# THE DIRAC OPERATOR AND THE STRUCTURE OF RIEMANN-CARTAN-WEYL SPACES

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## Relatório de Pesquisa

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**ABSTRACT** - In this paper we present a Clifford bundle approach to the geometry of a general Riemann-Cartan-Weyl space (RCWS). In our formulation, the so-called Dirac operators play a fundamental role. We first introduce one these operators in the context of a Riemannian space, calling it the fundamental Dirac operator ($\boldsymbol{\partial}$). We show, among other important results, that the fundamental Dirac operator can be written $\boldsymbol{\partial} = d - \delta$, where $d$ and $\delta$ are the familiar exterior derivative operator and the Hodge codifferential acting on sections of the Hodge bundle (interpreted as embedded in the Clifford bundle). In connection with the fundamental Dirac operator, we introduce the concepts of Dirac commutator and anticommutator and we investigate their geometrical interpretation. With the theory of the symmetric automorphisms of a Clifford algebra we introduce infinitely many other Dirac-like operators, one for each nondegenerate bilinear form field that can be defined on the metric manifold $\mathcal{M}$. We introduce also the concepts of Ricci and Einstein operators which are useful for intrinsic formulations of Einstein's gravitational theory. Later we generalize the fundamental Dirac operator and the Dirac commutators to a RCWS, showing their relations. In particular, we succeed in finding the correct generalization of the Hodge Laplacian to Riemann-Cartan spaces and we identify the natural wave operator ($\mathcal{L}_+$) for these spaces. Apart from a constant factor, $\mathcal{L}_+$ is the relativistic Hamiltonian operator which generates the theory of Markov processes of the Stochastic Mechanics. We obtain also new decompositions of the general affine connection $\nabla$ defining the RCWS structure and we identify new tensor objects. Our findings clear many results obtained in formulations of the flat space theory of gravitational field and of the theory of spinor fields in RCWS and suggest several generalizations of these theories. This subject is discussed in two following papers.

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**Abstract**

In this paper we present a Clifford bundle approach to the geometry of a general Riemann-Cartan-Weyl space (RCWS). In our formulation, the so-called Dirac operators play a fundamental role. We first introduce one these operators in the context of a Riemannian space, calling it the fundamental Dirac operator ($\partial$). We show, among other important results, that the fundamental Dirac operator can be written $\partial = d - \delta$, where $d$ and $\delta$ are the familiar exterior derivative operator and the Hodge codifferential acting on sections of the Hodge bundle (interpreted as embedded in the Clifford bundle). In connection with the fundamental Dirac operator, we introduce the concepts of Dirac commutator and anticommutator and we investigate their geometrical interpretation. With the theory of the symmetric automorphisms of a Clifford algebra we introduce infinitely many other Dirac-like operators, one for each nondegenerate bilinear form field that can be defined on the metric manifold $\mathcal{M}$. We introduce also the concepts of Ricci and Einstein operators which are useful for intrinsic formulations of Einstein's gravitational theory. Later we generalize the fundamental Dirac operator and the Dirac commutators to a RCWS, showing their relations. In particular, we succeed in finding the correct generalization of the Hodge Laplacian to Riemann-Cartan spaces and we identify the natural wave operator ($\mathcal{L}_+$) for these spaces. Apart from a constant factor, $\mathcal{L}_+$ is the relativistic Hamiltonian operator which generates the theory of Markov processes of the Stochastic Mechanics. We obtain also new decompositions of the general affine connection $\nabla$ defining the RCWS structure and we identify new tensor objects. Our findings clear many results obtained in formulations of the flat space theory of gravitational field and of the theory of spinor fields in RCWS and suggest several generalizations of these theories. This subject is discussed in two following papers.

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## Contents

## 1. Introduction

In this paper we give a Clifford bundle approach to the geometry of a general Riemann-Cartan-Weyl space.

In Sec. 2 we briefly recall the mathematical concepts necessary to develop the ideas of this work. The relation between exterior and Clifford algebras and how to work with the Clifford product is presented. In addition, we develop the theory of the symmetric automorphisms of a Clifford algebra, which plays an important role in our presentation, since it permits the definition of new Clifford products starting from a given one. Finally, we present in this section the relation between Cartan, Hodge and Clifford bundles and we recall Cartan's formulation of the differential geometry, extending it to a general Riemann-Cartan-Weyl space (hereafter denoted RCWS).

In Sec. 3 we formulate the geometry of a RCWS in the Clifford bundle $[\mathcal{C}\ell(\mathcal{M})]$. Here we introduce the concept of the fundamental Dirac operator $\partial$ (related to the Levi-Civita connection $D$ of an oriented Riemannian space with metric $\gamma$ ($\mathcal{M} = \langle M, D, \gamma\rangle$). Using this operator, we define the concepts of fundamental Dirac commutator and anticommutator and we discuss the very interesting geometrical meaning of these operators. In addition, using the theory of the symmetric automorphisms (Sec. 2), we introduce infinitely many other Dirac-like operators, one for each nondegenerate symmetric bilinear form field which can be defined on the metric manifold $\mathcal{M}$. We study relations between these operators and the fundamental Dirac operator.

Subsequently, we introduce the Dirac operator $\boldsymbol{\partial}$ of a general RCWS. This operator is related to the general affine connection $\nabla$ defining the RCWS structure. We then introduce the analogous of the concepts of Dirac commutator and anticommutator for $\boldsymbol{\partial}$ and explore their geometrical meaning. In particular, the fundamental Dirac commutator permits us to give the structure of a Lie algebra to the cotangent bundle, in analogy to the way that the bracket of vector fields defines a Lie algebra structure for the tangent bundle (the coefficients of the fundamental Dirac anticommutator—called the Killing coefficients—are related to the Lie derivative of the metric by a very interesting relation [Eq. 52]).

We obtain new decompositions of the general connection $\nabla$, exhibiting some tensor quantities which have not been identified in the literature. Our results sheds a new light on the flat space formulations of the theory of the gravitational field and on the theory of spinor fields in RCWS. The discussion of these subject will be presented in other publications.

Finally we study the square of the Dirac operator $\boldsymbol{\partial}$ in a Riemann-Cartan space. We succeed to determine the correct generalization of the Hodge Laplacian in such spaces, thereby identifying a wave operator $\mathcal{L}_+$ that (appart from a constant factor) is the relativistic Hamiltonian operator that describes the theory of Markov processes, as described recently by Rapoport.[1] Applications of the mathematical concepts developed here to the gravitational theory and to the theory of spinor fields in RCWS will be presented in two following papers.[2,3] (See also [4])

## 2. Mathematical Preliminaries

In this section we briefly recall some mathematical concepts necessary to develop the ideas of this work.

## 2.1. Exterior and Clifford Algebras

### a. Exterior and Grassmann Algebras

Let $V$ be an $n$-dimensional real vector space, $V^*$ its dual space, $T^rV$ the space of the $r$-contravariant tensors over $V$ ($r \geq 0$, $T^0V = \mathbb{R}$, $T^1V = V$) and let $TV$ be the tensor algebra of $V$.

We define the *exterior algebra* of $V$ as the quotient algebra

$$\Lambda V = \frac{TV}{J},$$

where $J \subset TV$ is the bilateral ideal in $TV$ generated by the elements of the form $u \otimes v + v \otimes u$, with $u, v \in V$. The elements of $\Lambda V$ will be called *multivectors*, or *multiforms* if $V$ is the dual of some previously specified vector space.

Let $\rho : TV \to \Lambda V$ be the canonical projection of $TV$ onto $\Lambda V$. Multiplication in $\Lambda V$ will be denoted as usually by $\wedge : \Lambda V \to \Lambda V$ and called *exterior product*. We have

$$A \wedge B = \rho(A \otimes B), \tag{1}$$

for every $A, B \in \Lambda V$, where $\otimes : TV \to TV$ is the usual tensor product.

We recall that $\Lambda V$ is a $2^n$-dimensional associative algebra with unity. In addition, it is a $\mathbb{Z}$-graded algebra, i.e.,

$$\Lambda V = \bigoplus_{r=0}^{n} \Lambda^r V,$$

with

$$\Lambda^r V \wedge \Lambda^s V \subset \Lambda^{r+s} V,$$

$r, s \geq 0$, where $\Lambda^r V = \rho(T^r V)$ is the $\binom{n}{r}$-dimensional subspace of the *r-vectors* on $V$. ($\Lambda^0 V = \mathbb{R}$; $\Lambda^1 V = V$; $\Lambda^r V = \{0\}$ if $r > n$) If $A \in \Lambda^r V$ for some fixed $r$ ($r = 0, \ldots, n$), then $A$ is said to be *homogeneous*. For any such multivectors we have:

$$A \wedge B = (-1)^{rs} B \wedge A, \tag{2}$$

$A \in \Lambda^r V$, $B \in \Lambda^s V$.

We recall also that any endomorphism $\psi : V \to V$ can be extended in unique way to a homomorphism $\Lambda\psi : \Lambda V \to \Lambda V$, satisfying:

$$\Lambda\psi \circ \rho = \rho \circ \psi. \tag{3}$$

The restriction of this map to the subspace $\Lambda^r V$ is denoted $\Lambda^r \psi : \Lambda^r V \to \Lambda^r V$ and it is called the *r-th exterior power* of the map $\psi$. We have:

$$\Lambda^r \psi(u_1 \wedge \ldots \wedge u_r) = \psi(u_1) \wedge \ldots \wedge \psi(u_r), \tag{4}$$

for every $u_1, \ldots, u_r \in V$. In particular, $\Lambda^0 \psi = 1$ is the identity map of $\mathbb{R}$, for any $\psi$.

Now let us suppose that $V$ is a metric vector space, that is, it is endowed with a non-degenerate metric tensor $\gamma \in T^2 V^*$ of signature $(p, q)$. We can use this metric tensor to induce an scalar product $\langle \, , \, \rangle : \Lambda V \times \Lambda V \to \mathbb{R}$ on $\Lambda V$, by letting

$$\langle A, B \rangle = \det(\gamma(u_i, v_j)), \tag{5}$$

for homogeneous multivectors $A = u_1 \wedge \ldots \wedge u_r \in \Lambda^r V$ and $B = v_1 \wedge \ldots \wedge v_r \in \Lambda^r V$, $u_i, v_i \in V$, $i = 1, \ldots, r$. This scalar product is extended to all of $\Lambda V$ due to linearity and orthogonality ($\langle A, B\rangle = 0$ if $A \in \Lambda^r V, B \in \Lambda^s V$, $r \neq s$). The algebra $\Lambda V$ endowed with this internal product is called the *Grassmann algebra* of $V$ and it will be denoted by $\Lambda(V,\gamma)$.

If the metric vector space $\langle V, \gamma\rangle$ is also endowed with an *orientation*, i.e., a *volume n-vector* $\tau \in \Lambda^n V$ such that

$$\langle \tau, \tau\rangle = (-1)^q, \tag{6}$$

then we can still introduce a natural isomorphism between the spaces $\Lambda^r V$ and $\Lambda^{n-r} V$ ($r = 0, \ldots, n$), called *Hodge star operator* (or *Hodge dual*) and denoted by $* : \Lambda^r V \to \Lambda^{n-r} V$. We have:

$$A \wedge *B = \langle A, B\rangle \tau, \tag{7}$$

for every $A, B \in \Lambda^r V$. Of course, this operator is naturally extended to an isomorphism $* : \Lambda V \to \Lambda V$ by linearity. The inverse $*^{-1} : \Lambda^{n-r} V \to \Lambda^r V$ of the Hodge star operator is given by:

$$*^{-1} = (-1)^{r(n-r)} \text{sgn}\ \bar{\gamma}\ *,$$

where $\text{sgn}\ \bar{\gamma} = \bar{\gamma}/|\bar{\gamma}|$ denotes the sign of the determinant $\bar{\gamma} = \det(\gamma_{\alpha\beta})$.

## b. Clifford Algebras

The *Clifford algebra* $C\ell(V,\gamma)$ of a metric vector space $\langle V, \gamma\rangle$ is defined as the quotient algebra[1]

$$C\ell(V,\gamma) = \frac{TV}{J_\gamma},$$

where $J_\gamma \subset TV$ is the bilateral ideal of $TV$ generated by the elements of the form $u \otimes v + v \otimes u - 2\gamma(u, v)$, with $u, v \in V \subset TV$. Clifford algebras generated by symmetric bilinear forms are sometimes referred as *orthogonal*, in order to be distinguished from the *symplectic* Clifford algebras, which are generated by skew-symmetric bilinear forms. (See, e.g., [7].)

Let $\rho_\gamma : TV \to C\ell(V,\gamma)$ be the natural projection of $TV$ onto the quotient algebra $C\ell(V,\gamma)$. Multiplication in $C\ell(V,\gamma)$ will be denoted as usually by juxtaposition and called *Clifford product*. We have:

$$AB = \rho_\gamma(A \otimes B), \tag{8}$$

$A, B \in C\ell(V,\gamma)$. The subspaces $\mathbb{R}, V \subset TV$ are identified with their images in $C\ell(V,\gamma)$. Then, in particular, for $u, v \in V \subset C\ell(V,\gamma)$, we have:

$$uv + vu = 2\gamma(u, v).$$

Note that since the ideal $J_\gamma \subset TV$ is inhomogeneous, of even grade, it induces a parity grading in the algebra $C\ell(V,\gamma)$, i.e.,

$$C\ell(V,\gamma) = C\ell^+(V,\gamma) \oplus C\ell^-(V,\gamma),$$

[1] For other possible definitions of Clifford algebras see, e.g., [5,6]. Here we shall be concerned only with Clifford algebras over *real* vector spaces, induced by *nondegenerate* bilinear forms.

with

$$\begin{aligned} \mathcal{C}\ell^{+}(V,\gamma) &= \rho_\gamma\left(\bigoplus_{r=0}^{\infty} T^{2r}V\right) \\ \mathcal{C}\ell^{-}(V,\gamma) &= \rho_\gamma\left(\bigoplus_{r=0}^{\infty} T^{2r+1}V\right). \end{aligned}$$

The elements of $\mathcal{C}\ell^{+}(V,\gamma)$ form a subalgebra of $\mathcal{C}\ell(V,\gamma)$, called *even subalgebra* of $\mathcal{C}\ell(V,\gamma)$.

We quote now a fundamental theorem concerning real Clifford algebras:[7]

**Universality of $\mathcal{C}\ell(V,\gamma)$** *If $\mathcal{A}$ is a real associative algebra with unity, then each linear mapping $\psi : V \to \mathcal{A}$ such that:*

$$(\psi(u))^2 = \gamma(u,u)$$

*for every $u \in V$, can be extended in a unique way to a homomorphism $C_\psi : \mathcal{C}\ell(V,\gamma) \to \mathcal{A}$, satisfying the relation:*

$$\psi = C_\psi \circ \rho_\gamma.$$

It follows from this result that if $\langle V,\gamma\rangle$ and $\langle V',\gamma'\rangle$ are two metric vector spaces and $\psi : V \to V'$ is a linear mapping satisfying:

$$\gamma'(\psi(u),\psi(v)) = \gamma(u,v) \tag{9}$$

for every $u, v \in V$, then there exists a homomorphism $C_\psi : \mathcal{C}\ell(V,\gamma) \to \mathcal{C}\ell(V',\gamma')$ between their Clifford algebras such that:

$$C_\psi \circ \rho_\gamma = \rho_{\gamma'} \circ \psi. \tag{10}$$

Moreover, if $\langle V,\gamma\rangle$ and $\langle V',\gamma'\rangle$ are metrically isomorphic[2] vector spaces, then their Clifford algebras are isomorphic. In particular, two Clifford algebras $\mathcal{C}\ell(V,\gamma)$ and $\mathcal{C}\ell(V,\gamma')$ with the same underlying vector space $V$ are isomorphic if and only if the bilinear forms $\gamma$ and $\gamma'$ which induce them have the same signature. Therefore, there is essentially one Clifford algebra for each signature on a given vector space $V$. We denote by $\mathcal{C}\ell_{p,q}$, $p+q=n$, the Clifford algebra of the vector space $\mathbb{R}^n$ endowed with a metric tensor of signature $(p,q)$.

Another important (indirect) consequence of the universality is that $\mathcal{C}\ell(V,\gamma)$ is isomorphic, as a vector space over $\mathbb{R}$, to the Grassmann algebra $\Lambda(V,\gamma)$. It is, then, $2^n$-dimensional and given $A \in \mathcal{C}\ell(V,\gamma)$ we can write:

$$A = \sum_{r=0}^{n} \langle A\rangle_r, \tag{11}$$

with $\langle A\rangle_r \in \Lambda^r V \subset \mathcal{C}\ell(V,\gamma)$.

The elements of $\mathcal{C}\ell(V,\gamma)$ will also be called *multivectors* (or *multiforms*, depending on $V$). Furthermore, if $A = \langle A\rangle_r$ for some fixed $r$, we say that $A$ is *homogeneous* of *grade* $r$.

[2] We call *metric isomorphism* a vector space isomorphism satisfying Eq. 9. The term *isometry* will be reserved to designate a metric isomorphism from a space onto itself.

In this case, we also write $A = A_r \in \Lambda^r V \subset C\ell(V,\gamma)$. The Clifford product of homogeneous multivectors $A_r, B_s \in C\ell(V,\gamma)$ is given by the relation:

$$A_r B_s = \langle A_r B_s\rangle_{|r-s|} + \langle A_r B_s\rangle_{|r-s|+2} + \ldots + \langle A_r B_s\rangle_{r+s} = \sum_{k=0}^{m} \langle A_r B_s\rangle_{|r-s|+2k}, \tag{12}$$

where $m = \frac{1}{2}(r + s - |r - s|)$.

We can introduce in $C\ell(V,\gamma)$ the following fundamental products: (cf. Hestenes[8])

**Dot product** $\cdot : C\ell(V,\gamma) \times C\ell(V,\gamma) \to C\ell(V,\gamma)$, defined, for homogeneous multivectors, by

$$A_r \cdot B_s = \langle A_r B_s\rangle_{|r-s|}. \tag{13}$$

**Exterior product** $\wedge : C\ell(V,\gamma) \times C\ell(V,\gamma) \to C\ell(V,\gamma)$, defined, for homogeneous multivectors, by

$$A_r \wedge B_s = \langle A_r B_s\rangle_{r+s}. \tag{14}$$

We observe that the exterior product is associative, i.e.,

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C = A \wedge B \wedge C. \tag{15}$$

For the dot product we have only the identities:

$$\begin{aligned} A_r \cdot (B_s \cdot C_t) &= (A_r \wedge B_s) \cdot C_t \ \text{ for } r + s \leq t, \ \ r, s > 0 \\ A_r \cdot (B_s \cdot C_t) &= (A_r \cdot B_s) \cdot C_t \ \text{ for } r + t \leq s. \end{aligned} \tag{16}$$

We can still introduce in the algebra $C\ell(V,\gamma)$ the *main automorphism* $^* : C\ell(V,\gamma) \to C\ell(V,\gamma)$; the *main antiautomorphism (reversion)* $^\dagger : C\ell(V,\gamma) \to C\ell(V,\gamma)$ and the *conjugation* $\tilde{}: C\ell(V,\gamma) \to C\ell(V,\gamma)$, given respectively by:

$$\begin{aligned} (AB)^* &= A^* B^*, \\ (AB)^\dagger &= B^\dagger A^\dagger, \\ \tilde{A} &= (A^\dagger)^*, \end{aligned} \tag{17}$$

for every $A, B \in C\ell(V,\gamma)$, with $A^* = A$ if $A \in \mathbb{R}$, $A^* = -A$ if $A \in V$ and $A^\dagger = A$ if $A \in \mathbb{R}$ or $A \in V$.

We present below some useful identities satisfied by the operations introduced above, which are valid for every $a, a_1, \ldots, a_r \in V$, $A_r \in \Lambda^r V$, $B_s \in \Lambda^s V$, $r, s \geq 0$: (see [8])

$$\begin{aligned} &\langle A^*\rangle_r = \langle A\rangle_r{}^* = (-1)^r \langle A\rangle_r \\ &\langle A^\dagger\rangle_r = \langle A\rangle_r{}^\dagger = (-1)^{r(r-1)/2} \langle A\rangle_r \\ &A_r \cdot B_s = (-1)^{r(s-1)} B_s \cdot A_r; \quad r \leq s \\ &A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r \\ &a \cdot A_r = \frac{1}{2}(aA_r - (-1)^r A_r a); \quad a \wedge A_r = \frac{1}{2}(aA_r + (-1)^r A_r a) \\ &aA_r = a \cdot A_r + a \wedge A_r; \quad A_r a = A_r \cdot a + A_r \wedge a \end{aligned} \tag{18}$$

$$\begin{aligned}
a\cdot(A_rB_s) &= (a\cdot A_r)B_s+(-1)^rA_r(a\cdot B_s)\\
&= (a\wedge A_r)B_s-(-1)^rA_r(a\wedge B_s)\\
a\wedge(A_rB_s) &= (a\wedge A_r)B_s-(-1)^rA_r(a\cdot B_s)\\
&= (a\cdot A_r)B_s+(-1)^rA_r(a\wedge B_s)
\end{aligned}$$

$$a\cdot(a_1\dots a_r)=\sum_{k=1}^{r}(-1)^{k+1}a\cdot a_k(a_1\dots a_{k-1}a_{k+1}\dots a_r)$$

$$a\cdot(a_1\wedge\dots\wedge a_r)=\sum_{k=1}^{r}(-1)^{k+1}a\cdot a_k(a_1\wedge\dots\wedge a_{k-1}\wedge a_{k+1}\wedge\dots\wedge a_r)$$

We note in addition that the Clifford interior product of multivectors with the same graduation is related to the Grassmann interior product by:

$$\langle A_r,B_r\rangle=A_r^\dagger\cdot B_r, \tag{19}$$

for every $A_r,B_r\in\Lambda^rV\subset\mathcal{C}\ell(V,\gamma), r>0$.

If the metric vector space $\langle V,\gamma\rangle$ is oriented, then we can also extend the Hodge star operator to the Clifford algebra of $V$, by letting $*:\mathcal{C}\ell(V,\gamma)\to\mathcal{C}\ell(V,\gamma)$ be given by:

$$*A=\sum_r *\langle A\rangle_r\,. \tag{20}$$

This operator satisfies, for every $A_r\in\Lambda^rV$ and $B_s\in\Lambda^sV$, $r,s\geq 0$:

$$\begin{aligned}
&A_r\wedge *B_s=B_s\wedge *A_r;\quad r=s\\
&A_r\cdot *B_s=B_s\cdot *A_r;\quad r+s=n\\
&A_r\wedge *B_s=(-1)^{r(s-1)}*(A_r^\dagger\cdot B_s);\quad r\leq s\\
&A_r\cdot *B_s=(-1)^{rs}*(A_r^\dagger\wedge B_s);\quad r+s\leq n\\
&*A_r=A_r^\dagger\cdot\tau=A_r^\dagger\,\tau\\
&*\tau=\operatorname{sgn}\bar{\gamma};\quad *1=\tau
\end{aligned} \tag{21}$$

## c. Symmetric Automorphisms and Orthogonal Clifford Products

Besides the "natural" Clifford product of $\mathcal{C}\ell(V,\gamma)$, we can introduce infinitely many other Clifford-like products on this same algebra, one for each symmetric automorphism of its underlying vector space. In what follows we are going to construct such new Clifford products, which will play an important role in the theory to be developed subsequently.[3]

First of all, let us recall that there is a one-to-one correspondence between the endomorphisms of $\langle V,\gamma\rangle$ and the bilinear forms over $V$. Indeed, to each endomorphism $\psi:V\to V$ we can associate a bilinear form $\Psi:V\times V\to\mathbb{R}$, by the relation:

$$\Psi(u,v)=\gamma(u,\psi(v)), \tag{22}$$

[3]This possibility of introducing different Clifford products in the same Clifford algebra was already established by Arcuri.[9] Our results possibly explains those obtained by her, but we shall not pay attention to this subject in this paper.

for every $u, v \in V$. Following the Hestenes' approach ([8], Sec. 3-7) we associate to the bilinear form $\Psi$ a "dot product" of vectors in the algebra $C\ell(V,\gamma)$, by letting

$$u \circ v \equiv \Psi(u,v) \equiv u \cdot \psi(v) \tag{23}$$

for every $u, v \in V \subset C\ell(V,\gamma)$.

An endomorphism $\psi : V \to V$ is said to be *symmetric* or *skew-symmetric* whether its associated bilinear form $\Psi$ is, respectively, symmetric or skew-symmetric. In the more general case we can write a bilinear form $\Psi$ as:

$$\Psi = \Psi_+ + \Psi_-,$$

with $\Psi_\pm(u,v) = \frac{1}{2}(\Psi(u,v) \pm \Psi(v,u))$, for every $u, v \in V$. Then, correspondently, its associated endomorphism $\psi$ will be written as the sum of a symmetric and a skew-symmetric endomorphism, i.e.,

$$\psi = \psi_+ + \psi_-,$$

with $\psi_+, \psi_- : V \to V$ standing for the endomorphisms associated to the bilinear forms $\Psi_+$ and $\Psi_-$, respectively.

If $\psi \equiv \psi_+$ is a symmetric automorphism (nonsingular endomorphism) of $\langle V,\gamma\rangle$, the bilinear form $\Psi$ associated to it has all the properties of a metric tensor on $V$, and it can be used to define a new Clifford algebra structure $C\ell(V,\Psi)$. It can be easily proved that $C\ell(V,\Psi)$ will be isomorphic to the original Clifford algebra $C\ell(V,\gamma)$ if and only if there exists an automorphism $\psi^{1/2} : V \to V$ such that:

$$u \cdot \psi(v) = \psi^{1/2}(u) \cdot \psi^{1/2}(v), \tag{24}$$

for every $u, v \in V$. Transformations satisfying this relation are called *positive* and $\psi^{1/2}$ is sometimes called the "square root" of $\psi$. It can be easily proved (see below) that every positive symmetric transformation possesses at most $2n$ square roots, all of them being symmetric transformations, but only one being itself positive.

If Eq. 24 is satisfied, we can reproduce the Clifford product of $C\ell(V,\Psi)$ into the algebra $C\ell(V,\gamma)$ defining an operation $\vee : C\ell(V,\gamma) \times C\ell(V,\gamma) \to C\ell(V,\gamma)$, by

$$A \vee B = \Lambda\psi^{-1/2}(\Lambda\psi^{1/2}(A)\Lambda\psi^{1/2}(B)), \tag{25}$$

for every $A, B \in C\ell(V,\gamma)$, where $\psi^{-1/2}$ is the inverse of the automorphism $\psi^{1/2}$ and by abuse of notation we have written $\Lambda\psi^{1/2} : C\ell(V,\gamma) \simeq \Lambda V \to \Lambda V \simeq C\ell(V,\gamma)$ for the mapping defined according to Eq. 3. In particular, if $u, v \in V \subset C\ell(V,\gamma)$ are vectors, then

$$u \vee v = u \circ v + u \wedge v.$$

In addition, the product $\vee : C\ell(V,\gamma) \times C\ell(V,\gamma) \to C\ell(V,\gamma)$ satisfies all the properties of a Clifford product which we have stated previously. Indeed, denoting by $\circ : C\ell(V,\gamma) \times C\ell(V,\gamma) \to C\ell(V,\gamma)$ the "dot product" induced by $\vee : C\ell(V,\gamma) \times C\ell(V,\gamma) \to C\ell(V,\gamma)$, i.e.,

$$A \circ B = \sum_{r,s>0} \langle A \vee B\rangle_{|r-s|}, \tag{26}$$

we obtain relations analogous to those given in the earlier section, with the usual dot product "$\circ$" replaced by this new one.

Furthermore, if we perform a change in the volume scale by introducing another volume $n$-vector $\tau_\Psi \in \Lambda^n V$ such that

$$\tau_\Psi^\dagger \circ \mu = (-1)^q,$$

then we can also define the analogous of the Hodge duality operation for this new Clifford product, by letting $\star : C\ell(V,\gamma) \to C\ell(V,\gamma)$ be given by:

$$A_r \wedge \star B_r = (A_r^\dagger \circ B_r)\tau_\Psi, \tag{27}$$

for every $A_r, B_r \in \Lambda^r V \subset C\ell(V,\gamma)$, $r = 0,\ldots,n$. Of course, the operation introduced in this way satisfies relations analogous to those given by Eqs. 21. In addition, it is related to the natural Hodge star operator of the algebra by:

$$\star = \Lambda\psi^{1/2} * \Lambda\psi^{-1/2}, \tag{28}$$

as can be easily verified.

Let us state our results in a more operational form. For this, we recall that every linear transformation can be expressed as composition of elementary transformations of the types $R_a : V \to V$ and $S_{ab} : V \to V$, defined by: (see [8], Sec. 3-6)

$$\begin{aligned} R_a(u) &= -aua^{-1} \\ S_{ab}(u) &= u + (u \cdot a)b, \end{aligned} \tag{29}$$

for every $u \in V$, where $a, b \in V$ are non-zero vectors parametrizing the transformation and $a^{-1} = a/a^2 = a/(a \cdot a)$. Transformations of the type $R_a$ are called elementary *reflections*. Remember that any isometry of $\langle V,\gamma\rangle$ can always be written as the composite of at most $n$ such transformations.

The skew-symmetric part of a transformation of the type "$S_{ab}$" will be denoted by $S_{[ab]}$. We have:

$$S_{[ab]}(u) = \frac{1}{2}\left(S_{ab}(u) - S_{ba}(u)\right) = \frac{1}{2} u \cdot (a \wedge b), \tag{30}$$

for every $u \in V$. By its turn, the symmetric part of a transformation of the type "$S_{ab}$" will be called a *strain*; it is a *shear* in the $a \wedge b$-plane if $a \cdot b = 0$, or a *dilation* along $a$, if $a \wedge b = 0$. Obviously, a dilation along a direction $a$ can be written more simply as:

$$S_a(u) = u + \xi(u \cdot a)\frac{a}{a^2}, \tag{31}$$

for every $u \in V$, where $\xi \in \mathbb{R}$, $\xi > -1$, is a scalar parameter. If $\xi = 0$, then $S_a$ is the identity map of $V$, for any $a \in V$. If $\xi \neq 0$, then $S_a$ is a contraction ($-1 < \xi < 0$) or a dilation ($\xi > 0$), in the direction of $a$, by a factor $1 + \xi$.

Every positive symmetric transformation can always be written as the composite of dilations along at most $n$ orthogonal directions. To see this, it is sufficient to remember that for any symmetric transformation $\psi$ we can find an orthonormal basis $\langle a_\mu\rangle$ of $V$ for which (aligned indices are not to be summed over)

$$\psi(a_\mu) = \lambda_{(\mu)} a_\mu,$$

where $\lambda_{(\mu)} \in \mathbb{R}$ is the *eigenvalue* of $\psi$ associated to the *eigenvector* $a_\mu$ ($\mu = 1, \ldots, n$). Then, defining

$$S_\mu(u) = u + \xi_{(\mu)}(u \cdot a_\mu)\frac{a_\mu}{a_\mu^2}, \tag{32}$$

for every $u \in V$, with $\xi_{(\mu)} = \lambda_{(\mu)} - 1$, we get:

$$\psi = S_1 \circ \cdots \circ S_n.$$

If the symmetric transformation $\psi$ is in addition positive, we have $a_\mu \cdot \psi(a_\mu) = \psi^{1/2}(a_\mu) \cdot \psi^{1/2}(a_\mu) = \lambda_{(\mu)} a_\mu \cdot a_\mu$. Then, since the signature of a bilinear form is preserverd by linear transformations (Sylvester's law of inertia), we conclude that:

$$\lambda_{(\mu)} = \frac{\psi^{1/2}(a_\mu) \cdot \psi^{1/2}(a_\mu)}{a_\mu \cdot a_\mu} > 0.$$

This means that in Eq. 32, $\xi_{(\mu)} = \lambda_{(\mu)} - 1 > 0$ and therefore it satisfies the definition of dilation given by Eq. 31. Note also that the positive square root of $\psi$ is given by $\psi^{1/2} = S_1^{1/2} \circ \cdots \circ S_n^{1/2}$, with

$$S_\mu^{1/2}(u) = u + \zeta_{(\mu)}(u \cdot a_\mu)\frac{a_\mu}{a_\mu^2}, \tag{33}$$

for every $u \in V$, where $\zeta_{(\mu)} = -1 + \sqrt{\lambda_{(\mu)}}$.

With these results it is trivial to give an operational form to the product defined through Eq. 25, although eventually this may demand a great deal of algebraic manipulation.

We stress finally that although we have considered only the positive symmetric transformations in the developments above, the formalism can be adapted to more general transformations. This is beyond the purposes of this paper and will be discussed elsewhere.

## 2.2. Cartan and Hodge Bundles

In this section, we discuss briefly the processes of differentiation taking place in Cartan and Hodge bundles and the formulation of differential geometry in these bundles. We shall follow basically the terminology and notation of Choquet-Bruhat.[10]

### a. Cartan Bundle

Let $M$ be a $n$-dimensional $C^\infty$ manifold, $T_x^*M$ the cotangent space of $M$ at a point $x \in M$ and $T^*M$ the cotangent bundle of $M$.

We define the *Cartan bundle* over the cotangent bundle of $M$ by:

$$\Lambda(T^*M) = \bigcup_{x \in M} \Lambda(T_x^*M) = \bigcup_{x \in M} \bigoplus_{r=0}^{n} \Lambda^r(T_x^*M),$$

where $\Lambda(T_x^*M)$, $x \in M$, is the exterior algebra of the vector space $T_x^*M$. The sub-bundle $\Lambda^r(T^*M) \subset \Lambda(T^*M)$ given by:

$$\Lambda^r(T^*M) = \bigcup_{x \in M} \Lambda^r(T_x^*M)$$

is called the *r-forms bundle* ($r = 0, \ldots, n$).

In the Cartan bundle, we can introduce the fundamental operator $d : \sec(\Lambda(T^*M)) \to \sec(\Lambda(T^*M))$,[4] called the *exterior derivative* and defined by the relations:

$$\begin{aligned} &\text{(i)} \quad d(A+B) = dA + dB; \\ &\text{(ii)} \quad d(A \wedge B) = dA \wedge B + A^* \wedge dB; \\ &\text{(iii)} \quad df(u) = u(f); \\ &\text{(iv)} \quad d^2 = 0, \end{aligned}$$

for every $A, B \in \sec(\Lambda(T^*M))$, $f \in \mathcal{F}(M)$ and $u \in \sec(TM)$.

## b. Hodge Bundle

Let us consider now an oriented metric manifold $\mathcal{M} = \langle M, \gamma, \tau \rangle$, i.e., we endow $M$ with a nondegenerate metric tensor field $\gamma \in \sec(T_0^2 M)$ of signature $(p,q)$ and with a volume $n$-form field $\tau \in \sec(\Lambda^n T^* M)$. We denote by $\gamma^{-1} \in \sec(T_2^0 M)$ the *reciprocal* of the metric tensor $\gamma$ and by $\langle\ ,\ \rangle : \sec(\Lambda(T^*M)) \times \sec(\Lambda(T^*M)) \to \mathcal{F}(M)$ the Grassmann product induced on $\Lambda(T^*M)$ by the metric tensor $\gamma^{-1}$.

We call *Hodge bundle* of the manifold $\mathcal{M}$, to the pair:

$$\Lambda(\mathcal{M}) = \langle \Lambda(T^*M), \langle\ ,\ \rangle \rangle .$$

In addition to the exterior derivative operator, we define in the Hodge bundle of $\mathcal{M}$ the *Hodge codifferential* operator $\delta : \sec(\Lambda(T^*M)) \to \sec(\Lambda(T^*M))$, given, for homogeneous multiforms, by:

$$\delta = (-1)^r *^{-1} d\, * \tag{34}$$

We can still introduce the operator $\Delta : \sec(\Lambda(T^*M)) \to \sec(\Lambda(T^*M))$, called the *Hodge Laplacian*, given by:[5]

$$\Delta = -(d\delta + \delta d). \tag{35}$$

The exterior derivative, the Hodge codifferential and the Hodge Laplacian satisfy the relations:

$$\begin{aligned} &dd = \delta\delta = 0; \quad \Delta = (d - \delta)^2 \\ &d\Delta = \Delta d; \quad \delta\Delta = \Delta\delta \\ &\delta * = (-1)^{r+1} * d; \quad * \delta = (-1)^r d * \\ &d\delta * = * \delta d; \quad * d\delta = \delta d *; \quad * \Delta = \Delta * \end{aligned} \tag{36}$$

## c. Differential Geometry in the Cartan and Hodge Bundles

Let us now endow the oriented metric manifold $\mathcal{M}$, with an arbitrary affine connection $\nabla$. The *torsion* and the *curvature* tensors of $\mathcal{M}$, associated to the connection $\nabla$, are given

[4] We denote by $\sec(X(M))$ the space of the sections of a bundle $X(M)$ and by $\mathcal{F}(M)$ the space of the differentiable, real-valued, functions on $M$.

[5] Our definition of the Hodge Laplacian differs by a sign from the definition given by Choquet-Bruhat.[10]

respectively by:

$$\begin{aligned} \boldsymbol{T}(\alpha, u, v) &= \alpha\left(\nabla_u v - \nabla_v u - [u, v]\right) \\ \boldsymbol{R}(w, \alpha, u, v) &= \alpha(\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w), \end{aligned} \tag{37}$$

for every $u, v, w \in \sec(TM)$ and $\alpha \in \sec(T^*M)$, where

$$[u, v] = uv - vu$$

is the *Lie bracket* of the vector fields $u$ and $v$.

Given an arbitrary moving frame $\langle e_\alpha \rangle$ on $TM$, we write:

$$\begin{aligned} [e_\alpha, e_\beta] &= c^\rho_{\alpha\beta} e_\rho \\ \nabla_{e_\alpha} e_\beta &= L^\rho_{\alpha\beta} e_\rho, \end{aligned}$$

where $c^\rho_{\alpha\beta}$ are the *structure coefficients* of the frame $\langle e_\alpha \rangle$ and $L^\rho_{\alpha\beta}$ are the *connection coefficients* in this frame. Then,the components of the torsion and curvature tensors are given, respectively, by:

$$\begin{aligned} T^\rho_{\alpha\beta} &= L^\rho_{\alpha\beta} - L^\rho_{\beta\alpha} - c^\rho_{\alpha\beta} \\ R_\mu{}^\rho{}_{\alpha\beta} &= e_\alpha(L^\rho_{\beta\mu}) - e_\beta(L^\rho_{\alpha\mu}) + L^\rho_{\alpha\sigma} L^\sigma_{\beta\mu} - L^\rho_{\beta\sigma} L^\sigma_{\alpha\mu} - c^\sigma_{\alpha\beta} L^\rho_{\sigma\mu}. \end{aligned} \tag{38}$$

Now, let $\langle \theta^\rho \rangle$ be the *dual frame* of $\langle e_\alpha \rangle$ (i.e., $\theta^\rho(e_\alpha) = \delta^\rho_\alpha$ ). Then we have:

$$\begin{aligned} d\theta^\rho &= -\frac{1}{2} c^\rho_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \\ \nabla_{e_\alpha} \theta^\rho &= -L^\rho_{\alpha\beta} \theta^\beta \end{aligned} \tag{39}$$

and we introduce the *connection 1-forms* $\omega^\rho_\beta \in \sec(T^*M)$, the *torsion 2-forms* $\Theta^\rho \in \sec(\Lambda^2(T^*M))$ and the *curvature 2-forms* $\Omega^\rho_\beta \in \sec(\Lambda^2(T^*M))$, by the relations:

$$\begin{aligned} \omega^\rho_\beta &= L^\rho_{\alpha\beta} \theta^\alpha, \\ \Theta^\rho &= \frac{1}{2} T^\rho_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \\ \Omega^\rho_\mu &= \frac{1}{2} R_\mu{}^\rho{}_{\alpha\beta} \theta^\alpha \wedge \theta^\beta. \end{aligned} \tag{40}$$

Multiplying Eqs. 38 by $\frac{1}{2}\theta^\alpha \wedge \theta^\beta$ and using Eqs. 39 and 40, we get the *Cartan's structure equations*:

$$\begin{aligned} \boldsymbol{d}\theta^\rho &\equiv d\theta^\rho + \omega^\rho_\beta \wedge \theta^\beta = \Theta^\rho \\ \boldsymbol{d}\omega^\rho_\mu &\equiv d\omega^\rho_\mu + \omega^\rho_\beta \wedge \omega^\beta_\mu = \Omega^\rho_\mu, \end{aligned} \tag{41}$$

where $\boldsymbol{d} : \sec(\Lambda(T^*M)) \to \sec(\Lambda(T^*M))$ denotes the *covariant exterior derivative* related to the connection $\nabla$.

Since we are dealing with a metric manifold, we must complete Cartan's structure equations with the equations stating the relation between the connection and the metric. For this, following the usual nomenclature,[11,12,13] we write:

$$Q_{\alpha\beta\mu} = -\nabla_\alpha \gamma_{\beta\mu} = -e_\alpha(\gamma_{\beta\mu}) + \gamma_{\sigma\mu} L^\sigma_{\alpha\beta} + \gamma_{\beta\sigma} L^\sigma_{\alpha\mu} \tag{42}$$

and we call *nonmetricity* to the tensor field having these components.[6] Correspondently, we introduce the *nonmetricity 2-forms*, by:

$$\boldsymbol{Q}^{\rho} = \frac{1}{2} Q^{\rho}_{[\alpha\beta]} \theta^{\alpha} \wedge \theta^{\beta}, \tag{43}$$

where $Q^{\rho}_{[\alpha\beta]} = \gamma^{\rho\mu}(Q_{\alpha\beta\mu} - Q_{\beta\alpha\mu})$. Multiplying Eq. 42 by $\theta^{\alpha} \wedge \theta^{\beta}$ and using Eq. 41a, we get:

$$\boldsymbol{d}\theta_{\mu} \equiv d\theta_{\mu} - \omega^{\beta}_{\mu} \wedge \theta_{\beta} = \Phi_{\mu}, \tag{44}$$

where $\langle \theta_{\mu} \rangle$ is the *reciprocal frame* of $\langle \theta^{\rho} \rangle$ (i.e., $\theta_{\mu} = \gamma_{\mu\nu}\theta^{\nu}$) and

$$\Phi_{\mu} = \Theta_{\mu} - \boldsymbol{Q}_{\mu}.$$

Eq. 44 can be used as the complement of Cartan's structure equations for the case of a metric manifold.

Differentiating Eqs. 41 and Eq. 44 we obtain the *Bianchi identities*:[7]

$$\begin{aligned} \boldsymbol{d}\Theta^{\rho} &= d\Theta^{\rho} + \omega^{\rho}_{\beta} \wedge \Theta^{\beta} = \Omega^{\rho}_{\beta} \wedge \theta^{\beta} \\ \boldsymbol{d}\Omega^{\rho}_{\mu} &= d\Omega^{\rho}_{\mu} - \Omega^{\rho}_{\beta} \wedge \omega^{\beta}_{\mu} + \omega^{\rho}_{\beta} \wedge \Omega^{\beta}_{\mu} = 0 \\ \boldsymbol{d}\Phi_{\mu} &= d\Phi_{\mu} - \omega^{\beta}_{\mu} \wedge \Phi_{\beta} = -\Omega^{\beta}_{\mu} \wedge \theta_{\beta}. \end{aligned} \tag{45}$$

Let us recall finally that any triple $\langle M, \gamma, \nabla \rangle$ where $\langle \gamma, \nabla \rangle$ is a gemetrical structure on $M$ such that

$$\nabla\gamma = 0 \qquad \text{and} \qquad \boldsymbol{T}[\nabla] \neq 0 \tag{46}$$

is called *Cartan space*. By another side, if

$$\nabla\gamma \neq 0 \qquad \text{and} \qquad \boldsymbol{T}[\nabla] = 0 \tag{47}$$

then $\langle M, \gamma, \nabla \rangle$ is called *Weyl space*. Finally, if torsion and nonmetricity are both null, i.e., if

$$\nabla\gamma = 0 \qquad \text{and} \qquad \boldsymbol{T}[\nabla] = 0, \tag{48}$$

then $\langle M, \gamma, \nabla \rangle$ is called *Riemann space* and the pair $\langle \gamma, \nabla \rangle$ is called *Riemannian structure*. In addition, if torsion and nonmetricity are both non-null, then $\langle M, \gamma, \nabla \rangle$ is called *Riemann-Cartan-Weyl space*.

For each metric tensor defined on the manifold $M$ there exists one and only one connection in the conditions of Eq. 48, which is called *Levi-Civita connection* of the metric considered. With the exception of the covariant derivative associated to the Levi-Civita connection (which will be denoted simply by $D$) and their coefficients (which will be denoted by $\Gamma^{\rho}_{\alpha\beta}$) any other quantity referring to a Riemannian structure will be denote by a hat over its usual symbol.

---

[6] We use the notation $\nabla_{\sigma} t^{\mu\dots}_{\nu\dots} \equiv (\nabla_{e_{\sigma}} t)^{\mu\dots}_{\nu\dots} \equiv (\nabla t)^{\mu\dots}_{\sigma\nu\dots}$ for the components of the covariant derivative of a tensor field $t$. This is not to be confused with $\nabla_{e_{\sigma}} t^{\mu\dots}_{\nu\dots} \equiv e_{\sigma}(t^{\mu\dots}_{\nu\dots})$, the derivative of the components of $t$ in the direction of $e_{\sigma}$.

[7] To our knowledge, Eqs. 44 and 45c are not found anywhere in the literature, although they appear to be the most natural extension of the structure equations for metric manifolds. We include Eq. 45c among the Bianchi identities for completeness.

## 3. Differential Geometry in the Clifford Bundle

We call *Clifford bundle* of the (oriented) metric manifold $\mathcal{M} = \langle M, \gamma, \tau \rangle$ to the vector bundle:

$$\mathcal{C}\ell(\mathcal{M}) = \frac{T(M)}{J_{\gamma^{-1}}} = \bigcup_{x \in M} \mathcal{C}\ell(T_x^* M, \gamma_x^{-1}),$$

where $T(M)$ denotes the tensor bundle of $M$, $J_{\gamma^{-1}} \subset T(M)$ is the ideal of $T(M)$ generated by the elements of the form $\alpha \otimes \beta + \beta \otimes \alpha - 2\gamma^{-1}(\alpha, \beta)$, with $\alpha, \beta \in \sec(T^* M) \subset T(M)$ and $\mathcal{C}\ell(T_x^* M, \gamma_x^{-1})$ is the Clifford algebra of the metric vector space $\langle T_x^* M, \gamma_x^{-1} \rangle$. It can be shown that:[14]

$$\mathcal{C}\ell(\mathcal{M}) = P_{SO_+(p,q)} \times_{Ad} \mathbb{R}_{p,q},$$

where $P_{SO_+(p,q)}$ is the principal bundle of orthonormal frames and $Ad$ is the *adjoint representation* of $\mathrm{Spin}_+(p,q)$, i.e., $Ad : SO_+(p,q) \to \mathrm{Aut}(\mathbb{R}_{p,q})$, $Ad : SO_+(p,q) \to \mathrm{Aut}(\mathbb{R}_{p,q})$, $u \mapsto Ad_u$, with $Ad_u x = u x u^{-1}$, $\forall u \in SO_+(p,q)$, $\forall x \in \mathbb{R}_{p,q}$.

It can be defined in the Clifford bundle a differential operator $\partial$, here called the fundamental Dirac operator, which is closely related to the Levi-Civita connection of $\mathcal{M}$. In what follows we shall study this operator and subsequently we will show that it can be generalized to connections other than the Levi-Civita, i.e., to connections defining a general Riemann-Cartan-Weyl geometry. Moreover, making use of the results developed in Sec. 2.1.c, we will show that it is possible to introduce infinitely many others Dirac-like operators, one for each bilinear form field on the manifold $\mathcal{M}$. These constructions will enable us to formulate the geometry of Riemann-Cartan-Weyl spaces in the Clifford bundle. Some new geometrical concepts, like the Dirac commutator and anticommutator, will be introduced and we will present a new decomposition of a general affine connection, identifying some new relevant tensors which are important for the clear understanding of the formulation of the gravitational theory in flat space and other related subjects appearing in the literature.

### 3.1. The Fundamental Dirac Operator

#### a. Definition; Basic Properties

Given $u \in \sec(TM)$ and $\tilde{u} \in \sec(T^*M) \subset \sec(\mathcal{C}\ell(\mathcal{M}))$, consider the tensorial mapping $\psi \mapsto \tilde{u} D_u \psi$, $\psi \in \sec(\mathcal{C}\ell(\mathcal{M}))$. Since $D_u J_{\gamma^{-1}} \subseteq J_\gamma^{-1}$, where $J_{\gamma^{-1}}$ is the ideal used in the definition of $\mathcal{C}\ell(\mathcal{M})$, the notion of covariant derivative (related to the Levi-Civita connection) pass to the quotient bundle $\mathcal{C}\ell(\mathcal{M})$ and we can define the *fundamental Dirac operator* (or *fundamental Dirac derivative*)

$$\partial = \mathrm{Tr}(\tilde{u}\, D_u).$$

If $\langle \theta^\alpha \rangle$ is a moving frame on $T^*M$, dual to the moving frame $\langle e_\alpha \rangle$ on $TM$, we have:

$$\partial = \theta^\alpha\, D_{e_\alpha} \tag{49}$$

For $A \in \sec(\mathcal{C}\ell(\mathcal{M}))$,

$$\partial A = \theta^\alpha (D_{e_\alpha} A) = \theta^\alpha \cdot (D_{e_\alpha} A) + \theta^\alpha \wedge (D_{e_\alpha} A)$$

and then we define:

$$\begin{aligned} \partial \cdot A &= \theta^\alpha \cdot (D_{e_\alpha} A) \\ \partial \wedge A &= \theta^\alpha \wedge (D_{e_\alpha} A), \end{aligned}$$

in order to have:

$$\partial = \partial \cdot + \partial \wedge \tag{50}$$

It is easily established that the operators $\partial\cdot$ and $\partial\wedge$ satisfy the following identities:

$$\begin{aligned} &\partial \wedge (A \wedge B) = (\partial \wedge A) \wedge B + A^* \wedge (\partial \wedge B) \\ &\partial \cdot (A_r \cdot B_s) = (\partial \wedge A_r) \cdot B_s + A_r{}^* \cdot (\partial \cdot B_s); \quad r+1 \leq s \\ &\partial \cdot * = (-1)^r * \partial \wedge; \quad * \partial \cdot = (-1)^{r+1} \partial \wedge * \end{aligned} \tag{51}$$

In addition to these identities, we have the following fundamental result:[14,15,16]

**Proposition 1** *The fundamental Dirac derivative $\partial$ is related to the exterior derivative $d$ and to the Hodge codifferential $\delta$ by:*

$$\partial = d - \delta, \tag{52}$$

*that is, we have $\partial\wedge = d$ and $\partial\cdot = -\delta$.*

**Proof** If $f$ is a function, $\partial \wedge f = \theta^\alpha \wedge D_{e_\alpha} f = e_\alpha(f)\theta^\alpha = df$ and $\partial \cdot f = \theta^\alpha \cdot D_{e_\alpha} f = 0$. For the 1-form fields $\theta^\rho$ of a moving frame on $T^*M$, we have $\partial \wedge \theta^\rho = \theta^\alpha \wedge D_{e_\alpha}\theta^\rho = -\Gamma^\rho_{\alpha\beta}\theta^\alpha \wedge \theta^\beta = -\hat{\omega}^\rho_\beta \wedge \theta^\beta = d\theta^\rho$.

Now, for a $r$-forms field $\omega = \frac{1}{r!}\omega_{\alpha_1 \ldots \alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r}$, we get, using Eq. 51a, $\partial \wedge \omega = \frac{1}{r!}(d\omega_{\alpha_1 \ldots \alpha_r} \wedge \theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_r} + \omega_{\alpha_1 \ldots \alpha_r} d\theta^{\alpha_1} \wedge \theta^{\alpha_2} \wedge \ldots \wedge \theta^{\alpha_r} + \cdots + (-1)^{r+1}\omega_{\alpha_1 \ldots \alpha_r}\theta^{\alpha_1} \wedge \ldots \wedge \theta^{\alpha_{r-1}} \wedge d\theta^{\alpha_r}) = d\omega$. Finally, using Eq. 36c and Eq. 51c, we get $\partial \cdot \omega = -\delta\omega$. ■

Note finally that given an arbitrary coordinate moving frame $\langle \theta^\rho = dx^\rho \rangle$ on $M$ ($x^\rho : U \to \mathbb{R}$, $U \subset M$, are coordinate functions), we have the following interesting relations:

$$\begin{aligned} \partial \cdot \theta^\rho &= \gamma^{\alpha\beta}\Gamma^\rho_{\alpha\beta} = \frac{1}{\sqrt{|\gamma|}}\partial_\sigma(\sqrt{|\gamma|}\gamma^{\rho\sigma}) \\ \partial \cdot \theta_\sigma &= \Gamma^\rho_{\rho\sigma} = \frac{1}{\sqrt{|\gamma|}}\partial_\sigma(\sqrt{|\gamma|}), \end{aligned} \tag{53}$$

where $\langle \partial_\sigma \rangle$ is the dual frame of $\langle dx^\rho \rangle$.

### b. Dirac Commutator and Dirac Anticommutator

Given the 1-form fields $\alpha, \beta \in \sec(T^*M) \subset \sec(\mathcal{C}\ell(\mathcal{M}))$, we define:

$$\begin{aligned} [\![\alpha, \beta]\!] &= (\alpha \cdot \partial)\beta - (\beta \cdot \partial)\alpha \\ \{\alpha, \beta\} &= (\alpha \cdot \partial)\beta + (\beta \cdot \partial)\alpha, \end{aligned} \tag{54}$$

where $\partial$ denotes the fundamental Dirac operator of the manifold. These operations will be called, respectively, the *Dirac commutator* and the *Dirac anticommutator* of the 1-form fields $\alpha$ and $\beta$.

We note that we have the identities:

$$
\begin{aligned}
[\![\alpha,\beta]\!] &= \partial\cdot(\alpha\wedge\beta)-[(\partial\cdot\alpha)\wedge\beta-\alpha\wedge(\partial\cdot\beta)] \\
\{\alpha,\beta\} &= \partial\wedge(\alpha\cdot\beta)-[(\partial\wedge\alpha)\cdot\beta-\alpha\cdot(\partial\wedge\beta)].
\end{aligned}
\tag{55}
$$

The algebraic meaning of these equations is clear: they state that the Dirac commutator and the Dirac anticommutator measure the amount by which the operators $\partial\cdot = -\delta$ and $\partial\wedge = d$ fail to satisfy the Leibniz rule when applied, respectively, to the exterior and to the dot product of 1-form fields.

Now, let $\langle e_\alpha\rangle$ be an arbitrary moving frame on $TM$, $\langle\theta^\rho\rangle$ its dual frame on $T^*M$ and $\langle\theta_\alpha\rangle$ the reciprocal frame of $\langle\theta^\rho\rangle$. From Eqs. 54 we obtain, respectively:

$$
\begin{aligned}
[\![\theta_\alpha,\theta_\beta]\!] &= D_{e_\alpha}\theta_\beta - D_{e_\beta}\theta_\alpha \\
&= (\Gamma^\rho_{\alpha\beta}-\Gamma^\rho_{\beta\alpha})\theta_\rho \\
&= c^\rho_{\alpha\beta}\theta_\rho,
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
\{\theta_\alpha,\theta_\beta\} &= D_{e_\alpha}\theta_\beta + D_{e_\beta}\theta_\alpha \\
&= (\Gamma^\rho_{\alpha\beta}+\Gamma^\rho_{\beta\alpha})\theta_\rho \\
&= b^\rho_{\alpha\beta}\theta_\rho,
\end{aligned}
\tag{57}
$$

where $\Gamma^\rho_{\alpha\beta}$ are the components of the Levi-Civita connection $D$ of $\gamma$, $c^\rho_{\alpha\beta}$ are the structure coefficients of the frame $\langle e_\alpha\rangle$ and we are introducing the notation $b^\rho_{\alpha\beta}=\Gamma^\rho_{\alpha\beta}+\Gamma^\rho_{\beta\alpha}$. (The meaning of these coefficients will be discussed below.)

Clearly, Eq. 56 states that the Dirac commutator is the analogous of the Lie bracket of vector fields. These operations have similar properties. In particular, the Dirac commutator satisfies the *Jacobi identity*:

$$
[\![\alpha,[\![\beta,\omega]\!]]\!]+[\![\beta,[\![\omega,\alpha]\!]]\!]+[\![\omega,[\![\alpha,\beta]\!]]\!]=0,
$$

$\alpha,\beta,\omega\in\sec(T^*M)\subset\mathcal{C}\ell(\mathcal{M})$. Therefore it gives to the cotangent bundle of $M$ the structure of a Lie algebra.

The geometrical meaning of the Dirac commutator and the Dirac anticommutator is also easily stated from Eqs. 56 and 57. In fact, Eq. 56 means that the Dirac commutator measures the amount by which the "vectors" $\theta_\alpha$ and $\theta_\beta$ and their infinitesimal lifts along the "integral lines" of each other fail to form a parallelogram. By its turn, Eq. 57 means that the Dirac anticommutator measures the rate of deformation of the frame $\langle\theta_\alpha\rangle$: $\{\theta_\alpha,\theta_\alpha\}$ gives the rate of dilation of the field $\theta_\alpha$ under dislocations along its own integral lines, while $\{\theta_\alpha,\theta_\beta\}$, $\alpha\neq\beta$, gives the rate of variation of the angle between $\theta_\alpha$ and $\theta_\beta$, under dislocations in the direction of each other.

We state now our second fundamental result:

**Proposition 2** *The coefficients $b^\rho_{\alpha\beta}$ of the Dirac anticommutator in a moving frame $\langle\theta_\alpha\rangle$ are given by:*

$$
b^\rho_{\alpha\beta} = -(\pounds_{e^\rho}\gamma)_{\alpha\beta},
\tag{58}
$$

*where $\pounds_{e^\rho}$ denotes the Lie derivative in the direction of the vector field $e^\rho$ and $\langle e^\rho \rangle$ is the dual frame of $\langle \theta_\alpha \rangle$.*

**Proof** The coefficients $\Gamma^\rho_{\alpha\beta}$ of the Levi-Civita connection of $\gamma$ are given by: (e.g., Choquet-Bruhat[10])

$$\begin{aligned} \Gamma^\rho_{\alpha\beta} &= \frac{1}{2}\gamma^{\rho\sigma}\left[e_\alpha(\gamma_{\beta\sigma}) + e_\beta(\gamma_{\sigma\alpha}) - e_\sigma(\gamma_{\alpha\beta})\right] \\ &+ \frac{1}{2}\gamma^{\rho\sigma}\left[\gamma_{\mu\alpha}c^\mu_{\sigma\beta} + \gamma_{\mu\beta}c^\mu_{\sigma\alpha} - \gamma_{\mu\sigma}c^\mu_{\alpha\beta}\right]. \end{aligned} \tag{59}$$

Hence,

$$b^\rho_{\alpha\beta} = \gamma^{\rho\sigma}\left[e_\beta(\gamma_{\alpha\sigma}) + e_\alpha(\gamma_{\sigma\beta}) - e_\sigma(\gamma_{\beta\alpha}) - \gamma_{\mu\alpha}c^\mu_{\beta\sigma} - \gamma_{\mu\beta}c^\mu_{\alpha\sigma}\right] \tag{60}$$

and the r.h.s. of Eq. 60 is just the negative of the components of the Lie derivative of the metric tensor $\gamma$ in the direction of $e^\rho = \gamma^{\rho\sigma}e_\sigma$. ∎

In view of the result stated by Eq. 58, the attempt to find (if existing) a moving frame for which $b^\rho_{\alpha\beta} = 0$ is equivalent to solve, locally, the Killing equations for the manifold. Because of this we shall refer to these coefficients as the *Killing coefficients* of the frame. Of course, since the solutions of the Killing equations are restricted by the structure of the metric as well as by the topology of the manifold, it will not be possible, in the more general case, to find any moving frame for which these coefficients are all null.

### c. Associated Dirac Operators

In view of the results stated in Sec. 2.1.c, it is clear that besides the fundamental Dirac operator we have just analyzed, we can also introduce in the Clifford bundle $\mathcal{C}\ell(\mathcal{M})$ infinitely many other Dirac-like operators, one for each nondegenerate symmetric bilinear form field that can be defined on the metric manifold $\mathcal{M}$.

Hereafter we convention to denote by $g \in \sec(T^2_0 M)$ an arbitrarily fixed nondegenerate *positive* symmetric bilinear form field on $\mathcal{M}$, by $g^{-1} \in \sec(T^0_2 M)$ its *reciprocal* bilinear form on $T^*M$ and by $h \in \sec(T^1_1 M)$ the "field of linear transformations" which induces $g$. Observe that we have:

$$g^{-1}(\alpha,\beta) = \alpha \cdot h^{-1}(\beta) = h^{-1/2}(\alpha) \cdot h^{-1/2}(\beta),$$

for every $\alpha, \beta \in \sec(T^*M)$, where $h^{-1}$ and $h^{-1/2}$ are the *reciprocals* of the transformations $h$ and $h^{1/2}$ respectively.[8] We stress that the bilinear form field $g$ is not to be confused with the metric tensor $\gamma$ of the manifold.

We also denote by $\vee : \mathcal{C}\ell(\mathcal{M}) \times \mathcal{C}\ell(\mathcal{M}) \to \mathcal{C}\ell(\mathcal{M})$ the "Clifford product" induced on $\mathcal{C}\ell(\mathcal{M})$ by the bilinear form field $g^{-1}$ and by $\circ : \mathcal{C}\ell(\mathcal{M}) \times \mathcal{C}\ell(\mathcal{M}) \to \mathcal{C}\ell(\mathcal{M})$ the "Clifford dot product" associated to "$\vee$."

We call Dirac operator *associated* to the bilinear form $g^{-1} \in \sec(T^2_0 M)$ the operator:

$$\check{\partial} \equiv \partial\vee = \mathrm{Tr}(\tilde{u} \vee D_u).$$

[8] We call *reciprocal* of a linear transformation $\psi : V \to V$ to the transformation $\psi^{-1} : V^* \to V^*$ defined by $\psi^{-1} \circ \gamma_* = \gamma_* \circ \psi$, where $(\gamma_* u)(v) = \gamma(u,v)$, for every $u, v \in V$. There is no risk of confusion with the *inverse* of the transformation $\psi$, since this last one is a transformation of $V$ onto $V$.

With respect to a moving frame $\langle\theta^\alpha\rangle$ of $T^*M$ we have:

$$\check{\boldsymbol{\partial}} = \theta^\alpha \vee D_{e_\alpha}, \tag{61}$$

where $\langle e_\alpha\rangle$ is the dual frame of $\langle\theta^\alpha\rangle$. We define also

$$\check{\boldsymbol{\partial}}\circ\ = \theta^\alpha \circ D_{e_\alpha},$$

so that

$$\check{\boldsymbol{\partial}} = \check{\boldsymbol{\partial}}\circ\ + \ \check{\boldsymbol{\partial}}\wedge = \check{\boldsymbol{\partial}}\circ\ + \ \boldsymbol{\partial}\wedge, \tag{62}$$

because the exterior part of the operator $\check{\boldsymbol{\partial}}$ coincides with that of the operator $\boldsymbol{\partial}$.

Of course, the properties of the operator $\check{\boldsymbol{\partial}}$ differ from those of the fundamental Dirac operator $\boldsymbol{\partial}$. It is enough to state the properties of the operator $\check{\boldsymbol{\partial}}\circ$ , which are obtained from the following proposition:

**Proposition 3** *The operators $\check{\boldsymbol{\partial}}\circ$ and $\boldsymbol{\partial}\cdot$ are related by:*

$$\check{\boldsymbol{\partial}}\circ\omega = \boldsymbol{\partial}\cdot\check{\omega} + s\cdot\check{\omega}, \tag{63}$$

*for every $\omega \in \sec(\mathcal{C}\ell(\mathcal{M}))$, where $s = g^{\rho\sigma}D_\rho g_{\sigma\mu}\theta^\mu \in \sec(T^*M)$ is called the dilation 1-form of the bilinear form $g$.*

**Proof** Given a $r$-forms field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r} \in \sec(\mathcal{C}\ell(\mathcal{M}))$, we have $D_{e_\rho}\omega = \frac{1}{r!}(D_\rho\omega_{\alpha_1\ldots\alpha_r})\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, with

$$D_\rho\omega_{\alpha_1\ldots\alpha_r} = e_\rho(\omega_{\alpha_1\ldots\alpha_r}) - \Gamma^\mu_{\rho\alpha_1}\omega_{\mu\alpha_2\ldots\alpha_r} - \cdots - \Gamma^\mu_{\rho\alpha_r}\omega_{\alpha_1\ldots\alpha_{r-1}\mu}. \tag{64}$$

Then, $\theta^\rho\circ D_{e_\rho}\omega = \frac{1}{r!}D_\rho\omega_{\alpha_1\ldots\alpha_r}\theta^\rho\circ(\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}) = \frac{1}{r!}D_\rho\omega_{\alpha_1\ldots\alpha_r}(g^{\rho\alpha_1}\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r}+\cdots+(-1)^{r+1}g^{\rho\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_{r-1}})$, or

$$\check{\boldsymbol{\partial}}\circ\omega = \frac{1}{(r-1)!}g^{\rho\sigma}D_\rho\omega_{\sigma\alpha_2\ldots\alpha_r}\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r}. \tag{65}$$

Now, taking into account the identities $g^{\rho\sigma}D_\rho\omega_{\sigma\alpha_2\ldots\alpha_r} = D_\rho(g^{\rho\sigma}\omega_{\sigma\alpha_2\ldots\alpha_r}) - (D_\rho g^{\rho\sigma})\omega_{\sigma\alpha_2\ldots\alpha_r}$; $g_{\sigma\mu}D_\rho g^{\rho\sigma} = -g^{\rho\sigma}D_\rho g_{\sigma\mu}$ and recalling also that $g^{\rho\sigma} = \gamma^{\rho\mu}\mathrm{g}^\sigma_\mu$, we conclude that

$$\begin{aligned}\check{\boldsymbol{\partial}}\circ\omega &= \frac{1}{(r-1)!}\gamma^{\rho\sigma}(D_\rho\mathrm{g}^\mu_\sigma\omega_{\mu\alpha_2\ldots\alpha_r})\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r}\\ &+ \frac{1}{(r-1)!}\gamma^{\rho\sigma}(g^{\alpha\beta}D_\alpha g_{\beta\rho})\mathrm{g}^\mu_\sigma\omega_{\mu\alpha_2\ldots\alpha_r}\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r}.\end{aligned}$$

Therefore, writing $\check{\omega}_{\sigma\alpha_2\ldots\alpha_r} = \mathrm{g}^\mu_\sigma\omega_{\mu\alpha_2\ldots\alpha_r}$ and $s_\rho = g^{\alpha\beta}D_\alpha g_{\beta\rho}$, we obtain the Eq. (63). ■

## 3.2. The Dirac Operator in Riemann-Cartan-Weyl Spaces

### a. Definition; Basic Properties

We now consider the manifold $\mathcal{M} = \langle M,\gamma,\tau\rangle$ endowed with an arbitrary affine connection $\nabla$. In this case, the notion of covariant derivative does not pass to the quotient bundle

$C\ell(\mathcal{M})$.[7] Despite this, it is still a well defined operation and in analogy with the earlier section, we can associate to it, acting on the sections of $C\ell(\mathcal{M})$, the operator:

$$\partial = \mathrm{Tr}(\tilde{u}\,\nabla u).$$

We call this operator simply the *Dirac operator* (or *Dirac derivative*). Also as before, if $\langle\theta^\alpha\rangle$ is a moving frame on $T^*M$, dual to the moving frame $\langle e_\alpha\rangle$ on $TM$, we have:

$$\partial = \theta^\alpha\,\nabla_{e_\alpha} \tag{66}$$

and we define:

$$\begin{aligned}\partial\cdot A &= \theta^\alpha\cdot(\nabla_{e_\alpha}A)\\ \partial\wedge A &= \theta^\alpha\wedge(\nabla_{e_\alpha}A),\end{aligned}$$

for every $A\in\sec(C\ell(\mathcal{M}))$, so that:

$$\partial = \partial\cdot + \partial\wedge \tag{67}$$

The operator $\partial\wedge$ satisfies, for every $A, B\in\sec(C\ell(\mathcal{M}))$:

$$\partial\wedge(A\wedge B) = (\partial\wedge A)\wedge B + A^*\wedge(\partial\wedge B), \tag{68}$$

what generalizes Eq. 51a. By its turn, Eq. 51c is generalized according to the following proposition:

**Proposition 4** *Let $Q^\rho$ be the nonmetricity 2-forms associated with the connection $\nabla$ in an arbitrary moving frame $\langle\theta^\rho\rangle$. Then we have, for homogeneous multiforms,*

$$\begin{aligned}(-1)^r *^{-1}\partial\cdot * &= \partial\wedge + Q^\rho\wedge i_\rho,\\ (-1)^{r+1} *^{-1}\partial\wedge * &= \partial\cdot - Q^\rho\cdot j_\rho,\end{aligned} \tag{69}$$

*where $i_\rho A = \theta_\rho\cdot A$ and $j_\rho A = \theta_\rho\wedge A$, for every $A\in\sec(C\ell(\mathcal{M}))$.*

**Proof** Let $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}\in\sec(\Lambda^r(\mathcal{M}))\subset\sec(C\ell(\mathcal{M}))$ be a $r$-form field on $M$. We have $(\theta_{\beta_1}\wedge\ldots\wedge\theta_{\beta_r})\wedge *\omega = ((\theta_{\beta_r}\wedge\ldots\wedge\theta_{\beta_1})\cdot\omega)\tau = \omega_{\beta_1\ldots\beta_r}\tau$ and it follows that $\nabla_{e_\sigma}[(\theta_{\beta_1}\wedge\ldots\wedge\theta_{\beta_r})\wedge *\omega] = e_\sigma(\omega_{\beta_1\ldots\beta_r})\tau$. But on the other hand, we also have $\nabla_{e_\sigma}[(\theta_{\beta_1}\wedge\ldots\wedge\theta_{\beta_r})\wedge *\omega] = \theta_{\beta_1}\wedge\ldots\wedge\theta_{\beta_r}\wedge\nabla_{e_\sigma} *\omega + (L^\rho_{\sigma\beta_1}\omega_{\rho\beta_2\ldots\beta_r}+\cdots+ L^\rho_{\sigma\beta_r}\omega_{\beta_1\ldots\beta_{r-1}\rho})\tau - (Q^\rho_{\sigma\beta_1}\omega_{\rho\beta_2\ldots\beta_r}+\cdots+Q^\rho_{\sigma\beta_r}\omega_{\beta_1\ldots\beta_{r-1}\rho})\tau$ and therefore we get, after some algebraic manipulation:

$$\nabla_{e_\sigma} *\omega = *\nabla_{e_\sigma}\omega + Q_{\sigma\mu\nu} *(\theta^\mu\wedge(\theta^\nu\cdot\omega)), \tag{70}$$

from which Eqs. 69 follow immediately. ■

Taking into account the result stated in the above proposition and the definition of the Hodge codifferential (Eq. 34), we are motivated to introduce in the Clifford bundle the *Dirac coderivative* operator, given, for homogeneous multiforms, by:

$$\tilde{\partial} = (-1)^r *^{-1}\partial\, * \tag{71}$$

Of course, we have:

$$\tilde{\partial} = (-1)^r *^{-1} \partial \cdot * \ + \ (-1)^r *^{-1} \partial \wedge *$$

and we can, then, define:

$$\begin{aligned} \tilde{\partial}\cdot &\equiv (-1)^r *^{-1} \partial \wedge * = -\partial \cdot + Q^\rho \cdot j_\rho \\ \tilde{\partial}\wedge &\equiv (-1)^r *^{-1} \partial \cdot * = \partial \wedge + Q^\rho \wedge i_\rho, \end{aligned} \tag{72}$$

so that:

$$\tilde{\partial} = \tilde{\partial} \cdot + \tilde{\partial}\wedge \tag{73}$$

The following identities are trivially established:

$$\begin{aligned} &\partial = (-1)^{r+1} *^{-1} \tilde{\partial} * \\ &*\partial = (-1)^{r+1}\tilde{\partial}*; \quad *\tilde{\partial} = (-1)^r \partial * \\ &\partial\tilde{\partial}* = *\tilde{\partial}\partial; \quad *\partial\tilde{\partial} = \tilde{\partial}\partial* \\ &*\partial^2 = -\tilde{\partial}^2*; \quad *\tilde{\partial}^2 = -\partial^2* \end{aligned} \tag{74}$$

In addition, we note that the Dirac coderivative permit us to generalize Eq. 51b in a very elegant way. In fact, in consequence of Prop. 4 we have:

**Corolary** *For every $A_r \in \sec(\Lambda^r(\mathcal{M})) \subset \sec(C\ell(\mathcal{M}))$ and $B_s \in \sec(\Lambda^s(\mathcal{M})) \subset \sec(C\ell(\mathcal{M}))$, with $r + 1 \leq s$, it holds:*

$$\partial \cdot (A_r \cdot B_s) = (\tilde{\partial} \wedge A_r) \cdot B_s + (-1)^r A_r \cdot (\partial \cdot B_s). \tag{75}$$

**Proof** Given a 1-form field $\alpha \in \sec(T^*M) \subset \sec(C\ell(\mathcal{M}))$ and a $r$-form field $\omega \in \sec(\Lambda^r(\mathcal{M})) \subset \sec(C\ell(\mathcal{M}))$, we have, from Eq. 70, that $\nabla_{e_\sigma} * (\alpha \cdot \omega) = *\nabla_{e_\sigma}(\alpha \cdot \omega + Q_{\sigma\mu\nu} * [\theta^\mu \wedge (\theta^\nu \cdot (\alpha \cdot \omega))]$. We have also that $\nabla_{e_\sigma} * (\alpha \cdot \omega) = (-1)^{r+1}\nabla_{e_\sigma}(\alpha \wedge *\omega) = *[(\nabla_{e_\sigma}\alpha) \cdot \omega + \alpha \cdot (\nabla_{e_\sigma}\omega + Q_{\sigma\mu\nu}\alpha \cdot (\theta^\mu \wedge (\theta^\nu \cdot \omega))]$, where we have used Eq. 70 once again. It follows that:

$$\nabla_{e_\sigma}(\alpha \cdot \omega) = (\nabla_{e_\sigma}\alpha) \cdot \omega + \alpha \cdot (\nabla_{e_\sigma}\omega) + Q_{\sigma\mu\nu}\alpha^\mu\theta^\nu \cdot \omega. \tag{76}$$

Then, recalling that $(\alpha_1 \wedge \ldots \wedge \alpha_r) \cdot \omega = \alpha_1 \cdot \ldots \cdot \alpha_r \cdot \omega$, with $\alpha_1, \ldots, \alpha_r \in \sec(T^*M), \omega \in \sec(\Lambda^s(\mathcal{M})), r \leq s + 1$, and applying Eq. 76 successively in this expression, we get Eq. 75. ∎

Another very important consequence of Prop. 4 states the relation between the operators $\partial$ and $\eth$:

**Proposition 5** *Let $\Phi^\rho = \Theta^\rho - Q^\rho$, where $\Theta^\rho$ and $Q^\rho$ denote, respectively, the torsion and the nonmetricity 2-forms of the connection $\nabla$ in an arbitrary moving frame $\langle\theta^\rho\rangle$. Then:*

$$\begin{aligned} \partial\cdot &= \eth\cdot - \Phi^\rho \cdot j_\rho, \\ \partial\wedge &= \eth\wedge - \Theta^\rho \wedge i_\rho, \end{aligned} \tag{77}$$

**Proof** If $f$ is a function, $\partial \wedge f = \theta^\alpha \wedge \nabla_{e_\alpha} f = e_\alpha(f)\theta^\alpha = df$ and $\partial \cdot f = \theta^\alpha \cdot \nabla_{e_\alpha} f = 0$. For the 1-form field $\theta^\rho$ of a moving frame on $T^*M$, we have $\partial \wedge \theta^\rho = \theta^\alpha \wedge \nabla_{e_\alpha}\theta^\rho = -L^\rho_{\alpha\beta}\theta^\alpha \wedge \theta^\beta = -\omega^\rho_\beta \wedge \theta^\beta = d\theta^\rho - \Theta^\rho$.

Now, for a $r$-form field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, we get $\partial\wedge\omega = \frac{1}{r!}(d\omega_{\alpha_1\ldots\alpha_r}\wedge\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r} + \omega_{\alpha_1\ldots\alpha_r}d\theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r} + \cdots + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_{r-1}}\wedge d\theta^{\alpha_r}) - \frac{1}{r!}(\omega_{\alpha_1\ldots\alpha_r}\Theta^{\alpha_1}\wedge\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r} + \cdots + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_{r-1}}\wedge\Theta^{\alpha_r}) = d\omega - \frac{1}{r!}\Theta^\rho\wedge(\omega_{\rho\alpha_2\ldots\alpha_r}\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r} + \cdots + (-1)^{r+1}\omega_{\alpha_1\ldots\alpha_{r-1}\rho}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_{r-1}}) = d\omega - \Theta^\rho\wedge i_\rho\omega$, what proves Eq. 77a.

Finally, from Eqs. 69b and 77a we obtain $\partial\wedge *\omega = (-1)^{r+1}*\partial\cdot\omega - (-1)^{r+1}*Q^\rho\cdot j_\rho\omega = \partial\wedge *\omega - \Theta^\rho\wedge *\omega = (-1)^{r+1}*\partial\cdot\omega - (-1)^{r+1}*\Theta^\rho\cdot j_\rho\omega$. Therefore, $\partial\cdot\omega = \partial\cdot\omega - \Phi^\rho\cdot j_\rho\omega$, what proves Eq. 77b. ∎

From Eqs. 77 we obtain the expressions of $\tilde{\partial}\cdot$ and $\tilde{\partial}\wedge$ in terms of $\partial\cdot$ and $\partial\wedge$:

$$\begin{aligned} \tilde{\partial}\cdot &= -\partial\cdot + Q^\rho\cdot j_\rho \\ \tilde{\partial}\wedge &= \partial\wedge - \Phi^\rho\wedge i_\rho. \end{aligned} \tag{78}$$

Obviously, the Dirac coderivative associated to the fundamental Dirac operator is given by:

$$\tilde{\partial} = d + \delta.$$

We observe finally that we can still introduce another Dirac operator, obtained by combining the arbitrary affine connection $\nabla$ with the algebraic structure induced by the generic bilinear form field $g \in \sec(T^2_0 M)$. With respect to an arbitrary moving frame $\langle\theta^\alpha\rangle$ on $T^*M$, this operator has the expression:

$$\partial\vee = \theta^\alpha \vee \nabla_{e_\alpha}. \tag{79}$$

It is clear that in the particular case where $\nabla$ is the Levi-Civita connection of $g$, the operator $\partial\vee$—which in this case is the fundamental Dirac operator associated to $g$—will satisfy the properties presented in Sec. 3.1.a, with the usual Clifford produtc exchanged by the product "$\vee$." In addition, for a more general connection we can apply the results of Sec. 3.2.a, once again with all the ocurrences of $\gamma$ replaced by $g$. (In particular, the fundamental Dirac operator associated to $\gamma$ is replaced by that associated to $g$.)

## b. Torsion, Strain, Shear and Dilation of a Connection

In analogy with the introduction of the Dirac commutator and the Dirac anticommutator, let us define the operations:

$$\begin{aligned} [\alpha,\beta] &= (\alpha\cdot\partial)\beta - (\beta\cdot\partial)\alpha - [\![\alpha,\beta]\!] \\ f\alpha,\beta g &= (\alpha\cdot\partial)\beta + (\beta\cdot\partial)\alpha - \{\alpha,\beta\}, \end{aligned} \tag{80}$$

for every $\alpha,\beta \in \sec(T^*M)$. We have subtracted the Dirac commutator and the Dirac anticommutator in the r.h.s. of these expressions in order to have objects which are independent of the structure of the fields on which they are applied.

If $\langle\theta_\alpha\rangle$ is the reciprocal of an arbitrary moving frame $\langle\theta^\rho\rangle$ on $T^*M$, we get, from Eq. 80a:

$$[\theta_\alpha,\theta_\beta] = (T^\rho_{\alpha\beta} - Q^\rho_{[\alpha\beta]})\theta_\rho,$$

where $T^\rho_{\alpha\beta}$ are the components of the usual torsion tensor (Eq. 38). Note from this last equation that the operation defined through Eq. 80a does not satisfy the Jacobi identity.

Indeed we have:

$$\sum_{[\alpha\beta\sigma]} [\theta_\alpha, [\theta_\beta, \theta_\sigma]] = \sum_{[\alpha\beta\sigma]} (T^\rho_{\alpha\mu} - Q^\rho_{[\alpha\mu]})(T^\mu_{\beta\sigma} - Q^\mu_{[\beta\sigma]})\theta_\rho,$$

where the summation in this equation is to be performed on the cyclic permutations of the indices $\alpha, \beta$ and $\sigma$.

From Eq. 80b, we get:

$$f\theta_\alpha, \theta_\beta = (S^\rho_{\alpha\beta} - Q^\rho_{(\alpha\beta)})\theta_\rho,$$

where $Q^\rho_{(\alpha\beta)} = \gamma^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\alpha\sigma})$ and we have written:

$$S^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} + L^\rho_{\beta\alpha} - b^\rho_{\alpha\beta}. \tag{81}$$

It can be easily shown that the object having these components is also a tensor. Using the nomenclature of the theories of continuum media,[17,18] we will call it the *strain tensor* of the connection. Note that it can be further decomposed into:

$$S^\rho_{\alpha\beta} = \tilde{S}^\rho_{\alpha\beta} + \frac{2}{n} s^\rho \gamma_{\alpha\beta} \tag{82}$$

where $\tilde{S}^\rho_{\alpha\beta}$ is its traceless part, which will be called the *shear* of the connection, and

$$s^\rho = \frac{1}{2}\gamma^{\mu\nu} S^\rho_{\mu\nu} \tag{83}$$

is its trace part, which will be called the *dilation* of the connection.

It is trivially established that:

$$L^\rho_{\alpha\beta} = \Gamma^\rho_{\alpha\beta} + \frac{1}{2}T^\rho_{\alpha\beta} + \frac{1}{2}S^\rho_{\alpha\beta}. \tag{84}$$

where $\Gamma^\rho_{\alpha\beta} = \frac{1}{2}(b^\rho_{\alpha\beta} + c^\rho_{\alpha\beta})$ are the components of the Levi-Civita connection of $\gamma$.[9]

Eq. 84 can be used to relate the covariant derivatives with respect to the connections $D$ and $\nabla$ of any tensor field on the manifold. In particular, recalling that $D_\alpha \gamma_{\beta\sigma} = e_\alpha(\gamma_{\beta\sigma}) - \gamma_{\mu\sigma}\Gamma^\mu_{\alpha\beta} - \gamma_{\beta\mu}\Gamma^\mu_{\alpha\sigma} = 0$, we get the expression of the nonmetricity tensor of $\nabla$ in terms of the torsion and the strain, namely,

$$Q_{\alpha\beta\sigma} = \frac{1}{2}(\gamma_{\mu\sigma}T^\mu_{\alpha\beta} + \gamma_{\beta\mu}T^\mu_{\alpha\sigma}) + \frac{1}{2}(\gamma_{\mu\sigma}S^\mu_{\alpha\beta} + \gamma_{\beta\mu}S^\mu_{\alpha\sigma}). \tag{85}$$

Eq. 85 can be inverted to yield the expression of the strain in terms of the torsion and the nonmetricity. We get:

$$S^\rho_{\alpha\beta} = \gamma^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) - \gamma^{\rho\sigma}(\gamma_{\beta\mu}T^\mu_{\alpha\sigma} + \gamma_{\alpha\mu}T^\mu_{\beta\sigma}). \tag{86}$$

[9] We note that the possibility of decomposing the connection coefficients into rotation (torsion), shear and dilation has already been suggested by Baekler *et al.*,[13] but in their work they do not arrive at the identification of a tensor-like quantity associated to these last two objects.

Indeed we have:

$$\sum_{[\alpha\beta\sigma]} [\theta_\alpha, [\theta_\beta, \theta_\sigma]] = \sum_{[\alpha\beta\sigma]} (T^\rho_{\alpha\mu} - Q^\rho_{[\alpha\mu]})(T^\mu_{\beta\sigma} - Q^\mu_{[\beta\sigma]})\theta_\rho,$$

where the summation in this equation is to be performed on the cyclic permutations of the indices $\alpha, \beta$ and $\sigma$.

From Eq. 80b, we get:

$$\mathrm{f}\theta_\alpha, \theta_\beta = (S^\rho_{\alpha\beta} - Q^\rho_{(\alpha\beta)})\theta_\rho,$$

where $Q^\rho_{(\alpha\beta)} = \gamma^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\alpha\sigma})$ and we have written:

$$S^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} + L^\rho_{\beta\alpha} - b^\rho_{\alpha\beta}. \tag{81}$$

It can be easily shown that the object having these components is also a tensor. Using the nomenclature of the theories of continuum media,[17,18] we will call it the *strain tensor* of the connection. Note that it can be further decomposed into:

$$S^\rho_{\alpha\beta} = \tilde{S}^\rho_{\alpha\beta} + \frac{2}{n} s^\rho \gamma_{\alpha\beta} \tag{82}$$

where $\tilde{S}^\rho_{\alpha\beta}$ is its traceless part, which will be called the *shear* of the connection, and

$$s^\rho = \frac{1}{2}\gamma^{\mu\nu} S^\rho_{\mu\nu} \tag{83}$$

is its trace part, which will be called the *dilation* of the connection.

It is trivially established that:

$$L^\rho_{\alpha\beta} = \Gamma^\rho_{\alpha\beta} + \frac{1}{2}T^\rho_{\alpha\beta} + \frac{1}{2}S^\rho_{\alpha\beta}. \tag{84}$$

where $\Gamma^\rho_{\alpha\beta} = \frac{1}{2}(b^\rho_{\alpha\beta} + c^\rho_{\alpha\beta})$ are the components of the Levi-Civita connection of $\gamma$.[9]

Eq. 84 can be used to relate the covariant derivatives with respect to the connections $D$ and $\nabla$ of any tensor field on the manifold. In particular, recalling that $D_\alpha \gamma_{\beta\sigma} = e_\alpha(\gamma_{\beta\sigma}) - \gamma_{\mu\sigma}\Gamma^\mu_{\alpha\beta} - \gamma_{\beta\mu}\Gamma^\mu_{\alpha\sigma} = 0$, we get the expression of the nonmetricity tensor of $\nabla$ in terms of the torsion and the strain, namely,

$$Q_{\alpha\beta\sigma} = \frac{1}{2}(\gamma_{\mu\sigma}T^\mu_{\alpha\beta} + \gamma_{\beta\mu}T^\mu_{\alpha\sigma}) + \frac{1}{2}(\gamma_{\mu\sigma}S^\mu_{\alpha\beta} + \gamma_{\beta\mu}S^\mu_{\alpha\sigma}). \tag{85}$$

Eq. 85 can be inverted to yield the expression of the strain in terms of the torsion and the nonmetricity. We get:

$$S^\rho_{\alpha\beta} = \gamma^{\rho\sigma}(Q_{\alpha\beta\sigma} + Q_{\beta\sigma\alpha} - Q_{\sigma\alpha\beta}) - \gamma^{\rho\sigma}(\gamma_{\beta\mu}T^\mu_{\alpha\sigma} + \gamma_{\alpha\mu}T^\mu_{\beta\sigma}). \tag{86}$$

[9] We note that the possibility of decomposing the connection coefficients into rotation (torsion), shear and dilation has already been suggested by Baekler *et al.*,[13] but in their work they do not arrive at the identification of a tensor-like quantity associated to these last two objects.

From Eqs. 85 and 86 it is clear that nonmetricity and strain can be used interchangeably in the description of the geometry of a Riemann-Cartan-Weyl space. In particular, we have the relation:

$$Q_{\alpha\beta\sigma} + Q_{\sigma\alpha\beta} + Q_{\beta\alpha\sigma} = S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha}, \tag{87}$$

where $S_{\alpha\beta\sigma} = \gamma_{\rho\sigma} S^{\rho}_{\alpha\beta}$. Thus, the strain tensor of a Weyl space satisfies the relation:

$$S_{\alpha\beta\sigma} + S_{\sigma\alpha\beta} + S_{\beta\sigma\alpha} = 0.$$

In order to simplify our next equations, let us introduce the notation:

$$\Delta^{\rho}_{\alpha\beta} = L^{\rho}_{\alpha\beta} - \Gamma^{\rho}_{\alpha\beta} = \frac{1}{2}(T^{\rho}_{\alpha\beta} + S^{\rho}_{\alpha\beta}). \tag{88}$$

From Eq. 86 it follows that:

$$\begin{aligned} \Delta^{\rho}_{\alpha\beta} &= -\frac{1}{2}\gamma^{\rho\sigma}(\nabla_{\alpha}\gamma_{\beta\sigma} + \nabla_{\beta}\gamma_{\sigma\alpha} - \nabla_{\sigma}\gamma_{\alpha\beta}) \\ &- \frac{1}{2}\gamma^{\rho\sigma}(\gamma_{\mu\alpha}T^{\mu}_{\sigma\beta} + \gamma_{\mu\beta}T^{\mu}_{\sigma\alpha} - \gamma_{\mu\sigma}T^{\mu}_{\alpha\beta}), \end{aligned} \tag{89}$$

where we have used that $Q_{\alpha\beta\sigma} = -\nabla_{\alpha}\gamma_{\beta\sigma}$. Note the similarity of this equation with that which gives the coefficients of a Riemannian connection (Eq. 59). Note also that for $\nabla\gamma = 0$, $\Delta^{\rho}_{\alpha\beta}$ is the so-called *contorsion tensor.*[10]

Returning to Eq. 84, we obtain now the relation between the curvature tensor $R_{\mu}{}^{\rho}{}_{\alpha\beta}$ associated to the connection $\nabla$ and the Riemann curvature tensor $\hat{R}_{\mu}{}^{\rho}{}_{\alpha\beta}$ of the Levi-Civita connection $D$ associated to the metric $\gamma$. We get, by a simple calculation:

$$R_{\mu}{}^{\rho}{}_{\alpha\beta} = \hat{R}_{\mu}{}^{\rho}{}_{\alpha\beta} + J_{\mu}{}^{\rho}{}_{[\alpha\beta]}, \tag{90}$$

where:

$$J_{\mu}{}^{\rho}{}_{\alpha\beta} = D_{\alpha}\Delta^{\rho}_{\beta\mu} - \Delta^{\rho}_{\beta\sigma}\Delta^{\sigma}_{\alpha\mu} = \nabla_{\alpha}\Delta^{\rho}_{\beta\mu} - \Delta^{\rho}_{\alpha\sigma}\Delta^{\sigma}_{\beta\mu} + \Delta^{\sigma}_{\alpha\beta}\Delta^{\rho}_{\sigma\mu}. \tag{91}$$

Multiplying both sides of Eq. 90 by $\frac{1}{2}\theta^{\alpha} \wedge \theta^{\beta}$ we get:

$$\Omega^{\rho}_{\mu} = \hat{\Omega}^{\rho}_{\mu} + J^{\rho}_{\mu}, \tag{92}$$

where we have written:

$$J^{\rho}_{\mu} = \frac{1}{2} J_{\mu}{}^{\rho}{}_{[\alpha\beta]}\theta^{\alpha} \wedge \theta^{\beta}. \tag{93}$$

From Eq. 90 we get also the relation between the Ricci tensors of the connections $\nabla$ and $D$. We have:

$$R_{\mu\alpha} = \hat{R}_{\mu\alpha} + J_{\mu\alpha}, \tag{94}$$

with

$$\begin{aligned} J_{\mu\alpha} &= D_{\alpha}\Delta^{\rho}_{\rho\mu} - D_{\rho}\Delta^{\rho}_{\alpha\mu} + \Delta^{\rho}_{\alpha\sigma}\Delta^{\sigma}_{\rho\mu} - \Delta^{\rho}_{\rho\sigma}\Delta^{\sigma}_{\alpha\mu} \\ &= \nabla_{\alpha}\Delta^{\rho}_{\rho\mu} - \nabla_{\rho}\Delta^{\rho}_{\alpha\mu} - \Delta^{\rho}_{\sigma\alpha}\Delta^{\sigma}_{\rho\mu} + \Delta^{\rho}_{\rho\sigma}\Delta^{\sigma}_{\alpha\mu}. \end{aligned} \tag{95}$$

[10] Eqs. 88 and 89 have appeared in the literature in two different contexts: with $\nabla\gamma = 0$, they have been used in the formulations of the theory of the spinor fields in Riemann-Cartan spaces[19,20] and with $T[\nabla] = 0$ they have been used in the formulations of the gravitational theory in a space endowed with a background metric.[21,22,23,24,25]

Observe that since the connection $\nabla$ is arbitrary, its Ricci tensor will not be generally symmetric. Then, since the Ricci tensor of $D$ is necessarily symmetric, we can split Eq. 94 into:

$$
\begin{aligned}
R_{[\mu\alpha]} &= J_{[\mu\alpha]} \\
R_{(\mu\alpha)} &= \hat{R}_{(\mu\alpha)} + J_{(\mu\alpha)}.
\end{aligned}
\tag{96}
$$

Now we specialize the above results for the case where the general connection $\nabla$ is the Levi-Civita connection of the bilinear form field $g \in \sec(T_0^2 M)$, i.e., $\boldsymbol{T}[\nabla] = 0$ and $\nabla g = 0$. The results that we are going to obtain generalize and clear up those found in the formulations of the gravitational theory in a background metric space.[21,22,23,24]

First of all, note that the connection $D$ plays with respect to the tensor field $g$ a role analogous to that played by the connection $\nabla$ with respect to the metric tensor $\gamma$ and in consequence we shall have similar equations relating these two pairs of objects. In particular, the strain of $D$ with respect to $g$ equals the negative of the strain of $\nabla$ with respect to $\gamma$, since we have:

$$S^{\rho}_{\ \alpha\beta} = L^{\rho}_{\ \alpha\beta} + L^{\rho}_{\ \beta\alpha} - b^{\rho}_{\ \alpha\beta} = -(\Gamma^{\rho}_{\ \alpha\beta} + \Gamma^{\rho}_{\ \beta\alpha} - d^{\rho}_{\ \alpha\beta}),$$

where $b^{\rho}_{\ \alpha\beta} = \Gamma^{\rho}_{\ \alpha\beta} + \Gamma^{\rho}_{\ \beta\alpha}$ and $d^{\rho}_{\ \alpha\beta} = L^{\rho}_{\ \alpha\beta} + L^{\rho}_{\ \beta\alpha}$ denote the Killing coefficients of the frame with respect to the tensors $\gamma$ and $g$ respectively. Furthermore, in view of Eq. 89, we can write $\Delta^{\rho}_{\ \alpha\beta} = \frac{1}{2} S^{\rho}_{\ \alpha\beta}$ as:

$$
\begin{aligned}
\Delta^{\rho}_{\ \alpha\beta} &= -\frac{1}{2}\gamma^{\rho\sigma}(\nabla_\alpha \gamma_{\beta\sigma} + \nabla_\beta \gamma_{\alpha\sigma} - \nabla_\sigma \gamma_{\alpha\beta}) \\
&= \frac{1}{2} g^{\rho\sigma}(D_\alpha g_{\beta\sigma} + D_\beta g_{\alpha\sigma} - D_\sigma g_{\alpha\beta}).
\end{aligned}
\tag{97}
$$

We introduce the notation:

$$\kappa = \sqrt{\frac{\bar{g}}{\bar{\gamma}}}. \tag{98}$$

Then we have the following very interesting relations:

$$
\begin{aligned}
\Delta^{\rho}_{\ \rho\sigma} &= -\frac{1}{2}\gamma^{\alpha\beta}\nabla_\sigma \gamma_{\alpha\beta} = \frac{1}{2} g^{\alpha\beta} D_\sigma g_{\alpha\beta} = \frac{1}{\kappa} e_\sigma(\kappa) \\
g^{\alpha\beta}\Delta^{\rho}_{\ \alpha\beta} &= -\frac{1}{\kappa} D_\sigma(\kappa g^{\rho\sigma}) \\
\gamma^{\alpha\beta}\Delta^{\rho}_{\ \alpha\beta} &= \frac{1}{\kappa^{-1}} \nabla_\sigma(\kappa^{-1}\gamma^{\rho\sigma}).
\end{aligned}
\tag{99}
$$

Another important consequence of the assumption that $\nabla$ is a Levi-Civita connection is that its Ricci tensor will then be symmetric. In view of Eqs. 96, this will be achieved if and only if the following equivalent conditions hold:

$$
\begin{aligned}
D_\alpha \Delta^{\rho}_{\ \rho\beta} &= D_\beta \Delta^{\rho}_{\ \rho\alpha} \\
\nabla_\alpha \Delta^{\rho}_{\ \rho\beta} &= \nabla_\beta \Delta^{\rho}_{\ \rho\alpha}.
\end{aligned}
\tag{100}
$$

**c. Structure Equations**

With the results stated above, we can write down the structure equations of the RCWS structure defined by the connection $\nabla$ in terms of the Riemannian structure defined by the metric $\gamma$. For this, let us write Eq. 84 in the form:

$$\omega^{\rho}_{\beta} = \hat{\omega}^{\rho}_{\beta} + w^{\rho}_{\beta} = \hat{\omega}^{\rho}_{\beta} + \tau^{\rho}_{\beta} + \sigma^{\rho}_{\beta}, \tag{101}$$

with $\omega^{\rho}_{\beta} = L^{\rho}_{\alpha\beta}\theta^{\alpha}$, $\hat{\omega}^{\rho}_{\beta} = \Gamma^{\rho}_{\alpha\beta}\theta^{\alpha}$, $w^{\rho}_{\beta} = \Delta^{\rho}_{\alpha\beta}\theta^{\alpha}$, $\tau^{\rho}_{\beta} = \frac{1}{2}T^{\rho}_{\alpha\beta}\theta^{\alpha}$ and $\sigma^{\rho}_{\beta} = \frac{1}{2}S^{\rho}_{\alpha\beta}\theta^{\alpha}$. Then, recalling Eq. 92 and the structure equations for both the RCWS and the Riemannian structures, we easily conclude that:

$$\begin{aligned} &w^{\rho}_{\beta} \wedge \theta^{\beta} = \Theta^{\rho} \\ &w^{\beta}_{\mu} \wedge \theta_{\beta} = -\Phi_{\mu} \\ &\hat{\boldsymbol{d}} w^{\rho}_{\mu} + w^{\rho}_{\beta} \wedge w^{\beta}_{\mu} = \boldsymbol{J}^{\rho}_{\mu}, \end{aligned} \tag{102}$$

where $\hat{\boldsymbol{d}}$ is the exterior covariant derivative associated to the Levi-Civita connection $D$ of $\gamma$. The third of these equations can also be written as:

$$\boldsymbol{d} w^{\rho}_{\mu} - w^{\rho}_{\beta} \wedge w^{\beta}_{\mu} = \boldsymbol{J}^{\rho}_{\mu}, \tag{103}$$

with $\boldsymbol{d}$ the exterior covariant derivative of the connection $\nabla$.

Now, the Bianchi identities for the RCWS structure are easily obtained by differentiating the above equations. We get:

$$\begin{aligned} &\hat{\boldsymbol{d}}\Theta^{\rho} = \boldsymbol{J}^{\rho}_{\beta} \wedge \theta^{\beta} - w^{\rho}_{\beta} \wedge \Theta^{\beta} \\ &\hat{\boldsymbol{d}}\Phi_{\mu} = \boldsymbol{J}^{\beta}_{\mu} \wedge \theta_{\beta} + w^{\beta}_{\mu} \wedge \Phi_{\beta} \\ &\hat{\boldsymbol{d}}\boldsymbol{J}^{\rho}_{\mu} = \Omega^{\rho}_{\beta} \wedge w^{\beta}_{\mu} - w^{\rho}_{\beta} \wedge \Omega^{\beta}_{\mu}, \end{aligned} \tag{104}$$

or equivalently,

$$\begin{aligned} &\boldsymbol{d}\Theta^{\rho} = \boldsymbol{J}^{\rho}_{\beta} \wedge \theta^{\beta} \\ &\boldsymbol{d}\Phi_{\mu} = \boldsymbol{J}^{\beta}_{\mu} \wedge \theta_{\beta} \\ &\boldsymbol{d}\boldsymbol{J}^{\rho}_{\mu} = \hat{\Omega}^{\rho}_{\beta} \wedge w^{\beta}_{\mu} - w^{\rho}_{\beta} \wedge \hat{\Omega}^{\beta}_{\mu}. \end{aligned} \tag{105}$$

# 4. The Square of the Dirac Operator

## 4.1. D'Alembertian, Ricci and Einstein Operators

As we have seen in the Sec. 3.1, the fundamental Dirac operator $\boldsymbol{\partial}$ of a manifold $\mathcal{M} = \langle M, \gamma, \tau \rangle$ is given by (Eq. 52)

$$\boldsymbol{\partial} = d - \delta.$$

Then the square of this operator, $\boldsymbol{\partial}^2 = \boldsymbol{\partial}\boldsymbol{\partial}$, will be given by:

$$\boldsymbol{\partial}^2 = (d - \delta)(d - \delta) = -(d\delta + \delta d) = \Delta, \tag{106}$$

that is, $\boldsymbol{\partial}^2$ is the usual Hodge Laplacian of the manifold (Eq. 35).

On the other hand, remembering also that (Eq. 49)

$$\boldsymbol{\partial} = \theta^\alpha \, De_\alpha,$$

where $\langle\theta^\alpha\rangle$ is an arbitrary reference frame on the manifold and $D$ is the Levi-Civita connection of the metric $\gamma$, we have:

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\theta^\alpha De_\alpha)(\theta^\beta De_\beta) = \theta^\alpha(\theta^\beta De_\alpha De_\beta + (De_\alpha\theta^\beta)De_\beta) \\ &= \gamma^{\alpha\beta}(De_\alpha De_\beta - \Gamma^\rho_{\alpha\beta}De_\rho) + \theta^\alpha\wedge\theta^\beta(De_\alpha De_\beta - \Gamma^\rho_{\alpha\beta}De_\rho).\end{aligned}$$

Then defining the operators:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \gamma^{\alpha\beta}(De_\alpha De_\beta - \Gamma^\rho_{\alpha\beta}De_\rho) \\ \boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \theta^\alpha\wedge\theta^\beta(De_\alpha De_\beta - \Gamma^\rho_{\alpha\beta}De_\rho),\end{aligned} \tag{107}$$

we can write:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial} \tag{108}$$

It is important to observe that the operators $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ e $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ do not have anything analogous in the formulation of the differential geometry in the Cartan and Hodge bundles.

The operator $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ can also be written as:

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = \frac{1}{2}\gamma^{\alpha\beta}\left[De_\alpha De_\beta + De_\beta De_\alpha - b^\rho_{\alpha\beta}De_\rho\right]. \tag{109}$$

Applying this operator to the 1-forms of the frame $\langle\theta^\alpha\rangle$, we get:

$$(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\theta^\mu = -\frac{1}{2}\gamma^{\alpha\beta}\hat{M}_\rho{}^\mu{}_{\alpha\beta}\theta^\rho, \tag{110}$$

where:

$$\hat{M}_\rho{}^\mu{}_{\alpha\beta} = e_\alpha(\Gamma^\mu_{\beta\rho}) + e_\beta(\Gamma^\mu_{\alpha\rho}) - \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\rho} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\rho} - b^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\rho}. \tag{111}$$

The proof that an object with these components is a tensor is a consequence of the following proposition:

**Proposition 6** *For every $r$-form field $\omega \in \sec(\Lambda^r M)$, $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, we have:*

$$(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\omega = \frac{1}{r!}\gamma^{\alpha\beta}D_\alpha D_\beta\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}, \tag{112}$$

*where $D_\alpha D_\beta\omega_{\alpha_1\ldots\alpha_r}$ are the components of the covariant derivative of $\omega$.*

**Proof** We have $De_\beta\omega = \frac{1}{r!}D_\beta\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, with $D_\beta\omega_{\alpha_1\ldots\alpha_r} = (e_\beta(\omega_{\alpha_1\ldots\alpha_r}) - \Gamma^\sigma_{\beta\alpha_1}\omega_{\sigma\alpha_2\ldots\alpha_r} - \cdots - \Gamma^\sigma_{\beta\alpha_r}\omega_{\alpha_1\ldots\alpha_{r-1}\sigma})$. Therefore, $De_\alpha De_\beta\omega = \frac{1}{r!}(e_\alpha(D_\beta\omega_{\alpha_1\ldots\alpha_r}) - \Gamma^\sigma_{\alpha\alpha_1}D_\beta\omega_{\sigma\alpha_2\ldots\alpha_r} - \cdots - \Gamma^\sigma_{\alpha\alpha_r}D_\beta\omega_{\alpha_1\ldots\alpha_{r-1}\sigma})\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$ and we conclude that:

$$(De_\alpha De_\beta - \Gamma^\rho_{\alpha\beta}De_\rho)\omega = \frac{1}{r!}D_\alpha D_\beta\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}.$$

Multiplying this equation by $\gamma^{\alpha\beta}$ and using the Eq. 107a, we get the Eq. 112. ∎

In view of Eq. 112, we shall call the operator $\boldsymbol{\partial} \cdot \boldsymbol{\partial}$ the *D'Alembertian*. Note that the D'Alembertian of the 1-forms $\theta^\mu$ can also be written as:

$$(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\theta^\mu = \gamma^{\alpha\beta} D_\alpha D_\beta \delta^\mu_\rho \theta^\rho = \frac{1}{2}\gamma^{\alpha\beta}(D_\alpha D_\beta \delta^\mu_\rho + D_\beta D_\alpha \delta^\mu_\rho)\theta^\rho$$

and therefore, taking into account the Eq. 110, we conclude that:

$$\hat{M}_\rho{}^\mu{}_{\alpha\beta} = -(D_\alpha D_\beta \delta^\mu_\rho + D_\beta D_\alpha \delta^\mu_\rho), \tag{113}$$

what proves our assertion that $\hat{M}_\rho{}^\mu{}_{\alpha\beta}$ are the components of a tensor.

By its turn, the operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ can also be written as:

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} = \frac{1}{2}\theta^\alpha \wedge \theta^\beta \left[D_{e_\alpha} D_{e_\beta} - D_{e_\beta} D_{e_\alpha} - c^\rho_{\alpha\beta} D_{e_\rho}\right]. \tag{114}$$

Applying this operator to the 1-forms of the frame $\langle\theta^\alpha\rangle$, we get:

$$(\boldsymbol{\partial} \wedge \boldsymbol{\partial})\theta^\mu = -\frac{1}{2}\hat{R}_\rho{}^\mu{}_{\alpha\beta}(\theta^\alpha \wedge \theta^\beta)\theta^\rho = -\hat{\Omega}^\mu_\rho \theta^\rho, \tag{115}$$

where $\hat{R}_\rho{}^\mu{}_{\alpha\beta}$ are the components of the curvature tensor of the connection $D$. From Eqs. 18f, we get:

$$\hat{\Omega}^\mu_\rho \theta^\rho = \hat{\Omega}^\mu_\rho \cdot \theta^\rho + \hat{\Omega}^\mu_\rho \wedge \theta^\rho.$$

The second term in the r.h.s. of this equation is identically null because of the Bianchi identity given by Eq. 45a for the particular case of a symmetric connection ($\Theta^\mu = 0$). Using Eqs. 18c and 18i we can write the first term in the r.h.s. as:

$$\begin{aligned}
\hat{\Omega}^\mu_\rho \cdot \theta^\rho &= \frac{1}{2}\hat{R}_\rho{}^\mu{}_{\alpha\beta}(\theta^\alpha \wedge \theta^\beta) \cdot \theta^\rho \\
&= -\frac{1}{2}\hat{R}_\rho{}^\mu{}_{\alpha\beta}(\gamma^{\rho\alpha}\theta^\beta - \gamma^{\rho\beta}\theta^\alpha) \\
&= -\gamma^{\rho\alpha}\hat{R}_\rho{}^\mu{}_{\alpha\beta}\theta^\beta = -\hat{R}^\mu_\beta \theta^\beta,
\end{aligned}$$

where $\hat{R}^\mu_\beta$ are the components of the Ricci tensor of the manifold. Thus we have:

$$(\boldsymbol{\partial} \wedge \boldsymbol{\partial})\theta^\mu = \hat{R}^\mu, \tag{116}$$

where $\hat{R}^\mu = \hat{R}^\mu_\beta \theta^\beta$ are the Ricci 1-forms of the manifold. Because of this relation, we will call the operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ the *Ricci operator* of the manifold.

The proposition below shows that the Ricci operator can be written in a purely algebraic way:

**Proposition 7** *The Ricci operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ satisfies the relation:*

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} = \hat{R}^\sigma \wedge i_\sigma + \hat{\Omega}^{\rho\sigma} \wedge i_\rho i_\sigma, \tag{117}$$

where $\hat{\Omega}^{\rho\sigma} = \gamma^{\rho\mu}\hat{\Omega}^{\sigma}_{\mu} = \frac{1}{2}\hat{R}^{\rho\sigma}{}_{\alpha\beta}\theta^{\alpha}\wedge\theta^{\beta}$.

**Proof** The Hodge Laplacian of an arbitrary $r$-form field $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$ is given by: (e.g., Choquet-Bruhat[10]—recall that our definition differs by a sign from that given there) $\Delta\omega = \boldsymbol{\partial}^2\omega = \frac{1}{r!}(\boldsymbol{\partial}^2\omega)_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, with:

$$\begin{aligned}(\boldsymbol{\partial}^2\omega)_{\alpha_1\ldots\alpha_r} &= \gamma^{\alpha\beta}D_\alpha D_\beta\omega_{\alpha_1\ldots\alpha_r} \\ &- \sum_p(-1)^p\hat{R}^{\sigma}_{\alpha_p}\omega_{\sigma\alpha_1\ldots\check{\alpha}_p\ldots\alpha_r} \\ &- 2\sum_{\substack{p,q\\p<q}}(-1)^{p+q}\hat{R}^{\rho}{}_{\alpha_q}{}^{\sigma}{}_{\alpha_p}\omega_{\rho\sigma\alpha_1\ldots\check{\alpha}_p\ldots\check{\alpha}_q\ldots\alpha_r},\end{aligned} \tag{118}$$

where the notation $\check{\alpha}$ means that the index $\alpha$ was exclude of the sequence.

The first term in the r.h.s. of this expression are the components of the D'Alembertian of the field $\omega$.

Now, recalling that $\boldsymbol{i}_\sigma\omega = \theta_\sigma\cdot\omega = \frac{1}{r!}(\omega_{\sigma\alpha_2\ldots\alpha_r}\theta^{\alpha_2}\wedge\ldots\wedge\theta^{\alpha_r}+\cdots+(-1)^{r+1}\omega_{\alpha_1\ldots\alpha_{r-1}\sigma}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_{r-1}})$, we obtain:

$$\hat{R}^{\sigma}\wedge\boldsymbol{i}_\sigma\omega = -\frac{1}{r!}\left[\sum_p(-1)^p\hat{R}^{\sigma}_{\alpha_p}\omega_{\sigma\alpha_1\ldots\check{\alpha}_p\ldots\alpha_r}\right]\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$$

and also,

$$\hat{\Omega}^{\rho\sigma}\wedge\boldsymbol{i}_\rho\boldsymbol{i}_\sigma\omega = -\frac{2}{r!}\left[\sum_{\substack{p,q\\p<q}}(-1)^{p+q}\hat{R}^{\rho}{}_{\alpha_q}{}^{\sigma}{}_{\alpha_p}\omega_{\rho\sigma\alpha_1\ldots\check{\alpha}_p\ldots\check{\alpha}_q\ldots\alpha_r}\right]\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}.$$

Hence, taking into account Eq. 108, we conclude that:

$$(\boldsymbol{\partial}\wedge\boldsymbol{\partial})\omega = \hat{R}^{\sigma}\wedge\boldsymbol{i}_\sigma\omega + \hat{\Omega}^{\rho\sigma}\wedge\boldsymbol{i}_\rho\boldsymbol{i}_\sigma\omega,$$

for every $r$-form field $\omega$. ■

Observe that applying the operator given by the second term in the r.h.s. of Eq. 117 to the dual of the 1-forms $\theta^\mu$, we get:

$$\begin{aligned}\hat{\Omega}^{\rho\sigma}\wedge\boldsymbol{i}_\rho\boldsymbol{i}_\sigma *\theta^\mu &= \hat{\Omega}_{\rho\sigma}\wedge\theta^\rho\cdot(\theta^\sigma\cdot *\theta^\mu)) \\ &= -\hat{\Omega}_{\rho\sigma}\wedge *(\theta^\rho\wedge\theta^\sigma\wedge\theta^\mu) \\ &= *(\hat{\Omega}_{\rho\sigma}\cdot(\theta^\rho\wedge\theta^\sigma\wedge\theta^\mu)),\end{aligned}$$

where we have used the Eqs. 21c and 21d. Then, recalling the definition of the curvature forms and using the Eq. 18j, we conclude that:

$$\hat{\Omega}^{\rho\sigma}\wedge\boldsymbol{i}_\rho\boldsymbol{i}_\sigma *\theta^\mu = 2*(\hat{R}^\mu - \frac{1}{2}\hat{R}\theta^\mu) = 2*\hat{G}^\mu, \tag{119}$$

where $\hat{R}$ is the scalar curvature of the manifold and $\hat{G}^\mu$ are the Einstein 1-form fields.

The observation above motivate us to introduce a new operator, $\Box$, which will be called the *Einstein operator*, defined by:

$$\Box = \frac{1}{2} *^{-1} (\hat{\Omega}^{\rho\sigma} \wedge i_\rho i_\sigma) * . \tag{120}$$

Obviously, we have:

$$\Box\theta^\mu = \hat{G}^\mu = \hat{R}^\mu - \frac{1}{2}\hat{R}\theta^\mu. \tag{121}$$

In addition, it is easy to verify that $*^{-1}(\partial \wedge \partial)* = -\partial \wedge \partial$ and $*^{-1}(\hat{R}^\sigma \wedge i_\sigma)* = \hat{R}^\sigma \cdot j_\sigma$. Thus we can also write the Einstein operator as:

$$\Box = -\frac{1}{2}(\partial \wedge \partial + \hat{R}^\sigma \cdot j_\sigma). \tag{122}$$

Another important result is given by the following proposition:

**Proposition 8** *Let $\hat{\omega}^\mu_\rho$ be the Levi-Civita connection 1-forms fields in an arbitrary moving frame $\langle\theta^\alpha\rangle$ on $\mathcal{M}$. Then:*

$$\begin{aligned} (\partial \cdot \partial)\theta^\mu &= -(\partial \cdot \hat{\omega}^\mu_\rho - \hat{\omega}^\sigma_\rho \cdot \hat{\omega}^\mu_\sigma)\theta^\rho \\ (\partial \wedge \partial)\theta^\mu &= -(\partial \wedge \hat{\omega}^\mu_\rho - \hat{\omega}^\sigma_\rho \wedge \hat{\omega}^\mu_\sigma)\theta^\rho, \end{aligned} \tag{123}$$

*that is,*

$$\partial^2\theta^\mu = -(\partial\hat{\omega}^\mu_\rho - \hat{\omega}^\sigma_\rho\hat{\omega}^\mu_\sigma)\theta^\rho. \tag{124}$$

**Proof** We have $\partial \cdot \hat{\omega}^\mu_\rho = \theta^\alpha \cdot D_{e_\alpha}(\Gamma^\mu_{\beta\rho}\theta^\beta) = \theta^\alpha \cdot (e_\alpha(\Gamma^\mu_{\beta\rho})\theta^\beta - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\alpha\beta}\theta^\beta) = \gamma^{\alpha\beta}(e_\alpha(\Gamma^\mu_{\beta\rho}) - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\alpha\beta})$ e $\hat{\omega}^\sigma_\rho \cdot \hat{\omega}^\mu_\sigma = (\Gamma^\sigma_{\beta\rho}\theta^\beta) \cdot (\Gamma^\mu_{\alpha\sigma}\theta^\alpha) = \gamma^{\beta\alpha}\Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\rho}$. Then, $-(\partial \cdot \hat{\omega}^\mu_\rho - \hat{\omega}^\sigma_\rho \cdot \hat{\omega}^\mu_\sigma)\theta^\rho = \gamma^{\alpha\beta}(e_\alpha(\Gamma^\mu_{\beta\rho}) - \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\rho} - \Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\rho})\theta^\rho = -\frac{1}{2}\gamma^{\alpha\beta}(e_\alpha(\Gamma^\mu_{\beta\rho}) + e_\beta(\Gamma^\mu_{\alpha\rho}) - \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\rho} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\rho} - b^\sigma_{\alpha\beta}\Gamma^\mu_{\sigma\rho})\theta^\rho = (\partial \cdot \partial)\theta^\mu$. The Eq. 123b is proved analogously. ■

### 4.2. The Square of the Dirac Operator

Let us now compute the square of the Dirac operator $\partial$ associated with an arbitrary Riemann-Cartan-Weyl connection $\nabla$. As in the earlier section, we have, by one side,

$$\begin{aligned} \partial^2 &= (\partial \cdot + \partial\wedge)(\partial \cdot + \partial\wedge) \\ &= \partial \cdot \partial \cdot + \partial \cdot \partial \wedge + \partial \wedge \partial \cdot + \partial \wedge \partial\wedge \end{aligned}$$

and we write $\partial \cdot \partial \cdot \equiv \partial^2 \cdot$, $\partial\wedge\partial\wedge \equiv \partial^2\wedge$ and

$$\mathcal{L}_+ = \partial \cdot \partial \wedge + \partial \wedge \partial \cdot, \tag{125}$$

so that:

$$\partial^2 = \partial^2 \cdot \; + \mathcal{L}_+ \; + \partial^2\wedge \tag{126}$$

The operator $\mathcal{L}_+$ corresponds, for the case of a Riemann-Cartan space, to the wave operator introduced by Rapoport[1] in his theory of Stochastic Mechanics. Obviously, for the case of the fundamental Dirac operator, $\mathcal{L}_+$ reduces to the usual Hodge Laplacian of the manifold.

Now, a similar calculation for the product $\boldsymbol{\partial}\tilde{\boldsymbol{\partial}}$ of the Dirac derivative and the Dirac coderivative yields:

$$\boldsymbol{\partial}\tilde{\boldsymbol{\partial}} = \boldsymbol{\partial} \cdot \tilde{\boldsymbol{\partial}} \cdot + \pounds_- + \boldsymbol{\partial} \wedge \tilde{\boldsymbol{\partial}} \wedge, \tag{127}$$

with:

$$\pounds_- = \boldsymbol{\partial} \cdot \tilde{\boldsymbol{\partial}} \wedge + \boldsymbol{\partial} \wedge \tilde{\boldsymbol{\partial}} \cdot \tag{128}$$

On the other hand, we have also:

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\theta^\alpha \nabla_{e_\alpha})(\theta^\beta \nabla_{e_\beta}) = \theta^\alpha(\theta^\beta \nabla_{e_\alpha}\nabla_{e_\beta} + (\nabla_{e_\alpha}\theta^\beta)\nabla_{e_\beta}) \\ &= \gamma^{\alpha\beta}(\nabla_{e_\alpha}\nabla_{e_\beta} - L^\rho_{\alpha\beta}\nabla_{e_\rho}) + \theta^\alpha \wedge \theta^\beta(\nabla_{e_\alpha}\nabla_{e_\beta} - L^\rho_{\alpha\beta}\nabla_{e_\rho})\end{aligned}$$

and we can then define:

$$\begin{aligned}\boldsymbol{\partial} \cdot \boldsymbol{\partial} &= \gamma^{\alpha\beta}(\nabla_{e_\alpha}\nabla_{e_\beta} - L^\rho_{\alpha\beta}\nabla_{e_\rho}) \\ \boldsymbol{\partial} \wedge \boldsymbol{\partial} &= \theta^\alpha \wedge \theta^\beta(\nabla_{e_\alpha}\nabla_{e_\beta} - L^\rho_{\alpha\beta}\nabla_{e_\rho})\end{aligned} \tag{129}$$

in order to have:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial} \cdot \boldsymbol{\partial} + \boldsymbol{\partial} \wedge \boldsymbol{\partial} \tag{130}$$

The operator $\boldsymbol{\partial} \cdot \boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial} \cdot \boldsymbol{\partial} &= \frac{1}{2}\theta^\alpha \cdot \theta^\beta(\nabla_{e_\alpha}\nabla_{e_\beta} - L^\rho_{\alpha\beta}\nabla_{e_\rho}) + \frac{1}{2}\theta^\beta \cdot \theta^\alpha(\nabla_{e_\beta}\nabla_{e_\alpha} - L^\rho_{\beta\alpha}\nabla_{e_\rho}) \\ &= \frac{1}{2}\gamma^{\alpha\beta}[\nabla_{e_\alpha}\nabla_{e_\beta} + \nabla_{e_\beta}\nabla_{e_\alpha} - (L^\rho_{\alpha\beta} + L^\rho_{\beta\alpha})\nabla_{e_\rho}]\end{aligned}$$

or,

$$\boldsymbol{\partial} \cdot \boldsymbol{\partial} = \frac{1}{2}\gamma^{\alpha\beta}(\nabla_{e_\alpha}\nabla_{e_\beta} + \nabla_{e_\beta}\nabla_{e_\alpha} - b^\rho_{\alpha\beta}\nabla_{e_\rho}) - s^\rho \nabla_{e_\rho} \tag{131}$$

By its turn, the operator $\boldsymbol{\partial} \wedge \boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial} \wedge \boldsymbol{\partial} &= \frac{1}{2}\theta^\alpha \wedge \theta^\beta(\nabla_{e_\alpha}\nabla_{e_\beta} - L^\rho_{\alpha\beta}\nabla_{e_\rho}) + \frac{1}{2}\theta^\beta \wedge \theta^\alpha(\nabla_{e_\beta}\nabla_{e_\alpha} - L^\rho_{\beta\alpha}\nabla_{e_\rho}) \\ &= \frac{1}{2}\theta^\alpha \wedge \theta^\beta[\nabla_{e_\alpha}\nabla_{e_\beta} - \nabla_{e_\beta}\nabla_{e_\alpha} - (L^\rho_{\alpha\beta} - L^\rho_{\beta\alpha})\nabla_{e_\rho}]\end{aligned}$$

or,

$$\boldsymbol{\partial} \wedge \boldsymbol{\partial} = \frac{1}{2}\theta^\alpha \wedge \theta^\beta(\nabla_{e_\alpha}\nabla_{e_\beta} - \nabla_{e_\beta}\nabla_{e_\alpha} - c^\rho_{\alpha\beta}\nabla_{e_\rho}) - \Theta^\rho \nabla_{e_\rho}. \tag{132}$$

## 5. Conclusions

We have presented a Clifford bundle formulation for the geometry of a general Riemann-Cartan-Weyl space (RCWS). The main ingredients of our presentation has been the introduction of: (i) the fundamental Dirac operator $\boldsymbol{\partial}$, related to the Levi-Civita connection $D$ of a Riemannian oriented space with metric $\gamma$ ($\mathcal{M} = \langle M, D, \gamma \rangle$); (ii) the infinitely many other Dirac-like operators (associated to $\boldsymbol{\partial}$), one for each nondegenerate symmetric bilinear form field which can be defined in $\mathcal{M}$ through the theory of the symmetric automorphisms of Clifford algebras (Sec. 2); (iii) the Dirac operators of a general RCWS, related to the general affine connection $\nabla$ defining the RCW structure.

We introduced the concept of fundamental Dirac commutator of 1-form fields, which enable us to give the structure of Lie algebra to the cotangent bundle, in analogy to the way that the bracket of vector fields defines a Lie algebra structure for the tangent bundle. We introduced also the notion of the fundamental Dirac anticommutator of 1-form fields and we showed that the coefficients $b^\rho_{\alpha\beta}$—the Killing coefficients—of the expansion $\{\theta_\alpha, \theta_\beta\} = b^\rho_{\alpha\beta}\theta_\rho$ are related to the Lie derivative of the metric $\gamma$ by $b^\rho_{\alpha\beta} = -(\pounds_{e^\rho}\gamma)_{\alpha\beta}$, which shows that to find (if existing) a moving frame for which $b^\rho_{\alpha\beta} = 0$ is equivalent to solve, locally, the Killing equations for the manifold. Also, introducing the concept of Dirac commutator and anticommutator of 1-form fields, we succeed in giving a simple decomposition of the general connection $\nabla$ associated with the Dirac operator in torsion, strain and dilation, identifying a new tensor-like object $S^\rho_{\alpha\beta}$ which has been used to relate the covariant derivatives with respect to the connections $D$ (the Levi-Civita connection of $\gamma$) and $\nabla$ (a general affine connection) of any tensor field on the manifold. We then got a simple way to relate the curvature tensor $R_\mu{}^\rho{}_{\alpha\beta}$ associated with the connection $\nabla$ and the Riemannian curvature tensor $\hat{R}_\mu{}^\rho{}_{\alpha\beta}$ of the Levi-Civita connection $D$ of $\gamma$. The results are particularly interestig when $\nabla$ is the Levi-Civita connection of a symmetric bilinear form field $g \in \sec(T^2_0 M)$ (introduced through the theory of the symmetric automorphims of the Clifford algebra), since we get several formulas that clear up many results appearing in the flat space formulations of Einstein's gravitational theory as, e.g., in [23, 24].

Finally, we have studied in detail the square of the Dirac operator $\partial^2$, which does not preserve the graduation of $r$-forms. We identified the natural wave operator in a general RCWS (which preserves the graduations) as the operator $\pounds_+$, which for the case of the fundamental Dirac operator reduces to the usual Hodge Laplacian operator.

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