1}}G_{a_{i2}} \end{bmatrix} \tag{5.23}$$

and

$$\boldsymbol{n_{ij}} = \begin{bmatrix} G_{g_{j1}}G_{b_{i1}} & G_{g_{j2}}G_{b_{i2}} & G_{g_{j2}}G_{b_{i1}} + G_{g_{j1}}G_{b_{i2}} \end{bmatrix} \tag{5.24}$$

respectively, for all $i, j = 1, 2, 3$, with

$$\left[\begin{array}{c|c} G_f & G_g \end{array}\right] = \sqrt{\frac{2}{3}} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ -1/2 & -\sqrt{3}/2 & \sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 & -\sqrt{3}/2 & -1/2 \end{array}\right] \tag{5.25}$$

and $G_a = \hat{P}_{R_{11}}G_f + \hat{P}_{R_{21}}G_g$, $G_b = \hat{P}_{R_{12}}G_f + \hat{P}_{R_{22}}G_g$ and $G_c = \hat{P}_{R_{13}}G_f + \hat{P}_{R_{23}}G_g$ matrices of order 3×2. The vectors $\boldsymbol{r_j} \in \mathbb{R}^{1\times 2}$ and $\boldsymbol{u_j} \in \mathbb{R}^{1\times 2}$ are formed by rows of G_f and G_g, respectively. They compose the associated matrices $\boldsymbol{r} \in \mathbb{R}^{3\times 2}$ and $\boldsymbol{u} \in \mathbb{R}^{3\times 2}$ and are defined of the following form

$$\boldsymbol{r_j} = e_j' G_f, \quad \boldsymbol{u_j} = e_j' G_g \tag{5.26}$$

with e_j being the j-th column of the third order identity matrix. Lastly, the element-vector $\boldsymbol{p_{ij}} \in \mathbb{R}^{1\times 2}$ composes the matrix $\boldsymbol{p} \in \mathbb{R}^{3\times 6}$ and is defined as

$$\boldsymbol{p_{ij}} = \left(e_i' G_c + e_j' \left(\hat{P}_{R_{31}}G_f + \hat{P}_{R_{32}}G_g\right)\right) / \sqrt{3} \tag{5.27}$$

The switching rule (5.20) is equivalent to (5.14) but computationally more efficient in terms of computational effort and, therefore, it was used in all the experimental essays.

5.4 Final Considerations

In this chapter, the theoretical results presented in Chapter 3 were specialized to cope with the control of a Permanent Magnet Synchronous Machine. The method proved to be effective ensuring asymptotic tracking of a desired trajectory and the conditions were less conservative than the ones proposed in (Egidio *et al.*, 2022a), presenting a lower guaranteed cost. An experimental arrangement was elaborated to validate the designed switching rule, which was simplified to become more efficient in terms of required computational effort, allowing higher switching frequency. The obtained measures were coincident with the simulation results, validating the experimental identified model. Also, the transient response put in evidence that the performance of this present control methodology overcomes the one obtained by the recent control technique proposed in (Egidio *et al.*, 2022a).

6 CONCLUSIONS

During the course of this work, two techniques to the control of switched systems, whose dynamic matrices vary periodically with a parameter θ, were developed. The nature of this parameter allowed us to treat, with the same control methodology, two classes of systems: the parameter-dependent switched affine system, when θ is exogenous, and the more challenging one, the nonlinear switched system, when it is state-dependent. First, a general switching rule was proposed to deal with the trajectory tracking problem ensuring a guaranteed cost of performance. Additionally, as this switching rule is sensitive to model uncertainties, another one with integral action was proposed to give robustness to the model. The design conditions for this last were more difficult to be obtained, since the associated dynamic matrix is rank deficient and the negative-definiteness of the time-derivative of the Lyapunov function is ensured only through conditions under the affine terms. Due to its complexity, the control with integral action is constrained to systems where $\dot{\theta}$ is constant and does not admit a quadratic performance index. All the obtained design conditions were expressed in terms of linear matrix inequalities, being simple to solve using off-the-shelf algorithms.

Further, the obtained control methodologies were applied in a three-phase AC-DC power converter, where θ is an exogenous parameter and represents the electrical angle of the converter. For this case, the design conditions ensured global asymptotic tracking with effective output voltage regulation and unitary power factor. The resulting guaranteed cost was smaller than the one obtained by the control technique of (Egidio *et al.*, 2020), putting in evidence that the conditions developed in this work are in fact less conservative. The control with integral action has successfully accomplished the task of assuring trajectory tracking of the system subjected to model uncertainties. The effectiveness of the solutions were illustrated through computational simulations.

Lastly, a permanent magnet synchronous machine was controlled by the developed switching function. In this case, θ is dependent on the rotational velocity of the machine, which is one of the state variable, making the system highly nonlinear. For the nonlinear case, the proposed conditions ensure local asymptotic tracking and a suitable guaranteed performance. An experimental arrangement was elaborated and the obtained measurements were coincident with the simulation ones validating the identified model. Moreover, the conditions also presented to be less conservative than the ones proposed in (Egidio *et al.*, 2022a), providing a smaller guaranteed cost and a fast transient response. Some aspects on the practical implementation were highlighted and discussed throughout the text.

About the perspectives for future work, some relevant topics are listed as follows:

- Obtain design conditions for the case where the switching frequency is bounded, making the problem more realistic from the practical application viewpoint.
- Investigate how to include control with integral to the case where θ is state-dependent and, as a consequence, the switched system is nonlinear.
- Propose a performance index, not necessarily quadratic, to the control with integral action of switched affine systems. This topic is important even for the simpler case where the dynamic matrices are constant.

BIBLIOGRAPHY

Bacha, S.; Munteanu, I.; Bratcu, A. I. *et al.* Power electronic converters modeling and control. **Advanced textbooks in control and signal processing**, v. 454, n. 454, 2014.

Baldi, S.; Papachristodoulou, A.; Kosmatopoulos, E. B. Adaptive pulse width modulation design for power converters based on affine switched systems. **Nonlinear Analysis: Hybrid Systems**, v. 30, p. 306–322, 2018.

Beerens, R.; Bisoffi, A.; Zaccarian, L.; Heemels, W.; Nijmeijer, H.; Wouw, N. van de. Reset integral control for improved settling of pid-based motion systems with friction. **Automatica**, v. 107, p. 483–492, 2019.

Beneux, G.; Riedinger, P.; Daafouz, J.; Grimaud, L. Adaptive stabilization of switched affine systems with unknown equilibrium points: Application to power converters. **Automatica**, v. 99, p. 82–91, 2019.

Bittanti, S.; Colaneri, P. **Periodic systems: filtering and control**. [S.l.]: Springer Science & Business Media, 2009.

Blondel, V.; Tsitsiklis, J. N. Np-hardness of some linear control design problems. **SIAM journal on control and optimization**, v. 35, n. 6, p. 2118–2127, 1997.

Bolzern, P.; Spinelli, W. Quadratic stabilization of a switched affine system about a nonequilibrium point. In: **IEEE Am. Contr. Conf.** [S.l.: s.n.], 2004. v. 5, p. 3890–3895.

Bouafia, A.; Krim, F.; Gaubert, J.-P. Fuzzy-logic-based switching state selection for direct power control of three-phase PWM rectifier. **IEEE Trans. Ind. Electron**, v. 56, n. 6, p. 1984–1992, 2009.

Boyd, S.; El Ghaoui, L.; Feron, E.; Balakrishnan, V. **Linear matrix inequalities in system and control theory**. [S.l.]: SIAM, 1994.

Clegg, J. A nonlinear integrator for servomechanisms. **Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry**, v. 77, n. 1, p. 41–42, 1958.

Costanzo, L. C.; Deaecto, G. S.; Egidio, L. N.; Barros, T. A. Nova metodologia de controle para conversores de potência trifásicos CC-CA. In: **Simpósio Brasileiro de Automação Inteligente-SBAI**. [S.l.: s.n.], 2021. p. 558–563.

Deaecto, G. S.; Costanzo, L. C.; Egidio, L. N. Trajectory tracking for a class of θ-periodic switched affine systems. **IEEE Transactions on Automatic Control**, 2022, submitted.

Deaecto, G. S.; Geromel, J. C.; Daafouz, J. Dynamic output feedback $\mathcal{H}_\infty$ control of switched linear systems. **Automatica**, v. 47, n. 8, p. 1713–1720, 2011.

Deaecto, G. S.; Geromel, J. C.; Daafouz, J. Switched state-feedback control for continuous time-varying polytopic systems. **International Journal of Control**, v. 84, n. 9, p. 1500–1508, 2011.

Deaecto, G. S.; Geromel, J. C.; Garcia, F. S.; Pomilio, J. A. Switched affine systems control design with application to DC–DC converters. **IET Control Theory & Applications**, v. 4, n. 7, p. 1201–1210, 2010.

DeCarlo, R. A.; Branicky, M. S.; Pettersson, S.; Lennartson, B. Perspectives and results on the stability and stabilizability of hybrid systems. **Proceedings of the IEEE**, v. 88, n. 7, p. 1069–1082, 2000.

Delpoux, R.; Hetel, L.; Kruszewski, A. Parameter-dependent relay control: Application to PMSM. **IEEE Transactions on Control Systems Technology**, v. 23, n. 4, p. 1628–1637, 2014.

Duan, C.; Wu, F. Analysis and control of switched linear systems via modified Lyapunov–Metzler inequalities. **International Journal of Robust and Nonlinear Control**, v. 24, n. 2, p. 276–294, 2014.

Egidio, L. N.; Daiha, H. R.; Deaecto, G. S. Global asymptotic stability of limit cycle and $\mathcal{H}_2/\mathcal{H}_\infty$ performance of discrete-time switched affine systems. **Automatica**, v. 116, p. 108927, 2020.

Egidio, L. N.; Deaecto, G. S. Novel practical stability conditions for discrete-time switched affine systems. **IEEE Trans. Autom. Control**, v. 64, n. 11, p. 4705–4710, 2019.

Egidio, L. N.; Deaecto, G. S.; Barros, T. A. Switched control of a three-phase AC–DC power converter. **IFAC-PapersOnLine**, v. 53, n. 2, p. 6471–6476, 2020.

Egidio, L. N.; Deaecto, G. S.; Hespanha, J. P.; Geromel, J. C. A nonlinear switched control strategy for permanent magnet synchronous machines. In: **IEEE Conference on Decision and Control**. [S.l.: s.n.], 2019. p. 3411–3416.

Egidio, L. N.; Deaecto, G. S.; Hespanha, J. P.; Geromel, J. C. Trajectory tracking for a class of switched nonlinear systems: Application to pmsm. **Nonlinear Analysis: Hybrid Systems**, v. 44, p. 101164, 2022.

Egidio, L. N.; Deaecto, G. S.; Jungers, R. Stabilization of rank-deficient continuous-time switched affine systems. **Automatica**, 2022, in press.

Fiacchini, M.; Girard, A.; Jungers, M. On the stabilizability of discrete-time switched linear systems: Novel conditions and comparisons. **IEEE Transactions on Automatic Control**, v. 61, n. 5, p. 1181–1193, 2015.

Fiacchini, M.; Jungers, M. Necessary and sufficient condition for stabilizability of discrete-time linear switched systems: A set-theory approach. **Automatica**, v. 50, n. 1, p. 75–83, 2014.

Gallier, J.; Xu, D. Computing exponentials of skew-symmetric matrices and logarithms of orthogonal matrices. **International Journal of Robotics and Automation**, v. 18, n. 1, p. 10–20, 2003.

Garcia, G.; Lopez-Santos, O.; Martinez-Salamero, L. A unified approach for the control of power electronics converters. part ii: Tracking. **Applied Sciences**, v. 11, n. 16, p. 7618, 2021.

Garcia, G.; Lopez Santos, O. A unified approach for the control of power electronics converters. part i—stabilization and regulation. **Applied Sciences**, v. 11, n. 2, p. 631, 2021.

Geromel, J. C.; Colaneri, P. Stability and stabilization of continuous-time switched linear systems. **SIAM Journal on Control and Optimization**, v. 45, n. 5, p. 1915–1930, 2006.

Geromel, J. C.; Colaneri, P. Stability and stabilization of continuous-time switched linear systems. **SIAM Journal on Control and Optimization**, v. 45, n. 5, p. 1915–1930, 2006.

Geromel, J. C.; Colaneri, P.; Bolzern, P. Dynamic output feedback control of switched linear systems. **IEEE Transactions on Automatic Control**, v. 53, n. 3, p. 720–733, 2008.

Geromel, J. C.; Deaecto, G. S. Switched state feedback control for continuous-time uncertain systems. **Automatica**, v. 45, n. 2, p. 593–597, 2009.

Guo, X.; Ren, H.-P. A switching control strategy based on switching system model of three-phase VSR under unbalanced grid conditions. **IEEE Transactions on Industrial Electronics**, v. 68, n. 7, p. 5799–5809, 2020.

Hadjeras, S.; Sanchez, C. A.; Gomez-Estern Aguilar, F.; Gordillo, F.; Garcia, G. Hybrid control law for a three-level NPC rectifier. In: **European Control Conference (ECC)**. [S.l.: s.n.], 2019. p. 281–286.

Hetel, L.; Fridman, E. Robust sampled–data control of switched affine systems. **IEEE Transactions on Automatic Control**, v. 58, n. 11, p. 2922–2928, 2013.

Khalil, H. **Nonlinear Systems**. [S.l.]: Prentice Hall, 2002. (Pearson Education).

Krause, P. C.; Wasynczuk, O.; Sudhoff, S. D.; Pekarek, S. D. **Analysis of electric machinery and drive systems**. [S.l.]: John Wiley & Sons, 2013. v. 75.

Krishnan, R. **Switched reluctance motor drives: modeling, simulation, analysis, design, and applications**. [S.l.]: CRC press, 2017.

Liberzon, D. **Switching in systems and control**. [S.l.]: Birkhäuser Boston, 2003.

Lin, H.; Antsaklis, P. J. Stability and stabilizability of switched linear systems: a survey of recent results. **IEEE Transactions on Automatic control**, v. 54, n. 2, p. 308–322, 2009.

Luenberger, D. **Introduction to Dynamic Systems: Theory, Models, and Applications Dynamic Systems**. [S.l.]: John Wiley & Sons Limited, 1979.

Patino, D.; Riedinger, P.; Iung, C. Practical optimal state feedback control law for continuous-time switched affine systems with cyclic steady state. **International Journal of Control**, v. 82, n. 7, p. 1357–1376, 2009.

Prieur, C.; Queinnec, I.; Tarbouriech, S.; Zaccarian, L. Analysis and synthesis of reset control systems. **Foundations and Trends in Systems and Control**, v. 6, n. 2-3, p. 117–338, 2018.

Repecho, V.; Biel, D.; Arias, A. Fixed switching period discrete-time sliding mode current control of a PMSM. **IEEE Transactions on Industrial Electronics**, v. 65, n. 3, p. 2039–2048, 2017.

Rodriguez, J.; Cortes, P. **Predictive control of power converters and electrical drives**. [S.l.]: John Wiley & Sons, 2012.

Scharlau, C. C.; Dezuo, T. J. M.; Trofino, A.; Reginatto, R. Switching rule design for inverter-fed induction motors. In: **IEEE Conference on Decision and Control**. [S.l.: s.n.], 2013. p. 4662–4667.

Scharlau, C. C.; De Oliveira, M. C.; Trofino, A.; Dezuo, T. J. Switching rule design for affine switched systems using a max-type composition rule. **Systems & Control Letters**, v. 68, p. 1–8, 2014.

Serieye, M.; Albea-Sanchez, C.; Seuret, A.; Jungers, M. Stabilization of switched affine systems via multiple shifted Lyapunov functions. In: **21st IFAC World Congress**. [S.l.]: Elsevier, 2020. v. 53, n. 2, p. 6133–6138.

Shorten, R.; Wirth, F.; Mason, O.; Wulff, K.; King, C. Stability criteria for switched and hybrid systems. **SIAM review**, v. 49, n. 4, p. 545–592, 2007.

Silva, J. A. Mesquita da; Deaecto, G. S.; Barros, T. A. d. S. Analysis and design aspects of min-type switching control strategies for synchronous buck–boost converter. **Energies**, v. 15, n. 7, p. 2302, 2022.

Slotine, J.; Slotine, J.; Li, W. **Applied Nonlinear Control**. [S.l.]: Prentice Hall, 1991.

Sun, Z. **Switched Linear Systems: Control and Design**. Springer London, 2006. (Communications and Control Engineering). ISBN 9781846281310.

Sun, Z.; Ge, S. S. **Stability theory of switched dynamical systems**. [S.l.]: Springer Science & Business Media, 2011.

Trofino, A.; Reginatto, R.; De Oliveira, J.; Scharlau, C.; Coutinho, D. A reference tracking strategy for affine switched systems. In: IEEE. **2009 IEEE International Conference on Control and Automation**. [S.l.], 2009. p. 1744–1750.

Vlassis, N.; Jungers, R. Polytopic uncertainty for linear systems: New and old complexity results. **Systems & Control Letters**, v. 67, p. 9–13, 2014.

Wu, B.; Narimani, M. **High-power converters and AC drives**. [S.l.]: John Wiley & Sons, 2017.

Zhai, G. Quadratic stabilizability of discrete-time switched systems via state and output feedback. In: IEEE. **Proceedings of the 40th IEEE Conference on Decision and Control (Cat. No. 01CH37228)**. [S.l.], 2001. v. 3, p. 2165–2166.

Zhao, G.; Wang, J. On l2 gain performance improvement of linear systems with lyapunov-based reset control. **Nonlinear Analysis: Hybrid Systems**, v. 21, p. 105–117, 2016.