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Topological Methods in the Study of Periodic Solutions of Non-Smooth Differential Equations

Métodos Topológicos no Estudo de Soluções Periódicas de Equações Diferenciais Não-Suaves

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Supervisor: Douglas Duarte Novaes

Este trabalho corresponde à versão final da Tese defendida pelo aluno Francisco Bruno Gomes da Silva e orientada pelo Prof. Dr. Douglas Duarte Novaes.

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"It is impossible to be a mathematician
without being a poet in soul."
Sofia Kovalevskaya

Resumo

Neste trabalho é realizado um estudo das soluções periódicas de equações diferenciais não-suaves através da Teoria *Averaging*, do grau de Brouwer e de equações de operadores em espaços de Banach. São fornecidas condições suficientes que garantem a persistência e também para a convergência de soluções periódicas de equações diferenciais tanto contínuas não-Lipschitz como Carathéodory descontínuas dependendo de um parâmetro pequeno. Apresenta-se ainda uma revisão das Teorias clássicas de Melnikov e *Averaging* para equações diferenciais periódicas suaves, como forma de motivar e expor os desafios do estudo realizado neste trabalho. Foram obtidos resultados consistentes com aqueles previamente estabelecidos na literatura e que os estendem para casos antes não contemplados, o que é devidamente evidenciado em exemplos.

Palavras-chave: Equações Diferenciais Não-Suaves. Equações Diferenciais Contínuas Não-Lipschitz. Equações Diferenciais Descontínuas Carathéodory. Teoria *Averaging*. Grau de Brouwer. Grau de Coincidência.

Abstract

In this work a study of periodic solutions of non-smooth differential equations is carried out by means of the Averaging theory, the Brouwer degree and operator equations in Banach spaces. Sufficient conditions that ensure the persistence and convergence of periodic solutions of both non-Lipschitz continuous and Carathéodory discontinuous differential equations depending upon a small parameter are provided. The classical Melnikov and Averaging Theories for periodic smooth differential equations are presented as a way to motivate and expose the challenges of the study undertaken herein. The main results proved in this work are consistent with those already established in the literature and extend them to cases not yet covered, as it is properly evidenced with examples.

Keywords: Non-Smooth Differential Equations. Continuous Non-Lipschitz Differential Equations. Discontinuous Carathéodory Differential Equations. Averaging Theory. Brouwer Degree. Coincidence Degree.

Contents

1 Introduction

In the qualitative study of dynamical systems, some objects carry important information about the behaviour of the phenomenon they model. One such type of object are the invariant sets, which, as the name suggests, are sets from which you can never get out of following the flow of the dynamical system in place. One such example in the context of ordinary differential equations, is a singularity, which is a stationary solution. Another example of invariant set, and the main subject of study of this work, is a periodic trajectory, which in turn corresponds to the the image set of a periodic solution.

In this work, we study the existence of periodic solutions of differential equations with non-smooth right-hand sides. More precisely, we intend to provide sufficient conditions for the existence of periodic solutions of non-smooth differential equations. In spite of using the term "non-smooth" we do not treat the prominent and interesting case of Filippov systems. Instead, we delve into those equations whose right-hand sides lack differentiability and even continuity. As a matter of fact, we deal at first with non-Lipschitz continuous differential equations which poses the absence of uniqueness of solutions as the main challenge. Later we further extend our results for systems that allow some discontinuities, which are the Carathéodory differential equations.

1.1 Brief historic background

In order to motivate the study carried out in this work, we discuss here a few established results that give conditions for detecting periodic solutions of differential equations while consistently relaxing smoothness hypotheses for their right-hand sides.

Consider at first the differential equation

$$x' = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{1.1}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$ and $R : \mathbb{R} \times D \times \mathbb{R} \to \mathbb{R}^n$ are $T-$periodic in t. A classical result for guaranteeing the existence of $T-$periodic solutions of this equations for $\varepsilon \neq 0$ small is the following theorem.

Theorem 1.1 ([30, Theorem 11.5]). *Consider the differential equation* (1.1). *Define the average*

$$f_1(z) = \frac{1}{T}\int_0^T F_1(s,z)\mathrm{d}s$$

and the averaged equation $z' = \varepsilon f_1(z)$. *Assume that*

- $F_1, R, \dfrac{\partial F_1}{\partial x}, \dfrac{\partial^2 F_1}{\partial x^2}$ *and* $\dfrac{\partial R}{\partial x}$ *are defined, continuous and bounded by a constant* M *(not depending on* ε*) in* $[0, \infty] \times D$*, for each* $0 \leqslant \varepsilon \leqslant \varepsilon_0$*;*

If p *is a critical point of the averaged equation satisfying* $\det \dfrac{\partial f_1}{\partial x}(p) \neq 0$*, then there exists a* $T-$*periodic solution* $\varphi(t, \varepsilon)$ *of equation* (1.1) *close to* p *such that* $\lim_{\varepsilon \to 0} \varphi(t, \varepsilon) = p$*.*

Notice that in this theorem, it is required that $F \in \mathcal{C}^2$ and $R \in \mathcal{C}^1$.

In [5], this theorem is generalized by first removing differentiability of F and R yielding the following theorem.

Theorem 1.2 ([5, Theorem 1.1])**.** *Consider the differential equation* (1.1)*, where* $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$ *and* $R \colon \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ *are continuous functions,* $T-$*periodic in the first variable and* D *is an open subset of* $\mathbb{R}^n$*. Assume that* F_1 *and* R *are locally Lipschitz in the second variable and that for* $a \in D$ *satisfying* $f_1(a) = 0$*, there exists a neighbourhood* V *of* a *such that* $\overline{V} \subset D$*,* $f_1(z) \neq 0$*,* $\forall\, z \in \overline{V} \backslash \{a\}$ *and* $d_B(f_1, V, 0) \neq 0$*. Then for* $|\varepsilon| > 0$ *sufficiently small there exists a* $T-$*periodic solution* $\varphi(t, \varepsilon)$ *to* (1.1) *such that* $\varphi(\cdot, \varepsilon) \to a$ *as* $\varepsilon \to 0$*.*

Observe that in this theorem, the authors only require F and R to be continuous and Lipschitz. Since differentiability has been dropped, the condition $\det \dfrac{\partial f_1}{\partial x}(p) \neq 0$ does not make sense and it is replaced by $d_B(f_1, V, 0) \neq 0$, which is the Brouwer degree of f_1 with respect to V and 0 (see Section 2.4).

In [5] the authors also outline the proof of a further extension of Theorem 1.1 by removing the Lipschitz assumption on F and R leaving just continuity. In this case, it is important to recall that uniqueness of solution is lost and, as we shall discuss in Section 1.2, this is an important property in classical approaches.

In [26], the author provides a first order analysis for Carathéodory differential equations (see Section 2.3) which allow some discontinuities in the time variable.

Consider now the differential equation

$$x' = \varepsilon F_1(t, x) + \cdots + \varepsilon^k F_k(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon), \tag{1.2}$$

where each $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ and $R : \mathbb{R} \times D \times \mathbb{R} \to \mathbb{R}^n$ are $T-$periodic in t.

For equation (1.2), the authors in [20, Theorem A] perform a higher order averaging analysis and give sufficient conditions for the persistence of periodic solutions assuming $F_i \in \mathcal{C}^{k-i}$ and that $\partial^{i-k} F_i$ and R are Lipschitz continuous.

In [18], the authors perform a second order ($k = 2$) analysis for Filippov differential equations assuming only Lipschitz continuity of the right-hand side in each zone. Later, in [19] this result is generalized for an arbitrary order.

The more recent advances in this field are part of this thesis and has already been published in [27]. In this paper, the authors consider the differential equation

$$x' = \varepsilon F(t, x, \varepsilon),$$

where F is $T-$periodic in t and continuous. In this setting, sufficient conditions are provided in order to guarantee the existence of $T-$periodic solutions for $\varepsilon \neq 0$. From this result, they derive a continuous higher order analysis for (1.2), which was unknown up to that point.

In this work, we shall present and discuss in more detail the results for continuous differential equations and also we propose further generalizations for Carathéodory differential equations.

1.2 The Melnikov method

Usually, when dealing with the problem of determining the existence of periodic solutions for ordinary differential equations, one resorts to the study of the displacement map associated to that system (see [5, 6, 20, 29]) via expansion in Taylor series. The displacement map measures how far a solution starting at some point is from being periodic. The approach we will briefly describe in the sequel is called the Melnikov method. It consists in performing the Taylor series expansion of the displacement map and then applying the Implicit Function Theorem to guarantee the existence of a branch of zeroes of that map.

More precisely, consider the differential equation

$$x' = \varepsilon F(t, x, \varepsilon), \qquad (t, x, \varepsilon) \in I \times D \times (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \tag{1.3}$$

where F is $T-$periodic in t, for some $T > 0$, at least Lipschitz continuous in x, and ε is a (usually small) parameter. For $(t_0, x_0, \varepsilon) \in I \times D \times (-\varepsilon_0, \varepsilon_0)$, let $\varphi = \varphi(t, t_0, x_0, \varepsilon)$ be the unique solution of equation (1.3) satisfying $\varphi(t_0, t_0, x_0, \varepsilon) = x_0$. Assume that $[t_0, t_0 + T] \subset I$. Define the Poincaré map $\pi : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ by setting

$$\pi(x, \varepsilon) = \varphi(t_0 + T, t_0, x, \varepsilon).$$

Note that the Poincaré map already encodes the information of whether a solution is $T-$periodic or not. Indeed, for each fixed $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, the fixed points of the map $\pi(\cdot, \varepsilon)$ are initial conditions for $T-$periodic solutions of equation (1.3), since $\pi(x_0, \varepsilon) = x_0$ if, and only if, $\varphi(t_0 + T, t_0, x_0, \varepsilon) = x_0 = \varphi(t_0, t_0, x_0, \varepsilon)$, so that φ can be extended to a $T-$periodic solution in $\mathbb{R}$.

Now, define the function $\Delta : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ by

$$\Delta(x, \varepsilon) = \pi(x, \varepsilon) - x. \tag{1.4}$$

For any pair $(x_0, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$ satisfying $\Delta(x_0, \varepsilon) = 0$, we have $\varphi(t_0, t_0, x_0, \varepsilon) = x_0 = \varphi(t_0 + T, t_0, x_0, \varepsilon)$, which implies that φ can be extended to a $T-$periodic solution in $\mathbb{R}$, since φ is C^1 in t. Conversely, if $\varphi(t, t_0, x_0, \varepsilon)$ is a $T-$periodic solution of equation (1.3), then $\Delta(x_0, \varepsilon) = 0$. We have thus established that there is an equivalence between the set of periodic solutions of equation (1.3) and the set of zeroes of the displacement function. Equivalently, it is straightforward that zeroes of the displacement function correspond to fixed points of the Poincaré map. Therefore, in this approach, one seeks conditions that guarantee the existence of zeroes of functions, especially the displacement function or some other function whose set of zeroes is contained in the set of zeroes of the displacement function. In other words, one reduces the problem of finding $T-$periodic solutions of some differential equation to the problem of finding zeroes of functions.

This is an important change of perspective and allows one to tackle the problem of finding periodic solutions of differential equations through the problem of finding zeroes of functions. It should be noted that determining zeroes of non-linear functions is not necessarily an easier problem, but allows for a whole new set of tools to be used, such as the Implicit Function Theorem and degree theory as we shall see in this work.

For equation (1.3), we note that for $\varepsilon = 0$ all the solutions are constant. In general, one is interested in periodic solutions branching from these constant solutions as ε ranges in some small interval around 0. Therefore, we usually look for branches of zeroes of Δ of the form $\Delta(x(\varepsilon), \varepsilon) = 0$ for $\varepsilon \neq 0$ small. One way to obtain this is using the Implicit Function Theorem. Indeed, if $\Delta(x_0, 0) = 0$ and $\det(\partial\Delta/\partial\varepsilon)(x_0, 0) \neq 0$, for some $x_0 \in D$, then the Implicit Function Theorem tells us that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ and a function $x : (-\varepsilon_1, \varepsilon_1) \to D$ such that $x(0) = x_0$ and $\Delta(x(\varepsilon), \varepsilon) = 0$ for every $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$. Notice that for this to work, we need the solution φ to be differentiable in ε.

The above procedure can be better elaborated. Assuming that F is C^k in (t, x, ε), then by the theorem of smooth dependence of solutions on the parameters, the solution φ is also C^k in ε, see [1, Chapter 9]. Therefore, in this case, we can expand both the right-hand side of equation (1.3) and the Poincaré map in powers of ε up to order k obtaining

$$x' = \varepsilon F(t, x, \varepsilon) = \varepsilon F(t, x, 0) + \varepsilon^2 \frac{1}{2}\frac{\partial F}{\partial \varepsilon}(t, x, 0) + \cdots + \varepsilon^k \frac{1}{k!}\frac{\partial^{k-1} F}{\partial \varepsilon^{k-1}}(t, x, 0) + \mathcal{O}(\varepsilon^{k+1})$$

and

$$\begin{aligned}\Delta(x, \varepsilon) &= \varepsilon \frac{\partial \Delta}{\partial \varepsilon}(x, 0) + \frac{\varepsilon^2}{2!}\frac{\partial^2 \Delta}{\partial \varepsilon^2}(x, 0) + \cdots + \frac{\varepsilon^k}{k!}\frac{\partial^k \Delta}{\partial \varepsilon^k}(x, 0) + \mathcal{O}(\varepsilon^{k+1}) \\ &=: \varepsilon M_1(x) + \varepsilon^2 M_2(x) + \cdots + \varepsilon^k M_k(x) + \mathcal{O}(\varepsilon^{k+1}).\end{aligned}$$

Thus equation (1.3) takes the form

$$x' = \varepsilon F_1(t, x) + \cdots + \varepsilon^k F_k(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon), \tag{1.5}$$

where

$$F_j(t,x) = \frac{1}{j!}\frac{\partial^j F}{\partial \varepsilon^j}(t,x,0),$$

for every $j \in \{1,\dots,k\}$, and R are differentiable functions in (t,x) and R is continuous in ε.

For each $j \in \{1,\dots,k\}$ the differentiable function

$$M_j(x) = \frac{1}{j!}\frac{\partial^j \Delta}{\partial \varepsilon^j}(x,0)$$

is called the $j-$th order Melnikov function.

Now by the Implicit Function Theorem we determine a branch of zeroes of Δ. If M_1 is not identically zero and x_0 is a zero of M_1 for which $\det DM_1(x_0) \neq 0$ (such a point is called a *simple zero* of M_1) then there exists $\varepsilon_1 \in (0,\varepsilon_0)$ and a smooth function $x : (-\varepsilon_1,\varepsilon_1) \to \mathbb{R}^n$ such that $x(0) = x_0$ and $\Delta(x(\varepsilon),\varepsilon) = 0$, for every $\varepsilon \in (-\varepsilon_1,\varepsilon_1)$. Indeed, consider

$$\delta(x,\varepsilon) = \frac{\Delta(x,\varepsilon)}{\varepsilon} = M_1(x) + \varepsilon M_2(x) + \cdots + \varepsilon^{k-1}M_k(x) + \mathcal{O}(\varepsilon^k).$$

Notice that the function δ can also be defined at $\varepsilon = 0$ due to the fact that the rightmost member of the above expression is well defined at $\varepsilon = 0$. Now, note that the pair $(x_0, 0)$ is a zero of δ. On the other hand,

$$\frac{\partial \delta}{\partial x}(x,\varepsilon) = DM_1(x) + \varepsilon DM_2(x) + \cdots + \varepsilon^{k-1}DM_k(x) + \mathcal{O}(\varepsilon^k),$$

so that

$$\frac{\partial \delta}{\partial x}(x_0,0) = DM_1(x_0).$$

Since, by hypothesis, $\det DM_1(x_0) \neq 0$, it follows that $\det(\partial\delta/\partial x)(x_0,0) \neq 0$. Then, the Implicit Function Theorem implies that there exists a smooth function $x : (-\varepsilon_1,\varepsilon_1) \to \mathbb{R}^n$ such that $x(0) = x_0$ and $\delta(x(\varepsilon),\varepsilon) = 0$. However, any zero of δ is also a zero of Δ. This proves that for ε sufficiently small there is always a point $x(\varepsilon)$ such that the pair $(x(\varepsilon),\varepsilon)$ is a zero of the displacement function. Taking into account the previous discussion, we have actually shown that any simple zero of the Melnikov function M_1 determines a $T-$periodic solution of equation (1.3).

In case some of the Melnikov functions M_j are identically zero, one can look at the first of these functions that is not so, say $M_\ell, 1 < \ell \leqslant k$, and the same argument above works using M_ℓ instead of M_1, except that the function δ should have its definition changed to

$$\delta(x,\varepsilon) = \frac{\pi(x,\varepsilon)}{\varepsilon^\ell} = M_\ell(x) + \varepsilon M_{\ell+1}(x) + \cdots + \varepsilon^{k-\ell}M_k(x) + \mathcal{O}(\varepsilon^{k-\ell+1}).$$

This methodology is called the higher order Melnikov analysis, because a higher order Melnikov function is needed. Thus, wrapping it all up, simple zeroes of the first non-identically zero Melnikov function gives rise to $T-$periodic solution of the underlying differential equation.

Another information we can get from this procedure is about the convergence of solutions with respect to the parameter ε. From what was shown in the previous discussion, $x(\varepsilon)$ is the initial condition of a $T-$periodic solution, denoted $\varphi(t, t_0, x(\varepsilon), \varepsilon)$, of equation (1.5), where $\varphi(t, t_0, x_0, 0) = x_0$ is a constant solution. Thus, by smoothness of x with respect to ε and the continuous dependence of φ with repect to the initial condition and parameters, it follows that $\varphi(\cdot, t_0, x(\varepsilon), \varepsilon) \to x_0$ uniformly as $\varepsilon \to 0$. Looking at the phase space of equation (1.5), this statement means that the constant solution $\varphi(t, t_0, x_0, 0) = x_0$ of equation (1.5) with $\varepsilon = 0$ persists for $\varepsilon \neq 0$ small and it starts at $x(\varepsilon)$, for each different ε.

Although we have changed the way we look at the problem, if we look closely, it has not been made easier, not only because finding zeroes of functions is also a hard problem, but mostly because to know the displacement function Δ or, equivalently, the Poincaré map π, and hence the Melnikov functions M_j, we would have to know the solution $\varphi(t, t_0, x_0, \varepsilon)$ explicitly, which is just as hard a problem as the one of determining periodic solutions directly by solving the differential equation itself. Hence, in order to enable this approach, more work is needed, namely, to find expressions for the Melnikov function in terms of the coefficient functions F_j.

By [20, 23], if $\varphi(t, x_0, \varepsilon)$ is the solution of equation (1.3) satisfying $\varphi(0, x_0, \varepsilon) = x_0$, then we expand φ in Taylor series about $\varepsilon = 0$ obtaining

$$\varphi(t, x, \varepsilon) = x + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t, x)}{i!} + \mathcal{O}(\varepsilon^{k+1}).$$

where the coefficient functions y_i are given by

$$\begin{aligned} y_1(t, x) &= \int_0^t F_1(s, x)\mathrm{d}s \\ y_i(t, x) &= \int_0^t \left(i! F_i(s, x) + \sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} \partial_x^m F_{i-j}(s, z) B_{j,m}(y_1, \dots, y_{j-m+1})(s, x) \right) \mathrm{d}s, \end{aligned}$$

for $i \in \{2, \dots, k\}$. For each p, q positive integers, $B_{p,q}$ is the Bell polynomial

$$B_{p,q}(x_1, \dots, x_{p-q+1}) = \sum_{b \in \mathcal{S}} \frac{p!}{b_1! b_2! \cdots b_{p-q+1}!} \prod_{j=1}^{p-q+1} \left(\frac{x_j}{j!} \right)^{b_j},$$

where $\mathcal{S} = \{b = (b_1, \dots, b_{p-q+1}) : b_1 + 2b_2 + \cdots + (p-q+1)b_{p-q+1} = p \text{ and } b_1 + \cdots + b_{p-q+1} = q\}$. In addition, $\partial_x^m F_{i-j}(s, x)$ denotes the derivative with respect to x calculated at x.

From the above development, we note that

$$\Delta(x,\varepsilon) = \varphi(T,x,\varepsilon) - x = \sum_{i=1}^{k} \varepsilon^i \frac{y_i(T,x)}{i!} + \mathcal{O}(\varepsilon^{k+1}),$$

which gives us that the Melnikov functions are precisely $M_i(x) = y_i(T,x)$ for every $i \in \{1,\ldots,k\}$. In particular, it follows that

$$\begin{aligned} M_1(x) &= \int_0^T F_1(s,x)\mathrm{d}s \\ M_2(x) &= 2!\int_0^T \left(F_2(s,x) + D_x F_1(s,x) y_1(s,x)\right)\mathrm{d}s. \end{aligned}$$

As i ranges through $\{1,\ldots,k\}$ the expressions for M_j get more and more cumbersome.

1.3 The Averaging method

Another approach to study the qualitative behaviour of dynamical systems is the averaging method. Consider a periodic differential equation in the form of equation (1.3). The averaging method consists of performing a change of coordinates, called a near-identity transformation, of the form

$$x = \mathcal{U}(t,y,\varepsilon) = y + \varepsilon u^1(t,y) + \varepsilon^2 u^2(t,y) + \cdots + \varepsilon^k u^k(t,y), \tag{1.6}$$

with $\mathcal{U}(0,y,\varepsilon) = y$, that takes the original equation into

$$y' = \varepsilon g_1(y) + \varepsilon^2 g_2(y) + \cdots + \varepsilon^k g_k(y) + \varepsilon^{k+1} r(t,y,\varepsilon), \tag{1.7}$$

and then provides long-time asymptotic estimates for the solutions of equation (1.5) based on the solutions of the truncated equation

$$y' = \varepsilon g_1(y) + \varepsilon^2 g_2(y) + \cdots + \varepsilon^k g_k(y), \tag{1.8}$$

see [29, Lemma 2.9.1 and Theorem 2.9.2]. That is to say, if $\varphi(t,\varepsilon)$ is a solution of equation (1.5) and $\psi(t,\varepsilon)$ is a solution of equation (1.8) such that $\varphi(0,\varepsilon) = \psi(0,\varepsilon)$, then $|\varphi(t,\varepsilon) - \psi(t,\varepsilon)| = \mathcal{O}(\varepsilon^k)$ for time $\mathcal{O}(1/\varepsilon)$, i.e., for $t \in [0, L/\varepsilon]$ for some positive constant L. The functions g_i in equation (1.7) are called the averaged functions.

In particular, when $k = 1$, equation (1.5) writes

$$x' = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{1.9}$$

then applying a corresponding near-identity transformation as in equation (1.6), we obtain the full averaged system

$$y' = \varepsilon g_1(y) + \varepsilon^2 r(t,y,\varepsilon), \tag{1.10}$$

with

$$g_1(x) = \frac{1}{T}\int_0^T F_1(s,x)\mathrm{d}s.$$

By considering the rescaling of time give by $\tau = \varepsilon t$ and truncating the resulting equation, equation (1.10) is taken into the so called *guiding system*

$$z' = g_1(z).$$

Now, if z_0 is an equilibrium of the guiding system such that $Dg_1(z_0)$ is non-singular, then there exists a smooth function $\varepsilon \mapsto z_\varepsilon$ defined for small ε such that $\varphi(t, z_\varepsilon, \varepsilon)$ is a $T-$periodic solution of equation (1.9) (see [29, Theorem 6.3.2]). This fact suggests that there exists some relation between the averaging and the Melnikov methods. And indeed there is.

1.4 Relating the averaging and Melnikov methods

We noting that, unlike for the Melnikov functions, in general there is no explicit formula to obtain the functions g_i in equation (1.7) in terms of the F_i, even though there exists an algorithmic process to achieve that. However, in [24] the author shows that the following relationship between the expressions of g_i and the Melnikov functions M_i hold

$$\begin{aligned} g_1(x) &= \frac{1}{T}M_1(x) \\ g_i(x) &= \frac{1}{T}\left(M_i(x) - \sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{1}{j!}d^m g_{i-j}(x)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,x)\mathrm{d}s\right), \end{aligned}$$

where $\tilde{y}_i(t,x)$ are polynomials in t defined recursively as

$$\begin{aligned} \tilde{y}_1(t,x) &= tg_1(x) \\ \tilde{y}_i(t,x) &= i!tg_i(x) + \sum_{j=1}^{i-1}\sum_{m=1}^{j}\frac{i!}{j!}d^m g_{i-j}(x)\int_0^T B_{j,m}(\tilde{y}_1,\dots,\tilde{y}_{j-m+1})(s,x)\mathrm{d}s. \end{aligned}$$

The above expressions, although cumbersome, provide a clear relationship between the methods of Melnikov and Averaging. In particular, if for some $\ell \in \{2,\dots,k\}$ either $M_1 = \cdots = M_{\ell-1} = 0$ or $g_1 = \cdots = g_{\ell-1} = 0$, then $M_i = Tg_i$ for every $i \in \{1,\dots,\ell\}$. Equivalently, this results says that $g_1 = \cdots = g_{\ell-1} = 0$ if, and only if, $M_1 = \cdots = M_{\ell-1} = 0$ and $M_\ell = Tg_\ell$. From this relation, the conclusion drawn before about the existence of periodic solutions being related to existence of simple zeroes of the first non-vanishing Melnikov function can be transposed to the averaged functions, that is, simple zeroes of the first non-vanishing averaged function, say g_ℓ, provide periodic solutions of the original differential equation, equation (1.5), [24].

1.5 Less regular systems

The results discussed in the previous sections work very well as long as the right-hand side of the equation is differentiable. However, if we lower the smoothness conditions on F (or the functions F_i) we may find it hard to apply the same rationale. Observe that in the above discussion, in order to guarantee the existence of periodic solutions of equation (1.3), the Lipschitz property of the right-hand side, F, is of paramount importance for the uniqueness of solutions, making it possible to define the Poincaré map and, consequently, the displacement function. Note that if we did not have this property, then there would be ambiguities in the choice of the solution used to define π. Moreover, the smoothness of F plays a crucial role by guaranteeing the differentiable dependence of the solutions with respect to the parameters and initial conditions, otherwise it would not be possible to extend the Poincaré function about $\varepsilon = 0$.

Thus, one can pursue this direction of finding the minimal regularity conditions under which it is possible to provide sufficient conditions for the existence of periodic solutions.

There has been already some interest in applying the averaging procedure to less regular systems. In [5] the authors consider the first order averaging for equation (1.9)

$$x' = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon),$$

removing, at first, the differentiability and assuming the locally Lipschitz property in the x variable of F_1 and R and later they outline the proof for the non-Lipschitz case. In both cases, the authors use the Brouwer degree to replace the condition $\det Df_1(z_0) \neq 0$ present in the classical averaging approach.

In [20] the authors develop an arbitrary order Melnikov analysis of equation (1.5)

$$x' = \varepsilon F_1(t, x) + \cdots + \varepsilon^k F_k(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon),$$

assuming that F_i is C^{k-i}, $\partial_x^{k-i} F_i$ and R are locally Lipschitz continuous in the x variable for each $i \in \{1, \ldots, k\}$, where $\partial_x^i F$ is the $i-$th derivative of F with respect to x. In this work, the authors provide explicit expressions for the Melnikov functions.

In [18] the authors consider a second order Melnikov analysis, providing the corresponding Melnikov functions, for the differential equation

$$x' = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon),$$

where each F_i is defined by parts in $\mathbb{S}^1 \times D$ and the boundary of each part is a piecewise smooth hypersurface. In this case, the trajectories that intersect the boundaries are given following the Filippov's convention.

It is important to note that in all of the above cases, uniqueness of solutions is guaranteed, allowing for a Taylor-like expansion of the displacement function, so that at the core, it is used a Melnikov analysis. In turn, the Brouwer degree is the main tool used to guarantee the continuation of zeroes of the corresponding averaged or Melnikov function.

1.6 The purpose of this work

From what we observed in the previous section, it is interesting to study an averaging- or Melnikov-like analysis for differential equations under low regularity conditions. The lack of Lipschitz continuity and differentiability suggests that the approach presented in Section 1.2 and Section 1.3 might not be directly applicable to provide existence results for this type of differential equation.

It can be seen that there are still some cases lacking a feasible approach to study their periodic solutions. For example, as far as we know there is no result providing sufficient conditions for the existence of periodic solution for equation (1.5)

$$x' = \varepsilon F_1(t,x) + \cdots + \varepsilon^k F_k(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

with arbitrary k when the functions F_i are continuous non-Lipschitz or Carathéodory.

We point out that in both cases the property of uniqueness of solutions is lost, so that the aforementioned approach of studying the displacement function of these systems is not quite feasible. Thus, in order to work around that, we formulate an operator equation defined in some Banach space whose solutions correspond to periodic solutions of the underlying differential equation. This approach implies the introduction of techniques that differ very much from those described in this chapter.

Thus the purpose of this work is to study the differential equations whose right-hand sides are not smooth. Therefore we carry out an analysis similar to the Melnikov method aiming at detecting periodic solutions of differential equations such as equation (1.3) and equation (1.5) where the functions on the right-hand side are not Lipschitz. We also study the case of Carathéodory right-hand sides, which generalizes the non-Lipschitz case and where continuity may fail too.

This work is structured as follows. In Chapter 2 we provide a few known results that will be used in this work for the sake of completeness. In Chapter 3, we address differential equations with continuous right-hand sides. In Chapter 4, we treat the case of Carathéodory right-hand sides. In Chapter 5 we study the convergence of the solutions with respect to the parameter ε. Finally, in Chapter 6 we apply our results to concrete examples of differential equations demonstrating the applicability of the theorems obtained.

2 Preliminaries

In this chapter, we present the already established results on functional analysis, measure theory, ordinary differential equations and related subjects that are used throughout this thesis. We make clear that the extensive explanation of such topics is not the main goal of this work, so that we do not spend too much energy proving the results that appear in this chapter, we rather refer the reader to the suitable references.

2.1 Relevant topics on functional analysis

In the sequel we present the main facts and theorems used in this work. We assume some basic knowledge of metric spaces and do not define all the concepts that show up.

2.1.1 Arzelá-Ascoli Theorem

The Arzelá-Ascoli Theorem is classical one when it comes to proving that a certain subset in a metric space has compact closure or, equivalently, that a certain sequence has a convergent subsequence. In particular, we shall use this theorem in the context of function spaces. This type of result will play a part when we discuss compact operators. For more details and a proof of this theorem, see [17, Chapter 8].

We start by making some definitions. Let (X, d) be a compact metric space and $\mathcal{F}$ a family of functions $\varphi : X \to \mathbb{R}^n$.

Definition 2.1. *We say that $\mathcal{F}$ is an* equicontinuous *family if for each $x \in X, \varepsilon > 0$, there exists $\delta > 0$, such that for every $y \in X$, with $d(x, y) < \delta$, we have $|\varphi(x) - \varphi(y)| < \varepsilon$, for each $\varphi \in \mathcal{F}$.*

Note that in this definition, δ depends only on x and ε, not on the particular element of the family $\mathcal{F}$.

Definition 2.2. *$\mathcal{F}$ is said to be a* pointwise bounded *family if for each $x \in X$, there exists a positive constant M such that $|\varphi(x)| < M$, for every $\varphi \in \mathcal{F}$.*

In the above definition, when the constant M does not depend on x, it is usual to call $\mathcal{F}$ a *uniformly bounded* family.

We can now state the Arzelá-Ascoli Theorem.

Theorem 2.1 ([17, Proposição 16]). *Let (X, d) be a compact metric space. Let $\mathcal{F}$ be a family of equicontinuous functions $\varphi : X \to \mathbb{R}^n$. If $\mathcal{F}$ is pointwise bounded, then every sequence $\{\varphi_n\}$ of elements of $\mathcal{F}$ has a uniformly convergent subsequence $\{\varphi_{n_k}\}$ in X.*

It is important to remark that the conclusion of the Arzelá-Ascoli Theorem is equivalent to saying that the family $\mathcal{F}$ has compact closure or, in other words, is relatively compact ([15]). We make this remark because both understandings of the Arzelá-Ascoli Theorem will be used in the text.

2.1.2 Normed spaces and operators

For this section, except otherwise mentioned, let X and Z be real normed spaces.

Definition 2.3 ([16, Chapter 2]). *The* codimension *of a vector subspace $Y \subset X$ is defined as* $\text{Codim } Y = \dim X/Y$.

Definition 2.4 ([12, Chapter 3]). *If $L\colon X \to Z$ is a linear operator, the space $Z/\text{Im } L$ is called the* cokernel *of L and is denoted by* $\text{Coker } L$.

Notice, in particular, that since the cokernel of a linear operator is given by a quotient, then there exists a canonical projection $\Pi : Z \to \text{Coker } L$ given by

$$\Pi(z) = [z] = z + \text{Im } L, \qquad z \in Z.$$

Moreover, Π is continuous. For this work, this is all the understanding one should have about quotient spaces. For the reader interested in more information on quotient spaces, we refer the reader to [22].

Proposition 2.1 ([8, Chapter 2, Proposition 7.10]). *Let X and Z be Banach spaces and $L : X \to Z$ a continuous linear operator. Then,* $\text{Im } L$ *is a closed subspace of Z if* $\text{Codim Im } L < \infty$.

Recall the definition of a bounded operator, even when this operator is not necessarily linear.

Definition 2.5. *Let $\Omega \subset X$ be any subset of X. An operator $N : \Omega \to Z$ (not necessarily linear) between Banach spaces is said to be* bounded *if it maps bounded sets into bounded sets, i.e., $N(A)$ is bounded for every $A \subset \Omega$ bounded.*

The following definition sets apart an important class of operators which play a major role in the future developments on this thesis.

Definition 2.6 ([8, Chapter 8]). *An operator $F : \Omega \to Z$ is said to be completely continuous if it is continuous and such that for each bounded set $B \subset \Omega$, the set $F(B)$ is relatively compact.*

The next Theorem tells us that the composition of a completely continuous map with a continuous one is still completely continuous.

Theorem 2.2 ([8]). *Let X, Y, Z be Banach spaces, $\Omega \subset X$ any subset of X, $K : \Omega \to Y$ a completely continuous operator and $F : Y \to Z$ any continuous map. Then, $F \circ K : \Omega \to Z$ is a completely continuous.*

2.1.3 Fredholm operators

In this subsection, we give a definition for Fredholm operators of index zero and discuss some facts associated to them.

Here we introduce a key notion for our exposition.

Definition 2.7 (Fredholm Operator, [8]). *Let X and Z be real Banach spaces. We say that $L : \text{dom } L \subset X \to Z$ is a Fredholm operator if $\text{Im } L$ is closed subspace of Z and $\dim \text{Ker } L$ and $\dim \text{Coker } L$ are finite.*

Associated to every Fredholm operator there is the concept of index which we define below.

Definition 2.8 (Index of a Fredholm Operator, [8]). *We define the Fredholm index of L as $\text{Ind } L = \dim \text{Ker } L - \dim \text{Coker } L$.*

In this work, we are going to work only with Fredholm operators of index zero, that is, $\dim \text{Ker } L = \dim \text{Coker } L$. It is known from linear algebra that one can always complement a space using the quotient space, see [14]. In particular, $Z = \text{Im } L \oplus (Z/\text{Im } L) = \text{Im } L \oplus \text{Coker } L$. In the case of a Fredholm operator of index zero, $\text{Ker } L$ is isomorphic to $\text{Coker } L$, so that $Z \simeq \text{Im } L \oplus \text{Ker } L$. Thus, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that the sequence

$$X \xrightarrow{P} \text{dom } L \xrightarrow{L} Z \xrightarrow{Q} Z \tag{2.1}$$

is exact, which means that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q$. Notice that, in particular, $\text{Ker } P \cap \text{Ker } L = \{0\}$, because if $x \in \text{Ker } P \cap \text{Ker } L$, then since $\text{Im } P = \text{Ker } L, x = Py$, for some $y \in X$, and since P is a projector and $x \in \text{Ker } P$, $0 = Px = P^2y = Py = x$. Furthermore, $X = \text{Ker } P \oplus \text{Ker } L$ and $Z = \text{Im } L \oplus \text{Im } Q$. Since $\text{Ker } P \cap \text{Ker } L = \{0\}$, it follows that $L_{|\text{Ker } P}$ is invertible. Define $K_P = L^{-1}_{|\text{Ker } P}$ and $K_{P,Q} = K_P(\text{Id} - Q)$. This last operator can be thought of as a generalized inverse of L. Indeed, since L is not necessarily invertible, K_P is only defined on the proper subset Im L of Z, so that the points of Z lying outside of Im L are then projected onto Im L = Ker Q through Id $- Q$.

We finish this section with another key definition.

Definition 2.9 ($L-$Compact Operator, [12])**.** *We say that the operator $N\colon \overline{\Omega} \to Z$ is $L-$Compact in $\overline{\Omega}$ if $\Pi N\colon \overline{\Omega} \to$ Coker L is a continuous and bounded operator and $K_{P,Q}N\colon \overline{\Omega} \to X$ is completely continuous.*

Assume that L is the identity map $L = \text{Id} : X \to X$ and $N : \Omega \subset X \to X$ is $L-$compact. First, notice that Ker $L = \{0\}$ and Coker $L = X/\text{Im } L = \{0\}$, showing that the identity map is a Fredholm operator of index zero. In this case, the canonical projection, $\Pi : Z \to$ Coker L, takes every element of Z to 0, so that $\Pi \equiv 0$, the projection $P : X \to X$ must have the image equal to the kernel of L, which yields $P \equiv 0$, and the kernel of $Q : Z \to Z$ must equal the image of L, thus $Q \equiv 0$. Hence, $\Pi N \equiv 0$ and $K_{P,Q} = \left(L_{|\text{Ker } P}\right)^{-1} (\text{Id} - Q) = \text{Id}$. Hence, $L-$compactness of N when $L = \text{Id}$ reduces to N being completely continuous (Definition 2.6). Therefore, the concept of $L-$compactness generalizes that of complete continuity.

The pair (L, N) where L is a linear Fredholm operator of index zero and N is $L-$compact is a fundamental piece in the development of the main contributions of this work and the role they play will become clear in Section 2.5 and even clearer in Chapter 3 and Chapter 4.

2.2 Essentials of measure theory

In this section, we enlist some definitions and results that will be used in Chapter 4 where the Carathéodory equations will be studied. The results present in this section can be found in the classical references on measure theory, for example [2, 10, 28].

Unless otherwise mentioned, in this section (X, μ) denotes a measure space and Y a topological space.

We start by the basic concept of a measurable function.

Definition 2.10 ([28])**.** *A function $f : X \to Y$ is said to be measurable if $f^{-1}(W)$ is a measurable set of X for every open set $W \subset Y$.*

The next proposition gives us a criterium to decide whether a given function is measurable or not.

Proposition 2.2 ([2, Chapter 5]). *If $f : X \to \mathbb{R}^n$ is measurable, $g : X \to \mathbb{R}$ is integrable and $|f| \leqslant |g|$, then f integrable and*

$$\int |f| \mathrm{d}\mu \leqslant \int |g| \mathrm{d}\mu.$$

The next theorem is the celebrated Lebesgue dominated convergence Theorem, which roughly speaking, gives us conditions under which the integral and limits signs can be interchanged.

Theorem 2.3 (Dominated Convergence Theorem). *Let $(f_m)_{m\in\mathbb{N}}$ be a sequence of integrable functions $f_m : X \to \mathbb{R}$ which converges almost everywhere to a real-valued measurable function $f : X \to \mathbb{R}$. If there exists an integrable function g such that $|f_m(x)| \leqslant |g(x)|$ for almost every $x \in X$ and all $m \in \mathbb{N}$, then f is integrable and*

$$\int f \mathrm{d}\mu = \lim \int f_m \mathrm{d}\mu.$$

We remark that if for each $m \in \mathbb{N}$, f_m is an integrable function from the measure space X into $\mathbb{R}^n$, then $f_m = (f_{m1}, \ldots, f_{mn})$, where each $f_{mj} : X \to \mathbb{R}$ is an integrable function. Then, for each $j \in \{1, \ldots, n\}$ we have a sequence $(f_{mj})_{m\in\mathbb{N}}$ of real-valued functions. Assuming that $\lim f_{mj} = F_j$ almost everywhere in $x \in X$ for each $j \in \{1, \ldots, n\}$ as $m \to \infty$. If, in addition, for each $j \in \{1, \ldots, n\}$ there exists an integrable real-valued function $g_j : X \to \mathbb{R}$ such that $|f_{mj}(x)| \leqslant g_j(x)$ for almost every $x \in X$ and every $m \in \mathbb{N}$, then Theorem 2.3 can be applied to each sequence $(f_{mj})_{m\in\mathbb{N}}$ with g_j and $f = (F_1, \ldots, F_n)$. Hence the following is true:

Theorem 2.4. *Let $(f_m)_{m\in\mathbb{N}}$ be a sequence of integrable functions $f_m : X \to \mathbb{R}^n$ which converges almost everywhere to a measurable function $f : X \to \mathbb{R}^n$. If there exists an integrable function g such that $|f_m(x)| \leqslant |g(x)|$ for almost every $x \in X$ and all $m \in \mathbb{N}$, then f is integrable and*

$$\int f \mathrm{d}\mu = \lim \int f_m \mathrm{d}\mu.$$

2.3 Some results on ordinary differential equations

Besides the well-known classical definitions of ordinary differential equations and their solutions, in this subsection we comment on an extended notion of ordinary differential equations and their solutions as well as provide existence and uniqueness results for these extended differential equations. We point out that this other notion of differential equation will be important when dealing with more general equations.

First, for completeness, let us recall the usual notion of solution of an ordinary differential equation and the classical existence and uniqueness theorem.

Consider the differential equation

$$x' = f(t, x, \varepsilon), \qquad (t, x, \varepsilon) \in I \times D \times (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}. \tag{2.2}$$

Then, by adjoining the initial condition

$$x(t_0) = x_0 \quad \text{for some} \quad t_0 \in I \quad \text{and} \quad x_0 \in D \tag{2.3}$$

to equation (2.2), we obtain what is called an initial value problem. A solution for the initial value problem (2.2)-(2.3) is a C^1 function $\varphi : I_0 \subset I \to \mathbb{R}^n$ defined on an interval I_0 containing t_0 such that

1. $\varphi(t_0) = x_0$;
2. $(t, \varphi(t)) \in I \times D$ for every $t \in I_0$; and
3. $\varphi'(t) = f(t, \varphi(t), \varepsilon)$ for every $t \in I_0$.

It follows directly from the definition above that φ is a solution of equation (2.2) if, and only if, it also satisfies the integral equation

$$\varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s), \varepsilon)\mathrm{d}s, \qquad \text{for every} \qquad t \in I_0.$$

Having that, we state the celebrated Picard-Lindelöf Theorem:

Theorem 2.5 (Picard-Lindelöf Theorem, [13])**.** *Consider the initial value problem* (2.2)*-*(2.3)*. If f is a continuous function and locally Lipschitz in the second variable, then there exists a unique solution φ for* (2.2)*-*(2.3) *for each tuple* $(t_0, x_0, \varepsilon) \in I \times D \times (-\varepsilon_0, \varepsilon_0)$.

In many applications, the above setting is adequate because the function f is smooth, thus, having continuous derivatives in both t, x and ε, so that f meets the hypotheses of Theorem 2.5. However, there are applications where the right-hand side of equation (2.2) is either not Lipschitz in x (as for instance in [11]) or continuous in t (see [3, Theorem 5.1.1]), or neither (see [25]). When the right-hand side of the differential equation (2.2) is continuous but lacks the Lipschitz property, we know that solutions exist, even though it is not guaranteed that they are unique.

The case where the differential equation (2.2) is not continuous may appear in many different fashions. In this work, we consider the case where $f(t, x, \varepsilon)$ is continuous in the (x, ε) variables for most of the values of t and may have some discontinuities in the variable t for fixed (x, ε). We shall elaborate on this in the sequel. An interesting fact, however, is that it is still possible to guarantee existence of solutions and even uniqueness, in some cases.

A central concept in this discussion is that of a Carathéodory function. These functions play a major role in this work, since Chapter 4 is entirely devoted to the study of the differential equations whose right-hand sides are Carathéodory functions. As we will see, this type of function may have discontinuities, but in some sense, not too many of them.

Definition 2.11 ([7]). *Let $I \subset \mathbb{R}$ be any open interval, $D \subset \mathbb{R}^n$ an open subset of $\mathbb{R}^n$ and $\varepsilon_0 > 0$. We say that a function $f : I \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a Carathéodory function if the following conditions are satisfied:*

C.1 *$(x, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0) \mapsto f(t, x, \varepsilon)$ is continuous for almost every $t \in I$;*

C.2 *$t \in I \mapsto f(t, x, \varepsilon)$ is measurable for every $(x, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$;*

C.3 *for each $r > 0$, there exists a positive integrable function $g_r : I \to \mathbb{R}$ such that for almost every $t \in I$, $|f(t, x, \varepsilon)| \leqslant g_r(t)$ whenever $|(x, \varepsilon)| \leqslant r$.*

The following result will allow us to decide whether some functions are Carathéodory.

Proposition 2.3. *Let $F : I \times D \times (-\varepsilon_0, \varepsilon_0) \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be such that*

$$F(t, x, \varepsilon) = p(t)h(x, \varepsilon),$$

where $p : I \to \mathbb{R}$ is a measurable function and $h : D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ is continuous. If there exists a positive integrable function $g : I \to \mathbb{R}$ such that

$$|p(t)| \leqslant g(t),$$

for every $t \in I$, then, F is a Carathéodory function.

Proof. We will check that F satisfies the conditions C.1–C.3. For each $t \in I$ fixed, $F(t, \cdot, \cdot)$ is a constant times h, thus it is continuous. So, C.1 holds. For each $(x, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$ fixed, $F(\cdot, x, \varepsilon)$ is a constant times p, thus it is measurable. Hence, C.2 holds. Given $r > 0$, define $g_r : I \to \mathbb{R}$ by setting

$$g_r(t) = g(t) \sup\{|h(x, \varepsilon)| : (x, \varepsilon) \in \overline{B(0, r)} \cap D\}.$$

Then, g_r is a positive integrable function and

$$|F(t, x, \varepsilon)| = |p(t)h(x, \varepsilon)| \leqslant g_r(t),$$

for every $(x, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$ such that $|(x, \varepsilon)| < r$ and every $t \in I$. Thus, C.3 holds. It follows that F is a Carathéodory function. □

Example 2.1. *Consider the function* $F : [0, 2\pi] \times (0, +\infty) \to \mathbb{R}$ *given by*

$$F(\theta, r) = \sin\theta \operatorname{sgn}(\sin\theta) \sqrt[3]{r^2 - 1}.$$

Notice that this function can be written as $F(\theta, r) = p(\theta)h(r)$, *where* $p(\theta) = \sin\theta \operatorname{sgn}(\sin\theta)$ *and* $h(r) = \sqrt[3]{r^2 - 1}$. *We have that* p *is an integrable and bounded function and* h *is continuous. Thus, taking* $g = |p|$, *by Proposition 2.3 it follows that* F *is a Carathéodory function.*

Example 2.2. *Consider* $F : [0, 2\pi] \times (0, +\infty) \to \mathbb{R}$ *given by*

$$F(\theta, r) = \sin\theta \operatorname{sgn}(\sin\theta) \operatorname{sgn}\left(r^2 - 1\right) \max\left\{0, \left(r^2 - \frac{1}{4}\right)\left(r^2 - \frac{9}{4}\right)\right\}.$$

This function can also be written as $F(\theta, r) = p(\theta)h(r)$, *with* $p(\theta) = \sin\theta \operatorname{sgn}(\sin\theta)$ *and* $h(r) = \operatorname{sgn}\left(r^2 - 1\right) \max\left\{0, \left(r^2 - \frac{1}{4}\right)\left(r^2 - \frac{9}{4}\right)\right\}$. *The function* p *is the same used in the previous example and it is integrable and bounded. On the other hand,* h *can be proved to be continuous for* $r > 0$. *Indeed, the only issue we might have is at* $r = 1$. *Notice that* $h(1) = 0$, *since* $\operatorname{sgn}(0) = 0$. *Taking* $r \in [3/4, 5/4] \subset (1/2, 3/2)$, *we see that* $h(r) = 0$, *which shows that* h *is continuous on* $[3/4, 5/4]$ *and, in particular, at* $r = 1$. *By Proposition 2.3, it follows that* F *is a Carathéodory function.*

For the sake of completeness, we recall the definition of an absolutely continuous function.

Definition 2.12 ([10, Chapter 3])**.** *A function* $\varphi : I \to \mathbb{R}^n$ *is said to be an absolutely continuous function if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any finite collection of closed disjoint intervals* $\{[a_j, b_j] : a_j < b_j\}_{j=1,\dots,m}$ *contained in* I,

$$\sum_{j=1}^{m} |b_j - a_j| < \delta$$

implies

$$\sum_{j=1}^{m} |\varphi(b_j) - \varphi(a_j)| < \varepsilon.$$

A useful characterization of absolute continuity is the following theorem:

Theorem 2.6 ([10, Chapter 3])**.** *A function* $\varphi : [a, b] \to \mathbb{R}^n$ *is absolutely continuous if, and only if, for every* $t \in [a, b]$

$$\varphi(t) = \varphi(a) + \int_a^t f(s)\mathrm{d}s$$

for some integrable function $f : [a, b] \to \mathbb{R}^n$.

From the above theorem, it is straightforward that an absolutely continuous function has derivatives for almost every point in its domain.

In the sequel, we extend our notion of solution for differential equations.

Definition 2.13 ([7, Chapter 2]). *A function $\varphi : I_0 \subset I \to \mathbb{R}^n$ is a solution of (2.2) in the extended sense if it is an absolutely continuous function satisfying*

1. $(t, \varphi(t)) \in I_0 \times D$ *for every* $t \in I_0$; *and*
2. $\varphi'(t) = f(t, \varphi(t), \varepsilon)$ *for almost every* $t \in I_0$.

Note that just like before, the second condition in the above definition is equivalent to

$$\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s), \varepsilon)\mathrm{d}s, \qquad \text{for every} \qquad t \in I_0.$$

Let $A = [t_0 - \tau, t_0 + \tau] \times \overline{B(x_0, \xi)} \times [-\varepsilon_1, \varepsilon_1]$, where $t_0 \in I_0, \tau > 0, \xi > 0$ and $0 < \varepsilon_1 < \varepsilon_0$ such that $A \subset I \times D \times (-\varepsilon_0, \varepsilon_0)$. The following theorem assures the existence of solutions in the extended sense (Definition 2.13) for equation (2.2) in A.

Theorem 2.7 ([7, Chapter 2]). *Let f be defined in A and assume that it satisfies the Carathéodory conditions C.1–C.3. Then there exists a solution φ of the initial value problem (2.2)–(2.3) in the extended sense defined on some interval $[t_0 - \beta, t_0 + \beta]$, with $0 < \beta \leqslant \tau$.*

Henceforth, we drop the term "in the extended sense", since it shall always be clear what type of solution we will be talking about in each context.

An important fact to establish about the solutions of Carathéodory differential equations is that they can be continued to maximal solutions in their domain of definition.

Theorem 2.8 ([9, Chapter 1]). *Let $V \subset \mathbb{R}^n$ be a bounded open set such that $\overline{V} \subset D$ and $x_0 \in V$. If f satisfies the Carathéodory condition, then any solution of initial value problem (2.2)-(2.3) can be continued on both sides up to the boundary of V.*

Although Carathéodory differential equations are a lot less regular than continuous ones, it is still possible to provide uniqueness results, so that one could try and follow the approach of studying the Poincaré map. One way to do that is requiring some sort of Lipschitz condition as shown in the following theorem.

Theorem 2.9 ([9, Chapter 1]). *Assume that there exists an integrable function $l : I \to \mathbb{R}$ such that for each (t, x, ε) and (t, y, ε) in $I \times D \times (-\varepsilon_0, \varepsilon_0)$ the following relation holds*

$$|f(t, x, \varepsilon) - f(t, y, \varepsilon)| \leqslant l(t)|x - y|.$$

Then there exists at most one solution of the initial value problem (2.2)-(2.3) *in* $I \times D \times (-\varepsilon_0, \varepsilon_0)$.

However, we are not that interested in having uniqueness of solutions, since we are able to provide a framework where this property is no longer necessary. In Chapter 4 we shall discuss in more detail the intricacies involved in dealing with Carathéodory differential equations.

2.4 Brouwer degree

In this section we present one of the central tools used to establish the results in this thesis.

As previously discussed in the Introduction, the study of periodic solutions of differential equations through zeroes of the Poincaré map works just fine when the right-hand side, and consequently the Poincaré map, of the differential equation is smooth, since the device used is the Implicit Function Theorem. However, if smoothness of the Poincaré map cannot be assured, the analysis could be compromised.

This is the point where the Brouwer degree comes into play. Roughly speaking, the Brouwer degree indicates whether there is a zero of a function in a certain domain or not. Thus, in practice, using the Brouwer degree, one can arrive at the same conclusions as those discussed in the Introduction without the use of the Implicit Function Theorem.

Consider the equation

$$f(x) = y_0, \quad x \in V, \tag{2.4}$$

where $f : \overline{V} \subset \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function defined on the closure of the open and bounded subset V of $\mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$ is an arbitrary constant vector.

The Brouwer degree is a function, denoted d_B, that assigns to every *admissible* triple (f, V, y_0) (Definition 2.14) an integer, $d_B(f, V, y_0)$, which gives information about the existence of solutions of equation (2.4).

Definition 2.14. *Consider a function* $f\colon X \to Y$, *an open bounded subset* V *of* X *such that* $\overline{V} \subset X$, *and a point* $y_0 \in Y$. *The triple* (f, V, y_0) *is said to be admissible if* f *is continuous and* $y_0 \notin f(\partial V)$.

The Brouwer degree is defined as the only integer-valued function satisfying the properties in the following theorem (see [4, 8]).

Theorem 2.10 ([8, Chapter 1]). *Let* $X = \mathbb{R}^n = Y$ *for some positive integer* n, $f : \overline{V} \subset X \to Y$ *a continuous function on* $\overline{V}$, *where* V *is a bounded open subset of* X *and* $y_0 \in Y$

such that (f, V, y_0) is admissible. For each triple (f, V, y_0) there corresponds an integer $d_B(f, V, y_0)$ satisfying the following conditions:

B.1 *If $d_B(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. Furthermore, if $f_0 : X \to Y$ is the identity function of X onto Y, then for any bounded open subset $V \subset X$ we have*

$$d_B(f_{0|V}, V, y_0) = 1.$$

B.2 *(Additivity) If $V_1, V_2 \subset V$ are disjoint open subsets of V such that $y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2))$, then*

$$d_B(f, V, y_0) = d_B(f_{|V_1}, V_1, y_0) + d_B(f_{|V_2}, V_2, y_0).$$

B.3 *(Invariance under Homotopy) Consider a homotopy $\{f_t : \overline{V} \to Y | t \in [0, 1]\}$. Let $\{y_t \in Y | t \in [0, 1]\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$, $\forall t \in [0, 1]$. Then $d_B(f_t, V, y_t)$ is constant in t.*

Observe that B.1 is the *raison d'être* of the Brouwer degree. It tells us, in particular, that when $d_B(f, V, y_0) \neq 0$ there exists at least one solution of equation (2.4) in V.

On the other hand, B.2 and B.3 are more computational properties. The former stating that the calculation of the Brouwer degree can be broken into smaller domains, whereas the latter says that as long as you can continuously deform one function into another in a way that zeroes do not escape nor enter through the boundary of V the Brouwer degree remains constant.

Example 2.3. *As an example, consider a differentiable function $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ defined on the open and bounded set D. If we assume that $\det f'(x) \neq 0$, for every $x \in f^{-1}(y_0)$, then the Brouwer degree for this class of functions is given by*

$$d_B(f, D, y_0) = \sum_{x \in f^{-1}(y_0)} \operatorname{sgn} \det f'(x).$$

Note that if f is invertible, there exists only one solution for $f(x) = y_0$, given by $x_0 = f^{-1}(y_0)$ and $d_B(f, D, y_0) = \pm 1$, depending on whether $f'(x_0)$ preserves or reverses orientation.

Although the Brouwer degree can be calculated for any smooth function as shown above, its usefulness comes from the fact that the function does not need to be differentiable, only continuous. The next examples illustrate this possibility through the use of the property of invariance under homotopies.

Before we go through the examples, we provide a technical result that will aid us in analysing them. For real functions defined on intervals, we can explicitly calculate the Brouwer degree according to the next lemma.

Lemma 2.1. *Let a, b be real numbers such that $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a)f(b) < 0$, then $d_B(f, [a, b], 0) = \operatorname{sgn}(f(b) - f(a))$.*

Proof. Indeed, let $g : [a, b] \to \mathbb{R}$ be the function whose graph is the straight line connecting the points $(a, f(a))$ and $(b, f(b))$, i.e.,

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Notice that $g(a) = f(a), g(b) = f(b)$ and since g is a smooth function, by Example 2.3 we get

$$d_B(g, [a, b], 0) = \operatorname{sgn}(g'(x^*)) = \operatorname{sgn}\left(\frac{f(b) - f(a)}{b - a}\right) = \operatorname{sgn}(f(b) - f(a)),$$

where x^* is the unique zero of g in the interval $[a, b]$, which exists since $f(a)f(b) < 0$. Define the homotopy $H : [a, b] \times [0, 1] \to \mathbb{R}$ by setting

$$H(x, t) = (1 - t)f(x) + tg(x).$$

First of all, H is clearly a continuous function. Moreover, $H(x, 0) = f(x), H(x, 1) = g(x)$ for every $x \in [a, b], H(a, t) = f(a)$ and $H(b, t) = f(b)$ for every $t \in [0, 1]$, so that $H(x, t) \neq 0$ for every $x \in \partial[a, b] = \{a, b\}$ and every $t \in [0, 1]$. Hence by property B.3 of invariance under homotopies, it follows that $d_B(H(\cdot, t), [a, b], 0)$ is constant in t. Thus,

$$\begin{aligned} d_B(f, [a, b], 0) &= d_B(H(\cdot, 0), [a, b], 0) = d_B(H(\cdot, 1), [a, b], 0) = d_B(g, [a, b], 0) \\ &= \operatorname{sgn}(f(b) - f(a)), \end{aligned}$$

as desired. □

As an application of this lemma, we work out explicitly the Brouwer degree of some functions.

Example 2.4. *Let $f : [0, 2] \to \mathbb{R}$ be defined by*

$$f(r) = \frac{r\sqrt[3]{1 - r^2}}{2}.$$

Observe that f is not differentiable at $r^* = 1$, which is a zero of f, see Figure 1. Therefore, we cannot apply the formula in Example 2.3 to calculate its Brouwer degree. On the other hand, since $f(r) > 0$ for $0 < r < 1$ and $f(r) < 0$ for $r > 1$, it follows from Lemma 2.1 that for any subinterval $[r_1, r_2] \subset [0, 2]$ such that $0 < r_1 < 1 < r_2 \leqslant 2$,

$$d_B(f, [r_1, r_2], 0) = \operatorname{sgn}(f(r_2) - f(r_1)) = -1.$$

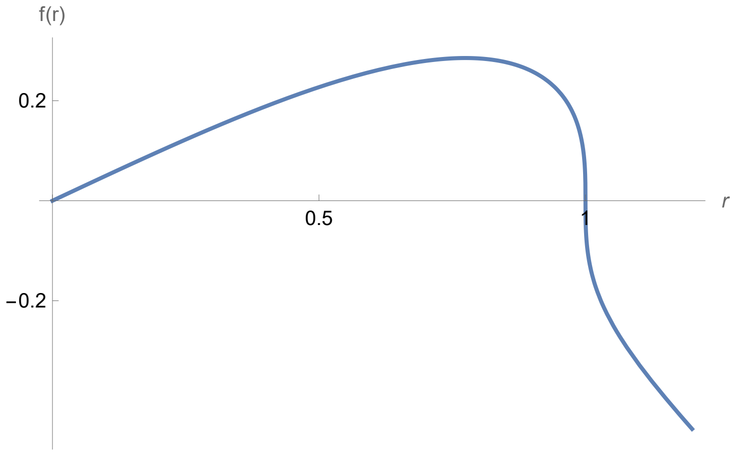

Figure 1 – The graph of $f(r) = \dfrac{r\sqrt[3]{r^2-1}}{2}$.

As we mentioned in Chapter 1, the Brouwer degree arises as a natural tool to replace the hypothesis on the derivative of the Melnikov function which allows one to apply the Implicit Function Theorem. The condition of some x_0 being a simple zero of, say, M_ℓ, that is, $M_\ell(x_0) = 0$ and $\det DM_\ell(x_0) \neq 0$ in particular guarantees that x_0 is an isolated zero of M_ℓ. So, even when M_ℓ is a smooth function, the Implicit Function Theorem cannot be directly applied if its zeroes are not isolated. The Brouwer degree also covers this case.

The next example illustrates the case where the set $f^{-1}(0)$ is a continuum.

Example 2.5. *Consider the function* $f : [0, 2] \to \mathbb{R}$

$$f(r) = \frac{2}{\pi}\,\mathrm{sgn}\left(1 - r^2\right)\max\left\{0, \left(r^2 - \frac{1}{4}\right)\left(r^2 - \frac{9}{4}\right)\right\},$$

whose graph is shown in Figure 2. From Lemma 2.1 follows that for any interval $[r_1, r_2] \subset [0, 2]$ *such that* $0 \leqslant r_1 < 1/2$ *and* $3/2 < r_2 \leqslant 2$,

$$d_B(f, [r_1, r_2], 0) = \mathrm{sgn}(f(2) - f(0)) = -1.$$

A handy property of the Brouwer degree is presented in the following proposition. It tells us that the calculation of the Brouwer degree may be carried out in a smaller domain provided there is no solution for the equation $f(x) = y_0$ outside this domain. It is actually a direct consequence of B.2 by taking $V_2 = \varnothing$ and declaring that the Brouwer degree over the empty set is zero, which is a fairly reasonable convention.

Proposition 2.4 (Excision Property, [8, Chapter 1])**.** *Let* $f : \overline{V} \to Y$ *be as in the above theorem. If* $V_1 \subset V$ *is open and* $y_0 \notin f(\overline{V}\backslash V_1)$, *then*

$$d_B(f, V, y_0) = d_B(f_{|V_1}, V_1, y_0).$$

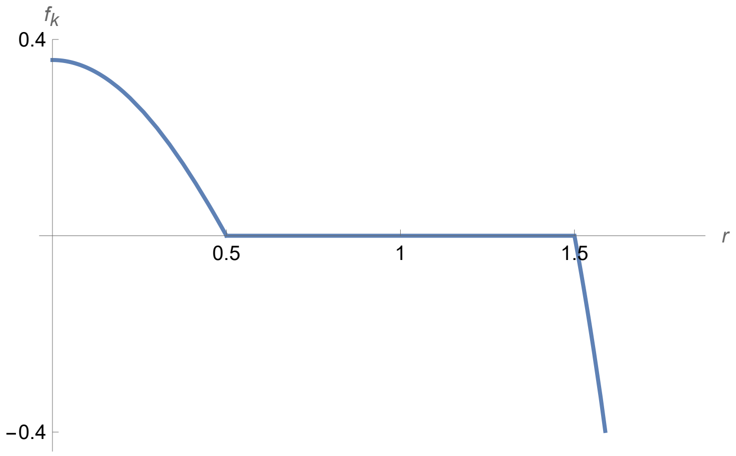

Figure 2 – The graph of $f(r) = -\frac{2}{\pi}\,\mathrm{sgn}\left(1-r^2\right)\max\left\{0, \left(r^2-\frac{1}{4}\right)\left(r^2-\frac{9}{4}\right)\right\}$.

The Brouwer degree also possesses the property of being constant in a neighbourhood of the function.

Proposition 2.5 (Local constancy, [8, Theorem 3.1 (d5)])**.** $d_B(g,V,y_0) = d_B(f,V,y_0)$ *for every continuous function* $g : \overline{V} \to \mathbb{R}^n$ *such that* $|g-f| = \sup_{x\in\overline{V}} |f(x)-g(x)| <$ $\mathrm{dist}\,(y_0, f(\partial V))$.

The next proposition is another invariance result of the Brouwer degree, which says that for a family of functions depending on a small parameter, the Brouwer degree does not vary.

Proposition 2.6 ([6])**.** *Let V be an open bounded subset of $\mathbb{R}^m$. Consider the continuous functions $f_i : \overline{V} \to \mathbb{R}^n$, $i = 0, 1, \cdots, k$, and $f, g, r : \overline{V} \times [0, \varepsilon_0] \to \mathbb{R}^n$ given by*

$$g(z,\varepsilon) = f_0(z) + \varepsilon f_1(z) + \cdots + \varepsilon^k f_k(z) \text{ and } f(z,\varepsilon) = g(z,\varepsilon) + \varepsilon^{k+1} r(z,\varepsilon).$$

Let $V_\varepsilon \subset V$, $R = \max\{|r(z,\varepsilon)| : (z,\varepsilon) \in \overline{V} \times [0,\varepsilon_0]\}$ and assume that $|g(z,\varepsilon)| > R|\varepsilon|^{k+1}$ for all $z \in \partial V_\varepsilon$ and $\varepsilon \in (0,\varepsilon_0]$. Then, for each $\varepsilon \in (0,\varepsilon_0]$ we have $d_B\left(f(\cdot,\varepsilon), V_\varepsilon, 0\right) = d_B\left(g(\cdot,\varepsilon), V_\varepsilon, 0\right)$.

2.5 Continuation theorem

In this section we introduce a key result. As we shall see in the next chapters, the study of the periodic solutions of differential equations will be undertaken by means of the analysis of associated operator equations in some suitable Banach space. Then it is essential to build some knowledge on the theory of operator equations in Banach spaces.

Let X and Z be real Banach spaces, L be a continuous linear Fredholm operator of index zero and N be continuous $L-$compact operator defined on $\overline{\Omega} \times [0,1]$, where Ω is an open and bounded subset of X. Then, the next theorem gives us sufficient conditions for the existence of solutions of the operator equation

$$Lx = \lambda N(x, \lambda), \qquad x \in \overline{\Omega}. \tag{2.5}$$

Theorem 2.11 ([12, Chapter 4])**.** *Suppose* $L : \text{dom } L \subset X \to Z$ *is a linear Fredholm operator of index zero, and that* $N : \overline{\Omega} \times [0,1] \subset X \times \mathbb{R} \to Z$ *is* $L-$*compact on* $\overline{\Omega}$*. If* $Q : Z \to Z$ *is a continuous projector such that* $\text{Im } L = \text{Ker } Q$, $J : \text{Im } Q \to \text{Ker } L$ *is an isomorphism and*

A.1 $Lx \neq \lambda N(x, \lambda)$, *for every* $x \in (\text{dom } L \cap \partial\Omega) \times (0, 1)$;

A.2 $QN(x, 0) \neq 0$, *for each* $x \in \text{Ker } L \cap \partial\Omega$.

Moreover, if

$$d_B(JQN(\cdot, 0)_{|\text{Ker } L\cap\partial\Omega}, \text{Ker } L \cap \partial\Omega, 0) \neq 0,$$

then equation (2.5) *with* $\lambda \in [0, 1)$

$$Lx = \lambda N(x, \lambda), \quad x \in \overline{\Omega},$$

has at least one solution on Ω *and*

$$Lx = N(x, 1), \quad x \in \overline{\Omega},$$

has at least one solution on $\overline{\Omega}$.

Although in this work, we shall not discuss all the intricate details of the proof of such theorem, it is worth mentioning a few facts about it. The proof of this theorem relies on the theory of the coincidence degree (see Theorem 2.13 below) developed by J. Mawhin in [21] (see also [12] for a systematic development of the theory and further properties and consequences).

2.5.1 Some comments on degree theory in infinite dimensions

In this subsection, we discuss some topics about the inner workings of Theorem 2.11, although we do not prove it. It is intended for the reader to have a better grasp of the techniques used in this work.

The coincidence degree is indeed a degree function since it satisfies properties that are very similar to B.1–B.3 of the Brouwer degree. Its definition involves the notion of Leray-Schauder degree, which is a generalization of the Brouwer degree for compact perturbations of the identity defined on infinite dimensional vector spaces. Indeed, we have the following theorem

Theorem 2.12 ([8, Theorem 8.1]). *Let X be a real Banach space and*

$$T = \{(\mathrm{Id} - M, \Omega, y) : \Omega \subset X \text{ open bounded, } M \text{ compact on } \overline{\Omega} \text{ and } y \notin (\mathrm{Id} - M)(\partial\Omega)\}.$$

Then there exists exactly one function $d_{LS} : T \to \mathbb{Z}$, the Leray-Schauder degree, satisfying

LS.1 *If $d_{LS}(\mathrm{Id} - M, \Omega, y) \neq 0$, then $y \in (\mathrm{Id} - M)(\Omega)$. Moreover, $d_{LS}(\mathrm{Id}, \Omega, y) = 1$ for every $y \in \Omega$;*

LS.2 *$d_{LS}(\mathrm{Id} - M, \Omega, y) = d_{LS}(\mathrm{Id} - M, \Omega_1, y) + d_{LS}(\mathrm{Id} - M, \Omega_2, y)$, whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin (\mathrm{Id} - M)(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$;*

LS.3 *$d_{LS}(\mathrm{Id} - H(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times \overline{\Omega} \to X$ is compact on $[0, 1] \times \overline{\Omega}$, $y : [0, 1] \to X$ is continuous and $y(t) \notin (\mathrm{Id} - H(t, \cdot))(\partial\Omega)$ on $[0, 1]$.*

The integer $d_{LS}(\mathrm{Id} - M, \Omega, y)$ is given by $d_B((\mathrm{Id} - M_1)_{|\Omega_1}, \Omega_1, y)$, where $M_1 : \overline{\Omega} \to X$ is any compact map such that $M_1(\overline{\Omega}) \subset X_1$ with X_1 any subspace of X such that $\dim X_1 < \infty, y \in X_1, \sup_{x\in\overline{\Omega}} |M_1x - Mx| < \mathrm{dist}\ (y, (\mathrm{Id} - M)(\partial\Omega))$ and $\Omega_1 = \Omega \cap X_1$.

Notice that the Leray-Schauder degree is obtained by approximating the original operator M by an operator M_1 whose image lies in a finite dimensional subspace of X and calculating the Brouwer degree of this approximating operator. The main property of this degree is LS.1, which asserts that whenever $d_{LS}(\mathrm{Id} - M,\ y) \neq 0$, there exists $x \in \Omega$ such that $(\mathrm{Id} - M)x = y$. Notice, in particular, that if $y = 0$, then $d_{LS}(\mathrm{Id} - M, \Omega, 0) \neq 0$ implies that $x = Mx$, so that x is a fixed point of M.

Now, consider the operator equation

$$Lx = Nx, \quad x \in \Omega, \tag{2.6}$$

where $L : \mathrm{dom}\ L \subset X \to Z$ and $N : \overline{\Omega} \subset X \to Z$ are a linear Fredholm operator of index zero and an $L-$compact operator, respectively, with X a Banach space, dom L a subspace of X and Ω an open and bounded subset of X. Using the same notation as in Subsection 2.1.3, in [12, Chapter 3] it is shown that the set of solutions of equation (2.6) is equal to the set of fixed points of the operator $M = P + (\Lambda\Pi + K_{P,Q})N$, where $\Lambda : \mathrm{Coker}\ L \to \mathrm{Ker}\ L$ is an isomorphism. Moreover, being L a linear Fredholm operator of index zero, N an $L-$compact operator and if $0 \notin (L - N)(\partial\Omega)$, then M is compact and the coincidence degree of the pair (L, N) with respect to Ω, denoted $d((L, N), \Omega)$, is defined by

$$d((L, N), \Omega) = d_{LS}(\mathrm{Id} - M, \Omega, 0).$$

Thus, by using the properties of the Leray-Schauder degree we have the following theorem.

Theorem 2.13 ([12, Chapter 3]). *Let* $L : \text{dom } L \subset X \to Z$ *be a linear Fredholm operator of index 0 and* $N : \overline{\Omega} \subset X \to Z$ *be an* $L-$*compact operator in* $\overline{\Omega}$*, where* Ω *is an open and bounded subset of* X *and* $0 \notin (L - N)(\partial\Omega)$*. Then*

CD.1 *If* $d((L, N), \Omega) \neq 0$*, then there exists* $x \in \text{dom } L \cap \Omega$ *such that* $Lx = Nx$*.*

CD.2 $d((L, N), \Omega) = d((L, N), \Omega_1) + d((L, N), \Omega_2)$*. whenever* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *such that* $0 \notin (L - N)(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$*.*

CD.3 $d((L, \tilde{N}(\cdot, \lambda)), \Omega)$ *is independent of* $\lambda \in [0, 1]$ *whenever* $\tilde{N} : \overline{\Omega} \times [0, 1] \to Z$ *is continuous and* $L-$*compact on* $\overline{\Omega} \times [0, 1]$ *and* $0 \notin (L - \tilde{N}(\cdot, \lambda))(\partial\Omega)$*,* $\forall\, \lambda \in [0, 1]$*.*

The main property of this degree is CD.1 which asserts that if $d((L, N), \Omega) \neq 0$, then there is at least one solution of equation (2.5) in Ω, as in LS.1 and B.1.

Continuation theorems like Theorem 2.11 are usually proved using an $L-$compact homotopy $H(\lambda, x)$ where $\lambda \in [0, 1]$. Generally, for $\lambda = 0, H(0, x)$ is simple and one can prove that $d((L, H(0, \cdot)), \Omega) = d_B(JQN(\cdot, 0)_{|\text{Ker } L \cap \Omega}, \text{Ker } L \cap \Omega, 0) \neq 0$, then by CD.3, it follows that $d((L, N(\lambda, \cdot)), \Omega)$ is constant in λ which in turn guarantees the existence of solution for the desired equation.

2.5.2 The boundary condition

Our aim in this subsection is to bring some familiarity with this condition. We shall carry out a more detailed discussion thereon in the next chapter where, in fact, we provide an equivalent formulation for it in the particular context we are mostly interested in. So here we make a few general comments.

We start by pointing out that the condition $0 \notin (L - N)(\partial\Omega)$ is indispensable in this framework. In words, it signifies that there is no solution of the operator equation (2.5) that lies in the boundary of Ω. Comparing this condition with **A.1** in Theorem 2.11, we see that it is the same condition except that in **A.1**, Ker L is excluded, but this is a technical detail that should not obscure our analysis. As a matter of fact, this case is treated in **A.2**. This hypothesis is very important to guarantee that the coincidence degree $d((L, N), \Omega)$ exists in the first place. Accordingly, it becomes important in this work in order to be able to apply the previous theorem.

It should be reinforced that requiring this condition in this context is just as natural as requiring that $y_0 \notin f(\partial V)$ when defining the Brouwer degree, Theorem 2.10. What we should keep in mind, however, is that in the context of infinite dimensional function spaces, the consequences of a condition like this can sometimes be surprising.

3 Averaging Theory for Periodic Solutions in Continuous Differential Systems

Periodic solutions in ordinary differential equations hold a central place in the understanding of dynamical systems, offering valuable insights into their behaviour. The Averaging method, a well-established tool, has played a crucial role in analysing differential equations with smooth right-hand sides. However, many real-world systems exhibit non-Lipschitz non-linearities, challenging the conventional applicability of the Averaging method.

Considerable progress has been achieved in extending the Averaging method to address Lipschitz differential equations and, notably, Piecewise Lipschitz differential equations. These developments have significantly augmented our comprehension of periodic solutions within specific system classes and order of perturbation, such as first order Lipschitz systems [5], arbitrary order in both Lipschitz systems [20] and piecewise Lipschitz systems [18]. However, a critical void persists in the examination of continuous non-Lipschitz differential perturbations of any order. The present chapter takes on the challenge of elucidating this hitherto unexplored territory, focusing on scenarios where the right-hand sides lack Lipschitz continuity. By applying the degree theory for operator equations on suitable function spaces, our objective is to contribute a pivotal segment to the evolving framework of Averaging theory, expanding its applicability across a broader range of differential equations, thereby augmenting the analytical toolkit for comprehending the dynamics of diverse systems.

This chapter introduces an extension of the Averaging method to address the challenges posed by differential equations with continuous but non-Lipschitz right-hand sides. The absence of Lipschitz continuity introduces difficulties, particularly regarding the uniqueness of solutions, necessitating novel analytical approaches. Our primary objectives in this chapter are to carry on an analysis for a very general setting and, as a consequence, derive new results for detecting periodic solutions that generalize those in [5, 20].

The theorems present in this chapter are published on [27].

3.1 Setting up

The starting point of our analysis is the general differential equation in the standard form of the averaging method. Thus, consider the following equation

$$x' = \varepsilon F(t, x, \varepsilon), \tag{3.1}$$

where $F : I \times D \times [0, \varepsilon_0] \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous function $T-$periodic in the variable t, with D being an open subset of $\mathbb{R}^n$ and $\varepsilon_0 > 0$.

Building upon the insights gained from equation (3.1), we derive profound results that extend beyond the general formulation. A notable case arises in the specific form

$$x' = \varepsilon F(t, x, \varepsilon) = \varepsilon F_1(t, x) + \cdots + \varepsilon^k F_k(t, x) + \varepsilon^{k+1} R(t, x, \varepsilon), \tag{3.2}$$

where $F_i : \mathbb{R} \times D \to \mathbb{R}^n$, for $i \in \{1, \ldots, k\}$, and $R : \mathbb{R} \times D \times [0, \varepsilon_0] \to \mathbb{R}^n$ are continuous functions $T-$periodic in the variable t.

This equation represents a distinctive instance, where the terms $F_1, \ldots, F_k$ characterize the system's behaviour, whereas the remainder term R encapsulates higher-order effects. Crucially, equation (3.2) emerges as a particular but highly significant case within the broader framework of equation (3.1). Despite its specificity, equation (3.2) often proves more amenable to analytical treatment, making it a key focal point in various practical applications.

3.2 Formulating the associated operator equation $Lx = N(x, \varepsilon)$

The approach adopted to demonstrate the existence of periodic solutions in differential equations, where the assurance of solution uniqueness is not guaranteed, involves a strategic transformation. We convert the original equation into an operator equation and leverage the tools of functional analysis, more specifically degree theory for operator equations, to employ effective methods for our analysis. This methodological shift enables a comprehensive exploration of the existence of periodic solutions, providing valuable insights into the dynamic behaviour of the system.

As discussed in Section 2.3, given the differential equation equation (3.1), a continuously differentiable function $x : I \subset \mathbb{R} \to \mathbb{R}^n$ is a solution for that equation if, and only if, it satisfies the integral equation

$$x(t) = x(0) + \int_0^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s, \qquad \forall\, t \in I. \tag{3.3}$$

In this work we denote by $C[0, T]$ the set of all continuous paths $x : [0, T] \to \mathbb{R}^n$ endowed with the sup $-$norm. We point out that $C[0, T]$ with this norm is a real Banach space. Now, we define two subspaces of $C[0, T]$ which play a fundamental role in what follows. Thus, let X and Z be given by

$$X = \{x \in C[0, T] : x(0) = x(T)\}, \quad Z = \{x \in C[0, T] : x(0) = 0\}$$

and note that X and Z are also real Banach spaces. These will be our ambient spaces. We must consider now an adequate subset of X for the solutions of the new problem we are about to define to lie in. Hence, for some open bounded subset $V \subset \mathbb{R}^n$ satisfying $\overline{V} \subset D$ we consider the set

$$\Omega = \{x \in X : x(t) \in V, \forall t \in [0, T]\}. \tag{3.4}$$

Let us now define the operators $L : X \to Z$ and $N : \overline{\Omega} \times [0, \varepsilon_0] \to Z$ by setting

$$Lx(t) = x(t) - x(0) \quad \text{and} \quad N(x, \varepsilon)(t) = \int_0^t \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s, \quad \forall\, t \in [0, T]. \tag{3.5}$$

Now taking into account equation (3.3) and equation (3.5), a function $x \in C[0, T]$ is a $T-$periodic solution of equation (3.1) if, and only if, it is a solution of the operator equation

$$Lx = N(x, \varepsilon), \;\; x \in \overline{\Omega}. \tag{3.6}$$

3.3 Some properties of L and N

Now that we have already established the equivalence between obtaining periodic solutions of equation (3.1) and solving an operator equation, our efforts must go in the direction of guaranteeing the existence of solutions for the latter type of equation. In order to do that, we must verify that L and N meet some minimal requirements. The whole point of this change of perspective is to be able to use Theorem 2.11 as a way to ensure the existence of solutions for equation (3.6). For this we prove some essential properties of L and N in the sequel.

We begin by showing that Ω defined in (3.4) is an open and bounded subset of X.

Proposition 3.1. *Ω is an open and bounded subset of X.*

Proof. To show that Ω is open, we take an element $x_0 \in \Omega$, then $x_0(t) \in V$, $\forall\, t \in [0, T]$. For each $t \in [0, T]$, let $\delta_t > 0$ be such that the open ball around $x_0(t)$ of radius δ_t is completely inside of V. This is possible since V is open. In this way, we get an open covering $\{B(x_0(t), \delta_t) \colon t \in [0, T]\}$ of $\{x_0(t) \colon t \in [0, T]\}$, which is a compact set and therefore, we can extract a finite subcovering $\{B(x_0(t_i), \delta_{t_i}) \colon i = 1, \ldots, m\}$. Now take $\delta < \min\{\delta_1, \ldots, \delta_m\}$ such that $\mathcal{T} = \cup_{t\in[0,T]} B(x_0(t), \delta) \subset V$. Now consider the set Ψ of all functions $x \colon [0, T] \to \mathbb{R}^n$ in X such that $x(t) \in \mathcal{T}$. Clearly, $\Psi \subset \Omega$ and the open ball around x_0 of radius $\delta, B(x_0, \delta) = \{x \in X \colon \sup_{t\in[0,T]} |x(t) - x_0(t)| < \delta\}$, is contained in Ψ. This shows that Ω is open. On the other hand, since V is a bounded subset of $\mathbb{R}^n$, there exists a constant $M > 0$, such that $V \subset B(0, M)$. Given any element of Ω, say x, we have $x(t) \in V$, $\forall\, t \in [0, T]$ and hence $\|x\| = \sup_{t\in[0,T]} |x(t)| \leqslant M$, which shows that Ω is bounded. □

Let us quickly verify that L is a well defined linear operator. For any $x \in X$ we notice that Lx is just a translation of x, so that Lx is continuous and, in addition, $Lx(0) = x(0) - x(0) = 0$ which implies that $Lx \in Z$. Furthermore, given $x_1, x_2 \in X$ and any $c \in \mathbb{R}$,

$$\begin{aligned} L(x_1 + cx_2)(t) &= (x_1 + cx_2)(t) - (x_1 + cx_2)(0) = (x_1(t) - x_1(0)) + c(x_2(t) - x_2(0)) \\ &= Lx_1(t) + cLx_2(t), \end{aligned}$$

for every $t \in [0, T]$, showing that $L(x_1 + cx_2) = Lx_1 + cLx_2$, thus L is linear. Moreover, L is a bounded, therefore continuous, operator, since

$$||Lx|| = \sup_{t\in[0,T]} |Lx(t)| = \sup_{t\in[0,T]} |x(t) - x(0)| \leqslant \sup_{t\in[0,T]} (|x(t)| + |x(0)|) \leqslant 2||x||.$$

It takes a little more effort to prove that N is well defined and continuous since it is not a linear operator. So we put these facts in a proposition.

Proposition 3.2. *N is a well defined continuous operator.*

Proof. Note that since F is continuous, then $N(x, \varepsilon)$ is a continuous function of t and that $N(x, \varepsilon)(0) = 0$, for every $(x, \varepsilon) \in \overline{\Omega} \times [0, \varepsilon_0]$ so that $N(x, \varepsilon) \in Z$. Now to see that it is continuous, fix $x \in X$ and $\varepsilon \in [0, \varepsilon_0]$, and consider a sequence $\{(x_m, \varepsilon_m)\}_{m\in\mathbb{N}}$ contained in $X \times [0, \varepsilon_0]$ converging to (x, ε). For each $m \in \mathbb{N}$ define the function $\Delta_m : [0, T] \to \mathbb{R}^n$ by

$$\Delta_m(t) = \varepsilon_m F(t, x_m(t), \varepsilon_m) - \varepsilon F(t, x(t), \varepsilon).$$

Observe that $\Delta_m(t) \to 0$ for every $t \in [0, T]$. Since F is continuous on a compact set, it is uniformly continuous and so is Δ_m. Thus, in particular,

$$\lim_{m\to\infty} \int_0^t |\Delta_m(s)| \mathrm{d}s = \int_0^t \lim_{m\to\infty} |\Delta_m(s)| \mathrm{d}s = 0, \tag{3.7}$$

for every $t \in [0, T]$. Now,

$$\begin{aligned} ||N(x_m, \varepsilon_m) - N(x, \varepsilon)|| &= \sup_{t\in[0,T]} |N(x_m, \varepsilon_m)(t) - N(x, \varepsilon)(t)| \\ &= \sup_{t\in[0,T]} \left| \int_0^t \varepsilon_m F(s, x_m(s), \varepsilon_m) \mathrm{d}s - \int_0^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s \right| \\ &= \sup_{t\in[0,T]} \left| \int_0^t \varepsilon_m F(s, x_m(s), \varepsilon_m) - \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s \right| \\ &\leqslant \sup_{t\in[0,T]} \int_0^t |\varepsilon_m F(s, x_m(s), \varepsilon_m) - \varepsilon F(s, x(s), \varepsilon)| \, \mathrm{d}s \\ &\leqslant \int_0^T |\Delta_m(s)| \, \mathrm{d}s. \end{aligned}$$

By (3.7), it follows that $N(x_m, \varepsilon_m) \to N(x, \varepsilon)$, showing that N is continuous. □

We move on now to the more intricate properties. The definitions for each of the properties we are about to check are all defined in Chapter 2.

We begin with the Fredholm property.

Lemma 3.1. *The linear operator* $L : X \to X$ *defined in equation* (3.5) *is a Fredholm operator of index* 0.

Proof. Indeed, we have

$$\text{Ker } L = \{x \in X : Lx = 0\} = \{x \in X : x(t) = x(0), \forall\, t \in [0,T]\},$$

that is, Ker L is the subspace of X consisting of constant paths. Hence, the map that assigns to each $z \in \mathbb{R}^n$ the constant function $x(t) \equiv z$ is an isomorphism between $\mathbb{R}^n$ and Ker L, so that the latter is a finite dimensional subspace of X and its dimension is n. Furthermore, the subspace Coker $L = Z/\text{Im } L$ is given by

$$\text{Coker } L = \{[u] = u + \text{Im } L : u \in Z\}.$$

Two elements $[u], [v]$ in Coker L are equal if and only $u - v \in \text{Im } L$, which means that there exists $x \in X$ such that $u - v = Lx$. In particular, evaluating at $t = T$, we obtain

$$(u - v)(T) = Lx(T) \Leftrightarrow u(T) - v(T) = x(T) - x(0) = 0 \Leftrightarrow u(T) = v(T).$$

In other words, $[u] = [v]$ if and only if $u(T) = v(T)$. Conversely, if two functions $u, v \in Z$ are such that $u(T) = v(T)$, then $L(u - v)(t) = (u - v)(t) - (u - v)(0) = (u - v)(t)$, hence, $u - v \in \text{Im } L$. Therefore, each element of Coker L is completely determined by its value at $t = T$ so that Coker L is isomorphic to $\mathbb{R}^n$. Hence, by Proposition 2.1, Im L is a closed subspace of Z. Since Ker L and Coker L are both finite dimensional and have the same dimension the proof is concluded. □

The Fredholm property holds significant importance within our context. Its possession of a finite-dimensional kernel becomes pivotal, as we will demonstrate later on. This feature empowers us to extend treatments typically reserved for operators on finite-dimensional spaces to those defined on infinite-dimensional counterparts. This extension facilitates a more straightforward formulation of our results, streamlining the exposition of our findings and enhancing the clarity and applicability of our theoretical framework.

As we saw in Chapter 2, since L is a linear Fredholm operator of index 0, there exists continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that the following sequence

$$X \xrightarrow{P} X \xrightarrow{L} Z \xrightarrow{Q} Z \tag{2.1}$$

is exact, meaning that Im P = Ker L and Im L = Ker Q. For our particular operator, we explicit the choices for P and Q. Thus, let $P : X \to X$ and $Q : Z \to Z$ be given by

$$Px(t) = x(0), \qquad Qy(t) = \frac{t}{T}y(T), \qquad \forall\, t \in [0,T]. \tag{3.8}$$

We claim that these operators make (2.1) exact. That Im P = Ker L it is clear, since Ker L consists of constant paths. On to the equality Im L = Ker Q, we note that Im $L = X \cap Z$, that is, the closed paths in $\mathbb{R}^n$ starting and ending at the origin of the coordinate system. On the other hand, an element $y \in Z$ is in Ker Q if, and only if, $y(T) = 0$. Consequently, Im $L \subset$ Ker Q. But since $y \in Z$, it also holds that $y(0) = 0$. As a result, Ker $Q \subset$ Im L, therefore, Ker Q = Im L, as desired. We point out that for this framework it is known ([12]) that the particular choice of P and Q does not interfere in the validity of the results, as long as the sequence (2.1) is exact.

Given these choices, notice that Ker $P = \{x \in X : x(0) = 0\}$, so that the restriction of L to Ker P is the identity operator. Therefore, it readily follows that $K_P = \text{Id}$, so that $K_{P,Q} = \text{Id} - Q$.

In the sequel we prove that N is $L-$compact (Definition 2.9), which is where the operators L and N connect.

Recall that we denote by $\Pi : Z \to \text{Coker } L$ the canonical projection from Z onto Coker L

$$\Pi(z) = [z] = z + \text{Im } L. \tag{3.9}$$

As shown above, since Coker $L \simeq \mathbb{R}^n$ and we can identify each element $[y] \in$ Coker L with the value of its representative at $t = T, y(T)$, we can think of the projection Π as $\Pi(y) = y(T)$.

Next we prove $L-$compactness of the operator N (Definition 2.9) which is a fundamental property for proving the a main theorems of this chapter.

Lemma 3.2. *$N : \overline{\Omega} \times [0, \varepsilon_0] \to Z$ is an $L-$compact operator on $\overline{\Omega}$ for each $\varepsilon \in [0, \varepsilon_0]$.*

Proof. For any $(x, \varepsilon) \in \overline{\Omega} \times [0, \varepsilon_0]$, since Π is a continuous linear map, there exists a constant $C > 0$ such that

$$\begin{aligned} |\Pi N(x,\varepsilon)| \leqslant C\,|N(x,\varepsilon)| &= C \sup_{t\in[0,T]} \left| \int_0^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s \right| \\ &\leqslant C \int_0^T |\varepsilon F(s, x(s), \varepsilon)|\, \mathrm{d}s \\ &\leqslant CT \max\{|\varepsilon F(t, z, \varepsilon)| : (t, z, \varepsilon) \in [0,T] \times \overline{V} \times [0, \varepsilon_0]\}, \end{aligned}$$

where the last inequality follows from the fact that F is continuous in the compact set $[0, T] \times \overline{V} \times [0, \varepsilon_0]$. This shows that the image of the operator ΠN is a bounded set, which implies that ΠN is a bounded operator.

Recall that $K_{P,Q} = \text{Id} - Q$. Since $K_{P,Q}N$ is composition of continuous operators, then it is also continuous. Consider the family $\Lambda = K_{P,Q}N(\overline{\Omega} \times [0, \varepsilon_0])$. To show that the closure of Λ is compact, we use the Arzelá-Ascoli Theorem, Theorem 2.1. Hence, we must check that Λ is equicontinuous and uniformly bounded. Let $y \in \Lambda$. Hence, there exist $x \in \overline{\Omega}$ and $\varepsilon \in [0, \varepsilon_0]$ such that

$$y(t) = (\mathrm{Id} - Q)N(x,\varepsilon)(t) = (\mathrm{Id} - Q)\left(\int_0^t \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s\right).$$

For $t, \tau \in [0, T]$ we have

$$\begin{aligned}
|y(t) - y(\tau)| &= |(\mathrm{Id} - Q)N(x,\varepsilon)(t) - (\mathrm{Id} - Q)N(x,\varepsilon)(\tau)| \\
&\leqslant |N(x,\varepsilon)(t) - N(x,\varepsilon)(\tau) - (QN(x,\varepsilon)(t) - QN(x,\varepsilon)(\tau))| \\
&\leqslant \left|\int_\tau^t \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s - \frac{t-\tau}{T}\int_0^T \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s\right|.
\end{aligned}$$

Using the continuity of F in $[0,T] \times \overline{V} \times [0, \varepsilon_0]$, we conclude that $M = \max\{|\varepsilon F(t, z, \varepsilon)| : (t, z, \varepsilon) \in [0, T] \times \overline{V} \times [0, \varepsilon_0]\} < \infty$. It follows that

$$\begin{aligned}
|y(t) - y(\tau)| &\leqslant \left|\int_\tau^t \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s - \frac{t-\tau}{T}\int_0^T \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s\right| \\
&\leqslant \int_\tau^t |\varepsilon F(s, x(s), \varepsilon)|\,\mathrm{d}s + \left|\frac{t-\tau}{T}\right|\int_0^T |\varepsilon F(s, x(s), \varepsilon)|\,\mathrm{d}s \\
&\leqslant |t-\tau|M + |t-\tau|\,M \\
&\leqslant 2M|t-\tau|.
\end{aligned}$$

Notice that the constant $M > 0$ does not depend on either x nor ε, consequently, it does not depend on y. Thus, given $\xi > 0$, whenever $|t - \tau| < \xi/(2M)$, we have $|y(t) - y(\tau)| < \xi$. Since y was taken arbitrarily in the family Λ it is then proved that Λ is an equicontinuous family.

On the other hand, since F is continuous on the compact set $[0,T] \times \overline{V} \times [0, \varepsilon_0]$ it follows that

$$\begin{aligned}
|y(t)| &= |(\mathrm{Id} - Q)N(x,\varepsilon)| \\
&\leqslant |(\mathrm{Id} - Q)||N(x,\varepsilon)| \\
&= \left|\int_0^t \varepsilon F(s, x(s), \varepsilon)\mathrm{d}s\right| \\
&\leqslant \int_0^t |\varepsilon F(s, x(s), \varepsilon)|\,\mathrm{d}s \\
&\leqslant TM,
\end{aligned}$$

showing that Λ is uniformly bounded. Thus, by applying the Theorem of Arzelá-Ascoli, Theorem 2.1, we conclude that the family has compact closure, implying the $L-$compactness of N on $\overline{\Omega}$ for every $\varepsilon \in [0, \varepsilon_0]$. □

3.4 The condition **H**

What we have proved so far provides the building blocks of our results. There is, however, an important piece that is still missing, once we will use degree theory to prove the theorems in the next section. When working with degree theory, we often come across some type of boundary condition. This kind of condition is necessary for the theory to work and usually does not imply a great compromise in generality. For instance, in the case of the Brouwer degree, as we saw in Section 2.4, for the triple (f, V, y_0), where $f : \overline{V} \subset \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function defined on the closure of the open and bounded subset V of $\mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$, one requires that $y_0 \notin f(\partial V)$.

In our case, we recall that behind the curtains, the results proved here rely on the theory of the coincidence degree, which is another degree function specialized in operator equations such as equation (2.5). Therefore, we present now a condition that is precisely the boundary condition associated to the coincidence degree, although at first it might not look like so.

Given an open and bounded subset V of $\mathbb{R}^n$, we say that the condition **H** holds on V if

H. there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for each $\lambda \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_1]$, any T-periodic solution of the differential equation

$$x' = \varepsilon\lambda F(t, x, \varepsilon),\ x \in \overline{V}, \tag{3.10}$$

is entirely contained in V.

This condition looks abstract and hard to check. However, it can be verified by a contradiction or contrapositive argument. In the sequel, we indicate how one proceeds to verify this condition. Later on, in Chapter 6, we apply this argument to verify condition **H** for a concrete example.

Proposition 3.3. *If condition **H** does not hold on an open and bounded subset V of $\mathbb{R}^n$, then there exist sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0, \varepsilon_0]$, $\{\lambda_m\}_{m\in\mathbb{N}} \in (0, 1)$ and $\{x_m\}_{m\in\mathbb{N}}$ of $T-$periodic solutions of equation (3.10) with $\varepsilon_m \to 0$ and $x_m \to z_0 \in \partial V$ uniformly on $[0, T]$ such that*

$$\int_0^T F(s, z_0, 0)\mathrm{d}s = 0. \tag{3.11}$$

Proof. Assume that **H** does not hold on the open and bounded subset V of $\mathbb{R}^n$. It follows immediately that for every $\varepsilon_1 \in (0, \varepsilon_0]$, there exist $\lambda \in (0, 1), \varepsilon \in (0, \varepsilon_1]$ and a $T-$periodic solution of the differential equation (3.10) with $\varepsilon = \varepsilon_m$ and $\lambda = \lambda_m$

$$x' = \varepsilon_m\lambda_m F(t, x, \varepsilon_m),\ x \in \overline{V}, \tag{3.12}$$

for which there exists $t \in [0,T]$ such that $x(t) \in \partial V$. Equivalently, consider any sequence $\{\hat{\varepsilon}_m\}_{m\in\mathbb{N}}$, contained in $(0,\varepsilon_0]$ converging to 0. For every $m \in \mathbb{N}$, there exist $\lambda_m \in (0,1), \varepsilon_m \in (0,\hat{\varepsilon}_m]$ and a $T-$periodic solution $x_m : [0,T] \to \mathbb{R}^n$ of equation (3.12) for which there exists $t_m \in [0,T]$ such that $x_m(t_m) \in \partial V$. Notice that since $\varepsilon_m \leqslant \hat{\varepsilon}_m$, then necessarily $\varepsilon_m \to 0$. Therefore, for each $m \in \mathbb{N}$

$$x_m(t) = x_m(0) + \int_0^t \varepsilon_m F(s, x_m(s), \varepsilon_m)\mathrm{d}s.$$

By periodicity $x_m(0) = x_m(T)$, consequently

$$\int_0^T F(s, x_m(s), \varepsilon_m)\mathrm{d}s = 0. \tag{3.13}$$

Consider the family $\Lambda = \{x_m : m \in \mathbb{N}\}$. Given $t, \tau \in [0,T]$,

$$\begin{aligned}
|x_m(t) - x_m(\tau)| &= \left| x_m(0) + \int_0^t \varepsilon_m \lambda_m F(s, x_m(s), \varepsilon_m)\mathrm{d}s \right. \\
&\qquad \left. -x_m(0) - \int_0^\tau \varepsilon_m \lambda_m F(s, x_m(s), \varepsilon_m)\mathrm{d}s \right| \\
&= \left| \int_\tau^t \varepsilon_m \lambda_m F(s, x_m(s), \varepsilon_m)\mathrm{d}s \right| \\
&\leqslant \int_\tau^t |F(s, x_m(s), \varepsilon_m)|\,\mathrm{d}s \\
&\leqslant |t - \tau| M,
\end{aligned}$$

where $M > 0$ is the maximum of $(t,z,\varepsilon) \mapsto |F(t,z,\varepsilon)|$ on $[0,T] \times \overline{V} \times [0,\varepsilon_0]$, and therefore does not depend upon x_m itself. We conclude that Λ is an equicontinuous family of functions. On the other hand, since V is a bounded set, there exists $r > 0$ such that $\overline{V} \subset B(0,r)$. Then, for each $t \in [0,T]$, $|x_m(t)| < r$ for every $m \in \mathbb{N}$, which shows that Λ is a uniformly bounded family. Then, by Theorem 2.1 (Arzelá-Ascoli) it follows that there exists a uniformly convergent subsequence of $\{x_m\}_{m\in\mathbb{N}}$, say $\{x_{k_m}\}_{m\in\mathbb{N}}$. Since $\varepsilon_m \to 0$ and F is uniformly continuous on the compact $[0,T] \times \overline{V} \times [0,\varepsilon_0]$, we conclude that

$$\begin{aligned}
\lim_{m\to 0} x_{k_m}(t) &= \lim_{m\to 0}\left(x_{k_m}(0) + \int_0^t \varepsilon_m F(s, x_{k_m}(s), \varepsilon_m)\mathrm{d}s \right) \\
&= \lim_{m\to 0} x_{k_m}(0) + \int_0^t \lim_{m\to 0} \varepsilon_m F(s, x_{k_m}(s), \varepsilon_m)\mathrm{d}s \\
&= \lim_{m\to 0} x_{k_m}(0) = z_0,
\end{aligned}$$

for some $z_0 \in \overline{V}$. Thus, the limit of the sequence of functions x_{k_m} is a constant function. Moreover, we claim that since each x_m touches the boundary of V at some time $t = t_m$, then z_0 must lie on ∂V. Indeed, if it were not so, that is, if z_0 were an interior point of V, then for $\xi > 0$ sufficiently small, the ball $B(z_0,\xi) \subset V$ and since $x_{k_m} \to z_0$, for $m \in \mathbb{N}$ large enough we would have $x_m(t) \in B(z_0,\xi)$, which contradicts $x_m(t_m) \in \partial V$. Thus, we must have $z_0 \in \partial V$. In addition, from equation (3.13) the uniform continuity of F and the uniform convergence of x_{k_m} we obtain

$$\begin{aligned} 0 = \lim_{m\to\infty}\int_0^T F(s, x_{k_m}(s), \varepsilon_{k_m})\mathrm{d}s &= \int_0^T \lim_{m\to\infty} F(s, x_{k_m}(s), \varepsilon_{k_m})\mathrm{d}s \\ &= \int_0^T F\left(s, \lim_{m\to\infty} x_{k_m}(s), \lim_{m\to\infty}\varepsilon_{k_m}\right)\mathrm{d}s \\ &= \int_0^T F(s, z_0, 0)\mathrm{d}s. \end{aligned} \tag{3.14}$$

□

To move forward from this point it depends on the particular equation one is dealing with. The aim of this discussion is to shed some light on the tractability of condition **H**. As a way to show that this condition is not too restrictive, in the next proposition, we prove that it is always valid for a class of differential equations.

Proposition 3.4. *Assume that equation* (3.1) *can be written as*

$$x' = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \qquad \text{for} \qquad (t, x, \varepsilon) \in [0, T] \times D \times [0, \varepsilon_0],$$

where F_1 and R are continuous functions and $T-$periodic in t. Assume that

$$f_1(z) = \frac{1}{T}\int_0^T F_1(s, z)\mathrm{d}s$$

*has an isolated zero z^*. If $V \subset \mathbb{R}^n$ is any neighbourhood of z^* such that $\overline{V} \subset D$ and $f_1(z) \neq 0$, for every $z \in \overline{V}\backslash\{z^*\}$, then condition **H** holds on V.*

Proof. Assume then that condition **H** does not hold on V. By the Proposition 3.3, it follows that there exist sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0, \varepsilon_1]$ and $\{x_m\}_{m\in\mathbb{N}}$ such that $\varepsilon_m \to 0$ as $m \to \infty$, $x_m \to z_0 \in \partial V$ uniformly on $[0, T]$ and

$$\int_0^T F(s, x_m(s), \varepsilon_m)\mathrm{d}s = 0,$$

for every $m \in \mathbb{N}$. Moreover, equation (3.14) implies

$$0 = \int_0^T F(s, z_0, 0)\mathrm{d}s = \int_0^T F_1(s, z_0)\mathrm{d}s = T f_1(z_0),$$

contradicting $f_1(z) \neq 0$ for every $z \in \overline{V}\backslash\{z^*\}$. Thus, condition **H** holds on V. □

We finish this section with a last comment on how to verify that condition **H** holds. Assume that the boundary of $V, \partial V$, is a smooth manifold. Intuitively, if the solutions of equation (3.10) all hit the boundary of V transversally, then no $T-$periodic solution of it can touch the boundary and stay inside.

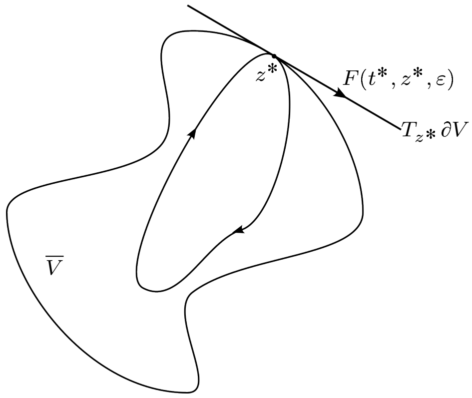

Figure 3 – Illustration of solution touching the boundary of the domain.

Proposition 3.5. *Let $V \subset \mathbb{R}^n$ be an open and bounded subset of $\mathbb{R}^n$ such that $\overline{V} \subset D$. If there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for each $(t, z, \varepsilon) \in [0, T] \times \partial V \times (0, \varepsilon_1]$, $F(t, z, \varepsilon)$ is transversal to the boundary of V, that is, $F(t, z, \varepsilon) \notin T_z\partial V$, then condition **H** holds on V.*

Proof. Assume that condition **H** does not hold on V. Then, for some $\varepsilon \in (0, \varepsilon_0]$, $\lambda \in (0, 1)$ there exists a $T-$periodic solution φ of equation (3.10) such that $\varphi(t) \in \overline{V}$ for every $t \in [0, T]$ and for some $t^* \in [0, T]$, $z^* = \varphi(t^*) \in \partial V$, see Figure 3. Since ∂V is a smooth $(n-1)-$manifold, there exists an open subset of $\mathbb{R}^n$, W, such that $W \cap \partial V$ is a parametrized neighbourhood of ∂V. In addition, this neighbourhood can be taken smaller if necessary, so that $W \cap \partial V$ is the inverse image of a regular value of some real function, say $W \cap \partial V = h^{-1}(0)$ where $h : W \to \mathbb{R}$ is a smooth function having 0 as a regular value. Notice that we can write $W = \{z \in \mathbb{R}^n : h(z) < 0\} \cup \partial V \cup \{z \in \mathbb{R}^n : h(z) > 0\}$ and we identify without loss of generality the points where $h(z) > 0$ with the points in the interior of V. Now, define $g = h \circ \varphi : [0, T] \to \mathbb{R}$. Of course, g is a smooth function. Moreover, since $\varphi(t) \in \overline{V}$ for every $t \in [0, T]$, it follows that $g(t) \geqslant 0$ for every $t \in [0, T]$. Hence, 0 is a minimum of g with minimizer t^*. Since g is smooth, we conclude that $g'(t^*) = 0$. On the other hand, $g'(t^*) = \langle \nabla h(z^*), F(t^*, z^*, \varepsilon)\rangle = 0$, showing that $F(t^*, z^*, \varepsilon)$ is tangent to ∂V at (t^*, z^*). □

3.5 Statements and proofs of main results

Building upon the foundations established thus far, this section engages in the precise statement and rigorous proof of the primary theorems that encapsulate the essence of our research. The preceding exposition has set the stage for a systematic exploration into the heart of our contributions.

The first theorem we present, Theorem A, is the most general one of this chapter, since it deals directly with equation (3.1). The next two theorems, Theorem B and Theorem C, treat equation (3.2).

Before we present the first theorem, we introduce the averaged function $f : \overline{V} \times [0, \varepsilon_0] \to \mathbb{R}^n$ given by

$$f(z, \varepsilon) = \frac{1}{T} \int_0^T \varepsilon F(s, z, \varepsilon) \mathrm{d}s.$$

In words, it correspond to the average of F over the interval $[0, T]$ for fixed $z \in \overline{V}$ and $\varepsilon \in [0, \varepsilon_0]$. The averaging method consists of extracting information about the solutions of equation (3.1) through f. In our particular case, roughly speaking, existence of zeroes of f determines the existence of periodic solutions of equation (3.1).

Theorem A. *Consider the continuous T-periodic non-autonomous differential equation (3.1). Assume that for a given open bounded subset $V \subset \mathbb{R}^n$, with $\overline{V} \subset D$, condition **H** holds,*

$$f(z, \varepsilon) \neq 0, \quad \textit{for all} \quad z \in \partial V \quad \textit{and} \quad \varepsilon \in (0, \varepsilon_1], \tag{3.15}$$

and $d_B(f(\cdot, \varepsilon^*), V, 0) \neq 0$, for some $\varepsilon^* \in (0, \varepsilon_1]$. Then, for each $\varepsilon \in (0, \varepsilon_1]$, there exists a $T-$periodic solution $\varphi(t, \varepsilon)$ of the differential equation (3.1) satisfying $\varphi(t, \varepsilon) \in \overline{V}$, for every $t \in [0, T]$.

Proof. The theorem will follow by applying Theorem 2.11. Let $N = N(x, \varepsilon)$. We shall use the conclusion of this theorem for $\lambda = 1$ in the operator equation

$$Lx = \lambda N(x, \varepsilon), \ (x, \lambda) \in \overline{\Omega} \times [0, 1], \ \varepsilon \in (0, \varepsilon_1]. \tag{3.16}$$

Recall the continuous projections $P : X \to X$ and $Q : Z \to Z$ defined in equation (3.8) given by

$$Px(t) = x(0) \quad \text{and} \quad Qy(t) = \frac{t\ y(T)}{T}, \quad \text{for} \quad t \in [0, T],$$

respectively. In Subsection 2.1.3 and Lemma 3.2 we proved that L is a linear Fredholm operator of index 0 and N is $L-$compact, respectively. Now, let us check the conditions of Theorem 2.11.

Claim 3.0.1. *Condition **H** implies that condition **A.1** of Theorem 2.11 holds, for each $\varepsilon \in (0, \varepsilon_1]$.*

To arrive at this conclusion, we must check that under **H**, for each $\varepsilon \in (0, \varepsilon_1]$, $Lx \neq \lambda N(x, \varepsilon)$, for every $x \in \text{dom } L \cap \partial\Omega$ and $\lambda \in (0, 1)$. Notice that $x \in \partial\Omega$ if, and only if, there exists $t_0 \in [0, T]$ such that $x(t_0) \in \partial V$, and that the equality $Lx = \lambda N(x, \varepsilon)$ means that x is a solution of equation (3.10). But since **H** holds, no solution of this equation can reach the boundary of V, therefore for $x \in \partial\Omega$, $Lx \neq \lambda N(x, \varepsilon)$, which is **A.1**.

Now we proceed to verify **A.2**. Note that for $x \in \text{Ker } L \cap \partial\Omega$, say, $x(t) \equiv z \in \partial V$,

$$QN(x, \varepsilon)(t) = \frac{t}{T} \int_0^T \varepsilon F(s, z, \varepsilon) \mathrm{d}s = t f(z, \varepsilon),$$

which, by Equation 3.15, is not the 0 constant function in Z. Therefore, condition **A.2** of Theorem 2.11 holds, for each $\varepsilon \in (0, \varepsilon_1]$.

Observe that $\text{Im } Q = \{y \in Z : y(t) = t\,v,\ v \in \mathbb{R}^n\}$ is isomorphic to $\mathbb{R}^n$ as well as $\text{Ker } L$. Thus, it is straightforward to see that the linear map $J : \text{Im } Q \to \text{Ker } L$ given by

$$Jy(t) = \frac{y(T)}{T} \tag{3.17}$$

is an isomorphism. If follows that for $x \in \text{Ker } L \cap \Omega, x(t) \equiv z \in V$

$$JQN(x, \varepsilon) = J\left(tf(z, \varepsilon)\right) = f(z, \varepsilon).$$

Thus,

$$d_B(JQN(\cdot, \varepsilon)_{|\text{Ker } L \cap \Omega}, \text{Ker } L \cap \Omega, 0) = d_B(f(\cdot, \varepsilon), V, 0).$$

To conclude the proof, we just need to show that $d_B(f(\cdot, \varepsilon), V, 0) \neq 0$ for each $\varepsilon \in (0, \varepsilon_1]$. For future reference, we make that into a claim.

Claim 3.0.2. $d_B(f(\cdot, \varepsilon), V, 0) \neq 0$ *for each* $\varepsilon \in (0, \varepsilon_1]$.

Let $\mathcal{E} = \{\varepsilon \in (0, \varepsilon_1] : d_B(f(\cdot, \varepsilon), V, 0) \neq 0\}$. By hypothesis, there exists $\varepsilon^* \in (0, \varepsilon_1]$ such that $d_B(f(\cdot, \varepsilon^*), V, 0) \neq 0$, so that $\mathcal{E} \neq \varnothing$. Recall that f is continuous on $\overline{V} \times (0, \varepsilon_1]$. Given any $\epsilon \in \mathcal{E}$, there exists $\delta > 0$ such that $[\epsilon - \delta, \varepsilon_1] \subset (0, \varepsilon_1]$. Now, since $\overline{V} \times [\epsilon - \delta, \varepsilon_1]$ is a compact set, f restricted to this set is uniformly continuous. From Equation 3.15, the function $(z, \varepsilon) \mapsto |f(z, \varepsilon)|$ has a minimum value that is positive, say $\mu = \min\{|f(z, \varepsilon)| : (z, \varepsilon) \in \partial V \times [\epsilon - \delta, \varepsilon_1]\} > 0$. Thus, that there exists a small open interval $\mathcal{I}$ such that $f(z, \varepsilon) \neq 0$ for all $z \in \partial V$ and $\varepsilon \in \mathcal{I}$. Using the uniform continuity of f, for any $\varepsilon \in \mathcal{I}$ sufficiently close to ϵ

$$|f(\cdot, \varepsilon) - f(\cdot, \epsilon)| = \sup_{z \in \overline{V}} |f(z, \varepsilon) - f(z, \epsilon)| < \mu = \text{dist }(0, f(\partial V, \epsilon)).$$

Therefore, by Proposition 2.5, $\mathcal{I}$ can be taken smaller if necessary so that $d_B(f(\cdot, \varepsilon), V, 0) = d_B(f(\cdot, \epsilon), V, 0) \neq 0$, for every $\varepsilon \in \mathcal{I}$. Thus, $\mathcal{I} \cap (0, \varepsilon_1] \subset \mathcal{E}$, from which follows that $\mathcal{E}$ is open in $(0, \varepsilon_1]$. Analogously, Proposition 2.5 also implies that $(0, \varepsilon_1] \backslash \mathcal{E} = \{\varepsilon \in (0, \varepsilon_1] : d_B(f(\cdot, \varepsilon), V, 0) = 0\}$ is open in $(0, \varepsilon_1]$ and, consequently, $\mathcal{E}$ is closed in $(0, \varepsilon_1]$. Hence, from the connectedness of $(0, \varepsilon_1]$, we obtain $\mathcal{E} = (0, \varepsilon_1]$.

Therefore, applying Theorem 2.11 for $\lambda = 1$ to equation (3.16) for each $\varepsilon \in (0, \varepsilon_1]$, we conclude that there exists a solution of equation (3.6) and, consequently, a $T-$periodic solution $\varphi(t, \varepsilon)$ of equation (3.1), for each $\varepsilon \in (0, \varepsilon_1]$, such that $\varphi(t, \varepsilon) \in \overline{V}$ for every $t \in [0, T]$. □

The above theorem, although of great theoretical value, is not very suitable to work with directly. It is more common in applications to be able to at least express

F as a polynomial in ε with coefficients $F_i(t,x)$ continuous functions of t and x, plus a remainder term $R(t,x,\varepsilon)$, as in equation (3.2). Since this equation is a particular instance of equation (3.1), the framework provided by Theorem A can be used to specialize this theorem for equation (3.2). Requiring a little more information on the structure of F, even though it particularizes the corresponding result, allows us to provide a more practical set of hypotheses.

Before stating and proving our second main theorem of this chapter, we recall a few definitions for the sake of clarity. equation (3.2) is given by

$$x' = \varepsilon F(t,x,\varepsilon) = \varepsilon F_1(t,x) + \cdots + \varepsilon^k F_k(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon),$$

where $F_i : [0,T] \times D \to \mathbb{R}^n$, for $i \in \{1,\ldots,k\}$, and $R : [0,T] \times D \times [0,\varepsilon_0] \to \mathbb{R}^n$ are continuous functions $T-$periodic in the variable t.

For each $i \in \{1,\ldots,k\}$ we define $f_i : D \to \mathbb{R}^n$ by setting

$$f_i(z) = \frac{1}{T}\int_0^T F_i(s,z)\mathrm{d}s.$$

Naturally, each f_i is a continuous function. We also define the $k-$truncated averaged function $\mathcal{F}_k : D \to \mathbb{R}^n$ and the averaged remainder $r : D \times [0,\varepsilon_0] \to \mathbb{R}^n$ by

$$\mathcal{F}_k(z,\varepsilon) = \varepsilon f_1(z) + \cdots + \varepsilon^k f_k(z) \qquad r(z,\varepsilon) = \frac{1}{T}\int_0^T R(s,z,\varepsilon)\mathrm{d}s.$$

Therefore, the full averaged function f can be written as

$$f(z,\varepsilon) = \frac{1}{T}\int_0^T \varepsilon F(s,z,\varepsilon)\mathrm{d}s = \mathcal{F}_k(z,\varepsilon) + \varepsilon^{k+1} r(z,\varepsilon).$$

Now we can proceed to the next theorem.

Theorem B. *Consider the continuous T-periodic non-autonomous differential equation (3.2). Assume that for a given open bounded subset $V \subset \mathbb{R}^n$, with $\overline{V} \subset D$, hypothesis* ***H*** *holds,*

$$\lim_{\varepsilon\to 0} \inf_{z\in\partial V} \left|\frac{\mathcal{F}_k(z,\varepsilon)}{\varepsilon^{k+1}}\right| > \max\{|r(z,\varepsilon)| : (z,\varepsilon) \in \overline{V} \times [0,\varepsilon_1]\}, \tag{3.18}$$

and $d_B(\mathcal{F}_k(\cdot,\varepsilon), V, 0) \neq 0$, for $\varepsilon > 0$ sufficiently small. Then, there exists $\varepsilon_V \in (0,\varepsilon_1]$ such that, for each $\varepsilon \in (0,\varepsilon_V]$, the differential equation (3.2) has a $T-$periodic solution $\varphi(t,\varepsilon)$ satisfying $\varphi(t,\varepsilon) \in \overline{V}$, for every $t \in [0,T]$.

Proof. As shown above, for equation (3.2), the full averaged function is given by

$$f(z,\varepsilon) = \mathcal{F}_k(z,\varepsilon) + \varepsilon^{k+1} r(z,\varepsilon).$$

The limit in equation (3.18) readily implies that there exists $\bar{\varepsilon} \in (0,\varepsilon_1]$ such that

$$\inf_{z\in\partial V} \left|\frac{\mathcal{F}_k(z,\varepsilon)}{\varepsilon^{k+1}}\right| > \max\{|r(w,\varepsilon)| : (w,\varepsilon) \in \overline{V} \times [0,\varepsilon_1]\},$$

for every $\varepsilon \in (0, \overline{\varepsilon}]$. In particular, for every $z \in \partial V$ and $\varepsilon \in (0, \overline{\varepsilon}]$

$$\left|\frac{\mathcal{F}_k(z,\varepsilon)}{\varepsilon^{k+1}}\right| > |r(z,\varepsilon)|.$$

This inequality implies Equation 3.15, since

$$\begin{aligned} |f(z,\varepsilon)| &= |\mathcal{F}_k(z,\varepsilon) + \varepsilon^{k+1} r(z,\varepsilon)| \\ &\geqslant |\mathcal{F}_k(z,\varepsilon)| - \varepsilon^{k+1}|r(z,\varepsilon)| > 0. \end{aligned}$$

Taking $V_\varepsilon = V$ in Proposition 2.6, we get that $d_B(f(\cdot,\varepsilon), V, 0) = d_B(\mathcal{F}_k(\cdot,\varepsilon), V, 0)$, for $\varepsilon \in (0, \overline{\varepsilon}]$ and since, by hypothesis, $d_B(\mathcal{F}_k(\cdot,\varepsilon), V, 0) \neq 0$ for $\varepsilon > 0$ sufficiently small, all the hypotheses of Theorem A are verified and we conclude that for every $\varepsilon \in (0, \overline{\varepsilon}]$, there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of equation (3.2) satisfying $\varphi(t,\varepsilon) \in \overline{V}$ for every $t \in [0,T]$, as desired. □

It is worth mentioning that in the above theorem, we are able to request that the Brouwer degree of the $k-$truncated averaged function be different from 0, instead of asking the same for the full averaged function. That by itself already refines Theorem A by presenting a more verifiable condition. Besides, $\mathcal{F}_k$ being polynomial in ε gives us more insight on what to expect of its behaviour.

We mention that since $\mathcal{F}_k(z,\varepsilon) = \varepsilon f_1(z) + \cdots + \varepsilon^k f_k(z)$ with ε small, the lowest order terms in its expression are those that control the behaviour of $\mathcal{F}_k$ regarding its Brouwer degree. This will be made clearer in the next theorem. This result is a particular case of Theorem B, when some, but not all, of the functions f_i are identically zero. In this case the hypotheses can be further simplified.

Theorem C. *Consider the continuous T-periodic non-autonomous differential equation* (3.2)*. Suppose that for some $\ell \in \{1, 2, \ldots, k\}$, $f_0 = \ldots = f_{\ell-1} = 0$, $f_\ell \neq 0$. Assume that there exists an open and bounded set $V \subset \mathbb{R}^n$ with $\overline{V} \subset D$ and $f_\ell(z) \neq 0$ for every $z \in \partial V$ for which condition **H** holds and $d_B(f_\ell, V, 0) \neq 0$. Then, there exists $\varepsilon_V \in (0, \varepsilon_1]$ such that, for each $\varepsilon \in (0, \varepsilon_V]$, the differential equation* (3.2) *has a $T-$periodic solution $\varphi(t,\varepsilon)$ satisfying $\varphi(t,\varepsilon) \in \overline{V}$, for every $t \in [0,T]$.*

Proof. Without loss of generality, we can assume that $\ell = k$, because if this is not the case, we can just consider the $\ell-$th truncated averaged function

$$\mathcal{F}_\ell(z,\varepsilon) = \varepsilon f_1(z) + \cdots + \varepsilon^\ell f_\ell(z) + \varepsilon^{\ell+1}(f_{\ell+1}(z) + \cdots + \varepsilon^{k-\ell+1} r(z,\varepsilon)),$$

where $f_{\ell+1}(z) + \cdots + \varepsilon^{k-\ell+1} r(z,\varepsilon)$ would be the remainder term. Since $f_0 = \cdots = f_{k-1} = 0$, then $\mathcal{F}_k(\cdot,\varepsilon) = \varepsilon^k f_k(z)$. In addition, we have $f_k(z) \neq 0$, for every $z \in \partial V$ and $r(z,\varepsilon)$ continuous, consequently, bounded on compact sets. As a result,

$$\liminf_{\varepsilon\to 0} \inf_{z\in\partial V} \left|\frac{\varepsilon^k f_k(z)}{\varepsilon^{k+1}}\right| = \liminf_{\varepsilon\to 0} \inf_{z\in\partial V} \left|\frac{f_k(z)}{\varepsilon}\right| = \infty > \max\{|r(z,\varepsilon)| : (z,\varepsilon) \in \overline{V} \times [0,\varepsilon_1]\}.$$

At last, for $\varepsilon \in (0, \varepsilon_1]$ it holds

$$d_B(\mathcal{F}_k(\cdot, \varepsilon), V, 0) = d_B(f_\ell, V, 0) \neq 0.$$

Thus, all the hypotheses of Theorem B hold and it follows that there exists $\varepsilon_V \in (0, \varepsilon_1]$ such that for every $\varepsilon \in (0, \varepsilon_V]$, there exists a $T-$periodic solution $\varphi(t, \varepsilon)$ of equation (3.2) satisfying $\varphi(t, \varepsilon) \in \overline{V}$ for every $t \in [0, T]$. □

Now we prove that in the first order case, that is $k = 1$, Theorem C follows without the need to assume condition **H**. This is the first step in showing that [5, Theorem 1.2] is a corollary of the results obtained in this work. We stress that the result in [5] does not require a hypothesis like condition **H** and it has also a convergence part. This part will be treated in Chapter 5.

Theorem D. *Consider the differential equation* (3.2) *with* $k = 1$, *that is,*

$$x' = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \qquad (t, x, \varepsilon) \in [0, T] \times D \times [0, \varepsilon_0], \tag{3.19}$$

where F_1 *and* R *are continuous functions, and* $T-$*periodic in* t. *Assume that there exist* $z^* \in D$ *and an open bounded set* $V \subset \mathbb{R}^n$ *satisfying* $\overline{V} \subset D, z^* \in V, f_1(z^*) = 0$ *and* $f_1(z) \neq 0$ *for every* $z \in \overline{V} \backslash \{z^*\}$. *If* $d_B(f_1, V, 0) \neq 0$, *then there exists* $\varepsilon_V \in (0, \varepsilon_1]$ *such that, for each* $\varepsilon \in (0, \varepsilon_V]$, *the differential equation* (3.19) *has a* $T-$*periodic solution* $\varphi(t, \varepsilon)$ *satisfying* $\varphi(t, \varepsilon) \in \overline{V}$, *for every* $t \in [0, T]$.

Proof. Since f_1 has an isolated zero, we get that $\ell = 1$. Then, it suffices to show that condition **H** holds on V, since all the other hypotheses of Theorem C match. But that was proved in Proposition 3.4. Therefore, by Theorem C, there exists $\varepsilon_V \in (0, \varepsilon_1]$ such that for every $\varepsilon \in (0, \varepsilon_V]$ there exists a $T-$periodic solution $\varphi(t, \varepsilon)$ of equation (3.19) satisfying $\varphi(t, \varepsilon) \in \overline{V}$ for every $t \in [0, T]$. □

In Chapter 6 we will see Theorem C applied to a concrete example.

4 Averaging Theory for Periodic Solutions in Carathéodory Differential Systems

The realm of ordinary differential equations defined by Carathéodory functions introduces a whole new layer of complexity, where the right-hand side may exhibit discontinuities. While existing literature has provided valuable insights into the existence and uniqueness of solutions for such equations, the permissibility of discontinuities within Carathéodory functions presents a set of novel challenges when compared to those of the previous chapter.

This chapter extends our exploration beyond the context of continuous functions by considering differential equations with Carathéodory right-hand sides. As we mentioned in Section 4.3 one may impose conditions to guarantee the uniqueness of solutions of Carathéodory equations, but in this chapter we shall not require that. In this context, the usual averaging o Melnikov techniques face limitations, necessitating innovative approaches to study the dynamics of solutions.

Our primary focus lies in addressing the challenges posed by discontinuities in Carathéodory functions. To overcome these difficulties, we leverage the powerful tools of degree theory for operator equations. Degree theory provides a systematic framework for analysing the solvability of equations involving non-linear operators, offering a robust foundation for understanding the existence of solutions in less regular scenarios.

Nevertheless, we point out that, naturally, the study of Carathéodory equations includes the continuous equations we worked with in the previous chapter. Therefore, this chapter provides a further extension of the study undertaken in Chapter 3. And as we shall see in the sequel, the theorems admit nearly identical statements, emphasizing a smooth transition from the continuous to this possibly discontinuous case.

4.1 Setting up

Consider the following differential equation in the standard form of the averaging method

$$x' = \varepsilon F(t, x, \varepsilon), \tag{4.1}$$

for $(t, x, \varepsilon) \in [0, T] \times D \times [-\varepsilon_0, \varepsilon_0]$, where F is a Carathéodory function. In Chapter 3 we showed that the Lipschitz property is not necessary when it comes to detecting periodic solutions of systems such as (4.1) for a continuous F. In particular, when F can be written as a polynomial in ε

$$x' = \varepsilon F(t, x, \varepsilon) = \varepsilon F_1(t, x) + \cdots + \varepsilon^{k-1} F_{k-1}(t, x) + \varepsilon^k F_k(t, x, \varepsilon), \tag{4.2}$$

for $(t, x, \varepsilon) \in \mathbb{R} \times D \times [\varepsilon_0, \varepsilon_0]$ the same result follows.

In this chapter, we further extend these results by relaxing the continuity hypothesis on F. In equation (4.1) and equation (4.2) we consider F to be a Carathéodory function (see Definition 2.11).

The strategy to arrive at those results is the same used in Chapter 3, however, since we are now dealing with functions that may not be continuous, some technical details must be filled in order to advance. That said, in order to prove the main results in this chapter, a common ground must be established and that is what will be done in the next sections. Then, in Section 4.6 we state and prove the main results.

4.2 What can be reused from the continuous case

In Chapter 3, we showed that looking for periodic solutions of a differential equation like equation (4.1) is equivalent to looking for solutions of an operator equation suitably defined. This rationale will be repeated here, and since the form of the equation itself is not changed, we use the same function spaces and operators. So we consider the spaces

$$X = \{x \in C[0, T] : x(0) = x(T)\}, \quad Z = \{x \in C[0, T] : x(0) = 0\},$$

with the sup $-$norm, the subset

$$\Omega = \{x \in X : x(t) \in V, \forall t \in [0, T]\},$$

where V is some open and bounded subset of $\mathbb{R}^n$ satisfying $\overline{V} \subset D$, and the operators $L : X \to Z$ and $N : \overline{\Omega} \times [0, \varepsilon_0] \to Z$ given by

$$Lx(t) = x(t) - x(0) \quad \text{and} \quad N(x, \varepsilon)(t) = \int_0^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s, \quad \forall\, t \in [0, T]. \tag{4.3}$$

Therefore, also by the integral form of solutions for Carathéodory differential equations presented in Section 2.3, it follows that an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ is a solution of equation (4.1) if, and only if, it is a solution of the operator equation

$$Lx = N(x, \varepsilon), \quad x \in \overline{\Omega}. \tag{4.4}$$

Because we are reusing the definitions from Chapter 3, some facts about these objects have already been proved therein. From Section 3.3, we know that Ω is an open and bounded subset of X and L is a continuous linear Fredholm operator of index zero. We can not, however, assert directly from the proof given in Section 3.3 that N is a well defined continuous $L-$compact operator, because now the right-hand side of equation (4.1) is a Carathéodory function, whose continuity in t is no longer guaranteed, thus necessitating a more careful treatment regarding its integrability.

4.3 N is L-compact

In the next two lemmas, we show that N is $L-$compact. Firstly, it needs to be shown that N is well defined and continuous.

Lemma 4.1. *The operator $N : \overline{\Omega} \times [0, \varepsilon_0] \to Z$ defined in equation (4.3) is a well-defined continuous operator.*

Proof. Since Ω is bounded, there exists $r > 0$ such that $|(x(s), \varepsilon)| \leqslant r$ for every $t \in [0, T]$ and $(x, \varepsilon) \in \overline{\Omega} \times [0, \varepsilon_0]$ and, by C.3, there exists an integrable function $g_r : [0, T] \to \mathbb{R}^n$ such that $|\varepsilon F(t, x(t), \varepsilon)| \leqslant g_r(t)$ for almost every $t \in [0, T]$. Hence, taking Proposition 2.2 into account, we see that the integral defining $N(x, \varepsilon)(t)$ is well-defined for every $t \in [0, T]$. To verify that $N(x, \varepsilon) \in Z$ we must see that it is a continuous function and $N(x, \varepsilon)(0) = 0$. Define $G_r : [0, T] \to \mathbb{R}$ by

$$G_r(t) = \int_0^t g_r(s) \mathrm{d}s, \tag{4.5}$$

and note that

$$|N(x, \varepsilon)(t)| \leqslant G_r(t), \quad \forall\, t \in [0, T], \tag{4.6}$$

which, in particular, implies that $N(x, \varepsilon)(0) = 0$. Furthermore, G_r is a uniformly continuous function. Given any $(x, \varepsilon) \in \overline{\Omega} \times [0, \varepsilon_0]$. For $t, \tau \in [0, T]$

$$\begin{aligned}
|N(x, \varepsilon)(t) - N(x, \varepsilon)(\tau)| &= \left| \int_0^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s - \int_0^\tau \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s \right| \\
&= \left| \int_\tau^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s \right| \leqslant \int_\tau^t |\varepsilon F(s, x(s), \varepsilon)| \, \mathrm{d}s \\
&\leqslant \int_\tau^t g_r(s) \mathrm{d}s = \int_0^t g_r(s) \mathrm{d}s - \int_0^\tau g_r(s) \mathrm{d}s \\
&\leqslant |G_r(t) - G_r(\tau)|
\end{aligned}$$

Therefore, by uniform continuity of G_r it follows that $N(x, \varepsilon)$ is a continuous function, concluding the proof that $N(x, \varepsilon) \in Z$.

We proceed now to prove that N is a continuous operator. For that matter, let $\{(x_m, \varepsilon_m)\}_{m \in \mathbb{N}}$ be any sequence converging to (x, ε). For each $m \in \mathbb{N}$ define

$$\Delta_m(t) = \varepsilon_m F(t, x_m(t), \varepsilon_m) - \varepsilon F(t, x(t), \varepsilon).$$

Observe that

$$\begin{aligned}
|N(x_m,\varepsilon_m)-N(x,\varepsilon)| &= \sup_{t\in[0,T]} |N(x_m,\varepsilon_m)(t)-N(x,\varepsilon)(t)| \\
&= \sup_{t\in[0,T]} \left| \int_0^t \varepsilon_m F(s,x_m(s),\varepsilon_m)\mathrm{d}s - \int_0^t \varepsilon F(s,x(s),\varepsilon)\mathrm{d}s \right| \\
&= \sup_{t\in[0,T]} \left| \int_0^t \varepsilon_m F(s,x_m(s),\varepsilon_m) - \varepsilon F(s,x(s),\varepsilon)\mathrm{d}s \right| \\
&\leqslant \sup_{t\in[0,T]} \int_0^t |\varepsilon_m F(s,x_m(s),\varepsilon_m) - \varepsilon F(s,x(s),\varepsilon)|\, \mathrm{d}s \\
&= \sup_{t\in[0,T]} \int_0^t |\Delta_m(s)|\, \mathrm{d}s.
\end{aligned}$$

Moreover, $\Delta_m(t) \to 0$ for almost every $t \in [0,T]$ and

$$|\Delta_m(t)| \leqslant |\varepsilon_m F(t,x_m(t),\varepsilon_m)| + |\varepsilon F(t,x(t),\varepsilon)| \leqslant 2g_r(t),$$

for almost every $t \in [0,T]$. Hence, each Δ_m is an integrable function and they are all dominated by $2g_r$. Thus, by the Dominated Convergence Theorem, Theorem 2.3

$$\lim \int_0^t \Delta_m(s)\mathrm{d}s = \int_0^t \lim \Delta_m(s)\mathrm{d}s = 0.$$

Therefore, for $m \in \mathbb{N}$ sufficiently large, $|N(x_m,\varepsilon_m) - N(x,\varepsilon)|$ can be made arbitrarily small, showing the continuity of N. □

Now the next lemma shows that N is an $L-$compact operator on $\overline{\Omega}$.

Lemma 4.2. *$N : \overline{\Omega} \times [0,\varepsilon_0] \to Z$ is an $L-$compact operator on $\overline{\Omega}$ for every $\varepsilon \in [0,\varepsilon_0]$.*

Proof. For any $(x,\varepsilon) \in \overline{\Omega} \times [0,\varepsilon_0]$, since Π is a continuous linear map, there exists a constant $C > 0$ such that

$$|\Pi N(x,\varepsilon)| \leqslant C\,|N(x,\varepsilon)| = C \sup_{t\in[0,T]} \left| \int_0^t \varepsilon F(s,x(s),\varepsilon)\mathrm{d}s \right|$$

Now since $x \in \Omega, x(t) \in \overline{V}$ for every $t \in [0,T]$, where V is a bounded subset of $\mathbb{R}^n$, hence there exists $r > 0$ for which $\overline{V} \times [0,\varepsilon_0] \subset B(0,r) \subset \mathbb{R}^{n+1}$. Using C.3, there exists an integrable positive function $g_r : [0,T] \to \mathbb{R}$ such that $|\varepsilon F(t,z,\varepsilon)| \leqslant g_r(t)$ for every $|(z,\varepsilon)| < r$. It follows that

$$\begin{aligned}
|\Pi N(x,\varepsilon)| &\leqslant C \int_0^t |\varepsilon F(s,x(s),\varepsilon)|\, \mathrm{d}s \\
&\leqslant C \int_0^t g_r(s)\mathrm{d}s.
\end{aligned}$$

Notice that g_r depends neither on x nor on ε. This shows that the image of the operator ΠN is a bounded set, which implies that ΠN is a bounded operator.

Recall that $K_{P,Q} = \text{Id} - Q$. Since $K_{P,Q}N$ is composition of continuous operators, then it is also continuous. Consider the family $\Lambda = K_{P,Q}N(\overline{\Omega} \times [0, \varepsilon_0])$. To show that the closure of Λ is compact, we use the Arzelá-Ascoli Theorem, Theorem 2.1. Hence, we must check that Λ is equicontinuous and uniformly bounded. Let $y \in \Lambda$. Hence, there exist $x \in \overline{\Omega}$ and $\varepsilon \in [0, \varepsilon_0]$ such that

$$y(t) = (\text{Id} - Q)N(x, \varepsilon)(t) = (\text{Id} - Q)\left(\int_0^t \varepsilon F(s, x(s), \varepsilon)\text{d}s\right).$$

For $t, \tau \in [0, T]$ we have

$$\begin{aligned}
|y(t) - y(\tau)| &= |(\text{Id} - Q)N(x, \varepsilon)(t) - (\text{Id} - Q)N(x, \varepsilon)(\tau)| \\
&\leqslant |N(x, \varepsilon)(t) - N(x, \varepsilon)(\tau) - (QN(x, \varepsilon)(t) - QN(x, \varepsilon)(\tau))| \\
&\leqslant \left|\int_\tau^t \varepsilon F(s, x(s), \varepsilon)\text{d}s - \frac{t - \tau}{T}\int_0^T \varepsilon F(s, x(s), \varepsilon)\text{d}s\right| \\
&\leqslant \int_\tau^t |\varepsilon F(s, x(s), \varepsilon)|\,\text{d}s + \left|\frac{t - \tau}{T}\right|\int_0^T |\varepsilon F(s, x(s), \varepsilon)|\,\text{d}s \\
&\leqslant \int_\tau^t g_r(s)\text{d}s + \left|\frac{t - \tau}{T}\right|\int_0^T g_r(s)\text{d}s \\
&\leqslant |G_r(t) - G_r(\tau)| + |t - \tau|\,\frac{G_r(T)}{T}.
\end{aligned}$$

Given $\xi > 0$, by the uniform continuity of G_r, there exists $\delta_1 > 0$ such that $|G_r(t) - G_r(\tau)| < \xi/2$. Hence, taking $\delta = \min\{\delta_1, \xi T/2G_r(T)\}$, we obtain that if $|t - \tau| < \delta$, then $|y(t) - y(\tau)| < \xi$. Since δ does not depend on the particular y, it follows that Λ is a equicontinuous family.

On the other hand,

$$\begin{aligned}
|y(t)| &= |(\text{Id} - Q)N(x, \varepsilon)| \\
&\leqslant |(\text{Id} - Q)||N(x, \varepsilon)| \\
&\leqslant \left|\int_0^t \varepsilon F(s, x(s), \varepsilon)\text{d}s\right| \\
&\leqslant \int_0^t |\varepsilon F(s, x(s), \varepsilon)|\,\text{d}s \\
&\leqslant \int_0^t g_r(s)\text{d}s \\
&\leqslant G_r(T),
\end{aligned}$$

showing that Λ is uniformly bounded. Thus, by applying the Theorem of Arzelá-Ascoli, Theorem 2.1, we conclude that the family has compact closure, implying the $L-$compactness of N on $\overline{\Omega}$ for every $\varepsilon \in [0, \varepsilon_0]$. □

4.4 Continuity of the averaged functions

For equation (4.1), we define the full averaged function $f : D \times [0, \varepsilon_0] \to \mathbb{R}^n$ by

$$f(z, \varepsilon) = \frac{1}{T} \int_0^T F(s, z, \varepsilon)\mathrm{d}s.$$

Proposition 4.1. *f is a continuous function.*

Proof. Let $\{(z_m, \varepsilon_m)\}_{m \in \mathbb{N}}$ be a sequence converging to $(z_0, \varepsilon) \in D \times [0, \varepsilon_0]$.

$$\begin{aligned} |f(z_m, \varepsilon_m) - f(z_0, \varepsilon)| &= \left| \int_0^T F(s, z_m, \varepsilon_m)\mathrm{d}s - \int_0^T F(s, z_0, \varepsilon)\mathrm{d}s \right| \\ &= \left| \int_0^T F(s, z_m, \varepsilon_m) - F(s, z_0, \varepsilon)\mathrm{d}s \right| \\ &= \int_0^T |F(s, z_m, \varepsilon_m) - F(s, z_0, \varepsilon)|\, \mathrm{d}s. \end{aligned}$$

By an almost identical argument as that used to prove that N is a continuous operator, we conclude that f is continuous. □

For the differential equation in (4.2)

$$x' = \varepsilon F(t, x, \varepsilon) = \varepsilon F_1(t, x) + \cdots + \varepsilon^{k-1} F_{k-1}(t, x) + \varepsilon^k F_k(t, x, \varepsilon),$$

where each $F_i : [0, T] \times D \to \mathbb{R}^n$ and $F_k : [0, T] \times D \times [0, \varepsilon_0] \to \mathbb{R}^n$ are Carathéodory functions $T-$periodic in t, we define for each $i \in \{1, \ldots, k\}$, $f_i : D \to \mathbb{R}^n$ by setting

$$\begin{aligned} f_i(z) &= \frac{1}{T} \int_0^T F_i(s, z)\mathrm{d}s, \quad i \in \{1, \ldots, k-1\}, \\ f_k(z) &= \frac{1}{T} \int_0^T F_k(s, z, 0)\mathrm{d}s. \end{aligned}$$

Proposition 4.2. *For each $i \in \{1, \ldots, k\}$, f_i is a continuous function.*

Proof. Consider at first $i \in \{1, \ldots, k-1\}$. Let $z_0 \in D$ be an arbitrary point in D and let $\{z_m\}_{m \in \mathbb{N}}$ be any sequence in D converging to z_0. Define

$$\Delta_m(t) = F_i(t, z_m) - F_i(t, z_0),$$

and notice that $\Delta_m(t) \to 0$ as $m \to \infty$ for almost every $t \in [0, T]$. Moreover, let $r > 0$ be such that $|z_0| < r$. Thus there exists $m_0 \in \mathbb{N}$ for which $m > m_0$ implies $|z_m| \leqslant r$. Thus, by C.3 there exists a positive integrable function $g_r : [0, T] \to \mathbb{R}$ such that $|F_i(t, z)| \leqslant g_r(t)$ for almost every $t \in [0, T]$. It follows that

$$|\Delta_m(t)| = |F_i(t, z_m) - F_i(t, z_0)| \leqslant 2g_r(t),$$

for almost every $t \in [0, T]$ and $m > m_0$. Now, we can apply the Dominated Convergence Theorem (Theorem 2.3) to conclude that

$$\lim_{m\to\infty} \int_0^T |\Delta_m(s)|\mathrm{d}s = \int_0^T \lim_{m\to\infty} |\Delta_m(s)|\mathrm{d}s = 0. \tag{4.7}$$

Now, since

$$\begin{aligned}
|f_i(z_m) - f_i(z_0)| &= \left| \frac{1}{T}\int_0^T F_i(s, z_m)\mathrm{d}s - \frac{1}{T}\int_0^T F_i(s, z_0)\mathrm{d}s \right| \\
&= \left| \frac{1}{T}\int_0^T F_i(s, z_m) - F_i(s, z_0)\mathrm{d}s \right| \\
&\leqslant \frac{1}{T}\int_0^T |F_i(s, z_m) - F_i(s, z_0)|\,\mathrm{d}s \\
&= \frac{1}{T}\int_0^T |\Delta_m(s)|\,\mathrm{d}s
\end{aligned}$$

by equation (4.7)

$$\lim_{m\to\infty} |f_i(z_m) - f_i(z_0)| \leqslant \lim_{m\to\infty} \frac{1}{T}\int_0^T |\Delta_m(s)|\,\mathrm{d}s = 0.$$

This shows that $f_i(z_m) \to f_i(z_0)$ as $m \to \infty$ for any sequence converging to z_0 which was also taken arbitrarily, so that f_i is continuous in D. Notice that the exact same argument follows for f_k. □

From the functions f_i defined above, let us now define the $k-$truncated averaged function $\mathcal{F}_k : D \to \mathbb{R}^n$ and the averaged remainder $r : D \times [0, \varepsilon_0] \to \mathbb{R}^n$ by

$$\mathcal{F}_k(z, \varepsilon) = \varepsilon f_1(z) + \cdots + \varepsilon^k f_k(z) \qquad r(z, \varepsilon) = \frac{1}{T}\int_0^T F_k(s, z, \varepsilon) - F_k(s, z, 0)\mathrm{d}s.$$

Therefore, the full averaged function f can be written as

$$f(z, \varepsilon) = \frac{1}{T}\int_0^T \varepsilon F(s, z, \varepsilon)\mathrm{d}s = \mathcal{F}_k(z, \varepsilon) + \varepsilon^{k+1} r(z, \varepsilon).$$

From the continuity of the f_is, we conclude that $\mathcal{F}_k$ and $r(z, \varepsilon)$ are continuous.

In Theorem F and Theorem G we provide conditions on the functions f_i and $\mathcal{F}_k$ in order to have $T-$periodic solutions of equation (4.2).

4.5 The condition **H** in the Carathéodory case

To extend the methodology for studying periodic solutions of Carathéodory differential equations, we use the same approach as we did in the last chapter, namely we convert the differential equation in an operator equation and use Theorem 2.11.

We have already discussed in Subsection 2.5.2 that a boundary condition is needed in order to apply Theorem 2.11. The Carathéodory case is no different from the continuous one for that matter, we also need a boundary condition. As it turns out, it is the same condition that was used in the continuous case, however it is important to mention that this condition is a little more strict in the Carathéodory context than it is in the continuous one.

Given an open and bounded subset V of $\mathbb{R}^n$, we say that the condition **H** holds on V if

H. there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that, for each $\lambda \in (0,1)$ and $\varepsilon \in (0, \varepsilon_1]$, any T-periodic solution of the differential equation

$$x' = \varepsilon\lambda F(t, x, \varepsilon),\ x \in \overline{V}, \tag{4.8}$$

is entirely contained in V.

To verify that this condition holds on a given set, one must follow a similar path as that described in Section 3.4, arguing by contradiction.

Notice that since V is bounded, there exists $r > 0$ such that $\overline{V} \subset B(0, r)$. Using C.3, there exists a positive integrable function $g_r : [0, T] \to \mathbb{R}$ such that $|F(t, z, \varepsilon)| \leqslant g_r(t)$ for almost every $t \in [0, T]$. Let $G_r : [0, T] \to \mathbb{R}$ be given by

$$G_r(t) = \int_0^t g_r(s)\mathrm{d}s.$$

Of course, G_r is also a positive function. Moreover, G_r is a continuous function on a compact set, thus G_r is uniformly continuous.

Proposition 4.3. *If condition **H** does not hold on an open and bounded subset V of $\mathbb{R}^n$, then there exist sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0, \varepsilon_0], \{\lambda_m\}_{m\in\mathbb{N}} \in (0,1)$ and $\{x_m\}_{m\in\mathbb{N}}$ of $T-$periodic solutions of equation (4.8) with $\varepsilon_m \to 0$ and $x_m \to z_0 \in \partial V$ uniformly on $[0, T]$ such that*

$$\int_0^T F(s, z_0, 0)\mathrm{d}s = 0. \tag{4.9}$$

Proof. The proof follows the same steps as the proof of Proposition 3.3. Thus, if we assume that **H** does not hold on V, then we get sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0, \varepsilon_0], \{\lambda_m\}_{m\in\mathbb{N}} \in (0,1)$ and $\{x_m\}_{m\in\mathbb{N}}$ of $T-$periodic solutions of equation (4.8) with $\varepsilon_m \to 0$ and $x_m(t_m) \in \partial V$, for some $t_m \in [0, T]$ and every $m \in \mathbb{N}$. We claim that as in the continuous case, we can pick the sequence $\{x_m\}_{m\in\mathbb{N}}$ so that $x_m \to z_0 \in \partial V$ uniformly on $[0, T]$. The proof is analogous and also makes use of the Arzelá-Ascoli Theorem. In proving that $\Lambda = \{x_m : m \in \mathbb{N}\}$ is equicontinuous and uniformly bounded, we must use property C.3 and the function G_r to conclude that for $t, \tau \in [0, T]$,

$$\begin{aligned}|x_m(t) - x_m(\tau)| &\leqslant \varepsilon_m \lambda_m \int_\tau^t |F(s, x_m(s), \varepsilon_m)| \,\mathrm{d}s \\ &\leqslant \int_\tau^t g_r(s)\mathrm{d}s \\ &\leqslant |G_r(t) - G_r(\tau)|,\end{aligned}$$

from where equicontinuity of Λ follows from the uniform continuity of G_r in $[0, T]$. In addition,

$$|x_m(t)| \leqslant |x_m(0)| + \int_0^t |\varepsilon_m \lambda_m F(s, x_m(s), \varepsilon_m)| \,\mathrm{d}s \leqslant r + G_r(T),$$

which shows that Λ is uniformly bounded. Moreover, since for every $m \in \mathbb{N}, x_m$ touches the boundary of $V, x_m \to z_0 \in \partial V$. Therefore $F(t, x_m(t), \varepsilon_m) \to F(t, z_0, 0)$ as $m \to \infty$ for almost every $t \in [0, T]$. From the periodicity of each x_m, we have

$$\int_0^T F(s, x_m(s), \varepsilon_m)\mathrm{d}s = 0.$$

By the Dominated Convergence Theorem (Theorem 2.3), it follows that

$$\lim_{m\to\infty} \int_0^T F(s, x_m(s), \varepsilon_m)\mathrm{d}s = \int_0^T \lim_{m\to\infty} F(s, x_m(s), \varepsilon_m)\mathrm{d}s = \int_0^T F(s, z_0, 0)\mathrm{d}s,$$

therefore

$$\int_0^T F(s, z_0, 0)\mathrm{d}s = 0, \tag{4.10}$$

completing the proof. □

Notice that the effect of negating condition **H** is the same, however it does not follow as naturally here as it does in Chapter 3 due to the lack of continuity.

As in Section 3.4, we show here that condition **H** also holds for the first order case.

Proposition 4.4. *Assume that equation (3.1) can be written as*

$$x' = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \qquad \text{for} \qquad (t, x, \varepsilon) \in [0, T] \times D \times [0, \varepsilon_0],$$

where F_1 and R are Carathéodory functions and $T-$periodic in t. Define $f_1 : D \to \mathbb{R}^n$ by

$$f_1(z) = \frac{1}{T} \int_0^T F_1(s, z)\mathrm{d}s.$$

*Let $V \subset \mathbb{R}^n$ be an open and bounded subset such that $\overline{V} \subset D$ and $f_1(z) \neq 0$, for every $z \in \partial V$. Then condition **H** holds on V.*

Proof. Assume then that condition **H** does not hold on V. By Proposition 4.3, it follows that there exist sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0, \varepsilon_1]$ and $\{x_m\}_{m\in\mathbb{N}}$ such that $\varepsilon_m \to 0$ as $m \to \infty$, $x_m \to z_0 \in \partial V$ uniformly on $[0,T]$ and

$$\int_0^T F(s, x_m(s), \varepsilon_m)\mathrm{d}s = 0,$$

for every $m \in \mathbb{N}$. Moreover, equation (4.9) implies

$$0 = \int_0^T F(s, z_0, 0)\mathrm{d}s = \int_0^T F_1(s, z_0)\mathrm{d}s = T f_1(z_0),$$

contradicting $f_1(z) \neq 0$ for every $z \in \partial V$. Thus, condition **H** holds on V. □

On the other hand, we do not have an analogue of Proposition 3.5 for the Carathéodory case, that is, when the boundary of $V, \partial V$, is a smooth manifold and the the right-hand side F is transversal to ∂V, it is not true that condition **H** holds. We show that via an example.

Let $\mathcal{D} = [0, 2\pi] \times \mathbb{R} \times [0, \varepsilon_0]$ and define $f : \mathcal{D} \to \mathbb{R}$ by

$$f(t,x,\varepsilon) = \begin{cases} 1, & t \in \left[0, \dfrac{\pi}{2}\right) \cup \left[\dfrac{3\pi}{2}, 2\pi\right] \\ -1, & t \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right) \end{cases}, \qquad \forall\, (x,\varepsilon) \in \mathbb{R} \times [0, \varepsilon_0].$$

and note that it is clearly a Carathéodory function.

Let $V = (-1, 1) \subset \mathbb{R}$. The solutions of $x' = \varepsilon\lambda f(t,x,\varepsilon), x \in \overline{V}$, for each $\varepsilon \in (0, \varepsilon_0]$ and $\lambda \in (0,1)$ fixed have the form shown in Figure 4.

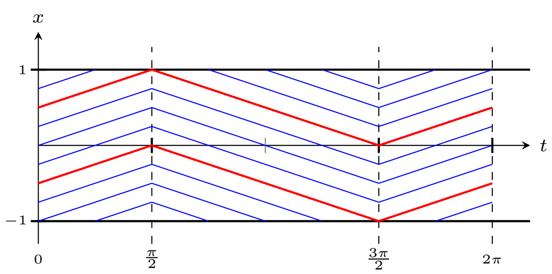

Figure 4 – Solutions of $x' = \varepsilon\lambda f(t,x,\varepsilon)$.

Notice that $\partial V = \{-1, 1\}$ and $f(t, \pm 1, \varepsilon) \neq 0$ for every $(t, \varepsilon) \in [0, 2\pi] \times [0, \varepsilon_0]$. Furthermore, since the solutions of Carathéodory functions can have a few points where the derivative may not exist, it is possible to have a 2π−periodic solution of $x' = \varepsilon\lambda f(t,x,\varepsilon)$ with $x \in \overline{V}$ such that $x(t^*) \in \partial V$ for some $t^* \in [0, 2\pi]$ even if ∂V is a smooth manifold and $f(t,x,\varepsilon)$ is transversal to ∂V for every $t \in [0,T], x \in \partial V, \varepsilon \in (0, \varepsilon_0]$.

4.6 Statements and proofs of main results

Now, we can state and prove the main results of this chapter. Recall that given equation (4.1), we define the full averaged function $f : D \times [0, \varepsilon_0] \to \mathbb{R}^n$ as

$$f(z, \varepsilon) = \frac{1}{T} \int_0^T \varepsilon F(s, z, \varepsilon) \mathrm{d}s$$

and by Proposition 4.1 it is continuous.

Theorem E. *Consider the non-autonomous Carathéodory $T-$periodic differential equation (4.1). Suppose that there exists an open bounded subset $V \subset \mathbb{R}^n$, with $\overline{V} \subset D$, and $\varepsilon_1 \in (0, \varepsilon_0]$ for which **H** holds. Assume that*

$$f(z, \varepsilon) \neq 0, \quad (z, \varepsilon) \in \partial V \times (0, \varepsilon_1], \tag{4.11}$$

and that $d_B(f(\cdot, \varepsilon^*), V, 0) \neq 0$ for some $\varepsilon^* \in (0, \varepsilon_1]$. Then, for each $\varepsilon \in (0, \varepsilon_1]$, there exists a $T-$periodic solution $\varphi(t, \varepsilon)$ of (4.1) such that $\varphi(t, \varepsilon) \in \overline{V}$, for every $t \in [0, T]$.

Proof. We will apply Theorem 2.11. We start by showing that **H** implies **A.1**. Indeed, if **A.1** does not hold, then there exists $x \in \operatorname{dom} L \cap \partial\Omega, \varepsilon \in [0, \varepsilon_0]$ and $\lambda \in (0, 1)$ for which $Lx = \lambda N(x, \varepsilon)$. This is equivalent to saying that there exists a solution of $x' = \lambda \varepsilon F(t, x, \varepsilon), x \in \overline{V}$ lying in $\partial\Omega$, meaning that $x(t^*) \in \partial V$ for some $t^* \in [0, T]$. Then **H** does not hold. This shows the desired implication.

Now let $x \in \operatorname{Ker} L \cap \partial\Omega$. Let $z \in \mathbb{R}^n$ be such that $x(t) \equiv z$. Working out $QN(x, \varepsilon)$ for each $\in [0, \varepsilon_0]$ we get

$$QN(x, \varepsilon) = \frac{t}{T} \int_0^T \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s = t f(z, \varepsilon).$$

Then, by (4.11), we conclude that the condition **A.2** holds. Using Claim 3.0.2 from the proof of Theorem A, we show that $d_B(f(\cdot, \varepsilon), V, \varepsilon^*) \neq 0$ for some $\varepsilon^* \in (0, \varepsilon_1]$ implies that $d_B(f(\cdot, \varepsilon), V, \varepsilon) \neq 0$ for every $\varepsilon \in (0, \varepsilon_1]$. Then, by Theorem 2.11, there exist $\varepsilon_V \in (0, \varepsilon_0]$ and a $T-$periodic solution $\varphi : [0, T] \times (0, \varepsilon_V] \to \mathbb{R}^n$ such that $\varphi(t, \varepsilon) \in \overline{V}$ for every $t \in [0, T]$. □

Notice that the proof of the above theorem is a lot shorter than that of Theorem A. This is due to the fact that many of the technical machinery has already been developed for it and is being reused in this context.

Theorem F. *Consider the non-autonomous Carathéodory $T-$periodic differential equation (4.2). Suppose that there exists and open bounded subset $V \subset \mathbb{R}^n$, with $\overline{V} \subset D$, and $\varepsilon_1 \in (0, \varepsilon_0]$ for which condition **H** holds. Assume that*

$$\lim_{\varepsilon \to 0^+} \inf_{z \in \partial V} \frac{|\mathcal{F}_k(z, \varepsilon)|}{\varepsilon^k} > \max \{ |r(z, \varepsilon)|, \forall\, (z, \varepsilon) \in \partial V \times [0, \varepsilon_1] \}, \tag{4.12}$$

and that $d_B(\mathcal{F}_k(\cdot,\varepsilon),V,0) \neq 0$ for every $\varepsilon \in (0,\varepsilon_1]$. Then, there exists $0 < \varepsilon_V \leqslant \varepsilon_1$ such that for each $\varepsilon \in (0,\varepsilon_V]$, there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of (4.1) such that $\varphi(t,\varepsilon) \in \overline{V}$, for every $t \in [0,T]$.

Proof. Note that by equation (4.12) there exists $\varepsilon_2 > 0$ such that

$$\frac{|\mathcal{F}_k(z,\varepsilon)|}{\varepsilon^k} > |r(z,\varepsilon)|, \quad \text{for} \quad (z,\varepsilon) \in \partial V \times (0,\varepsilon_2].$$

The above inequality implies

$$|f(z,\varepsilon)| = |\mathcal{F}_k(z,\varepsilon) + \varepsilon^{k+1} r(z,\varepsilon)| \geqslant |\mathcal{F}_k(z,\varepsilon)| - \varepsilon^{k+1}|r(z,\varepsilon)| > 0,$$

showing that hypothesis (4.11) of Theorem E is met. We have also that

$$d_B(f(\cdot,\varepsilon),V,0) = d_B(\mathcal{F}_k(\cdot,\varepsilon),V,0) \neq 0.$$

Thus, applying Theorem E the result follows. □

Finally, we consider the case when some of the function f_i are identically zero.

Theorem G. *Consider the non-autonomous Carathéodory $T-$periodic differential equation (4.2). Suppose that, for some $\ell \in \{1,2,\ldots,k\}$, $f_0 = \ldots = f_{\ell-1} = 0$, $f_\ell \neq 0$. Assume that there exists an open bounded set $V \subset \mathbb{R}^n$, with $\overline{V} \subset D$, for which condition **H** holds, $f_\ell(z) \neq 0$, for every $z \in \partial V$, and $d_B(f_\ell,V,0) \neq 0$. Then, there exists $\varepsilon_V > 0$ such that, for each $\varepsilon \in [0,\varepsilon_V]$, the differential equation (4.1) has a $T-$periodic solution $\varphi(t,\varepsilon)$ satisfying $\varphi(t,\varepsilon) \in \overline{V}$, for every $t \in [0,T]$.*

Proof. We have

$$f(z,\varepsilon) = \mathcal{F}_\ell(z,\varepsilon) + \varepsilon^{\ell+1} r(z,\varepsilon),$$

so that

$$\liminf_{\varepsilon\to 0}\inf_{z\in\partial V} \frac{\mathcal{F}_k(z,\varepsilon)}{\varepsilon^{\ell+1}} = \liminf_{\varepsilon\to 0}\inf_{z\in\partial V} \frac{f_\ell(z,\varepsilon)}{\varepsilon} = +\infty.$$

Hence, by continuity of $r(z,\varepsilon)$, it is clear that (4.12) holds. Since

$$d_B(f_\ell,V,0) = d_B(\mathcal{F}_\ell(\cdot,\varepsilon),V,0) \neq 0,$$

for $\varepsilon > 0$ sufficiently small, then Theorem F can be applied and the result follows. □

As for the continuous case, to perform a first order analysis for Carathéodory differential equations it is not needed to assume condition **H**.

Theorem 4.1. *Consider the non-autonomous Carathéodory $T-$periodic differential equation*

$$x' = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \qquad (t,x,\varepsilon) \in [0,T] \times D \times [0,\varepsilon_0]. \tag{4.13}$$

Let $f_1 : D \to \mathbb{R}^n$ be given by

$$f_1(z) = \frac{1}{T}\int_0^T F_1(s,z)\mathrm{d}s.$$

Let $V \subset \mathbb{R}^n$ be an open and bounded subset such that $\overline{V} \subset D$ and $f_1(z) \neq 0$, for every $z \in \partial V$. If $d_B(f_1, V, 0) \neq 0$, there exists $\varepsilon_V > 0$ such that, for each $\varepsilon \in [0, \varepsilon_V]$, equation (4.13) *has a $T-$periodic solution $\varphi(t,\varepsilon)$ satisfying $\varphi(t,\varepsilon) \in \overline{V}$, for every $t \in [0,T]$.*

Proof. By Proposition 4.4, we have that condition **H** holds on V. Since by hypothesis, $f_1 \neq 0$ and $d_B(f_1, V, 0) \neq 0$, by Theorem G the conclusions follows immediately. □

5 Convergence of solutions

In the exploration of ordinary differential equations, unravelling the behaviour of solutions concerning the system's parameters stands as a cornerstone in understanding the robustness of the underlying dynamical system. This chapter delves into the intricate interplay between the solutions of differential equations and the parameters that govern their evolution. The overarching theme revolves around the convergence of solutions, shedding light on a kind of regularity exhibited by these solutions concerning changes in the parameter space.

Our primary objective is to discern the manner in which solutions evolve with variations in the parameters. This nuanced perspective extends beyond the mere existence of solutions, delving into the broader question of their convergence as the parameters undergo systematic transformations.

As we have seen in Chapter 1, in the literature, the convergence of solutions is obtained by the Implicit Function Theorem. Indeed, we obtain a branch of zeroes, $x : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^n$, of the displacement function, $\Delta(x, \varepsilon) = \pi(x, \varepsilon) - x$, by the Implicit Function Theorem such that $x(0) = x_0$ and $\Delta(x(\varepsilon), \varepsilon) = 0$, the function $x(\varepsilon)$ is known to be differentiable and, therefore, $x(\varepsilon) \to x_0$ as $\varepsilon \to 0$ showing that the solutions starting at $x(\varepsilon)$ tend to the solution starting at x_0 as $\varepsilon \to 0$.

However in the non-smooth case, where the Implicit Function Theorem is not available, the convergence of solutions is not as straightforward. Nonetheless, even in this scenario, it is still possible to arrive at convergence results.

The main tool used in this work specially as a replacement to the condition on the derivative of the averaged functions is the Brouwer degree. Besides providing a criterium to know whether a function has a zero in some domain or not, the Brouwer degree also has the excision property Proposition 2.4 which will be fundamental in the next results.

5.1 Statements and proofs of main results

In this section the reader will notice that we state the results mainly driven to the more general case of Carathéodory equations. However, it should be noted that this case contains the non-Lipschitz one and because of that, in the next theorems we reference inside parentheses the continuous non-Lipschitz counterpart of the underlying Carathéodory equation.

The first result regarding convergence in the parameter is the following.

Theorem H. *In addition to the hypotheses of Theorem E (Theorem A), assume that for each* $\varepsilon \in (0, \varepsilon_1]$ *there exists a unique* $z_\varepsilon \in V$ *such that* $f(z_\varepsilon, \varepsilon) = 0$, $z_\varepsilon \to z^* \in V$ *as* $\varepsilon \to 0$, *and that there exists a family* $\{W_\mu : \mu \in (0, \mu_0]\}$ *of open subsets of* V *containing* z^* *such that* $\operatorname{diam} W_\mu \to 0$ *as* $\mu \to 0$ *on each of which condition* ***H*** *holds. Then, there exists a* $T-$*periodic solution* $\varphi(t, \varepsilon)$ *of* (4.1) *satisfying* $\varphi(\cdot, \varepsilon) \to z^*$ *uniformly as* $\varepsilon \to 0$.

Proof. Condition (4.11) still holds if we change V for W_μ. From the convergence $\lim_{\varepsilon \to 0} z_\varepsilon = z^*$ it follows that for each $\mu \in (0, \mu_0]$, there exists $\varepsilon_\mu \in (0, \varepsilon_1]$ such that for each $\varepsilon \in (0, \varepsilon_\mu]$, $z_\varepsilon \in W_\mu$. By uniqueness of z_ε, we conclude using the excision property of the Brouwer degree, Proposition 2.4, that given $\mu \in (0, \mu_0]$,

$$d_B(f(\cdot, \varepsilon), W_\mu, 0) = d_B(f(\cdot, \varepsilon), V, 0) \neq 0,$$

for every $\varepsilon \in (0, \varepsilon_\mu]$. In addition, **H** holds on each W_μ, therefore, by Theorem E, for each $\mu \in (0, \mu_0]$ there exists $\varepsilon'_{W_\mu} \in (0, \varepsilon_1]$ such that for every $\varepsilon \in (0, \varepsilon'_{W_\mu}]$ there exists a $T-$periodic solution $\varphi^\mu(t, \varepsilon)$ of (4.1) satisfying

$$\varphi^\mu(t, \varepsilon) \in W_\mu, \text{ for every } t \in [0, T] \text{ and } \varepsilon \in (0, \varepsilon'_{W_\mu}]. \tag{5.1}$$

In order to construct a solution $\varphi(t, \varepsilon)$ converging to z^*, we consider a sequence $\{\mu_m\}_{m \in \mathbb{N}}$ whose terms lie in $(0, \mu_0]$ satisfying $\mu_m \to 0$ as $m \to \infty$. This sequence automatically gives us two other sequences $\{\epsilon_m\}_{m \in \mathbb{N}}$ and $\{\varphi_m\}_{m \in \mathbb{N}}$, with $\epsilon_m := \varepsilon'_{W_{\mu_m}}$ and each $\varphi_m := \varphi^{\mu_m}$ is a $T-$periodic solution of equation (4.1) satisfying $\varphi_m(t, \varepsilon) \in W_{\mu_m}$ for every $t \in [0, T], \varepsilon \in (0, \epsilon_m]$ and $m \in \mathbb{N}$. Since a convergent subsequence of $\{\epsilon_m\}_{m \in \mathbb{N}}$ can be extracted so that $\epsilon_{m_k} \to \varepsilon^* \in [0, \varepsilon_1]$, we assume that the sequence itself converges to ε^*. We treat the cases $\varepsilon^* > 0$ and $\varepsilon^* = 0$ separately. Assume first that $\varepsilon^* > 0$. Since, $\epsilon_m > 0$ for every $m \in \mathbb{N}$, there exists $\bar{\varepsilon} > 0$ such that $[0, \bar{\varepsilon}] \subset [0, \epsilon_m]$. By (5.1), it follows that $\varphi_m \to z^*$ uniformly as $m \to \infty$. Thus, define $\varphi(t, \varepsilon) := z^*$. We claim that $\varphi(t, \varepsilon)$ is a solution of (4.1) for every $\varepsilon \in [0, \bar{\varepsilon}]$. Indeed, in order to prove this claim, we must verify that this function satisfies the integral equation

$$x(t) = x(0) + \int_0^t \varepsilon F(s, x(s), \varepsilon) \mathrm{d}s,$$

for every $t \in [0, T]$. Since $\varphi(t, \varepsilon)$ is constant in t, the above equation is equivalent to

$$\int_0^t \varepsilon F(s, z^*, \varepsilon) \mathrm{d}s = 0, \quad \forall\, t \in [0, T].$$

For every $m \in \mathbb{N}$, we have

$$\varphi_m(t, \varepsilon) = \varphi_m(0, \varepsilon) + \int_0^t \varepsilon F(s, \varphi_m(s, \varepsilon), \varepsilon) \mathrm{d}s.$$

Making $m \to \infty$ and using that $\varphi_m(t, \varepsilon) \to z^*$ for every $(t, \varepsilon) \in [0, T] \times (0, \bar{\varepsilon}]$, we get that

$$0 = \lim \int_0^t \varepsilon F(s, \varphi_m(s, \varepsilon), \varepsilon) \mathrm{d}s.$$

Define $p_m(t,\varepsilon) = \varepsilon F(t,\varphi_m(t,\varepsilon),\varepsilon)$, for each $m \in \mathbb{N}$. By C.3 and Proposition 2.2 we conclude that each p_m is an integrable function. As it is clear, $p_m(t,\varepsilon) \to p(t,\varepsilon) := \varepsilon F(t,z^*,\varepsilon)$, which by C.2 is a measurable function for each fixed ε. Finally, by C.3, for $r > 0$ large enough, there exists an integrable function g_r satisfying $|p_m(t,\varepsilon)| \leqslant g_r(t)$, for almost every $t \in [0,T]$ and every $m \in \mathbb{N}$. Hence, by the Lebesgue Dominated Convergence Theorem, Theorem 2.3, we obtain the following

$$0 = \lim \int_0^t \varepsilon F(s,\varphi_m(s,\varepsilon),\varepsilon)\mathrm{d}s = \int_0^t \lim \varepsilon F(s,\varphi_m(s,\varepsilon),\varepsilon)\mathrm{d}s = \int_0^t \varepsilon F(s,z^*,\varepsilon)\mathrm{d}s,$$

proving the claim.

Now, if $\varepsilon^* = 0$, set $\epsilon_{\max} = \max\{\epsilon_m : m \in \mathbb{N}\}$ and for each $\varepsilon \in (0,\epsilon_{\max}]$, define $m_\varepsilon = \max\{m \in \mathbb{N} : \varepsilon \leqslant \epsilon_m\}$. Note that $m_\varepsilon \to \infty$ as $\varepsilon \to 0$. We then put $\varphi(t,\varepsilon) := \varphi_{m_\varepsilon}(t,\varepsilon)$. One readily sees that for each $\varepsilon \in (0,\epsilon_{\max}]$, $\varphi(t,\varepsilon) \in W_{\mu_{m_\varepsilon}}$ for every $t \in [0,T]$, which once $\operatorname{diam} W_{\mu_{m_\varepsilon}} \to 0$ as $\mu_{m_\varepsilon} \to 0$ and $\mu_{m_\varepsilon} \to 0$ as $\varepsilon \to 0$ yields $\varphi(\cdot,\varepsilon) \to z^*$ uniformly as $\varepsilon \to 0$. □

Now, by specializing equation (4.1) to equation (4.2), we can provide a condition that depends on the $k-$truncated averaged function instead of the full averaged.

Theorem I. *In addition to the hypotheses of Theorem F (Theorem B), assume that for each $\varepsilon \in (0,\varepsilon_1]$ there exists a unique $z_\varepsilon \in V$ such that $\mathcal{F}_k(z_\varepsilon,\varepsilon) = 0$, $z_\varepsilon \to z^* \in V$ as $\varepsilon \to 0$, and that there exists a family $\{W_\mu : \mu \in (0,\mu_0]\}$ of open subsets of V containing z^* such that $\operatorname{diam} W_\mu \to 0$ as $\mu \to 0$ on each of which condition **H** holds. Then, there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of (4.1) satisfying $\varphi(\cdot,\varepsilon) \to z^*$ uniformly as $\varepsilon \to 0$.*

Proof. Note that

$$f(z,\varepsilon) = \mathcal{F}_k(z,\varepsilon) + \varepsilon^{k+1} r(z,\varepsilon)$$

alongside equation (4.12) imply that Proposition 2.6 can be applied on W_μ. Therefore, by the excision property of the Brouwer degree, Proposition 2.4, and Proposition 2.6 we obtain $d_B(f(\cdot,\varepsilon),V,0) = d_B(f(\cdot,\varepsilon),W_\mu,0) = d_B(\mathcal{F}_k(\cdot,\varepsilon),W_\mu,0) \neq 0$. Hence, Theorem F can be used to conclude that for each $\mu \in (0,\mu_0]$ there exists $\varepsilon_\mu \in (0,\varepsilon_1]$ such that for every $\varepsilon \in (0,\varepsilon_\mu]$ there exists a $T-$periodic solution $\varphi^\mu(t,\varepsilon)$ of equation (4.2) satisfying $\varphi^\mu(t,\varepsilon) \in W_\mu$ for every $t \in [0,T]$. Now the exact same argument used in the proof of Theorem H can be applied to conclude the proof of this result. □

Moreover, when some of the averaged functions vanish, the previous theorem can be further simplified.

Theorem J. *In addition to the hypotheses of Theorem G (Theorem C), if there exists $z^* \in V$ such that $f_\ell(z^*) = 0$, $f_\ell(z) \neq 0$ for every $z \in \overline{V}\backslash\{z^*\}$, and there exists a family*

$\{W_\mu : \mu \in (0, \mu_0]\}$ *of open subsets of* V *containing* z^* *such that* $\operatorname{diam} W_\mu \to 0$ *as* $\mu \to 0$ *on each of which* ***H*** *holds, then* $\varphi(\cdot, \varepsilon) \to z^*$*, uniformly, as* $\varepsilon \to 0$.

Proof. We have that $\mathcal{F}_\ell(z, \varepsilon) = \varepsilon^\ell f_\ell(z)$. Hence, by Theorem I the result follows immediately. □

Now using Theorem J we can finish the proof of the first order case.

Corollary 5.0.1. *Consider the non-autonomous* $T-$*periodic Carathéodory differential equation*

$$x' = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon).$$

In addition to the hypotheses of Theorem D, if there exists $z^* \in V$ *such that* $f_1(z^*) = 0$, $f_1(z) \neq 0$ *for every* $z \in \overline{V} \backslash \{z^*\}$*, then* $\varphi(\cdot, \varepsilon) \to z^*$*, uniformly, as* $\varepsilon \to 0$.

Proof. The results follows immediately from Theorem J with $k = \ell = 1$. □

5.2 Further comments

We note that the theorems present in this section provide a correction to the convergence part of [27, Theorem C]. Indeed, in that paper, for the convergence part of the result, the authors only require that condition **H** hold on V and not on a family of open subsets of V containing z^*. The watchful reader will notice, however, that the proof makes a tacit use of this stronger condition where the family consists of the open balls around z^* contained in V. Moreover, the example present in that paper is still valid since condition **H** is shown to hold on every subinterval $(1 - \alpha, 1 + \alpha)$, for $\alpha \in (0, 1)$.

6 Applications

In this chapter we apply the results obtained thus far in order to show that they really extend the range of application of the averaging theory. We stress the verification of condition **H** in both the continuous and Carathéodory case.

6.1 Continuous Non-Lipschitz perturbation of a harmonic oscillator

Consider the continuous higher order non-Lipschitz perturbation of a harmonic oscillator

$$\ddot{x} = -x + \varepsilon\left(x^2 + \dot{x}^2\right) + \varepsilon^k\,\dot{x}\sqrt[3]{x^2 + \dot{x}^2 - 1} + \varepsilon^{k+1}E(x, \dot{x}, \varepsilon), \tag{6.1}$$

where k is a positive integer and E is a continuous function on $\mathbb{R}^3$. One can readily see that the right-hand side of this differential equation is not Lipschitz in any neighbourhood of $\mathbb{S}^1 = \{(x, \dot{x}) \in \mathbb{R}^2 : x^2 + \dot{x}^2 = 1\}$ due to the presence of the term $\sqrt[3]{x^2 + \dot{x}^2 - 1}$.

The goal of this section is to prove the following theorem

Theorem 6.1. *For any positive integer k and $|\varepsilon| \neq 0$ sufficiently small, the differential equation (6.1) admits a periodic solution $x(t;\varepsilon)$ satisfying $(x(t;\varepsilon), \dot{x}(t;\varepsilon)) \to \mathbb{S}^1$ uniformly as $\varepsilon \to 0$.*

The first step is to put equation (6.1) in the standard form as in equation (3.1). For that, we start by writing equation (6.1) as a first order differential equation by introducing a new variable y

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x + \varepsilon\left(x^2 + y^2\right) + \varepsilon^k y\sqrt[3]{x^2 + y^2 - 1} + \varepsilon^{k+1}E(x, y, \varepsilon). \end{cases}$$

Applying a polar change of variables

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta), \quad r > 0, \quad \theta \in [0, 2\pi],$$

we obtain

$$\begin{cases} \dot{r} &= \varepsilon\sin\theta\left(r^2 + \varepsilon^{k-1}\sqrt[3]{r^2 - 1} + \varepsilon^k E(r\cos\theta, r\sin\theta, \varepsilon)\right), \\ \dot{\theta} &= -1 + \dfrac{\varepsilon\cos\theta\left(r^2 + \varepsilon^{k-1}\sqrt[3]{r^2 - 1} + \varepsilon^k E(r\cos\theta, r\sin\theta, \varepsilon)\right)}{r}. \end{cases}$$

Observe that for ε sufficiently small, $\dot{\theta} \neq 0$. Therefore, we can take θ as the new independent variable and

$$\begin{aligned} r' = \frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\dot{r}}{\dot{\theta}} \\ &= \frac{\varepsilon r\sin\theta\left(r^2 + \varepsilon^{k-1}\sqrt[3]{r^2 - 1} + \varepsilon^k E(r\cos\theta, r\sin\theta, \varepsilon)\right)}{-r + \varepsilon\cos\theta\left(r^2 + \varepsilon^{k-1}\sqrt[3]{r^2 - 1} + \varepsilon^k E(r\cos\theta, r\sin\theta, \varepsilon)\right)}, \end{aligned}$$

which expresses the differential equation for r in the form

$$r' = \varepsilon F(\theta, r, \varepsilon), \tag{6.2}$$

where

$$F(\theta, r, \varepsilon) = \frac{r^3 \sin\theta + \varepsilon^{k-1} r \sin\theta \sqrt[3]{r^2-1} + \varepsilon^k r \sin\theta E(r\cos\theta, r\sin\theta, \varepsilon)}{-r + \varepsilon r^2 \cos\theta + \varepsilon^k \cos\theta \sqrt[3]{r^2-1} + \varepsilon^{k+1} \cos\theta E(r\cos\theta, r\sin\theta, \varepsilon)},$$

which is $2\pi-$periodic in θ. Calculating the Taylor series expansion of order $k-1$ of F about $\varepsilon = 0$, we obtain

$$\begin{aligned} F(\theta, r, \varepsilon) &= \varepsilon F_1(\theta, r) + \cdots + \varepsilon^k F_k(\theta, r) + \mathcal{O}(\varepsilon^{k+1}) \\ &= -\sum_{i=1}^{k-1} \varepsilon^{i-1} r^{i+1} \cos^{i-1}\theta \sin\theta \\ &\qquad -\varepsilon^{k-1} r \left(\sqrt[3]{r^2-1} \sin\theta + r^k \cos^{k-1}\theta\right) \sin\theta + \mathcal{O}(\varepsilon^k), \end{aligned} \tag{6.3}$$

with

$$\begin{aligned} F_i(\theta, r) &= -r^{i+1} \cos^{i-1}\theta \sin\theta, \qquad i \in \{1, \ldots, k-1\}, \\ F_k(\theta, r) &= -r \left(\sqrt[3]{r^2-1} \sin\theta + r^k \cos^{k-1}\theta\right) \sin\theta. \end{aligned}$$

Notice that for $\varepsilon \neq 0, F$ is not Lipschitz in any neighbourhood of $r = 1$, so that no known version of the averaging methodology can be applied in this case to detect any periodic solution around $r = 1$ since a first order analysis in this case is not possible and the higher order results available require at least Lipschitz continuity of the right-hand side of the equation.

Let $V_\alpha = (1-\alpha, 1+\alpha)$ for some $0 < \alpha < 1$. The averaged functions $f_i : \overline{V_\alpha} \to \mathbb{R}$ are then given by

$$\begin{aligned} f_i(r) &= \frac{1}{2\pi} \int_0^{2\pi} -r^{i+1} \cos^{i-1}\theta \sin\theta \mathrm{d}\theta = 0, \qquad i \in \{1, 2, \ldots, k-1\}, \\ f_k(r) &= \frac{1}{2\pi} \int_0^{2\pi} -r \left(\sqrt[3]{r^2-1} \sin\theta + r^k \cos^{k-1}\theta\right) \sin\theta \mathrm{d}\theta = -\frac{r\sqrt[3]{r^2-1}}{2}. \end{aligned}$$

From the above expressions it is clear that f_k has a unique zero $r^* = 1$ in $\overline{V}_\alpha$. In addition, its Brouwer degree was calculated in Example 2.4, thus $d_B(f_k, V_\alpha, 0) = -1 \neq 0$. Since $f_1 = 0$, a first order analysis is not applicable, therefore, in order to apply any of the theorems proved in Chapter 3 and Chapter 5, we have to show that condition **H** holds on V_α. As it turns out, we are actually able to prove that **H** holds on V_α for every $\alpha \in (0, 1)$.

Proposition 6.1. *For any positive integer k, condition **H** holds on V_α for every $\alpha \in (0, 1)$.*

Proof. In what follows we shall assume that $\varepsilon > 0$. The result for $\varepsilon < 0$ can be obtained analogously just by considering $-F(\theta, r, \varepsilon)$. As discussed in Section 3.4, the negation of condition **H** provides numerical convergent sequences $(\varepsilon_m)_{m\in\mathbb{N}} \subset (0, \varepsilon_0]$ and $(\lambda_m)_{m\in\mathbb{N}} \subset (0, 1)$, such that $\varepsilon_m \to 0$ as $m \to \infty$, and a sequence $(r_m)_{m\in\mathbb{N}}$ of 2π-periodic solutions of

$$r' = \varepsilon_m \lambda_m F(\theta, r, \varepsilon_m), \quad r \in \overline{V}, \tag{6.4}$$

for which there exists $\theta_m \in [0, 2\pi]$ such that $r_m(\theta_m) \in \partial V$ for each $m \in \mathbb{N}$ uniformly converging to a constant function $r_0 \in \partial V = \{1 \pm \alpha\}$ and, in particular,

$$\int_0^{2\pi} F(\theta, r_m(\theta), \varepsilon_m) \mathrm{d}\theta = 0, \qquad m \in \mathbb{N}. \tag{6.5}$$

For each $i, j, m \in \mathbb{N}$, define

$$G_m^{i,j} = \int_0^{2\pi} r_m^i(\theta) \cos^j(\theta) \sin(\theta) \mathrm{d}\theta. \tag{6.6}$$

Henceforth, despite $(\varepsilon_m)_{m\in\mathbb{N}}$ being a *numerical* sequence, we borrow the Landau's symbol notation $h_m = \mathcal{O}(\varepsilon_m^p)$, for some $p \in \mathbb{N}$, to mean that there exists a positive constant C such that $|h_m| \leqslant C|\varepsilon_m^p|$, for m sufficiently large. Thus, by Equation 6.5 and equation (6.3), we have

$$\begin{aligned}
0 &= \int_0^{2\pi} F(\theta, r_m(\theta), \varepsilon_m) \mathrm{d}\theta \\
&= \int_0^{2\pi} \Bigg(- \sum_{i=1}^{k-1} \varepsilon_m^{i-1} r_m^{i+1}(\theta) \cos^{i-1}\theta \sin\theta \\
&\qquad - \varepsilon_m^{k-1} r_m(\theta) \left(\sqrt[3]{r_m^2(\theta) - 1} \sin\theta + r_m^k(\theta) \cos^{k-1}\theta \right) \sin\theta \Bigg) \mathrm{d}\theta + \mathcal{O}(\varepsilon_m^k) \\
&= - \sum_{i=1}^{k-1} \varepsilon_m^{i-1} \int_0^{2\pi} r_m^{i+1}(\theta) \cos^{i-1}\theta \sin\theta \mathrm{d}\theta \\
&\qquad - \varepsilon_m^{k-1} \int_0^{2\pi} r_m(\theta) \left(\sqrt[3]{r_m^2(\theta) - 1} \sin\theta + r_m^k(\theta) \cos^{k-1}\theta \right) \sin\theta \mathrm{d}\theta + \mathcal{O}(\varepsilon_m^k) \\
&= - \sum_{i=1}^{k-1} \varepsilon_m^{i-1} G_m^{i+1,i-1} - \varepsilon_m^{k-1} \left(\int_0^{2\pi} r_m(\theta) \sqrt[3]{r_m^2(\theta) - 1} \sin^2\theta \mathrm{d}\theta + G_m^{k+1,k-1} \right) + \mathcal{O}(\varepsilon_m^k),
\end{aligned}$$

which, upon rearrangement, implies

$$\int_0^{2\pi} r_m(\theta) \sqrt[3]{r_m(\theta)^2 - 1} \sin^2\theta \mathrm{d}\theta = \frac{1}{\varepsilon_m^{k-1}} \sum_{i=1}^{k} \varepsilon_m^{i-1} G_m^{i+1,i-1} + \mathcal{O}(\varepsilon_m). \tag{6.7}$$

Our aim here is to provide some estimate on left-hand side of the above equality and study its behaviour as $m \to \infty$. To achieve that, we apply integration by parts on $G_m^{i,j}$ for arbitrary $i, j, m \in \mathbb{N}$.

$$\begin{aligned}
G_m^{i,j} &= \int_0^{2\pi} r_m^i(\theta) \cos^j\theta \sin\theta \mathrm{d}\theta \\
&= \left(- \frac{r_m^i(\theta) \cos^{j+1}\theta}{j+1} \Bigg|_0^{2\pi} \right) + \frac{i}{j+1} \int_0^{2\pi} r_m(\theta)^{i-1} \cos^{j+1}\theta r_m'(\theta) \mathrm{d}\theta \\
&= \frac{i}{j+1} \int_0^{2\pi} r_m(\theta)^{i-1} \cos^{j+1}\theta r_m'(\theta) \mathrm{d}\theta,
\end{aligned}$$

where the last equality follows from periodicity of r_m. Since $r_m(\theta)$ satisfies equation (6.4), we conclude that

$$\begin{aligned} G_m^{i,j} &= -\frac{i}{j+1}\sum_{l=1}^{k-1}\lambda_m\varepsilon_m^l\int_0^{2\pi} r_m(\theta)^{j+l}\cos^{j+l}(\theta)\sin\theta\ \mathrm{d}\theta + \mathcal{O}(\varepsilon_m^k) \\ &= -\frac{i}{j+1}\sum_{l=1}^{k-1}\lambda_m\varepsilon_m^l G_m^{i+l,j+l} + \mathcal{O}(\varepsilon_m^k) \\ &= \mathcal{O}(\varepsilon_m). \end{aligned}$$

Therefore, it follows that $G_m^{i+l,j+l} = \mathcal{O}(\varepsilon_m)$. Plugging this into the above expression, we get that $G_m^{i,j} = \mathcal{O}(\varepsilon_m^2)$. Applying this procedure recursively, we conclude that $G_m^{i,j} = \mathcal{O}(\varepsilon_m^k)$. Thus, from (6.7), we get that

$$\int_0^{2\pi} r_m(\theta)\sqrt[3]{r_m(\theta)^2-1}\sin^2\theta\mathrm{d}\theta = \mathcal{O}(\varepsilon_m).$$

Since $r_m \to r_0 \in \{1 \pm \alpha\}$ uniformly, we compute the limit of the integral above as

$$\begin{aligned} 0 &= \lim_{m\to\infty}\int_0^{2\pi} r_m(\theta)\sqrt[3]{r_m(\theta)^2-1}\sin^2\theta\mathrm{d}\theta \\ &= \int_0^{2\pi}\lim_{m\to\infty} r_m(\theta)\sqrt[3]{r_m(\theta)^2-1}\sin^2\theta\mathrm{d}\theta \\ &= \int_0^{2\pi} r_0\sqrt[3]{r_0^2-1}\sin^2\theta\mathrm{d}\theta, \end{aligned}$$

which is an absurd, because

$$\int_0^{2\pi} r_0\sqrt[3]{r_0^2-1}\sin^2\theta\mathrm{d}\theta = \pi r_0\sqrt[3]{r_0^2-1} \neq 0,$$

for $r_0 \neq 1$. Thus, we obtain that condition **H** holds on V. □

Finally, Theorem 6.1 can be proved.

Proof of Theorem 6.1. Since $f_1 = \cdots = f_{k-1} = 0, f_k \neq 0, f_k(r) \neq 0$, for $r \in \partial V$ and $d_B(f_k, V_\alpha, 0) \neq 0$, by Theorem C, with $\ell = k$, it follows that there exists $\varepsilon_{V_\alpha} > 0$ such that for every $\varepsilon \in (0, \varepsilon_{V_\alpha}]$ there exists a $2\pi-$periodic solution $\varphi(\theta,\varepsilon)$ of equation (6.1) satisfying $\varphi(\theta,\varepsilon) \in \overline{V}_\alpha$ for every $\theta \in [0, 2\pi]$ and, since condition **H** holds on V_α for every $\alpha \in (0,1)$, then **H** holds on every open subset of $(0,2)$ containing $r^* = 1$. Thus, by Theorem J, it follows that $\varphi(\theta,\varepsilon) \to r^* = 1$ uniformly as $\varepsilon \to 0$. □

In order to illustrate the existence of the periodic solution ensured by Theorem 6.1 for some values of k and ε, we ran some numerical simulations. Using the software application MATHEMATICA® we show the displacement function obtained for some of these simulations, which has its zero corresponding to a periodic solution.

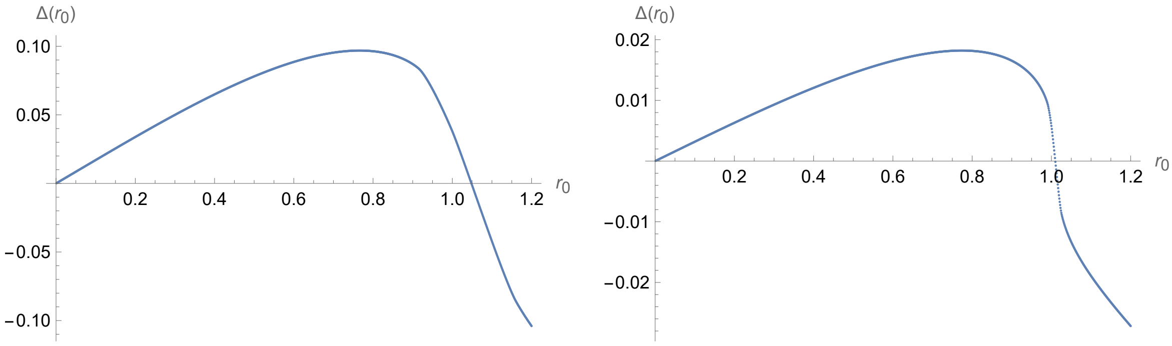

Figure 5 – Displacement function of differential equation (6.2) assuming $k = 1$ and $E = 0$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

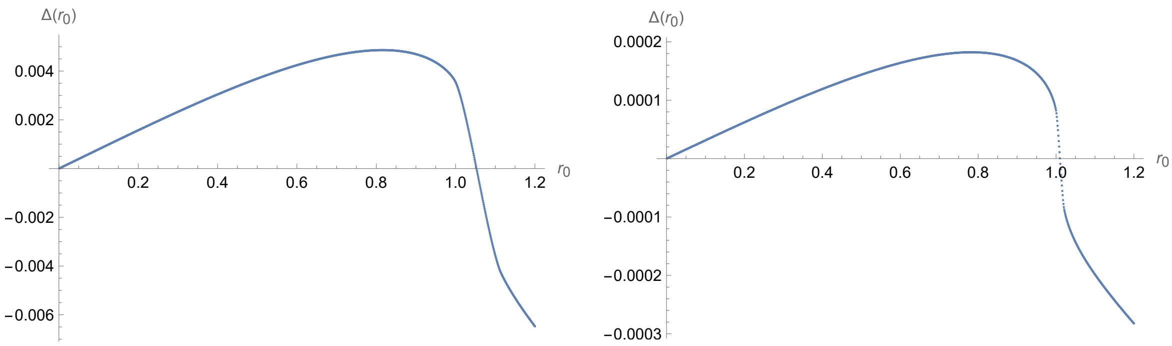

Figure 6 – Displacement function of differential equation (6.2) assuming $k = 2$ and $E = 0$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

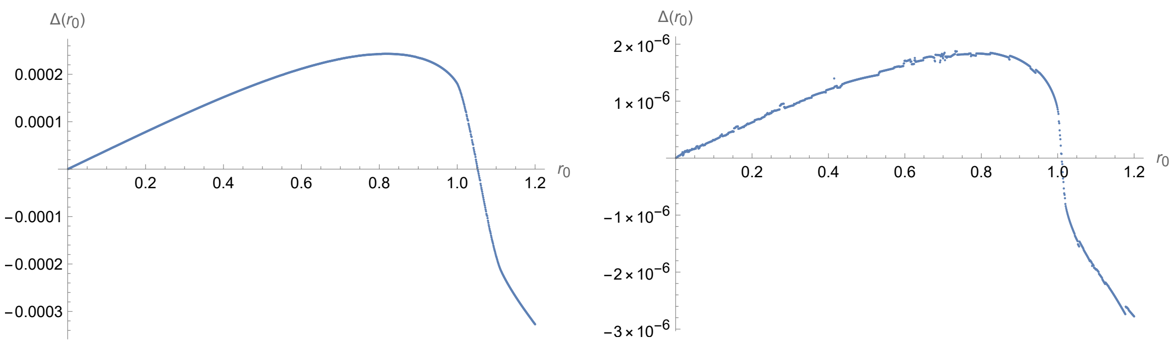

Figure 7 – Displacement function of differential equation (6.2) assuming $k = 3$ and $E = 0$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

6.2 Discontinuous perturbations of a harmonic oscillator

6.2.1 First example

Consider the following modification of equation (6.1)

$$\ddot{x} = -x + \varepsilon\left(x^2 + \dot{x}^2\right) + \varepsilon^k \operatorname{sgn}(\dot{x})\sqrt[3]{x^2 + \dot{x}^2 - 1} + \varepsilon^{k+1} E(x, \dot{x}, \varepsilon), \tag{6.8}$$

where $k > 1$ is a positive integer and E is a continuous function on $\mathbb{R}^3$. Notice the presence of the sign function, $\text{sgn}(\dot{x})$, multiplying the cube root term.

In this section, we will prove the following result.

Theorem 6.2. *For any positive integer k and $|\varepsilon| \neq 0$ sufficiently small, the differential equation (6.8) admits a $2\pi-$periodic solution $x(t, \varepsilon)$ satisfying $(x(t, \varepsilon), \dot{x}(t, \varepsilon)) \to \mathbb{S}^1$ uniformly as $\varepsilon \to 0$.*

We start by noting that the above differential equation possesses a discontinuity introduced by the term $\text{sgn}(\dot{x})$. Moreover, it is also important to notice, just as in the previous section that this right-hand side is not a Lipschitz function in any neighbourhood of the unit circle $\mathbb{S}^1$ centred at the origin of the phase space $(x, \dot{x})$. Therefore, none of the Melnikov-type or Averaging-type theorems known so far could be applied to study this system.

Observe that equation (6.8) is not in the form of equation (1.3). A procedure similar to that employed in the previous section can be applied to transform equation (6.8) into an equation in the standard form, such as equation (1.3). Thus after performing the same steps as in the previous section, the above system reads

$$r' = \frac{\varepsilon r \sin\theta \left(r^2 + \varepsilon^{k-1}\,\text{sgn}(\sin\theta)\sqrt[3]{r^2-1} + \varepsilon^k E(r\cos\theta, r\sin\theta, \varepsilon)\right)}{-r + \varepsilon r^2\cos\theta + \varepsilon^k \cos\theta\,\text{sgn}(\sin\theta)\sqrt[3]{r^2-1} + \varepsilon^{k+1}\cos\theta E(r\cos\theta, r\sin\theta, \varepsilon)}.$$

We point out that since $r > 0$ we omitted it from the argument of the sgn function and wrote just $\text{sgn}(\sin\theta)$ instead of $\text{sgn}(r\sin\theta)$. We shall follow with this notation henceforth. Again, we have expressed equation (6.8) as

$$r' = \varepsilon F(\theta, r, \varepsilon), \tag{6.9}$$

with F a $2\pi-$periodic function in θ given by

$$F(\theta, r, \varepsilon) = \frac{r \sin\theta \left(r^2 + \varepsilon^{k-1}\,\text{sgn}(\sin\theta)\sqrt[3]{r^2-1} + \varepsilon^k E(r\cos\theta, r\sin\theta, \varepsilon)\right)}{-r + \varepsilon r^2\cos\theta + \varepsilon^k \cos\theta\,\text{sgn}(\sin\theta)\sqrt[3]{r^2-1} + \varepsilon^{k+1}\cos\theta E(r\cos\theta, r\sin\theta, \varepsilon)}.$$

Now expanding F in Taylor series up to order $k-1$ about $\varepsilon = 0$ we get

$$r' = -\sum_{j=1}^{k-1} \varepsilon^j r^{j+1}\cos^{j-1}\theta\sin\theta - \varepsilon^k\Big(r^{k+1}\cos^{k-1}\theta\sin\theta + \sin\theta\,\text{sgn}(\sin\theta)\sqrt[3]{r^2-1}\Big) + \mathcal{O}(\varepsilon^{k+1}) \tag{6.10}$$

Observe that this equation is in the standard form we have been using all along

$$r' = \sum_{j=1}^{k} \varepsilon^j F_j(\theta, r) + \mathcal{O}(\varepsilon^{k+1}),$$

where

$$F_j(\theta, r) = -r^{j+1} \cos^{j-1}\theta \sin\theta, \qquad \text{for each} \qquad j \in \{1, \ldots, k-1\}$$

and

$$F_k(\theta, r) = -r^{k+1} \cos^{k-1}\theta \sin\theta - \sin\theta \,\mathrm{sgn}(\sin\theta) \sqrt[3]{r^2 - 1}.$$

Notice that F_i is continuous for every $i \in \{1, \ldots, k-1\}$ and F_k is a sum of a continuous function with a Carathéodory function, as shown in Example 2.1. We can now apply the methodology described in this work. Let $V_\alpha = (1-\alpha, 1+\alpha)$ for some $0 < \alpha < 1$. Integrating each F_j over $[0, 2\pi]$, we obtain the averaged functions $f_i : \overline{V}_\alpha \to \mathbb{R}$ for each $j \in \{1, \ldots, k-1\}$

$$f_j(r) = \frac{1}{2\pi} \int_0^{2\pi} F_j(\theta, r) \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} -r^{j+1} \cos^{j-1}\theta \sin\theta \mathrm{d}\theta = 0$$

and

$$\begin{aligned} f_k(r) &= \frac{1}{2\pi} \int_0^{2\pi} F_k(\theta, r) \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} -r^{k+1} \cos^{k-1}\theta \sin\theta - \sin\theta \,\mathrm{sgn}(\sin\theta) \sqrt[3]{r^2 - 1} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} -\sin\theta \,\mathrm{sgn}(\sin\theta) \sqrt[3]{r^2 - 1} \mathrm{d}\theta \\ &= \frac{2\sqrt[3]{1 - r^2}}{\pi}. \end{aligned}$$

Notice that f_k has an isolated zero at $r^* = 1$ and that its Brouwer degree is the same as the one calculated in Example 2.4, so $d_B(f_k, V_\alpha, 0) = -1 \neq 0$ for every $\alpha \in (0, 1)$. Now in order to apply Theorem G and Theorem J we only have to show that condition **H** holds.

Proposition 6.2. *For any positive integer k, condition **H** holds on V_α for every $\alpha \in (0, 1)$.*

Proof. The proof that condition **H** holds goes along the same lines as that of Proposition 6.1. There are only a few changes that must be noted. Let $\alpha \in (0, 1)$ be fixed. By negating item **H** we get numerical convergent sequences $(\varepsilon_m)_{m \in \mathbb{N}} \subset (0, \varepsilon_0]$ and $(\lambda_m)_{m \in \mathbb{N}} \subset (0, 1)$, such that $\varepsilon_m \to 0$ as $m \to \infty$, and a sequence $(r_m)_{m \in \mathbb{N}}$ of 2π-periodic solutions of

$$r' = \varepsilon_m \lambda_m F(\theta, r, \varepsilon_m), \quad r \in \overline{V}_\alpha, \tag{6.11}$$

for which there exists $\theta_m \in [0, 2\pi]$ such that $r_m(\theta_m) \in \partial V_\alpha$ for each $m \in \mathbb{N}$ uniformly converging to a constant function $r_0 \in \partial V_\alpha = \{1 \pm \alpha\}$ and, in particular,

$$\int_0^{2\pi} F(\theta, r_m(\theta), \varepsilon_m) \mathrm{d}\theta = 0, \qquad m \in \mathbb{N}. \tag{6.12}$$

By the exact same arguments, we conclude that

$$\int_0^{2\pi} \sin\theta \operatorname{sgn}(\cos\theta + \sin\theta)\sqrt[3]{r_m(\theta)^2 - 1}\mathrm{d}\theta = \mathcal{O}(\varepsilon_m). \tag{6.13}$$

Thus, by taking the limit of the above equality as $m \to \infty$, we obtain

$$\begin{aligned} 0 &= \int_0^{2\pi} \sin\theta \operatorname{sgn}(\cos\theta + \sin\theta)\sqrt[3]{r_0^2 - 1}\mathrm{d}\theta \\ &= 2\sqrt{2}\sqrt[3]{r_0^2 - 1} \neq 0, \end{aligned}$$

which is an absurd. And since α was taken arbitrarily, it follows that condition **H** holds on V_α for every $\alpha \in (0, 1)$. □

Now Theorem 6.2 can be proved.

Proof of Theorem 6.2. Since $f_1 = \cdots = f_{k-1} = 0, f_k \neq 0, f_k(r) \neq 0$, for $r \in \partial V$ and $d_B(f_k, V_\alpha, 0) \neq 0$, by Theorem G, with $\ell = k$, it follows that there exists $\varepsilon_{V_\alpha} > 0$ such that for every $\varepsilon \in (0, \varepsilon_{V_\alpha}]$ there exists a $2\pi-$periodic solution $\varphi(\theta, \varepsilon)$ of equation (6.8) satisfying $\varphi(\theta, \varepsilon) \in \overline{V}_\alpha$ for every $\theta \in [0, 2\pi]$ and, since condition **H** holds on V_α for every $\alpha \in (0, 1)$, then **H** holds on every open subset of $(0, 2)$ containing $r^* = 1$. Thus, by Theorem J, it follows that $\varphi(\theta, \varepsilon) \to r^* = 1$ uniformly as $\varepsilon \to 0$. □

Similar numerical simulations were run for equation (6.9) and we can observe a curious behaviour. The reader will notice that in each figure there are two branches of points, one that is clearly a straight line and the other consisting of a few points near the x axis. These points near the x axis show that the displacement function reaches values very close to zero, which indicate the presence of $2\pi-$periodic solutions. The other branch appears due to the lack of uniqueness of solutions.

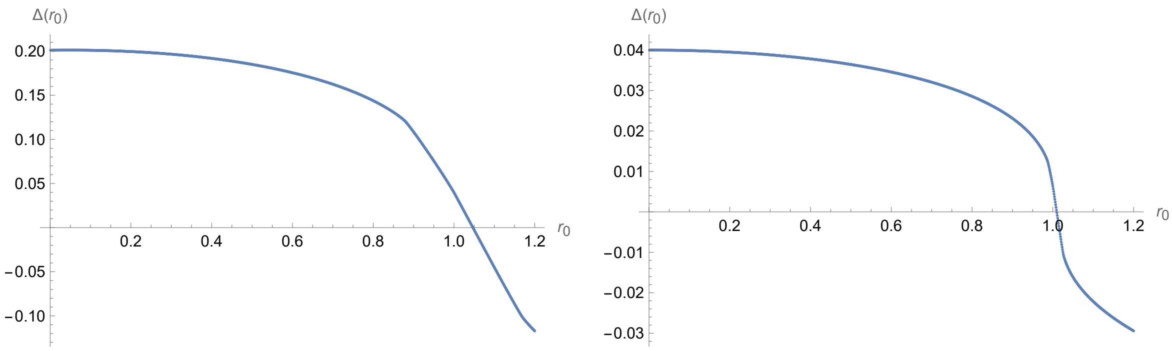

Figure 8 – Displacement function of differential equation (6.2) assuming $k = 1$ and $E = 0$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

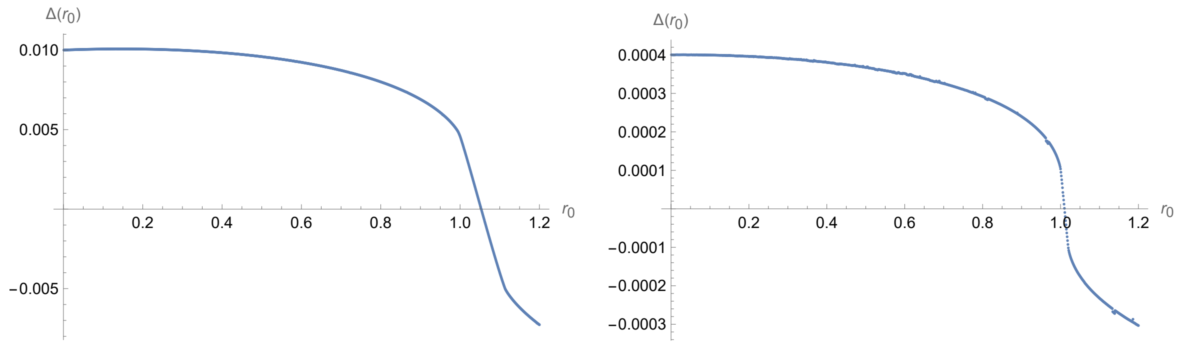

Figure 9 – Displacement function of differential equation (6.2) assuming $k = 2$ and $E = 0$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

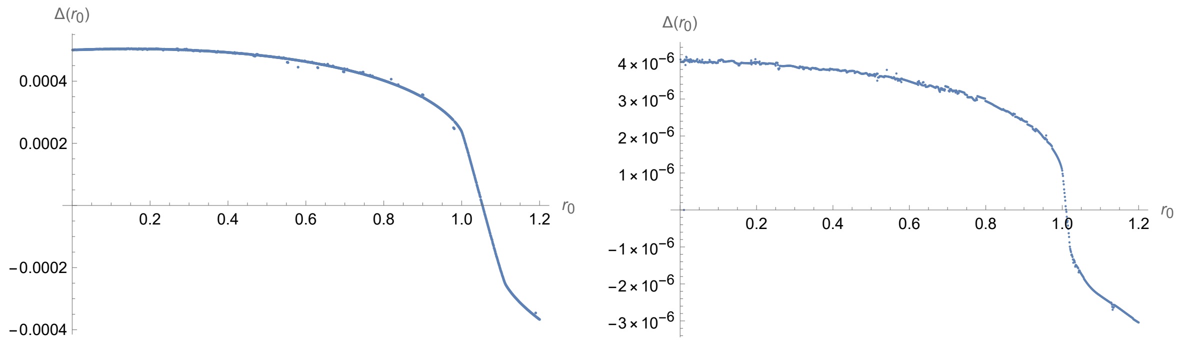

Figure 10 – Displacement function of differential equation (6.2) assuming $k = 3$ and $E = 0$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

6.2.2 Second example

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$g(x,y) = \operatorname{sgn}(y(x^2+y^2-1)) \max\left\{0, \left(x^2+y^2-\frac{1}{4}\right)\left(x^2+y^2-\frac{9}{4}\right)\right\}$$

and consider the following differential equation

$$\ddot{x} = -x + \varepsilon(x^2+\dot{x}^2) + \varepsilon^k g(x,\dot{x}) + \varepsilon^{k+1}E(x,\dot{x},\varepsilon), \tag{6.14}$$

where E is a continuous function on $\mathbb{R}^3$. The function $g(x,\dot{x})$ multiplying the ε^k term is clearly a discontinuous one.

We will prove the following result.

Theorem 6.3. *For $\alpha \in (1/2, 1)$, let $\mathcal{A}_\alpha = \{(x,y) \in \mathbb{R}^2 : 1-\alpha < x^2+y^2 < 1+\alpha\}$. For any positive integer k and $|\varepsilon| \neq 0$ sufficiently small, the differential equation (6.14) admits a $2\pi-$periodic solution $x(t,\varepsilon)$ satisfying $(x(t,\varepsilon), \dot{x}(t,\varepsilon)) \in \mathcal{A}_\alpha$ for every $t \in [0, 2\pi]$.*

Applying once more the same procedure employed in the previous sections of this chapter, namely perform a polar change to variables (r,t) and taking θ as the new time variable we obtain the system

$$r' = \frac{r\sin\theta(\varepsilon r^2 + \varepsilon^k g(r\cos\theta, r\sin\theta) + \varepsilon^{k+1}E(r\cos\theta, r\sin\theta, \varepsilon))}{-r + \varepsilon r^2\cos\theta + \varepsilon^k\cos\theta g(r\cos\theta, r\sin\theta) + \varepsilon^{k+1}\cos\theta E(r\cos\theta, r\sin\theta, \varepsilon)}, \tag{6.15}$$

which is a $2\pi-$periodic differential equation. By expanding the tight-hand side of the above equation in Taylor series up to order k about $\varepsilon = 0$, we get

$$\begin{aligned} r' &= \varepsilon F_1(\theta, r) + \cdots + \varepsilon^k F_k(\theta, r) + \mathcal{O}(\varepsilon^{k+1}) \\ &= -\sum_{i=1}^{k-1} \varepsilon^i r^{i+1}\cos^{i-1}\theta\sin\theta \\ &\qquad - \varepsilon^k\left(r^{k+1}\cos^{k-1}\theta\sin\theta + \sin\theta g(r\cos\theta, r\sin\theta)\right) + \mathcal{O}(\varepsilon^{k+1}). \end{aligned}$$

Notice that the right-hand side of this equation is a Carathéodory function, since it is a sum of continuous functions with the term $\sin\theta g(r\cos\theta, r\sin\theta)$ which is Carathéodory, as shown in Example 2.2. Let $1/2 < \alpha < 1$ and define $V_\alpha = (1-\alpha, 1+\alpha)$. The averaged functions $f_i : \overline{V} \to \mathbb{R}$ are given by

$$\begin{aligned} f_i(r) &= \frac{1}{2\pi}\int_0^{2\pi} -r^{i+1}\cos^{i-1}\theta\sin\theta d\theta = 0, \quad i \in \{1, \ldots, k-1\} \\ f_k(r) &= \frac{1}{2\pi}\int_0^{2\pi} -r^{i+1}\cos^{i-1}\theta\sin\theta - g(r\cos\theta, r\sin\theta)\sin\theta d\theta \\ &= -\frac{2}{\pi}\operatorname{sgn}\left(r^2-1\right)\max\left\{0, \left(r^2-\frac{1}{4}\right)\left(r^2-\frac{9}{4}\right)\right\}. \end{aligned}$$

Hence, we have $f_1 = \cdots = f_{k-1} = 0$ and $f_k \neq 0$ in $\overline{V}$. In addition, the degree of f_k was calculated in Example 2.5 so that $d_B(f_k, V_\alpha, 0) = -1 \neq 0$.

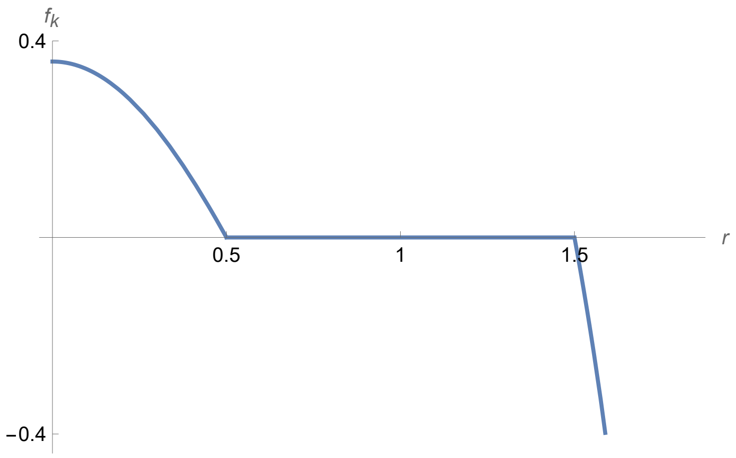

Figure 11 – Graph of the $k-$th averaged function.

In order to verify condition **H**, we follow the same procedure employed in the last sections.

Proposition 6.3. *For any positive integer k, and any $\alpha \in (1/2, 1)$, condition **H** holds on V_α.*

Proof. Negating condition **H**, we obtain numerical sequences $\{\varepsilon_m\}_{m\in\mathbb{N}}, \{\lambda_m\}_{m\in\mathbb{N}}$ such that $\varepsilon_m \to 0$ as $m \to \infty$, and a sequence $(r_m)_{m\in\mathbb{N}}$ of 2π-periodic solutions of

$$r' = \varepsilon_m \lambda_m F(\theta, r, \varepsilon_m), \quad r \in \overline{V}_\alpha, \tag{6.16}$$

for which there exists $\theta_m \in [0, 2\pi]$ such that $r_m(\theta_m) \in \partial V_\alpha$ for each $m \in \mathbb{N}$ uniformly converging to a constant function $r_0 \in \partial V_\alpha = \{1 \pm \alpha\}$ and, in particular,

$$\int_0^{2\pi} F(\theta, r_m(\theta), \varepsilon_m) \mathrm{d}\theta = 0, \qquad m \in \mathbb{N}. \tag{6.17}$$

By the exact same arguments used in the previous sections, we conclude that

$$\int_0^{2\pi} \sin\theta \,\mathrm{sgn}(r(r^2-1)\sin\theta) \max\{0, (r^2 - 1/4)(r^2 - 9/4)\} \mathrm{d}\theta = \mathcal{O}(\varepsilon_m). \tag{6.18}$$

Thus, by taking the limit of the above equality as $m \to \infty$, we obtain

$$\begin{aligned} 0 &= \int_0^{2\pi} \sin\theta \,\mathrm{sgn}(r(r^2-1)\sin\theta) \max\{0, (r^2 - 1/4)(r^2 - 9/4)\} \mathrm{d}\theta \\ &= -4 \,\mathrm{sgn}\left(r_0^2 - 1\right) \max\left\{0, \left(r_0^2 - \frac{1}{4}\right)\left(r_0^2 - \frac{9}{4}\right)\right\} \neq 0, \end{aligned}$$

which is an absurd. And since α was taken arbitrarily, it follows that condition **H** holds on V_α for every $\alpha \in (0, 1)$. □

Thus, we can prove Theorem 6.3.

Proof of Theorem 6.3. We have $f_1 = \cdots = f_{k-1} = 0$ and $f_k \neq 0$. In addition, condition **H** holds on V_α and $d_B(f_k, V_\alpha, 0) \neq 0$ for every $\alpha \in (1/2, 1)$. Therefore, by Theorem G there exists $\varepsilon_{V_\alpha} \in (0, \varepsilon_0]$ such that for every $\varepsilon \in (0, \varepsilon_{V_\alpha})$ there exists a $2\pi-$periodic solution $\varphi(t, \varepsilon)$ of equation (6.14) satisfying $\varphi(t, \varepsilon) \in \overline{V}_\alpha$ for every $t \in [0, 2\pi]$. □

This is an example where we can guarantee the existence of periodic solutions, but not the convergence of these solutions as $\varepsilon \to 0$.

Below we show some numerical simulations carried out with $E(x, y, \varepsilon) = y - y^3$. We note, that for most cases these simulations clearly indicate the presence of 3 periodic solutions.

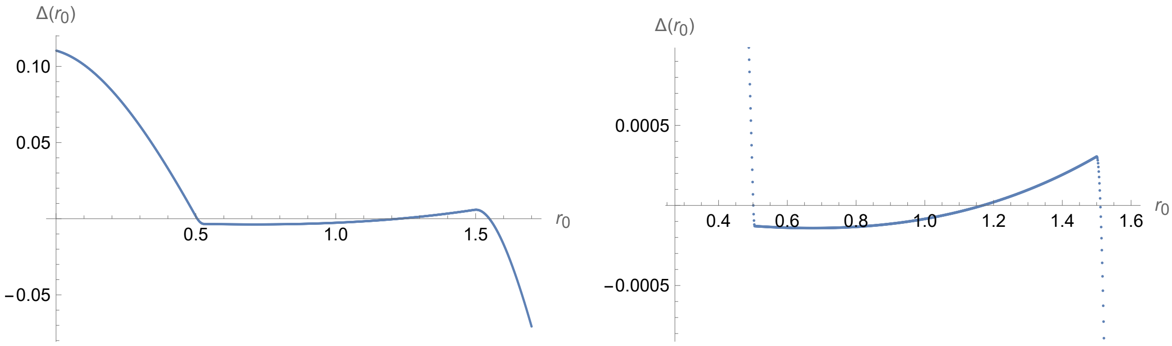

Figure 12 – Displacement function of differential equation (6.2) assuming $k = 1$ and $E(x, y, \varepsilon) = y - y^3$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

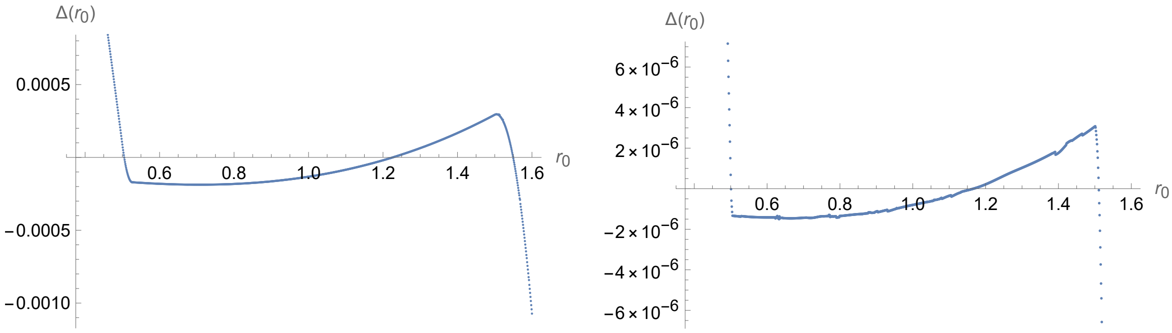

Figure 13 – Displacement function of differential equation (6.2) assuming $k = 2$ and $E(x, y, \varepsilon) = y - y^3$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

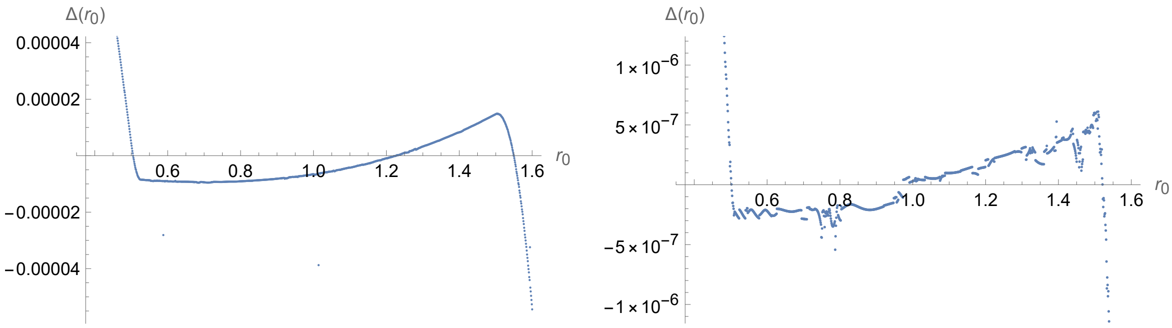

Figure 14 – Displacement function of differential equation (6.2) assuming $k = 3$ and $E(x, y, \varepsilon) = y - y^3$ for $\varepsilon = 1/20$ (left) and $\varepsilon = 1/100$ (right).

7 Conclusion and future works

In this work, we have studied the problem of detecting periodic solutions of ordinary differential equations with discontinuous right-hand sides. More precisely, we have considered differential equations whose right-hand sides are functions satisfying the Carathéodory conditions.

The major challenge in dealing with less regular differential equations is the lack of uniqueness of solutions. This property is very important in order to properly define a Poincaré map and study its fixed points. Since this is not possible in the context of this work, we had to look for a feasible approach.

The main technique used for proving existence results for non-smooth differential equations was topological degree theory for operator equations in infinite dimensional spaces. We have used the integral characterization of solutions of differential equations to define operators on suitable Banach spaces and studied the resulting equivalent operator equation. We are then able to use the average of the right-hand side as bifurcation functions.

Convergence of solutions was also studied. In this way, we see that the periodic solutions detected for $\varepsilon \neq 0$ converge to some solution of the unperturbed differential equation as $\varepsilon \to 0$.

The development of this work opened many possibilities for further research. We briefly discuss some of these topics in the sequel.

7.1 Averaging for Filippov systems

Aiming at considering as much cases as possible, it is natural to also consider the case of Filippov systems.

Consider the differential equation

$$x' = \varepsilon F(t,x,\varepsilon) = \varepsilon \sum_{j=1}^{m} \chi_{D_i}(x) F_j(t,x,\varepsilon), \quad (t,x,\varepsilon) \in [0,T] \times D \times (-\varepsilon_0, \varepsilon_0),$$

where χ_A is the characteristic function of the subset A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$, and $F_j : [0,T] \times D_j \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ for each $j \in \{1, \ldots, m\}$ is a Carathéodory function $T-$periodic in t and $D = \bigcup_{j=1}^{m} D_j$.

It is then interesting to investigate whether it is possible to apply the methodology used in this work to provide sufficient conditions for the existence of periodic solutions for the Filippov system above.

7.2 Bifurcation from a non-degenerate family of periodic solutions

In this work, the differential equation studied is reduced to $x' = 0$ by taking $\varepsilon = 0$. This unperturbed system has all of its solutions as constants, thus periodic.

An interesting problem is consider the differential equation

$$x' = F_0(t, x) + \varepsilon R(t, x, \varepsilon), \tag{7.1}$$

where $F_0 : [0, T] \times D \to \mathbb{R}^n$, and $R : [0, T] \times D \times [0, \varepsilon_f] \to \mathbb{R}^n$ are continuous function, $T-$periodic in t. Let us assume that F_0 is Lipschitz continuous. Consider the unperturbed problem

$$x' = F_0(t, x). \tag{7.2}$$

Let $k \leqslant n$, be a non-negative integer, $V \subset \mathbb{R}^k$ an open bounded subset, $\beta_0 : \overline{V} \to \mathbb{R}^{n-k}$ a C^1 function and consider the set

$$\mathcal{Z} = \{z_\alpha = (\alpha, \beta_0(\alpha)) : \alpha \in \overline{V}\}.$$

We pose the following condition

$\boldsymbol{H}_1$ for each $z_\alpha \in \mathcal{Z}$, the solution $\varphi_\alpha(t) = \varphi(t, z_\alpha)$ of (7.2) such that $\varphi_\alpha(0) = z_\alpha$ is $T-$periodic.

The goal here is to provide an averaging like result using topological degree theory to give conditions that guarantee that equation (7.1) has a $T-$periodic solution for $\varepsilon > 0$ sufficiently small.

7.3 Stability of periodic solutions

When studying the bifurcation of periodic solutions, a natural question arises, namely, whether we can get some information about the stability of these solutions.

The analysis of stability of periodic solutions of differential equations usually employ techniques such as te study of the Poincaré map. However, as we have seen, for equations with regularity as low as those considered in this work, we cannot properly define a Poincaré map. Thus it remains an open problem to provide sufficient conditions that guarantee the stability of the periodic solutions detected by the methods used in this work.

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