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On the self-similar blowup for the generalized SQG equation

Estudo de soluções autossimilares da equação SQG generalizada

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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"Não vim até aqui pra desistir agora
Entendo você se você quiser ir embora
Não vai ser a primeira vez nas últimas 24 horas
Mas eu não vim até aqui pra desistir agora"
Engenheiros do Hawaii

Resumo

Questões relacionadas à existência de soluções suaves globais no tempo ou à possibilidade de desenvolverem singularidades em tempo finito para determinadas equações na dinâmica dos fluidos podem ser desafiadoras e complexas. Neste trabalho, analisamos possíveis cenários de formação de singularidade autossimilar para a equação quase geostrófica de superfície generalizada (gSQG) em duas dimensões. Mostramos que, sob uma condição no crescimento da norma L^r do perfil autossimilar e seu gradiente, o perfil autossimilar é identicamente zero ou seu comportamento assintótico L^p pode ser caracterizado, para p e r adequados, em intervalos apropriados do parâmetro autossimilar. Este resultado generaliza e melhora o resultado análogo provado para a equação SQG em [76], e recupera os resultados provados em [9], relativos às soluções globalmente autossimilares da equação gSQG. Também analisamos a equação gSQG com dissipação fracionária, generalizando o resultado provado em [13], o qual exclui a possibilidade de singularidade globalmente autossimilar em tempo finito para a equação SQG dissipativa, sob certas condições no perfil. Mais precisamente, assumindo que o gradiente do perfil autossimilar decai a zero no infinito e a parte simétrica do gradiente da velocidade autossimilar é limitada nos pontos de máximo do gradiente do perfil autossimilar, provamos que o perfil autossimilar é identicamente nulo em $\mathbb{R}^2$.

Palavras-chave: Equação gSQG, Laplaciano Fracionário, Potencial de Riesz, Soluções Autossimilares, Formação de Singularidade.

Abstract

Questions related to the existence of global smooth solutions over time or the possibility of developing singularities in finite time for certain equations in fluid dynamics can be challenging and complex. In this work, we analyze possible scenarios of self-similar singularity formation for the generalized surface quasi-geostrophic equation (gSQG) in two dimensions. We show that, under a condition on the growth of the L^r norm of the self-similar profile and its gradient, the self-similar profile is identically zero or its asymptotic L^p behavior can be characterized, for suitable p and r, in appropriate intervals of the self-similar parameter. This result generalizes and improves the analogous result proven for the SQG equation in [76], and recovers the results proven in [9], related to globally self-similar solutions of the gSQG equation. We also analyze the gSQG equation with fractional dissipation, generalizing the result proven in [13], which excludes the possibility of globally self-similar singularity in finite time for the dissipative SQG equation, under certain conditions on the profile. More precisely, assuming that the gradient of the self-similar profile decays to zero at infinity and the symmetric part of the self-similar velocity gradient is bounded at the maximum points of the self-similar profile, we prove that the self-similar profile is identically zero in $\mathbb{R}^2$.

Keywords: gSQG equation, Fractional Laplacian, Riesz Potential, Self-similar solution, Blowup.

List of symbols

$\mathbb{N}_0, \mathbb{N}, \mathbb{Z}, \mathbb{R}$	set of non-negative integers, positive integers, integers, and real numbers, respectively.
$\mathbb{R}^n$	real Euclidean space of dimension n.
$\mathbb{S}^{n-1}$	unit sphere in $\mathbb{R}^n$.
$\|\alpha\|$	size of the multi-index $\alpha \in \mathbb{N}_0^n$, $\|\alpha\| = \sum_{i=1}^n \alpha_i$.
x^α	power of $x \in \mathbb{R}^n$ of order $\|\alpha\|$ with $\alpha \in \mathbb{N}_0^n$, $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$.
$x \cdot y$	inner product of points $x, y \in \mathbb{R}^n$, $x \cdot y = \sum_{i=1}^n x_i y_i$.
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
$\partial^\alpha u$	partial derivative of u of order $\|\alpha\|$, α multi-index, $\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u$.
$\partial_t u$	partial derivative of u with respect to t.
∇u	Jacobian matrix of the vector field $u = (u_1, u_2, \ldots, u_n)$.
$\nabla \cdot u$	divergence of the vector field $u = (u_1, u_2, \ldots, u_n)$.
$f\|_K$	restriction of the function f to the set K.
χ_Ω	characteristic function of Ω.
$\hat{f}$, $\mathcal{F}(\mathrm{f})$	Fourier transform of the function f.
$f * g$	convolution between the functions/distributions f and g.
Λ^s	Fractional Laplacian of order $s \geqslant 0$.
$(-\Delta)^{-\frac{s}{2}}$	Riesz potential of order $s \geqslant 0$.
$x^\perp$	perpendicular vector of $x \in \mathbb{R}^n$.
$C_c^\infty(\mathbb{R}^n)$	space of smooth functions in $\mathbb{R}^n$ with compact support.
$C^\lambda(\Omega)$	space of λ-Hölder functions in Ω, $0 < \lambda < 1$.
$\mathcal{C}^{1,\lambda}(\mathbb{R}^n)$	space of functions that are $\mathcal{C}^1(\mathbb{R}^n)$ with Hölder continuous first derivatives of order α for $0 < \lambda < 1$.
$\mathcal{C}^k(\mathbb{R}^n)$	space of functions f with $\partial^\alpha f$ continuous for all $\|\alpha\| \leqslant k, k \in \mathbb{N}$.

$W^{k,p}(\Omega)$	Sobolev space in Ω.
$\mathcal{S}(\mathbb{R}^n)$	space of Schwartz functions in $\mathbb{R}^n$.
$\mathcal{S}'(\mathbb{R}^n)$	space of tempered distributions in $\mathbb{R}^n$.
$L^p(\mathbb{R}^n)$	Lebesgue space of p-integrable functions in $\mathbb{R}^n$.
$L^p_{\mathrm{loc}}(\mathbb{R}^n)$	Lebesgue space of locally p-integrable functions in $\mathbb{R}^n$.
$C([0,T];\mathbf{X})$	space of time-continuous functions with values in the Banach space $\mathbf{X}$.
$A \lesssim B$	means that $A \leqslant CB$ for some constant $C > 0$.
$A \sim B$	means that both $A \lesssim B$ and $B \lesssim A$ hold.

Contents

1 Introduction

Fluid dynamics is an interdisciplinary field that uses concepts from physics and mathematical tools to describe the flow of fluids, whether in a liquid or gaseous state, under various conditions and subject to the influence of external forces. Essential for understanding meteorological and oceanographic phenomena, fluid dynamics enables the description of fluid behavior over time, see [65].

For certain equations in fluid dynamics, answering questions about the existence of global smooth solutions on time or the possibility of developing singularities in finite time can be challenging and complex. Among these equations, the 3D incompressible Navier-Stokes equation stands out, modeling the movement of viscous incompressible fluids, proposed by Claude-Louis Navier and George Gabriel Stokes in the 19th century. The global well-posedness of these equations is one of the seven Millennium Prize Problems proposed by the Clay Mathematics Institute (see [34]) in 2000, highlighting it as one of the most significant challenges in the analysis of Partial Differential Equations and attracting the attention of notorious researchers in this field.

Considering zero viscosity in the Navier-Stokes equations, we obtain the 3D incompressible Euler equations, which characterize the movement of ideal fluids which are incompressible fluids that offer no resistance to shear forces and have constant density. The problem of global well-posedness for the incompressible 3D Euler equations also remains open and is of great relevance. Recently, it was proven in [32] that solutions in $\mathcal{C}^{1,\alpha}$ can develop singularity in finite time. For a well-detailed summary of the available results, we refer to [31]. Furthermore, various numerical experiments indicate the possibility of singularity formation in finite time, as seen in [38, 50, 51, 57, 58].

The issue of global well-posedness for the Euler equations in three dimensions can be explored by studying equations that share similar analytical and geometric properties. A notable example is the surface quasi-geostrophic (SQG) equation in $\mathbb{R}^2$, which models the temperature or buoyancy of a strongly stratified fluid in a rapidly rotating regime in $\mathbb{R}^2$ (see [2, 40, 45, 55, 65]). Its similarity to the 3D Euler equations for incompressible fluids has motivated its analytical study, which was initiated in [21]. Furthermore, the problem of global well-posedness remains open (see [18, 23])

To advance the understanding of these equations, initial studies in [15] and [29] started to analyze intermediate equations between the vorticity equation for incompressible and ideal flows in $\mathbb{R}^2$ and the SQG equation. This investigation led to the development of the generalized surface quasi-geostrophic equations in $\mathbb{R}^2$. This reflects the complexity involved in studying the global well-posedness of solutions in fluid dynamics, emphasizing the relevance of the study conducted in this project on self-similar solutions. In Section 2.3, it is observed that these types of solutions satisfy the blowup criteria in the sense can that develop singularity in finite time.

We consider the (dissipative) generalized surface quasi-geostrophic equation (gSQG) in $\mathbb{R}^2$ given by

$$\begin{cases} \theta_t + \mathbf{u} \cdot \nabla\theta + \kappa\Lambda^\eta\theta = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \mathbf{u} = -\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta, & x \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{1.0.1}$$

where $\beta, \eta \in (0, 2)$ are fixed parameters, $\kappa \geqslant 0$, $\theta = \theta(x, t)$ is an unknown scalar function, and $\mathbf{u} = \mathbf{u}(x, t)$ denotes a velocity field. The latter is given in terms of θ according to the second equation in (1.0.1), where $\nabla^\perp = (-\partial_2, \partial_1)$, $(-\Delta)^{-s/2}$, $0 < s < 2$, is the Riesz potential and Λ^η is the Fractional Laplacian (see Section 2.2.2 for definition). Invoking the definition of the Riesz potential, we can also rewrite $\mathbf{u}$ as

$$\mathbf{u}(x, t) = C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x - y)\theta(y, t)dy, \tag{1.0.2}$$

where

$$K_\beta(x) = \frac{x^\perp}{|x|^{2+\beta}}, \quad x \in \mathbb{R}^2\backslash\{0\}, \tag{1.0.3}$$

and C_β is a constant depending only on β, see Section 2.2.2 for details.

To facilitate the understanding of notation during the development of the project, we emphasize that in the dissipative case, that is, when $\kappa > 0$, we will refer to equation (1.0.1) as the dissipative gSQG equation and as the dissipative SQG when $\beta = 1$. In the case where $\kappa = 0$, we will name it the gSQG equation, and when $\beta = 1$, the SQG equation.

Note that when $\kappa \geqslant 0$ $\beta = 0$ and $\eta = 2$, equation (1.0.1) reduces to the vorticity formulation of the Navier-Stokes equations in $\mathbb{R}^2$, which is well-posed for any initial data in $L^2(\mathbb{R}^2)$ (see [41, 54]). Now, if $\kappa = 0$, considering $\beta = 0$, equation (1.0.1) reduces to the vorticity formulation of the 2D incompressible Euler equations, which is well-posed for initial data belonging to $H^s(\mathbb{R}^2)$ with $s > 2$ (see [30, 78]).

Among the available results, local existence and uniqueness for the Cauchy problem associated with gSQG equation in the range $\beta \in (1, 2)$ was shown in [14] for any initial data in $H^4(\mathbb{R}^2)$, and later improved in [43] to any initial data in $H^s(\mathbb{R}^2)$, with $s > 1 + \beta$. An analogous local well-posedness result in $H^s(\mathbb{R}^2)$, $s > 1 + \beta$, for the more regular case $\beta \in (0, 1]$ was shown in detail in [44, 77].

The dissipative gSQG equation was first introduced by Chae, Constantin, and Wu [16], where they studied the global regularity of the intermediate equations between the 2D Navier-Stokes equation and the dissipative SQG equation. Recently, this equation has been intensively studied, and several results regarding local/global well-posedness have been obtained. Specifically, in [60], the global well-posedness for the diagonal $\eta = \beta$ was established. In [61], global well-posedness was achieved for the regime $1 < \beta < 2$ and $2\beta - 2 < \eta < \beta$. A recent work regarding the local well-posedness of the dissipative gSQG equation was established in [46], which proved the existence and uniqueness of solutions for arbitrary initial conditions in $H^{1+\beta-\eta}(\mathbb{R}^2)$, where $\eta \in (0, 1)$ and $\beta \in (1, 2)$. For further information, references [16, 14, 56, 19] are recommended, especially [46] for a well-detailed diagram summarizing the available results.

As mentioned earlier, an important class of (1.0.1) is when $\kappa > 0$ and $\beta = 1$, in which case the system is reduced to the dissipative SQG equations.

In some functional spaces of interest, such as $H^s(\mathbb{R}^2)$ and $L^\infty(\mathbb{R}^2)$, the scale-invariance of the norm of the solution of the dissipative SQG equation occurs when $\eta = 1$. For this reason, the study of this system is usually divided into three dissipation ranges: subcritical ($1 < \alpha < 2$), critical ($\eta = 1$), and supercritical ($0 < \eta < 1$). The subcritical case was well understood by Resnick [67], proving the existence of weak solutions and the global well-posedness of the solution. Later, Constantin and Wu established in [24] that any solution with a smooth initial value is smooth for all time. In the critical case, the global well-posedness of the weak solution was established in [18, 23, 53], using different approaches. We also refer to[7, 52]. Finally, in the supercritical case, the problem of global well-posedness remains an open problem (see [23, 26, 46]). However, local well-posedness for large data and global well-posedness for small data has been established in [25, 48, 62] and [26, 47], respectively. In this direction, an important recent development is a result by Coti-Zelati and Vicol in [79], which proves the existence of a range for η, dependent on the $H^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ norms of the initial data, where the initial value issue for this data is globally well-posed.

In addition to these analytical results, several computational studies were developed to numerically investigate the possibility of finite-time singularity formation for the equation (1.0.1) in specific scenarios. Starting with the SQG equation, [21] indicated a possible finite-time singularity in the form of a hyperbolic closing saddle, a suggestion that was later contested in [20, 22, 64] via further numerical tests, and eventually theoretically ruled out in [27, 28]. On the other hand, in [70, 71], analyzing an alternative scenario proposed by [42, 66], the authors found numerical evidence of a singularity occurring as a self-similar cascade of filament instabilities. Regarding the gSQG equation, numerical simulations were performed in [29, 59, 72] focusing on the evolution of patch-like initial data, i.e. given by the indicator function of a spatial domain with smooth boundary [14, 36, 68, 69].

While a rigorous proof of the formation of such singularities is still not available, these numerical studies provide a strong motivation to investigate further solutions of the equation (1.0.1) that develop a finite-time singularity of self-similar type. In the dissipative case, such solutions are defined with respect to the invariance under the following scaling transformation

$$x, t, \theta \mapsto \lambda x, \lambda^\eta t, \lambda^{\eta-\beta}\theta,$$

with $\lambda \in \mathbb{R}^+$, $\eta \in (0, 2)$; i.e. if θ is a solution of (1.0.1), then

$$\theta_\lambda(x, t) = \lambda^{\eta-\beta}\theta(\lambda x, \lambda^\eta t),$$

is also a solution. In the case $\kappa = 0$, the equation (1.0.1) is invariant under the scaling transformation $x, t, \theta \mapsto \lambda x, \lambda^{1+\alpha} t, \lambda^{1+\alpha-\beta}\theta$, with $\lambda \in \mathbb{R}^+$, $\alpha \in \mathbb{R}$. Note that, in this case, one of the scaling parameters does not come from the equation (1.0.1). We refer Section 2.3 for further details.

Precisely, we say that a solution θ of the dissipative gSQG on $\mathbb{R}^2$ and on a time interval $(0,T)$ is *(globally) self-similar* if $\theta(x,t) = \theta_\lambda(x,t)$ for all $(x,t) \in \mathbb{R}^2 \times (0,T)$ and for all $\lambda > 0$. This implies that θ can be expressed as

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{\eta-\beta}{\eta}}}\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right) \quad \text{for all } (x,t) \in \mathbb{R}^2 \times (0,T), \tag{1.0.4}$$

for some function $\Theta : \mathbb{R}^2 \to \mathbb{R}$, which is called an associated *self-similar profile*. Similarly, if $\kappa = 0$, the *(globally) self-similar* of the gSQG equation, is given by

$$\theta(x,t) = \frac{1}{t^{\frac{1+\alpha-\beta}{1+\alpha}}}\Theta\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right) \quad \text{for all } (x,t) \in \mathbb{R}^2 \times (0,T), \tag{1.0.5}$$

where Θ is the self-similar profile and $\alpha > -1$ is called the scaling parameter.

To the best of our knowledge, the only available result concerning globally self-similar solutions of the dissipative SQG equation is due to Chae in [13]. In this work, under the assumption that the gradient of the self-similar profile decays to zero at infinity and the symmetric part of the self-similar velocity gradient is bounded at the points of the maximum gradient of the self-similar profile, Chae proves that the profile associated with the globally self-similar solution is constant in $\mathbb{R}^2$. We emphasize that in Chapter 4, we extend this result to cover all $(\eta, \beta) \in (0,2) \times (0,2)$.

Some results on the nonexistence of nontrivial globally self-similar solutions for the SQG and gSQG equations were obtained in [12, 13] and [9], respectively, by imposing suitable assumptions on the profile Θ and showing that $\Theta \equiv 0$ as a consequence, thus excluding the possibility of finite-time singularity of this type. More precisely, [12] assumed $\Theta \in L^{p_1}(\mathbb{R}^2) \cap L^{p_2}(\mathbb{R}^2)$ with $p_1, p_2 \in [1,\infty]$ and $p_1 < p_2$, whereas [13] considered $\Theta \in C^1(\mathbb{R}^2)$ such that $\lim_{|x|\to\infty} |\Theta(x)| = 0$. Both [12] and [13] utilize a particle trajectory and back-to-labels map approach to establish that $\Theta \equiv 0$. In [9], the authors analyze the gSQG equation in the case $\beta \in [0,1]$ and obtain an analogous result as in [12] while relying on a different technique centered on a local L^p inequality satisfied by the profile Θ.

We also mention the recent work [37], where the construction of a class of non-radial globally self-similar solutions with infinite energy of the gSQG in the case $\beta \in (0,1)$ was obtained via suitable perturbations of a stationary solution. See additionally [10], where the authors consider solutions of the SQG equation in $\mathbb{R}^2$ of the form $\theta(x_1, x_2, t) = x_2 f_{x_1}(x_1, t)$ and construct a self-similar solution for the one-dimensional equation satisfied by f which yields an infinite-energy solution for the SQG equation.

Beyond globally self-similar solutions, this thesis also studies the more general scenario of solutions θ of the gSQG equation that satisfies an equality as in (1.0.5) only locally in space, namely with $(x,t) \in B_\rho(x_0) \times (0,T)$, for some $\rho > 0$. Here, $B_\rho(x_0)$ denotes the ball in

$\mathbb{R}^2$ centered at 0 and with radius ρ. In fact, since, for any $x_0 \in \mathbb{R}^2$, $\tilde{\theta}(x,t) = -\theta(x-x_0, T-t)$, $(x,t) \in \mathbb{R}^2 \times (0,T)$, is also a solution of (1.0.1) due to its spatial translation and time reversal symmetries, we may consider more generally solutions θ of (1.0.1) that satisfy

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{1+\alpha-\beta}{1+\alpha}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right) \quad \text{for all } (x,t) \in B_\rho(x_0) \times (0,T), \tag{1.0.6}$$

for some $\rho > 0$ and some profile function $\Theta : \mathbb{R}^2 \to \mathbb{R}$. We refer to such θ as a *locally self-similar solution*.

Besides the several numerical computations, the continuation criteria available for the dissipative SQG equation and gSQG equation, when $\beta \in (0,1]$, also provide a strong motivation to further investigate self-similar blowup. Namely, for the gSQG equation, the global regularity criterion in the case $\beta \in (0,1]$ was obtained in [15] for the norm of a given solution in β-Hölder spaces, which generalizes a previous regularity criterion established for the SQG in [21]. Specifically, [15] shows that $[0,T)$ is a maximal interval of existence for a solution θ of the gSQG equation within the class $\mathcal{C}^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, with $\sigma > 1$ and $q > 1$, if

$$\lim_{t \to T} \int_0^t \|\theta(\cdot, s)\|_{\mathcal{C}^\beta(\mathbb{R}^2)} ds = \infty, \tag{1.0.7}$$

where $\mathcal{C}^\gamma(\mathbb{R}^2)$, $0 < \gamma \leqslant 1$, denotes the space of γ-Hölder continuous functions on $\mathbb{R}^2$. For the dissipative case, as far as known, the only available continuation criteria were obtained in [16, Theorem 3.1]. Precisely, $[0,T)$ is the maximal interval of existence for a solution θ of the dissipative SQG equation within the class $C([0,T); H^s(\mathbb{R}^2))$, $s > 2$, with $T < \infty$, if and only if there exist p, r with $\frac{2}{\eta} < p < \infty$ and $1 < r < \infty$ such that

$$\lim_{s \to T} \int_0^s ||\nabla_x^\perp \theta||_{L^p}^r dt = \infty \quad \text{with} \quad \frac{2}{p} + \frac{\eta}{r} \leqslant \eta. \tag{1.0.8}$$

A straightforward computation shows that if $\Theta \in \mathcal{C}^\beta(\mathbb{R}^2)$ then condition (1.0.6) is indeed consistent with the regularity criterion from [15] when $\beta \in (0,1]$. Namely, (1.0.6) implies (1.0.7), and hence T represents a finite blowup time for θ in the class $\mathcal{C}^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, with $\sigma > 1$ and $q > 1$. Similarly, this argument holds for the globally self-similar solution (1.0.5) and the continuation criteria proved in [16, Theorem 3.1]. For further details on this topic, we refer to Section 2.3.

Up to the present moment, no results have been obtained regarding the non-existence of locally self-similar blowup or the asymptotic behavior of the profile in $L^p(\mathbb{R}^2)$ space for the dissipative gSQG equation. A significant challenge in this context is the analysis of non-local operators, especially for the term $\Lambda^\eta \theta$ in the equation (1.0.1). Since the Fractional Laplacian is a non-local operator in $\mathbb{R}^2$, and the condition (1.0.4) applies only within the self-similar region, estimating $\Lambda^\eta \theta$ in the L^p norm outside the self-similar region becomes challenging when relying on assumptions about the profile. Furthermore, in contrast to globally self-similar solutions, formulating an equation that describes the profile directly from (1.0.1) seems not possible.

We point out that there are results concerning locally self-similar singularity scenarios for the N-dimensional incompressible Euler equations, for $N \geqslant 3$, see [6, 17, 75]. These self-similar solutions for the Euler system are broadly applied in numerical simulations, for instance, the works [3, 49, 59] indicate the potential occurrence of self-similar blowups at a finite time through the examination of vortex filament models or high symmetric flows. For the Navier-Stokes equations, the question about the self-similar solutions was raised by Leray in 1934, and the mathematical proof for the nonexistence of self-similar solutions was only established, in 1996, in [63]. We refer to [8] for a summary of the available results.

The analysis of self-similar blowup scenarios is more recent for the SQG equation. We mention the result obtained in [76] that yields, similarly to the previous works for Euler Equations [6, 75], suitable conditions on the self-similar profile under which the existence of nontrivial Θ is only possible within an explicitly identified range of α, i.e. Θ must necessarily be zero for α outside of this range. Moreover, any nontrivial profile corresponding to a value of α in this range must satisfy a certain asymptotic characterization of its L^p average over sufficiently large regions in the spatial domain, for some $p > 1$. As a consequence, this allows one to automatically exclude the existence of locally self-similar solutions with sufficiently fast decaying profiles, while also guaranteeing the aforementioned asymptotic characterization of the L^p average of certain non-decaying types of Θ.

This thesis is organized as follows, Chapter 2 is dedicated to recalling classical analysis results regarding L^p space, Sobolev space, Singular integral, and others. Chapter 3 is based on the paper [5], which was developed by the author in collaboration with his supervisor and co-supervisor. This chapter is dedicated to the study of the locally self-similar solution of the gSQG equation and it is divided into two sections, depending on the range of values of the parameter β. Section 3.1 is committed to studying the case $\beta \in (1, 2)$. Our goal is to prove Theorem 3.1, which provides, under an assumption on the L^r growth of the self-similar profile and its gradient, intervals of the scaling parameter where the profile is identically zero or its asymptotic L^p behavior can be characterized, for suitable p and r. Next, in Section 3.2, we present a similar result for $\beta \in (0, 1]$ in Theorem 3.2, assuming only the growth bound for the profile, Θ. We emphasize that Theorem 3.2, also yields an improvement in comparison to the analogous result from [76] regarding $\beta = 1$. Specifically, we allow for a larger range of possible values of the parameters r and γ under which assumption (3.2.1) must be verified. Additionally, Theorem 3.2 includes as special case the aforementioned result established in [9] concerning globally self-similar solutions of the gSQG equation for $\beta \in (0, 1]$, since every globally self-similar solution is also locally self-similar.

The division of Chapter 3 into two cases, based on the parameter β, is necessary because of the different conditions required for Θ in each case. This distinction arises from the specific approach required for estimating the component of $\mathbf{u}$ in (3.1.11), where the integrand is restricted to the self-similar region, as detailed in Lemma 6.2 and Lemma 6.3. For the less singular scenario, $\beta \in (0, 1]$, we observe that the kernel K_β, associated with the velocity field $\mathbf{u}$, satisfies $||K_\beta * f||_{L^q(B\rho(0))} \lesssim ||f||_{L^q(\mathbb{R}^2)}$ for every $f \in L^q(\mathbb{R}^2)$ and $1 < q < \infty$. For $\beta \in (0, 1)$, this result is obtained by invoking Young's convolution inequality, together with the integrability of K_β near the origin. In the case $\beta = 1$, this follows from the fact that K_β is a Calderón-Zygmund operator (see Section 2.2.1). On the other hand, for $\beta \in (1, 2)$, K_β is neither a Calderón-Zygmund operator nor integrable near the origin. To address this challenge, we write $K_\beta(x) = \nabla^\perp(|x|^{-\beta})$ and apply integration by parts in $K_\beta * \theta$, thus transferring one derivative to θ, and hence to Θ, when restricting θ to the self-similar region. Therefore, in this case, stronger assumptions on the profile are required than in the case $\beta \in (0, 1]$. Specifically, an additional growth condition on the L^r norm of $\nabla\Theta$ is needed.

In Chapter 4, we study the globally self-similar solution to the dissipative gSQG equation and extend the result proved by Chae in [13, Theorem 3.1] to cover all $\beta \in (0, 2)$, beyond of the specific case of $\beta = 1$. More precisely, in Theorem 4.1, under the assumption that the gradient of the self-similar profile decays to zero at infinity and the symmetric part of the self-similar velocity gradient is bounded at the points of the maximum gradient of the profile, we conclude that the profile is identically null in $\mathbb{R}^2$, and hence, the non-existence of globally self-similar blowup. In this case, the approach is made possible thanks to the assumption that θ is a globally self-similar solution. Thus, by invoking (1.0.4) in (1.0.1), we can derive the equation for the profile, Θ.

In Chapter 6, we provide a comprehensive proof of Lemma 6.2 and Lemma 6.3, as well as the proof of the local L^p inequality (3.1.4), which are referenced in the proofs of Theorem 3.1 and Theorem 3.2 in Chapter 3. Although the local L^p inequality is also utilized in [76] and [9], its proof is not included in these references.

Finally, in Chapter 5, we summarize the original results obtained in this thesis and discuss some interesting problems associated with self-similar solutions of the dissipative gSQG equation that we plan to investigate further.

2 Preliminary

In this chapter, we revisit several preliminary results employed in this thesis. We commence by fixing the notation and revisiting fundamental concepts associated with L^p and Sobolev spaces. The subsequent section delves into the fractional Laplacian operator, the Riesz operator and the Riesz transform. Later, we bring up some results related to these operators. For further details, see [4, 33, 35, 39, 73, 74].

2.1 Functional Spaces

Fix $\Omega \subset \mathbb{R}^n$ an open set and $1 \leqslant p \leqslant \infty$. We denote by $L^p(\Omega)$ the set of all measurable functions for which the integral of the p-power of its absolute value is finite. More precisely,

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \ \mid f \text{ is Lebesgue measurable with } ||f||_{L^p(\Omega)} < \infty\},$$

where

$$||f||_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, & \text{if} \quad 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}_\Omega |f|, & \text{if} \quad p = \infty. \end{cases}$$

The space $L^p(\Omega)$ endorsed with the norm $|| \cdot ||$ is a Banach space.

The Sobolev spaces $W^{m,p}(\Omega)$ with $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, consist of all locally integrable functions $f : \Omega \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \leqslant m$, $D^\alpha f$, exists in the weak sense and belongs to $L^p(\Omega)$. Well recall that $W^{m,p}(\Omega)$ is a Banach space when endowed

$$\|f\|_{W^{m,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leqslant m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} & 1 \leqslant p < \infty; \\ \max_{|\alpha| \leqslant m} \|D^\alpha f\|_{L^\infty(\Omega)} & p = \infty \end{cases},$$

Another important space in the study of partial differential equations is the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$. This space consists of smooth functions that, together with their derivatives, decay faster than the reciprocal of any polynomial at infinity. More precisely, the Schwartz space is defined as

$$\mathcal{S}(\mathbb{R}^n) := \{f \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \forall \ \alpha, \beta \in \mathbb{N}^n, \|f\|_{\alpha,\beta} < \infty\},$$

where $\mathcal{C}^\infty(\mathbb{R}^n)$ is the function space of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$ and

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left|x^\alpha (D^\beta f)(x)\right|$$

are the so-called Schwartz seminorms. We have that $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space with the topology defined by the Schwartz seminorms. We denote $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$. The element of $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distribuitons.

2.2 Classic Analysis Results

In this section, we review some analysis results related to convolution of functions, L^p inequality, and Sobolev Space embeddings. Let us initiate our discussion by recalling the definition of convolution of two functions.

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. We define the convolution of f and g, denoted by $f * g$, as the following integral:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy, \ x \in \mathbb{R}^n, \tag{2.2.1}$$

whenever this integral exists.

The mollifiers are a useful tool in mathematical analysis for smoothing out and approximating functions with irregularities or singularities via convolution. To define them, consider $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ given by

$$\phi(x) = \begin{cases} Ce^{-1/(1-|x|^2)}, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geqslant 1. \end{cases}$$

where C is a constant such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Now, for each $\varepsilon > 0$, consider the following function

$$\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right). \tag{2.2.2}$$

We call ϕ the standard mollifier and $\{\phi_\varepsilon\}_{\varepsilon>0}$ the mollifying family. The functions ϕ_ε are $\mathcal{C}^\infty$ and satisfy

$$\int_{\mathbb{R}^n} \phi_\varepsilon(x)dx = 1,$$

and

$$\operatorname{supp}(\phi_\varepsilon) \subset B(0, \varepsilon).$$

For $f \in L^1(\mathbb{R}^n)$, we define

$$f_\varepsilon(x) = (\phi_\varepsilon * f)(x) = \int_{\mathbb{R}^n} \phi_\varepsilon(x-y)f(y)dy, \ x \in \mathbb{R}^n.$$

The family $\{f_\varepsilon\}_{\varepsilon>0}$ satisfies the following properties:

a) $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$.

b) $f_\varepsilon \to f$ a.e. as $\varepsilon \to 0$.

c) If $f \in \mathcal{C}(\mathbb{R}^n)$, then $f_\varepsilon \to f$ uniformly on compacts subsets of $\mathbb{R}^n$ as $\varepsilon \to 0$.

d) If $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.

Next, let us see several inequalities regarding the $L^p(\mathbb{R}^n)$ space. Let us start with Young's convolution inequality.

Proposition 2.2.1. ***(Hölder Inequality)*** *Assume $1 \leqslant p, q \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, if $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, we have $uv \in L^1(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |uv| dx \leqslant ||u||_{L^p(\mathbb{R}^n)} ||v||_{L^q(\mathbb{R}^n)}.$$

Proposition 2.2.2. ***(Interpolation inequality)*** *If $1 \leqslant p < q < r \leqslant \infty$, then $L^p(\mathbb{R}^n) \cap L^r(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and*

$$||f||_{L^q(\mathbb{R}^n)} \leqslant ||f||^{\lambda}_{L^p(\mathbb{R}^n)} ||f||^{1-\lambda}_{L^r(\mathbb{R}^n)}, \tag{2.2.3}$$

where $\lambda = (\frac{1}{q} - \frac{1}{r})(\frac{1}{p} - \frac{1}{r})^{-1} \in (0, 1)$.

Proposition 2.2.3. ***(Young's convolution inequality)*** *If $1 \leqslant p, q, r \leqslant \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g \in L^r(\mathbb{R}^n)$ for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ and*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leqslant \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Before presenting some results regarding the Sobolev embedding, let us recall the definition of continuous embedding between two normed vector spaces.

Definition 2.2.4. Let X and Y be two normed vector spaces, with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively, such that $X \subseteq Y$. If the inclusion map (identity function) is continuous, i.e., if there exists a constant $C > 0$ such that

$$\|x\|_Y \leqslant C\|x\|_X, \quad \forall\ x \in X,$$

then X is said to be continuously embedded in Y and we write $X \hookrightarrow Y$.

The next theorems are classical results of the Measure theory.

Theorem 2.1 (Dominated Convergence Theorem). *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions such that*

$$f_k \to f \quad a.e.$$

Also, suppose

$$|f_k| \leqslant g \quad a.e., \forall\ k,$$

for some integrable function g. Then

$$\int_{\mathbb{R}^n} f_k(x) dx \to \int_{\mathbb{R}^n} f(x) dx$$

Theorem 2.2 (Lebesgue Differentiation Theorem). *Suppose $f \in L^1_{loc}(\mathbb{R}^n)$. For almost every $x \in \mathbb{R}^n$, we have*

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)| dy = 0,$$

and

$$\lim_{r\to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)dy = f(x),$$

where $B_r(x)$ represents the ball of radius r centered at x, and $|B_r(x)|$ represents the volume of this ball.

Now, we recall the Poincaré inequality.

Theorem 2.3 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open subset with a C^1 boundary $\partial\Omega$. Assume $1 \leqslant p \leqslant \infty$. Then there exists a constant C, depending only on n, p and Ω, such that for each $u \in W^{1,p}(\Omega)$ holds*

$$||u - (u)_\Omega||_{L^p(\Omega)} \leqslant C||Du||_{L^p(\Omega)}, \tag{2.2.4}$$

where $(u)_\Omega = \frac{1}{|\Omega|}\int_\Omega u(y)dy$.

Now, let us introduce the Fourier transform and recall the Plancherel Theorem.

Definition 2.2.5. Suppose $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f, denoted by $\hat{f}$ or $\mathcal{F}(f)$, is defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{R^n} e^{-2\pi i\xi\cdot x} f(x)dx, \ \xi \in \mathbb{R}^n.$$

Theorem 2.4 (Plancherel theorem). *Let $f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Then, $\hat{f} \in L^2(\mathbb{R}^n)$, and the following equality holds:*

$$\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi.$$

Thus, since $\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an isometric application and $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is a dense subset in $L^2(\mathbb{R}^n)$, it follows by density that there is a unique extension of the Fourier transform on $L^2(\mathbb{R}^n)$, which is also denoted by $\mathcal{F}$. Therefore, we conclude that for all $f \in L^2(\mathbb{R}^n)$ holds

$$||\mathcal{F}(f)||_{L^2(\mathbb{R}^n)} = ||f||_{L^2(\mathbb{R}^n)}.$$

We refer to [39] for more details.

From the definition of the Fourier transform, we may define the Sobolev H^s space, for $s \geqslant 0$, as follows

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) \mid [(1 + |\xi|^2)^{\frac{s}{2}}]\hat{f} \in L^2(\mathbb{R}^n)\},$$

and

$$||u||_{H^s(\mathbb{R}^n)} := ||(1 + |y|^2)^{\frac{s}{2}}\hat{u}||_{L^2(\mathbb{R}^n)}.$$

The Sobolev space $H^s(\mathbb{R}^2)$ is continuously embedded in $L^p(\mathbb{R}^2)$ for all $s \geqslant 1 - \frac{2}{p}$ and $p \in [2, \infty)$ (see [1, Theorem 7.58]), i.e., there exists a constant $c > 0$ such that for any $f \in H^s(\mathbb{R}^2)$ holds

$$||f||_{L^p(\mathbb{R}^2)} \leqslant c||f||_{H^s(\mathbb{R}^2)}. \tag{2.2.5}$$

We emphasize that given that if $f \in H^s(\mathbb{R}^2)$ $s \geqslant 0$, it follows that

$$f_\varepsilon \to f \quad \text{in } H^s(\mathbb{R}^2), \quad \text{as } \varepsilon \to 0.$$

where f_ε is the mollifying function of f. Moreover, it holds

$$||f_\varepsilon||_{H^s(\mathbb{R}^2)} \leqslant c||f||_{H^s(\mathbb{R}^2)}. \tag{2.2.6}$$

for some constant $c > 0$.

Finally, let us recall the Cauchy Principal Value ($P.V.$) of an integral. Let ϕ, be a continuous function on $\mathbb{R}^n$ except at the origin, then the Cauchy principal value of the integral of ϕ is defined as

$$P.V. \int_{\mathbb{R}^n} \phi(x) dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \phi(x) \, \mathrm{d}x. \tag{2.2.7}$$

where $B_\varepsilon(0)$ denotes the ball in $\mathbb{R}^n$ centered at the origin and with radius ε.

2.2.1 Singular Integral

Our goal in this subsection is to introduce the Calderón-Zygmund kernel and then state the Calderón-Zygmund Theorem. Additionally, we demonstrate that the kernel K_β, associated with the velocity field, does not qualify as a Calderón-Zygmund kernel if $\beta \in (0, 2) \setminus \{1\}$. For further details, we refer to [74].

Definition 2.2.6. We say that a function K, locally integrable away from the origin, is a Calderón-Zygmund singular kernel provided it verifies the following three properties:

1. For each $0 < \varepsilon < N$ holds
$$\left| \int_{\varepsilon < |\xi| < N} K(x) \, dx \right| \leqslant c_1,$$
where c_1 is a constant independent of ε, N. Additional, $\lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < N} K(x) \, dx$ exists for each fixed N.

2. For all $R > 0$ holds
$$\int_{|x| \leqslant R} |x||K(x)| \, dx \leqslant c_2 R, \tag{2.2.8}$$
with c_2 independent of R.

3. If $y \neq 0$, then
$$\int_{|x|\geqslant 2|y|} |K(x-y)-K(x)|\, dx \leqslant c_3,$$
with c_3 independent of y.

Now, we see that if K is a Calderón-Zygmund kernel, then the operator $T = K * f$, with $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$, is a continuous operator in $L^p(\mathbb{R}^n)$.

Theorem 2.5 (**Calderón-Zygmund Theorem**). *Suppose that K is a Calderon-Zygmund kernel and for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ let*
$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} K(x-y)f(y)dy,\ \varepsilon > 0.$$
Then there is a constant $c = c_p$, depending only on p, so that
$$\|T_\varepsilon f\|_p \leqslant c\|f\|_p. \tag{2.2.9}$$
Moreover, $Tf = \lim_{\varepsilon\to 0} T_\varepsilon f$ exists in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, and it also satisfies the estimate (2.2.9). *Furthermore, $T_\varepsilon f$ and Tf are well-defined for arbitrary f in $L^p(\mathbb{R}^n)$, and the norm inequality* (2.2.9) *holds for these functions as well T_ε and T are called the Calderón-Zygmund singular operator associated to K.*

We highlight that the kernel associated with the velocity field $\mathbf{u}$, given by $K_\beta = x^\perp |x|^{-(2+\beta)}$ for $x \in \mathbb{R}^2$, is a Calderón-Zygmund kernel if and only if $\beta = 1$. To analyze the condition outlined in (2.2.8). Fix $\beta \in (0,2)$. It follows, by using polar coordinates in $\mathbb{R}^2$, that
$$\int_{|x|\leqslant R} |x||K_\beta(x)|\, dx = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|\leqslant R} \frac{1}{|x|^\beta} = \lim_{\varepsilon\to 0}\int_\varepsilon^R\int_0^{2\pi} \frac{1}{r^\beta} r dr d\phi = 2\pi \lim_{\varepsilon\to 0}\int_\varepsilon^R r^{1-\beta}dr.$$
If $\beta \neq 1$, we have
$$\int_{|x|\leqslant R} |x||K_\beta(x)|\, dx = 2\pi \lim_{\varepsilon\to 0}\left[\frac{R^{2-\beta}}{2-\beta} - \frac{\varepsilon^{2-\beta}}{2-\beta}\right] = \frac{2\pi}{2-\beta}R^{2-\beta}. \tag{2.2.10}$$
Now, if $\beta = 1$, it follows from directly computation that
$$\int_{|x|\leqslant R} |x||K(x)|\, dx = R.$$
Combining the last two results, we conclude that
$$\int_{|x|\leqslant R} |x||K_\beta|\, dx = \begin{cases} \frac{2\pi}{2-\beta}R^{2-\beta}, & \text{se } \beta \neq 1, \\ R, & \text{se } \beta = 1. \end{cases}$$
Therefore, notice that if $\beta \neq 1$, it is not possible to obtain (2.2.8), and hence, we conclude the statement.

2.2.2 Non-local Operators

Non-local operators are characterized by their ability to gather information across the entire domain of a function, unlike local operators, which only obtain information from specific points or neighborhoods. A prime example of a non-local operator is the integral operator, which analyzes the entire function through integration, while strong derivatives are a typical example of local operators, focusing on point variations of the function.

In this section, we aim to introduce the Fractional Laplacian and the Riesz Potential in $\mathbb{R}^n$ and use their properties to derive the equation (1.0.2).

For a given $\eta \geqslant 0$, the fractional Laplacian of order η, represented as $\Lambda^\eta := (-\Delta)^{\frac{\eta}{2}}$, extends the power of the classical Laplacian operator Δ to non-integer values. This operator can be defined in various forms, including through the Fourier transform. Specifically, for a function $f \in \mathcal{S}(\mathbb{R}^n)$, it is established that

$$\mathcal{F}[\Lambda^\eta(f)](\xi) = |\xi|^\eta \mathcal{F}(f)(\xi), \ \xi \in \mathbb{R}^n.$$

For $\eta \in (0,2)$, the Fractional Laplacian can also be interpreted as a singular integral operator, expressed as

$$\Lambda^\eta f(x) = d_{\eta,n} P.V. \int\limits_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+\eta}}\, dy, \quad x \in \mathbb{R}^n \text{ and } f \in \mathcal{S}(\mathbb{R}^n),$$

where $P.V.$ represents the Value Principal, defined in (2.2.7), and the constant $d_{\eta,n}$ is given by

$$d_{\eta,n} = \frac{4^\eta \Gamma(n/2+\eta)}{\pi^{\frac{n}{2}} |\Gamma(-\eta)|}.$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^t dt.$$

is the Gamma function. When $n = 2$, we write $d_{\eta,2} := d_\eta$.

Given $\alpha \in (0,2)$ and $f \in \mathcal{S}(\mathbb{R}^n)$, we recall that the Riesz operator can be defined by Fourier transform as

$$\mathcal{F}[(-\Delta)^{-\frac{\alpha}{2}} f](\xi) = |\xi|^{-\alpha} \mathcal{F}(f)(\xi), \ \xi \in \mathbb{R}^n.$$

We can also define the Riesz operator as an integral operator

$$(-\Delta)^{-\frac{\alpha}{2}} f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad x \in \mathbb{R}^n \text{ and } f \in \mathcal{S}(\mathbb{R}^n), \tag{2.2.11}$$

where the constant $c_{n,\alpha}$ is expressed as

$$c_{n,\alpha} = \pi^{\frac{n}{2}} 2^\alpha \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

When $n = 2$, we used the notation $c_{2,\alpha} := c_\alpha$.

Now, let us briefly discuss the expression (1.0.2). Firstly, we emphasize that by applying $\nabla^\perp$ in (2.2.11) and invoking a consequence of the Dominated Convergence Theorem (see [35, Theorem 2.27]), we have that the velocity field $\mathbf{u}$ can be represented as the following integral operator:

$$\begin{aligned}\mathbf{u}(x,t) &= -\nabla_x^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta(x,t) = -\nabla_x^\perp\left(c_\beta\int_{\mathbb{R}^2}\frac{\theta(y,t)}{|x-y|^\beta}dy\right)\\ &= c_\beta \text{P.V.}\int_{\mathbb{R}^2}\theta(y,t)\nabla_x^\perp\left(\frac{1}{|x-y|^{2+\beta}}\right)dy\\ &= C_\beta \text{P.V.}\int_{\mathbb{R}^2}K_\beta(x-y)\theta(y,t)dy, \quad (x,t)\in\mathbb{R}^2\times(0,T), \qquad (2.2.12)\end{aligned}$$

where $C_\beta = -\beta c_\beta$ and

$$K_\beta(x) := \frac{x^\perp}{|x|^{2+\beta}}, \quad x\in\mathbb{R}^2\backslash\{0\}. \qquad (2.2.13)$$

Furthermore, we emphasize that $\nabla\cdot\mathbf{u} = 0$.

2.2.3 Auxiliary results for the gSQG equation

We start by revisiting the Positivity Lemma of the Fractional Laplacian. This lemma is an essential tool for establishing the maximum principle for the equation (1.0.1). For further detailed of the proof, we refer [26, Lemma 2.5].

Lemma 2.1 ([26], Lemma 2.5)**.** *Let $0\leqslant\eta\leqslant 2$. If $\theta, \Lambda^\eta\theta\in L^p(\mathbb{R}^2)$ with $1\leqslant p<\infty$, then*

$$\int_{\mathbb{R}^2}|\theta|^{p-2}\theta\Lambda^\eta\theta dx \geqslant 0.$$

The next result establishes that the L^p norms of the solution of (4.1), are conserved for all time.

Lemma 2.2 ([67], Lemma 3.2)**.** *Let θ be a smooth solutions of the dissipative gSQG equation (1.0.1) on $\mathbb{R}^2\times[0,T)$, with initial data $\theta_0=\theta(0)$. Then, for all $2\leqslant p<\infty$, it holds*

$$||\theta(t)||_{L^p}\leqslant||\theta_0||_{L^p}, \quad \forall\ t\in[0,T).$$

Proof. Let us start the proof by multiplying (1.0.1) by $|\theta|^{p-2}\theta$ and then integrating it over the spatial variables. This yields

$$\int_{\mathbb{R}^2}(\partial_t\theta)|\theta|^{p-2}\theta dx + \int_{\mathbb{R}^2}|\theta|^{p-2}\theta(\mathbf{u}\cdot\nabla)\theta dx + \kappa\int_{\mathbb{R}^2}|\theta|^{p-2}\theta\Lambda^\eta\theta dx = 0. \qquad (2.2.14)$$

For the first term on the left-hand side, we have

$$\int_{\mathbb{R}^2}(\partial_t\theta)|\theta|^{p-2}\theta dx = \frac{1}{p}\int_{\mathbb{R}^2}\partial_t(|\theta|^p)dx = \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^2}|\theta|^p dx = \frac{1}{p}\frac{d}{dt}||\theta||_{L^p}^p.$$

Then, we can rewrite (2.2.14) as follows

$$\frac{d}{dt}||\theta||_{L^p}^p = -p\int_{\mathbb{R}^2} |\theta|^{p-2}\theta(\mathbf{u}\cdot\nabla)\theta dx - \kappa p\int_{\mathbb{R}^2} |\theta|^{p-2}\theta\Lambda^\eta\theta dx. \tag{2.2.15}$$

Now, since $\nabla\cdot\mathbf{u} = 0$, we deduce

$$p\int_{\mathbb{R}^2} |\theta|^{p-2}\theta(\mathbf{u}\cdot\nabla)\theta dx = \int_{\mathbb{R}^2} (\mathbf{u}\cdot\nabla)|\theta|^p dx = 0, \tag{2.2.16}$$

where we used integration by parts. For the last term on the right-hand side of (2.2.15), it follows from Lemma 2.1 that

$$-p\kappa\int_{\mathbb{R}^2} |\theta|^{p-2}\theta\Lambda^\eta\theta dx \leqslant 0. \tag{2.2.17}$$

Plugging (2.2.16) and (2.2.17) into (2.2.15), we obtain that

$$\frac{d}{dt}||\theta||_{L^p}^p \leqslant 0.$$

Therefore, we conclude the proof. $\square$

In the following, we summarize the available results regarding the local well-posedness of solutions of the gSQG equation, as proved in [77, Theorem 1.1] for $0 < \beta \leqslant 1$ and in [43, Theorem 1.1] for $1 < \beta < 2$.

Theorem 2.6 ([77, Theorem 1.1], [43, Theorem 1.1])**.** *Let $0 < \beta < 2$ and $\theta_0 \in H^s(\mathbb{R}^2)$ for $s > 1+\beta$. Then there exists $T = T(||\theta_0||_{H^s}) > 0$, such that the gSQG equation admits a unique solution $\theta \in \mathcal{C}([0,T), H^s(\mathbb{R}^2))$ satisfying $\theta(0) = \theta_0$.*

For the local well-posedness of the fractionally dissipative gSQG equation, we compile several results available in the literature in the following Theorem (see [16, 24, 27, 56, 60, 67, 79]). We also refer to [46] for a well-detailed summary of the available results.

Theorem 2.7. *Let $(\eta,\beta) \in (0,2)\times(0,2)$ and $\theta_0 \in H^s(\mathbb{R}^2)$ for $s > 1+\beta-\eta$. Then there exists $T = T(||\theta_0||_{H^s}) > 0$, such that the dissipative gSQG equation admits a unique solution*

$$\theta \in \mathcal{C}([0,T), H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\frac{\eta}{2}}(\mathbb{R}^2)),$$

satisfying $\theta(0) = \theta_0$.

Now, let us recall some continuation criteria available for the equation (1.0.1). We start with the continuation criterion for the gSQG equation, as proved in [15], which generalizes the result established in [21] for $\beta = 1$ as follows:

Theorem 2.8 ([15], Theorem 1.5)**.** *Consider $0 \leqslant \beta \leqslant 1$. Let θ be a solution of the gSQG equation corresponding to the initial data $\theta_0 \in \mathcal{C}^\delta(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ with $\delta > 1$ and $q > 1$. Let $T > 0$. If θ satisfies*

$$\int_0^T ||\theta(\cdot,t)||_{\mathcal{C}^\beta(\mathbb{R}^2)} dt < \infty, \tag{2.2.18}$$

then θ remains in $\mathcal{C}^\delta(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ on the time interval $[0,T]$.

For the dissipative gSQG equation, the first continuation criterion was proved in [11, Theorem 1.1] when $\beta = 1$, and later, observed in [16, Theorem 3.1] that it holds for all $\beta \in [0, 1]$, as follows:

Theorem 2.9 ([16], Theorem 3.1)**.** *Fix $\beta \in [0,1]$ and $\eta \in (0,1]$. Let θ be a solution of* (1.0.1) *with initial data $\theta_0 \in H^m(\mathbb{R}^2)$ with $m > 2$. If there are indices p, r with $\frac{2}{\eta} < p < \infty$ and $1 < r < \infty$ respectively such that*

$$\nabla^{\perp}\theta \in L^r(0,T; L^p(\mathbb{R}^2)) \text{ for some } p,r \text{ with } \frac{2}{p} + \frac{\eta}{r} \leqslant \eta. \tag{2.2.19}$$

then θ remains in $H^m(\mathbb{R}^2)$ on [0,T].

2.3 Self-Similar solutions

Numerical simulations, as previously discussed, encourage further investigation of solutions to the gSQG equation which exhibits a self-similar blowup. Therefore, this section formally defines the locally and globally self-similar solutions to the gSQG equation and dissipative gSQG equation. Here, we also analyze the continuation criteria mentioned in Section 2.2.3.

Let us start by analyzing the self-similar solution of the gSQG equation. Such solutions are defined regarding the invariance of gSQG equation under the following scaling transformation $(x, t, \theta) \mapsto (\lambda x, \lambda^{1+\alpha} t,\ \lambda^{1+\alpha-\beta}\theta)$, with $\lambda \in \mathbb{R}^+$, $\alpha \in \mathbb{R}$; i.e. if θ is a solution of (1.0.1) with $\kappa = 0$, then $\theta_\lambda(x,t) = \lambda^{1+\alpha-\beta}\theta(\lambda x,\ \lambda^{1+\alpha}t)$ is also a solution. Specifically, we say that, for a fixed scaling parameter $\alpha > -1$, a solution θ of (1.0.1) on $\mathbb{R}^2$ and on a time interval $(0,T)$ is *(globally) self-similar* if $\theta(x,t) = \theta_\lambda(x,t)$ for all $(x,t) \in \mathbb{R}^2 \times (0,T)$ and for all $\lambda > 0$. This is equivalent to being able to write θ as

$$\theta(x,t) = \frac{1}{t^{\frac{1+\alpha-\beta}{1+\alpha}}}\Theta\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right) \quad \text{for all } (x,t) \in \mathbb{R}^2 \times (0,T), \tag{2.3.1}$$

for some function $\Theta : \mathbb{R}^2 \to \mathbb{R}$, which is called the associated *self-similar profile.*

In this work, we focus on studying the more general case of solution of the gSQG equation that satisfies an equality as in (2.3.1) only locally in space, namely with $(x,t) \in B_\rho(0) \times (0,T)$, for some $\rho > 0$ [1]. Here, $B_\rho(0)$ denotes the ball in $\mathbb{R}^2$ centered at 0 and with radius ρ. Since, for any $x_0 \in \mathbb{R}^2$, it follows that $\tilde{\theta}(x,t) := -\theta(x - x_0, T - t)$, $(x,t) \in \mathbb{R}^2 \times (0,T)$ is also a solution of the SQG equation in view of its spatial translation and time reversal symmetries, we may consider a more general form of solution of the gSQG equation as follows

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{1+\alpha-\beta}{1+\alpha}}}\Theta\left(\frac{x - x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right) \quad \text{for all } (x,t) \in B_\rho(x_0) \times (0,T), \tag{2.3.2}$$

[1] Note that, in contrast to the identity $\theta(x,t) = \theta_\lambda(x,t)$ for all $(x,t) \in \mathbb{R}^2 \times (0,T)$ and $\lambda > 0$ satisfied by θ in the globally self-similar case, the local condition (2.3.2) with $x_0 = 0$ implies instead that $\theta(x,t) = \theta_\lambda(x,t)$ for all $x \in B_{\min\{\rho, \rho/\lambda\}}(0)$, $t \in \left(0, \min\left\{T, \frac{T}{\lambda^{1+\alpha}}\right\}\right)$, and $\lambda > 0$.

where $x_0 \in \mathbb{R}^2$.

Similarly, for the dissipative gSQG equation, such solutions are defined concerning the invariance of (1.0.1) with $\kappa > 0$, under the following scaling transformation $(x, t, \theta) \mapsto (\lambda x, \lambda^\eta t, \lambda^{\eta-\beta}\theta)$, with $\lambda \in \mathbb{R}^+$, $\eta \in (0, 2)$; i.e. if θ is a solution of (1.0.1) with $\kappa > 0$, then $\theta_\lambda(x, t) = \lambda^{\eta-\beta}\theta(\lambda x, \lambda^\eta t)$ is also a solution. Repeating the same argument as before, we can consider the more general globally self-similar solution, given by

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{\eta-\beta}{\eta}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{\eta}}}\right) \quad \text{for all } (x,t) \in \mathbb{R}^2 \times (0,T), \tag{2.3.3}$$

where $x_0 \in \mathbb{R}^2$.

Now, using the continuation criteria mentioned in Section 2.2.3, let us prove that self-similar solutions can develop finite-time singularity at time T. Assume that θ is a locally self-similar solution of the gSQG equation and that the self-similar profile $\Theta \in \mathcal{C}^\beta(\mathbb{R}^2)$ is not null. We commence by recalling that

$$\|\theta(\cdot,s)\|_{\mathcal{C}^\beta(\mathbb{R}^2)} = \begin{cases} \sup\limits_{x,y\in\mathbb{R}^2} |\theta(x,s)|, & \text{if } \beta = 0, \\ \sup\limits_{\substack{x,y\in\mathbb{R}^2 \\ x\neq y}} \dfrac{|\theta(x,s)-\theta(y,s)|}{|x-y|^\beta}, & \text{if } \beta \in (0,1), \\ \sup\limits_{x,y\in\mathbb{R}^2} |\nabla\theta(x,s)|, & \text{if } \beta = 1. \end{cases}$$

Without loss of generality, suppose $\beta \in (0, 1)$. Thus, it follows from Theorem 2.8 that

$$\begin{aligned} \lim_{t\to T}\int_0^t \|\theta(\cdot,s)\|_{\mathcal{C}^\beta(\mathbb{R}^2)}ds &\geqslant \lim_{t\to T}\int_0^t \|\theta(\cdot,s)\|_{\mathcal{C}^\beta(B_\rho(x_0))}ds \\ &= \lim_{t\to T}\int_0^t \sup_{\substack{x,y\in B_\rho(x_0)\\ x\neq y}} \frac{|\theta(x,s)-\theta(y,s)|}{|x-y|^\beta}ds \\ &= \lim_{t\to T}\int_0^t (T-s)^{\frac{\beta-\alpha-1}{1+\alpha}} \sup_{\substack{x,y\in B_\rho(x_0)\\ x\neq y}} \frac{\left|\Theta\left(\frac{x-x_0}{(T-s)^{\frac{1}{1+\alpha}}}\right)-\Theta\left(\frac{y-x_0}{(T-s)^{\frac{1}{1+\alpha}}}\right)\right|}{|x-y|^\beta}ds \\ &= \lim_{t\to T}\int_0^t (T-s)^{-1} \sup_{\substack{x,y\in B_\rho(x_0)\\ x\neq y}} \frac{\left|\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right)-\Theta\left(\frac{y-x_0}{(T-s)^{\frac{1}{1+\alpha}}}\right)\right|}{\left|\frac{x-x_0}{(T-s)^{\frac{1}{1+\alpha}}}-\frac{y-x_0}{(T-s)^{\frac{1}{1+\alpha}}}\right|^\beta}ds. \end{aligned}$$

Since $x, y \in B_\rho(x_0)$ and $s \in [0, T)$, it follows that $\tilde{x} = (x - x_0)(T - s)^{-\frac{1}{1+\alpha}}$ and $\tilde{y} = (y - x_0)((T - s)^{-\frac{1}{1+\alpha}})$ belong to $B_{\pi(s)}(0)$, where $\pi(s) = \rho(T - s)^{-\frac{1}{1+\alpha}}$. Thus, we have

$$\begin{aligned} \lim_{t\to T}\int_0^t \|\theta(\cdot,s)\|_{\mathcal{C}^\beta(\mathbb{R}^2)}ds &= \lim_{t\to T}\int_0^t (T-s)^{-1} \sup_{\substack{\tilde{x},\tilde{y}\in B_{\pi(s)}\\ x\neq y}} \frac{|\Theta(\tilde{x})-\Theta(\tilde{y})|}{|\tilde{x}-\tilde{y}|}ds \\ &\geqslant \lim_{t\to T}\int_0^t (T-s)^{-1}||\Theta||_{\mathcal{C}^\beta(B_{\pi(0)})}ds \\ &= ||\Theta||_{\mathcal{C}^\beta(B_{\pi(0)})} \lim_{t\to T}(\ln(T-t)-\ln T) = +\infty, \text{ if } ||\Theta||_{\mathcal{C}^\beta(B_{\pi(0)})} \neq 0. \end{aligned} \tag{2.3.4}$$

where we used that $\Theta \in \mathcal{C}^{\beta}(\mathbb{R}^2)$. Therefore, we conclude the proof of the statement.

Now, in view of Theorem 2.9, let us analyze the globally self-similar solutions of the fractionally dissipative gSQG equation for $\beta \in [0,1]$. Assume that θ is a globally self-similar solution to the dissipative gSQG equation and that the self-similar profile $\Theta \in W^{1,p}(\mathbb{R}^2)$ with $1 \leqslant p < \infty$ is not null. Thus, invoking (2.3.3) with $x_0 = 0$, it follows that

$$\nabla_x^{\perp}\theta(x,t) \;=\; \nabla_x^{\perp}\left(\frac{1}{(T-t)^{\frac{\eta-\beta}{\eta}}}\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)\right) = \frac{1}{(T-t)^{\frac{\eta-\beta+1}{\eta}}}(\nabla^{\perp}\Theta)\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right).$$

Then, by changing the variable $y = x(T-t)^{-\frac{1}{\eta}}$, we have

$$\begin{aligned} ||\nabla_x{}^{\perp}\theta(x,t)||_{L^p}^r &= \left(\int_{\mathbb{R}^2} \frac{1}{(T-t)^{\frac{(\eta-\beta+1)p}{\eta}}}\left|(\nabla^{\perp}\Theta)\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)\right|^p dx\right)^{\frac{r}{p}} \\ &= \frac{(T-t)^{\frac{2r}{p\eta}}}{(T-t)^{\frac{(\eta-\beta+1)r}{\eta}}}\left(\int_{\mathbb{R}^2}|\nabla^{\perp}\Theta(y)|^p dy\right)^{\frac{r}{p}} \\ &= (T-s)^{\frac{r}{\eta}(\frac{2}{p}-\eta+\beta-1)}||\nabla^{\perp}\Theta||_{L^p}^r. \end{aligned}$$

Supposing that $0 < ||\nabla^{\perp}\Theta||_{L^p} < \infty$ and integrating on time, we obtain

$$\int_0^t ||\nabla^{\perp}\theta(\cdot,t)||_{L^p}^r ds = ||\nabla^{\perp}\Theta||_{L^p}\int_0^t (T-s)^{\frac{r}{\eta}(\frac{2}{p}-\eta+\beta-1)}ds,$$

since $0 \leqslant \beta \leqslant 1$ and $\frac{2}{p} < \eta$, we can deduce that

$$\lim_{t\to T}\int_0^t ||\nabla^{\perp}\theta||_{L^p}^r dt = \infty.$$

Therefore, we conclude that T is a potential instance of a finite-time blowup.

3 Locally self-similar solutions of the gSQG equation

The results presented in this chapter are based on the paper [5], which was developed by the author during his Ph.D. program in collaboration with his supervisor and co-supervisor. Here, our focus is on analyzing the locally self-similar solution of the gSQG equation in $\mathbb{R}^2$, given by

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{1+\alpha-\beta}{1+\alpha}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{1+\alpha}}}\right) \quad \text{for all } (x,t) \in B_\rho(x_0) \times (0,T), \tag{3.0.1}$$

where $\alpha > -1$ is a scaling parameter. As usual, $B_\rho(0)$ denotes the ball of radius ρ centered at the origin, and $\Theta : \mathbb{R}^2 \to \mathbb{R}$ the self-similar profile. According to Theorem 2.6, we can consider the solution, θ, of the gSQG equation within the class $\mathcal{C}([0,T), H^s(\mathbb{R}^2))$ with $s > 1+\beta$.

In this chapter, we employ the techniques developed by [9] and [76] to study the globally self-similar solution and locally self-similar solution to the SQG equation, respectively. Specifically, we extend these results to the gSQG equation for all $\beta \in (0,2)$. Our main results are split between the cases $\beta \in (0,1]$ in Theorem 3.2, and $\beta \in (1,2)$ in Theorem 3.1, with each one requiring different conditions on Θ. These differences manifest in the estimates of the component of the velocity field in the self-similar region, as outlined in (3.2.5) for $0 < \beta \leqslant 1$, and (3.1.12) for $1 < \beta < 2$. As mentioned in Chapter 1, when $\beta \in (1,2)$, $K_\beta = x^\perp |x|^{-(2+\beta)}$ is neither a Calderón-Zygmund operator (see Section 2.2.1) or integrable near the origin. To circumvent this issue, we write $K_\beta(x) = \nabla^\perp(|x|^{-\beta})$ and integrate by parts in (1.0.2), thus transferring one derivative to θ, and hence to Θ inside the self-similar region. Now, in the case $\beta \in (0,1]$, this is achieved thanks to the fact that the kernel K_β from (1.0.2) satisfies $\|K_\beta * f\|_{L^q(B_\rho(0))} \lesssim \|f\|_{L^q(\mathbb{R}^2)}$ for every $f \in L^q(\mathbb{R}^2)$ and $1 < q < \infty$. Finally, in the case $\beta = 1$, this follows from the fact that K_β is a Calderón-Zygmund operator, whereas for $\beta \in (0,1)$ this is a consequence of Young's convolution inequality together with K_β being integrable near the origin. Therefore, it is not necessary to impose assumptions on the gradient of the profile.

We commence this chapter addressing the results for $1 < \beta < 2$, followed by the results for $0 < \beta \leqslant 1$.

3.1 Case $1 < \beta < 2$

In this section, we address the gSQG equation in the scenario where $1 < \beta < 2$, aiming to analyze possible scenarios of self-similar blowup. More precisely, our goal is to prove that under an L^r growth assumption on the self-similar profile and its gradient, we can identify appropriate ranges of the scaling parameter where the profile is either identically zero, which excludes blowup, or its L^p asymptotic behavior can be characterized for suitable values of r and p, see (Theorem 3.1.)

Later, we present Corollary 3.2, which gives different conditions for the profile and its gradient, as decaying at infinity or growth bound on the gradient, as that ensures that assumptions in Theorem 3.1 holds. To prove this result, we also employ the techniques developed by Xue in [76, Theorem 1.2].

Theorem 3.1. *Fix $\beta \in (1,2)$. Suppose $\theta \in \mathcal{C}([0,T);H^s(\mathbb{R}^2)) \cap L^\infty(0,T;L^1(\mathbb{R}^2))$, with $s > 1+\beta$, is a solution to the gSQG equation* (1.0.1) *that is locally self-similar in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, with scaling parameter $\alpha > -1$ and profile $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Fix also $p \geqslant 1$, and suppose that for some $r > p$, $\gamma_1 \in [0, r(\beta-1)+2)$, and $\gamma_0 \in [0, \gamma_1 + r]$, it holds*

$$\int_{|y|\leqslant L} |\Theta(y)|^r dy \lesssim L^{\gamma_0}, \tag{3.1.1}$$

and

$$\int_{|y|\leqslant L} |\nabla\Theta(y)|^r dy \lesssim L^{\gamma_1} \tag{3.1.2}$$

for all L sufficiently large. Under these conditions, it follows that if $\alpha > \beta + \frac{2}{p} - 1$ or $-1 < \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$ then $\Theta \equiv 0$. Moreover, if $\alpha \in \left[\beta - 1 + \frac{2-\gamma_0}{r}, \beta - 1 + \frac{2}{p}\right]$ then either $\Theta \equiv 0$ or Θ is a nontrivial profile, and it satisfies

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \sim L^{2-p(1+\alpha-\beta)} \tag{3.1.3}$$

for all L sufficiently large.

As mentioned earlier, the proof is guided by a similar approach to that proposed in [9, 76] and also starts from the local L^p equality satisfied by any solution $\theta \in \mathcal{C}([0,T);H^s(\mathbb{R}^2)) \cap L^\infty(0,T;L^p(\mathbb{R}^2))$, with $s > 1+\beta$, of (1.0.1). Namely, for fixed $0 < t_1 < t_2 < T$ and $p \in [1,\infty)$,

$$\begin{aligned}\int_{\mathbb{R}^2} |\theta(x,t_2)|^p \eta(x,t_2) dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \eta(x,t_1) dx \\ = \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(x,t)|^p \partial_t \eta(x,t) dx dt + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(x,t)|^p (\mathbf{u}\cdot\nabla)\eta(x,t) dx dt,\end{aligned} \tag{3.1.4}$$

for every smooth and compactly supported test function η on $\mathbb{R}^2 \times [0,\infty)$, i.e. $\eta \in \mathcal{C}_c^\infty([0,\infty)\times\mathbb{R}^2,\mathbb{R})$. See Section 6.2 for details of the proof.

The proof of Theorem 3.1 is divided into three cases, each corresponding to a particular range of α, as described in the statement. In the scenarios where $\alpha > \beta + \frac{2}{p} - 1$ or $-1 < \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$, the aim is to establish that

$$\int_{|y|\lesssim L} |\Theta(y)|^p dy \lesssim L^\sigma \quad \text{for some } \sigma < 0. \tag{3.1.5}$$

which implies that taking $L \to \infty$ it follows that $\Theta \equiv 0$. In the first case, (3.1.5) is directly derived from the local self-similarity condition, (3.0.1), combined with the maximum principle Lemma 2.2. For another case, a similar inequality to (3.1.5) is obtained by establishing a fundamental local L^p inequality from the local equality (3.1.4), see (3.1.22) below. This is derived by suitably employing cut-off functions to split the velocity field $\mathbf{u}$ into its restrictions to the self-similar region and the corresponding exterior. With the help of assumptions (3.1.1), (3.1.2), and Lemma 6.2, we then estimate the terms on the right-hand side of (3.1.22) to yield an upper bound for $\int_{|y|\lesssim L} |\Theta(y)|^p dy$. Next, we redo the estimates by using this new upper bound and bootstrap on this argument until we eventually arrive at an upper bound as in (3.1.5) with a negative power of L. Redoing the estimates by using this new upper bound and proceeding via bootstrapping on this argument, we eventually arrive at an upper bound as in (3.1.5) with a negative power of L.

In the final scenario, where $\beta - 1 + \frac{2-\gamma_0}{r} \leqslant \alpha \leqslant \beta - 1 + \frac{2}{p}$, it is necessary to prove that every nontrivial profile Θ satisfies (3.1.3). The upper bound is guaranteed from the estimate derived for the first range of α, whereas for the lower bound we proceed by contradiction. Namely, assuming that such lower estimate does not hold, it follows that Θ must satisfy the same local L^p inequality as in the second case, (3.1.22). Proceeding with a similar analysis from this case, we then arrive at the contradiction that $\Theta \equiv 0$.

Now, let us start the proof:

Proof. Without loss of generality, we may assume $x_0 = 0$. The proof is divided into three different cases, each corresponding to a particular range for α within the interval $(-1, \infty)$.

Case 1: In the first scenario, let us prove that if $\alpha > \beta + \frac{2}{p} - 1$, then $\Theta \equiv 0$ in $\mathbb{R}^2$.

Fix $t \in [0, T)$ and denote $L = \rho(T-t)^{\frac{-1}{1+\alpha}}$. Invoking the local self-similarity of θ, namely (3.0.1), and changing variables, it follows that

$$\begin{aligned} \int_{|x|\leqslant\rho} |\theta(x,t)|^p dx &= \frac{1}{(T-t)^{\frac{p(1+\alpha-\beta)}{1+\alpha}}} \int_{|x|\leqslant\rho} \left| \Theta\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}} \right) \right|^p dx \\ &= C L^{p(1+\alpha-\beta)-2} \int_{|y|\leqslant L} |\Theta(y)|^p dy. \end{aligned} \tag{3.1.6}$$

Since $s > 1$, it follows by Sobolev embedding that $\theta(0) \in H^s(\mathbb{R}^2) \subset L^{\tilde{p}}(\mathbb{R}^2)$ for every $\tilde{p} \geqslant 2$. This implies that

$$\|\theta(t)\|_{L^{\tilde{p}}} = \|\theta(0)\|_{L^{\tilde{p}}} \quad \text{for all } t \in [0,T) \text{ and } \tilde{p} \geqslant 2, \tag{3.1.7}$$

see e.g. [67, Theorem 3.3]. Thus, by Hölder's inequality 2.2.1, it follows that for all $p \in [1, \infty)$ and $\tilde{p} \geqslant \max\{2, p\}$, we have

$$\int_{|x|\leqslant\rho} |\theta(x,t)|^p dx \leqslant C\|\theta(0)\|_{L^{\tilde{p}}} \quad \text{for all } t \in [0,T).$$

Hence, we obtain from (3.1.6) that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{2-p(1+\alpha-\beta)}. \tag{3.1.8}$$

Since $2-p(1+\alpha-\beta)<0$, taking the limit as $t\to T$ in (3.1.8), which implies $L\to\infty$, we deduce that $\Theta\equiv 0$ in $\mathbb{R}^2$.

Case 2: In this scenario, we assume that $-1<\alpha<\beta-1+\frac{2-\gamma_0}{r}$. Here we once again show that $\Theta\equiv 0$ in $\mathbb{R}^2$.

Take cut-off functions $\phi_{\frac{\varrho}{4}},\phi_\rho\in\mathcal{C}^\infty(\mathbb{R}^2)$ with $0\leqslant\phi_{\frac{\varrho}{4}},\phi_\rho\leqslant 1$, $\phi_{\frac{\varrho}{4}}\equiv 1$ in $B_{\rho/8}(0)$, $\phi_{\frac{\varrho}{4}}\equiv 0$ in $B^c_{\rho/4}(0)$, and $\phi_\rho\equiv 1$ in $B_{\rho/2}(0)$, $\phi_\rho\equiv 0$ in $B^c_\rho(0)$. Fix $t_1,t_2\in[0,T)$. From (3.1.4), we have in particular that

$$\int_{\mathbb{R}^2}|\theta(x,t_2)|^p\phi_{\frac{\varrho}{4}}(x)dx-\int_{\mathbb{R}^2}|\theta(x,t_1)|^p\phi_{\frac{\varrho}{4}}(x)dx=\int_{t_1}^{t_2}\int_{\mathbb{R}^2}(\mathbf{u}(x,t)\cdot\nabla\phi_{\frac{\varrho}{4}}(x))|\theta(x,t)|^p dx\,dt. \tag{3.1.9}$$

We proceed to analyze each term in (3.1.9), starting with the first two terms on the left-hand side. By the local self-similarity of θ, (3.0.1), it follows that for $i=1,2$

$$\begin{aligned}\int_{\mathbb{R}^2}|\theta(x,t_i)|^p\phi_{\frac{\varrho}{4}}(x)dx&=\frac{1}{(T-t_i)^{\frac{p(1+\alpha-\beta)}{1+\alpha}}}\int_{\mathbb{R}^2}\left|\Theta\left(\frac{x}{(T-t_i)^{\frac{1}{1+\alpha}}}\right)\right|^p\phi_{\frac{\varrho}{4}}(x)dx\\&=\frac{1}{(T-t_i)^{\frac{p(1+\alpha-\beta)-2}{1+\alpha}}}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\varrho}{4}}(y(T-t_i)^{\frac{1}{1+\alpha}})dy\\&=l_i^{p(1+\alpha-\beta)-2}\int_{|y|\leqslant\frac{\varrho}{4}l_i}|\Theta(y)|^p\phi_{\frac{\varrho}{4}}(yl_i^{-1})dy,\end{aligned} \tag{3.1.10}$$

where $l_i=(T-t_i)^{-\frac{1}{1+\alpha}}$, $i=1,2$.

To analyze the term in the right-hand side of (3.1.9), we first decompose the velocity field $\mathbf{u}$ into a term involving the self-similarity region and another one outside of it. More precisely, recalling that

$$\mathbf{u}(x,t)=C_\beta P.V.\int_{\mathbb{R}^2}K_\beta(x-y)\theta(y,t)dy,$$

where

$$K_\beta(x):=\frac{x^\perp}{|x|^{2+\beta}},\quad x\in\mathbb{R}^2\backslash\{0\}.$$

Thus, it follows that

$$
\begin{aligned}
&\mathbf{u}(x,t)\\
&= C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)\phi_\rho(y)dy + C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)(1-\phi_\rho(y))dy\\
&= C_\beta P.V. \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\beta}}\theta(y,t)\phi_\rho(y)dy + C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)(1-\phi_\rho(y))dy\\
&= \frac{1}{\beta} C_\beta P.V. \int_{\mathbb{R}^2} \nabla_y^\perp\left(\frac{1}{|x-y|^\beta}\right)\theta(y,t)\phi_\rho(y)dy + C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)(1-\phi_\rho(y))dy\\
&= -\frac{1}{\beta} C_\beta P.V. \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta}\nabla^\perp\theta(y,t)\phi_\rho(y)dy - \frac{1}{\beta} C_\beta P.V. \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta}\theta(y,t)\nabla^\perp\phi_\rho(y)dy\\
&\qquad + C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)(1-\phi_\rho(y))dy\\
&=: \mathbf{u}^{(1)}(x,t) + \mathbf{u}^{(2)}(x,t) + \mathbf{u}^{(3)}(x,t),
\end{aligned}
\tag{3.1.11}
$$

where the second to last equality follows by integration by parts. We now analyze each of these terms. By the local self-similarity of θ, (3.0.1), we get

$$
\begin{aligned}
\mathbf{u}^{(1)}(x,t) &= -\frac{1}{\beta} C_\beta P.V. \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta}\nabla^\perp\theta(y,t)\phi_\rho(y)dy\\
&= -\frac{C_\beta}{\beta(T-t)^{\frac{2+\alpha-\beta}{1+\alpha}}} P.V. \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta}\nabla^\perp\Theta\left(\frac{y}{(T-t)^{\frac{1}{1+\alpha}}}\right)\phi_\rho(y)dy\\
&= -\frac{C_\beta}{\beta(T-t)^{\frac{\alpha-\beta}{1+\alpha}}} P.V. \int_{\mathbb{R}^2} \frac{1}{|x-(T-t)^{\frac{1}{1+\alpha}}y|^\beta}\nabla^\perp\Theta(y)\phi_\rho(y(T-t)^{\frac{1}{1+\alpha}})dy\\
&= -\frac{C_\beta}{\beta(T-t)^{\frac{\alpha}{1+\alpha}}} P.V. \int_{\mathbb{R}^2} \frac{1}{|(T-t)^{\frac{-1}{1+\alpha}}x-y|^\beta}\nabla^\perp\Theta(y)\phi_\rho(y(T-t)^{\frac{1}{1+\alpha}})dy\\
&= -\frac{C_\beta}{\beta(T-t)^{\frac{\alpha}{1+\alpha}}} V^{(1)}\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}},t\right),
\end{aligned}
\tag{3.1.12}
$$

where

$$
V^{(1)}(x,t) := P.V. \int_{\mathbb{R}^2} \frac{1}{|x-y|^\beta}\nabla^\perp\Theta(y)\phi_\rho(y(T-t)^{\frac{1}{1+\alpha}})dy. \tag{3.1.13}
$$

Next, we analyze $\mathbf{u}^{(2)}(x,t)$. Note that due to the presence of $\nabla\phi_{\frac{\rho}{4}}(x)$ in the right-hand side of (3.1.9), it suffices to consider $x \in \mathbb{R}^2$ with $\rho/8 \leqslant |x| \leqslant \rho/4$. Then, since for each such x we have $|x-y| \geqslant \frac{|y|}{2}$ for every $|y| \geqslant \rho/2$, it follows that

$$
\begin{aligned}
|\mathbf{u}^{(2)}(x,t)| &\leqslant C\int_{\frac{\rho}{2}\leqslant|y|\leqslant\rho} \frac{1}{|x-y|^\beta}|\theta(y,t)||\nabla^\perp\phi_\rho(y)|dy\\
&\leqslant C\int_{|y|\geqslant\frac{\rho}{2}} \frac{|\theta(y,t)|}{|y|^\beta}dy\\
&\leqslant C\|\theta\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leqslant C\|\theta(0)\|_{L^2},
\end{aligned}
\tag{3.1.14}
$$

where in the last line we applied Hölder's inequality and (3.1.7) with $\tilde{p}=2$.

Finally, for the last term in (3.1.11), $\mathbf{u}^{(3)}(x,t)$, we proceed similarly as was done for $\mathbf{u}^{(2)}(x,t)$ in (3.1.14) and obtain that

$$\begin{aligned} |\mathbf{u}^{(3)}(x,t)| &= C_\beta \int_{|y|\geqslant \rho/2} \frac{1}{|x-y|^{\beta+1}} |\theta(y,t)|(1-\phi_\rho(y))dy \\ &\leqslant C \int_{|y|\geqslant \frac{\rho}{2}} \frac{|\theta(y,t)|}{|y|^{\beta+1}} dy \leqslant C\|\theta(0)\|_{L^2}. \end{aligned} \tag{3.1.15}$$

From (3.1.12), (3.1.14) and (3.1.15), we may then estimate the term in the right-hand side of (3.1.9) as

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta(x,t)|^p (\mathbf{u}(x,t)\cdot \nabla \phi_{\frac{\rho}{4}}(x)) dx\, dt \right| \\ &\leqslant \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\mathbf{u}_1(x,t)||\nabla \phi_{\frac{\rho}{4}}(x)||\theta(x,t)|^p dx\, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\mathbf{u}_2(x,t)||\nabla \phi_{\frac{\rho}{4}}(x)||\theta(x,t)|^p dx\, dt \\ &\qquad + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\mathbf{u}_3(x,t)||\nabla \phi_{\frac{\rho}{4}}(x)||\theta(x,t)|^p dx\, dt \\ &\leqslant C \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{\alpha+p(1+\alpha-\beta)}{1+\alpha}}} \int_{\mathbb{R}^2} \left| V^{(1)}\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}, t \right) \right| \left| \Theta\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}} \right) \right|^p |\nabla \phi_{\frac{\rho}{4}}(x)| dx\, dt \\ &\qquad + C \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{p(1+\alpha-\beta)}{1+\alpha}}} \int_{\mathbb{R}^2} \left| \Theta\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}} \right) \right|^p |\nabla \phi_{\frac{\rho}{4}}(x)| dx\, dt \\ &\leqslant C \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{\alpha-2+p(1+\alpha-\beta)}{1+\alpha}}} \int_{\mathbb{R}^2} |V^{(1)}(y,t)||\Theta(y)|^p |\nabla \phi_{\frac{\rho}{4}}(y(T-t)^{\frac{1}{1+\alpha}})| dy\, dt \\ &\qquad + C \int_{t_1}^{t_2} \frac{1}{(T-t)^{\frac{p(1+\alpha-\beta)-2}{1+\alpha}}} \int_{\mathbb{R}^2} |\Theta(y)|^p |\nabla \phi_{\frac{\rho}{4}}(y(T-t)^{\frac{1}{1+\alpha}})| dy\, dt. \end{aligned} \tag{3.1.16}$$

In view of the support of $\nabla\phi_{\frac{\rho}{4}}$, we may restrict the integrands in (3.1.16) to $(y,t) \in \mathbb{R}^2 \times [t_1,t_2]$ such that $\rho/8 \leqslant |y|(T-t)^{\frac{1}{1+\alpha}} \leqslant \rho/4$. In particular, each such y satisfies $\rho l_1/8 \leqslant |y| \leqslant \rho l_2/4$, where we recall that $l_i = (T-t_i)^{-\frac{1}{1+\alpha}}$, $i=1,2$. Then, for each fixed $y \in \mathbb{R}^2$ with $\rho l_1/8 \leqslant |y| \leqslant \rho l_2/4$, we define the set

$$A_y := \left\{ t \in [t_1,t_2] \, : \, \frac{\rho}{8}\frac{1}{|y|} \leqslant (T-t)^{\frac{1}{1+\alpha}} \leqslant \frac{\rho}{4}\frac{1}{|y|} \right\}. \tag{3.1.17}$$

After rearrangement, it is easy to see that

$$A_y \subset \left[T - \left(\frac{\rho}{4|y|}\right)^{1+\alpha}, T - \left(\frac{\rho}{8|y|}\right)^{1+\alpha} \right],$$

so that its length satisfies $|A_y| \leqslant c_{\alpha,\rho}/|y|^{1+\alpha}$. Thus, denoting by $\mathbb{1}_{A_y}$ the indicator function of the set A_y, it follows from (3.1.16) that

$$\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^2}|\theta(x,t)|^p(\mathbf{u}(x,t)\cdot\nabla\phi_{\frac{\varrho}{4}}(x))dx\,dt\right| \\
&\qquad\leqslant C\int_{t_1}^{t_2}\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|V^{(1)}(y,t)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}\mathbb{1}_{A_y}(t)dy\,dt \\
&\qquad\qquad+C\int_{t_1}^{t_2}\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{2-p(1+\alpha-\beta)}}\mathbb{1}_{A_y}(t)dy\,dt \\
&\qquad\leqslant C\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}\int_{t_1}^{t_2}|V^{(1)}(y,t)|\mathbb{1}_{A_y}(t)dt\,dy \\
&\qquad\qquad+C\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{2-p(1+\alpha-\beta)}}\int_{t_1}^{t_2}\mathbb{1}_{A_y}(t)dt\,dy \\
&\qquad\leqslant C\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy+C\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy, \qquad (3.1.18)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{V}^{(1)}(y) &:= \int_{t_1}^{t_2}|V^{(1)}(y,t)|\mathbb{1}_{A_y}(t)dt \\
&= \int_{t_1}^{t_2}\left|\int_{\mathbb{R}^2}\frac{1}{|y-z|^\beta}\nabla^\perp\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz\right|\mathbb{1}_{A_y}(t)dt. \qquad (3.1.19)
\end{aligned}$$

Plugging (3.1.18) into (3.1.9) and recalling (3.1.10), yields

$$\begin{aligned}
&\left|l_2^{p(1+\alpha-\beta)-2}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\varrho}{4}}(yl_2^{-1})dy-l_1^{p(1+\alpha-\beta)-2}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\varrho}{4}}(yl_1^{-1})dy\right| \\
&\qquad\leqslant C\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy+C\int_{\frac{\varrho}{8}l_1\leqslant|y|\leqslant\frac{\varrho}{4}l_2}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy. \qquad (3.1.20)
\end{aligned}$$

Note that, by Hölder's inequality and assumption (3.1.1), it follows that

$$\begin{aligned}
l_2^{p(1+\alpha-\beta)-2}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\varrho}{4}}(yl_2^{-1})dy &\leqslant Cl_2^{p(1+\alpha-\beta)-2}\left(\int_{|y|\leqslant\frac{\varrho}{4}l_2}|\Theta(y)|^rdy\right)^{\frac{p}{r}}l_2^{2\left(1-\frac{p}{r}\right)} \\
&\leqslant Cl_2^{p(1+\alpha-\beta)+(\gamma_0-2)\frac{p}{r}}.
\end{aligned}$$

Since, by the current assumption on α, we have $(1+\alpha-\beta)+(\gamma_0-2)/r<0$, then

$$l_2^{p(1+\alpha-\beta)-2}\int_{\mathbb{R}^2}|\Theta(y)|^p\phi_{\frac{\varrho}{4}}(yl_2^{-1})dy\to 0 \quad \text{as } l_2\to\infty. \qquad (3.1.21)$$

Thus, denoting $L:=\frac{\varrho}{8}l_1$ and taking the limit in (3.1.20) as $t_2\to T$, so that $l_2\to\infty$, we obtain

$$\frac{1}{L^{2-p(1+\alpha-\beta)}}\int_{|y|\leqslant L}|\Theta(y)|^pdy\leqslant C\int_{|y|\geqslant L}\frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy+C\int_{|y|\geqslant L}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy. \qquad (3.1.22)$$

In what follows, we always assume that L is sufficiently large (equivalently, t_1 is sufficiently close to T), so that assumptions (3.1.1) and (3.1.2) can be applied.

We now further estimate each of the terms on the right-hand side of (3.1.22) by splitting the integrals according to a dyadic decomposition. For the first term, we make use of Lemma 6.2 below, which yields a control on the L^r norm of the function $\widetilde{V}^{(1)}$ on a dyadic shell under assumption (3.1.2). We obtain

$$\begin{aligned}
&\int_{|y|\geqslant L} \frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \\
&\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} \int_{2^kL\leqslant|y|\leqslant 2^{k+1}L} |\widetilde{V}^{(1)}(y)||\Theta(y)|^p dy \\
&\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} \left(\int_{|y|\sim 2^kL} |\widetilde{V}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} \left(\int_{|y|\sim 2^kL} |\Theta(y)|^r dy\right)^{\frac{p}{r}} (2^kL)^{2\left(1-\frac{p+1}{r}\right)} \\
&\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} (2^kL)^{1-\alpha-\beta+\frac{\gamma_1}{r}} (2^kL)^{\gamma_0\frac{p}{r}} (2^kL)^{2-\frac{2p}{r}-\frac{2}{r}} \\
&\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-2-\beta+3+\frac{\gamma_1-2}{r}+\frac{(\gamma_0-2)p}{r}} \\
&\leqslant CL^{p(1+\alpha-\beta)-2-\beta+3+\frac{\gamma_1-2}{r}+\frac{(\gamma_0-2)p}{r}}, \qquad (3.1.23)
\end{aligned}$$

where in the last inequality we used the hypotheses that $\alpha < \beta - 1 + \frac{2-\gamma_0}{r}$ and $\gamma_1 < r(\beta-1)+2$.

For the second term in the right-hand side of (3.1.22), applying again the dyadic decomposition together with Hölder's inequality, yields

$$\begin{aligned}
\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy &\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}} \int_{|y|\sim 2^kL} |\Theta(y)|^p dy \\
&\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}} \left(\int_{|y|\sim 2^kL} |\Theta(y)|^r dy\right)^{\frac{p}{r}} (2^kL)^{2\left(1-\frac{p}{r}\right)} \\
&\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-3-\alpha} (2^kL)^{\gamma_0\frac{p}{r}} (2^kL)^{2\left(1-\frac{p}{r}\right)} \\
&\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-2-\alpha+1+(\gamma_0-2)\frac{p}{r}} \\
&\leqslant CL^{p(1+\alpha-\beta)-2+1-\alpha+\frac{(\gamma_0-2)p}{r}}, \qquad (3.1.24)
\end{aligned}$$

where we used that $-1 < \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$. Combining (3.1.23) and (3.1.24) with (3.1.22), we deduce that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{3-\beta+\frac{(\gamma_1-2)}{r}+\frac{(\gamma_0-2)p}{r}} + CL^{1-\alpha+\frac{(\gamma_0-2)p}{r}} \leqslant CL^{a_0}, \qquad (3.1.25)$$

where

$$a_0 := \max\left\{1-\alpha+\frac{(\gamma_0-2)p}{r}, 3-\beta+\frac{(\gamma_1-2)}{r}+\frac{(\gamma_0-2)p}{r}\right\}. \qquad (3.1.26)$$

Note that, if $a_0 < 0$, we conclude that $\Theta \equiv 0$ on $\mathbb{R}^2$, i.e., no locally self-similar blowup occurs and the proof is finished. Otherwise, if $a_0 \geqslant 0$, we improve the estimates in (3.1.23) and (3.1.24) by making use of the new upper bound in (3.1.25). In particular, for the first term in the right-hand side of (3.1.22), we leverage (3.1.25) via a suitable interpolation inequality. Namely, in view of the assumption on γ_1, we may take $q \in (p, r)$ sufficiently close to p such that

$$0 \leqslant \gamma_1 < r\left(\beta - 1 - 2\left(1 - \frac{p}{q}\right)\right) + 2, \tag{3.1.27}$$

and

$$0 \leqslant \gamma_0 < \frac{(r-p)q}{(q-p)p}\left(\beta - 1 + \frac{2-\gamma_1}{r} - 2\left(1 - \frac{p}{q}\right)\right). \tag{3.1.28}$$

Then, by interpolation, we have

$$\begin{aligned} \int_{|y|\leqslant L} |\Theta(y)|^q dy &\leqslant \left(\int_{|y|\leqslant L} |\Theta(y)|^p dy\right)^{\delta} \left(\int_{|y|\leqslant L} |\Theta(y)|^r dy\right)^{1-\delta} &(3.1.29)\\ &\leqslant C L^{a_0\delta + (1-\delta)\gamma_0}, \quad \text{with } \delta := \frac{r-q}{r-p} \in (0,1). &(3.1.30) \end{aligned}$$

Next, employing once again the dyadic decomposition and Hölder's inequality, we derive via (3.1.25), (3.1.30), and Lemma 6.2 that

$$\begin{aligned} &\int_{|y|\geqslant L} \frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}} dy \\ &\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-\alpha-p(1+\alpha-\beta)}} \left(\int_{|y|\sim 2^k L} |\Theta(y)|^q dy\right)^{\frac{p}{q}} \left(\int_{|y|\sim 2^k L} |\widetilde{V}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} (2^k L)^{2(1-\frac{p}{q}-\frac{1}{r})} \\ &\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-\alpha-p(1+\alpha-\beta)}} (2^k L)^{\frac{p}{q}(a_0\delta + (1-\delta)\gamma_0)} (2^k L)^{\frac{\gamma_1}{r}+1-\alpha-\beta} (2^k L)^{2(1-\frac{p}{q}-\frac{1}{r})} \\ &\leqslant C\sum_{k=0}^{\infty} (2^k L)^{p(1+\alpha-\beta)-2+a_0+\frac{p}{q}(1-\delta)(\gamma_0-a_0)+\frac{\gamma_1-2}{r}+1-\beta+2\left(1-\frac{p}{q}\right)-a_0\left(1-\frac{p}{q}\right)} \\ &\leqslant C\sum_{k=0}^{\infty} (2^k L)^{p(1+\alpha-\beta)-2+a_0-a_1}, \end{aligned} \tag{3.1.31}$$

where

$$a_1 := \frac{p}{q}(1-\delta)(a_0 - \gamma_0) + \beta - 1 + \frac{2-\gamma_1}{r} - 2\left(1 - \frac{p}{q}\right) + a_0\left(1 - \frac{p}{q}\right). \tag{3.1.32}$$

Recall from (3.1.23) and (3.1.24) that $a_0 + p(1+\alpha-\beta) - 2 < 0$. Then, to obtain a finite sum in (3.1.31), it suffices to show that $a_1 \geqslant 0$. Firstly, assume that $a_0 = 1 - \alpha + \frac{(\gamma_0-2)p}{r}$. From (3.1.26), it follows that $-1 < \alpha \leqslant \beta - 2 + \frac{2-\gamma_1}{r}$. Moreover, since $1-\delta = (q-p)/(r-p)$, we have

$$a_1 = \frac{p(q-p)}{q(r-p)}(a_0-\gamma_0) + \beta - 1 + \frac{2-\gamma_1}{r} - 2\frac{(q-p)}{q} + a_0\frac{(q-p)}{q}$$
$$= \frac{r(q-p)}{q(r-p)}a_0 - \frac{p(q-p)}{q(r-p)}\gamma_0 + \beta - 1 + \frac{2-\gamma_1}{r} - 2\frac{(q-p)}{q} \tag{3.1.33}$$
$$= \frac{r(q-p)}{q(r-p)}(-1-\alpha) + \beta - 1 + \frac{2-\gamma_1}{r} > 0, \tag{3.1.34}$$

where the inequality follows by using that $r > q$, which implies $r(q-p) < q(r-p)$.

Now, if $a_0 = 3 - \beta + \frac{(\gamma_1-2)}{r} + \frac{(\gamma_0-2)p}{r}$, then from (3.1.26) we have $\beta - 2 + \frac{2-\gamma_1}{r} \leqslant \alpha < \beta - 1 + \frac{2-\gamma_0}{r}$. Hence, from (3.1.33),

$$a_1 = \frac{r(q-p)}{q(r-p)}\left(3 - \beta + \frac{(\gamma_1-2)}{r} + \frac{(\gamma_0-2)p}{r}\right) - \frac{p(q-p)}{q(r-p)}\gamma_0 + \beta - 1 + \frac{2-\gamma_1}{r} - 2\frac{(q-p)}{q}$$
$$= \frac{r(q-p)}{q(r-p)}\left(3 - \beta + \frac{(\gamma_1-2)}{r} - \frac{2p}{r}\right) + \beta - 1 + \frac{2-\gamma_1}{r} - 2\frac{(q-p)}{q}$$
$$= \left(\beta - 1 + \frac{2-\gamma_1}{r}\right)\left(1 - \frac{r(q-p)}{q(r-p)}\right) = \left(\beta - 1 + \frac{2-\gamma_1}{r}\right)\frac{p(r-q)}{q(r-p)} > 0, \tag{3.1.35}$$

where we used that $r > q > p$ and $\gamma_1 < r(\beta-1) + 2$.

Therefore, $a_1 > 0$, and it follows from (3.1.31) that

$$\int_{|y|\geqslant L} \frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \leqslant CL^{p(1+\alpha-\beta)-2+a_0-a_1}. \tag{3.1.36}$$

Similarly, since $a_0 + p(1+\alpha-\beta) - 2 < 0$ and $\alpha > -1$, we obtain for the second term in the right-hand side of (3.1.22) that

$$\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy \leqslant \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3+\alpha-p(1+\alpha-\beta)}} \int_{|y|\sim 2^k L} |\Theta(y)|^p dy$$
$$\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3+\alpha-p(1+\alpha-\beta)}}(2^k L)^{a_0}$$
$$\leqslant CL^{p(1+\alpha-\beta)-2+a_0-(1+\alpha)}. \tag{3.1.37}$$

Plugging (3.1.36) and (3.1.37) into (3.1.22), we deduce that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0-a_1} + cL^{a_0-(1+\alpha)}$$
$$\leqslant CL^{a_0-b_0}, \quad \text{where } b_0 := \min\{a_1, 1+\alpha\} > 0. \tag{3.1.38}$$

Again, if $a_0 - b_0 < 0$ then the proof is finished. Otherwise, we proceed with the bootstrap argument by now leveraging (3.1.38) to obtain improved estimates. To put this argument into a more general form, suppose that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\sigma} \quad \text{with } \sigma \leqslant a_0.$$

From the interpolation inequality (3.1.29), we have

$$\int_{|y|\leqslant L} |\Theta(y)|^q dy \leqslant CL^{\sigma\delta+\gamma_0(1-\delta)}, \tag{3.1.39}$$

where we recall that $\delta = (r-q)/(r-p)$. Then, proceeding similarly as in (3.1.31) and recalling the definition of a_1 in (3.1.32), we obtain

$$\begin{aligned}
&\int_{|y|\geqslant L} \frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \\
&\quad\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}}(2^kL)^{\frac{p}{q}(\sigma\delta+\gamma_0(1-\delta))}(2^kL)^{1-\alpha-\beta+\frac{\gamma_1}{r}}(2^kL)^{2\left(1-\frac{p}{q}-\frac{1}{r}\right)} \\
&\quad\leqslant C\sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-2+\sigma\frac{\delta p}{q}+\gamma_0(1-\delta)\frac{p}{q}+1-\beta+\frac{\gamma_1-2}{r}+2\left(1-\frac{p}{q}\right)} \\
&\quad\leqslant C\sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-2+a_0-a_1+\sigma\frac{\delta p}{q}-a_0\frac{\delta p}{q}} \\
&\quad\leqslant CL^{p(1+\alpha-\beta)-2+a_0-a_1+\frac{\delta p}{q}(\sigma-a_0)},
\end{aligned} \tag{3.1.40}$$

where the last inequality follows from the fact that $a_0 + p(1+\alpha-\beta) - 2 < 0$, $a_1 > 0$, and $\sigma \leqslant a_0$.

Next, similarly as in (3.1.37), we obtain for the second term in the right-hand side of (3.1.22) that

$$\begin{aligned}
\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy &\leqslant C\sum_{k=0}^{\infty}\frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}}(2^kL)^{\sigma} \\
&\leqslant CL^{p(1+\alpha-\beta)-2+\sigma-(1+\alpha)},
\end{aligned} \tag{3.1.41}$$

where the last inequality is justified by the fact that $p(1+\alpha-\beta)-2+\sigma \leqslant p(1+\alpha-\beta)-2+a_0 < 0$, and $\alpha > -1$. Therefore, combining (3.1.40) and (3.1.41) with (3.1.22), yields

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0-a_1+(\sigma-a_0)\delta_p} + CL^{\sigma-(1+\alpha)}, \tag{3.1.42}$$

where

$$\delta_p := \frac{\delta p}{q} \in (0,1).$$

Note that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant \begin{cases} CL^{\sigma-(1+\alpha)} & \text{if } \ a_1-(1+\alpha) \geqslant (a_0-\sigma)(1-\delta_p), \quad (3.1.43) \\ CL^{a_0-a_1+(\sigma-a_0)\delta_p} & \text{if } \ a_1-(1+\alpha) < (a_0-\sigma)(1-\delta_p). \quad (3.1.44) \end{cases}$$

Let us now specialize this estimate to the case $\sigma = a_0 - b_0$, as in (3.1.38), where we recall that $b_0 = \min\{a_1, 1+\alpha\}$. Firstly, suppose $b_0 = a_1$, so that $a_1 \leqslant 1+\alpha$. Since $\delta_p < 1$, it follows from (3.1.44) with $\sigma = a_0 - a_1$ that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0-a_1(1+\delta_p)}. \tag{3.1.45}$$

If $a_0 - a_1(1+\delta_p) < 0$, then $\Theta \equiv 0$ in $\mathbb{R}^2$. Otherwise, i.e. if $a_0 - a_1(1+\delta_p) \geqslant 0$, we invoke (3.1.44) with $\sigma = a_0 - a_1(1+\delta_p)$ and obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0 - a_1\left(1+\delta_p+\delta_p^2\right)}. \tag{3.1.46}$$

Hence, repeating this process n times, for any given $n \in \mathbb{N}$, we arrive at

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0 - a_1(1+\delta_p+\delta_p^2+\ldots+\delta_p^n)} = CL^{a_0 - a_1\left(\frac{1-\delta_p^{n+1}}{1-\delta_p}\right)}. \tag{3.1.47}$$

Taking the limit as $n \to \infty$ and observing that $\frac{1-\delta_p^{n+1}}{1-\delta_p} \to \frac{1}{1-\delta_p} = \frac{q(r-p)}{r(q-p)}$ as $n \to \infty$, we obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0 - a_1\frac{q(r-p)}{r(q-p)}}. \tag{3.1.48}$$

Now, recalling the definition of a_1 in (3.1.32), and particularly (3.1.33), we have

$$\begin{aligned} a_0 &- a_1\frac{q(r-p)}{r(q-p)} \\ &= a_0 - \frac{q(r-p)}{r(q-p)}\left[\frac{r(q-p)}{q(r-p)}a_0 - \frac{p(q-p)}{q(r-p)}\gamma_0 + \beta - 1 + \frac{2-\gamma_1}{r} - 2\frac{(q-p)}{q}\right] \\ &= \frac{p}{r}\left[\gamma_0 - \frac{q(r-p)}{p(q-p)}\left(\beta - 1 + \frac{2-\gamma_1}{r} - 2\frac{(q-p)}{q}\right)\right] < 0, \end{aligned} \tag{3.1.49}$$

where the inequality follows from (3.1.28). Therefore, it follows from (3.1.48) that $\Theta \equiv 0$ in $\mathbb{R}^2$.

Next, let us consider the case when $b_0 = 1+\alpha$, so that $a_1 \geqslant 1+\alpha$. We apply (3.1.43)-(3.1.44) with $\sigma = a_0 - (1+\alpha)$ and obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant \begin{cases} CL^{a_0 - 2(1+\alpha)} & \text{if } \; a_1 - (1+\alpha) \geqslant (1+\alpha)(1-\delta_p), \quad (3.1.50) \\ CL^{a_0 - a_1 - (1+\alpha)\delta_p} & \text{if } \; a_1 - (1+\alpha) < (1+\alpha)(1-\delta_p). \quad (3.1.51) \end{cases}$$

If the powers of L in both (3.1.50) and (3.1.51) are negative, then we conclude the proof. Otherwise, we proceed to improve on the upper bound of $\int_{|y|\leqslant L} |\Theta(y)|^p dy$ again via bootstrapping. To this end, we start by taking $m_0 \in \{1, 2, \ldots\}$ as the smallest integer such that

$$a_1 - (1+\alpha) < m_0(1+\alpha)(1-\delta_p). \tag{3.1.52}$$

If $m_0 = 1$, then (3.1.51) holds. On the other hand, if $m_0 \geqslant 2$ then

$$(m_0 - 1)(1+\alpha)(1-\delta_p) \leqslant a_1 - (1+\alpha). \tag{3.1.53}$$

and (3.1.50) holds. In the latter case, we may repeat this computation (m_0-1)-times, where at each kth time with $k = 1, \ldots, m_0 - 2$, we invoke (3.1.43) with $\sigma = a_0 - (k+1)(1+\alpha)$, and at $k = m_0 - 1$ we invoke (3.1.44) with $\sigma = a_0 - m_0(1+\alpha)$. We then arrive at

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0 - b_1}, \quad \text{where } b_1 := a_1 + m_0(1+\alpha)\delta_p. \tag{3.1.54}$$

Note that $b_1 > a_1$, and that (3.1.54) in fact holds for all $m_0 \geqslant 1$.

If $a_0 - b_1 < 0$, the proof is finished. Otherwise, we proceed similarly as before and apply (3.1.43)-(3.1.44) with $\sigma = a_0 - b_1$, which yields

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant \begin{cases} CL^{a_0-b_1-(1+\alpha)} & \text{if } \ a_1-(1+\alpha)-b_1(1-\delta_p) \geqslant 0, \qquad (3.1.55) \\ CL^{a_0-a_1-b_1\delta_p} & \text{if } \ a_1-(1+\alpha)-b_1(1-\delta_p) < 0. \qquad (3.1.56) \end{cases}$$

Then, if necessary, we proceed by taking $m_1 \in \{0, 1, 2, \ldots\}$ the smallest integer such that

$$a_1 - (1+\alpha) - b_1(1-\delta_p) < m_1(1+\alpha)(1-\delta_p).$$

After repeating this process m_1 times, we obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0-b_2}, \quad \text{where } b_2 := a_1 + (b_1 + m_1(1+\alpha))\delta_p.$$

Here we note that $b_2 > a_1 + a_1\delta_p$.

We may keep on iteratively repeating the same argument if necessary and denote by $m_n \in \{0, 1, 2, \ldots\}$, for each $n \in \mathbb{N}$, $n = 2, 3, \ldots$, the smallest integer such that

$$a_1 - (1+\alpha) - b_n(1-\delta_p) < m_n(1+\alpha)(1-\delta_p)$$

to obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{a_0-b_{n+1}}, \quad \text{where } b_{n+1} := a_1 + (b_n + m_n(1+\alpha))\delta_p. \tag{3.1.57}$$

Moreover, we have

$$b_{n+1} > a_1(1 + \delta_p + \ldots + \delta_p^n).$$

Therefore, by the same argument from (3.1.47)-(3.1.49), we deduce by taking the limit as $n \to \infty$ in (3.1.57) that $\Theta \equiv 0$ in $\mathbb{R}^2$. This concludes the proof of this case.

Case 3: Finally, suppose that $\beta - 1 + \frac{2-\gamma_0}{r} \leqslant \alpha \leqslant \beta - 1 + \frac{2}{p}$. In this case, we prove that either $\Theta \equiv 0$ in $\mathbb{R}^2$, or $\Theta \not\equiv 0$ and (3.1.3) holds.

Assume $\Theta \not\equiv 0$. From the first case, and particularly (3.1.8), it follows that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \lesssim L^{2-p(1+\alpha-\beta)} \quad \text{for all } L \gg 1. \tag{3.1.58}$$

Therefore, it only remains to show that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \gtrsim L^{2-p(1+\alpha-\beta)} \quad \text{for all } L \gg 1. \tag{3.1.59}$$

Suppose by contradiction that (3.1.59) does not hold. Then, there exists a sequence of positive numbers L_i, $i \in \mathbb{N}$, such that $L_i \to \infty$ as $i \to \infty$ and

$$\frac{1}{L_i^{2-p(1+\alpha-\beta)}} \int_{|y|\leqslant L_i} |\Theta(y)|^p dy \to 0 \quad \text{as } i \to \infty.$$

Taking $l_2 = 4L_i/\rho$ and $L := \rho l_1/8 \gg 1$ in (3.1.20), it follows after taking $i \to \infty$ that

$$\frac{1}{L^{2-p(1+\alpha-\beta)}}\int_{|y|\leqslant L}|\Theta(y)|^p dy \leqslant C\int_{|y|\geqslant L}\frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + C\int_{|y|\geqslant L}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy. \tag{3.1.60}$$

We now proceed similarly as in (3.1.29)-(3.1.38). Namely, we fix $q \in (p, r)$ such that (3.1.27)-(3.1.28) hold. Then, by interpolation, (3.1.58) and assumption (3.1.1), we have

$$\int_{|y|\leqslant L}|\Theta(y)|^q dy \leqslant \left(\int_{|y|\leqslant L}|\Theta(y)|^p dy\right)^{\delta}\left(\int_{|y|\leqslant L}|\Theta(y)|^r dy\right)^{1-\delta} \leqslant CL^{(2-p(1+\alpha-\beta))\delta+\gamma_0(1-\delta)},$$

where $\delta = (r-q)/(r-p)$.

Proceeding analogously as in (3.1.31) and recalling that $\delta_p := \delta p/q$, we estimate

$$\begin{aligned}
&\int_{|y|\geqslant L}\frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \\
&\leqslant \sum_{k=0}^{\infty}\frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}}\left(\int_{|y|\sim 2^kL}|\Theta(y)|^q dy\right)^{\frac{p}{q}}\left(\int_{|y|\sim 2^kL}|\widetilde{V}^{(1)}(y)|^r dy\right)^{\frac{1}{r}}(2^kL)^{2(1-\frac{p}{q}-\frac{1}{r})} \\
&\leqslant C\sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)+\alpha-2}(2^kL)^{\frac{p}{q}(2-p(1+\alpha-\beta))\delta+\gamma_0\frac{p}{q}(1-\delta)}(2^kL)^{1-\alpha-\beta+\frac{\gamma_1}{r}}(2^kL)^{2(1-\frac{p}{q}-\frac{1}{r})} \\
&\leqslant CL^{(p(1+\alpha-\beta)-2)(1-\delta_p)+\gamma_0\frac{p}{q}(1-\delta)+1-\beta+\frac{\gamma_1-2}{r}+2\left(1-\frac{p}{q}\right)},
\end{aligned} \tag{3.1.61}$$

where in the last inequality we used that $\alpha \leqslant \beta - 1 + \frac{2}{p}$ and condition (3.1.28).

Moreover, analogously as in (3.1.37), we obtain

$$\begin{aligned}
\int_{|y|\geqslant L}\frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy &\leqslant \sum_{k=0}^{\infty}\frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}}\int_{|y|\sim 2^kL}|\Theta(y)|^p dy \\
&\leqslant C\sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-3-\alpha}(2^kL)^{2-p(1+\alpha-\beta)} \leqslant CL^{-(1+\alpha)},
\end{aligned} \tag{3.1.62}$$

where we recall that $\alpha > -1$.

Thus, from (3.1.60),

$$\frac{1}{L^{2-p(1+\alpha-\beta)}}\int_{|y|\leqslant L}|\Theta(y)|^p dy \leqslant CL^{(p(1+\alpha-\beta)-2)(1-\delta_p)+\gamma_0\frac{p}{q}(1-\delta)+1-\beta+\frac{\gamma_1-2}{r}+2\left(1-\frac{p}{q}\right)} + CL^{-(1+\alpha)}.$$

Note that the power of L in the first term from the right-hand side cannot be smaller than the power of L in the second term. Indeed, since $1-\delta_p = \frac{r(q-p)}{q(r-p)}$, $\alpha \geqslant \beta - 1 + \frac{2-\gamma_0}{r}$, and $\gamma_0 \leqslant \gamma_1 + r$, we deduce that

$$\begin{aligned}
&(p(1+\alpha-\beta)-2)(1-\delta_p) + \gamma_0\frac{p}{q}(1-\delta) + 1 - \beta + \frac{\gamma_1-2}{r} + 2\left(1-\frac{p}{q}\right) + 1 + \alpha \\
&\geqslant \frac{p(q-p)}{q(r-p)}(2-\gamma_0) - 2\frac{r(q-p)}{q(r-p)} + \gamma_0\frac{p(q-p)}{q(r-p)} + 2 - \beta + \frac{\gamma_1-2}{r} + 2\left(1-\frac{p}{q}\right) \\
&\qquad\qquad + \beta - 1 + \frac{2-\gamma_0}{r} \\
&\geqslant \frac{2(p-r)(q-p)}{q(r-p)} + 1 + \frac{\gamma_1}{r} + 2\left(1-\frac{p}{q}\right) - \frac{\gamma_0}{r} = 1 + \frac{\gamma_1}{r} - \frac{\gamma_0}{r} \geqslant 0.
\end{aligned} \tag{3.1.63}$$

Hence,

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{2-p(1+\alpha-\beta)-d_0}, \tag{3.1.64}$$

where

$$d_0 := (2-p(1+\alpha-\beta))(1-\delta_p) - \gamma_0\frac{p}{q}(1-\delta) - 1 + \beta + \frac{2-\gamma_1}{r} - 2\left(1-\frac{p}{q}\right) > 0. \tag{3.1.65}$$

If $2-p(1+\alpha-\beta)-d_0 < 0$ then (3.1.64) implies that $\Theta \equiv 0$ in $\mathbb{R}^2$, which yields a contradiction and finishes the proof. Otherwise, for $2-p(1+\alpha-\beta)-d_0 \geqslant 0$, we repeat the above argument by using now the improved estimate (3.1.64). This gives

$$\int_{|y|\leqslant L} |\Theta(y)|^q dy \leqslant CL^{(2-p(1+\alpha-\beta))\delta - d_0\delta + (1-\delta)\gamma_0} \tag{3.1.66}$$

and

$$\begin{aligned}
&\frac{1}{L^{2-p(1+\alpha-\beta)}}\int_{|y|\leqslant L} |\Theta(y)|^p dy \\
&\quad\leqslant C\sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)+\alpha-2}(2^kL)^{\delta_p(2-p(1+\alpha-\beta))-d_0\delta_p+\gamma_0\frac{p}{q}(1-\delta)+1-\alpha-\beta+\frac{\gamma_1}{r}+2\left(1-\frac{p}{q}-\frac{1}{r}\right)} \\
&\qquad + C\sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-3-\alpha}(2^kL)^{2-p(1+\alpha-\beta)-d_0} \\
&\quad\leqslant C\sum_{k=0}^{\infty}(2^kL)^{(p(1+\alpha-\beta)-2)(1-\delta_p)+\gamma_0\frac{p}{q}(1-\delta)+1-\beta+\frac{\gamma_1-2}{r}+2\left(1-\frac{p}{q}\right)-d_0\delta_p} \\
&\qquad + C\sum_{k=0}^{\infty}(2^kL)^{-(1+\alpha)-d_0} \\
&\quad\leqslant CL^{-d_0(1+\delta_p)} + CL^{-(1+\alpha)-d_0} \leqslant CL^{-d_0(1+\delta_p)},
\end{aligned}$$

where we used that $0 < d_0 \leqslant 1+\alpha$, according to (3.1.63), (3.1.65). Thus,

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{2-p(1+\alpha-\beta)-d_0(1+\delta_p)}.$$

We may repeat this process for as many n times, $n \in \mathbb{N}$, as necessary, to obtain that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{2-p(1+\alpha-\beta)-d_0(1+\delta_p+\delta_p^2+\ldots+\delta_p^n)} = CL^{2-p(1+\alpha-\beta)-d_0\left(\frac{1-\delta_p^{n+1}}{1-\delta_p}\right)}.$$

Since $\frac{1-\delta_p^{n+1}}{1-\delta_p} \to \frac{1}{1-\delta_p} = \frac{q(r-p)}{r(q-p)}$ as $n \to \infty$, then

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{2-p(1+\alpha-\beta)-d_0\frac{q(r-p)}{r(q-p)}}. \tag{3.1.67}$$

From the definition of d_0 in (3.1.65) and condition (3.1.28), we have

$$\begin{aligned} &2 - p(1+\alpha-\beta) - d_0 \frac{q(r-p)}{r(q-p)} \\ &\quad = 2 - p(1+\alpha-\beta) - \frac{q(r-p)}{r(q-p)} \left[(2 - p(1+\alpha-\beta)) \frac{r(q-p)}{q(r-p)} - \gamma_0 \frac{p(q-p)}{q(r-p)} - 1 \right. \\ &\qquad\qquad \left. + \beta + \frac{2-\gamma_1}{r} - 2\left(1 - \frac{p}{q}\right) \right] \\ &\quad = \frac{p}{r} \left[\gamma_0 - \frac{q(r-p)}{p(q-p)} \left(\beta - 1 + \frac{2-\gamma_1}{r} - 2\left(1 - \frac{p}{q}\right) \right) \right] < 0. \end{aligned} \tag{3.1.68}$$

Therefore, we deduce from (3.1.68) that $\Theta \equiv 0$ in $\mathbb{R}^2$, which is a contradiction with our starting assumption that $\Theta \not\equiv 0$ in $\mathbb{R}^2$. This concludes the proof.

$\square$

The subsequent result provides a criterion for the profile to exclude self-similar blow-up by assuming a specific decay behavior at infinity. Furthermore, under the growth bound on the profile and its gradient, we obtain an appropriate range for α, where the characterization of the L^p norm as in (3.1.3) for possible types of blowup profiles.

Corollary 3.1. *Fix $\beta \in (1,2)$. Suppose $\theta \in \mathcal{C}([0,T]; H^s(\mathbb{R}^2)) \cap L^\infty(0,T; L^1(\mathbb{R}^2))$, with $s > 1+\beta$, is a solution to the gSQG equation* (1.0.1) *that is locally self-similar in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, with scaling parameter $\alpha > -1$ and profile $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Then, the following statements hold:*

(i) If there exist some $\sigma_0 > 0$ and $\sigma_1 > 0$ such that $|\Theta(y)| \lesssim |y|^{-\sigma_0}$ and $|\nabla\Theta(y)| \lesssim |y|^{-\sigma_1}$ for all $|y| \gg 1$, then $\Theta \equiv 0$ in $\mathbb{R}^2$.

(ii) Suppose that $|\Theta(y)| \gtrsim 1$ for all $|y| \gg 1$, and that there exists a real number $0 \leqslant \sigma_1 < \beta - 1$ such that $|\nabla\Theta(y)| \lesssim |y|^{\sigma_1}$ for all $|y| \gg 1$. Then the values of α admitting nontrivial profiles belong to the interval $[\beta - 2 - \sigma_1, \beta - 1]$ and for each such α the corresponding profile Θ satisfies

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \sim L^{2-p(1+\alpha-\beta)}$$

for every $p \in [1,\infty)$ and for all L sufficiently large.

Proof. We start with the proof of (i). Let M be a positive constant such that $|\Theta(y)| \lesssim |y|^{-\sigma_0}$ and $|\nabla\Theta(y)| \lesssim |y|^{-\sigma_1}$ for all $|y| \geqslant M$. Thus, since $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$, it follows that for all $L > 0$ and $r > \max\left\{\frac{2}{\sigma_0}, \frac{2}{\sigma_1}\right\}$, we have

$$\int_{|y|\leqslant L} |\Theta(y)|^r dy \leqslant \int_{|y|\leqslant M} |\Theta(y)|^r dy + \int_{|y|\geqslant M} \frac{1}{|y|^{r\sigma_0}} dy \leqslant C,$$

and

$$\int_{|y|\leqslant L} |\nabla\Theta(y)|^r dy \leqslant \int_{|y|\leqslant M} |\nabla\Theta(y)|^r dy + \int_{|y|\geqslant M} \frac{1}{|y|^{r\sigma_1}} dy \leqslant C.$$

Let $p_1 := \max\left\{1, \frac{2}{\sigma_0}, \frac{2}{\sigma_1}\right\}$. Then, the assumptions of Theorem 3.1 are satisfied with the following parameter choices: $p = p_1$, $r = p_1 + 1$, $\gamma_0 = 0$, and $\gamma_1 = 0$. Consequently, the values of α that admit a nontrivial corresponding profile Θ belong to the interval $\left[\beta - 1 + \frac{2}{p_1+1}, \beta - 1 + \frac{2}{p_1}\right]$. On the other hand, note that the assumptions of Theorem 3.1 are also satisfied with $p = p_1 + k$, $r = p_1 + k + 1$, $\gamma_0 = 0$, and $\gamma_1 = 0$, for any $k > 0$. This implies that the values of α admitting nontrivial profiles must also belong to the interval $\left[\beta - 1 + \frac{2}{p_1+k+1}, \beta - 1 + \frac{2}{p_1+k}\right]$ for any $k > 0$. In particular, for $k \geqslant 2$, we obtain that $\alpha \in \left[\beta - 1 + \frac{2}{p_1+k+1}, \beta - 1 + \frac{2}{p_1+k}\right] \cap \left[\beta - 1 + \frac{2}{p_1+1}, \beta - 1 + \frac{2}{p_1}\right] = \varnothing$. Therefore, we conclude that $\Theta \equiv 0$ in $\mathbb{R}^2$.

We proceed to prove (ii). Let $M > 0$ such that $|\nabla\Theta(y)| \lesssim |y|^{\sigma_1}$ for all $|y| \geqslant M$, and fix any $p \in [1, \infty)$. Observe that, for all $L \gg 1$ and $r > p$,

$$\begin{aligned}\int_{|y|\leqslant L} |\nabla\Theta(y)|^r dy &\leqslant \int_{|y|\leqslant M} |\nabla\Theta(y)|^r dy + \int_{M\leqslant|y|\leqslant L} |y|^{\sigma_1 r} dy \\ &\leqslant C \int_{|y|\leqslant M} dy + L^{\sigma_1 r} \int_{M\leqslant|y|\leqslant L} dy \leqslant C L^{\sigma_1 r + 2},\end{aligned} \tag{3.1.69}$$

where we used that $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Then, it follows by Sobolev embedding that

$$\int_{|y|\leqslant L} |\Theta(y)|^r dy \leqslant C L^{(\sigma_1+1)r+2}.$$

Therefore, the assumptions of Theorem 3.1 are satisfied by setting $\gamma_1 = \sigma_1 r + 2$ and $\gamma_0 = (\sigma_1 + 1)r + 2 = \gamma_1 + r$. It follows that the values of α admitting nontrivial profiles belong to the interval $\left[\beta - 2 - \sigma_1, \beta - 1 + \frac{2}{p}\right]$ and the corresponding profile satisfies

$$C_1 L^{2-p(1+\alpha-\beta)} \leqslant \int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant C_2 L^{2-p(1+\alpha-\beta)} \quad \text{for all } L \gg 1, \tag{3.1.70}$$

for some positive constants C_1, C_2. On the other hand, since $|\Theta(y)| \gtrsim 1$ for $|y| \gg 1$, it follows that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \geqslant C L^2 \quad \text{for } L \gg 1. \tag{3.1.71}$$

Combining the upper bound in (3.1.70) with (3.1.71), we must have $1 + \alpha - \beta \leqslant 0$, which implies that the values of α admitting nontrivial profiles in fact belong to $[\beta - 2 - \sigma_1, \beta - 1]$. This completes the proof. □

3.2 Case $0 < \beta \leqslant 1$

This section is dedicated to studying the locally self-similar solution of the gSQG equation for the case when, $0 < \beta \leqslant 1$. Here, differently from the case $1 < \beta < 2$, we only assume a L^r growth in the self-similar profile, and analogous to Theorem 3.1, we obtain certain ranges of the scaling parameter where the profile is either identically zero, excluding any locally self-similar blowup, or, for appropriate values of r and p, its L^p asymptotic behavior can be characterized.

In the case $\beta = 1$, Theorem 3.2 improves upon the results previously proven by Xue [76], offering greater flexibility for the parameters r and γ. Furthermore, our assumptions on the parameters r and γ for (3.2.1) to hold, namely $r > p$ and $\gamma \in [0, r+2)$, are weaker than those in [76, Theorem 1.1], where it is assumed that $r \geqslant p+1$ and $\gamma \in [0, r-p)$. Moreover, for $0 < \beta < 1$, Theorem 3.2 recovers the result established in [9] concerning globally self-similar solutions of the gSQG equation, with weaker assumptions, since any globally self-similar solution is also locally self-similar.

Theorem 3.2. *Fix $\beta \in (0,1]$. Suppose $\theta \in \mathcal{C}([0,T]; H^s) \cap L^\infty([0,T]; L^1)$, with $s > 1+\beta$, is a solution to the gSQG equation (1.0.1) that is locally-self-similar in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, with scaling parameter $\alpha > -1$ and profile $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Fix also $p > 1$, and suppose that for some $r > p$ and $\gamma \in [0, \beta r + 2)$, it holds*

$$\int_{|y| \leqslant L} |\Theta(y)|^r dy \lesssim L^\gamma \tag{3.2.1}$$

for all L sufficiently large. Under these condition, it follows that if $\alpha > \beta - 1 + \frac{2}{p}$ or $-1 < \alpha < \beta - 1 + \frac{2-\gamma}{r}$ then $\Theta \equiv 0$. Moreover, if $\alpha \in \left[\beta - 1 + \frac{2-\gamma}{r}, \beta - 1 + \frac{2}{p}\right]$ then either $\Theta \equiv 0$ or Θ is a nontrivial profile and it satisfies

$$\int_{|y| \leqslant L} |\Theta(y)|^p dy \sim L^{2-p(1+\alpha-\beta)}, \tag{3.2.2}$$

for all L sufficiently large.

Proof. The proof follows a structure similar to that of Theorem 3.1 and is also divided based on three distinct ranges for α.

Case 1:

The first case is when $\alpha > \beta + \frac{2}{p} - 1$, and the proof is identical to the firt case in Theorem 3.1.

Case 2:

The second case is when $-1 < \alpha < \beta - 1 + \frac{2-\gamma}{r}$. Our goal is to prove that $\Theta \equiv 0$ in $\mathbb{R}^2$. To this end, let us recall that from (6.2.1), we derive the local L^p equality that

$$\int_{\mathbb{R}^2} |\theta(x,t_2)|^p \phi_{\frac{\rho}{4}}(x) dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \phi_{\frac{\rho}{4}}(x) dx = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} (\mathbf{u}(x,t) \cdot \nabla \phi_{\frac{\rho}{4}}(x)) |\theta(x,t)|^p dx\, dt, \tag{3.2.3}$$

where $\phi_{\frac{\rho}{4}} \in \mathcal{C}^\infty(\mathbb{R}^2)$ with $0 \leqslant \phi_{\frac{\rho}{4}} \leqslant 1$ such that $\phi_{\frac{\rho}{4}} \equiv 1$ in $B_{\rho/8}$, and $\phi_{\frac{\rho}{4}} \equiv 0$ in $B^c_{\rho/4}$.

Proceeding to estimate each term individually. The distinction from the case $1 < \beta < 2$, arises in the estimation of $\mathbf{u}$, which is attributed to the distinct integrability characteristics of K_β within the self-similar region for $0 < \beta < 1$. In the special case of $\beta = 1$, K_β is a Calderón-Zygmund operator. To facilitate this analysis, consider the following cut-off function: $\phi_\rho \in \mathcal{C}^\infty(\mathbb{R}^2)$ with $0 \leqslant \phi_\rho \leqslant 1$ such that $\phi_\rho \equiv 1$ in $B_{\rho/2}$, and $\phi_\rho \equiv 0$ in B^c_ρ.

Fix $t_1, t_2 \in [0, T)$. Let us decompose $\mathbf{u}$ into two components: one within the self-similarity region and the other outside it. Thus,

$$\begin{aligned}\mathbf{u}(x,t) &= C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)\phi_\rho(y)dy + C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)(1-\phi_\rho(y))dy \\ &=: \tilde{\mathbf{u}}^{(1)}(x,t) + \tilde{\mathbf{u}}^{(2)}(x,t).\end{aligned} \tag{3.2.4}$$

Invoking the local self-similarity of θ, given in (3.0.1), and repeating the computation in (3.1.12), we obtain

$$\begin{aligned}&\mathbf{u}^{(1)}(x,t) \\ &= C_\beta P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\theta(y,t)\phi_\rho(y)dy \\ &= C_\beta (T-t)^{\frac{\beta-\alpha-1}{1+\alpha}} \mathrm{P.V.} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+\beta}} \Theta\left(\frac{y}{(T-t)^{\frac{1}{1+\alpha}}}\right)\phi_\rho(y)dy \\ &= C_\beta (T-t)^{\frac{-(\alpha+2)}{1+\alpha}} P.V. \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{(T-t)^{\frac{1}{\alpha}}} \frac{|x-y|^{-(2+\beta)}}{(T-t)^{\frac{-(2+\beta)}{\alpha}}} \Theta\left(\frac{y}{(T-t)^{\frac{1}{1+\alpha}}}\right)\phi_\rho(y)dy \\ &= C_\beta (T-t)^{\frac{-\alpha}{1+\alpha}} P.V. \int_{\mathbb{R}^2} \left(\frac{x}{(T-t)^{\frac{1}{\alpha}}} - y\right)^\perp \left|\frac{x}{(T-t)^{\frac{1}{\alpha}}} - y\right|^{-(2+\beta)} \Theta(y)\phi_\rho(y(T-t)^{\frac{1}{1+\alpha}})dy \\ &= \frac{C_\beta}{(T-t)^{\frac{\alpha}{1+\alpha}}} U^{(1)}\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}, t\right),\end{aligned} \tag{3.2.5}$$

where

$$U^{(1)}(x,t) := P.V. \int_{\mathbb{R}^2} K_\beta(x-y)\Theta(y)\phi_\rho(y(T-t)^{\frac{1}{1+\alpha}})dy.$$

Given that K_β is square integrable in any region excluding the origin, for all $\beta > 0$, it follows that when $\rho/8 \leqslant |x| \leqslant \rho/4$, we derive that

$$|\tilde{\mathbf{u}}^{(2)}(x,t)| \leqslant C_\beta \int_{|y| \geqslant \frac{\rho}{2}} \frac{|\theta(y,t)|}{|y|^{\beta+1}} dy \leqslant C_\beta ||\theta(0)||_{L^2}. \tag{3.2.6}$$

Repeating the same computation as in (3.1.16) - (3.1.18), it leads to the conclusion that the right-hand side of (3.2.3) can be estimated by

$$\begin{aligned}&\left| \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta(x,t)|^p (\mathbf{u}(x,t) \cdot \nabla \phi_{\frac{\rho}{4}}(x)) dx\, dt \right| \\ &\qquad \leqslant C \int_{\frac{\rho}{8} l_1 \leqslant |y| \leqslant \frac{\rho}{4} l_2} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}} dy + C \int_{\frac{\rho}{8} l_1 \leqslant |y| \leqslant \frac{\rho}{4} l_2} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}} dy,\end{aligned} \tag{3.2.7}$$

where

$$\widetilde{U}^{(1)}(y) = \int_{t_1}^{t_2} \left| P.V. \int_{\mathbb{R}^2} K_\beta(y-z)\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz \right| \mathbb{1}_{A_y}(t)dt. \tag{3.2.8}$$

The indicator function $\mathbb{1}_{A_y}$ is associated with the set A_y, which is defined for each $y \in \mathbb{R}^2$, satisfying $\rho l_1/8 \leqslant |y| \leqslant \rho l_2/4$, as follows

$$A_y := \left\{ t \in [t_1, t_2] \, : \, \frac{\rho}{8}\frac{1}{|y|} \leqslant (T-t)^{\frac{1}{1+\alpha}} \leqslant \frac{\rho}{4}\frac{1}{|y|} \right\}. \tag{3.2.9}$$

where the length of the set A_y is bounded by $|A_y| \leqslant c_{\alpha,\rho}/|y|^{1+\alpha}$. Plugging this back into (3.2.3) and recalling that

$$\int_{\mathbb{R}^2} |\theta(x,t_i)|^p \phi_{\frac{\varrho}{4}}(x)dx = l_i^{p(1+\alpha-\beta)-2} \int_{|y| \leqslant \frac{\varrho}{4}l_i} |\Theta(y)|^p \phi_{\frac{\varrho}{4}}(yl_i^{-1})dy,$$

where $l_i = (T-t_i)^{-\frac{1}{1+\alpha}}$, $i=1,2$, we obtain that

$$\begin{aligned} &\left| l_2^{p(1+\alpha-\beta)-2} \int_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\varrho}{4}}(yl_2^{-1})dy - l_1^{p(1+\alpha-\beta)-2} \int_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\varrho}{4}}(yl_1^{-1})dy \right| \\ &\qquad \leqslant \int_{\frac{\varrho}{8}l_1 \leqslant |y| \leqslant \frac{\varrho}{4}l_2} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + \int_{\frac{\varrho}{8}l_1 \leqslant |y| \leqslant \frac{\varrho}{4}l_2} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy, \end{aligned} \tag{3.2.10}$$

Note that, invoking Hölder's inequality and assumption (3.2.1), we have that

$$\begin{aligned} l_2^{p(1+\alpha-\beta)-2} \int_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\varrho}{4}}(yl_2^{-1})dy &\leqslant l_2^{p(1+\alpha-\beta)-2} \left(\int_{|y| \leqslant \frac{\varrho}{4}l_2} |\Theta(y)|^r dy \right)^{\frac{p}{r}} l_2^{2\left(1-\frac{p}{r}\right)} \\ &\leqslant C l_2^{p(1+\alpha-\beta)+(\gamma-2)\frac{p}{r}}. \end{aligned}$$

Since $\alpha < \beta - 1 + \frac{2-\gamma}{r}$, it follows that $(1+\alpha-\beta)+(\gamma-2)/r < 0$. Hence,

$$l_2^{p(1+\alpha-\beta)-2} \int_{\mathbb{R}^2} |\Theta(y)|^p \phi_{\frac{\varrho}{4}}(yl_2^{-1})dy \to 0 \quad \text{as } l_2 \to \infty. \tag{3.2.11}$$

Setting $L := \frac{\varrho}{8}l_1$ and taking the limit in (3.2.10) as $t_2 \to T$, so that $l_2 \to \infty$, we obtain

$$\frac{1}{L^{2-p(1+\alpha-\beta)}} \int_{|y| \leqslant L} |\Theta(y)|^p dy \leqslant C \int_{|y| \geqslant L} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy + C \int_{|y| \geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy. \tag{3.2.12}$$

We now start to estimate each term on the right-hand side, employing the dyadic decomposition together with Lemma 6.3 and Holder inequality. More precisely,

$$\begin{aligned}
&\int_{|y|\geqslant L} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \\
&\quad\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} \int_{2^kL\leqslant|y|\leqslant 2^{k+1}L} |\widetilde{U}^{(1)}(y)||\Theta(y)|^p dy \\
&\quad\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} \left(\int_{|y|\sim 2^kL} |\widetilde{U}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} \left(\int_{|y|\sim 2^kL} |\Theta(y)|^r dy\right)^{\frac{p}{r}} (2^kL)^{2\left(1-\frac{p+1}{r}\right)} \\
&\quad\leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} (2^kL)^{\frac{1}{r}(\gamma-r(\alpha+\beta))} (2^kL)^{\gamma\frac{p}{r}} (2^kL)^{2\left(1-\frac{p+1}{r}\right)} \\
&\quad\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-2+\frac{(p+1)(\gamma-2)}{r}-\beta+2} \\
&\quad\leqslant CL^{p(1+\alpha-\beta)-2+\frac{(p+1)(\gamma-2)}{r}-\beta+2}
\end{aligned} \tag{3.2.13}$$

where in the last inequality we used the assumption that $\alpha < \beta - 1 + \frac{2-\gamma}{r}$ and $\gamma < \beta r + 2$.

To address the second term on the right-hand side of (3.2.12), we once again apply the dyadic decomposition, in combination with Hölder's inequality, to obtain

$$\begin{aligned}
\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy &\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}} \int_{|y|\sim 2^kL} |\Theta(y)|^p dy \\
&\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}} \left(\int_{|y|\sim 2^kL} |\Theta(y)|^r dy\right)^{\frac{p}{r}} (2^kL)^{2\left(1-\frac{p}{r}\right)} \\
&\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-3-\alpha} (2^kL)^{\gamma\frac{p}{r}} (2^kL)^{2\left(1-\frac{p}{r}\right)} \\
&\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-2-\alpha+1+(\gamma-2)\frac{p}{r}} \\
&\leqslant CL^{p(1+\alpha-\beta)-2+1-\alpha+\frac{(\gamma-2)p}{r}},
\end{aligned} \tag{3.2.14}$$

where we used the fact that $-1 < \alpha < \beta - 1 + \frac{2-\gamma}{r}$. Combining (3.2.13) and (3.2.14) with (3.2.12), and recalling that $\alpha < \beta - 1 + \frac{2-\gamma}{r}$, we deduce that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\frac{(p+1)(\gamma-2)}{r}-\beta+2} + CL^{1-\alpha+\frac{(\gamma-2)p}{r}} \leqslant CL^{\tilde{a}_0}, \tag{3.2.15}$$

where

$$\tilde{a}_0 := 1 - \alpha + \frac{p(\gamma-2)}{r}. \tag{3.2.16}$$

Observe that, if $\tilde{a}_0 < 0$, we conclude that Θ is identically zero on $\mathbb{R}^2$, then the proof is complete. Otherwise, if $\tilde{a}_0 \geqslant 0$, let us improve the estimates of the terms on the right-hand side of (3.2.10) by using the new upper bound established in (3.2.15). To accomplish this, we highlight that since $r > p$, we may take $q \geqslant 1$ such that

$$p < q < \frac{r(p+1)}{r+1} < r. \tag{3.2.17}$$

Then, by interpolation, it follows that

$$\begin{aligned}\int_{|y|\leqslant L}|\Theta(y)|^q dy &\leqslant \left(\int_{|y|\leqslant L}|\Theta(y)|^p dy\right)^{\delta}\left(\int_{|y|\leqslant L}|\Theta(y)|^r dy\right)^{1-\delta}\\ &\leqslant CL^{\tilde{a}_0\delta+(1-\delta)\gamma}, \quad \text{with } \delta := \frac{r-q}{r-p}\in(0,1). \end{aligned}\tag{3.2.18}$$

Next, by applying the dyadic decomposition once more and using Hölder's inequality, we obtain from (3.2.16), (3.2.18), and Lemma 6.3 that

$$\begin{aligned}
&\int_{|y|\geqslant L}\frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy\\
&\leqslant \sum_{k=0}^{\infty}\frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}}\left(\int_{|y|\sim 2^kL}|\Theta(y)|^q dy\right)^{\frac{p}{q}}\left(\int_{|y|\sim 2^kL}|\widetilde{U}^{(1)}(y)|^q dy\right)^{\frac{1}{q}}(2^kL)^{2\left(1-\frac{p+1}{q}\right)}\\
&\leqslant \sum_{k=0}^{\infty}\frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}}(2^kL)^{\frac{p}{q}(\tilde{a}_0\delta+\gamma(1-\delta))}(2^kL)^{\frac{1}{q}(\tilde{a}_0\delta+\gamma(1-\delta)-q(\alpha+\beta))}(2^kL)^{2\left(1-\frac{p+1}{q}\right)}\\
&\leqslant \sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-2+\alpha+\tilde{a}_0\frac{(p+1)\delta}{q}+\gamma\frac{(p+1)(1-\delta)}{q}-\alpha-\beta+2\left(1-\frac{p+1}{q}\right)}\\
&\leqslant \sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-2+\tilde{a}_0+\tilde{a}_0\left(\frac{(p+1)\delta}{q}-1\right)-\beta+\gamma\frac{(p+1)(1-\delta)}{q}+2\left(1-\frac{p+1}{q}\right)}\\
&\leqslant \sum_{k=0}^{\infty}(2^kL)^{p(1+\alpha-\beta)-2+\tilde{a}_0-\tilde{a}_1},
\end{aligned}\tag{3.2.19}$$

where

$$\tilde{a}_1 := \tilde{a}_0\left(1-\frac{(p+1)\delta}{q}\right)+\beta-\frac{(p+1)(1-\delta)\gamma}{q}-2\left(1-\frac{p+1}{q}\right).\tag{3.2.20}$$

Recalling from (3.2.14) that $\tilde{a}_0+p(1+\alpha-\beta)-2<0$. Then, to ensure that (3.2.19) is bounded, it suffices to prove that $\tilde{a}_1\geqslant 0$. Thus, since $\alpha<\beta-1+\frac{2-\gamma}{r}$, $\tilde{a}_0=1-\alpha+\frac{\gamma-2}{r}$, and $1-\delta=\frac{q-p}{r-p}$, it follows that

$$\begin{aligned}
\tilde{a}_1 &= \tilde{a}_0\left(1-\frac{(p+1)\delta}{q}\right)+\beta-\frac{(p+1)(1-\delta)\gamma}{q}-2\left(1-\frac{p+1}{q}\right)\\
&= \left(1-\alpha+\frac{\gamma-2}{r}\right)\left(1-\frac{(p+1)\delta}{q}\right)+\beta-\frac{(p+1)(1-\delta)\gamma}{q}-2\left(1-\frac{p+1}{q}\right)\\
&> \left(2-\beta+\frac{(p+1)(\gamma-2)}{r}\right)\left(1-\frac{(p+1)\delta}{q}\right)+\beta-\frac{(p+1)(1-\delta)\gamma}{q}-2\left(1-\frac{p+1}{q}\right)\\
&= \frac{2(p+1)(1-\delta)}{q}+\frac{(p+1)(\gamma-2)}{r}\left(1-\frac{\delta(p+1)}{q}\right)+\beta\delta\frac{(p+1)}{q}-\frac{(p+1)(1-\delta)\gamma}{q}\\
&= \frac{(p+1)(2-\gamma)}{q}\left[1-\delta-\frac{q-\delta(p+1)}{r}\right]+\beta\delta\frac{(p+1)}{q}\\
&= \frac{(p+1)(2-\gamma)}{q}\left(\frac{r-q}{r(r-p)}\right)+\beta\frac{(p+1)(r-q)}{(r-p)q}\\
&= \frac{(p+1)(r-q)}{rq(r-p)}(2-\gamma+\beta r)>0
\end{aligned}\tag{3.2.21}$$

where in the last inequality we used that $\gamma < \beta r + 2$. Therefore, we conclude from (3.2.19) that

$$\int_{|y|\geqslant L} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}} dy \leqslant CL^{p(1+\alpha-\beta)-2+\tilde{a}_0-\tilde{a}_1}. \tag{3.2.22}$$

For another term, invoking (3.2.15) and recalling that $\tilde{a}_0 + p(1+\alpha-\beta) - 2 < 0$ and $-1 < \alpha$, it follows that

$$\begin{aligned}
\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}} dy &\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3+\alpha-p(1+\alpha-\beta)}} \int_{|y|\sim 2^k L} |\Theta(y)|^p dy \\
&\leqslant C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3+\alpha-p(1+\alpha-\beta)}} (2^k L)^{3+\alpha-p(1+\alpha-\beta)+\tilde{a}_0} \\
&\leqslant CL^{p(1+\alpha-\beta)-2+\tilde{a}_0-(1+\alpha)}.
\end{aligned} \tag{3.2.23}$$

Plugging (3.2.22) and (3.2.23) into (3.2.12), we deduce that

$$\begin{aligned}
\int_{|y|\leqslant L} |\Theta(y)|^p dy &\leqslant CL^{\tilde{a}_0-\tilde{a}_1} + cL^{\tilde{a}_0-(1+\alpha)} \\
&\leqslant CL^{\tilde{a}_0-b_0}, \quad \text{where } \tilde{b}_0 := \min\{\tilde{a}_1, 1+\alpha\} > 0.
\end{aligned} \tag{3.2.24}$$

Clearly, if $\tilde{a}_0 - \tilde{b}_0 < 0$, we finished the proof. Otherwise, analogous to (3.1.42), we proceed to obtain a more general form for the profile estimate in L^p. Let us assume that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\sigma} \quad \text{with } \sigma \leqslant \tilde{a}_0. \tag{3.2.25}$$

From the interpolation inequality (3.2.18), we have

$$\int_{|y|\leqslant L} |\Theta(y)|^q dy \leqslant CL^{\sigma\delta+\gamma(1-\delta)}. \tag{3.2.26}$$

Then, proceeding similarly as in (3.2.19) and recalling the definition of $\tilde{a}_1$ in (3.2.20), we obtain

$$\begin{aligned}
&\int_{|y|\geqslant L} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}} dy \\
&\quad\leqslant C \sum_{k=0}^{\infty} \frac{1}{(2^k L)^{2-\alpha-p(1+\alpha-\beta)}} (2^k L)^{\frac{p}{q}(\sigma\delta+\gamma(1-\delta))} (2^k L)^{\frac{1}{q}(\sigma\delta+(1-\delta)\gamma-q(\alpha+\beta))} (2^k L)^{2\left(1-\frac{p+1}{q}\right)} \\
&\quad\leqslant C \sum_{k=0}^{\infty} (2^k L)^{p(1+\alpha-\beta)-2+\sigma\frac{\delta(p+1)}{q}+\gamma\frac{(1-\delta)(p+1)}{q}-\beta+2\left(1-\frac{p+1}{q}\right)} \\
&\quad\leqslant C \sum_{k=0}^{\infty} (2^k L)^{p(1+\alpha-\beta)-2+\tilde{a}_0-\tilde{a}_1+\sigma\frac{\delta(p+1)}{q}-\tilde{a}_0\frac{\delta(p+1)}{q}} \\
&\quad\leqslant CL^{p(1+\alpha-\beta)-2+\tilde{a}_0-\tilde{a}_1+\frac{\delta(p+1)}{q}(\sigma-\tilde{a}_0)},
\end{aligned} \tag{3.2.27}$$

where we used that $\tilde{a}_0 + p(1+\alpha-\beta) - 2 < 0$, $\tilde{a}_1 > 0$, and $\sigma \leqslant \tilde{a}_0$. For another term, it follows that

$$\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}} dy \leqslant C\sum_{k=0}^{\infty} \frac{1}{(2^k L)^{3+\alpha-p(1+\alpha-\beta)}} (2^k L)^{\sigma}$$
$$\leqslant CL^{p(1+\alpha-\beta)-2+\sigma-(1+\alpha)}, \tag{3.2.28}$$

where we used that $p(1+\alpha-\beta) - 2 + \sigma \leqslant p(1+\alpha-\beta) - 2 + \tilde{a}_0 < 0$, and $\alpha > -1$. Therefore, combining (3.2.27) and (3.2.28) with (3.2.12), we obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-\tilde{a}_1+(\sigma-\tilde{a}_0)\delta_p} + CL^{\sigma-(1+\alpha)}, \tag{3.2.29}$$

where

$$\delta_p := \frac{\delta(p+1)}{q} = \frac{(p+1)(r-q)}{q(r-p)} > 1.$$

The inequality above holds due to the choice of q in (3.2.17). We emphasize that the condition $\delta_p > 1$ implies that the power on the right-hand side in (3.2.29) will be more negative. This will simplify the calculations that follow, in contrast to the case $1 < \beta < 2$, where $\delta_p < 1$.

Let us analyze the general estimate (3.2.29) to the case $\sigma = \tilde{a}_0 - b_0$, as in (3.2.24). Starting with the assumption that $b_0 = 1+\alpha$, we can deduce from (3.2.29) with $\sigma = \tilde{a}_0 - (1+\alpha)$ that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-\tilde{a}_1-(1+\alpha)\delta_p} + CL^{\tilde{a}_0-2(1+\alpha)}$$
$$\leqslant CL^{\tilde{a}_0-2(1+\alpha)}, \tag{3.2.30}$$

where we used the fact that $\tilde{b}_0 = 1+\alpha \leqslant \tilde{a}_1$ and $\delta_p > 1$. It is important to note that if $\tilde{a}_0 - 2(1+\alpha) < 0$, then Θ is identically zero in $\mathbb{R}^2$. On the other hand, if $\tilde{a}_0 - 2(1+\alpha) \geqslant 0$, we apply (3.2.29) with $\sigma = \tilde{a}_0 - 2(1+\alpha)$ to obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-\tilde{a}_1-2(1+\alpha)\delta_p} + CL^{\tilde{a}_0-3(1+\alpha)}$$
$$\leqslant CL^{\tilde{a}_0-3(1+\alpha)}. \tag{3.2.31}$$

Once again, we use the fact that $1+\alpha \leqslant \tilde{a}_1$ and $\delta_p > 1$. Therefore, if needed, by repeating this process n times for any given $n \in \mathbb{N}$, and setting $\sigma = \tilde{a}_0 - k(1+\alpha)$ at each kth iteration, where $k \in 3, \ldots, n$, and arrive at

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-n(1+\alpha)\delta_p} \tag{3.2.32}$$

Taking the limit as $n \to \infty$ and observing that $1+\alpha \geqslant 0$, we conclude that $\Theta \equiv 0$ on $\mathbb{R}^2$.

In the secondary scenario where $\tilde{b}_0 = \tilde{a}_1$ and $\tilde{a}_1 \leqslant 1+\alpha$, we proceed similarly to the case $b_0 = 1+\alpha$ in the proof of Theorem 3.1. We invoke (3.2.29) with $\sigma = \tilde{a}_0 - \tilde{a}_1$ to obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant \begin{cases} CL^{\tilde{a}_0-\tilde{a}_1-(1+\alpha)} & \text{if } \tilde{a}_1\delta_p \geqslant (1+\alpha), \quad (3.2.33)\\ CL^{\tilde{a}_0-\tilde{a}_1(1+\delta_p)} & \text{if } \tilde{a}_1\delta_p < (1+\alpha). \quad (3.2.34)\end{cases}$$

The proof concludes if the exponents of L in both (3.2.33) and (3.2.34) are negative. Otherwise, we can consider $m_0 \in \{1,2,\ldots\}$ be the smallest integer such that

$$\tilde{a}_1\delta_p^{m_0} \geqslant 1+\alpha.$$

If $m_0 = 1$, then (3.2.33) is satisfied. Otherwise, for $m_0 \geqslant 2$, the minimality of m_0 implies that

$$\tilde{a}_1\delta_p^{m_0-1} < 1+\alpha$$

and (3.2.34) holds. In this scenario, the computation can be iteratively conducted (m_0-1) times. At each kth iteration with $k = 1,\ldots,m_0-2$, we invoke (3.2.29) with $\sigma = \tilde{a}_0 - \tilde{a}_1(1+\delta_p+\ldots+\delta_p^k)$, and, for $k = m_0-1$, we invoke (3.2.29) with $\sigma = \tilde{a}_0-\tilde{a}_1(1+\delta_p+\ldots+\delta_p^{m_0-1})$. This approach results in

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-\tilde{b}_1}, \quad \text{where } \tilde{b}_1 := \tilde{a}_1(1+\delta_p+\ldots+\delta_p^{m_0-1}) + (1+\alpha), \quad (3.2.35)$$

which is valid for all $m_0 \geqslant 1$.

Now, if $\tilde{a}_0 - \tilde{b}_1 > 0$, we emphasize that, in contrast to the proof of case 2 in Theorem (3.1), which required the construction of a sequence of the smallest integers m_n associated with b_n to ultimately prove that the profile is null, such construction is unnecessary in this instance. Since $\delta_p > 1$, it accelerates the process of making the exponents negative. More precisely, let us apply (3.2.29) n times, for any given $n \in \mathbb{N}$, where at the k-th iteration, we set $\sigma = \tilde{a}_0 - \tilde{b}_1 - (k-1)(1+\alpha)$ to derive

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-\tilde{a}_1-(\tilde{b}_1+(k-1)(1+\alpha))\delta_p} + CL^{\tilde{a}_0-\tilde{b}_1-k(1+\alpha)}, \quad (3.2.36)$$

recalling that $\tilde{b}_1 := \tilde{a}_1(1+\delta_p+\ldots+\delta_p^{m_0-1}) + (1+\alpha)$ and $\tilde{a}_1\delta_p^{m_0} \geqslant 1+\alpha$, we obtain that

$$\begin{aligned}
&\tilde{a}_1 + (\tilde{b}_1 + (k-1)(1+\alpha))\delta_p - [\tilde{b}_1 + k(1+\alpha)]\\
&\quad = \tilde{a}_1 + \tilde{b}_1(\delta_p - 1) + (1+\alpha)((k-1)\delta_p - k)\\
&\quad = \tilde{a}_1 + [\tilde{a}_1(1+\delta_p+\ldots+\delta_p^{m_0-1}) + (1+\alpha)](\delta_p-1) + (1+\alpha)((k-1)\delta_p - k)\\
&\quad = \tilde{a}_1(\tilde{a}_1 + \tilde{a}_1\delta_p + \ldots + \tilde{a}_1\delta_p^{m_0-1})(\delta_p-1) + (1+\alpha)((k-1)\delta_p - k)\\
&\quad = \tilde{a}_1 + \tilde{a}_1(\delta_p^{m_0} - 1) + (1+\alpha)(k\delta_p - (k+1))\\
&\quad > \tilde{a}_1\delta_p^{m_0} - (1+\alpha) \geqslant 0
\end{aligned}$$

the second term in the the right-hand side of (3.2.36) is larger, so that $k = n$, we arrive at

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\tilde{a}_0-\tilde{b}_1-n(1+\alpha)}. \tag{3.2.37}$$

Since $\alpha > -1$, there exists n sufficiently large such that $\tilde{a}_0 - \tilde{b}_1 - n(1+\alpha) < 0$, which implies that $\Theta \equiv 0$ and concludes the proof of this case.

Case 3:

Finally, in the case $\beta - 1 + \frac{2-\gamma}{r} \leqslant \alpha \leqslant \beta - 1 + \frac{2}{p}$, let us prove that either $\Theta \equiv 0$ in $\mathbb{R}^2$, or $\Theta \not\equiv 0$ and (3.2.2) is satisfied.

We start by supposing that $\Theta \not\equiv 0$ and recalling that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \lesssim L^{2-p(1+\alpha-\beta)} \quad \text{for all } L \gg 1. \tag{3.2.38}$$

Thus, we only must establish that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \gtrsim L^{2-p(1+\alpha-\beta)} \quad \text{for all } L \gg 1. \tag{3.2.39}$$

Next, let us replicate the argument by contradiction argument employed in **Case 3** of Theorem 3.1. By assuming that (3.2.39) does not hold, it follows that there exists a sequence of positive numbers L_i, with $i \in \mathbb{N}$, such that $L_i \to \infty$ as $i \to \infty$ and

$$\frac{1}{L_i^{2-p(1+\alpha-\beta)}} \int_{|y|\leqslant L_i} |\Theta(y)|^p dy \to 0 \quad \text{as } i \to \infty.$$

Setting $l_2 = 4L_i/\rho$, $L := \rho l_1/8 \gg 1$ and later taking $L_i \to \infty$ in (3.2.10), we obtain

$$\frac{1}{L^{2-p(1+\alpha-\beta)}} \int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant C \int_{|y|\geqslant L} \frac{|\widetilde{V}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}} dy + C \int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}} dy. \tag{3.2.40}$$

We now proceed similarly as previously approach. Namely, we fix $q \in (p, r)$ such that condition (3.2.17) holds. Then, by interpolation, and considering both (3.2.38) and assumption (3.2.1), we obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^q dy \leqslant \left(\int_{|y|\leqslant L} |\Theta(y)|^p dy\right)^{\delta} \left(\int_{|y|\leqslant L} |\Theta(y)|^r dy\right)^{1-\delta} \leqslant CL^{(2-p(1+\alpha-\beta))\delta+\gamma(1-\delta)},$$

where $\delta = (r-q)/(r-p)$. Proceeding analogously to (3.2.19) and recalling that $\delta_p :=$

$\delta(p+1)/q > 1$, it follows that

$$\begin{aligned}
&\int_{|y|\geqslant L} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \\
&\quad\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{2-\alpha-p(1+\alpha-\beta)}} \left(\int_{|y|\sim 2^kL} |\Theta(y)|^q dy\right)^{\frac{p}{q}} \left(\int_{|y|\sim 2^kL} |\widetilde{U}^{(1)}(y)|^q dy\right)^{\frac{1}{q}} (2^kL)^{2\left(1-\frac{p+1}{q}\right)} \\
&\quad\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)+\alpha-2} (2^kL)^{\frac{p}{q}((2-p(1+\alpha-\beta))\delta+\gamma(1-\delta))} (2^kL)^{\frac{1}{q}((2-p(1+\alpha-\beta))\delta+\gamma(1-\delta)-q(\alpha+\beta))} \\
&\qquad \times (2^kL)^{2\left(1-\frac{p+1}{q}\right)} \\
&\quad\leqslant C\sum_{k=0}^{\infty} (2^kL)^{(p(1+\alpha-\beta)-2)(1-\delta_p)+\gamma\frac{(p+1)(1-\delta)}{q}-\beta+2\left(1-\frac{p+1}{q}\right)}, \qquad (3.2.41)
\end{aligned}$$

since $\alpha \geqslant \beta - 1 + \frac{2-\gamma}{r}$, $\delta_p > 1$, and $\gamma < \beta r + 2$, we deduce that

$$\begin{aligned}
&(p(1+\alpha-\beta)-2)\,(1-\delta_p) + \gamma\frac{(p+1)(1-\delta)}{q} - \beta + 2\left(1-\frac{p+1}{q}\right) \\
&\quad\leqslant (1-\delta_p)\left(-\gamma\frac{p}{r} + 2\left(\frac{p}{r}-1\right)\right) + \gamma\frac{(p+1)(1-\delta)}{q} - \beta + 2\left(1-\frac{p+1}{q}\right) \\
&\quad= \gamma\left(\frac{(p+1)(1-\delta)}{q} - \frac{p}{r}(1-\delta_p)\right) + 2\left(\frac{p}{r}-1\right)(1-\delta_p) - \beta + 2\left(1-\frac{p+1}{q}\right) \\
&\quad= \gamma\left(\frac{(p+1)(q-p)}{q(r-p)} - \frac{p}{r}\left(1-\frac{(p+1)(r-q)}{q(r-p)}\right)\right) - \beta \\
&\quad+ 2\left[\left(\frac{p}{r}-1\right)\left(1-\frac{(p+1)(r-q)}{q(r-p)}\right) + 1 - \frac{p+1}{q}\right] \\
&\quad= \frac{\gamma}{r} - \frac{2}{r} - \beta < 0,
\end{aligned}$$

where we used that $1-\delta = \frac{q-p}{r-p}$ and $\delta_p = \frac{p+1}{q}\delta = \frac{(p+1)(r-q)}{q(r-p)}$. Plugging this back into (3.2.41), we conclude

$$\int_{|y|\geqslant L} \frac{|\widetilde{U}^{(1)}(y)||\Theta(y)|^p}{|y|^{2-\alpha-p(1+\alpha-\beta)}}dy \leqslant CL^{(p(1+\alpha-\beta)-2)(1-\delta_p)+\gamma\frac{(p+1)(1-\delta)}{q}-\beta+2\left(1-\frac{p+1}{q}\right)}. \qquad (3.2.42)$$

Now, for another term, we have that

$$\begin{aligned}
\int_{|y|\geqslant L} \frac{|\Theta(y)|^p}{|y|^{3+\alpha-p(1+\alpha-\beta)}}dy &\leqslant \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{3+\alpha-p(1+\alpha-\beta)}} \int_{|y|\sim 2^kL} |\Theta(y)|^p dy \\
&\leqslant C\sum_{k=0}^{\infty} (2^kL)^{p(1+\alpha-\beta)-3-\alpha} (2^kL)^{2-p(1+\alpha-\beta)} \\
&\leqslant CL^{-(1+\alpha)}. \qquad (3.2.43)
\end{aligned}$$

Plugging (3.2.42) and (3.2.43) in (3.2.40), we obtain

$$\frac{1}{L^{2-p(1+\alpha-\beta)}} \int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{(p(1+\alpha-\beta)-2)(1-\delta_p)+\gamma\frac{(p+1)(1-\delta)}{q}-\beta+2\left(1-\frac{p+1}{q}\right)} + CL^{-(1+\alpha)},$$

which implies that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{D-\tilde{d}_0} + CL^{D-(1+\alpha)}, \tag{3.2.44}$$

where

$$D := 2 - p(1+\alpha-\beta),$$

and

$$\tilde{d}_0 := D\,(1-\delta_p) - \gamma\frac{(p+1)(1-\delta)}{q} + \beta - 2\left(1 - \frac{p+1}{q}\right) > 0.$$

This estimate will be used repeatedly to improve the upper bound of $\int_{|y|\leqslant L} |\Theta(y)|^p dy$. To present this in a more organized manner, assume that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{\sigma} \quad \text{with } \sigma \leqslant 2 - p(1+\alpha-\beta).$$

Then, based on estimates similar to (3.1.39)-(3.1.41), we obtain from (3.2.10) the following general estimate

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{D-\tilde{d}_0+(\sigma-D)\delta_p} + CL^{\sigma-(1+\alpha)}, \tag{3.2.45}$$

Let us follow the analysis setting $\sigma = D$ in (3.2.45) to recover (3.2.44). Note that, if both power indexes in (3.2.44) are positive, we conclude that $\Theta \equiv 0$ in $\mathbb{R}^2$, which yields a contradiction and finishes the proof. Otherwise, we commence to analyze each power separately.

To start, assume that $\tilde{d}_0 \geqslant 1+\alpha$, then from (3.2.44), it holds

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{D-(1+\alpha)}.$$

Setting $\sigma = D - (1+\alpha)$ in (3.2.45), we obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{D-2(1+\alpha)},$$

Therefore, this process can be iteratively applied n times, as required, for any given $n \in \mathbb{N}$, to obtain

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant CL^{D-n(1+\alpha)},$$

Since $\alpha > -1$, we can find n sufficiently large such that $D - n(1+\alpha) < 0$, which consequently infers that $\Theta \equiv 0$.

If $D - \tilde{d}_0 < 0$, the proof is finished. Otherwise, if $D - \tilde{d}_0 \geqslant 0$, we apply (3.2.45) with $\sigma = D - \tilde{d}_0$ to arrive at

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant \begin{cases} CL^{D-\tilde{d}_0-(1+\alpha)} & \text{if } \delta_p\tilde{d}_0 \geqslant 1+\alpha, \qquad (3.2.46) \\ CL^{D-\tilde{d}_0(1+\delta_p)} & \text{if } \delta_p\tilde{d}_0 < 1+\alpha. \qquad (3.2.47) \end{cases}$$

Then, if necessary, we proceed by taking $m_0 \in \{1, 2, \ldots\}$ the smallest integer such that

$$\delta_p^{m_0} \tilde{d}_0 \geqslant 1 + \alpha.$$

Thus,

$$\delta_p^{m_0 - 1} \tilde{d}_0 < 1 + \alpha.$$

We can iterate this computation $m_0 - 1$ times and obtain that

$$\int_{|y| \leqslant L} |\Theta(y)|^p dy \leqslant C L^{D - \tilde{d}_1},$$

where $\tilde{d}_1 := \tilde{d}_0 (1 + \delta_p + \delta_p^2 + \ldots + \delta_p^{m_0 - 1}) + (1 + \alpha)$. If $D - \tilde{d}_1 < 0$, the proof is finished. Otherwise, we apply (3.2.45) n times, for any given $n \in \mathbb{N}$, where at the k-th iteration we take $\sigma = D - \tilde{d}_1 - (k-1)(1+\alpha)$ and use that

$$\tilde{d}_0 + (\tilde{d}_1 + (k-1)(1+\alpha))\delta_p - [\tilde{d}_1 + k(1+\alpha)] = \tilde{d}_0 \delta_p^{m_0} + (1+\alpha)(k\delta_p - (k+1)) > \tilde{d}_0 \delta_p^{m_0} - (1+\alpha) \geqslant 0,$$

This process leads to

$$\int_{|y| \leqslant L} |\Theta(y)|^p dy \leqslant C L^{D - \tilde{d}_1 - n(1+\alpha)}.$$

Since $\alpha > -1$, there exists n sufficiently large such that $D - \tilde{d}_1 - n(1 + \alpha) < 0$, which implies $\Theta \equiv 0$.

Thus, in each case, we proved that $\Theta \equiv 0$ in $\mathbb{R}^2$, which is a contradiction with our initial assumption that $\Theta \not\equiv 0$ in $\mathbb{R}^2$. This concludes the proof. $\square$

The following result resembles Corollary 3.1, with subtle changes to the profile assumptions, since we no longer have assumptions on $\nabla\Theta$.

Corollary 3.2. *Fix $\beta \in (0, 1]$. Suppose $\theta \in \mathcal{C}([0, T); H^s(\mathbb{R}^2)) \cap L^\infty(0, T; L^1(\mathbb{R}^2))$, with $s > 1 + \beta$, is a locally self-similar solution to the gSQG equation that is locally self-similar in a ball $B_\rho(x_0) \subset \mathbb{R}^2$, with scaling parameter $\alpha > -1$ and profile $\Theta \in \mathcal{C}^\beta(\mathbb{R}^2)$. Then, the following statements hold:*

(i) If there exists some $\sigma > 0$ such that $|\Theta(y)| \lesssim |y|^{-\sigma}$ for all $|y| \gg 1$, then no locally self-similar blowup occurs, i.e., $\Theta \equiv 0$ in $\mathbb{R}^2$.

(ii) If there exists some $\sigma \in (0, \beta)$ such that $1 \lesssim |\Theta(y)| \lesssim |y|^\sigma$ for all $|y| \gg 1$, then the values of α admitting nontrivial profiles belong to $[\beta - 1 - \sigma, \beta - 1]$, and for each such α the corresponding profile Θ satisfies

$$\int_{|y| \leqslant L} |\Theta(y)|^p dy \sim L^{2 - p(1 + \alpha - \beta)},$$

for every $p \in [1, \infty)$ and for all L sufficiently large.

Proof. We follow a similar proof as in Corollary 3.1 and make the appropriate modifications. As such, to prove (i), we first take M to be a positive constant such that $|\Theta(y)| \lesssim |y|^{-\sigma}$ for all $|y| \geqslant M$. Since $\Theta \in \mathcal{C}^\beta(\mathbb{R}^2)$, we obtain that for all $L > 0$ and $r > \frac{2}{\sigma}$

$$\int_{|y|\leqslant L} |\Theta(y)|^r dy \leqslant \int_{|y|\leqslant M} |\Theta(y)|^r dy + \int_{|y|\geqslant M} \frac{1}{|y|^{r\sigma}} dy \leqslant C.$$

Then, denoting $p_1 := \max\left\{1, \frac{2}{\sigma}\right\}$, it follows that the assumptions of Theorem 3.2 are satisfied with $p = p_1$, $r = p_1+1$ and $\gamma = 0$. As a consequence, the values of α admitting nontrivial profiles Θ must belong to the interval $\left[\beta - 1 + \frac{2}{p_1+1}, \beta - 1 + \frac{2}{p_1}\right]$. Moreover, since the assumptions of Theorem 3.2 are also verified with $p = p_1 + k$, $r = p_1 + k + 1$ and $\gamma = 0$, for any $k > 0$, then such α must also belong to $\left[\beta - 1 + \frac{2}{p_1+k+1}, \beta - 1 + \frac{2}{p_1+k}\right]$ for any $k > 0$. Taking $k \geqslant 2$, it follows that $\alpha \in \left[\beta - 1 + \frac{2}{p_1+k+1}, \beta - 1 + \frac{2}{p_1+k}\right] \cap \left[\beta - 1 + \frac{2}{p_1+1}, \beta - 1 + \frac{2}{p_1}\right] = \varnothing$, and we deduce that $\Theta \equiv 0$ in $\mathbb{R}^2$, as desired.

Regarding item (ii), let us now consider $M > 0$ such that $|\Theta(y)| \lesssim |y|^\sigma$ for all $|y| \geqslant M$, and fix an arbitrary $p \in [1, \infty)$ and $r > p$. Similarly as in (3.1.69), we have that for $L \gg 1$

$$\int_{|y|\leqslant L} |\Theta(y)|^r dy \leqslant \int_{|y|\leqslant M} |\Theta(y)|^r dy + \int_{M\leqslant|y|\leqslant L} |y|^{r\sigma} dy \leqslant CL^{\sigma r+2}, \tag{3.2.48}$$

where we again used the fact that Θ is a continuous function in $\mathbb{R}^2$ to bound the first integral.

Setting $\gamma = \sigma r + 2$ and recalling that $\sigma \in (0, \beta)$, we have from Theorem 3.2 that α may admit nontrivial profiles belongs to the interval $\left[\beta - 1 - \sigma, \beta + \frac{2}{p} - 1\right]$ and the corresponding profile satisfies

$$C_1 L^{2-p(1+\alpha-\beta)} \leqslant \int_{|y|\leqslant L} |\Theta(y)|^p dy \leqslant C_2 L^{2-p(1+\alpha-\beta)}, \text{ for all } L \gg 1, \tag{3.2.49}$$

for some positive constant C_1, C_2. On the other hand, since $|\theta(y)| \gtrsim 1$ for $|y| \gtrsim 1$, we obtain that

$$\int_{|y|\leqslant L} |\Theta(y)|^p dy \geqslant CL^2, \text{ for } L \gg 1. \tag{3.2.50}$$

Combining the upper bound (3.2.49) with (3.2.50), we must have that $1 + \alpha - \beta < 0$, which implies that the values of α admitting nontrivial profiles belongs to the interval $[\beta - 1 - \sigma, \beta - 1]$. Therefore, we conclude the proof. $\square$

4 Global self-similar solutions of the dissipative gSQG equation

In this chapter, we focus on studying the globally self-similar solutions of the fractionally dissipative gSQG equation. Our main result, presented in Theorem 4.1, extends the result proved by Chae in [13, Theorem 3.1] to cover all $\beta \in (0,2)$, beyond the specific case of $\beta = 1$. Theorem 4.1 states that, under the assumptions that the gradient of the self-similar profile decays to zero at infinity and the symmetric part of the self-similar velocity gradient is bounded at the points of maximum gradient of the self-similar profile, any possibility of a globally self-similar blowup in finite time is excluded.

We recall that the fractionally dissipative generalized surface quasi-geostrophic equation in $\mathbb{R}^2$ is given by

$$\begin{cases} \theta_t + \mathbf{u}\cdot\nabla\theta + \kappa\Lambda^\eta\theta = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \mathbf{u} = -\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta, & x \in \mathbb{R}^2,\ t > 0, \end{cases} \tag{4.1}$$

where $\eta \in (0,2]$ and $\beta \in (0,2)$ are given parameters, $\kappa > 0$, $\theta(x,t)$ is an unknown scalar function, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t)) \in \mathbb{R}^2$ denotes a velocity field, and $(-\Delta)^{-s/2}$, $0 < s < 2$, is the Riesz potential, and Λ^η is the fractional Laplacian.

Now, let us recall the globally self-similar solutions of dissipative gSQG equation, as follows:

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{\eta-\beta}{\eta}}}\Theta\left(\frac{x-x_0}{(T-t)^{\frac{1}{\eta}}}\right), \quad (x,t) \in \mathbb{R}^2 \times [0,T), \tag{4.2}$$

for some function $\Theta : \mathbb{R}^2 \to \mathbb{R}$, which is called an associated *self-similar profile*.

Solutions that are globally self-similar preserve their self-similar characteristics throughout the entire spatial domain. This property facilitates the analysis of the non-local operators present in the equation (4.1), such as the Fractional Laplacian and the Riesz operator. Additionally, this particularity enables us to deduce the equation for the profile Θ, by invoking the globally self-similarity assumption of θ in (4.1).

To obtain the equation for the profile, Θ, let us analyze separately each term in (4.1). Let us commence with the Fractional Laplacian and, afterward, the velocity field $\mathbf{u}$. Without loss of generality, assume that $x_0 = 0$. Then, we have that

$$
\begin{aligned}
&\Lambda^\eta\theta(x,t)\\
&= d_\eta \mathrm{P.V.}\int_{\mathbb{R}^2}\frac{\theta(x,t)-\theta(y,t)}{|x-y|^{2+\eta}}dy\\
&= d_\eta (T-t)^{\frac{\beta-\eta}{\eta}}\mathrm{P.V.}\int_{\mathbb{R}^2}\left[\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)-\Theta\left(\frac{y}{(T-t)^{\frac{1}{\eta}}}\right)\right]|x-y|^{-(2+\eta)}dy\\
&= d_\eta (T-t)^{\frac{\beta-2\eta-2}{\eta}}\mathrm{P.V.}\int_{\mathbb{R}^2}\left[\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)-\Theta\left(\frac{y}{(T-t)^{\frac{1}{\eta}}}\right)\right]\frac{|x-y|^{2+\eta}}{(T-t)^{\frac{2+\eta}{\eta}}}dy\\
&= d_\eta (T-t)^{\frac{\beta-\eta}{\eta}-1}\mathrm{P.V.}\int_{\mathbb{R}^2}\left[\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)-\Theta(z)\right]\left|\frac{x}{(T-t)^{\frac{1}{\eta}}}-z\right|^{-(2+\eta)}dz\\
&= (T-t)^{\frac{\beta-\eta}{\eta}-1}\Lambda^\eta\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right).
\end{aligned}
\tag{4.3}
$$

Now, invoking (2.2.12), (4.2) and employing the same computation above, we have that the velocity field can be expressed as

$$
\begin{aligned}
u(x,t) &= C_\beta \mathrm{P.V.}\int_{\mathbb{R}^2}K_\beta(x-y)\theta(y,t)dy\\
&= C_\beta (T-t)^{\frac{\beta-\eta}{\eta}}\mathrm{P.V.}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^{2+\beta}}\Theta\left(\frac{y}{(T-t)^{\frac{1}{\eta}}}\right)dy\\
&= C_\beta (T-t)^{\frac{1-\eta}{\eta}}\mathrm{P.V.}\int_{\mathbb{R}^2}\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}-z\right)^\perp\left|\frac{x}{(T-t)^{\frac{1}{\eta}}}-z\right|^{-(2+\beta)}\Theta(z)dz\\
&= (T-t)^{\frac{1-\eta}{\eta}}U\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right),
\end{aligned}
\tag{4.4}
$$

where $U(x) = -\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\Theta(x)$. Finally, invoking again (4.2), we obtain that

$$
\begin{aligned}
\partial_t\theta(x,t) = {}& \frac{(\eta-\beta)}{\eta}(T-t)^{\frac{\beta-\eta}{\eta}-1}\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)\\
&+\frac{1}{\eta}(T-t)^{\frac{\beta-2\eta-1}{\eta}}\nabla\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)\cdot\frac{x}{(T-t)^{\frac{1}{\eta}}},
\end{aligned}
$$

and

$$
\nabla\theta(x,t) = (T-t)^{\frac{\beta-1-\eta}{\eta}}\nabla\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right). \tag{4.5}
$$

Plugging (4.3) - (4.5) into (4.1), it follows that Θ satifies

$$
\begin{aligned}
&\frac{(\eta-\beta)}{\eta}(T-t)^{\frac{\beta-\eta}{\eta}-1}\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)+\frac{1}{\eta}(T-t)^{\frac{\beta-\eta}{\eta}-1}\frac{x}{(T-t)^{\frac{1}{\eta}}}\cdot\nabla\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)\\
&+(T-t)^{\frac{\beta-\eta}{\eta}-1}(U\cdot\nabla\Theta)\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)+(T-t)^{\frac{\beta-\eta}{\eta}-1}\Lambda^\eta\Theta\left(\frac{x}{(T-t)^{\frac{1}{\eta}}}\right)=0,
\end{aligned}
$$

setting $y = x(T-t)^{-\frac{1}{\eta}}$, we derive the equation describing the profile Θ as follows:

$$\frac{(\eta-\beta)}{\eta}\Theta(y) + \frac{1}{\eta} y\cdot\nabla\Theta(y) + (U\cdot\nabla)\Theta(y) + \Lambda^\eta\Theta(y) = 0, \quad \text{for all } y\in\mathbb{R}^2, \tag{4.6}$$

where $U(y) = -\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\Theta(y)$ is the velocity field associate with the self-similar profile Θ.

Next, let us see the following technical lemma:

Lemma 4.1. *Let $W:\mathbb{R}^2\to\mathbb{R}^2$ and $f:\mathbb{R}^2\to\mathbb{R}$ be differentiable functions. If $\nabla\cdot W = 0$, then it holds*

$$\nabla^\perp((W\cdot\nabla)f)\cdot\nabla^\perp f = \frac{1}{2}(W\cdot\nabla)|\nabla^\perp f|^2 - \nabla^\perp f\cdot[S_W\nabla^\perp f],$$

where $S_W = \frac{\nabla W + (\nabla W)^t}{2}$ is the symmetric part of the velocity gradient matrix ∇W.

Proof. Let us start the proof by computing the term on the left-hand side. Then,

$$\begin{aligned}
\nabla^\perp((W\cdot\nabla)f(y)) &= (-\partial_2(W_1\partial_1 f + W_2\partial_2 f), \partial_1(W_1\partial_1 f + W_2\partial_2 f))\\
&= (-W_1\partial_2\partial_1 f - W_2\partial^2_{22} f, W_1\partial^2_{11} f + W_2\partial_1\partial_2 f)\\
&+ (-\partial_2 W_1\partial_1 f - \partial_2 W_2\partial_2 f, \partial_1 W_1\partial_1 f + \partial_1 W_2\partial_2 f)\\
&= (W\cdot\nabla)\nabla^\perp f + (-\partial_2 W_1\partial_1 f - \partial_2 W_2\partial_2 f, \partial_1 W_1\partial_1 f + \partial_1 W_2\partial_2 f)\\
&= (W\cdot\nabla)\nabla^\perp f + (-\partial_2 W_1\partial_1 f - \partial_2 W_2\partial_2 f, -\partial_2 W_2\partial_1 f + \partial_1 W_2\partial_2 f),
\end{aligned}$$

where in the last inequality, we employed the fact that div $W = 0$, which leads to $\partial_1 W_1 = -\partial_2 W_2$. Next, by taking the inner product with $\nabla^\perp f$, it yields

$$\begin{aligned}
&\nabla^\perp((W\cdot\nabla)f(y))\cdot\nabla^\perp f\\
&= \frac{1}{2}(W\cdot\nabla)|\nabla^\perp f|^2 + (-\partial_2 W_1\partial_1 f - \partial_2 W_2\partial_2 f, -\partial_2 W_2\partial_1 f + \partial_1 W_2\partial_2 f)\cdot\nabla^\perp f\\
&= \frac{1}{2}(W\cdot\nabla)|\nabla^\perp f|^2 + \partial_2 W_1\partial_1 f\partial_2 f + \partial_2 W_2\partial_2 f\partial_2 f - \partial_2 W_2\partial_1 f\partial_1 f + \partial_1 W_2\partial_2 f\partial_1 f\\
&= \frac{1}{2}(W\cdot\nabla)|\nabla^\perp f|^2 + \partial_2 W_2(\partial_2 f\partial_2 f - \partial_1 f\partial_1 f) + \partial_2 W_1\partial_1 f\partial_2 f + \partial_1 W_2\partial_1 f\partial_2 f.
\end{aligned}\tag{4.7}$$

On the other hand, we have that

$$S_W(x) = \frac{\nabla W + (\nabla W)^t}{2} = \frac{1}{2}\begin{bmatrix} 2\partial_1 W_1(x) & \partial_2 W_1(x) + \partial_1 W_2(x)\\ \partial_1 W_2(x) + \partial_2 W_1(x) & 2\partial_2 W_2(x)\end{bmatrix}.$$

Then,

$$\begin{aligned}
S_W\nabla^\perp f &= \frac{1}{2}\begin{bmatrix} 2\partial_1 W_1 & \partial_2 W_1 + \partial_1 W_2\\ \partial_1 W_2 + \partial_2 W_1 & 2\partial_2 W_2\end{bmatrix}\cdot\begin{bmatrix}-\partial_2 f\\ \partial_1 f\end{bmatrix}\\
&= \frac{1}{2}(-2\partial_1 W_1\partial_2 f + \partial_2 W_1\partial_1 f + \partial_1 W_2\partial_1 f, -\partial_1 W_2\partial_2 f - \partial_2 W_1\partial_2 f + 2\partial_2 W_2\partial_1 f)\\
&= \frac{1}{2}(\partial_2 W_2\partial_2 f + \partial_2 W_1\partial_1 f, -\partial_2 W_1\partial_2 f + \partial_2 W_2\partial_1 f)\\
&+ \frac{1}{2}(\partial_2 W_2\partial_2 f + \partial_1 W_2\partial_1 f, -\partial_1 W_2\partial_2 f + \partial_2 W_2\partial_1 f),
\end{aligned}$$

where in the last inequality we used that $\partial_1 W_1 = -\partial_2 W_2$. Now, taking the inner product with $\nabla^\perp f$, we can conclude that

$$\begin{aligned}\nabla^\perp f \cdot S_W \nabla^\perp f &= \frac{1}{2}(\partial_2 W_2 \partial_2 f + \partial_2 W_1 \partial_1 f, -\partial_2 W_1 \partial_2 f + \partial_2 W_2 \partial_1 f) \cdot \nabla^\perp f \\ &+ \frac{1}{2}(\partial_2 W_2 \partial_2 f + \partial_1 W_2 \partial_1 f, -\partial_1 W_2 \partial_2 f + \partial_2 W_2 \partial_1 f) \cdot \nabla^\perp f \\ &= \partial_2 W_2(\partial_1 f \partial_1 f - \partial_2 f \partial_2 f) - \partial_2 W_1 \partial_1 f \partial_2 f - \partial_1 W_2 \partial_1 f \partial_2 f.\end{aligned}$$

We conclude the result by plugging this back into (4.7). □

Next, we establish a straightforward result concerning the positivity of the inner product between a vector field and its fractional Laplacian at the point of maximum of the norm of the vector field.

Lemma 4.2. *Let $\eta \in (0,2)$ and $f \in H^{1+\eta}(\mathbb{R}^2)$. Then, for all*

$$\tilde{y} \in \mathcal{M} := \{y \in \mathbb{R}^2 \ / \ |\nabla^\perp f(y)| = \sup_{x \in \mathbb{R}^2} |\nabla^\perp f(x)|\},$$

we have that

$$\nabla^\perp f(\tilde{y}) \cdot \Lambda^\eta(\nabla^\perp f(\tilde{y})) \geqslant 0. \tag{4.8}$$

Proof. Let us start by recalling the definition of the fractional Laplacian, which is given by

$$\Lambda^\eta(\nabla^\perp f(\tilde{y})) = P.V. \int_{\mathbb{R}^2} \frac{\nabla^\perp f(\tilde{y}) - \nabla^\perp f(z)}{|\tilde{y} - z|^{2+\eta}} \, dz.$$

Taking the inner product with $\nabla^\perp f(\tilde{y})$, we obtain

$$\begin{aligned}\nabla^\perp f(\tilde{y}) \cdot \Lambda^\eta \nabla^\perp f(\tilde{y}) &= P.V. \int_{\mathbb{R}^2} \frac{\nabla^\perp f(\tilde{y}) \cdot \nabla^\perp f(\tilde{y}) - \nabla^\perp f(\tilde{y}) \cdot \nabla^\perp f(z)}{|\tilde{y} - z|^{2+\eta}} \, dz \\ &= P.V. \int_{\mathbb{R}^2} \frac{|\nabla^\perp f(\tilde{y})|^2 - \nabla^\perp f(\tilde{y}) \cdot \nabla^\perp f(z)}{|\tilde{y} - z|^{2+\eta}} \, dz \\ &\geqslant P.V. \int_{\mathbb{R}^2} \frac{|\nabla^\perp f(\tilde{y})|(|\nabla^\perp f(\tilde{y})| - |\nabla^\perp f(z)|)}{|\tilde{y} - z|^{2+\eta}} \, dz \geqslant 0,\end{aligned}$$

where we use that $\tilde{y}$ is a maximum point of the function $|\nabla^\perp f|$. Hence, this concludes the proof. □

The following theorem generalizes the result proven in [13, Theorem 3.1] for $\beta = 1$, for all $\beta \in (0,2)$.

Theorem 4.1. *Fix $\eta, \beta \in (0,2)$. Suppose $\theta \in \mathcal{C}([0,T); H^s(\mathbb{R}^2))$[1], with $s > 1 + \beta - \eta$, is a solution to the gSQG equation (4.1) that is globally self-similar in $\mathbb{R}^2$, with self-similar profile Θ. Assume that $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$ is a solution to the equation (4.6) satisfying*

$$\lim_{|y| \to \infty} |\nabla^\perp \Theta(y)| = 0. \tag{4.9}$$

[1] According to Theorem 2.7, we can consider the solution θ of the dissipative gSQG equation within the class $\mathcal{C}([0,T), H^s(\mathbb{R}^2))$ with $s > 1 + \beta - \eta$.

Furthermore, suppose that there exists $\tilde{y} \in \mathcal{M} := \{y \in \mathbb{R}^2 \ / \ |\nabla^\perp \Theta(y)| = \sup_{x \in \mathbb{R}^2} |\nabla^\perp \Theta(x)|\}$ *such that*

$$B(\tilde{y}) < \frac{\eta - \beta + 1}{\eta}. \tag{4.10}$$

$$B(y) = \begin{cases} \frac{\nabla^\perp \Theta(y)}{|\nabla^\perp \Theta(y)|} \cdot S_U(y) \frac{\nabla^\perp \Theta(y)}{|\nabla^\perp \Theta(y)|}, & \text{if } \nabla^\perp \Theta(y) \neq 0 \\ 0, & \text{if } \nabla^\perp \Theta(y) = 0, \end{cases} \tag{4.11}$$

where U is the velocity field associated with Θ and $S_U = \frac{\nabla U + (\nabla U)^t}{2}$ is the symmetric part of the velocity gradient matrix ∇U. Under these conditions, Θ is identically null.

Proof. This proof employs the technique developed by Chae in [13, Teorema 3.1] to exclude the globally self-similar solution of the dissipative SQG equation.

Let us commence the proof by revisiting the equation for the profile, given by

$$\frac{(\eta - \beta)}{\eta} \Theta(y) + \frac{1}{\eta}(y \cdot \nabla \Theta(y)) + (U \cdot \nabla)\Theta(y) + \Lambda^\eta \Theta(y) = 0,$$

Applying $\nabla^\perp$ to the equation above and then taking the inner product with $\nabla^\perp \Theta$, we obtain

$$\begin{aligned} \frac{(\eta - \beta)}{\eta} |\nabla^\perp \Theta(y)|^2 + \frac{1}{\eta} \nabla^\perp (y \cdot \nabla \Theta(y)) \cdot \nabla^\perp \Theta + \nabla^\perp ((U \cdot \nabla)\Theta(y)) \cdot \nabla^\perp \Theta \\ + \nabla^\perp \Theta \cdot \Lambda^\eta \nabla^\perp \Theta(y) = 0. \end{aligned} \tag{4.12}$$

Let us now analyze each term separately, beginning with the second term. Hence,

$$\begin{aligned} \nabla^\perp (y \cdot \nabla \Theta(y)) &= \nabla^\perp (y_1 \partial_1 \Theta + y_2 \partial_2 \Theta) \\ &= (-y_1 \partial_2 \partial_1 \Theta - \partial_2 \Theta - y_2 \partial^2_{22} \Theta, \partial_1 \Theta + y_1 \partial^2_{11} \Theta + y_2 \partial_1 \partial_2 \Theta) \\ &= \nabla^\perp \Theta + (-y_1 \partial_2 \partial_1 \Theta - y_2 \partial^2_{22} \Theta, y_1 \partial^2_{11} \Theta + y_2 \partial_1 \partial_2 \Theta) \\ &= \nabla^\perp \Theta + (y \cdot \nabla) \nabla^\perp \Theta. \end{aligned}$$

Then, we deduce that

$$\nabla^\perp (y \cdot \nabla \Theta(y)) \cdot \nabla^\perp \Theta = |\nabla^\perp \Theta|^2 + \frac{1}{2}(y \cdot \nabla)|\nabla^\perp \Theta|^2 \tag{4.13}$$

For the third term in (4.12), we invoke the Lemma (4.1) with $f = \Theta$ and $W = U$, we obtain that

$$\nabla^\perp ((U \cdot \nabla)\Theta(y)) \cdot \nabla^\perp \Theta = \frac{1}{2}(U \cdot \nabla)|\nabla^\perp \Theta|^2 - \nabla^\perp \Theta \cdot S_U \nabla^\perp \Theta. \tag{4.14}$$

Therefore, plugging (4.13) and (4.14) into (4.12), we obtain

$$\begin{aligned} &\frac{(\eta - \beta)}{\eta} |\nabla^\perp \Theta(y)|^2 + \frac{1}{\eta} |\nabla \Theta(y)|^2 + \frac{1}{2\eta}(y \cdot \nabla)|\nabla^\perp \Theta|^2 + \frac{1}{2}(U \cdot \nabla)|\nabla^\perp \Theta|^2 \\ &- \nabla^\perp \Theta(y) \cdot S_U \nabla^\perp \Theta(y) + \nabla^\perp \Theta(y) \cdot \Lambda^\eta (\nabla^\perp \Theta)(y) = 0. \end{aligned} \tag{4.15}$$

Here, we highlight that in [13, Theorem 3.1], the fourth term on the right-hand side in (4.15) is missing. Afterward, the correct equation was obtained. We point out this detail to avoid any confusion that may arise if readers directly compare the results presented here with those in [13, Teorema 3.1].

Now, using (4.11), the equality above can be reformulated as

$$\frac{(\eta-\beta+1)}{\eta}|\nabla^{\perp}\Theta(y)|^2+\frac{1}{2\eta}(y\cdot\nabla)|\nabla^{\perp}\Theta(y)|^2+\frac{1}{2}(U\cdot\nabla)|\nabla^{\perp}\Theta(y)|^2-|\nabla^{\perp}\Theta|^2B(y)$$
$$+\nabla^{\perp}\Theta(y)\cdot\Lambda^{\eta}(\nabla^{\perp}\Theta)(y)=0. \tag{4.16}$$

Evaluating (4.16) at $\tilde{y}\in\mathcal{M}$, we obtain

$$|\nabla^{\perp}\Theta(\tilde{y})|^2\left(\frac{\eta-\beta+1}{\eta}-B(\tilde{y})\right)=-\nabla^{\perp}\Theta(\tilde{y})\cdot\Lambda^{\eta}(\nabla^{\perp}\Theta)(\tilde{y}). \tag{4.17}$$

Invoking Lemma (4.2) with $f=\Theta$, we have that

$$\nabla^{\perp}\Theta(\tilde{y})\cdot\Lambda^{\eta}(\nabla^{\perp}\Theta)(\tilde{y})\geqslant 0.$$

Plugging this inequality back into (4.17) and recalling that $B(\tilde{y})<\frac{\eta-\beta+1}{\eta}$, it follows that $|\nabla^{\perp}\Theta(\tilde{y})|^2=0$ and consequently $|\nabla^{\perp}\Theta(y)|=0$ for all $y\in\mathbb{R}^2$. Thus, we conclude that Θ is constant.

Now, given that θ is a globally self-similar solution, as expressed in (4.2), we can deduce that θ is spatially constant. More precisely, there is $c\in\mathbb{R}$ such that $\theta(t)=c(T-t)^{\frac{\eta-\beta}{\eta}}$, for all $t\in[0,T)$. On the other hand, since $\theta(t)\in L^2(\mathbb{R}^2)$, for $t\in[0,T)$, we conclude that $c\equiv 0$, and hence, we conclude the proof.

□

5 Conclusion

This thesis presented significant results regarding the locally self-similar solutions of the gSQG equation and the globally self-similar solutions of the dissipative gSQG equation. In Chapter 3, we conducted a rigorous study concerning locally self-similar solutions of the gSQG equation, proving that under growth conditions on the profile and its gradient, it is possible to identify ranges of the scaling parameter where the profile is identically zero, or its L^p asymptotic behavior is characterized, for suitable r, p.

One direction to advance this research involves refining the estimates for the velocity field in the self-similar region, notably when $1 < \beta < 2$, where is required the assumption of growth condition on the profile and its gradient. This approach could involve exploring new methods for estimating the function $\widetilde{V}^{(1)}$, as defined in (3.1.19) and established by Lemma 6.2, without resorting the assumption of the Theorem 3.1 that could yield weaker conditions on the profile as presented in Theorem 3.1, and might also enable us to identify a larger range of the parameter scaling α, where the profile remains identically null, thereby avoiding self-similar blowup.

In Chapter 4, our focus shifted to the globally self-similar solutions of the dissipative gSQG equation for η and β within the range $(0, 2)$. Assuming that the gradient of the self-similar profile decays to zero at infinity and the symmetric part of the self-similar velocity gradient is bounded at the points of the maximum gradient of the self-similar profile, we established in Theorem 4.1 that the profile is identically null in $\mathbb{R}^2$, excluding any possibility of globally self-similar blowup. Looking forward, our future research aims to develop new techniques that can achieve similar results under weaker conditions for the self-similar profile. This includes exploring the equation for the profile Θ, given in (4.6), to obtain further insights into the profile's behavior.

We highlight that, to date, no results have been found in the literature regarding the non-existence of locally self-similar blow-up for the dissipative gSQG equation, nor on the asymptotic behavior of the profile in L^p. Therefore, this field presents numerous significant challenges, offering ample opportunities to advance our understanding of self-similar blowup to the dissipative gSQG equation.

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6 Appendix

This appendix presents the proofs of Lemma 6.2 and Lemma 6.3, as well the proof of the local L^p equality shown in (6.2.1).

We highlight that in other studies that employed the local L^p equality (6.2.1), such as in [9] and [76], the proof of this result is not provided. The case when $p \geqslant 2$ is classical and follows from standard approximate arguments. On the other hand, when $1 \leqslant p < 2$, a suitable adjustment is necessary, see Section 6.2 for details.

6.1 Auxiliary Lemmas

This section is dedicated to proving the auxiliary lemmas used in this work. Let us start with Lemma 6.1, invoked in the proof of Corollary 3.1, which establishes a growth bound for the profile based on the growth bound for its gradient.

Lemma 6.1. *Let $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$ and $1 \leqslant q < \infty$. Suppose that there is $\sigma > 0$ such that*

$$\left(\int_{|y|\leqslant M} |\nabla^{\perp}\Theta(y)|^q dy\right)^{\frac{1}{q}} \leqslant M^{\sigma}, \tag{6.1.1}$$

for some $M > 1$. Then, it holds

$$\left(\int_{|y|\leqslant M} |\Theta(y)|^q dy\right)^{\frac{1}{q}} \leqslant CM^{1+\frac{\sigma}{q}} \tag{6.1.2}$$

Proof. We start by pointing out that since $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$, it follows from Poincaré inequality that

$$||\Theta||_{L^p(B_r(0))} \leqslant C||\nabla\Theta||_{L^p(B_r(0))} + \frac{1}{r}\int_{B_r(0)} \Theta(y)dy < C + ||\nabla\Theta||_{L^p(B_r(0))},$$

where $B_r(0) \subset \mathbb{R}^n$ denotes the ball of radius r centered at the origin and $C := \int_{B_r(0)} \Theta(y)dy$. Thus, by applying change of variables and invoking the above inequality, we have

$$\begin{aligned}
\left(\int_{|y|\leqslant M} |\Theta(y)|^q dy\right)^{\frac{1}{q}} &= \left(\int_{|y|\leqslant 1} |M^{\frac{2}{q}}\Theta(My)|^q dy\right)^{\frac{1}{q}} \leqslant \left(\int_{|y|\leqslant 1} |M^{\frac{2}{q}+1}(\nabla\Theta)(My)|^q dy\right)^{\frac{1}{q}} \\
&= C + M^{\frac{2}{q}+1}\left(\int_{|y|\leqslant 1} |(\nabla\Theta)(My)|^q dy\right)^{\frac{1}{q}} \\
&\leqslant C + M^{\frac{2}{q}+1-\frac{2}{q}}\left(\int_{|y|\leqslant M} |\nabla\Theta(y)|^q dy\right)^{\frac{1}{q}} \\
&\leqslant \tilde{C}M^{1+\frac{\sigma}{q}}.
\end{aligned}$$

where we used that $M > 1$, $\square$

Next, let us prove Lemma 6.2 and Lemma 6.3 used in the proof of Theorem 3.1 and Theorem 3.2 respectively. In Lemma 6.2, we establish the upper bound for the function $\widetilde{V}^{(1)}$, as expressed in (3.1.19), which was utilized at multiple instances in the proof of Theorem 3.1 for the case $\beta \in (1,2)$. More precisely, this result shows that the growth assumption (3.1.2), on the L^r norm of the gradient of the profile $\nabla\Theta$, results in an upper bound on the L^r norm of $\widetilde{V}^{(1)}$ over a certain annulus in $\mathbb{R}^2$. Later, Lemma 6.3 presents a similar outcome for the case $\beta \in (0,1]$ regarding $\widetilde{U}^{(1)}$, by relying on the assumption outlined in (3.1.1) on Θ, which was applied in the proof of Theorem 3.2.

Before proceeding, let us recall the definition of the set A_y given in (3.1.17), namely

$$\begin{aligned} A_y &:= \left\{ t \in [t_1, t_2] \,:\, \frac{\rho}{8}\frac{1}{|y|} \leqslant (T-t)^{\frac{1}{1+\alpha}} \leqslant \frac{\rho}{4}\frac{1}{|y|} \right\} \\ &= \left\{ t \in [t_1, t_2] \,:\, T - \frac{(\rho/4)^{1+\alpha}}{|y|^{1+\alpha}} \leqslant t \leqslant T - \frac{(\rho/8)^{1+\alpha}}{|y|^{1+\alpha}} \right\}, \end{aligned} \tag{6.1.3}$$

for fixed $y \in \mathbb{R}^2 \backslash \{0\}$ and $0 < t_1 < t_2 < T$.

Lemma 6.2. *Let $\beta \in (1,2)$ and $\Theta \in \mathcal{C}^1(\mathbb{R}^2)$. Suppose that for some $r \in [1,\infty)$ and $\gamma \in \mathbb{R}$, it holds*

$$\int_{|y|\leqslant L} |\nabla\Theta|^r dy \lesssim L^{\gamma} \quad \textit{for all } L \gg 1. \tag{6.1.4}$$

Then, the function $\widetilde{V}^{(1)}$ defined by

$$\widetilde{V}^{(1)}(y) = \int_{t_1}^{t_2} \left| \int_{\mathbb{R}^2} \frac{1}{|y-z|^{\beta}} \nabla^{\perp}\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz \right| \mathbb{1}_{A_y}(t)dt, \tag{6.1.5}$$

with $0 < t_1 < t_2 < T$ and A_y as given in (6.1.3)*, satisfies the following estimate*

$$\int_{L\leqslant|y|\leqslant 2L} |\widetilde{V}^{(1)}(y)|^r dy \lesssim L^{\gamma + r(1-\alpha-\beta)} \quad \textit{for all } L \gg 1. \tag{6.1.6}$$

Proof. Denote by $\overline{K}_\beta$ the kernel from (6.1.5), i.e.,

$$\overline{K}_\beta(y) = \frac{1}{|y|^{\beta}}, \quad y \in \mathbb{R}^2\backslash\{0\}.$$

From the definitions of $\widetilde{V}^{(1)}$ and A_y, we have

$$\begin{aligned} &\left(\int_{L\leqslant|y|\leqslant 2L} |\widetilde{V}^{(1)}(y)|^r dy \right)^{\frac{1}{r}} \\ &= \left(\int_{L\leqslant|y|\leqslant 2L} \left(\int_{t_1}^{t_2} \left| \int_{\mathbb{R}^2} \overline{K}_\beta(y-z)\nabla^{\perp}\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz \right| \mathbb{1}_{A_y}(t)dt \right)^r dy \right)^{\frac{1}{r}} \\ &\leqslant \left(\int_{L\leqslant|y|\leqslant 2L} \left(\int_{T-\frac{(\rho/4)^{1+\alpha}}{|y|^{1+\alpha}}}^{T-\frac{(\rho/8)^{1+\alpha}}{|y|^{1+\alpha}}} \left| \int_{\mathbb{R}^2} \overline{K}_\beta(y-z)\nabla^{\perp}\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz \right| dt \right)^r dy \right)^{\frac{1}{r}} \\ &= \left(\int_{L\leqslant|y|\leqslant 2L} \left(\int_{T-\frac{(\rho/4)^{1+\alpha}}{|y|^{1+\alpha}}}^{T-\frac{(\rho/8)^{1+\alpha}}{|y|^{1+\alpha}}} \left| \int_{\mathbb{R}^2} \left(\overline{K}_\beta \mathbb{1}_{B_{18L}(0)}\right)(y-z)\nabla^{\perp}\Theta\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz \right| dt \right)^r dy \right)^{\frac{1}{r}}. \end{aligned}$$

As usual, $B_{18L}(0)$ denotes the ball of radius $18L$ centered at the origin, and we utilized the fact that for $|y| \leqslant 2L$, $|z| \leqslant \rho(T-t)^{-\frac{1}{1+\alpha}}$ and $t \leqslant T - (\frac{\rho/8}{|y|})^{1+\alpha}$, it holds that $|y-z| \leqslant 2L + \rho(T-t)^{-\frac{1}{1+\alpha}} \leqslant 2L + 8|y| \leqslant 18L$.

Applying Minkowski and Young's convolution inequality, we obtain

$$\begin{aligned}
&\left(\int_{L\leqslant|y|\leqslant 2L} |\widetilde{V}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} \\
&\leqslant \int_{T-(\frac{\rho/8}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} \left(\int_{L\leqslant|y|\leqslant 2L} \left|\int_{\mathbb{R}^2} \left(\overline{K}_\beta \mathbb{1}_{B_{18L}(0)}\right)(y-z)\nabla^\perp\Theta(z)\phi_\rho((T-t)^{\frac{1}{1+\alpha}}z)dz\right|^r dy\right)^{\frac{1}{r}} dt \\
&\leqslant \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} \left(\int_{\mathbb{R}^2} |[(\overline{K}_\beta \mathbb{1}_{B_{18L}(0)}) * ((\nabla^\perp\Theta\phi_\rho((\cdot)(T-t)^{\frac{1}{1+\alpha}}))](y)|^r dy\right)^{\frac{1}{r}} dt \\
&\leqslant \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} ||\overline{K}_\beta \mathbb{1}_{B_{18L}(0)}||_{L^1(\mathbb{R}^2)} ||\nabla^\perp\Theta\phi_\rho((\cdot)(T-t)^{\frac{1}{1+\alpha}})||_{L^r(\mathbb{R}^2)} dt \\
&\lesssim \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} \left(\int_{|y|\leqslant 18L} \frac{1}{|y|^\beta} dy\right)\left(\int_{|y|\leqslant\rho(T-t)^{-\frac{1}{1+\alpha}}} |\nabla^\perp\Theta(y)|^r dy\right)^{\frac{1}{r}} dt \\
&\lesssim \left(\int_{|y|\leqslant 18L} \frac{1}{|y|^\beta} dy\right) \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} \left(\int_{|y|\leqslant 16L} |\nabla\Theta(y)|^r dy\right)^{\frac{1}{r}} dt \\
&\lesssim L^{2-\beta} \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} \left(\int_{|y|\leqslant 16L} |\nabla\Theta(y)|^r dy\right)^{\frac{1}{r}} dt.
\end{aligned} \tag{6.1.7}$$

Invoking assumption (6.1.4), it follows that for all L sufficiently large

$$\left(\int_{L\leqslant|y|\leqslant 2L} |\widetilde{V}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} \lesssim L^{2-\beta} \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}} L^{\frac{\gamma}{r}} dt \lesssim L^{\frac{\gamma}{r}+1-\alpha-\beta}.$$

Therefore, we conclude (6.1.6). □

Lemma 6.3. *Let $\beta \in (0,1]$ and $\Theta \in \mathcal{C}^\beta(\mathbb{R}^2)$. Suppose that for some $r \in [1,\infty)$ and $\gamma \in \mathbb{R}$, it holds*

$$\int_{|y|\leqslant L} |\Theta(y)|^r dy \lesssim L^\gamma, \quad \text{for all } L \gg 1. \tag{6.1.8}$$

Then, the function $\widetilde{U}^{(1)}$ defined by

$$\widetilde{U}^{(1)}(y) = \int_{t_1}^{t_2} P.V. \int_{\mathbb{R}^2} \left|\frac{(y-z)^\perp}{|y-z|^{2+\beta}}\Theta(z)\phi_\rho(z(T-t)^{\frac{1}{1+\alpha}})dz\right| \mathbb{1}_{A_y}(t)dt \tag{6.1.9}$$

where $0 < t_1 < t_2 < T$ and A_y as given in (6.1.3), satisfies the following estimate

$$\int_{L\leqslant|y|\leqslant 2L} |\widetilde{U}^{(1)}(y)|^r dy \lesssim L^{\gamma - r(\alpha+\beta)}, \quad \text{for all } L \gg 1. \tag{6.1.10}$$

Proof. Let us initiate by considering $\beta \in (0,1)$. It should be noted that in this scenario, the kernel $K_\beta(x) = y^\perp|y|^{-(2+\beta)}$, $y \in \mathbb{R}^2\backslash\{0\}$, is integrable near the origin. Namely,

$$\int_{|y|\lesssim L} |K_\beta(y)|dy \leqslant \int_{|y|\lesssim L} \frac{1}{|y|^{1+\beta}} dy \lesssim L^{1-\beta}. \tag{6.1.11}$$

We may thus apply similar arguments as in the proof of Lemma 6.2 to arrive at

$$
\begin{aligned}
\left(\int_{L\leqslant|y|\leqslant 2L}|\widetilde{U}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} &\leqslant \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}}\left(\int_{\mathbb{R}^2}|[K_\beta*((\Theta\phi_\rho((\cdot)(T-t)^{\frac{1}{1+\alpha}}))](y)|^r dy\right)^{\frac{1}{r}}dt \\
&\leqslant \int_{T-(\frac{\rho/4}{L})^{1+\alpha}}^{T-(\frac{\rho/16}{L})^{1+\alpha}}||K_\beta||_{L^1(\mathbb{R}^2)}||\Theta\phi_\rho((\cdot)(T-t)^{\frac{1}{1+\alpha}})||_{L^r(\mathbb{R}^2)}dt \\
&\leqslant \left(\int_{|y|\leqslant 18L}\frac{1}{|y|^{1+\beta}}dy\right)\int_{T-\left(\frac{\rho/4}{L}\right)^{1+\alpha}}^{T-\left(\frac{\rho/16}{L}\right)^{1+\alpha}}\left(\int_{|y|\leqslant 16L}|\Theta(z)|^r dz\right)^{\frac{1}{r}}dt.
\end{aligned}
$$

Thus, it follows from (6.1.11) and assumption (6.1.8) that

$$
\left(\int_{L\leqslant|y|\leqslant 2L}|\widetilde{U}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} \lesssim L^{1-\beta}\int_{T-\left(\frac{\rho/4}{L}\right)^{1+\alpha}}^{T-\left(\frac{\rho/16}{L}\right)^{1+\alpha}} L^{\frac{\gamma}{r}}dt \lesssim L^{\frac{\gamma}{r}-\alpha-\beta},
$$

as desired.

Regarding the case $\beta = 1$, the proof is detailed in [76, Lemma 2.2], and it presents nuanced differences from the preceding case, primarily because (6.1.11) no longer holds. However, in this specific situation, the kernel K_β is a Calderón-Zygmund operator (see Section 2.2.1), and hence $\|K_\beta * f\|_{L^q(\mathbb{R}^2)} \lesssim \|f\|_{L^q(\mathbb{R}^2)}$ for any $f \in L^q(\mathbb{R}^2)$ and $1 < q < \infty$. By replacing the use of Young's convolution inequality in (6.1.7) by Theorem 2.5, we can derive

$$
\begin{aligned}
&\left(\int_{L\leqslant|y|\leqslant 2L}|\widetilde{U}^{(1)}(y)|^r dy\right)^{\frac{1}{r}} \\
&\leqslant \int_{T-\left(\frac{\rho/4}{L}\right)^{1+\alpha}}^{T-\left(\frac{\rho/16}{L}\right)^{1+\alpha}}\left(\int_{L\leqslant|y|\leqslant 2L}\left|P.V.\int_{\mathbb{R}^2}K_\beta^{\perp}(y-z)\Theta(z)\phi(z(T-t)^{\frac{1}{1+\alpha}})dz\right|^r dy\right)^{\frac{1}{r}}dt \\
&\lesssim \int_{T-\left(\frac{\rho/4}{L}\right)^{1+\alpha}}^{T-\left(\frac{\rho/16}{L}\right)^{1+\alpha}}\left(\int_{\mathbb{R}^2}|\Theta(z)\phi(z(T-t)^{\frac{1}{1+\alpha}})dz|^r\right)^{\frac{1}{r}}dt \\
&\lesssim L^{-(1+\alpha)}\left(\int_{|z|\leqslant 16L}|\Theta(z)|^r dz\right)^{\frac{1}{r}} \\
&\lesssim L^{\frac{\gamma}{r}-(1+\alpha)}.
\end{aligned} \tag{6.1.12}
$$

This concludes the proof.

□

6.2 Proof of the local L^p inequality

The goal of this section is to prove the local L^p equality, which is fundamental to the proofs of Theorem 3.1 and Theorem 3.2. More precisely, if $\theta \in \mathcal{C}([0,T); H^s(\mathbb{R}^2))$ with $s > 1 + \beta$, is a solution of gSQG equation (1.0.1), then for every $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times [0,T))$, it holds

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\theta(x,t_2)|^p \eta(x,t_2)dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \eta(x,t_1)dx \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(x,t)|^p \partial_t \eta(x,t)dxdt + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(x,t)|^p (\mathbf{u}\cdot\nabla)\eta(x,t)dxdt, \qquad (6.2.1)
\end{aligned}
$$

where $0 \leqslant t_1 < t_2, T$, $p \in [1,\infty)$. The proof of (6.2.1) will be divided into two cases, each corresponding to $p \in [1,2)$ and $p \in [2,\infty)$ due to the approach employed. This method involves multiplying the equation (1.0.1) with $\kappa = 0$, by $\theta_\varepsilon|\theta_\varepsilon|^{p-2}$, with $\theta_\varepsilon = \rho_\varepsilon * \theta$ where ρ_ε represents the standard mollifier. Thus, when $p \in [1,2]$, we need to modify our approach to avoid division by zero, as there may be values of $x \in \mathbb{R}^2$ where $\theta_\varepsilon(x) = 0$.

Before proceeding with the proof, let us recall that

$$
H^s(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2), \quad \text{if} \quad s \geqslant 1 - \frac{2}{p} \ \text{ and } \ p \geqslant 2, \qquad (6.2.2)
$$

and

$$
H^s(\mathbb{R}^2) \hookrightarrow \mathcal{C}_0^k(\mathbb{R}^2), \qquad \text{if } s > k+1. \qquad (6.2.3)
$$

Thus, since $s > 1+\beta$, it follows that $\theta(t) \in L^p(\mathbb{R}^2)$ for all $p \in [2,\infty)$ and $\theta(t) \in \mathcal{C}_0(\mathbb{R}^2)$ for all $t \in (0,T)$. Now, observe that since θ is a solution of the gSQG equation, then

$$
\partial_t \theta^\varepsilon + (u^\varepsilon \cdot \nabla)\theta^\varepsilon = (u^\varepsilon \cdot \nabla)\theta^\varepsilon - \rho_\varepsilon * [(u\cdot\nabla)\theta]. \qquad (6.2.4)
$$

Next, let us commence with the proof of (6.2.1).

Case 1: $2 \leqslant p < \infty$.

Let us start by multiplying (6.2.4) by $\theta^\varepsilon|\theta^\varepsilon|^{p-2}\ \eta$, which is well-defined since $p \geqslant 2$. Subsequently, integrating over the spatial variable and applying integration by parts, we obtain

$$
\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^2} \eta(x,t)\partial_t|\theta^\varepsilon(x,t)|^p dx - \frac{1}{p}\int_{\mathbb{R}^2} |\theta^\varepsilon(x,t)|^p (u^\varepsilon\cdot\nabla)\eta(x,t)dx \\
&= \int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta(x,t)(u^\varepsilon\cdot\nabla)\theta^\varepsilon dx - \int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta(x,t)[\rho_\varepsilon * (u\cdot\nabla)\theta]dx. \qquad (6.2.5)
\end{aligned}
$$

where we used in the second term on the left-hand side that $\nabla\cdot\mathbf{u} = 0$. Now, by integrating on time $t_1, t_2 \in [0,T)$ and later applying integration by parts to the initial term on the left-hand side, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\theta^\varepsilon(x,t_2)|^p \eta(x,t_2)dx - \int_{\mathbb{R}^2} |\theta^\varepsilon(x,t_1)|^p \eta(x,t_1)dx \\
&- \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta^\varepsilon(x,t)|^p \partial_t\eta(x,t)dx - \int_{\mathbb{R}^2} |\theta^\varepsilon(x,t)|^p (u^\varepsilon\cdot\nabla)\eta(x,t)dx \\
&= p\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}(u^\varepsilon\cdot\nabla)\theta^\varepsilon dxdt - p\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta\rho_\varepsilon * ((u\cdot\nabla)\theta dxdt \qquad (6.2.6)
\end{aligned}
$$

We proceed to analyze each of these terms separately. Focusing on the first two and third terms on the left-hand side of (6.2.6), and recalling that $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times [0,T))$ and $\theta^\varepsilon \to \theta$ in L^p for all $p \in [2,\infty)$, we deduced that for all $t \in [0,t)$

$$\begin{aligned} \int_{\mathbb{R}^2} |\theta^\varepsilon(x,t_2)|^p \eta(x,t_2)dx &\longrightarrow \int_{\mathbb{R}^2} |\theta(x,t_2)|^p \eta(x,t_2)dx \\ \int_{\mathbb{R}^2} |\theta^\varepsilon(x,t_1)|^p \eta(x,t_1)dx &\longrightarrow \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \eta(x,t_1)dx, \end{aligned} \tag{6.2.7}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta^\varepsilon(x,t)|^p \partial_t \eta(x,t)dxdt \longrightarrow \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta(x,t)|^p \partial_t \eta(x,t)dxdt. \tag{6.2.8}$$

For the last term on the left-hand side of (6.2.6), we start by recalling that since $\theta \in \mathcal{C}([0,T); H^s(\mathbb{R}^2))$ with $s > 1+\beta$, it follows that

$$\theta^\varepsilon(t) \to \theta(t) \text{ in } H^s(\mathbb{R}^2), \text{ as } \varepsilon \to 0, \ \forall\, t \in [0,T)$$

Moreover, since $s > 1+\beta$, it follows from Theorem 2.4 that

$$\|u\|_{L^2(\mathbb{R}^2)} = \|-\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta\|_{L^2(\mathbb{R}^2)} = \||\xi|^{\frac{\beta}{2}}\mathcal{F}(\theta)\|_{L^2(\mathbb{R}^2)} = ||\Lambda^{\frac{\beta}{2}}\theta||_{L^2(\mathbb{R}^2)} \leqslant \|\theta\|_{H^s(\mathbb{R}^2)},$$

which implies that

$$u^\varepsilon(t) \to u(t) \text{ in } L^2(\mathbb{R}^2), \text{ as } \varepsilon \to 0, \ \forall\, t \in [0,T). \tag{6.2.9}$$

Then, for all $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times [0,T))$, we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} |\theta^\varepsilon|^p (u^\varepsilon \cdot \nabla)\eta dx - \int_{\mathbb{R}^2} |\theta|^p (u \cdot \nabla)\eta dx \right| \\ &\quad \leqslant |||\theta^\varepsilon|^p (u^\varepsilon \cdot \nabla)\eta - |\theta|^p (u \cdot \nabla)\eta||_{L^1} \\ &\quad \leqslant |||\theta^\varepsilon|^p (u^\varepsilon \cdot \nabla)\eta - |\theta^\varepsilon|^p (u \cdot \nabla)\eta||_{L^1} + |||\theta^\varepsilon|^p (u \cdot \nabla)\eta - |\theta|^p (u \cdot \nabla)\eta||_{L^1} \\ &\quad \leqslant |||\theta^\varepsilon|^p ((u^\varepsilon - u) \cdot \nabla)\eta||_{L^1} + ||(|\theta^\varepsilon|^p - |\theta|^p)(u \cdot \nabla)\eta||_{L^1} \\ &\quad \leqslant ||\theta||_{L^{2p}}^p ||\nabla\eta||_{L^\infty} ||u^\varepsilon - u||_{L^2} + |||\theta^\varepsilon|^p - |\theta|^p||_{L^2} ||u||_{L^2} ||\nabla\eta||_{L^\infty}. \end{aligned} \tag{6.2.10}$$

Since that $\theta(t) \in L^{2p}(\mathbb{R}^2)$, $\nabla\eta(t) \in L^\infty(\mathbb{R}^2)$, and $\mathbf{u}^\varepsilon(t) \to u(t)$ in $L^2(\mathbb{R}^2)$ for all $t \in [0,t)$, the first term on the right-hand side of (6.2.10) goes to zero. For the second term, notice that

$$\int_{\mathbb{R}^2} (|\theta^\varepsilon|^p - |\theta|^p|)^2 dx = \int_{R^2} |\theta^\varepsilon|^{2p} dx - 2\int_{\mathbb{R}^2} |\theta^\varepsilon|^p |\theta|^p dx + \int_{\mathbb{R}^2} |\theta|^{2p} dx, \tag{6.2.11}$$

since $\theta_\varepsilon(t) \to \theta(t)$ in $L^q(\mathbb{R}^2)$ for all $q \in [2,\infty)$ and $t \in (0,T)$, it follows that $|\theta^\varepsilon|^p \rightharpoonup |\theta|^p$ in $L^2(\mathbb{R}^2)$. Hence,

$$\int_{\mathbb{R}^2} |\theta^\varepsilon|^{2p} dx \longrightarrow \int_{\mathbb{R}^2} |\theta|^{2p} dx \qquad \text{and} \qquad \int_{\mathbb{R}^2} |\theta^\varepsilon|^p |\theta|^p dx \longrightarrow \int_{\mathbb{R}^2} |\theta|^{2p} dx, \tag{6.2.12}$$

which implies that

$$|||\theta^\varepsilon|^p - |\theta|^p||_{L^2} \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

Plugging this back into (6.2.10) and taking $\varepsilon \to 0$, we conclude that

$$\int_{\mathbb{R}^2} |\theta^\varepsilon|^p (u^\varepsilon \cdot \nabla)\eta dx \longrightarrow \int_{\mathbb{R}^2} |\theta|^p (u \cdot \nabla)\eta dx, \quad \text{for all } p \in [2,\infty). \tag{6.2.13}$$

Setting $g(t) := \int_{\mathbb{R}^2} (|\theta^\varepsilon|^p (u^\varepsilon \cdot \nabla)\eta - |\theta|^p (u \cdot \nabla)\eta) dx$, with $t \in [t_1, t_2]$, it follows from (6.2.10) and the Lemma 2.2 that g is uniformly bounded by $c\|\theta_0\|^p_{L^{2p}} \|\theta\|_{L^\infty(t_1,t_2,H^s)}$, which is integrable in $[t_1, t_2]$. Hence, it follows by the Theorem 2.1 that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta^\varepsilon|^p (u^\varepsilon \cdot \nabla)\eta dx dt \longrightarrow \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta|^p (u \cdot \nabla)\eta dx dt, \quad \text{for all } p \in [2,\infty). \tag{6.2.14}$$

Combining (6.2.7), (6.2.8), (6.2.13) and (6.2.14) with (6.2.6), we establish that for all $p \in [2,\infty)$, it holds

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\theta(x,t_2)|^p \eta(x,t_2) dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \eta(x,t_1) dx \\
&- \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \partial_t \eta |\theta|^p dx dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta|^p (u \cdot \nabla)\eta dx dt \\
&= p \lim_{\varepsilon \to 0} \int_{t_1}^{t_2} \left[\int_{\mathbb{R}^2} \theta^\varepsilon |\theta^\varepsilon|^{p-2} \eta (u^\varepsilon \cdot \nabla)\theta^\varepsilon dx - \int_{\mathbb{R}^2} \theta^\varepsilon |\theta^\varepsilon|^{p-2} \eta [\rho_\varepsilon * (u \cdot \nabla)\theta] dx \right] dt
\end{aligned} \tag{6.2.15}$$

Now, note that the right-hand side of (6.2.15) converges to zero as $\epsilon \to 0$. Indeed, since $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times [0,T))$ and $\theta(\cdot,t) \in \mathcal{C}_0(\mathbb{R}^2)$ for all $t \in [0,T)$, from (6.2.3), it follows that for each $t \in [0,T), \theta^\varepsilon(\cdot,t) \to \theta(\cdot,t)$ uniformly on compact subsets of $\mathbb{R}^2$. Then, we deduce that $|\theta^\varepsilon(t)|^{p-1}\eta(t) \in L^\infty(U)$, for all $t \in [0,T)$, where U is a subset of $\operatorname{supp} \eta(x,t) \times [0,T)$. Therefore, applying Hölder and Young's convolution inequalities, it is established that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} \theta^\varepsilon |\theta^\varepsilon|^{p-2} \eta (u^\varepsilon \cdot \nabla)\theta^\varepsilon dx - \int_{\mathbb{R}^2} \theta^\varepsilon |\theta^\varepsilon|^{p-2} \eta [\rho_\varepsilon * (u \cdot \nabla)\theta] dx \right| \\
&= \left| \int_{\mathbb{R}^2} \theta^\varepsilon |\theta^\varepsilon|^{p-2} \eta ((u^\varepsilon \cdot \nabla)\theta^\varepsilon - [\rho_\varepsilon * (u \cdot \nabla)\theta]) dx \right| \\
&\leqslant |||\theta^\varepsilon|^{p-1}\eta||_{L^\infty(\mathbb{R}^2)} ||(u^\varepsilon \cdot \nabla)\theta^\varepsilon - [\rho_\varepsilon * (u \cdot \nabla)\theta]||_{L^1(\mathbb{R}^2)} \\
&\lesssim ||(u^\varepsilon \cdot \nabla)\theta^\varepsilon - (u^\varepsilon \cdot \nabla)\theta||_{L^1} + ||(u^\varepsilon \cdot \nabla)\theta - (u \cdot \nabla)\theta||_{L^1} + ||(u \cdot \nabla)\theta - \rho_\varepsilon * (u \cdot \nabla)\theta||_{L^1} \\
&\lesssim ||u^\varepsilon||_{L^2} ||\nabla(\theta^\varepsilon - \theta)||_{L^2} + ||u^\varepsilon - u||_{L^2} ||\nabla\theta||_{L^2} + ||(u \cdot \nabla)\theta - \rho_\varepsilon * (u \cdot \nabla)\theta||_{L^1} \\
&\lesssim ||\theta||_{H^s} ||\nabla(\theta^\varepsilon - \theta)||_{L^2} + ||u^\varepsilon - u||_{L^2} ||\nabla\theta||_{L^2} + ||(u \cdot \nabla)\theta - \rho_\varepsilon * (u \cdot \nabla)\theta||_{L^1},
\end{aligned} \tag{6.2.16}$$

where in the last inequality we used that since $s > 1 + \beta$ holds

$$\begin{aligned}
||u^\varepsilon||_{L^2(\mathbb{R}^2)} &= ||\mathcal{F}(\nabla^\perp(-\Delta)^{-1+\frac{\beta}{2}}\theta^\varepsilon)||_{L^2} = |||\xi|^{\frac{\beta}{2}}\mathcal{F}(\theta^\varepsilon)||_{L^2} = ||\mathcal{F}(\Lambda^{\frac{\beta}{2}}\theta^\varepsilon)||_{L^2} \\
&= ||\Lambda^{\frac{\beta}{2}}\theta^\varepsilon||_{L^2} \leqslant ||\theta^\varepsilon||_{H^s} \leqslant ||\theta||_{H^s}.
\end{aligned} \tag{6.2.17}$$

Therefore, invoking (2.2.6) and (6.2.17), we obtain that $||u^\varepsilon - u||_{L^2} \to 0$ and $||\nabla(\theta^\varepsilon - \theta)||_{L^2} \to 0$ as $\varepsilon \to 0$. For the last term in (6.2.16), we observe that $(u \cdot \nabla)\theta \in L^1$. Indeed, from Hölder's inequality, it follows that

$$||(u \cdot \nabla)\theta||_{L^1} \leqslant ||u||_{L^2}||\nabla\theta||_{L^2} \leqslant ||\Lambda^{\beta-1}\theta||_{L^2}||\nabla\theta||_{L^2} \leqslant ||\theta||_{H^s}^2 < \infty.$$

Hence,

$$\rho^\varepsilon * (u \cdot \nabla)\theta \to (u \cdot \nabla)\theta, \text{ in } L^1, \text{ as } \varepsilon \to 0.$$

Plugging all information above in (6.2.16) and taking $\varepsilon \to 0$, we obtain

$$\left| \int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta(x,t)(u^\varepsilon \cdot \nabla)\theta^\varepsilon dx - \int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta(x,t)[\rho_\varepsilon * (u \cdot \nabla)\theta]dx \right| \to 0.$$

Furthermore, it follows from (6.2.16) and the maximum principle (Lemma 2.2) that

$$h(t) = \int_{\mathbb{R}^2} \theta(x,t)^\varepsilon(x,t)|\theta^\varepsilon(x,t)|^{p-2}\eta(x,t)(u^\varepsilon \cdot \nabla)\theta^\varepsilon(x,t)dx, \ t \in [t_1, t_2],$$

is uniformly bounded in time by $c\|\theta_0\|_{L^{2p}}^p\|\theta\|_{L^\infty(t_1,t_2,H^s)}$. Then, by the Theorem 2.1, we obtain that

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta(u^\varepsilon \cdot \nabla)\theta^\varepsilon dxdt - \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \theta^\varepsilon|\theta^\varepsilon|^{p-2}\eta[\rho_\varepsilon * (u \cdot \nabla)\theta]dxdt \to 0. \quad (6.2.18)$$

Combining (6.2.15) and (6.2.18), and taking $\varepsilon \to 0$, we obtain (6.2.1), i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^2} |\theta(x,t_2)|^p\eta(x,t_2)dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p\eta(x,t_1)dx \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta(x,t)|^p\partial_t\eta(x,t)dxdt + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} |\theta|^p(x,t)(u \cdot \nabla)\eta(x,t)dxdt. \quad (6.2.19)\end{aligned}$$

Case 2: $1 \leqslant p < 2$.

Here, we proceed with the proof of the local L^p inequality (6.2.1), as outlined for the range $1 \leqslant p < 2$. Caution is necessary when multiplying (6.2.4) by $\theta_\varepsilon|\theta_\varepsilon|^{p-2}$, as the mollified function θ^ε might have real roots. Therefore, it is crucial to adjust the proof to accommodate this specific aspect of this case.

Fix $t_1, t_2 \in [0, T)$. Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times [0, T))$ and define the set K as follows:

$$K = \overline{\{(x,t) \ \mathbb{R}^2 \times [0,T) \ / \ \eta(x,t) \neq 0\}} := \operatorname{supp}\eta,$$

since η is a continuous function, it follows that $\eta(y) \equiv 0, \ \forall \ y \in \partial K$.

Now, for each $\delta \geqslant 0$, we define

$$K_{\varepsilon,\delta} := \{(x,t) \in K \cap \operatorname{supp}\theta_\varepsilon \ : \ \operatorname{dist}((x,t), \partial(K \cap \operatorname{supp}\theta_\varepsilon)) \geqslant \delta\}, \quad (6.2.20)$$

and

$$\psi_{\varepsilon,\delta} = \begin{cases} 1, & if \ (x,t) \in K_{\varepsilon,\delta}; \\ 0, & if \ (x,t) \in (K \cap \operatorname{supp}\theta_\varepsilon)^c. \end{cases} \quad (6.2.21)$$

Let us extend $\psi_{\varepsilon,\delta}$ to all of $\mathbb{R}^2 \times [0,T)$ as a continuous and affine function. Namely, for each $(x,t) \in K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp}\theta)$, take $(x',t') \in K_{\varepsilon,\delta}$ such that $(x',t') = \inf_{(y,s)\in K_{\varepsilon,\delta}} |(y,s)-(x,t)|$, which is guaranteed by the fact that $K_{\varepsilon,\delta}$ is a compact set. Now take the line passing through the points (x',t') and (x,t), and let (x'',t'') be the first point where it intersects the boundary of $K \cap \operatorname{supp}\theta_\varepsilon$, denoted as $\partial(K \cap \operatorname{supp}\theta_\varepsilon)$.

Note that $(x'',t'') = \inf_{(y,s)\in\partial(K\cap\operatorname{supp}\theta_\varepsilon)} |(y,s)-(x',t')|$. Indeed, suppose that there is another point $(x_0,t_0) \in \partial(K \cap \operatorname{supp}\theta_\varepsilon)$ such that

$$\operatorname{dist}((x_0,t_0),(x',t')) < \operatorname{dist}((x'',t''),(x',t')) = \delta,$$

then it implies based on the characterization of $\partial K_{\varepsilon,\delta}$ that $(x_0,t_0) \in int(K_{\varepsilon,\delta})$, which is a contradiction.

Now, let $\lambda_{(x,t)} \in [0,1]$ be such that

$$(x,t) = (x',t') + \lambda_{(x,t)}[(x'',t'') - (x',t')],$$

i.e.

$$\lambda_{(x,t)} = \frac{((x,t)-(x',t'))\cdot((x'',t'')-(x',t'))}{|(x'',t'')-(x',t')|^2} = \frac{((x,t)-(x',t'))\cdot((x'',t'')-(x',t'))}{\delta^2},$$

where we utilized the fact that since $(x'',t'') \in \partial(K \cap \operatorname{supp}\theta_\varepsilon)$ and $(x',t') \in \partial K_{\varepsilon,\delta}$, it follows that $|(x'',t'')-(x',t')| = \delta$. Therefore, for each $(x,t) \in K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp}\theta)$, we can define

$$\begin{aligned} \psi_{\varepsilon,\delta}(x,t) &:= \psi_{\varepsilon,\delta}(x',t') + \lambda_{(x,t)}[\psi_{\varepsilon,\delta}(x'',t'') - \psi_{\varepsilon,\delta}(x',t')] \\ &= 1 - \lambda_{(x,t)}, \text{ for all } x \in K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp}\theta), \end{aligned}$$

Note that, since $(x,t) \mapsto \lambda_{(x,t)}$ is an injective function, this ensures that $\psi_{\varepsilon,\delta}(x,t)$ is a well-defined function for all $(x,t) \in K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp}\theta_\varepsilon)$. Therefore, we can extend $\psi_{\varepsilon,\delta}$ to all of $\mathbb{R}^2 \times [0,T)$ as follows:

$$\psi_{\varepsilon,\delta} = \begin{cases} 1, & \text{if } (x,t) \in K_{\varepsilon,\delta} \\ 1-\lambda_{(x,t)}, & \text{if } (x,t) \in K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp}\theta_\varepsilon) \\ 0, & \text{if } (x,t) \in (K \cap \operatorname{supp}\theta_\varepsilon)^c. \end{cases} \tag{6.2.22}$$

Additionally, based on straightforward calculations, we can deduce that

$$|D_{(x,t)}\psi_{\varepsilon,\delta}(x,t)| = \frac{1}{\delta}, \text{ for all } (x,t) \in K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp}\theta_\varepsilon), \tag{6.2.23}$$

and

$$\psi_{\varepsilon,\delta}(x,t) \to \mathbb{1}_{K\cap\operatorname{supp}\theta_\varepsilon}(x,t) \ \text{ as } \delta \to 0, \text{ for all } (x,t) \in \mathbb{R}^2 \times [0,T). \tag{6.2.24}$$

Let us recall that θ_ε satisfies

$$\partial_t\theta_\varepsilon + [\mathbf{u}_\varepsilon \cdot \nabla]\theta_\varepsilon = (\mathbf{u}_\varepsilon \cdot \nabla)\theta_\varepsilon - \rho_\varepsilon * [(\mathbf{u}\cdot\nabla)\theta]. \tag{6.2.25}$$

Thus, we multiply (6.2.25) by $\theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}$, which is well-defined since $\operatorname{supp}\psi_{\varepsilon,\delta} \subset \operatorname{supp}\theta_\varepsilon$, integrate over x in $\mathbb{R}^2$ and later, apply integration by parts.

$$
\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^2} \eta(x,t)\psi_{\varepsilon,\delta}(x,t)\partial_t|\theta_\varepsilon(x,t)|^p dx - \frac{1}{p}\int_{\mathbb{R}^2} |\theta_\varepsilon(x,t)|^p(\mathbf{u}_\varepsilon\cdot\nabla)(\eta(x,t)\psi_{\varepsilon,\delta}(x,t))dx\\
&= \int_{\mathbb{R}^2} \theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}(\mathbf{u}_\varepsilon\cdot\nabla)\theta_\varepsilon dx - \int_{\mathbb{R}^2} \theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}[\rho_\varepsilon * (\mathbf{u}\cdot\nabla)\theta]dx,
\end{aligned}
$$

where we used on the second term on the left-hand side, that $\nabla\cdot\mathbf{u} = 0$. Thus, by using the product rule to the second term on the left-hand side[1] , we obtain

$$
\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^2} \eta\psi_{\varepsilon,\delta}\partial_t|\theta_\varepsilon|^p dx - \frac{1}{p}\int_{\mathbb{R}^2} \psi_{\varepsilon,\delta}|\theta_\varepsilon|^p(\mathbf{u}_\varepsilon\cdot\nabla)\eta dx - \frac{1}{p}\int_{\mathbb{R}^2} \eta|\theta_\varepsilon|^p(\mathbf{u}_\varepsilon\cdot\nabla)\psi_{\varepsilon,\delta}dx\\
&= \int_{\mathbb{R}^2} \theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}(\mathbf{u}_\varepsilon\cdot\nabla)\theta_\varepsilon dx - \int_{\mathbb{R}^2} \theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}[\rho_\varepsilon * (\mathbf{u}\cdot\nabla)\theta]dx. \qquad (6.2.26)
\end{aligned}
$$

where in the last expression we integrate over $t \in [t_1, t_2]$ and subsequently apply integration by parts, together with the product rule to the first term on the left-hand side and obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \eta(x,t_2)\psi_{\varepsilon,\delta}(x,t_2)|\theta(x,t_2)|^p dx - \int_{\mathbb{R}^2} \eta(x,t_1)\psi_{\varepsilon,\delta}(x,t_1)|\theta(x,t_1)|^p dx\\
&- \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \eta|\theta_\varepsilon|^p\partial_t\psi_{\varepsilon,\delta}dxdt - \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \psi_{\varepsilon,\delta}|\theta_\varepsilon|^p\partial_t\eta dxdt\\
&- \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \psi_{\varepsilon,\delta}|\theta_\varepsilon|^p(\mathbf{u}_\varepsilon\cdot\nabla)\eta dxdt - \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \eta|\theta_\varepsilon|^p(\mathbf{u}_\varepsilon\cdot\nabla)\psi_{\varepsilon,\delta}dxdt\\
&= p\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}(\mathbf{u}_\varepsilon\cdot\nabla)\theta_\varepsilon dxdt - p\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \theta_\varepsilon|\theta_\varepsilon|^{p-2}\eta\psi_{\varepsilon,\delta}[\rho_\varepsilon * (\mathbf{u}\cdot\nabla)\theta]dxdt.
\end{aligned}
\qquad (6.2.27)
$$

The next goal is to prove that both the third and sixth terms on the left-hand side of (6.2.27) are null. Let us start by analyzing the third term. Remember that since $\overline{K^c_{\varepsilon,\delta}\cap(K\cap\operatorname{supp}\theta_\varepsilon)}$ is a compact set, any cover of the set admits a finite subcover. Hence, without loss of generality, we can assume there are a finite number of points $(x_1,t_1),\ldots,(x_n,t_n) \in \partial(K\cap\operatorname{supp}\theta_\varepsilon), n \in \mathbb{N}$, such that

$$
K^c_{\varepsilon,\delta}\cap(K\cap\operatorname{supp}\theta_\varepsilon) \subset \overline{K^c_{\varepsilon,\delta}\cap(K\cap\operatorname{supp}\theta_\varepsilon)} \subset \bigcup_{i=1}^n B_{2\delta}(x_i,t_i),
$$

where $B_\delta(x_i,t_i)$, for $i = 1,\ldots,n$, is a ball centered at (x_i,t_i) with radius δ. We now recall that the Lebesgue Differentiation Theorem (LDT) (see Theorem 2.2) holds for almost everywhere $(x,t) \in \mathbb{R}^2\times[0,T)$. Consequently, the set $\Omega := \{(x,t) \in \mathbb{R}^2\times[0,T) \mid \text{LDT holds}\}$ is a dense subset in $\mathbb{R}^2\times[0,T)$. Therefore, for each $i = 1,\ldots,n$, we choose $(\tilde{x}_i,\tilde{t}_i) \in$

[1] Note that the product $\eta\psi_{\varepsilon,\delta}$ may not be derivable because $\psi_{\varepsilon,\delta}$ is not a smooth function. Therefore, if necessary, we take a sequence of mollifying functions $\psi^\zeta_{\varepsilon,\delta} = \rho_\zeta * \psi_\delta$, where ρ_ζ is a standard mollifier and later, we take the limit as $\zeta \to 0$, to obtain (6.2.26).

$\Omega \cap B_{2\delta}(x_i, t_i)$, which implies that the union of the sets $B_{3\delta}(\tilde{x}_i, \tilde{t}_i)$ forms a new cover for $K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp} \theta_\varepsilon)$. Thus

$$K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp} \theta_\varepsilon) \subset \bigcup_{i=1}^{n} B_{3\delta}(\tilde{x}_i, \tilde{t}_i). \tag{6.2.28}$$

Thus, recalling that $\operatorname{supp} \psi_{\varepsilon,\delta} \subset K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp} \theta_\varepsilon)$, invoking (6.2.23) and (6.2.28), and applying the Lebesgue Differentiation Theorem, we deduce that

$$\begin{aligned}
&\lim_{\delta \to 0} \left| \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \eta(x,t) |\theta_\varepsilon(x,t)|^p \partial_t \psi_{\varepsilon,\delta}(x,t) dx dt \right| \\
&\quad \leqslant \lim_{\delta \to 0} \left[\frac{1}{\delta} \int_{K^c_{\varepsilon,\delta} \cap (K \cap \operatorname{supp} \theta_\varepsilon)} \eta(x,t) |\theta_\varepsilon(x,t)|^p dx dt \right] \\
&\quad \leqslant C \lim_{\delta \to 0} \left[\delta^2 \sum_{i=1}^{n} \frac{1}{|B_{3\delta}(\tilde{x}_i, \tilde{t}_i)|} \int_{B_{3\delta}(\tilde{x}_i, \tilde{t}_i)} \eta(x,t) |\theta_\varepsilon(x,t)|^p dx dt \right] \\
&\quad = C \lim_{\delta \to 0} (\delta^2) \sum_{i=1}^{n} \eta(\tilde{x}_i, \tilde{t}_i) |\theta_\varepsilon(\tilde{x}_i, \tilde{t}_i)|^p \\
&\quad = 0.
\end{aligned} \tag{6.2.29}$$

Hence, we obtain

$$\lim_{\delta \to 0} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \eta(x,t) |\theta_\varepsilon(x,t)|^p \partial_t \psi_{\varepsilon,\delta}(x,t) dx dt = 0.$$

The approach for the last term in the left-hand side of (6.2.27) is identical to the approach previously applied. Then, we also can conclude that

$$\lim_{\delta \to 0} \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \eta(x,t) |\theta_\varepsilon(x,t)|^p (\mathbf{u}_\varepsilon \cdot \nabla) \psi_{\varepsilon,\delta}(x,t) dx dt = 0. \tag{6.2.30}$$

Plugging (6.2.29) and (6.2.30) in (6.2.27) and taking the limit $\delta \to 0$, we obtain

$$\begin{aligned}
&\int_{K \cap \operatorname{supp} \theta_\varepsilon} \eta(x,t_2) |\theta_\varepsilon(x,t_2)|^p dx - \int_{K \cap \operatorname{supp} \theta_\varepsilon} \eta(x,t_1) |\theta_\varepsilon(x,t_1)|^p dx \\
&- \int_{t_1}^{t_2} \int_{K \cap \operatorname{supp} \theta_\varepsilon} \partial_t \eta(x,t) |\theta_\varepsilon(x,t)|^p dx dt - \int_{\mathbb{R}^2} |\theta_\varepsilon(x,t)|^p (\mathbf{u}_\varepsilon \cdot \nabla) \eta(x,t) dx dt \\
&= p \int_{t_1}^{t_2} \int_{K \cap \operatorname{supp} \theta_\varepsilon} \theta_\varepsilon(x,t) |\theta_\varepsilon(x,t)|^{p-2} \eta(x,t) (\mathbf{u}_\varepsilon \cdot \nabla) \theta_\varepsilon(x,t) dx dt \\
&- p \int_{t_1}^{t_2} \int_{K \cap \operatorname{supp} \theta_\varepsilon} \theta_\varepsilon(x,t) |\theta_\varepsilon(x,t)|^{p-2} \eta(x,t) [\rho_\varepsilon * ((\mathbf{u} \cdot \nabla) \theta(x,t))] dx dt
\end{aligned} \tag{6.2.31}$$

It should be noted that (6.2.31) is analogous to (6.2.6) for the case where $2 \leqslant p < \infty$. Consequently, this enables us to replicate the same argument, leading to the conclusion that for all $1 \leqslant p < 2$, it holds

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\theta(x,t_2)|^p \eta(x,t_2) dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p \eta(x,t_1) dx \\
&= \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta(x,t)|^p \partial_t \eta(x,t) dx dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta(x,t)|^p (\mathbf{u} \cdot \nabla) \eta(x,t) dx dt.
\end{aligned} \tag{6.2.32}$$

where $t_1, t_2 \in [0,T)$ and $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \times [0,T))$. Therefore, we conclude the proof of (6.2.1) for all $1 \leqslant p < \infty$.

We highlight that, choosing $\eta(x,t) = h(x)\psi(t)$ for all $(x,t) \in \mathbb{R}^2 \times [0,T)$, where $h \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ and $\psi \in \mathcal{C}_c^\infty((0,T))$ such that $\psi(t) = 1$ for all $t \in [t_1, t_2]$, we obtain from (6.2.19) the equality (6.2.33) invoked in the proof of Theorem 3.1 and Theorem 3.2. More precisely,

$$\int_{\mathbb{R}^2} |\theta(x,t_2)|^p h(x) dx - \int_{\mathbb{R}^2} |\theta(x,t_1)|^p h(x) dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\theta(x,t)|^p (u \cdot \nabla) h(x) dx dt, \quad (6.2.33)$$

where $p \in [2, \infty)$.