

R. 2793

RT-IMECC
IM/4074

ON THE EQUIVALENCE OF DIRAC AND
MAXWELL EQUATIONS, AND
QUANTUM MECHANICS

Jayme Vaz, Jr.
and
Waldyr A. Rodrigues, Jr.

Novembro

RP 58/92

Relatório de Pesquisa

**Instituto de Matemática
Estatística e Ciência da Computação**



**UNIVERSIDADE ESTADUAL DE CAMPINAS
Campinas - São Paulo - Brasil**

R.P.
IM/58/92

ABSTRACT – In this paper we present an analysis of the possible equivalence of Dirac and Maxwell equations using the Clifford bundle formalism and compare it with Campollataro's approach which uses the traditional tensor calculus and the standard Dirac covariant spinor field. We show that the intricate Campollataro's calculations can be proved in a few lines in our formalism. We briefly discuss the implications of our findings for the interpretation of quantum mechanics.

IMECC – UNICAMP
Universidade Estadual de Campinas
CP 6065
13081 Campinas SP
Brasil

O conteúdo do presente Relatório de Pesquisa é de única responsabilidade dos autores.

Novembro – 1992

On the Equivalence of Dirac and Maxwell Equations, and Quantum Mechanics

Jayme Vaz, Jr. and Waldyr A. Rodrigues, Jr.

Departamento de Matemática Aplicada, IMECC-UNICAMP
Universidade Estadual de Campinas
13083 Campinas, SP, Brazil.

Abstract. In this paper we present an analysis of the possible equivalence of Dirac and Maxwell equations using the Clifford bundle formalism and compare it with Campollataro's approach which uses the traditional tensor calculus and the standard Dirac covariant spinor field. We show that the intrincated Campollataro's calculations can be proved in a few lines in our formalism. We briefly discuss the implications of our findings for the interpretation of quantum mechanics.

1. Introduction

The majority of physicist's community would probably agree that Maxwell equations and Dirac equation are ones of the most celebrated equations of physics. Several presentations of Maxwell equations in (matrix) Dirac's like "spinor" - form can be found in the literature (see (Rodrigues and Oliveira, 1990) for discussions), some of them motivated to give a "first quantization" interpretation of Maxwell fields. The possibility of existing an intimate relationship between Maxwell and Dirac fields is an object of serious speculation since it could provide an answer to a secular question: what is an electron?

That possibility has been considered by us in two previous papers (Vaz and Rodrigues, 1991; Rodrigues and Vaz, 1991) by using the Clifford bundle formalism. Among the many advantages of this approach we can consider its simplicity and the fact that with it Maxwell and Dirac fields are represented by objects of the same mathematical nature. In (Vaz and Rodrigues, 1991) we show that starting from the *free* Maxwell equations and by writing¹

$$F = b\psi\gamma^1\gamma^2\psi^* \quad (1)$$

for the electromagnetic field (where ψ is a Dirac-Hestenes (DH) spinor field) that the nonlinear DH equation²

$$\gamma^\mu\partial_\mu\psi\gamma^1\gamma^2 + \mathcal{F}(\psi) = 0, \quad (2)$$

with

$$\mathcal{F}(\psi) = \gamma^\mu\psi\gamma^1\gamma^2(\partial_\mu\psi^*)\psi(\psi^*\psi)^{-1}, \quad (3)$$

is equivalent to the free Maxwell equations, and that there are two solutions ψ of eq.(2) which satisfy the DH equation (which is the representative of the standard Dirac equation in the Clifford bundle over Minkowski spacetime). These two solutions were naturally identified as "electron" and "positon" solutions. In (Rodrigues and Vaz, 1991) we generalize our approach in order to obtain *localized* "electron" solutions. In section 2 we briefly review our approach.

The possibility of existing an intimate correspondence between Maxwell and Dirac fields has also been considered by Campolattaro (1980a; 1980b; 1990). In Campolattaro (1980a) deduced a nonlinear Dirac-like spinor equation (for the usual standard Dirac covariant spinor field) equivalent to Maxwell equations. For the case where the electromagnetic current $J = 0$ the equation is

¹This is always possible according to the theorem of Rainich-Misner-Wheeler (see appendix B).

²In eqs.(1,2,3), etc. γ^μ are the generators of the local Clifford algebra $\mathcal{R}_{1,3}$ of the Clifford bundle. The γ^μ in eq.(4), etc. are the usual Dirac matrices in the standard representation. We shall use the same symbol for both objects since no confusion appears in the text.

$$\gamma^\mu \partial_\mu \Psi = -i\gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \{Im(\partial_\mu \bar{\Psi} \Psi) - \gamma^5 Im(\partial_\mu \bar{\Psi} \gamma^5 \Psi)\} \Psi \quad (4)$$

where

$$Im(\partial_\mu \bar{\Psi} \gamma^5 \Psi) = -\frac{1}{2}(\partial_\mu \bar{\Psi} i \gamma^5 \Psi - \bar{\Psi} i \gamma^5 \partial_\mu \Psi), \quad (5)$$

$$Im(\partial_\mu \bar{\Psi} \Psi) = \frac{1}{2i}(\partial_\mu \bar{\Psi} \Psi - \bar{\Psi} \partial_\mu \Psi), \quad (6)$$

and α is the “complexion” (Misner and Wheeler, 1957) of the field $F^{\mu\nu}$ in the given point. We also have

$$\cos \alpha = \frac{\bar{\Psi} \Psi}{\rho}, \quad \sin \alpha = \frac{\bar{\Psi} \gamma^5 \Psi}{\rho}, \quad (7)$$

$$\rho^2 = (\bar{\Psi} \Psi)^2 + (\bar{\Psi} \gamma^5 \Psi)^2, \quad (8)$$

so that $\Omega_1 = \bar{\Psi} \Psi$ and $\Omega_2 = \bar{\Psi} \gamma^5 \Psi$ are the invariants in Dirac theory.

Now if we compare eq.(2) and eq.(4) we see that they look very different. However Campolattaro started his demonstration of the equivalence between eq.(4) and Maxwell equations by proving that any given electromagnetic field $F^{\mu\nu}$ can be written as

$$F^{\mu\nu} = \frac{1}{2} \bar{\Psi} \gamma^{[\mu} \gamma^{\nu]} \Psi, \quad (9)$$

where $\gamma^{[\mu} \gamma^{\nu]} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$.

Despite the apparent difference between eq.(2) and eq.(4) and the apparent similarity between eq.(1) and eq.(9), a difference exists. In fact, while ψ in eq.(1)-(2) is a DH spinor field, i.e., an operator spinor field according to Figueiredo et. al. (1990a; 1990b), Ψ in eq.(4),(9) is a standard Dirac covariant (SDC) spinor field, according to Figueiredo et. al. (1990a; 1990b). Thus $\bar{\Psi}$ is the usual Dirac conjugate $\bar{\Psi} = \psi^\dagger \gamma^0$. In (Rodrigues and Oliveira, 1990) it is shown that Ψ can be identified with $\Psi = \psi e$, where $e = \frac{1}{2} (1 + \gamma^0) \in \text{sec } \mathcal{CL}(M, \hat{g})$ is a global idempotent field.

It is reasonable to suppose that eq.(2) and eq.(4) are different representations for the same equation, and indeed this is the case! We prove this explicitly in section 3, and since these two equations are the same, then eq.(4) exhibit *plane wave* solutions, as we shall see. It is very difficult to see this fact directly from eq.(4), but from eq.(2) this is a trivial task. Also, with that proof, our theory (Vaz and Rodrigues, 1991; Rodrigues and Vaz, 1991) can now be translated into “traditional” mathematical tolls.

2. The Clifford Bundle Approach

Let $\mathcal{Cl}(m, \hat{g})$ be the Clifford bundle of differential forms over Minkowski spacetime. The spacetime algebra is the typical fiber of the bundle (Rodrigues et. al., 1989, Rodrigues and Figueiredo, 1990; Maia et. al., 1990). Let be $\{e_\mu\} \in \sec TM$ a basis of TM and $\{\gamma^\mu\} \in \sec T^*M = \sec \Lambda^1 M \subset \sec \mathcal{Cl}(M, \hat{g})$ the dual basis, satisfying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$; $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The Dirac operator ∂ acting on sections of $\mathcal{Cl}(M, \hat{g})$ is $\partial = d - \delta$, where d is the differential and δ the Hodge codifferential operators, and $\partial = d - \delta = \gamma^\mu \nabla_\mu$, where ∇ is this Levi-Civita connection of $g = \eta_{\mu\nu} \gamma^\mu \otimes \gamma^\nu$ ($\hat{g} = \eta^{\mu\nu} e_\mu \otimes e_\nu$). We can choose for simplicity $\{\gamma^\mu\}$ such that $\nabla_\mu = \partial_\mu$; thus $\partial = \gamma^\mu \partial_\mu$. Now, in this formalism, the free Maxwell equations $dF = 0$ and $\delta F = 0$ can be written as:

$$\partial F = 0, \quad (10)$$

where the electromagnetic field $F \in \sec \Lambda^2(M) \subset \sec \mathcal{Cl}(M, \hat{g})$. This form of Maxwell equations is originally due to Riesz (1958). On the other hand, the Dirac equation for a free electron can be written in this formalism as:

$$\partial \psi \gamma^1 \gamma^2 + \frac{mc}{\hbar} \psi \gamma^0 = 0, \quad (11)$$

which is due to Hestenes (1966; 1967; 1975). The DH spinor field $\psi \in \sec(\Lambda^0(M) + \Lambda^2(M) + \Lambda^4(M)) \subset \sec \mathcal{Cl}(M, \hat{g})$ can be written in the canonical form:

$$\psi = \rho^{1/2} e^{\gamma^5 \beta/2} R, \quad (12)$$

where $\rho, \beta \in \sec \Lambda^0(M) \subset \sec \mathcal{Cl}(M, \hat{g})$, and $\forall x \in M$, $R \in Spin_+(1, 3) \simeq SL(2, \mathcal{C})$, i.e., $RR^* = R^*R = 1$, where $*$ (called reversion) is the principal antiautomorphism in $\mathbb{R}_{1,3}$, Figueiredo et. al., (1990a; 1990b). Finally $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$ is the volume element.

Now, if we look for a solution of the free Maxwell equations (10) having the form given by eq.(12) a simple substitution of eq.(12) in eq.(10) gives eq.(8). Despite the fact that eq.(2) is a nonlinear equation, we have shown in (Vaz and Rodrigues, 1991) that it possess plane wave solutions that satisfy DH equation (11), namely:

$$\psi_- = \rho^{1/2} e^{\gamma^5 \beta/2} e^{-\gamma^2 \gamma^1 (p \cdot x / \hbar)}, \quad (13)$$

$$\psi_+ = \rho^{1/2} e^{\gamma^5 \beta/2} \gamma^1 \gamma^2 e^{\gamma^2 \gamma^1 (p \cdot x / \hbar)}, \quad (14)$$

which were identified with $\beta = 0$ for ψ_+ and $\beta = \pi$ for ψ_- as the “electron” and “positon” solutions, respectively. We have also proved that ψ_{e-} and ψ_{e+} rotate around the streamlines of “electromagnetic fluid” with the same frequency $\omega_0 = 2mc^2/\hbar$ in

different rotation senses and that such a rotation motion is the origin of mass in this theory – a conclusion also obtained by Hestenes (1991) but with a different point of view. In (Rodrigues and Vaz, 1991) we have generalized our approach in order to obtain localized “electron” solutions by proving that each component $\phi_0 e^{-i(p \cdot x/\hbar)}$ of the DH spinor field satisfies a nonlinear Klein-Gordon equation with nonlinearity of quantum potential type, that is:

$$\square\psi + \left(\frac{mc}{\hbar}\right)^2\psi = \frac{\square\phi_0}{\phi_0}\psi. \quad (15)$$

Our approach to these localized solutions is to be compared with the ones of Gueret and Vigier (1982a; 1982b; 1982c), Mackinnon (1978; 1981) and Barut (1990). Moreover, since the ψ field in our approach is of electromagnetic nature, we have an intimate relationship between ψ and the behaviour of phase-locked cavities studied by Jennison (1978) and which have the inertial properties of classical particles.

3. Proof of Equivalence Between Equations (2) and (4)

In order to prove that eq.(2) and eq.(4) are the same equation we must prove that the components of the spinor field ψ satisfying eq.(2) and the components of the spinor field Ψ satisfying eq.(4) satisfy the same equation. Thus we shall first write eq.(4) in terms of its components, then we shall write eq.(2) in terms of its components, and finally we compare them.

Let us calculate the term in braces in eq.(4). For

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, -\bar{\psi}_3, -\bar{\psi}_4), \quad (16)$$

we have that (the bar over the components denotes complex conjugation)

$$\begin{aligned} Im(\partial_\mu \bar{\Psi} \Psi) &= \frac{-i}{2} [(\partial_\mu \bar{\psi}_1 \psi_1 - \bar{\psi}_1 \partial_\mu \psi_1) + (\partial_\mu \bar{\psi}_2 \psi_2 - \bar{\psi}_2 \partial_\mu \psi_2) - \\ &\quad - (\partial_\mu \bar{\psi}_3 \psi_3 - \bar{\psi}_3 \partial_\mu \psi_3) - (\partial_\mu \bar{\psi}_4 \psi_4 - \bar{\psi}_4 \partial_\mu \psi_4)] \equiv \frac{-i}{2} \xi \end{aligned} \quad (17)$$

$$\begin{aligned} Im(\partial_\mu \bar{\Psi} \gamma^5 \Psi) &= \frac{-1}{2} [(\partial_\mu \bar{\psi}_1 \psi_3 + \bar{\psi}_3 \partial_\mu \psi_1) + (\partial_\mu \bar{\psi}_2 \psi_4 + \bar{\psi}_4 \partial_\mu \psi_2) - \\ &\quad - (\partial_\mu \bar{\psi}_3 \psi_1 - \bar{\psi}_1 \partial_\mu \psi_3) - (\partial_\mu \bar{\psi}_4 \psi_2 + \bar{\psi}_2 \partial_\mu \psi_4)] \equiv \frac{-1}{2} \eta \end{aligned} \quad (18)$$

and then

$$\begin{aligned} & \{Im(\partial_\mu \bar{\Psi} \Psi) - \gamma^5 Im(\partial_\mu \bar{\Psi} \gamma^5 \Psi)\} \Psi = \\ & = \frac{-i}{2} \left[\xi \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} + \eta \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \right]. \end{aligned} \quad (19)$$

Thus eq.(4) is explicitly:

$$\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = -\frac{1}{2} \gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \begin{pmatrix} \xi \psi_1 + \eta \psi_3 \\ \xi \psi_2 + \eta \psi_4 \\ \xi \psi_3 + \eta \psi_1 \\ \xi \psi_4 + \eta \psi_2 \end{pmatrix} = -\frac{1}{2} \gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix} \quad (20)$$

Now we shall write eq.(2) in terms of components. In the appendix we show that a DH spinor field has the following matrix representation:

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix}, \quad \psi^* = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\psi_4 & \psi_3 \\ -\psi_3 & -\psi_4 & \psi_1 & \psi_2 \\ -\psi_4 & \psi_3 & -\psi_2 & \psi_1 \end{pmatrix}. \quad (21)$$

Since we also have

$$\gamma^1 \gamma^2 = -i \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad (22)$$

the calculation of $\mathcal{F}(\psi)$ is a simple task. We have

$$\psi \gamma^1 \gamma^2 = -i \begin{pmatrix} \psi_1 & \bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & -\bar{\psi}_1 & \psi_4 & \bar{\psi}_3 \\ \psi_3 & -\bar{\psi}_4 & \psi_1 & \bar{\psi}_2 \\ \psi_4 & \bar{\psi}_3 & \psi_2 & -\bar{\psi}_1 \end{pmatrix}. \quad (23)$$

Also,

$$\begin{aligned} \psi \gamma^1 \gamma^2 \partial_\mu \psi^* &= -i \begin{pmatrix} \psi_1 & \bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & -\bar{\psi}_1 & \psi_4 & \bar{\psi}_3 \\ \psi_3 & -\bar{\psi}_4 & \psi_1 & \bar{\psi}_2 \\ \psi_4 & \bar{\psi}_3 & \psi_2 & -\bar{\psi}_1 \end{pmatrix} \begin{pmatrix} \partial_\mu \bar{\psi}_1 & \partial_\mu \bar{\psi}_2 & -\partial_\mu \bar{\psi}_3 & -\partial_\mu \bar{\psi}_4 \\ -\partial_\mu \psi_2 & \partial_\mu \psi_1 & -\partial_\mu \psi_4 & \partial_\mu \psi_3 \\ -\partial_\mu \psi_3 & -\partial_\mu \psi_4 & \partial_\mu \psi_1 & \partial_\mu \psi_2 \\ -\partial_\mu \psi_4 & \partial_\mu \psi_3 & -\partial_\mu \psi_2 & \partial_\mu \psi_1 \end{pmatrix} = \\ &= -i \begin{pmatrix} A & B & C & D \\ \bar{B} & -\bar{A} & -\bar{D} & \bar{C} \\ C & D & A & B \\ -\bar{D} & \bar{C} & \bar{B} & -\bar{A} \end{pmatrix} \end{aligned} \quad (24)$$

where we have defined:

$$A = \psi_1 \partial_\mu \bar{\psi}_1 - \bar{\psi}_2 \partial_\mu \psi_2 - \psi_3 \partial_\mu \bar{\psi}_3 + \bar{\psi}_4 \partial_\mu \psi_4, \quad (25)$$

$$B = \psi_1 \partial_\mu \bar{\psi}_2 + \bar{\psi}_2 \partial_\mu \psi_1 - \psi_3 \partial_\mu \bar{\psi}_4 - \bar{\psi}_4 \partial_\mu \psi_3, \quad (26)$$

$$C = -\psi_1 \partial_\mu \bar{\psi}_3 - \bar{\psi}_2 \partial_\mu \psi_4 + \psi_3 \partial_\mu \bar{\psi}_1 + \bar{\psi}_4 \partial_\mu \psi_2, \quad (27)$$

$$D = -\psi_1 \partial_\mu \bar{\psi}_4 + \bar{\psi}_2 \partial_\mu \psi_3 + \psi_3 \partial_\mu \bar{\psi}_2 - \bar{\psi}_4 \partial_\mu \psi_1. \quad (28)$$

Since $\psi \gamma^2 \gamma^1 = -\psi \gamma^1 \gamma^2$ we have using eq.(24) and eq.(23):

$$\begin{aligned} \psi \gamma^1 \gamma^2 (\partial_\mu \psi^*) \psi \gamma^2 \gamma^1 &= \begin{pmatrix} A & B & C & D \\ \bar{B} & -\bar{A} & -\bar{D} & \bar{C} \\ C & D & A & B \\ -\bar{D} & \bar{C} & \bar{B} & -\bar{A} \end{pmatrix} \begin{pmatrix} \psi_1 & \bar{\psi}_2 & \psi_3 & -\bar{\psi}_4 \\ \psi_2 & -\bar{\psi}_1 & \psi_4 & \bar{\psi}_3 \\ \psi_3 & -\bar{\psi}_4 & \psi_1 & \bar{\psi}_2 \\ \psi_4 & +\bar{\psi}_3 & \psi_2 & -\bar{\psi}_1 \end{pmatrix} = \\ &= \begin{pmatrix} E & F & G & H \\ -\bar{F} & \bar{E} & \bar{H} & -\bar{G} \\ G & H & E & F \\ \bar{H} & -\bar{G} & -\bar{F} & \bar{E} \end{pmatrix} = \Phi \end{aligned} \quad (29)$$

where we used

$$E = A\psi_1 + B\psi_2 + C\psi_3 + D\psi_4, \quad (30)$$

$$F = A\bar{\psi}_2 - B\bar{\psi}_1 - C\bar{\psi}_4 + D\bar{\psi}_3, \quad (31)$$

$$G = A\psi_3 + B\psi_4 + C\psi_1 + D\psi_2, \quad (32)$$

$$H = -A\bar{\psi}_2 + B\bar{\psi}_3 + C\bar{\psi}_2 - D\bar{\psi}_1. \quad (33)$$

Finally, since $\psi \psi^* = \rho e^{\gamma^5 \beta}$ and using eq.(29) we have for eq.(2);

$$\gamma^\mu \partial_\mu \psi = -\gamma^\mu \frac{e^{-\gamma^5 \beta}}{\rho} \Phi. \quad (34)$$

Now if we introduce the expressions for A, B, C, D into the ones for E, F, G, H and remember that

$$\Omega_1 = \psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2 - \psi_3 \bar{\psi}_3 - \psi_4 \bar{\psi}_4, \quad (35)$$

$$\Omega_2 = i(\psi_1 \bar{\psi}_3 + \psi_2 \bar{\psi}_4 - \psi_3 \bar{\psi}_1 - \psi_4 \bar{\psi}_2), \quad (36)$$

then after a straightforward but tedious calculation we arrive at

$$\phi_1 = E = \xi\psi_1 + \eta\psi_3 + \Omega_1\partial_\mu\psi_1 + i\Omega_2\partial_\mu\psi_3, \quad (37)$$

$$\phi_2 = -\bar{F} = \xi\psi_2 + \eta\psi_4 + \Omega_1\partial_\mu\psi_2 + i\Omega_2\partial_\mu\psi_4, \quad (38)$$

$$\phi_3 = G = \xi\psi_3 + \eta\psi_1 + \Omega_1\partial_\mu\psi_3 + i\Omega_2\partial_\mu\psi_1, \quad (39)$$

$$\phi_4 = \bar{H} = \xi\psi_4 + \eta\psi_2 + \Omega_1\partial_\mu\psi_4 + i\Omega_2\partial_\mu\psi_2, \quad (40)$$

and

$$\begin{aligned} \Phi &= \begin{pmatrix} \phi_1 & -\bar{\phi}_2 & \phi_3 & -\bar{\phi}_4 \\ \phi_2 & \bar{\phi}_1 & \phi_4 & -\bar{\phi}_3 \\ \phi_3 & \bar{\phi}_4 & \phi_1 & -\bar{\phi}_2 \\ \phi_4 & -\bar{\phi}_3 & \phi_2 & \bar{\phi}_1 \end{pmatrix} = \\ &= \begin{pmatrix} \chi_1 & -\bar{\chi}_2 & \chi_3 & \bar{\chi}_4 \\ \chi_2 & \bar{\chi}_1 & \chi_4 & -\bar{\chi}_3 \\ \chi_3 & \bar{\chi}_4 & \chi_1 & -\bar{\chi}_2 \\ \chi_4 & -\bar{\chi}_3 & \chi_2 & \bar{\chi}_1 \end{pmatrix} + \Omega_1\partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} + \\ &\quad + i\Omega_2\partial_\mu \begin{pmatrix} \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \\ \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \end{pmatrix}. \end{aligned} \quad (41)$$

The last matrix can be written as

$$\begin{pmatrix} \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \\ \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \end{pmatrix} = \begin{pmatrix} & 1 & 0 \\ & 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} \quad (42)$$

in such a way that the sum of the two latter matrices in eq.(41) is

$$\begin{aligned} \Omega_1\partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} + \Omega_2 \begin{pmatrix} & i & 0 \\ & 0 & i \\ i & 0 \\ 0 & i \end{pmatrix} \partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} = \\ = (\Omega_1 - \gamma^5\Omega_2)\partial_\mu\psi. \end{aligned} \quad (43)$$

Now if we use eq.(21) we find that

$$\psi\psi^* = \begin{pmatrix} \Omega_1 & 0 & i\Omega_2 & 0 \\ 0 & \Omega_1 & 0 & i\Omega_2 \\ i\Omega_2 & 0 & \Omega_1 & 0 \\ 0 & i\Omega_2 & 0 & \Omega_1 \end{pmatrix} = \Omega_1 - \gamma^5 \Omega_2, \quad (44)$$

and using eq.(44) in eq.(43) and then in eq.(41) we have that

$$\Phi = \chi + (\psi\psi^*)\partial_\mu\psi = \chi + \rho e^{\gamma^5\beta}\partial_\mu\psi, \quad (45)$$

where χ is the first matrix on the r.h.s. of eq.(41).

Introducing this result in eq.(32) we get

$$\gamma^\mu\partial_\mu\psi = -\frac{1}{2}\gamma^\mu\frac{e^{-\gamma^5\beta}}{\rho}\chi, \quad (46)$$

or, in terms of components:

$$\gamma^\mu\partial_\mu\begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} = -\frac{1}{2}\gamma^\mu\frac{e^{-\gamma^5\beta}}{\rho}\begin{pmatrix} \chi_1 & -\bar{\chi}_2 & \chi_3 & \bar{\chi}_4 \\ \chi_2 & \bar{\chi}_1 & \chi_4 & -\bar{\chi}_3 \\ \chi_3 & \bar{\chi}_4 & \chi_1 & -\bar{\chi}_2 \\ \chi_4 & -\bar{\chi}_3 & \chi_2 & \bar{\chi}_1 \end{pmatrix}. \quad (47)$$

Now we compare eq.(20) and eq.(47): they are the same equation^(*)! Thus we have proved that eq.(2) and eq.(4) are one and the same equation. The reader is invited to compare our approach to eq.(2) in (Vaz and Rodrigues, 1991) – just one line of calculus – with the one of Campolattaro^[4] to eq.(4). – six pages of calculus.

Note the identification of the “complexion” α with β whose values $\beta = 0$ and $\beta = \pi$ distinguishes electrons from positons in Dirac theory (Vaz and Rodrigues, 1991; Hestenes, 1991).

4. Plane wave Solutions of the Nonlinear Equation

It is only a matter of verification to see that plane waves are solutions of eq.(20) or eq.(47). To simplify the calculations we shall consider the rest frame. The equations for the components are:

$$\partial_0\psi_1 = -\frac{1}{2\rho}[\cos\beta(\xi\psi_1 + \eta\psi_3) - i\sin\beta(\xi\psi_3 + \eta\psi_1)], \quad (48)$$

^{2(*)} more precisely each column of eq.(17) contains the same information of eq.(20).

$$\partial_0 \psi_2 = -\frac{1}{2\rho} [\cos \beta (\xi \psi_2 + \eta \psi_4) - i \sin \beta (\xi \psi_4 + \eta \psi_2)], \quad (49)$$

$$\partial_0 \psi_3 = -\frac{1}{2\rho} [\cos \beta (\xi \psi_3 + \eta \psi_1) - i \sin \beta (\xi \psi_1 + \eta \psi_3)], \quad (50)$$

$$\partial_0 \psi_4 = -\frac{1}{2\rho} [\cos \beta (\xi \psi_4 + \eta \psi_2) - i \sin \beta (\xi \psi_2 + \eta \psi_4)], \quad (51)$$

where, in accordance to eq.(17)-(18):

$$\begin{aligned} \xi &= (\partial_\mu \bar{\psi}_1 \psi_1 - \bar{\psi}_1 \partial_\mu \psi_1) + (\partial_\mu \bar{\psi}_2 \psi_2 - \bar{\psi}_2 \partial_\mu \psi_2) \\ &- (\partial_\mu \bar{\psi}_3 \psi_3 - \bar{\psi}_3 \partial_\mu \psi_3) - (\partial_\mu \bar{\psi}_4 \psi_4 - \bar{\psi}_4 \partial_\mu \psi_4), \end{aligned} \quad (52)$$

$$\begin{aligned} \eta &= (\partial_\mu \bar{\psi}_1 \psi_3 - \bar{\psi}_3 \partial_\mu \psi_1) + (\partial_\mu \bar{\psi}_2 \psi_4 - \bar{\psi}_4 \partial_\mu \psi_2) \\ &- (\partial_\mu \bar{\psi}_3 \psi_1 + \bar{\psi}_1 \partial_\mu \psi_3) - (\partial_\mu \bar{\psi}_4 \psi_2 + \bar{\psi}_2 \partial_\mu \psi_4). \end{aligned} \quad (53)$$

Now it is trivial to verify that

$$\Psi_1^{(-)} = \sqrt{\rho} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2 t/\hbar}; \quad \Psi_2^{(-)} = \sqrt{\rho} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2 t/\hbar} \quad (54)$$

are solutions of the above equations (48)-(51) provided that

$$\beta = 0, \quad (55)$$

and that

$$\Psi_1^{(+)} = \sqrt{\rho} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imc^2 t/\hbar}; \quad \Psi_2^{(+)} = \sqrt{\rho} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imc^2 t/\hbar} \quad (56)$$

are also solutions of eq.(48)-(51) provided that

$$\beta = \pi. \quad (57)$$

These solutions (54)-(56) are written in the spacetime algebra as

$$\psi_1^{(-)} = \sqrt{\rho} e^{\gamma^5 \beta/2} e^{-\gamma^2 \gamma^1 mc^2 t/\hbar} \Big|_{\beta=0}, \quad (58)$$

$$\psi_2^{(-)} = \sqrt{\rho} e^{\gamma^5 \beta/2} \gamma^3 \gamma^1 e^{-\gamma^2 \gamma^1 m c^2 t/\hbar} \Big|_{\beta=0}, \quad (59)$$

$$\psi_1^{(+)} = \sqrt{\rho} e^{\gamma^5 \beta/2} \gamma^1 \gamma^2 e^{\gamma^2 \gamma^1 m c^2 t/\hbar} \Big|_{\beta=\pi}, \quad (60)$$

$$\psi_2^{(+)} = \sqrt{\rho} e^{\gamma^5 \beta/2} \gamma^3 \gamma^1 \gamma^1 \gamma^2 e^{\gamma^2 \gamma^1 m c^2 t/\hbar} \Big|_{\beta=\pi}, \quad (61)$$

which can be easily verified by right multiplication by the idempotent $e = \frac{1}{2}(1 + \gamma^0)$ according to the method described in (Rodrigues and Oliveira, 1990).

5. Conclusions

We cannot but be surprised by the fact that eq.(4) exhibits plane solutions. Indeed it is very difficult to deal with it, so that the proof requires an extensive calculus. On the other hand, the same equation in the Clifford bundle, eq.(2), is simple, and the fact that it exhibit plane wave solution can be easily seen in this case. The essential difference between our method and the one of Campolattaro is that a DH spinor field can be represented as a matrix which has an inverse, while the SDC spinor field used by Campolattaro is represented by a one-column matrix which does not have an inverse. Thus Campolattaro's method is expected to be much more complicated. Moreover, due to this fact, some essential aspects of the problem could be easily obscured.

Acknowledgments: The authors are grateful to Professor E. Recami and Drs. J.R.R. Zeni and Q.A.G. de Souza for discussions. W.A.R. is grateful to CNPq for a continuity of a research grant.

Appendix A: Matrix Representation of the Dirac-Hestenes Spinor Field

In this appendix we give a matrix representation for DH spinor field. In Dirac representation we have

$$\gamma^0 = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ; \quad \gamma^1 = \begin{pmatrix} & 0 & 1 & \\ & 1 & 0 & \\ 0 & -1 & & \\ -1 & 0 & & \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} ,$$

$$\gamma^2 = \begin{pmatrix} & 0 & -i & \\ & i & 0 & \\ 0 & i & & \\ -i & 0 & & \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} ; \quad \gamma^3 = \begin{pmatrix} & 1 & 0 & \\ & 0 & -1 & \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} .$$

Since $\psi \in \sec(\Lambda^0(M) + \Lambda^2(M) + \Lambda^4)) \subset \sec \mathcal{C}\ell(M, \hat{g})$ we have that

$$\psi = a_1 + a_{01}\gamma^0\gamma^1 + a_{02}\gamma^0\gamma^2 + a_{03}\gamma^0\gamma^3 + a_{12}\gamma^1\gamma^2 + a_{13}\gamma^1\gamma^3 + a_{23}\gamma^2\gamma^3 + a_{0123}\gamma^0\gamma^1\gamma^2\gamma^3 .$$

but in this representation:

$$\begin{aligned} \gamma^0\gamma^1 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} ; \quad \gamma^0\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} ; \quad \gamma^0\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} ; \\ \gamma^1\gamma^2 &= -i \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} ; \quad \gamma^1\gamma^3 = i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} ; \quad \gamma^2\gamma^3 = -i \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} ; \\ \gamma^0\gamma^1\gamma^2\gamma^3 &= \gamma^5 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \end{aligned}$$

Thus we have:

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix}$$

where:

$$\begin{aligned} \psi_1 &= a_0 - ia_{12} , \\ \psi_2 &= -a_{13} - ia_{23} , \\ \psi_3 &= a_{03} - ia_{0123} , \\ \psi_4 &= a_{01} + ia_{02} . \end{aligned}$$

For $\psi^* = \gamma^0 \psi^\dagger \gamma^0$ we have then

$$\psi^* = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\bar{\psi}_2 & \bar{\psi}_1 & -\bar{\psi}_4 & \bar{\psi}_3 \\ -\bar{\psi}_3 & -\bar{\psi}_4 & \bar{\psi}_1 & \bar{\psi}_2 \\ -\bar{\psi}_4 & \bar{\psi}_3 & -\bar{\psi}_2 & \bar{\psi}_1 \end{pmatrix}$$

Appendix B: Theorem of Rainich-Misner-Wheeler

In this appendix we give a proof in terms of spacetime algebra of the theorem of Rainich-Misner-Wheeler (Rainich; 1927; Misner and Wheeler, 1957) because of its fundamental importance for this work.

If we define an extremal field as a field for which the magnetic (electric) field is zero and the electric (magnetic) field is parallel to one coordinate axis, the theorem of Rainich-Misner-Wheeler says that: “At any point of Minkowski spacetime any nonnull electromagnetic field can be reduced to an extremal field by a Lorentz transformation and a duality rotation”.

Let $F = \frac{1}{2} F_{\mu\nu} \gamma^\mu \wedge \gamma^\nu \in \sec \Lambda^2(M) \subset \sec \mathcal{Cl}(M, \hat{g})$ be the electromagnetic field. The invariants of F are given by $F^2 = F \cdot F + F \wedge F$. In terms of Pauli algebra $\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^+$ we have $F = \vec{E} + \hat{i} \vec{H}$, where $\vec{E} = E_i \vec{\sigma}_i$, $\vec{H} = H_i \vec{\sigma}_i$, $\vec{\sigma}^i = \gamma^i \gamma^0$, $\hat{i} = \vec{\sigma}^1 \vec{\sigma}^2 \vec{\sigma}^3$ (the $\vec{\sigma}^i$ are the generators of $\mathbb{R}_{3,0}$), we have

$$F \cdot F = \vec{E}^2 - \vec{H}^2 ; \quad F \wedge F = 2\hat{i} \vec{E} \cdot \vec{H}.$$

Let us consider a duality rotation of F by an angle α , that is:

$$F' = e^{\gamma^5 \alpha} F = \cos \alpha F + \gamma^5 F \sin \alpha.$$

The invariants of the fields are changed under a duality rotation in such a way that from

$$F'^2 = e^{2\gamma^5 \alpha} F^2$$

we have

$$F' \cdot F' = \cos 2\alpha (F \cdot F) + \sin 2\alpha \gamma^5 (F \wedge F),$$

$$F' \wedge F' = \cos 2\alpha (F \wedge F) + \sin 2\alpha \gamma^5 (F \cdot F),$$

or by writing $F = \vec{E}' + \hat{i} \vec{H}'$:

$$\vec{E}'^2 - \vec{H}'^2 = \cos 2\alpha (\vec{E}^2 - \vec{H}^2) - \sin 2\alpha 2(\vec{E} \cdot \vec{H}),$$

$$2(\vec{E}' \cdot \vec{H}') = \cos 2\alpha 2(\vec{E} \cdot \vec{H}) - \sin 2\alpha (\vec{E}^2 - \vec{H}^2).$$

Now we can choose α in such a way that

$$\vec{E}' \cdot \vec{H}' = 0,$$

that is

$$\tan 2\alpha = \frac{\gamma^5 (F \wedge F)}{(F \cdot F)} = \frac{2(\vec{E} \cdot \vec{H})}{\vec{H}^2 - \vec{E}^2}$$

and then for $F' \cdot F'$ we have

$$\vec{E}'^2 - \vec{H}'^2 = \pm \sqrt{(\vec{E}^2 - \vec{H}^2)^2 + 4(\vec{E} \cdot \vec{H})^2},$$

where the different signs come from the fact that $\tan \phi$ has period π and $\cos(\phi + \pi) = -\cos \phi$, so that the angles ϕ and $\phi + \pi$ correspond to $\vec{E}'^2 - \vec{H}'^2 < 0$ and $\vec{E}'^2 - \vec{H}'^2 > 0$. Indeed, since $2\alpha = \phi + \pi$ and $\alpha = \phi/2 + \pi/2$ so that $e^{\gamma^5 \alpha} = e^{\gamma^5 \phi/2} e^{\gamma^5 \pi/2} = e^{\gamma^5 \phi/2} \gamma^5$, the duality rotation by $\pi/2$, i.e. $e^{\gamma^5 \pi/2} = \gamma^5$ transforms an electric field in a magnetic one and vice-versa.

Now, a well-known (Doubrovine et. al., 1987) theorem says that: "If $\vec{E}' \cdot \vec{H}' = 0$ then there exists a Lorentz transformation R such that $F'' = RF'R^* = \vec{E}'' + \hat{i}\vec{H}''$ we have: (a) if $\vec{E}'^2 - \vec{H}'^2 > 0$ then $\vec{E}'' \neq 0$ and $\vec{H}'' = 0$; (b) if $\vec{E}'^2 - \vec{H}'^2 < 0$ then $\vec{E}'' = 0$ and $\vec{H}'' \neq 0$." Let us find this transformation explicitly. Consider $\vec{v} = v\vec{\sigma}^1$; then for a Lorentz transformation ($\gamma = (1 - \beta^2)^{-1/2}$):

$$E_1'' = E_1'; \quad E_2'' = \gamma(E_2' + \beta H_3'); \quad E_3'' = \gamma(E_3' - \beta H_2'); \quad$$

$$H_1'' = H_1'; \quad H_2'' = \gamma(H_2' - \beta E_3'); \quad H_3'' = \gamma(H_3' + \beta E_2').$$

Considering a duality rotation such that $\vec{E}' \cdot \vec{H}' = 0$ and $\vec{E}'^2 - \vec{H}'^2 < 0$ and choosing $\vec{\sigma}^2$ and $\vec{\sigma}^3$ such that $\vec{E} = E\vec{\sigma}^2$ and $\vec{H} = H\vec{\sigma}^3$ and $\beta = \frac{v}{c} = \frac{E}{H}$ we have

$$E_1'' = E_2'' = E_3'' = 0; \quad H_1'' = H_1' = 0; \quad H_3'' = \sqrt{H'^2 - E'^2}.$$

Thus, by defining

$$\bar{\alpha} = -\alpha, \quad \bar{R} = R^*,$$

we have shown that

$$e^{\gamma^5 \bar{\alpha}} \bar{R}^* F \bar{R} = -\hat{i} H \vec{\sigma}^3 = H \gamma^1 \gamma^2,$$

from which it follows that

$$F = e^{\gamma^5 \bar{\alpha}} \bar{R} (H \gamma^1 \gamma^2) \bar{R}^*,$$

or, by writing

$$H = b\rho, \quad \psi = \sqrt{\rho} e^{\gamma^5 \bar{\alpha}/2} \bar{R},$$

that

$$F = b\psi \gamma^1 \gamma^2 \psi^*.$$

References

- Barut, A.O., (1990), *Physics Letters A*, **143**, 349.
- Campolattaro, A.A., (1980a), *International Journal of Theoretical Physics*, **19**, 99.
- Campolattaro, A.A., (1980b), *International Journal of Theoretical Physics*, **19**, 127.
- Campolattaro, A.A., (1990), *International Journal of Theoretical Physics*, **29**, 141.
- Doubrovine, B., Novikov, S., and Fomenko, A., (1987), *Géométrie Contemporaine, Méthodes et Applications, 1^e partie*, Editions Mir, Moscow, (1987).
- Figueiredo, V.L., de Oliveira, E.C., and Rodrigues, W.A., Jr. (1990a), *International Journal of Theoretical Physics*, **29**, 371.
- Figueiredo, V.L., de Oliveira, E.C., and Rodrigues, W.A., Jr. (1990b), *Algebras Groups, and Geometries*, **7**, 153.
- Gueret, Ph., “La Théorie de la Double Solution de de Broglie Revisitée”, preprint Università di Bari (1988).
- Gueret, Ph., and Vigier, J.P., (1982a), *Lettere Nuovo Cimento*, **35**, 256.
- Gueret, Ph., and Vigier, J.P., (1982b), *Lettere Nuovo Cimento*, **35**, 260.
- Gueret, Ph., and Vigier, J.P., (1982b), *Lettere Nuovo Cimento*, **38**, 125.
- Hestenes, D., (1966), *Space-Time Algebra*, Gordon and Breach, N.Y.
- Hestenes, D., (1967), *Journal of Mathematical Physics* **8**, 789.
- Hestenes, D., (1975), *Journal of Mathematical Physics* **16**, 556.
- Hestenes, D., (1991), *Foundations of Physics*, **20**, 1213.
- Jennison, R.C., (1978), *Journal of Physics*, A **11**, 1525.
- Mackinnon, L., (1978), *Foundations of Physics*, **8**, 157.
- Mackinnon, L., (1981), *Lettere. Nuovo Cimento*, **32**, 311.
- Maia, A., Jr., Recami, E., Rosa, M.A.F., and Rodrigues, W.A., Jr. (1990), *Journal of Mathematical Physics*, **31**, 502.

- Misner, C.W., and Wheeler, J.A., (1957), *Annals of Physics*, **2**, 525.
- Rainich, G.Y., (1925), *Transactions of the American Mathematical Society*, **27**, 106.
- Riesz, M., (1958), *Clifford Number and Spinors*, Lecture series No. 38, The Institute for Fluid Mechanics and Applied Mathematics, University of Maryland.
- Rodrigues, W.A., Jr. and Figueiredo, V.L., (1990), *International Journal of Theoretical Physics*, **29**, 413.
- Rodrigues, W.A., Jr. and de Oliveira, E.C., (1990), *International Journal of Theoretical Physics*, **29**, 397.
- Rodrigues, W.A., Jr. Faria-Rosa, M.A., Maia, A., Jr. and Recami, E., (1989), *Hadronic Journal*, **12**, 187.
- Rodrigues, W.A., Jr. and Vaz, J., Jr. (1991), "Nondispersive de Broglie Wavepackets from the Free Maxwell equations", sub. for publication.
- Vaz, J., Jr. and Rodrigues, W.A., Jr. (1991), "A Spinor Representation of Maxwell Equations and its Relationship with Dirac Equation", subm. for publication.

RELATÓRIOS DE PESQUISA — 1992

- 01/92 Uniform Approximation the: Non-locally Convex Case — *João B. Prolla.*
- 02/92 Compactificação de $L_{\omega}^{\tau}(Q)$ com τ Finito — *A. M. Sette and J. C. Cifuentes.*
- 03/92 Um Modelo para Aquisição da Especificação — *Cecilia Inés Sosa Arias and Ariadne Carvalho.*
- 04/92 Convergence Estimates for the Wavelet Galerkin Method — *Sônia M. Gomes and Elsa Cortina.*
- 05/92 Optimal Chemotherapy: A Case Study with Drug Resistance, Saturation Effect and Toxicity — *M. I. S. Costa, J. L. Boldrini and R. C. Bassanezi.*
- 06/92 On the Paper “Cauchy Completeness of Elementary Logic” of D. Mundici and A. M. Sette — *J. C. Cifuentes.*
- 07/92 What is the EM Algorithm for Maximum Likelihood Estimation in PET and How to Accelerate it — *Alvaro R. De Pierro.*
- 08/92 Bifurcation from infinity and multiple solutions for an elliptic system — *Raffaele Chiappinelli and Djairo G. de Figueiredo.*
- 09/92 Approximation Processes for Vector-Valued Continuous Functions — *João B. Prolla.*
- 10/92 Aplicação do Método de Fraissé à Compactificação de Lógicas com Quantificadores Co-filtro — *A. M. Sette and J. C. Cifuentes.*
- 11/92 Absolutely Summing Holomorphic Mappings — *Mário C. Matos.*
- 12/92 The Feynman-Dyson Proof of Maxwell Equations and Magnetic Monopoles — *Adolfo A. Jr. and Waldyr A. R. Jr.*
- 13/92 A Generalized Dirac’s Quantization Condition for Phenomenological Non-abelian Magnetic Monopoles — *Adolfo M. Jr. and Waldyr A. R. Jr.*
- 14/92 Multiplicity Results for the 1-Dimensional Generalized p -Laplacian — *Pedro Ubilla.*
- 15/92 Nowhere Vanishing Torsion Closed Curves Always Hide Twice — *Sueli R. Costa. and Maria Del Carmen R. Fuster.*
- 16/92 Uniform Approximation of Continuous Convex-Cone-Valued Functions — *João B. Prolla.*
- 17/92 Monotonically Dominated Operators on Convex Cones — *A. O. Chiacchio, J. B. Prolla, M. L. B. Queiroz and M. S. M. Roversi.*
- 18/92 Testing the Concept of a Photon as an Extended Object in a Variation of Franson’s Experiment — *V. Buonomano, A. J. R. Madureira and L. C. B. Ryff.*
- 19/92 A New Trust Region Algorithm for Boun Constrained Minimization — *Ana Friedlander, José Mario Martínez and Sandra A. Santos.*

- 20/92 **A Priori Estimates for Positive Solutions of Semilinear Elliptic Systems Via Hardy-Sobolev Inequalities** — *Ph. Clément, D.G. de Figueiredo and E. Mitidieri.*
- 21/92 **On the Resolution of Linearly Constrained Convex Minimization Problems** — *Ana Friedlander, José Mario Martínez and Sandra A. Santos.*
- 22/92 **Convergência no Espectro Real de um Anel** — *J. C. Cifuentes.*
- 23/92 **Parallel Methods for Large Least Norm Problems** — *Alvaro R. De Pierro. and Jose M. Lopes.*
- 24/92 **A Generalization of Dirac Non-Linear Electrodynamics, and Spinning Charged Particles** — *Waldyr A. Rodrigues Jr., Jayme Vaz Jr. and Erasmo Recami.*
- 25/92 **Bifurcation of Certain Singularities of Sliding Vector Fields** — *Marco Antonio Teixeira.*
- 26/92 **Density of Infimum-Stable Convex Cone** — *João B. Prolla.*
- 27/92 **An Extended Decomposition Through Formalization in Product Spaces** — *Alvaro R. De Pierro.*
- 28/92 **Accelerating Iterative Algorithms for Symmetric Linear Complementarity Problems** — *Alvaro R. De Pierro and José Marcos Lopes.*
- 29/92 **Free Maxwell Equations, Dirac Equation, and Non-Dispersive de Broglie Wave-packets** — *Waldyr A. Rodrigues Jr., Jayme Vaz Jr. and Erasmo Recami.*
- 30/92 **New Results on the Equivalence of Regularization and Truncated Iteration for Ill-posed Problems** — *Reginaldo J. Santos.*
- 31/92 **Topological Defects: A Distribution Theory Approach** — *Patricio S. Letelier and Anzhong Wang.*
- 32/92 **Some Applications of the Generalized Khintchine's Inequality** — *Mário C. Matos.*
- 33/92 **A Technique for Finding The Center of A Polytope** — *Antonio Carlos Moretti.*
- 34/92 **On The Convergence of the Column-Updating Method** — *José Mario Martínez.*
- 35/92 **Post-Linear Formalism For Gravitating Strings: Crossed Straight Strings Collision** — *D. V. Gal'tsov, Yu. V. Grats and P. S. Letelier.*
- 36/92 **A Simple Model of Life Expectancy With Subjective Parameters** — *R. C. Bassanezi and L. C. Barros.*
- 37/92 **On the Growth of Caiman Crocodilus Yacare: A Strategy for Optimal Farming** — *Alejandro B. Engel and R. C. Bassanezi.*
- 38/92 **The Dirac Operator and the Structure of Riemann-Cartan-Weyl Spaces** — *Quintino A. G. de Souza and Waldyr A. Rodrigues, Jr.*
- 39/92 **On the Nature of the Gravitational Field** — *Waldyr A. Rodrigues, Jr. and Quintino A. G. de Souza.*
- 40/92 **Spinning Strings and Cosmic Dislocations** — *D. V. Gal'tsov and P. S. Letelier.*
- 41/92 **Natural Ultrabornological, Non-Complete, Normed Function Spaces** — *Antonio Gilioli (edited by Klaus Floret and Chaim S. Höning).*
- 42/92 **Parallel and Semi-Parallel Immersions Into Space Forms** — *Francesco Mercuri.*

- 43/92 Periodic Motions of Sucker Rod Pumping Systems — *Aloisio Freiria Neves and José Luiz Boldrini.*
- 44/92 Algorithms for Solving Nonlinear Systems of Equations — *José Mario Martínez.*
- 45/92 Spin and Electron Structure — *Matej Pavšič, Erasmo Recami, Waldyr A. Rodrigues Jr. and G. Daniele Maccarrone, Fabio Raciti, Giovanni Salesi.*
- 46/92 The Initial Value Problem For the Boussinesq Equations in a Time-Dependt Domain — *Carlos Conca Rozende and Marko Rojas Medar.*
- 47/92 A New Expansion for Jacobi Function — *E. Capelas de Oliveira.*
- 48/92 On the Equivalence of Convergences of Fuzzy Sets — *Marko Rojas Medar and Heriberto Román Flores.*
- 49/92 Monotonicity and Symmetry of Solutions of Elliptic Systems in General Domains — *Djairo G. de Figueiredo.*
- 50/92 Tendências em Programação Não Linear de Grande Porte — *José Mario Martínez.*
- 51/92 Teoría y Práctica en Optimización Numérica — *José Mario Martínez.*
- 52/92 The Clifford Bundle and the Nature of the Gravitational Field — *Waldyr A. Rodrigues Jr. and Quintino A.G. de Souza.*
- 53/92 Completamento e Compactificação de Espaços Zero-Dimensionais — *J. C. Cifuentes.*
- 54/92 Some Results on Polarized Partition Relations of Higher Dimension — *Walter Alexandre Carnielli and Carlos Augusto Di Prisco.*
- 55/92 The Generalized Wave Equation: Revisited — *G. C. Ducati and E. Capelas de Oliveira.*
- 56/92 The Four-Vertex Theorem for Constant Curvature Manifolds — *Sueli I. R. Costa and Marcelo Firer.*
- 57/92 The Riemann-Cartan-Weyl Geometry Generated by a Dirac-Hestenes Spinor Field — *Diego Lucio Rapoport-Campodónico, Waldyr A. Rodrigues, Jr., Quintino A. G. de Souza and Jayme Vaz Jr.*