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Invariant tori, boundedness of solutions, and periodic orbits in a class of non-smooth oscillators

Toros invariantes, limitação de soluções e órbitas periódicas em uma classe de osciladores não suaves

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

Supervisor: Douglas Duarte Novaes

Este trabalho corresponde à versão final da Tese defendida pelo aluno Luan Vinicio de Mattos Ferreira Silva e orientada pelo Prof. Dr. Douglas Duarte Novaes.

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“Oh Olalla on the borderline
There is a world down on its knees
For better times
Oh Olalla don’t you fear the night
There’s only time left to believe
To believe”
(Blanco White - Olalla)

Abstract

Since the 1960s, the boundedness of solutions of the Duffing-type equations $\ddot{x} + g(x) = p(t)$ has been a significant research focus in dynamical systems. This is primarily attributed to mathematician John E. Littlewood, who proposed investigating conditions on the functions $g(x)$ and $p(t)$ to determine the boundedness of all solutions to the Duffing-type equation. In light of this, a crucial strategy is to establish the existence of invariant tori in the extended phase space of the differential equation, confining all the dynamics in the interiors of the regions delimited by them.

Assuming $p(t)$ to be a σ-periodic function, we aim to study the existence of invariant tori for the family of second-order discontinuous differential equations

$$(\mathcal{F}): \quad \ddot{x} + \mu \operatorname{sgn}(x) = \theta x + \varepsilon\, p(t),$$

where θ and ε are real parameters, $\operatorname{sgn}(x)$ represents the usual sign function, and $\mu \in \{-1, 1\}$ is a modal parameter. We focus on cases where the unperturbed equation ($\varepsilon = 0$) admits a ring of periodic orbits. More precisely, assuming $\theta \neq 0$, we employ KAM theory to investigate the existence of invariant tori for $(\mathcal{F})$. In this case, $p(t)$ is required to be sufficiently differentiable. For $\theta = 0$, considering $p(t)$ as a Lebesgue-integrable function with vanishing average, we establish the existence of invariant tori through a constructive and non-perturbative method. These results provide conditions for the boundedness of solutions that initiate either on these tori or in the interiors of the regions delimited by them, as well as conditions for the existence of periodic orbits.

Finally, for the sake of completeness, we perform a Melnikov analysis on a more general class of differential equations given by $\ddot{x} + \alpha \operatorname{sgn}(x) = \theta x + \varepsilon\, f(t, x, \dot{x})$, where $\alpha \neq 0$ and $f(t, x, \dot{x})$ is a function of class $\mathcal{C}^1$ and σ-periodic in t, aiming to detect bifurcating periodic orbits of the differential equation.

Keywords: Non-smooth differential equations, Carathéodory equations, Filippov systems, KAM Theory, Invariant tori, Boundedness of solutions, Melnikov method, Periodic solutions.

Resumo

Desde os anos 60, a limitação das soluções das equações de Duffing $\ddot{x} + g(x) = p(t)$ tem sido um importante objeto de pesquisa em sistemas dinâmicos. Isso se deve, principalmente, ao matemático John E. Littlewood, que propôs a investigação de condições sobre as funções $g(x)$ e $p(t)$ para determinar a limitação de todas as soluções da equação de Duffing. Nesse contexto, uma estratégia crucial é determinar a existência de toros invariantes no espaço de fase estendido da equação diferencial, confinando toda a dinâmica na região delimitada pelos mesmos.

Assumindo $p(t)$ como uma função σ-periódica, pretendemos estudar a existência de toros invariantes para a família de equações diferenciais descontínuas de segunda ordem

$$(\mathcal{F}): \quad \ddot{x} + \mu\,\mathrm{sgn}(x) = \theta x + \varepsilon\, p(t),$$

em que θ e ε são parâmetros reais, $\mathrm{sgn}(x)$ representa a função sinal usual e $\mu \in \{-1, 1\}$ é um parâmetro modal, nos casos em que a equação não perturbada ($\varepsilon = 0$) admite um anel de órbitas periódicas. Mais precisamente, assumindo $\theta \neq 0$, recorremos à Teoria KAM para investigar a existência de toros invariantes de $(\mathcal{F})$. Neste caso, é necessário que $p(t)$ seja suficientemente diferenciável. Para $\theta = 0$, tomando $p(t)$ como uma função Lebesgue integrável com média zero, constatamos a existência de toros invariantes por meio de um método construtivo e não perturbativo. Tais resultados fornecem condições para a limitação de todas as soluções que se iniciam em tais toros ou nas regiões delimitadas pelos mesmos, bem como condições para a existência de órbitas periódicas.

Por fim, para efeito de completude, desenvolvemos uma análise de Melnikov para a classe mais geral de equações diferenciais dadas por $\ddot{x} + \alpha\,\mathrm{sgn}(x) = \theta x + \varepsilon\, f(t, x, \dot{x})$, em que $\alpha \neq 0$ e $f(t, x, \dot{x})$ é uma função de classe $\mathcal{C}^1$ e σ-periódica em t, com o objetivo de detectar órbitas periódicas que bifurcam dos anéis periódicos da equação diferencial.

Palavras-chave: Equações diferenciais não-suaves, Equações de Catarathéodory, Sistemas de Filippov, Teoria KAM, Toros invariantes, Limitação de soluções, Método de Melnikov, Soluções periódicas.

Contents

1 Introduction

Since the 1960s, researchers have been investigating the boundedness of solutions of the Duffing-type equations

$$\ddot{x} + g(x) = p(t). \tag{1.1}$$

The interest in such investigation is mainly due to Littlewood, who, after demonstrating in [38, 39] the existence of unbounded solutions of (1.1) when $p(t)$ is assumed to be bounded and $g(x)$ (saturation function) satisfies specific asymptotic conditions, proposed in [40] to investigate conditions on $p(t)$ and $g(x)$ that would ensure the boundedness of all solutions of (1.1). Morris, in [42], was the first to provide an example to Littlewood's proposal, assuming $p(t)$ to be periodic and continuous, and $g(x) = 2x^3$. Subsequently, Morris had its result extended by Dieckerhoff and Zehnder in [18] to the class of equations

$$\ddot{x} + x^{2n+1} + \sum_{j=0}^{2n} x^j p_j(t) = 0, \quad n \geqslant 1,$$

under the assumption that p_j are periodic $\mathcal{C}^\infty$ functions. Also, by considering $g(x)$ to be piecewise smooth, boundedness of solutions has been analyzed in the non-smooth context in [20, 34, 53, 55] . It is noteworthy that in the literature, boundedness of all solutions is also referred to as Lagrange stability, as described in [67].

1.1 A class of non-smooth Duffing-type equations

In this work, considering sgn as the standard sign function defined as

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -1, & \text{if} \quad x < 0, \end{cases}$$

we aim to study qualitative aspects of the class of differential equations

$$\ddot{z} + \alpha \operatorname{sgn}(z) = \theta z + \tilde{\varepsilon} \tilde{f}(t, z, \dot{z}),$$

where $\alpha \neq 0$, θ and $\tilde{\varepsilon}$ are real parameters, and $\tilde{f}(t, z, \dot{z})$ is a function σ-periodic in the variable t. It is worthy mentioning that, by considering the change of variables $z = |\alpha| x$, the differential equation is reduced to

$$\ddot{x} + \mu \operatorname{sgn}(x) = \theta x + \varepsilon f(t, x, \dot{x}), \tag{1.2}$$

with $\varepsilon = \tilde{\varepsilon}/|\alpha|$, $f(t, x, \dot{x}) = \tilde{f}(t, |\alpha| x, |\alpha| \dot{x})$, and $\mu = \operatorname{sgn}(\alpha) \in \{-1, 1\}$. We notice that $f(t, x, \dot{x})$ and $\tilde{f}(t, z, \dot{z})$ have the same periodicity in t. For this reason, we reduce our analysis to the differential equation (1.2).

The most significant aspect of this work consists in determining the existence of invariant tori (see Definitions 1.1.1 and 1.1.2) and the boundedness of solutions for the differential equation (1.2) in the case $f(t,x,\dot{x}) = p(t)$, with $p(t)$ being a σ-periodic function. Thus, we consider the subclass of differential equations given by

$$\ddot{x} + \mu\,\mathrm{sgn}(x) = \theta x + \varepsilon\, p(t). \tag{1.3}$$

For the the sake of completeness, we perform a Melnikov analysis on the equation (1.2) in order to determine the existence of periodic orbits.

The importance in studying non-smooth differential equations primarily lies in understanding phenomena observed in both natural and engineering domains characterized by sudden changes. For instance, such study can be used to comprehend biological and climate models involving abrupt changes [7, 13, 35, 56], collisions in mechanical systems [10, 31, 32], electronic circuits in the presence of a relay [30], and automatic pilots for ships [3].

Important studies have been conducted on the family of differential equations (1.3). For example, in [34], Kunze et al. showed that, for $\theta = -1$ and $\mu = 1$, all the solutions of (1.3) are bounded provided that ε is sufficiently small. Furthermore, they established the existence of infinitely many periodic orbits. Regarding this matter, Silva et al. in [15] assumed $p(t)$ to be periodically continuous and provided conditions on μ and θ to ensure the existence of periodic solutions for (1.3) through direct computations. In terms of practical applications, equation (1.3) is very useful in describing, for instance, the states of a simple automatic pilot for ship [3] and a dry-friction oscillator [32] through the interaction laws

$$\ddot{x} + x = \mathrm{sgn}(x) \quad \text{and} \quad \ddot{x} + x = \sin(\omega t) - A_F \mathrm{sgn}(\dot{x}),$$

respectively, where ω is the frequency of the forcing term and A_F is the magnitude of the Coulomb friction force.

As previously mentioned, this work focuses on investigating the boundedness of solutions for the differential equation (1.3). To this end, the primary objective is to establish the existence of a collection of invariant tori for equation (1.3), particularly in the scenarios where the unperturbed equation ($\varepsilon = 0$) admits a period annulus. For a better understanding of the concept of solution and invariant torus of the differential equation (1.3), it is convenient to consider the change of variables $y = \dot{x}$, and, then, transform equation (1.3) into the differential system

$$X_{\theta,\mu}(t,x,y;\varepsilon) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu\,\mathrm{sgn}(x) + \varepsilon\, p(t). \end{cases} \tag{1.4}$$

The solutions of the differential system (1.4) will be considered according to the Filippov convention, which exists for every initial condition if $p(t)$ is assumed to be, at least, a Lebesgue-integrable function (see Section 2.1.1). Thus, we will refer to (1.4) as a perturbed Filippov system. Now, taking into account the function $\operatorname{sgn}(x)$, we notice that $\Sigma' = \{(t, x, y) \in \mathbb{R}^3 : x = 0\}$ corresponds to the switching plane of the Filippov system (1.4), which, in turn, can be decomposed into the differential systems

$$X^+_{\theta,\mu}(t, x, y; \varepsilon) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu + \varepsilon\, p(t), \end{cases} \quad \text{and} \quad X^-_{\theta,\mu}(t, x, y; \varepsilon) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x + \mu + \varepsilon\, p(t), \end{cases} \tag{1.5}$$

for $x \geqslant 0$ and $x \leqslant 0$, respectively. The regularities of $X^+_{\theta,\mu}$ and $X^-_{\theta,\mu}$ depend on the regularity of the function $p(t)$. We shall see that, depending on the parameters θ and μ, distinct regularities for $p(t)$ will be assumed in our study. In any case, local uniqueness of solutions is guaranteed if we assume $p(t)$ to be at least Lebesgue-integrable (see Example 2.1.12). We shall focus our attention on solutions of the differential systems in (1.5) that intersect the region of discontinuity Σ' transversely. Under these conditions, solutions of (1.4) are obtained by concatenating solutions of (1.5) along Σ', which establishes the global uniqueness property in such cases.

Another important aspect of the Filippov system (1.4) to be used throughout this work is its piecewise Hamiltonian structure associated with the function

$$H_{\theta,\mu}(x, y, t; \varepsilon) = \frac{y^2}{2} + G_{\theta,\mu}(x) - \varepsilon x p(t), \tag{1.6}$$

where $G_{\theta,\mu}(x) = -\theta \dfrac{x^2}{2} + \mu|x|$. We shall see that such characteristic plays a crucial role in constructing coordinate changes that transform the Hamiltonian (1.6) into a nearly integrable one, which, roughly speaking, consists in a perturbation of an integrable Hamiltonian.

Since $p(t)$ is assumed to be σ-periodic, the Filippov system (1.4) can be seen as the vector field

$$\mathcal{X}_{\theta,\mu}(\phi, x, y; \varepsilon) = \begin{cases} \phi' = 1, \\ \mathbf{x}' = X_{\theta,\mu}(\phi, \mathbf{x}; \varepsilon), \end{cases} \tag{1.7}$$

in the extended phase space $(\phi, \mathbf{x}) \in \mathbb{S}_\sigma \times \mathbb{R}^2$, where $\mathbb{S}_\sigma = \mathbb{R}/\sigma\mathbb{Z}$. Let us denote the flow associated with (1.7) by Φ^τ, with τ being the time.

Definition 1.1.1. *We say that $A \subset \mathbb{S}_\sigma \times \mathbb{R}^2$ is an invariant set of $\mathcal{X}_{\theta,\mu}$ if $\Phi^\tau(a) \in A$, for every $a \in A$ and $\tau \in \mathcal{I}_a$, where $\mathcal{I}_a$ denotes the maximal interval of definition of $\Phi^\tau(a)$.*

Definition 1.1.2. *A set $\mathcal{T} \subset \mathbb{S}_\sigma \times \mathbb{R}^2$ is said to be a torus of (1.7), if the intersection of $\mathcal{T}$ with $\{\phi\} \times \mathbb{R}^2$ is homeomorphically a circle, for every $\phi \in \mathbb{S}_\sigma$, and the intersections with $\{\phi\} \times \mathbb{R}^2$ and $\{\phi + \sigma\} \times \mathbb{R}^2$ coincide in the quotient space $\mathbb{S}_\sigma \times \mathbb{R}^2$, for every $\phi \in \mathbb{S}_\sigma$.*

Therefore, when we refer to a set as an invariant torus of either (1.3) or (1.4), we essentially regard it as an invariant torus of (1.7) (see Figure 1). Moreover, in cases where $p(t)$ is a σ-periodic function of class $\mathcal{C}^r$, we denote this property by $p \in \mathcal{C}^r(\mathbb{S}_\sigma)$.

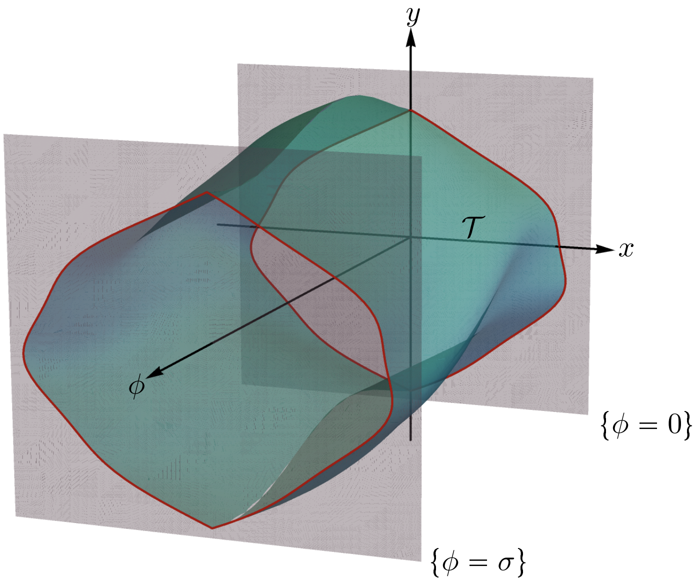


Figure 1 – $\mathcal{T}$ represents a torus of (1.7) in the quotient space $\mathbb{S}_\sigma \times \mathbb{R}^2$, meaning that $\mathcal{T} \cap (\{0\} \times \mathbb{R}^2) \equiv \mathcal{T} \cap (\{\sigma\} \times \mathbb{R}^2)$.

In order to discuss the main results of this thesis concerning existence of invariant tori, boundedness of solutions, and periodic orbits, we analyze the unperturbed Filippov system associated with (1.4).

1.1.1 Unperturbed Filippov system

For $\varepsilon = 0$, the Filippov system (1.4) writes as

$$\mathbf{X}_{\theta,\mu}(x,y) = X_{\theta,\mu}(t,x,y;0) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu \operatorname{sgn}(x), \end{cases}$$

which matches

$$\mathbf{X}^+_{\theta,\mu}(x,y) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu, \end{cases} \quad \text{and} \quad \mathbf{X}^-_{\theta,\mu}(x,y) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x + \mu, \end{cases}$$

when restricted to the semi-planes $x \geqslant 0$ and $x \leqslant 0$, respectively. The line $\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$ represents the set of discontinuity of $\mathbf{X}_{\theta,\mu}$. Moreover, except for $y = 0$, it corresponds to a crossing region of $\mathbf{X}_{\theta,\mu}$, which means that the solutions of $\mathbf{X}_{\theta,\mu}$ with initial conditions distinct from $(x,y) = (0,0)$ are given by the concatenation of the solutions of $\mathbf{X}^+_{\theta,\mu}$ and $\mathbf{X}^-_{\theta,\mu}$ along Σ. By varying the parameters μ and θ, we describe the phase portraits of $\mathbf{X}_{\theta,\mu}$, as follows:

(C1) $\theta > 0$ and $\mu = 1$: In this case, the points $p_0 = (0,0)$, $p^+ = (1/\theta, 0)$, and $p^- = (-1/\theta, 0)$ are the singularities of $\mathbf{X}_{(\mathrm{C1})}$, with p_0 being an invisible fold-fold point and also a center-type. The points p^+ and p^- are both admissible linear saddles (see Figure 2);

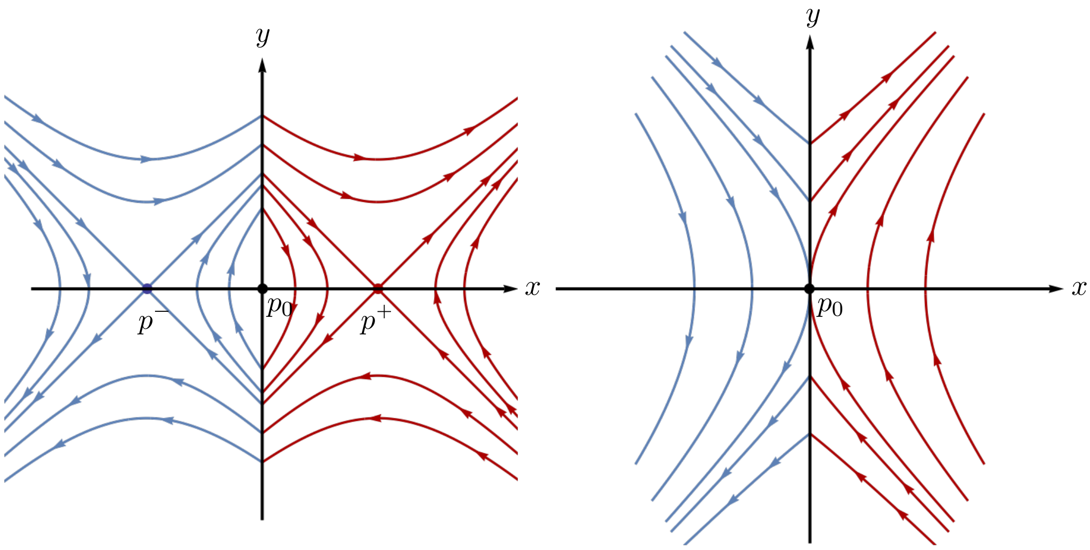
(C2) $\theta > 0$ and $\mu = -1$: In this case, the only singularity of $\mathbf{X}_{(\mathrm{C2})}$ is p_0 and it is a visible fold-fold (see Figure 2);

(C3) $\theta = 0$ and $\mu = 1$: In this case, the only singularity of $\mathbf{X}_{(\mathrm{C3})}$ is p_0, which corresponds to an invisible fold-fold as well as a center (see Figure 3);

(C4) $\theta = 0$ and $\mu = -1$: In this case, p_0 is a visible fold-fold of $\mathbf{X}_{(\mathrm{C4})}$ and also its only singularity (see Figure 3);

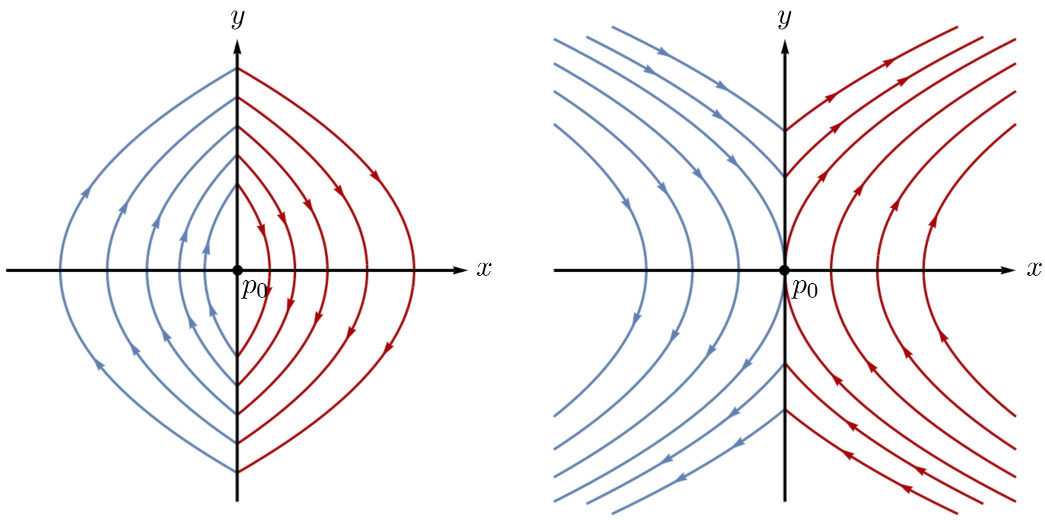
(C5) $\theta < 0$ and $\mu = 1$: In this case, the point p_0 is an invisible fold-fold and also a center of $\mathbf{X}_{(\mathrm{C5})}$ (see Figure 4);

(C6) $\theta < 0$ and $\mu = -1$: In this case, the point p_0 is a visible fold-fold, and the points p^+ and p^- are both linear centers of $\mathbf{X}_{(\mathrm{C6})}$. These points are the only singularities of $\mathbf{X}_{(\mathrm{C6})}$ (see Figure 4).



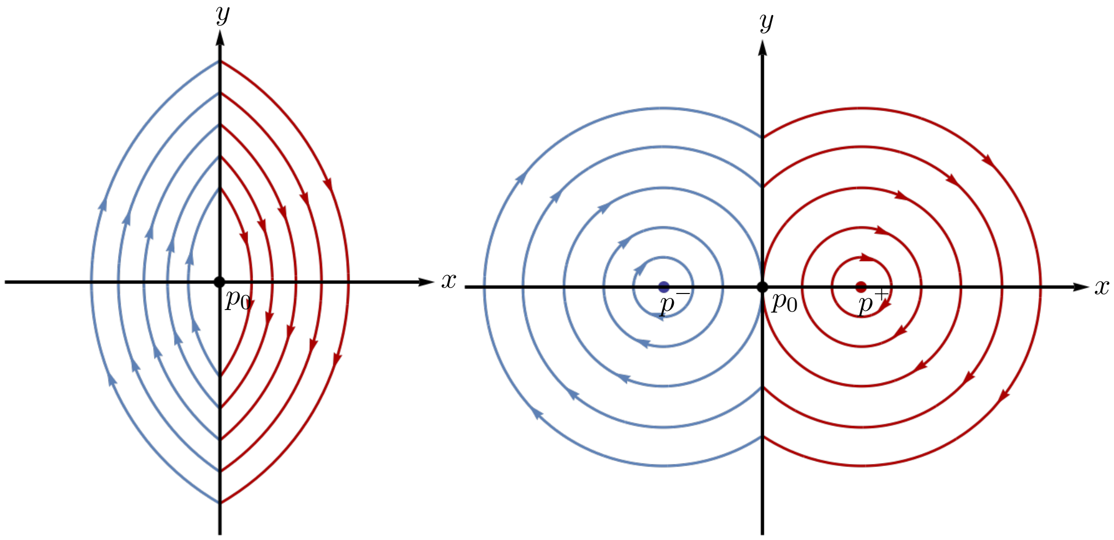
(C1) **(C2)**

Figure 2 – Phase portraits of the cases **(C1)** and **(C2)**.



(C3) **(C4)**

Figure 3 – Phase portraits of the cases **(C3)** and **(C4)**.



(C5) **(C6)**

Figure 4 – Phase portraits of the cases **(C5)** and **(C6)**.

1.1.2 Cases of period annulus: $\mathcal{A}_0 - \mathcal{A}_3$

We notice that in the cases **(C1)**, **(C3)**, **(C5)**, and **(C6)**, there is are regions known as *period annuli*, consisting exclusively of periodic orbits (see Section 6.1 and Figure 5). Since we are only concerned with these cases, we rename them according to their appearance in this work in the following way:

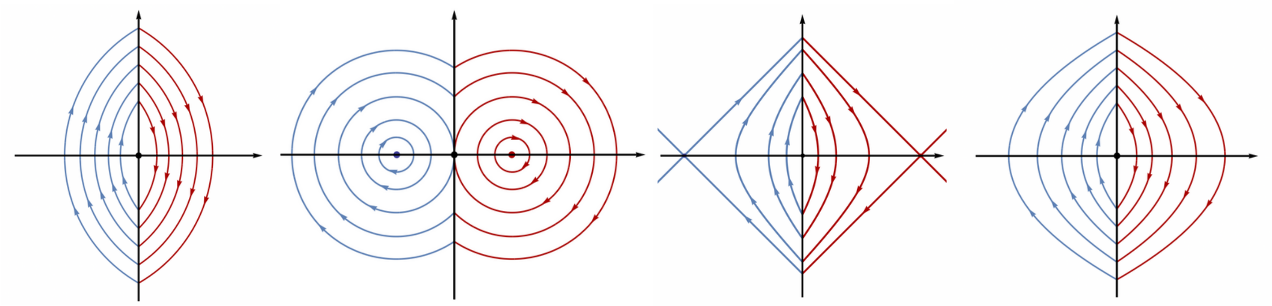
- $\mathcal{A}_0$ = **(C5)**: $\theta < 0$ and $\mu = 1$;
- $\mathcal{A}_1$ = **(C6)**: $\theta < 0$ and $\mu = -1$;
- $\mathcal{A}_2$ = **(C1)**: $\theta > 0$ and $\mu = 1$;
- $\mathcal{A}_3$ = **(C3)**: $\theta = 0$ and $\mu = 1$.

We rename the case **(C5)** as $\mathcal{A}_0$ because, in the context of the existence of invariant tori and boundedness of solutions, this case has already been studied in [34].

On the other hand, taking $\mathbf{H}_{\mathcal{A}_2}(x, y) = \frac{y^2}{2} + G_{\mathcal{A}_2}(x) = H_{\mathcal{A}_2}(x, y, t; 0)$ to be the unperturbed Hamiltonian associated with (1.3) in the case $\mathcal{A}_2$, we notice that $L_\theta \backslash \{(0,0)\}$, where

$$L_\theta := \left\{(x, y) \in \mathbb{R}^2 : 0 \leqslant \mathbf{H}_{\mathcal{A}_2}(x, y) < \frac{1}{2\theta} \text{ and } |x| < \frac{1}{\theta}\right\}, \tag{1.8}$$

corresponds to the period annulus in case $\mathcal{A}_2$. It is noteworthy that the case $\mathcal{A}_2$ is the only case among the aforementioned ones where the period annulus is bounded (see Figure 5). For $\varepsilon \neq 0$, we adopt the same labels for the corresponding cases.



$\mathcal{A}_0$ $\mathcal{A}_1$ $\mathcal{A}_2$ $\mathcal{A}_3$

Figure 5 – Cases of the unperturbed equation (1.3) featuring period annuli.

1.2 Main goals

In [34], Kunze et al. demonstrated that, for sufficiently small $\varepsilon > 0$, all solutions of (1.3) in the case $\mathcal{A}_0$ are bounded, provided that $p \in \mathcal{C}^6(\mathbb{S}_\sigma)$. This result is a consequence of the existence of an infinite collection of closed curves of the time-σ-map associated with (1.3), which is directly related to the existence of a sequence of nested invariant tori for the differential equation (1.3) in the extended phase space, confining any solution of the system. In order to obtain aforementioned closed curves, some coordinate changes are necessary to overcome the lack of regularity of the time-σ-map and to fit it into the conditions of the Invariant Curve Theorem, a variant version of the Moser's Twist Map Theorem present in the KAM theory.

The main goal of our work is to provide conditions over $p(t)$ that ensure the existence of a family of invariant tori in the remaining cases $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ of the differential equation (1.3), and consequently the boundedness of solutions whose initial conditions lies within such invariant tori. According to the nature of the unperturbed differential equation (1.3), we expect boundedness of all solutions in the cases $\mathcal{A}_1$ and $\mathcal{A}_3$, but not in the case $\mathcal{A}_2$, since the period annulus, for $\varepsilon = 0$, in such a case is bounded.

Specifically, by adopting the framework provided by Kunze et al. in [34], we address the existence of invariant tori for the differential equation (1.3) in the case $\mathcal{A}_1$ by means of the Invariant Curve Theorem, while case $\mathcal{A}_2$ is treated with Moser's Twist Map Theorem. In both cases, $p(t)$ is required to be sufficiently smooth. In contrast to the former cases, we study the existence of invariant tori for the differential equation (1.3) in case $\mathcal{A}_3$ through a simpler constructive method by assuming $p(t)$ to be a Lebesgue-integrable function with vanishing average. More specifically, we construct an explicit family of invariant tori of (1.3) whose union of their interiors covers the enlarged three-dimensional space $(t, x, \dot{x}) \in \mathbb{S}_\sigma \times \mathbb{R}^2$. In this case, the Carathéodory theory for differential equations must also be considered in order to address the uniqueness of solutions.

The reason for not using KAM theory in the study of case $\mathcal{A}_3$ is that, since $\theta = 0$, the saturation $g(x)$ as in (1.1) becomes a bounded function. This characteristic generates a subtle twist at infinity, making it challenging to apply the standard versions of Moser's Twist Map Theorem to obtain global results regarding the existence of invariant tori. On the other hand, our reasoning for case $\mathcal{A}_3$ allows us to significantly reduce the regularity of $p(t)$ by assuming a vanishing average for it. This differs from cases $\mathcal{A}_1$ and $\mathcal{A}_2$, where smoother conditions on $p(t)$ are crucial for the techniques employed. Additionally, the results obtained for case $\mathcal{A}_3$ have a non-perturbative character, meaning that their accuracy is independent of ε. For the non-vanishing average case, we present a work in progress to deal with the existence of invariant tori and boundedness of all solutions of (1.3), primarily relying on the parametrization method (see [27], for instance).

Besides the existence of invariant tori, we will use three distinct approaches to deal with the existence of periodic solutions of (1.3). The first one is grounded on a topological method for detecting fixed points in twist area-preserving maps, and it shall be applied to cases $\mathcal{A}_1$ and $\mathcal{A}_2$. The second one consists in finding periodic solutions of (1.3) in the case $\mathcal{A}_3$ through direct computations. Finally, the third one relies on the Melnikov method for detecting periodic orbits, which is going to be addressed to the general class of differential equations (1.2).

1.3 Structure of the thesis

Chapter 2 is dedicated to present some basic notions to be used throughout this work such as Carathéodory equations, Filippov systems, Hamiltonian systems, and some results in KAM Theory.

In Chapter 3 we present a result concerning the existence of infinitely many invariant tori of the differential equation (1.3) in the case $\mathcal{A}_1$ provided $p \in \mathcal{C}^6(\mathbb{S}_\sigma)$ and ε is sufficiently small (Theorem A). As a direct consequence of the main result of this chapter, we shall see that, under the previous assumptions, all the solutions of (1.3) are bounded (Corollary 3.1.1). Furthermore, by means of the Poincaré-Birkhoff Theorem, we will see that (1.3) has infinitely many periodic orbits (Theorem B). The strategy followed to prove Theorem A is mainly based on [34], where the key idea is to obtain closed curves for a Poincaré-map, which, in turn, is conjugated to the time-σ-map of (1.3), fulfilling the conditions of the Invariant Curve Theorem.

Chapter 4 is devoted to addressing the existence of invariant tori for (1.3) in case $\mathcal{A}_2$. Given that the period annulus is bounded in this case, we adopt a different approach than in case $\mathcal{A}_1$. Here, we demonstrate the existence of a finite collection of invariant tori using Moser's Twist Map Theorem. Accordingly, for $p \in \mathcal{C}^5(\mathbb{S}_\sigma)$ and $K \subset L_\theta$ a compact subset, where L_θ is the set defined in (1.8), we prove the existence of an invariant torus of (1.3) whose intersection with the time section $\{0\} \times \mathbb{R}^2$ encloses K and it is contained in L_θ, whenever ε is sufficiently small (Theorem C). As an immediate consequence of Theorem C, we have that every solution of (1.3) initiating in K must be bounded for sufficiently small values of ε (Corollary 4.1.1). Moreover, given $n \in \mathbb{N}$, Theorem C is used to obtain a $\varepsilon^*_(n,p) > 0$ and to construct a family of n invariant tori of (1.3), $0 \leqslant \varepsilon \leqslant \varepsilon^*_(n,p)$, whose the intersection of the n-th term of such a family with the time section $\{t = 0\}$ converges to the boundary of L_θ, as n goes to infinity (Theorem D). Finally, for each $n \in \mathbb{N}$, we prove the existence of $n - 1$ periodic solutions of (1.3) (Corollary 4.1.2).

In Chapter 5, which corresponds to the published manuscript [51], we present Theorem E as the main result of the chapter, which addresses the existence of an infinite collection of invariant tori for (1.3) in the case $\mathcal{A}_3$. This result is a consequence of a Fundamental Lemma (Lemma 5.2.1) that provides sufficient conditions for the existence of an invariant torus of (1.3) by assuming that $p(t)$ is a Lebesgue-integrable function with vanishing average. Furthermore, the invariant tori provided by Theorem E are foliated by periodic orbits, representing a highly exceptional phenomenon. This stands in contrast to the tori obtained through the KAM theory, which typically carry quasi-periodic motions. As a final result of the chapter, Proposition 5.3.3 stands for a simple approach to the existence of invariant tori of (1.3) in the case that $p(t)$ is a L^∞-function, instead of just Lebesgue-integrable. The chapter is ended with further directions concerning case $\mathcal{A}_3$ when $p(t)$ has a non-vanishing average.

Finally, Chapter 6, corresponding to the manuscript [50], is dedicated to presenting a Melnikov analysis applied to the more general class of differential equations (1.2). Specifically, by considering $f(t, x, \dot{x})$ to be a $\mathcal{C}^1$-function and σ-periodic in t, we provide a Melnikov function (Theorem F) for each one of the cases $\mathcal{A}_{\mathbf{0}}$, $\mathcal{A}_{\mathbf{1}}$, $\mathcal{A}_{\mathbf{2}}$, and $\mathcal{A}_{\mathbf{3}}$, whose simple zeros imply the existence of periodic solutions bifurcating from the corresponding period annulus.

2 Basic Concepts

This chapter is devoted to shortly introduce some important concepts and classical results to be used throughout this work, such as discontinuous differential equations, Hamiltonian systems, and KAM theory.

2.1 Some concepts and results of differential equations: continuous, Carathéodory, and Filippov

When exploring differential equations with discontinuous right-hand side, different notions of solutions may arise, creating a challenge in investigating fundamental mathematical concepts such as existence and uniqueness of solutions.

Before we proceed with the discontinuous right-hand side differential equations, let us revisit some crucial aspects of the continuous ones.

Consider the non-autonomous Cauchy problem

$$\begin{cases} \dot{x} = f(t,x), \\ x(t_0) = x_0, \end{cases} \qquad (t,x) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n. \tag{2.1}$$

A classical solution of (2.1) is a curve $x(t)$ defined in an interval $\mathcal{I} \subset \mathbb{R}$ containing t_0, which is differentiable for every $t \in \mathcal{I}$, $(t, x(t)) \in \mathcal{U}$ for every $t \in \mathcal{I}$, and satisfies (2.1) for every $t \in \mathcal{I}$. In the case that f is continuous in the open set $\mathcal{U}$, the Cauchy problem (2.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\mathrm{d}s, \tag{2.2}$$

meaning that a differentiable curve $\varphi : \mathcal{I} \to \mathcal{U}$, whose graph is contained in $\mathcal{U}$, is a solution to (2.1) if and only if it is a solution of (2.2).

Some of the basic results concerning the existence and uniqueness of solutions in the classical sense are presented below.

Theorem 2.1.1. *[59, Peano's Theorem] Suppose that for $(t_0, x_0) \in \mathcal{U}$ there exist a and b positive numbers such that $R = R(t_0, x_0, a, b) \subset \mathcal{U}$, where*

$$R(t_0, x_0, a, b) = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t_0 \leqslant t \leqslant t_0 + a \ \text{ and } \ \|x - x_0\| \leqslant b\}, \tag{2.3}$$

and f is continuous on R. Then, there exists a solution to the Cauchy problem (2.1) defined on $[t_0, t_0 + \bar{a}]$, where $\bar{a} = \inf\{a, b/M\}$ and $M = \|f\|_\infty = \sup_{(t,x)\in R} \{|f(t,x)|\}$.

Theorem 2.1.2. *[59, Theorem 3 §5] Suppose that $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n$ is an open set and $f : \mathcal{U} \to \mathbb{R}^n$ is a continuous function. If $x(t)$ is a solution to the Cauchy problem (2.1) on some interval, then there exists a continuation of $x(t)$ to a maximal interval of existence. Moreover, if (ω_-, ω_+) is the maximal interval of existence of $x(t)$, then $g(t) = (t, x(t))$ tends to the boundary of $\mathcal{U}$ as $t \to \omega_-$ and $t \to \omega_+$. That is, for each compact $K \subset \mathcal{U}$ there exists a neighborhood $\mathcal{V}$ of $\omega_\pm$ such that $g(t) \notin K$ for every $t \in \mathcal{V}$.*

Remark 2.1.3. *As a direct consequence of the previous result, we have every maximal solution of (2.1) confined in an invariant compact set of $\dot{x} = f(t, x)$ must be defined for every $t \in \mathbb{R}$.*

Theorem 2.1.4. *[59, Uniqueness of classical solutions] For $(t_0, x_0) \in \mathcal{U}$, suppose that there exist a and b such that f is Lipschitiz continuous on $R = R(t_0, x_0, a, b)$. If $\|f\|_\infty \leqslant M$ in R, then there exists a unique solution to (2.1) defined on $[t_0, t_0 + \bar{a}]$, where $\bar{a} = \inf\{a, b/M\}$.*

For the purposes of this work, our focus lies specifically on the concepts of Carathéodory and Filippov solutions for differential equations with discontinuous right-hand sides, which are intrinsically related with the concept of absolutely continuous functions.

2.1.1 Carathéodory differential equations

The essence of Carathéodory's theory of differential equations is to extend the concept of a solution to a wider class of initial value problems of the form

$$\begin{cases} \dot{x} = f(t, x), \\ x(t_0) = x_0, \end{cases} \qquad (t, x) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n, \tag{2.4}$$

and investigate their properties under less restrictive assumptions over f.

In what follows, we present the concept of absolutely continuous function, as well as the definition of a Carathéodory differential equation.

Definition 2.1.5. *A function $u : [a, b] \subset \mathbb{R} \to \mathbb{R}^n$ is called absolutely continuous on the closed interval $[a, b]$, if there exists a Lebesgue-integrable function $v : [a, b] \to \mathbb{R}^n$ satisfying*

$$u(x) = u(a) + \int_a^x v(s)\mathrm{d}s,$$

for every $x \in [a, b]$. In such cases, we say that u is differentiable for almost every $x \in [a, b]$ and $u'(x) = v(x)$ almost everywhere.

Remark 2.1.6. *Every absolute continuous function is continuous, but the converse is not true. For instance, the function*

$$u(t) := \begin{cases} t \sin\left(\dfrac{1}{t}\right), & \textit{if} \quad t \neq 0, \\ 0, & \textit{if} \quad t = 0, \end{cases}$$

is continuous, but not absolute continuous.

Definition 2.1.7. *Let $\mathcal{I} \times \mathcal{V} \subset \mathbb{R} \times \mathbb{R}^n$ be an open set. We say that a function $f : \mathcal{I} \times \mathcal{V} \to \mathbb{R}^n$ satisfies the Carathéodory conditions if:*

(i) For every $x \in \mathcal{V}$, the function $t \mapsto f(t, x)$ is measurable;

(ii) For almost every $t \in \mathcal{I}$, $x \mapsto f(t, x)$ is continuous;

(iii) For every $(t_0, x_0) \in \mathcal{I} \times \mathcal{V}$, there exist positive numbers a and b, and a non-negative summable function $\rho_{(t_0,x_0)} : [t_0, t_0 + a] \to \mathbb{R}$ satisfying

(a) $R(t_0, x_0, a, b) \subset \mathcal{U}$;

(b) $\|f(t, x)\| \leqslant \rho_{(t_0,x_0)}(t)$ for every $(t, x) \in R(t_0, x_0, a, b)$;

with $R(t_0, x_0, a, b)$ being the set defined in (2.3).

Then, if f satisfies the Carathéodory conditions, we call (2.4) a Carathéodory differential equation.

Definition 2.1.8. *A Carathéodory solution of (2.4) is an absolutely continuous function satisfying (2.2) whose graph is contained in $\mathcal{U}$.*

Evidently, every classical solution is a Carathéodory solution.

Example 2.1.9. *Consider $f : \mathbb{R} \to \mathbb{R}$ the piecewise function defined by*

$$f(x) = \begin{cases} 1, & \textit{if} \quad x > 0, \\ \dfrac{1}{3}, & \textit{if} \quad x = 0, \\ -1, & \textit{if} \quad x < 0. \end{cases}$$

The differential equation $\dot{x} = f(x)$ has no classical solutions at $x_0 = 0$. However, there exist two Carathéodory solutions initiating in 0 and having $[0, +\infty)$ as the interval of definition, namely, $x_1(t) = t$ and $x_2(t) = -t$.

Now, we provide some sufficient conditions for the existence and uniqueness of Carathéodory solutions. We emphasize that different assumptions over f may be considered in order to establish such properties (see [14], for instance).

Theorem 2.1.10. *[22, Existence of Carathéodory solutions] For a given $(t_0, x_0) \in I \times V$, let us assume that f satisfies the Carathéodory conditions on $R(t_0, x_0, a, b) \subset I \times V$ for some positive number a and b. Then, on a closed interval $[t_0, t_0 + d]$ there exists a Carathéodory solution to the initial value problem (2.4), where d is a constant satisfying $\int_{t_0}^{t_0+d} \rho_{(t_0,x_0)}(t)\mathrm{d}t \leqslant b$.*

Theorem 2.1.11. *[22, Uniqueness of Carathéodory solutions] Given $(t_0, x_0) \in \mathcal{I} \times \mathcal{V}$, let us assume that $R(t_0, x_0, a, b) \subset \mathcal{I} \times \mathcal{V}$, for some positive numbers a and b, and that there exists an integrable function $\lambda_{(t_0,x_0)} : [t_0, t_0 + a] \to \mathbb{R}$ satisfying*

$$\|f(t, x_1) - f(t, x_2)\| \leqslant \lambda_{(t_0,x_0)}(t)\|x_1 - x_2\|, \tag{2.5}$$

for every $(t, x_1), (t, x_2) \in R(t_0, x_0, a, b)$. Then, there exists a unique Carathéodory solution to the initial value problem (2.4).

In the following example, we study the uniqueness of solutions of the differential systems in (1.5).

Example 2.1.12. *Assuming $p(t)$ to be Lebesgue integrable, let us revisit the differential systems in (1.5). Assume that $x_0 > 0$, and take $b = x_0/2$. Then, for each $(t, x, y) \in R = R(t_0, x_0, y_0, a, b)$, we have that $0 < x_0/2 \leqslant x \leqslant 3x_0/2$, for any $y_0 \in \mathbb{R}$ and $a > 0$. This implies that $X^+_{\theta,\mu}(\cdot; \varepsilon) : R \to \mathbb{R}^2$ is well defined. Besides that, for each $(t, x_1, y_1), (t, x_2, y_2) \in R = R(t_0, x_0, y_0, a, b)$ and for a fixed $\varepsilon > 0$, we get*

$$\|X^+_{\theta,\mu}(t, x_1, y_1; \varepsilon) - X^+_{\theta,\mu}(t, x_2, y_2; \varepsilon)\| \leqslant \|(y_1 - y_2, \theta(x_1 - x_2))\| \leqslant \max\{1, |\theta|\}\|(x_1, y_1) - (x_2, y_2)\|,$$

which means that, by defining $\lambda_{(t_0,x_0,y_0)}(t) = \max\{1, |\theta|\}$, relationship (2.5) holds. We conclude that, for any $x_0 > 0$, $y_0 \in \mathbb{R}$, and $t_0 \in \mathbb{R}$, there exists a unique local solution to the initial value problem

$$\begin{cases} (\dot{x}, \dot{y}) = X^+_{\theta,\mu}(t, x, y; \varepsilon), \\ (x(t_0), y(t_0)) = (x_0, y_0). \end{cases}$$

The proof of the uniqueness of the local trajectories of $X^-_{\theta,\mu}(t, x, y; \varepsilon)$ is entirely analogous to that of $X^+_{\theta,\mu}(t, x, y; \varepsilon)$.

2.1.2 Filippov differential equations

We saw in the previous section that when (2.4) exhibits discontinuities in t, we can use Carathéodory's theory for differential equations to understand the flow generated by it in such case. Nevertheless, when such discontinuities occurs with respect to x, the associated vector field may undergo substantial variations near a specific point, making it impossible to construct a Carathéodory solution in this situation. The Filippov theory for differential equations with discontinuous right-hand side aims to describe the solutions of a equation of the form

$$\dot{x} = f(t, x), \quad (t, x) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n, \tag{2.6}$$

by considering the behavior of the associated vector field in a neighborhood of each point (t, x) in its domain U.

In this section, we briefly introduce the Filippov formalism and discuss some important results concerning this theory. For further details on this topic, we recommend [22].

For $x \in \mathbb{R}^n$ and $\delta > 0$, let us denote by $\mathcal{B}(x, \delta)$ the ball centered in x of radio δ. Assume that the differential equation (2.6) is discontinuous in x in a set of measure zero and define the set-valued map

$$\mathcal{F}[f](t,x) := \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{\text{co}}\{f(t, \mathcal{B}(x,\delta)\backslash S)\},$$

with $\overline{\text{co}}$ standing for the convex closure and μ for the Lebesgue measure. We notice that, if $f(t,\cdot)$ is continuous in x, then $\mathcal{F}[f](t,x) = \{f(t,x)\}$.

Definition 2.1.13. *Let $F : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a set-valued function. A function $\varphi : I \to U$ is called a solution of the differential inclusion*

$$\dot{x} \in F(t,x), \tag{2.7}$$

if it is absolutely continuous and it satisfies (2.7) for almost every $t \in I$.

More informations about differential inclusions can be found in [57].

Definition 2.1.14. *A curve $x(t)$ defined in some interval $\mathcal{I} \subset \mathbb{R}$ containing t_0 is said to be a Filippov solution of*

$$\begin{cases} \dot{x} = f(t,x), \\ x(t_0) = x_0, \end{cases} \qquad (t,x) \in \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n, \tag{2.8}$$

if $(t, x(t)) \in \mathcal{U}$ for every $t \in U$ and it is a solution to the differential inclusion

$$\begin{cases} \dot{x} \in \mathcal{F}[f](t,x), \\ x(t_0) = x_0. \end{cases} \tag{2.9}$$

In general, Carathéodory and Filippov solutions are not related, as we shall see in the following example.

Example 2.1.15. *Assume that $t_0 = 0$ and $x_0 = 0$, and consider the initial value problem (2.8) with*

$$f(t,x) = f(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

There are two Carathéodory solutions to the initial value problem with the maximal interval of definition being $[0, +\infty)$. Namely, $x_1(t) = 0$ and $x_2(t) = t$. When we adopt the Filippov convention for the initial value problem, we have that $\mathcal{F}[f] : \mathbb{R} \to 2^{\mathbb{R}}$ is given by $\mathcal{F}[f](x) = \{1\}$, and consequently, $x_2(t) = t$ is the only solution to the initial value problem (2.8).

In what follows, we present a definition that extends the concept of continuity of vector-valued functions to set-valued functions.

Definition 2.1.16. *Let $F : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a set-valued map, that is, each point $(t,x) \in \mathcal{U}$ is assigned to a subset $F(t,x) \subset \mathbb{R}^n$. We say the F is upper semi-continuous (resp. lower semi-continuous) on $x \in \mathbb{R}^n$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(y) \subset F(x) + \mathcal{B}(0,\varepsilon)$ (resp. $F(x) \subset F(y) + \mathcal{B}(0,\varepsilon)$) for every $y \in \mathcal{B}(x,\delta)$. The set-valued function F is said to be continuous if it is both upper and lower semi-continuous.*

Sufficient conditions for the existence of solutions for a given differential inclusion of the form $\dot{x} \in F(t,x)$ is given in the following result.

Theorem 2.1.17. *[22, Theorem 1 §1] Assume that for each $(t,x) \in \mathcal{U}$, the set $F(t,x) \subset \mathbb{R}^n$ is non-empty, closed, convex, and bounded. If F is upper semi-continuous in $\mathcal{U}$, then for each (t_0,x_0) there exists a solution to the problem*

$$\begin{cases} \dot{x} \in F(t,x), \\ x(t_0) = x_0. \end{cases}$$

Back to the differential equation (2.4), some conditions over f can be assumed in order to the associated Filippov set-valued function $\mathcal{F}[f]$ fulfills the conditions of Theorem 2.1.17.

Definition 2.1.18. *A function $f : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is called locally essentially bounded in $\mathcal{U}$ if, for each $(t_0,x_0) \in \mathcal{U}$, there exist $\delta_1 > 0$, $\delta_2 > 0$, and a positive integrable function $m_{(t_0,x_0)} : [t, t+\delta_2] \to \mathbb{R}$ satisfying*

$$\|f(t,x)\| \leqslant m_{(t_0,x_0)}(t)$$

for almost every $t \in [t_0, t_0 + \delta_2]$ and almost every $x \in \mathcal{B}(x_0,\delta_1)$ in the sense of Lebesgue measure.

Proposition 2.1.19. *[22, §7] Let $f : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a locally essentially bounded function. Then, for each $(t,x) \in \mathcal{U}$, the set $\mathcal{F}[f](t,x)$ is non-empty and bounded. Moreover, the set-valued function $\mathcal{F}[f]$ is upper semi-continuous.*

As direct consequence of the previous result, we have the following.

Theorem 2.1.20. *[22, Theorem 8 §7] Let $f : \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a locally essentially bounded function. Thus, for each $(t_0,x_0) \in \mathcal{U}$, there exists a solution to the initial value problem (2.8).*

Example 2.1.21. *Let us consider the differential system $X_{\theta,\mu}(t,x,y;\varepsilon) = (y, \theta x - \mu\,\mathrm{sgn}(x) + \varepsilon\, p(t))$ introduced in (1.4), with θ and ε being real parameters, and $\mu \in \{-1,1\}$. We notice that*

$$\|X_{\theta,\mu}(t,x,y;\varepsilon)\| \leqslant |y| + |\theta||x| + 1 + |\varepsilon||p(t)|.$$

Then, given $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^2$, it follows that, for any $\delta_1 > 0$,

$$\|X_{\theta,\mu}(t, x, y; \varepsilon)\| \leqslant \delta_1 + |y_0| + |\theta|(\delta_1 + |x_0|) + 1 + |\varepsilon||p(t)|,$$

for every $(x, y) \in \mathcal{B}((x_0, y_0), \delta_1)$. Therefore, by taking $p(t)$ to be Lebesgue-integrable, it follows that $m_{(t_0,x_0,y_0)}(t) := \delta_1 + |y_0| + |\theta|(\delta_1 + |x_0|) + 1 + |\varepsilon||p(t)|$ is a positive Lebesgue-integrable function defined on $[t_0, t_0 + \delta_2]$, for any $\delta_2 > 0$. This implies that $X_{\theta,\mu}(t, x, y; \varepsilon)$ is locally essentially bounded provided $p(t)$ is a Lebesgue-integrable function. It results from Theorem 2.1.20 that, for any given initial condition $(t_0, x_0, y_0) \in \mathbb{R} \times \mathbb{R}^2$, there exists a local solution of $X_{\theta,\mu}(t, x, y; \varepsilon)$ passing through it.

In the Filippov context of solutions, there exists a similar result concerning continuous extension to the boundary of maximal solutions.

Theorem 2.1.22. *[22, Theorem 9 §7] Under the conditions of Theorem 2.1.20, any maximal solution existing within a specified closed and bounded domain is extended on both sides until it reaches the boundary of the domain.*

The task of describing the trajectories of a differential equation is significantly facilitated when the set of discontinuity of a given vector-valued function corresponds to a codimension one smooth manifold being the preimage of a regular value. For instance, let us assume that $f : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ is a Filippov vector field discontinuous on $\Sigma = \{x \in \mathcal{V} : h(x) = 0\}$, where h is a smooth submersion. Then, Σ separates the set $\mathcal{V}$ into two subsets, namely

$$\mathcal{V}^+ = \{x \in \mathcal{V} : h(x) > 0\} \quad \text{and} \quad \mathcal{V}^- = \{x \in \mathcal{V} : h(x) < 0\},$$

and the vector-valued function $f(x)$ can be expressed as follows

$$f(x) = \begin{cases} f^+(x) & \text{if } \ h(x) \geqslant 0, \\ f^-(x) & \text{if } \ h(x) < 0. \end{cases}$$

If $x_0 \in \mathcal{V}^+$ (resp. $x_0 \in \mathcal{V}^-$), then the local trajectory of f having point x_0 as initial condition is determined by the trajectory of f^+ (resp. f^-) passing through x_0. For $x_0 \in \Sigma$, we use the geometrical characteristics of Σ to determine the local trajectory passing though a given point. The set Σ can be separated into three disjoint subsets

$$\begin{aligned} \Sigma^c &:= \{x \in \Sigma : f^+h(x) \cdot f^-h(x) > 0\}, \\ \Sigma^e &:= \{x \in \Sigma : f^+h(x) > 0, f^-h(x) < 0\}, \\ \Sigma^s &:= \{x \in \Sigma : f^+h(x) < 0, f^-h(x) > 0\}, \end{aligned}$$

where $f^{\pm}h(x) := \langle f^{\pm}(x), \nabla h(x)\rangle$ denotes the Lie derivative of h in the direction of the vector fields $f^{\pm}$. The sets above are usually referred as *crossing*, *escaping*, and *sliding* regions, respectively. If

$x_0 \in \Sigma^c$, the local trajectory of f having x_0 as initial condition is given by the concatenation of the local trajectories of f^+ and f^- passing through x_0. When $x \in \Sigma^e \dot{\cup} \Sigma^s$, then it is necessary to study the sliding vector field given by

$$f^s(p) = \frac{f^-h(p)f^+(p) - f^+h(p)f^-(p)}{f^-h(p) - f^+h(p)}$$

to understand the local behavior of solutions of the vector field f, and in such case, uniqueness is not expected for the local solutions. The points $x \in \Sigma$ where either $f^+h(x) = 0$ or $f^-h(x) = 0$ are called *tangency points* and are one of the singular points of Filippov vector fields.

Since we are primarily concerned with crossing solutions in our work, we are not extending the discussion on Filippov vector fields having smooth manifolds as their set of discontinuity. For the readers interested in this topic, we recommend [22, 24].

Example 2.1.23. *We revisit the vector field* (1.4). *Let us denote* $\mathcal{X}_{\theta,\mu}(\phi, x, y; \varepsilon) = (1, X_{\theta,\mu}(\phi, x, y; \varepsilon))$ *the Filippov vector field associated with* (1.4). *The set of discontinuity of* $\mathcal{X}_{\theta,\mu}(\phi, x, y; \varepsilon)$ *is* $\Sigma' = h^{-1}(0)$, *where* $h : \mathbb{R}^3 \to \mathbb{R}$ *is defined by* $h(\phi, x, y) = x$. *Taking into account the decomposed differential systems in* (1.5), *we have that*

$$\mathcal{X}_{\theta,\mu}(\phi, x, y; \varepsilon) = \begin{cases} \mathcal{X}^+_{\theta,\mu}(\phi, x, y; \varepsilon) = (1, X^+_{\theta,\mu}(\phi, x, y; \varepsilon)), & \text{if } x > 0, \\ \mathcal{X}^-_{\theta,\mu}(\phi, x, y; \varepsilon) = (1, X^-_{\theta,\mu}(\phi, x, y; \varepsilon)), & \text{if } x < 0. \end{cases}$$

Thus, for a fixed ε, *it follows that*

$$\mathcal{X}^\pm_{\theta,\mu}h(\phi, x, y; \varepsilon) = y,$$

which implies that, except for $y = 0$, *the plane* Σ' *corresponds to a crossing region of* $\mathcal{X}_{\theta,\mu}(\phi, x, y; \varepsilon)$, *since* $\mathcal{X}^+_{\theta,\mu}h(\phi, x, y; \varepsilon) \cdot \mathcal{X}^-_{\theta,\mu}h(\phi, x, y; \varepsilon) = y^2 > 0$.

2.2 Hamiltonian systems

Hamiltonian differential systems are a powerful mathematical tool for describing mechanical dynamics, especially those related to celestial mechanics. Its development is mainly attributed to the Irish mathematician W. R. Hamilton, who in [26] proposed a new way of dealing with the interaction laws of I. Newton introduced in [45]. In this section, we briefly introduce Hamiltonian systems and present some interesting properties concerning these objects. For further details on this theory, we suggest consulting [6].

Let us consider the differential system

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y}(x, y), \\ \dot{y} = -\dfrac{\partial H}{\partial x}(x, y), \end{cases} \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2.10}$$

where $H : \mathcal{U} \subset \mathbb{R}^{2n} \to \mathbb{R}$, with $\mathcal{U}$ being open, is a function of class $\mathcal{C}^r$, $r \geqslant 1$. We refer to (2.10) as Hamiltonian system with n-degrees of freedom associated with the Hamiltonian H. The variables x and y are usually referred to as position and conjugated momentum, respectively.

A distinctive characteristic of Hamiltonian systems is the invariance of the level surfaces of the Hamiltonian function under its generated flow. In other words, if $(x(t), y(t))$ is an integral curve of (2.10), then, for some $h \in \mathbb{R}$, $H(x(t), y(t)) = h$ for every $t \in I$, with I denoting the maximal interval of definition of $(x(t), y(t))$. In the case of a Hamiltonian system with one degree of freedom, the level curves of H carry the solutions of the originating differential system, providing a full description of its phase portrait.

Hamiltonian systems are also at the core of symplectic geometry, a branch of differential topology and differential geometry that deals with symplectic manifolds. This area of mathematics provides a modern language to study classical mechanics through the Hamiltonian formulation. More on this theory can be found in [61].

For our purposes, we concentrate on the exact sympletic character of Poincaré maps associated with Hamiltonian systems. Therefore, we only present some aspects concerning the symplectic geometry.

Let $\mathcal{V}$ be a $2n$-dimensional vector space and $\omega : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ a 2-form. We say that the pair $(\mathcal{V}, \omega)$ is a *sympletic vector space* if ω is a closed skew-symmetric form satisfying a non-degeneracy condition given by

$$\omega\,(u_1, u_2) = 0 \quad \text{for every} \quad u_1 \in V, \quad \text{then} \quad u_2 = 0.$$

The pair (M, ω) is called *symplectic manifold*, if M is a $2n$-dimensional manifold and ω is closed 2-form for which $(T_x M, \omega_x)$ is a symplectic vector space for every $x \in M$.

Definition 2.2.1. *A diffeomorphism f between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is called symplectomorphism if it satisfies $f^*\omega_2 = \omega_1$. Under this condition, (M_1, ω_1) and (M_2, ω_2) are said to be symplectomorph.*

Let us consider $(\mathbb{R}^{2n}, \eta)$ with the 2-form $\eta = \sum_{j=1}^{n} dp_j \wedge dq_j$ and $(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$ the usual coordinates of $\mathbb{R}^{2n}$. The pair $(\mathbb{R}^{2n}, \eta)$ is a symplectic manifold, with η being called *canonical symplectic form*. According to Darboux theorem (see [12, p.40]), if (M, ω) is a symplectic manifold, then, for every $x \in M$ there exist $\mathcal{U}_x \subset M$ and $\mathcal{V}_0 \subset \mathbb{R}^{2n}$ open neighborhoods of x and 0, respectively, and a symplectomorphism $\Phi_x : \mathcal{U}_x \to \mathcal{V}_0$ such that $\Phi_x^*\eta = \omega$.

In what follows, we provide a way to define Hamiltonian systems in the context of symplectic geometry.

Definition 2.2.2. *Let (M, ω) be a sympletic manifold and $H : M \to \mathbb{R}$ a smooth function. The function H defines a 1-form in M denoted $\mathrm{d}H$. From the non-degeneracy of ω, there exists a unique vector field*

X_H satisfying

$$\omega(X_H, \cdot) = \mathrm{d}H.$$

The triple (M, ω, H) is called Hamiltonian system with n-degree of freedom.

Remark 2.2.3. *From Darboux Theorem, it follows that, in local coordinates, a Hamiltonian vector field X_H assumes the form* (2.10).

Let $\rho_t^H : M \to M$ denotes the flow generated by a Hamiltonian system (M, ω, H). Assume that ρ_t^H is complete. Then, for every $t \in \mathbb{R}$, $(\rho_t^H)^*\omega = \omega$, meaning that the flow associated with a Hamiltonian system (M, ω, H) defines a group of symplectomorphisms on M.

Hamiltonian systems can indeed depend on time. In such cases, the extended phase space of a Hamiltonian system is considered as $M \times \mathbb{R}$, or $M \times \mathbb{S}_\sigma$ when the Hamiltonian is σ-periodically dependent on time. We refer to this as a $n^1\!/\!_2$- degree of freedom Hamiltonian system.

Notice that when the symplectic structure ω is exact, meaning there exists a 1- form ν for which $\omega = \mathrm{d}\nu$, any symplectomorphism $f : (M, \omega) \to (M, \omega)$ defines a closed 1-form $\tilde{\nu} = \nu - f^*\nu$. If $\tilde{\nu}$ is exact, then f is called an exact symplectic map. On this subject, we have the following interesting result.

Proposition 2.2.4. *[61, Proposition 1.1] Let (M, ω, H) be a σ-periodic non-autonomous Hamiltonian system, and let $P_H : M \times \{\tau_0\} \to M \times \{\tau_0 + \sigma\}$ denote the time-σ-map associated with (M, ω, H), for some $\tau_0 \in \mathbb{R}$. If ω is exact, then P_H is an exact sympletic map.*

As a consequence of the previous result we have that Poincaré maps associated with periodic non-autonomous Hamiltonian systems are volume preserving.

In the following sections, we present important results concerning planar Hamiltonian systems, such as the period function for closed trajectories and canonical transformations for Hamiltonian systems featuring period annuli.

2.2.1 Period function for symmetric one degree of freedom Hamiltonian systems

For Hamiltonian differential systems whose trajectories exhibit symmetry with respect to the x-axis expressed by $\mathbf{H}(x, y) = \frac{y^2}{2} + G(x)$, the period of a closed trajectory lying on the level curve $\mathbf{H}(x, y) = h$, with $h \in \mathbb{R}$, can be determined using the function

$$T(h) = 2\int_b^a \frac{\mathrm{d}u}{\sqrt{2(h - G(u))}},$$

with $(a, 0)$ and $(b, 0)$, $b < 0 < a$, representing the points resulting from the intersection between the corresponding level curve and the x-axis (see [25, p. 203]). If $G(x) = G(-x)$ for every $x \in \mathbb{R}$, then the level curves of $\mathbf{H}$ are also symmetric with respect to the y-axis. This implies that the period function $T(h)$ can be simplified to

$$T(h) = 4\int_0^a \frac{\mathrm{d}u}{\sqrt{2(h - G(u))}},$$

since $b = -a$.

2.2.2 Action-angle coordinates for non-autonomous planar Hamiltonian systems

In this section, we will outline a procedure involving the action-angle transformation for planar time-dependent Hamiltonian systems. This technique is a common method used to modify the original Hamiltonian into a version where the main component depends only on the action variable. The discussion present in this section is mainly based on [36, §2]. It is also important to mention [5] as a standard literature on this subject.

The formal definition of the action-angle variables is as follows: By taking time as a parameter, we consider the Hamiltonian

$$H(x, y, t) = \frac{y^2}{2} + G(x, t). \tag{2.11}$$

Assume that $h_0(I, t)$ corresponds to the value of H on the level curve that encloses the area I in the (x, y)-plane. Implicitly, the function $h_0(I, t)$ can be expressed by the following integral

$$I = \oint\limits_{H(x,y,t)=h_0(I,t)} y\mathrm{d}x.$$

Remark 2.2.5. *Let $h_0(I, t) = h$ be the energy for which the corresponding level curve encloses the region with area I, and let $A(h, t)$ be the inverse of h_0 in I, that is, for a fixed t, $A(h_0(I, t), t) = I$. As discussed in [5, p. 282], the derivative of A with respect to h corresponds to the period of the level curve $H(x, y, t) = h$ in terms of h, which yields*

$$T(h, t) = \partial_h A(h, t) = \frac{1}{\partial_I h_0(I, t)}.$$

By considering γ as the curve connecting the y-axis to the point (x, y) on the level curve $H(x, y, t) = h_0(I, t)$ oriented clockwise, we define the generating function

$$S(x, I, t) = \oint\limits_{\gamma} y\mathrm{d}x.$$

Then, the *action-angle* transformation $(\phi, I, t) \mapsto (x, y, t)$ is defined through the relations

$$y = \partial_x S(x, I, t) \quad \text{and} \quad \phi = \partial_I S(x, I, t). \tag{2.12}$$

We notice that (2.12) defines a sympletic transformation, since

$$\begin{aligned}\mathrm{d}\phi \wedge \mathrm{d}I &= \mathrm{d}\left(\partial_I S(x,I,t)\right) \wedge \mathrm{d}I = \partial_x\partial_I S(x,I,t)\mathrm{d}x \wedge \mathrm{d}I,\\ \mathrm{d}x \wedge \mathrm{d}y &= \mathrm{d}x \wedge \mathrm{d}\left(\partial_x S(x,I,t)\right) = \partial_I\partial_x S(x,I,t)\mathrm{d}x \wedge \mathrm{d}I.\end{aligned}$$

This implies that the transformation provided by (2.12) between (x, y, t) and (ϕ, I, t) does not change the Hamiltonian character of H, now given by

$$\mathcal{H}(\phi, I, t) = h_0(I,t) + \mathcal{H}_1(\phi, I, t),$$

where $\mathcal{H}_1(\phi, I, t) = \partial_t S(x, I, t)$ and $x = x(\phi, I, t)$ is implicitly given by (2.12).

For the Hamiltonian (2.11), we notice that, for $y \geqslant 0$,

$$y = \sqrt{2\left(h_0(I,t) - G(x,t)\right)},$$

which means that the curve γ defining the generating function $S(x, I, t)$ can be parameterized by $\gamma(s) = (s, \sqrt{2\left(h_0(I,t) - G(s,t)\right)})$, with $s \in [0, x]$, yielding

$$S(x,I,t) = \int_0^x \sqrt{2\left(h_0(I,t) - G(s,t)\right)}\mathrm{d}s.$$

Consequently, from the Leibniz Integral rule, the angle variable is given by

$$\phi(x,h,t) = \partial_I h_0(I,t) \int_0^x \frac{\mathrm{d}s}{\sqrt{2\left(h_0(I,t) - G(s,t)\right)}} = \frac{1}{T(h,t)} \int_0^x \frac{\mathrm{d}s}{\sqrt{2\left(h - G(s,t)\right)}}, \tag{2.13}$$

with the second equality above resulting from Remark 2.2.5.

2.2.3 Time and energy as the new position and momentum

For the following coordinate transformation, we employ the well-known technique introduced by Arnold in [4], which has been consistently reiterated in works such as [34, 36, 37, 64]. Roughly speaking, this coordinate transformation consists in exchange the roles of position and time in the Hamiltonian systems in the region where the derivative of the corresponding Hamiltonian function with respect to the momentum variable does not vanish.

In order to establish such coordinate change, let us first consider $\mathcal{V} \subseteq \mathbb{R}^3$ as an open set and $\mathcal{H} : \mathcal{V} \to \mathbb{R}$ a non-autonomous Hamiltonian function of class $\mathcal{C}^1$. The Hamiltonian system associated with $\mathcal{H}$ in its extended form is given by

$$\begin{cases} \dot{q} = \dfrac{\partial \mathcal{H}}{\partial p}(q,p,t),\\[2mm] \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial q}(q,p,t),\\[2mm] \dot{t} = 1, \end{cases} \tag{2.14}$$

with *dot* denoting the derivative with respect to s.

We assume that there exists an open set $\mathcal{U} \subset \mathcal{V}$ such that $\dfrac{\partial \mathcal{H}}{\partial p}(q, p, t) > 0$ for every $(q, p, t) \in \mathcal{U}$. Then, we have that (2.14) is equivalent with

$$\begin{cases} q' = 1, \\ p' = -\dfrac{\partial \mathcal{H}}{\partial q}(q, p, t) / \dfrac{\partial \mathcal{H}}{\partial p}(q, p, t), \\ t' = 1 / \dfrac{\partial \mathcal{H}}{\partial p}(q, p, t), \end{cases} \tag{2.15}$$

where $' = {}^{\mathrm{d}}/_{\mathrm{d}q}$. Fixing q and t as parameters and taking account $\dfrac{\partial \mathcal{H}}{\partial p}(q, p, t) > 0$, it follows that $\mathcal{H}(q, \cdot, t)$ is invertible, for every $(q, p, t) \in \mathcal{U}$. This leads us to the following coordinate change

$$\begin{array}{rccc} \Phi : & \mathcal{U} & \longrightarrow & \Phi(\mathcal{U}) \\ & (q, p, t) & \longmapsto & (Q, P, \tau), \end{array}$$

where $Q(q, p, t) = t$, $P(q, p, t) = \mathcal{H}(q, p, t)$, and $\tau(q, p, t) = q$. Let us denote by $\mathscr{H}(Q, P, t)$ the inverse of $\mathcal{H}$ in p, that is, $\mathcal{H}(\tau, \mathscr{H}(Q, P, \tau), Q) = P$, for every $(Q, P, t) \in \Phi(\mathcal{U})$. By performing Φ to the differential system (2.15), we have that

$$\begin{cases} Q' = 1 / \dfrac{\partial \mathcal{H}}{\partial p}(\tau, \mathscr{H}(Q, P, \tau), Q), \\ P' = \dfrac{\partial \mathcal{H}}{\partial t}(\tau, \mathscr{H}(Q, P, \tau), Q) / \dfrac{\partial \mathcal{H}}{\partial p}(\tau, \mathscr{H}(Q, P, \tau), Q), \\ \tau' = 1, \end{cases}$$

which is Hamiltonian with the function $\mathscr{H}(Q, P, \tau)$, since

$$\frac{\partial \mathscr{H}}{\partial P}(Q, P, \tau) = \frac{1}{\frac{\partial \mathcal{H}}{\partial p}(\tau, \mathscr{H}(Q, P, \tau), Q)} \quad \text{and} \quad \frac{\partial \mathscr{H}}{\partial Q}(Q, P, \tau) = -\frac{\frac{\partial \mathcal{H}}{\partial t}(\tau, \mathscr{H}(Q, P, \tau), Q)}{\frac{\partial \mathcal{H}}{\partial p}(\tau, \mathscr{H}(Q, P, \tau), Q)}.$$

2.3 KAM theory

The KAM theory comprises a collection of results concerning the persistence of quasi-periodic motions in nearly integrable systems under specific Diophantine and non-degeneracy conditions. It is named in honor of the Russian mathematicians Kolmogorov, Arnold, and Moser, who, in the 1950s and 1960s, introduced a novel approach to tackle the n-body problem initially posed by Newton. This problem remained unsolved for almost two centuries, primarily due to issues arising from the appearance of the so-called "small divisors" when attempting to solve functional equations through series expansions.

Among the most celebrated results in this theory are the KAM Theorem [6], which addresses the existence of invariant tori in nearly integrable Hamiltonian systems; Arnold's Theorem

[66], dealing with the existence of analytical conjugacies between pure rotations and close to pure rotations on the circle; and Moser's Twist Map Theorem [43], addressing the existence of invariant curves for twist regular maps defined on the annulus. The main idea behind the proof of the KAM results mainly relies on two points: The first one involves linearizing the functional equation around an approximate solution and solving the linearized problem, and the second is based on an adaptation of Newton's method to approximate the solution of the original problem by the solutions of the linearized one. This procedure, together with the symplectic structure and the Diophantine character required in the KAM results, enables the construction of a convergent sequence with a square rate of convergence, overcoming the harmful effects of the "small divisors".

In this section, our primary focus is on Moser's Twist Map Theorem and its variations. For readers interested in KAM theory, we recommend consulting [19].

2.3.1 Diophantine numbers

This section is devoted to briefly introduce a special class of irrational numbers called Diophantine.

Definition 2.3.1. *We say that a number $\omega \in \mathbb{R}\backslash\mathbb{Q}$ is Diophantine if there exist two positive constants $\nu > 2$ and $\gamma > 0$ such that*

$$\left|\omega - \frac{p}{q}\right| \geqslant \frac{\gamma}{|q|^{\nu}},$$

for every $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$.

Remark 2.3.2. *A direct consequence of the previous definition is that if $\omega \in \mathbb{R}\backslash\mathbb{Q}$ is a Diophantine number, then so it is $\omega + k$ for every $k \in \mathbb{Z}$.*

In what follows we present a particular class of Diophantine numbers.

Definition 2.3.3. *A number $\omega \in \mathbb{R}\backslash\mathbb{Q}$ is said to be of constant type if the quantity*

$$\Omega := \inf \left\{ q^2 \left|\omega - \frac{p}{q}\right| : \ p \in \mathbb{Z} \ \ \textit{and} \ \ q \in \mathbb{N} \right\}$$

is positive. In such cases, Ω is called Markoff constant associated with ω.

Equivalently, irrational numbers of constant type can be characterized through their continued fraction expansions. In particular, for a given $\omega \in \mathbb{R}\backslash\mathbb{Q}$, we express it as

$$\omega = [a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}.$$

This continued fraction expansion is unique for every real number, and ω is irrational if, and only if the terms a_i are infinitely determined. Such characterization leads to the following result.

Proposition 2.3.4. *[28, §3.5] An irrational number $\omega = [a_0; a_1, a_2, \dots]$ is of constant type if, and only if* $\sup\{a_i : i \in \mathbb{N}\} = C < \infty$. *The Markoff constant Ω associated with ω satisfies*

$$\frac{1}{C+2} \leqslant \Omega \leqslant \frac{1}{C}.$$

The previous proposition is crucial for finding irrational numbers of constant type in any real interval and determine the bounds for their Markoff constants.

Proposition 2.3.5. *[53, Lemma 4.4] For each interval $[a, b] \subseteq [0, 1]$ with $b - a \geqslant \epsilon > 0$, there exists an irrational number $\alpha^* \in [a, b]$ of constant type such that the corresponding Markoff constant Ω^* satisfies $\epsilon/16 \leqslant \Omega^* \leqslant \epsilon/4$.*

2.3.2 Moser's Twist Map Theorem

Let A denote an annulus defined by $A = \left\{(\bar{\theta}, r) : \bar{\theta} \in \mathbb{S}_1 \text{ and } a \leqslant r \leqslant b\right\}$, whose universal cover is the strip $\mathcal{A} = \{(\theta, r) : \theta \in \mathbb{R} \text{ and } a \leqslant r \leqslant b\}$.

Definition 2.3.6. *The maps of the form*

$$\begin{aligned} M : A &\longrightarrow \mathbb{S}_1 \times \mathbb{R}, \\ (\bar{\theta}, r) &\longmapsto \left(\bar{\theta} + \beta + \alpha(r), r\right), \end{aligned}$$

where $\alpha : [a, b] \to \mathbb{R}$ is a smooth function with $\alpha'(r) > 0$ for every $r \in [a, b]$, are called twist maps.

Definition 2.3.7. *Let $\phi : \mathbb{S}_1 \to \mathbb{S}_1$ be an orientation preserving diffeomorphism. The rotation number of ϕ is defined as the limit*

$$\rho(\phi) = \lim_{n \to \infty} \frac{\phi^n(x) - x}{n},$$

when such a limit exists.

Remark 2.3.8. *Important aspects regarding the rotation number include the independence of the limit in Definition 2.3.7 on the choice of $x \in \mathbb{S}_1$ and its invariance under topological conjugations. Furthermore, if f is a C^r-diffeomorphism, with $r \geqslant 2$, defined on a curve topologically equivalent to a circle and possessing an irrational rotation number ω, then f is topologically conjugated to a pure rotation of the circle*

$$\begin{aligned} R_\omega : \ \mathbb{S}_1 &\longrightarrow \mathbb{S}_1 \\ \theta &\longmapsto \theta + \omega. \end{aligned}$$

The term "twist maps", as defined in Definition 2.3.6, is appropriate because, for fixed angles $\theta_0 \in \mathbb{S}_1$, the lines $\mathscr{L}_{\theta_0} := \{\theta_0\} \times [a, b]$ are twisted when subjected to the map M. Additionally, it is noteworthy that the curves $\mathscr{C}_{r_0} := \mathbb{S}_1 \times \{r_0\}$, where $r_0 \in [a, b]$, remain invariant under M. Moreover, the restriction $M\big|_{\mathscr{C}_{r_0}}$ has rotation number $\rho(M\big|_{\mathscr{C}_{r_0}}) = \beta + \alpha(r_0)$. A natural question arising from these points is: What happens if M is slightly perturbed? Is there any invariant curve for such perturbation? The answer for the second question, in general, is no, as noticed in the following example.

Example 2.3.9. *Consider the map $M(\theta, r) = (\theta + r, (1-\varepsilon)r)$, defined on the annulus $\mathbb{S}_1 \times (0, \bar{r})$, with $\varepsilon \in (0,1)$ and $\bar{r} > 0$. The map M does not have any invariant curve in $\mathbb{S}_1 \times (0, \bar{r})$. Indeed, suppose that there exists an invariant curve $\mathscr{C}$ for M in $\mathbb{S}_1 \times (0, \bar{r})$. Then, there would exist $r_1, r_2 \in (0, \bar{r})$ such that $\mathscr{C}$ is contained in the region enclosed by the curves $\mathscr{C}_{r_1}$ and $\mathscr{C}_{r_2}$, which is going to be denoted $\mathcal{D}$. However, when we apply M iterativelly to $\mathcal{D}$, this region is shrunk to $\mathbb{S}_1 \times \{0\}$, and, thus, the curve $\mathscr{C}$ cannot be invariant. This implies that M cannot have invariant curves.*

Definition 2.3.10. *A map $M : \mathbb{S}_1 \times [a,b] \to \mathbb{R}^2$ has the intersection property in $\mathbb{S}_1 \times [a,b]$ if $M(\Gamma) \cap \Gamma \neq \varnothing$ for any Jordan curve $\Gamma \subset \mathbb{S}_1 \times [a,b]$, which is homotopic to $\mathscr{C}_{r_0}$, for $r_0 \in [a,b]$.*

Certainly, the map provided in Example 2.3.9 does not have the intersection property. On the other hand, in certain maps, such a property follows from previously assumed geometric conditions, such as area-preserving maps and *Poincaré-maps* derived from Hamiltonian systems. For the latter, we present the following result, provided by R. Dieckerhoff and E. Zehnder in [18], which is a consequence of the exact symplectic character presented in Proposition 2.2.4.

Theorem 2.3.11. *[18, Lemma 5] Let $P : \mathbb{S}_1 \times [a,b] \to \mathbb{R}^2$ be a Poincaré map associated to a non-autonomous periodic Hamiltonian system. Then P has the intersection property.*

It is not difficult to see that the intersection property is a topologically invariant condition, as stated in the following proposition.

Proposition 2.3.12. *Consider the maps $P : \mathcal{D} \to \mathcal{D}$ and $\bar{P} : \overline{\mathcal{D}} \to \overline{\mathcal{D}}$, along with a conjugation $\Psi : \mathcal{D} \to \overline{\mathcal{D}}$ between P and $\bar{P}$. If P exhibits the intersection property, then such a property also holds for $\bar{P}$.*

In order to present the *Moser's Twist Map Theorem*, let us introduce the $\mathcal{C}^r$- norm. For $f \in \mathcal{C}^r(A)$, we denote by $\|f\|_{\mathcal{C}^r(A)}$ the norm

$$\|f\|_{\mathcal{C}^r(A)} = \max_{0 \leqslant m+n \leqslant r} \left\| \frac{\partial^{m+n} f}{\partial_x^m \partial_y^n} \right\|_\infty, \tag{2.16}$$

where

$$\left\| \frac{\partial^{m+n} f}{\partial_x^m \partial_y^n} \right\|_\infty = \sup_{(x,y) \in A} \left| \frac{\partial^{m+n} f}{\partial_x^m \partial_y^n}(x,y) \right|.$$

Theorem 2.3.13. *[44, Moser's Twist Map Theorem] Let $M : \mathbb{S}_1 \times [a,b] \to \mathbb{R}^2$ be a twist map with the intersection property. We assume that its lift can be written in the form*

$$\begin{aligned} \mathcal{M} : \mathbb{R} \times [a,b] &\longrightarrow \mathbb{R} \times \mathbb{R}, \\ (\theta, r) \quad &\longmapsto (\theta + \beta + \alpha(r) + \varphi_1(\theta, r), r + \varphi_2(\theta, r)), \end{aligned}$$

where $\alpha \in \mathcal{C}^4([a,b])$ is a function satisfying $\alpha'(r) > 0$ for every $r \in [a,b]$, and φ_1 and φ_2 are of class $\mathcal{C}^4$ in $\mathbb{R} \times [a,b]$ and 1-periodic in θ. Then, there exists $\kappa > 0$ depending on $b-a$ and α such that if

$$\|\varphi_1\|_{\mathcal{C}^4(\mathbb{S}_1 \times [a,b])} + \|\varphi_2\|_{\mathcal{C}^4(\mathbb{S}_1 \times [a,b])} < \kappa,$$

then M has invariant curves.

Remark 2.3.14. *As noticed in [44, Thereom 2.11], the invariant curves mentioned in the previous result are directly associated with irrational numbers satisfying the Diophantine condition 2.3.1. Specifically, if the map M satisfies the conditions of Moser's Twist Map Theorem, then there exist an invariant curve Γ of M and a Diophantine number ω such that $M|_\Gamma$ is conjugated to the pure rotation of the circle R_ω. Furthermore, any irrational number within the image of the function $\alpha(r)$ satisfying the same Diophantine condition as ω leads to the existence of an invariant curve of M distinct of Γ.*

2.3.3 Invariant Curve Theorem

As one of the variations of *Moser's Twist Map Theorem*, the following theorem was presented by Kunze et al. [34], and it builds upon the results established by M. Herman in [29]. As a particular case of the Diophantine numbers, this result requires that the associated rotation number to be irrational of constant type. Additionally, we will notice that a small condition on the twist is allowed in this case, justifying the alternative name that this result can be referred to: Small Twist Map Theorem.

Theorem 2.3.15. *[34, Invariant Curve Theorem] Consider the map $P : \mathbb{R} \times [-6,-3] \to \mathbb{R}^2$ defined as*

$$P(u,v) = (u + \beta + \delta v + \delta F_1(u,v), v + \delta F_2(u,v)), \tag{2.17}$$

where F_1 and F_2 are both $\mathcal{C}^5$ functions defined in $\mathbb{R} \times [-6,-3]$, and δ lies within the interval $(0,2)$. It is also assumed that P satisfies the intersection property, it is one-to-one, and 1-periodic in u. Furthermore, let us assume that β is an irrational number of constant-type with Markoff constant Ω that fulfills the condition

$$\Omega \leqslant \delta \leqslant M\Omega, \tag{2.18}$$

for some fixed positive number M. Then, there exists a constant M^*, depending only on M, such that if

$$\|F_1\|_{\mathcal{C}^5(\mathbb{R}\times[-6,-3])} + \|F_2\|_{\mathcal{C}^5(\mathbb{R}\times[-6,-3])} \leqslant M^*, \tag{2.19}$$

then there exists a function $\gamma \in \mathcal{C}^3(\mathbb{S}_1)$ that parameterizes a closed curve $\Gamma = \{(u,\gamma(u)) : u \in \mathbb{S}_1\}$ which is invariant under P and for which $\rho\left(P|_\Gamma\right) = \beta$.

2.4 Periodic points for maps on the annulus

In this section, we present a specific condition for a twist map to have periodic points. We shall see that the existence of such points is guaranteed by a version of the Poincaré-Birkhoff Theorem, concerned with maps with the intersection property satisfying a twist condition on the boundaries.

Definition 2.4.1. *Let $f : \mathbb{S}_1 \times [a,b] \to \mathbb{S}_1 \times [a,b]$ be an area-preserving homeomorphism. We say that f leaves the boundaries of $\mathbb{S}_1 \times [a,b]$ invariant if $f(\mathbb{S}_1 \times \{a\}) = \mathbb{S}_1 \times \{a\}$ and $f(\mathbb{S}_1 \times \{b\}) = \mathbb{S}_1 \times \{b\}$.*

Definition 2.4.2. *Let $F = (F_\theta, F_r) : \mathbb{R} \times [a,b] \to \mathbb{R} \times [a,b]$ be a lift of $f : \mathbb{S}_1 \times [a,b] \to \mathbb{S}_1 \times [a,b]$. The map f satisfy a twist condition on the boundaries of $\mathbb{S}_1 \times [a,b]$ if either $F_\theta(u,a) > u$ and $F_\theta(u,b) < u$ or $F_\theta(u,b) > u$ and $F_\theta(u,a) < u$, for every $u \in \mathbb{R}$.*

Theorem 2.4.3. *[9, Poincaré Birkhoff Theorem] Suppose that $f : \mathbb{S}_1 \times [a,b] \to \mathbb{S}_1 \times [a,b]$ is an area-preserving homeomorphism that leaves the boundaries of $\mathbb{S}_1 \times [a,b]$ invariant and satisfies a twist condition on the boundaries. Then, f has at least two fixed points in $\mathbb{S}_1 \times [a,b]$.*

Recently, in [65], the Poincaré-Birkhoff Theorem was extended to maps with the intersection property.

Theorem 2.4.4. *[65, Theorem 1] Let $f : \mathbb{S}_1 \times [a,b] \to \mathbb{S}_1 \times [a,b]$ be a homeomorphism with the intersection property and satisfying the twist condition on the boundaries. Then, f has at least one fixed point in $\mathbb{S}_1 \times [a,b]$.*

The following theorem is an adaptation of the result presented in [18, Proof of Theorem 3].

Theorem 2.4.5. *Assume that $G : \mathbb{S}_1 \times [0,1] \to \mathbb{S}_1 \times [0,1]$ is a diffeomorphism of class $\mathcal{C}^2$ with the intersection property, leaving the boundaries of $\mathbb{S}_1 \times [0,1]$ invariant. Suppose that $\rho\left(G\big|_{\mathbb{S}_1 \times \{0\}}\right) = \omega_1$ and $\rho\left(G\big|_{\mathbb{S}_1 \times \{1\}}\right) = \omega_2$, with $\omega_1, \omega_2 \in \mathbb{R}\backslash\mathbb{Q}$, satisfying $\omega_2 > \omega_1$. Then, there exists $q^* \in \mathbb{N}$ such that, for each $q \in \mathbb{N}$ satisfying $q \geqslant q^*$, the map G has at least one periodic point of period q.*

Proof. From assumption, both $G\big|_{\mathbb{S}_1 \times \{0\}}$ and $G\big|_{\mathbb{S}_1 \times \{1\}}$ are $\mathcal{C}^2$-diffeomorphisms on the circle having irrational rotation numbers. From Denjoy's Theorem ([17]), there exists two orientation preserving homeomorphisms $h_1 : \mathbb{S}_1 \to \mathbb{S}_1$ and $h_2 : \mathbb{S}_1 \to \mathbb{S}_1$ that conjugate $G\big|_{\mathbb{S}_1 \times \{0\}}$ and $G\big|_{\mathbb{S}_1 \times \{1\}}$ to the pure rotations of the circle R_{ω_1} and R_{ω_2}, respectively, that is,

$$G\big|_{\mathbb{S}_1 \times \{0\}} = h_1 \circ R_{\omega_1} \circ h_1^{-1} \quad \text{and} \quad G\big|_{\mathbb{S}_1 \times \{1\}} = h_2 \circ R_{\omega_2} \circ h_2^{-1}.$$

Since the group of orientation preserving homeomorphisms of the circle is path connected, it follows that there exists a path of orientation preserving homeomorphisms $\rho_r : \mathbb{S}_1 \to \mathbb{S}_1$, with $r \in [0,1]$, between h_1 and h_2 satisfying $\rho_0 = h_1$ and $\rho_1 = h_2$. We define the transformation

$$\begin{array}{rccc} \chi: & \mathbb{S}_1 \times [0,1] & \longrightarrow & \mathbb{S}_1 \times [0,1] \\ & (\theta, r) & \longmapsto & (\rho_r(\theta), r), \end{array}$$

which is a homeomorphism from $\mathbb{S}_1 \times [0,1]$ onto itself, as it is a continuous bijection between a compact and a Hausdorff space. Now, let us consider the conjugated map $\bar{G} = \chi^{-1} \circ G \circ \chi : \mathbb{S}_1 \times [0,1] \longrightarrow \mathbb{S}_1 \times [0,1]$. We notice that

$$\begin{aligned} \bar{G}(\theta,0) &= \chi^{-1} \circ G \circ \chi(\theta,0) = \chi^{-1} \circ G(h_1(\theta),0) = \chi^{-1}(h_1 \circ R_{\omega_1}(\theta),0) = (\theta+\omega_1,0), \\ \bar{G}(\theta,1) &= \chi^{-1} \circ G \circ \chi(\theta,1) = \chi^{-1} \circ G(h_2(\theta),1) = \chi^{-1}(h_2 \circ R_{\omega_2}(\theta),1) = (\theta+\omega_2,1), \end{aligned}$$

with $\omega_1, \omega_2 \in \mathbb{R}\backslash\mathbb{Q}$ satisfying $\omega_2 - \omega_1 > 0$. This implies that there exists $q^* \in \mathbb{N}$ such that, for every $q \geqslant q^*$, $q(\omega_2 - \omega_1) > 1$. Let us fix $q \geqslant q^*$. Then, by denoting $\lfloor q\omega_1 \rfloor$ as the integer part of $q\omega_1$, we notice that $(q\omega_1 - \lfloor q\omega_1 \rfloor, q\omega_2 - \lfloor q\omega_1 \rfloor)$ is a positive interval, satisfying $(q\omega_1 - \lfloor q\omega_1 \rfloor, q\omega_2 - \lfloor q\omega_1 \rfloor) \cap \mathbb{Z} \neq \varnothing$. Take $p \in (q\omega_1 - \lfloor q\omega_1 \rfloor, q\omega_2 - \lfloor q\omega_1 \rfloor) \cap \mathbb{Z}$ and define the induced function $\bar{G}_q := s^{-p-\lfloor q\omega_1 \rfloor} \circ \bar{G}^q$, where

$$\begin{array}{rccc} s: & \mathbb{S}_1 \times [0,1] & \longrightarrow & \mathbb{S}_1 \times [0,1] \\ & (u_1, u_2) & \longmapsto & (u_1+1, u_2). \end{array}$$

The map $\bar{G}_q$ leaves the boundary of $\mathbb{S}_1 \times [0,1]$ invariant. Indeed,

$$\bar{G}_q(u_1,0) = s^{-p-\lfloor q\omega_1 \rfloor} \circ \bar{G}^q(u_1,0) = (u_1 + q\omega_1 - p - \lfloor q\omega_1 \rfloor, 0),$$

and

$$\bar{G}_q(u_1,1) = s^{-p-\lfloor q\omega_1 \rfloor} \circ \bar{G}^q(u_1,1) = (u_1 + q\omega_2 - p - \lfloor q\omega_1 \rfloor, 1).$$

Moreover, from equations above and taking $\bar{g}_{1,q}$ and $\bar{g}_{2,q}$ to be the components of $\bar{G}_q$, we have that

$$\bar{g}_{1,q}(u_1,0) - u_1 = -p + q\omega_1 - \lfloor q\omega_1 \rfloor < 0 < -p + q\omega_2 - \lfloor q\omega_1 \rfloor = \bar{g}_{1,q}(u_1,1) - u_1,$$

because $q\omega_1 - \lfloor q\omega_1 \rfloor < p < q\omega_2 - \lfloor q\omega_1 \rfloor$. This implies that $\bar{G}_q$ is satisfying the twist condition on the boundaries of $\mathbb{S}_1 \times [0,1]$. Moreover, the map $\bar{G}_q$ has the intersection property, since it is conjugated with G (Proposition 2.3.12). Thus, from Theorem 2.4.4, it follows that $\bar{G}_q$ has at least one fixed point, which is associated with a periodic point of G of period q. This concludes the proof of the theorem. □

3 Analysis of the Case $\mathcal{A}_1$

In this chapter, our focus is to study the existence of invariant tori and boundedness of all solutions for the subclass of differential equation

$$\ddot{x} - \operatorname{sgn}(x) = \theta x + \varepsilon p(t), \tag{3.1}$$

where $\theta < 0$ and $p(t)$ is a σ-periodic function of class $\mathcal{C}^6$. This equation corresponds to the particular case $\mathcal{A}_1$ of (1.3).

As previously mentioned, a similar case of equation (3.1) was considered in [34], where, under the same assumptions for $p(t)$, the authors showed that all solutions of

$$\ddot{x} + x + \operatorname{sgn}(x) = \varepsilon p(t), \tag{3.2}$$

are bounded when ε is sufficiently small. The equation (3.2), when non-perturbed, features two virtual linear centers, in contrast to the two admissible linear centers for the unperturbed equation $(3.1)_{\varepsilon=0}$. In both equations, (3.1) and (3.2), a symmetric behavior can be observed in the unperturbed scenario (see Figure 4). By taking $p \in \mathcal{C}^4(\mathbb{S}_1)$, the boundedness of solutions is also investigated in [53] for the forced asymmetric oscillator

$$\ddot{x} + ax^+ - bx^- = 1 + p(t),$$

where $x^\pm := \max\{\pm x, 0\}$, and a and b are distinct positive real numbers. Thus, the investigation presented in this chapter complements the findings from [34] and [53].

It is worthy mentioning that unbounded solutions arise in (3.2) under a Landesman-Lazer-like condition on $p(t)$. This condition is explored in [33], where a differential inclusion analysis demonstrates that all solutions of (3.2) become unbounded if the first Fourier coefficient of $p(t)$ satisfies

$$\varepsilon \left| \int_0^\sigma p(t) e^{\frac{\pi i t}{\sigma}} \mathrm{d}t \right| > 4\sigma.$$

We believe that a similar analysis can be conducted on (3.1) in order to determine unbounded solutions.

This chapter is structured as follows: Section 3.1 introduces our main result (Theorem A) and the corollaries resulting from it. Section (3.2) is devoted to present some preliminary results concerning the unperturbed equation $(3.1)_{\varepsilon=0}$. In Sections 3.3 and 3.4, we provide some coordinate changes in order to fit our problem into the conditions needed to apply the Invariant Curve Theorem. The proof of Theorem A is done in Section 3.5. Finally, we devote Section 3.6 to present the proofs of the technical lemmas mentioned in the preceding sections.

3.1 Main results

The main result of this chapter stands for the existence of a family of nested invariant tori of (3.1) whose union of their interiors cover all the extended phase space $\mathbb{S}_\sigma \times \mathbb{R}^2$, as follows.

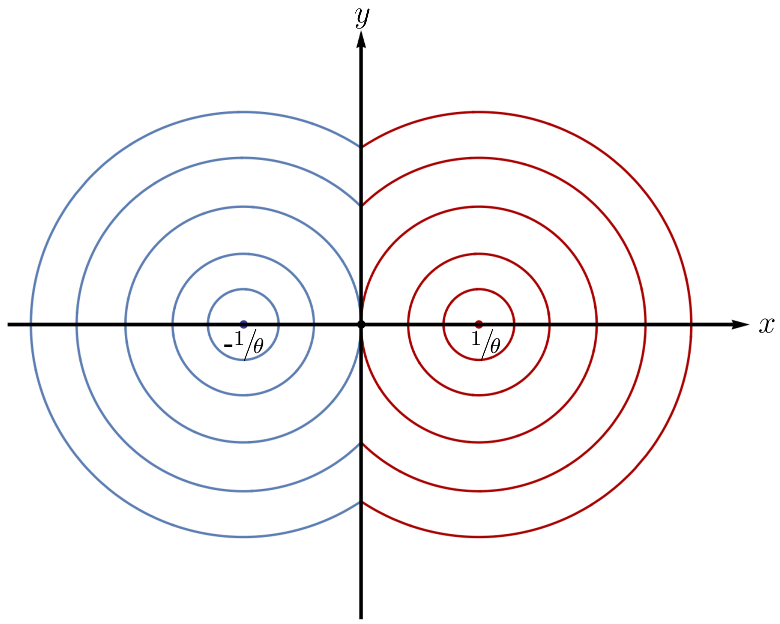


Figure 6 – Trajectories of (3.1) when $\varepsilon = 0$.

Theorem A. *For each $p \in \mathcal{C}^6(\mathbb{S}_\sigma)$ and $\theta < 0$, there exists $\varepsilon^*_{(p,\theta)} > 0$ such that, if $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$ then there exists an infinity collection $\{\mathcal{T}^i_\varepsilon\}_{i\in\mathbb{N}}$ of nested invariant tori of* (3.1)*, carrying quasi-periodic motion and satisfying*

$$\bigcup_{i\in\mathbb{N}} \mathrm{int}(\mathcal{T}^i_\varepsilon) = \mathbb{S}_\sigma \times \mathbb{R}^2,$$

where $\mathrm{int}(\mathcal{T}^i_\varepsilon)$ corresponds to the open region in $\mathbb{S}_\sigma \times \mathbb{R}^2$ enclosed by the torus $\mathcal{T}^i_\varepsilon$.

The proof of Theorem A primarily relies on the techniques and results presented in [34].

As can be noticed in the unperturbed scenario, the trajectories of $(3.1)_{\varepsilon=0}$ are either singular points or periodic orbits (see Figure 6). This implies that every solution of (3.1) is bounded when $\varepsilon = 0$. From Theorem A, it follows that, even under regularly small periodic perturbations, the boundedness property of the solutions of (3.1) persists. In fact, given $p \in \mathcal{C}^6(\mathbb{S}_\sigma)$ and $\theta < 0$, there exists $\varepsilon^*_{(p,\theta)} > 0$ such that for each $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$ there exists a sequence of nested invariant tori $\{\mathcal{T}^i_\varepsilon\}_{i\in\mathbb{N}}$ of (3.1) whose union of their interiors cover all the phase space $\mathbb{S}_\sigma \times \mathbb{R}^2$. Consequently, for any initial condition $(0, x(0), \dot{x}(0)) \in \mathbb{S}_\sigma \times \mathbb{R}^2$, there exists $k \in \mathbb{N}$ such that $(0, x(0), \dot{x}(0)) \in \overline{\mathrm{int}(\mathcal{T}^{k+1}_\varepsilon)\backslash\mathrm{int}(\mathcal{T}^k_\varepsilon)}$.

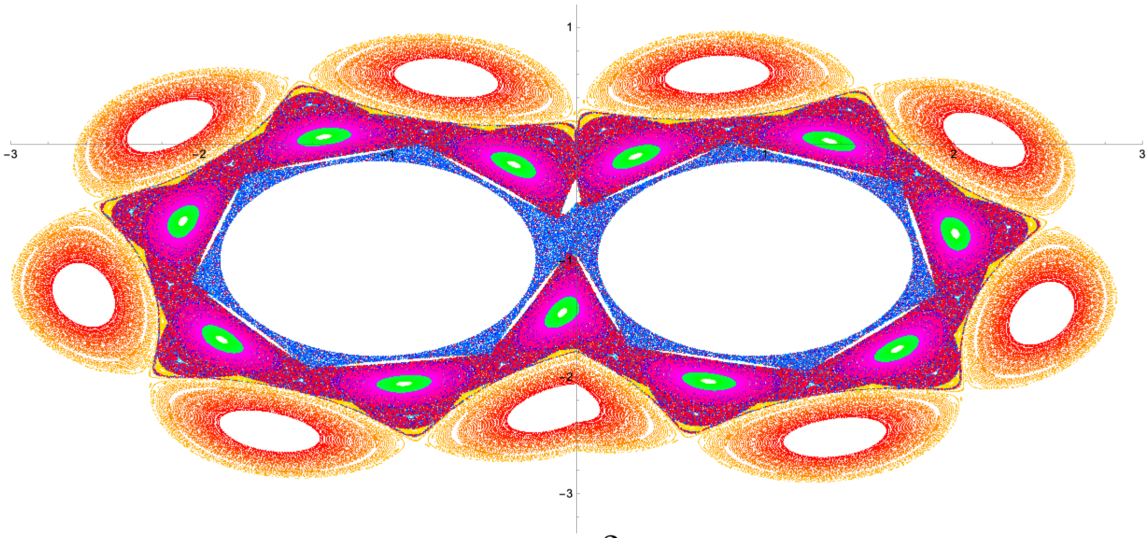
Since $\overline{\text{int}(\mathcal{T}_\varepsilon^{k+1})\backslash\text{int}(\mathcal{T}_\varepsilon^{k})}$ is invariant under the flow of (3.1), it follows that the trajectory whose initial condition is $(0, x(0), \dot{x}(0))$ must perpetually remain in $\overline{\text{int}(\mathcal{T}_\varepsilon^{k+1})\backslash\text{int}(\mathcal{T}_\varepsilon^{k})}$, which, in turn, is a bounded set of $\mathbb{S}_\sigma \times \mathbb{R}^2$. This leads us to the following result.

Corollary 3.1.1. *For every $p \in \mathcal{C}^6(\mathbb{S}_\sigma)$ and $\theta < 0$, there exists $\varepsilon^*_{(p,\theta)} > 0$ such that if $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$, then all solutions of (3.1) are bounded.*

By using Theorems A and 2.4.5, we get the existence of infinitely many periodic orbits of (3.1). This result is stated below, and its proof is deferred to Section 3.5.

Theorem B. *For any $p \in \mathcal{C}^6(\mathbb{S}_\sigma)$ and $\theta > 0$, there exists an $\varepsilon^*_{(p,\theta)} > 0$ such that equation (3.1) has infinitely many periodic solutions whenever $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$.*

In order to illustrate the main result of this chapter (Theorem A), we present some numerical simulations concerning the solutions of the differential equation (3.1). Specifically, assuming $\theta = -1$ and $p(t) = \cos(2\pi t) + 3\sin(2\pi t)$, and choosing a specific value for ε in (3.1), we consider several initial conditions for the differential equation (3.1). Subsequently, we plot 1000 points for each of them on the time section $(\{t = 1\} \times \mathbb{R}^2)$, as $p(t)$ is 1-periodic in this case.



$\boldsymbol{\varepsilon = 2}$

Figure 7 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (3.1), is indicated by the concentration of colors in the figure.

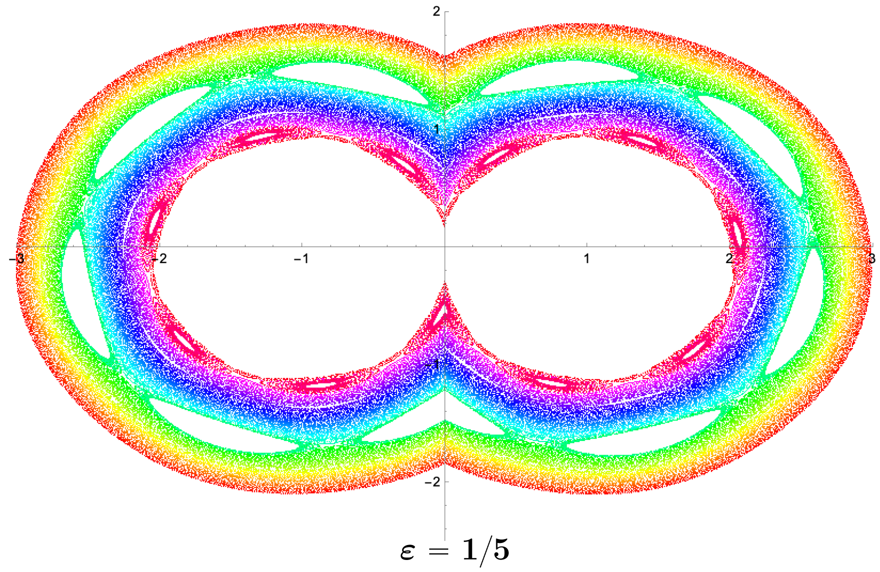


Figure 8 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (3.1), is indicated by the concentration of colors in the figure.

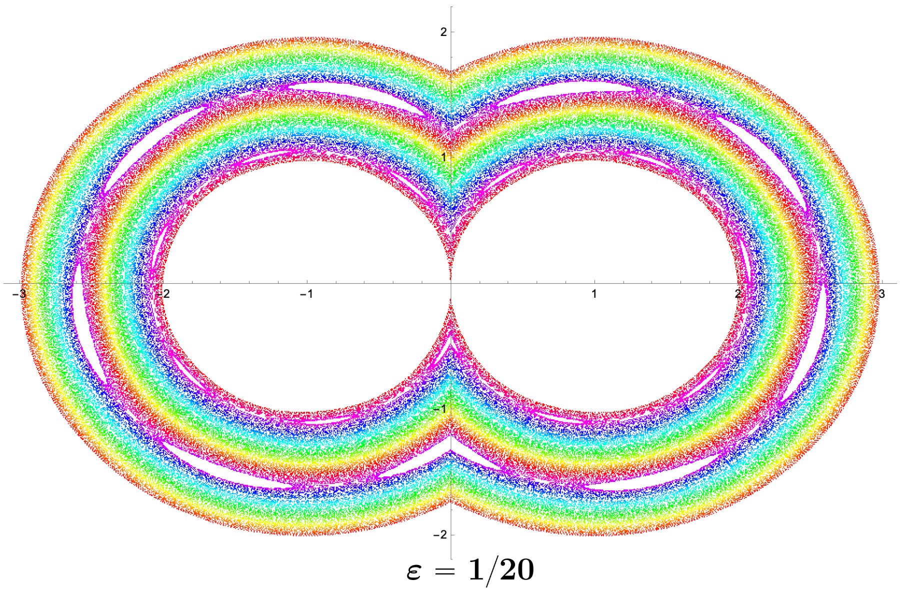


Figure 9 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (3.1), is indicated by the concentration of colors in the figure.

3.2 Preliminary results

This section is dedicated to the study of some objects related to the unperturbed equation $(3.1)_{\varepsilon=0}$. Such objects play a crucial role in constructing the transformations needed to fit our problem into the conditions of the Invariant Curve Theorem.

For the sake of simplicity, we are assuming that $\theta > 0$ and changing its sign at the corresponding system, that is,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \operatorname{sgn}(x) - \theta x + \varepsilon\, p(t). \end{cases} \tag{3.3}$$

Additionally, since θ remains constant, we will omit its dependence on the subsequent elements. In this configuration, we remind that (3.3) is Hamiltonian with

$$H(x, y, t; \varepsilon) = H_{\mathcal{A}_1}(x, y, t; \varepsilon) = \frac{y^2}{2} + G(x) - \varepsilon x p(t) \tag{3.4}$$

where $G(x) = G_{\mathcal{A}_1}(x) = -|x| + \theta\dfrac{x^2}{2}$. We notice that $H(x, y, t; \varepsilon)$ is continuous but not smooth on the plane $\{x = 0\}$. By taking $\varepsilon = 0$, we notice that the Hamiltonian H becomes an one degree of freedom piecewise smooth Hamiltonian to be denoted $\mathbf{H}(x, y) = H(x, y, t; 0)$. Thus, in the unperturbed scenario, the trajectories of $(3.3)_{\varepsilon=0}$ lie in the level curves $\mathcal{C}_h := \{(x, y) \in \mathbb{R}^2 : \mathbf{H}(x, y) = h\}$, for $h > -1/(2\theta)$ (see Figure 6).

The points resulting from the intersection between $\mathcal{C}_h$ and the x-axis are given by $\{-a^-(h), -a^+(h), a^-(h), a^+(h)\}$, where $a^{\pm}(h) = (1 \pm \sqrt{2h\theta + 1})/\theta$. In order to compute the period of a trajectory that lies in the level curve $\mathcal{C}_h$, we consider the value

$$a(h) = a^+(h) = \frac{1 + \sqrt{2h\theta + 1}}{\theta},$$

for $h > 0$, which corresponds to the maximum value among $\{-a^-(h), -a^+(h), a^-(h), a^+(h)\}$, for $h > 0$.

By selecting the values $h > 0$, we effectively eliminate the possibility of level curves for $\mathbf{H}$ being completely situated within the half-planes $x \geqslant 0$ and $x \leqslant 0$ (see Figure 11). This scenario can be more precisely outlined by considering the complement in $\mathbb{R}^2$ of the region

$$\mathcal{D} := \{(x, y) \in \mathbb{R}^2 : -1/(2\theta) \leqslant \mathbf{H}(x, y) \leqslant 0\}. \tag{3.5}$$

Remark 3.2.1. *It is important to mention that the intersections between the invariant tori provided by Theorem A and the section time $\{t = 0\}$ must be closed curves contained in the set $\{(0, x, y) : (x, y) \in \mathbb{R}^2 \backslash \mathcal{D}\}$. This is fact will be clarified along the proof of Theorem A.*

Period function. Since $\mathbf{H}(-x, y) = \mathbf{H}(x, y)$ and $\mathbf{H}(x, -y) = \mathbf{H}(x, y)$ for every $(x, y) \in \mathbb{R}^2$, we notice that the energy curves of the Hamiltonian $\mathbf{H}(x, y)$ exhibits symmetry with respect to both x-axis and y-axis. Thus, as a consequence of the discussion provided in Section 2.2.1, we find that the period

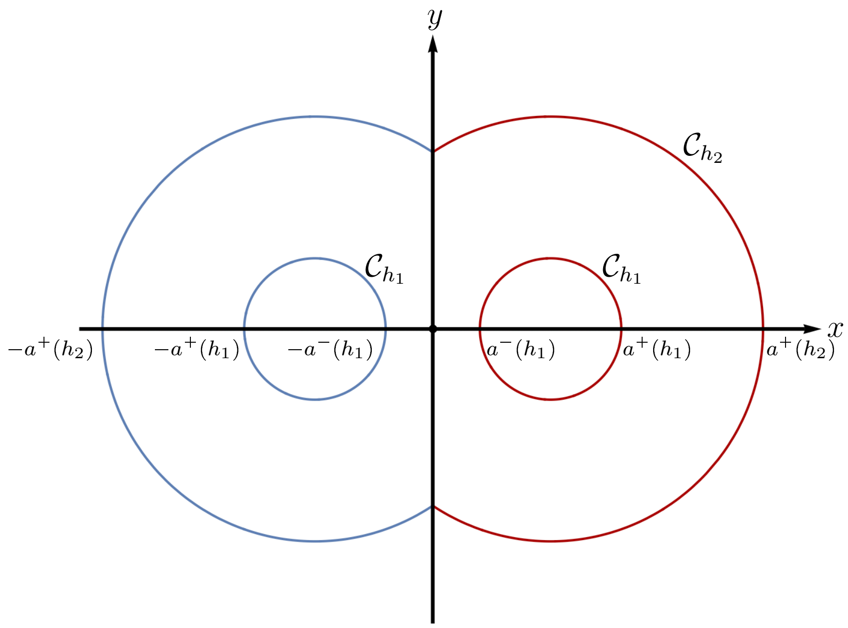


Figure 10 – Intersection between the level curves $\mathcal{C}_{h_1}$ and $\mathcal{C}_{h_2}$ with the $x-$axis. We notice that, if $-1/(2\theta) < h_1 < 0$, then the intersection results in four points, while for $h_2 \in (0, \infty)$, then the intersection must consist in only two points, since the other two are virtual.

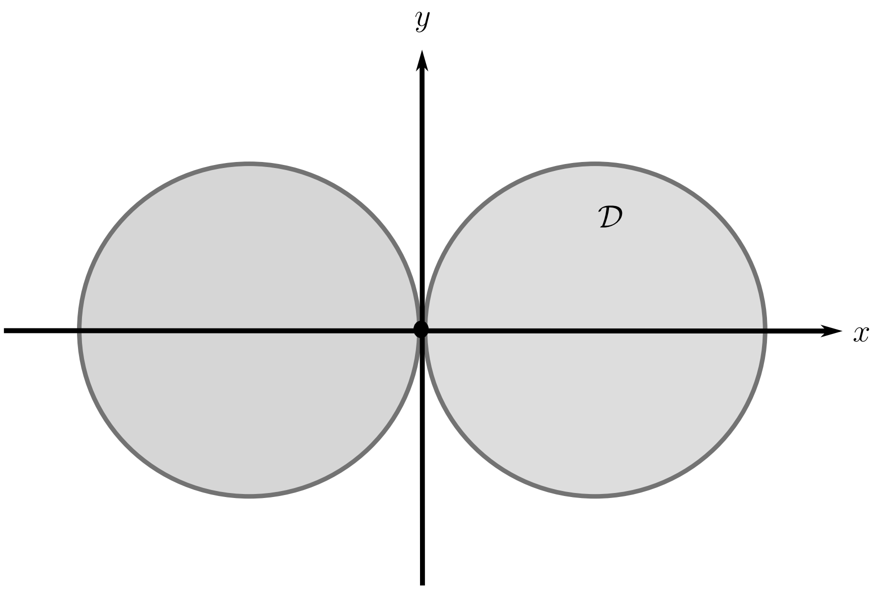


Figure 11 – The shaded area corresponds to the set $\mathcal{D}$.

of a solution lying on the level curve $\mathcal{C}_h$, for $h > 0$, is given by

$$\begin{aligned} T(h) &= 4\int_0^{a(h)} \frac{\mathrm{d}u}{\sqrt{2(G(a(h)) - G(u))}} \\ &= \frac{4}{\sqrt{\theta}}\left(\frac{\pi}{2} + \arcsin\left(\frac{1}{\sqrt{2\theta h + 1}}\right)\right) \\ &= \frac{4}{\sqrt{\theta}}\left(\arccos\left(-\frac{1}{\sqrt{2\theta h + 1}}\right)\right), \end{aligned} \tag{3.6}$$

where we are using the identity $\pi/2 + \arcsin(x) = \arccos(-x)$.

In the next result, we provide some properties of the period function T and we postpone their proofs to Section 3.6.

Lemma 3.2.2. *Let $T : (0, \infty) \to (2\pi/\sqrt{\theta}, 4\pi/\sqrt{\theta})$ be the period function given by (3.6). Then, T is a strictly decreasing smooth function and its inverse is given by*

$$T^{-1}(\rho) = \frac{1}{2\theta}\tan^2\left(\frac{\sqrt{\theta}\rho}{4}\right).$$

In addition, for $h > 0$ sufficiently large and $|2\pi - \sqrt{\theta}\rho|$ sufficiently small, we have

$$c_i h^{-1/2-i} \leqslant |D^i T(h)| \leqslant C_i h^{-1/2-i}, \quad i \geqslant 1, \tag{3.7}$$

$$c_i|2\pi - \sqrt{\theta}\rho|^{-2} \leqslant T^{-1}(\rho) \leqslant C_i|2\pi - \sqrt{\theta}\rho|^{-2}, \tag{3.8}$$

$$c_i|2\pi - \sqrt{\theta}\rho|^{-(i+2)} \leqslant |D^i T^{-1}(\rho)| \leqslant C_i|2\pi - \sqrt{\theta}\rho|^{-(i+2)}, \quad i \geqslant 1, \tag{3.9}$$

where c_i and C_i are universal positive constants depending on θ , and D^i denotes the i-th total derivative of the corresponding function.

Remark 3.2.3. *Let us consider a positive geometric sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with ratio strictly smaller than $1/2$. By defining the numbers $b_n^+ := T^{-1}(2\pi/\sqrt{\theta} + \lambda_n/2)$ and $b_n^- := T^{-1}(2\pi/\sqrt{\theta} + \lambda_n)$, we notice that, for each $n \in \mathbb{N}$, the interval $\mathcal{I}_n := [b_n^-, b_n^+]$ is well defined and satisfies the properties*

(i) $b_n^-, b_n^+, b_n^+ - b_n^- \underset{n\to+\infty}{\longrightarrow} +\infty$;

(ii) *For each $n \in \mathbb{N}$, $\mathcal{I}_n \cap \mathcal{I}_{n+1} = \varnothing$.*

These properties essentially ensure that the intervals $\mathcal{I}_n$ are pairwise disjoint and their lengths increase as n approaches to infinity.

3.3 Action-angle transformation

For the construction of the so-called *action-angle* variables, we adhere to the procedure discussed in Section 2.2.2. Then, by noticing that the level curves of $\mathbf{H}(x, y)$ are piecewise smooth, we partition the process of obtaining the generating function into zones where this approach is applicable. Hence, the **angle-function** is given by

$$\phi(x,h) = \begin{cases} \phi_1(x,h) & \text{if} \quad x, y \geqslant 0, \\ \pi - \phi_1(x,h) & \text{if} \quad x \geqslant 0,\ y \leqslant 0, \\ \pi + \phi_1(-x,h) & \text{if} \quad x, y \leqslant 0, \\ 2\pi - \phi_1(-x,h) & \text{if} \quad x \leqslant 0, y \geqslant 0, \end{cases}$$

where

$$\phi_1(x,h) = \frac{2\pi}{T(h)}\int_0^x \frac{\mathrm{d}u}{\sqrt{2(G(a(h))-G(u))}}$$
$$= \frac{2\pi}{\sqrt{\theta}T(h)}\left(\arcsin\left(\frac{1}{\sqrt{2h\theta+1}}\right)+\arcsin\left(\frac{\theta x-1}{\sqrt{2h\theta+1}}\right)\right),$$

as given in (2.13). The **action-function** is then given by

$$\begin{aligned} A(h) :=& 4\int_0^{a(h)} \sqrt{2h-2x-\theta x^2}\mathrm{d}x \\ =& \frac{\pi+2\pi h\theta+2\sqrt{2h\theta}+2(1+2h\theta)\operatorname{arccsc}(\sqrt{1+2h\theta})}{\theta^{3/2}}. \end{aligned} \tag{3.10}$$

The value $A(h)$ corresponds to the area of the region enclosed by the level curve $\mathcal{C}_h$, while $\phi(x,h)$ is bijectively equivalent to the angle formed between the positive part of the y-axis and the line connecting the origin with the point (x,y) that lies on the level curve $\mathcal{C}_h$, as we can see in the Figure 12.

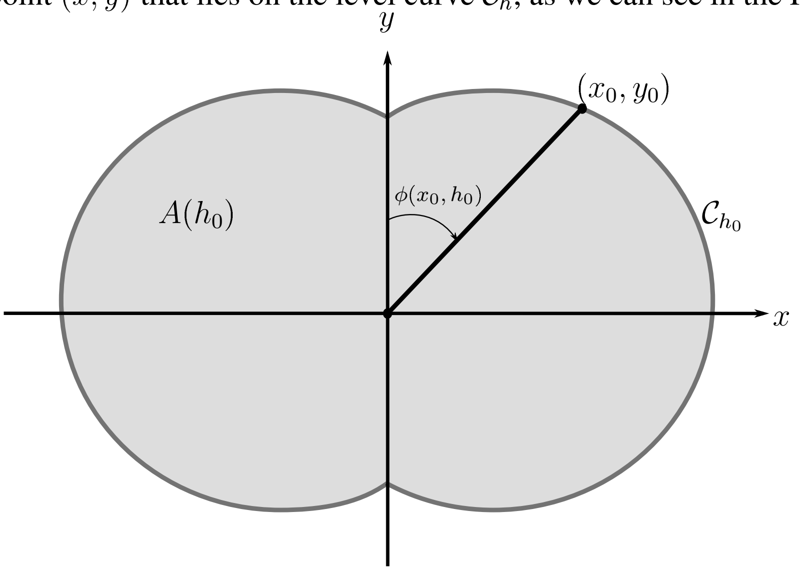


Figure 12 – The shaded area enclosed by the level curve $\mathcal{C}_{h_0}$ is represented by the value $A(h_0)$. The value $\phi(x_0,h_0)$ uniquely corresponds to the angle between the positive y-axis and the line originating at $(0,0)$ and passing through (x_0,y_0).

Lemma 3.3.1. *Let us consider the functions* $A : (0,+\infty) \to (2\pi/\theta^{3/2}, +\infty)$ *and* $T : (0,+\infty) \to (2\pi/\sqrt{\theta}, 4\pi/\sqrt{\theta})$ *defined in* (3.10) *and* (3.6)*, respectively. Then,* A *is an invertible smooth function and satisfies* $A'(h) = T(h)$ *for* $h \in (0,+\infty)$*. Moreover, if we denote by* $h_0 : (2\pi/\theta^{3/2}, +\infty) \to (0,+\infty)$ *the inverse of* A*, then* h_0 *is smooth, and for* h *and* $I = A(h)$ *sufficiently large, the following estimates holds*

$$c_0 h \leqslant A(h) \leqslant C_0 h, \quad c_1 \leqslant A'(h) \leqslant C_1, \quad c_i h^{1/2-i} \leqslant D^i A(h) \leqslant C_i h^{1/2-i} \quad i \geqslant 2, \tag{3.11}$$

$$c_0 I \leqslant h_0(I) \leqslant C_0 I, \quad c_1 \leqslant h_0'(I) \leqslant C_1, \quad |D^i h_0(I)| \leqslant C_i I^{1/2-i} \quad i \geqslant 2. \tag{3.12}$$

Next lemma provides a homeomorphism between the complementary of $\mathcal{D}$ (see equation (3.5)) and the infinite cylinder $\mathbb{S}_{2\pi} \times (2\pi/\theta^{3/2}, +\infty)$.

Lemma 3.3.2. *Let $\mathbf{H}(x, y)$ be the unperturbed Hamiltonian associated with (3.4). The transformation $\Phi_1 : \mathbb{R}^2 \backslash \mathcal{D} \to \mathbb{S}_{2\pi} \times (2\pi/\theta^{3/2}, +\infty)$, defined by*

$$\Phi_1(x, y) = (\phi\,(x, \mathbf{H}(x, y))\,, A\,(\mathbf{H}(x, y))) = (\phi(x, h), A(h)),$$

is a homeomorphism. Furthermore, if $x(\phi, I)$ and $y(\phi, I)$ are the ones satisfying $\Phi_1^{-1}(\phi, I) = (x(\phi, I), y(\phi, I))$, then x is smooth in I and, for I sufficiently large, we have

$$|\partial_I^i x(\phi, I)| \leqslant C_i\, I^{1/2 - i} \quad \textit{for} \quad i \geqslant 0, \tag{3.13}$$

with $C_i > 0$ being constants.

As outlined in Section 2.2.3, the transformation Φ_1 applied to H does not change the Hamiltonian character of (3.3), leading us to a new Hamiltonian

$$\mathcal{H}(\phi, I, t; \varepsilon) = h_0(I) - \varepsilon x(\phi, I) p(t), \tag{3.14}$$

which is smooth in I, continuous and 2π-periodic in ϕ, and of class $\mathcal{C}^6$ and σ-periodic in t.

3.4 Angle and energy as new time and position

We notice that, by differentiating (3.14) with respect to I, while keeping ϕ and t fixed, we have

$$\partial_I \mathcal{H}(\phi, I, t; \varepsilon) = h_0'(I) - \varepsilon \partial_I x(\phi, I) p(t).$$

Then, taking into account (3.12) and (3.13), it follows that

$$\partial_I \mathcal{H}(\phi, I, t; \varepsilon) > 0,$$

for sufficiently large I and sufficiently small ε. This implies that, with ϕ and t fixed, $\mathcal{H}(\phi, \cdot, t; \varepsilon)$ is invertible for sufficiently large values of I. This leads us to the following transformation:

$$\Phi_2 : (\phi, I, t) \mapsto (\beta, r, \tau) := (t, \mathcal{H}(\phi, I, t; \varepsilon), \phi).$$

Thus, by performing Φ_2 to $\mathcal{H}(\phi, I, t; \varepsilon)$, we have the new Hamiltonian

$$\mathscr{H}(\beta, r, \tau; \varepsilon) = [\mathcal{H}(\tau, \cdot, \beta; \varepsilon)]^{-1}(r),$$

which, in turn, can be written in the form

$$\mathscr{H}(\beta, r, \tau; \varepsilon) = A(r) + \varepsilon\, \mathscr{H}_1(\beta, r, \tau; \varepsilon), \tag{3.15}$$

with $\mathscr{H}_1$ being defined implicitly by the formula above. It is worthy noting that r plays the role previously attributed to the energy h.

Lemma 3.4.1. *For r sufficiently large and sufficiently small ε, we have*

$$\left|\partial_\beta^i \partial_r^j \mathscr{H}_1(\beta, r, \tau; \varepsilon)\right| \leqslant C_{i,j} r^{1/2-j} \quad \text{for} \quad 0 \leqslant i+j \leqslant 6,$$

where $C_{i,j}$ are positive constants depending on $\|p\|_{C^6(\mathbb{S}_\sigma)}$ and θ.

We recall that $A'(r) = T(r)$ for all $r \in (0, +\infty)$. Thus, the Hamiltonian system originating from (3.15) is given by

$$\begin{cases} \dfrac{\mathrm{d}\beta}{\mathrm{d}\tau} = \dfrac{\partial \mathscr{H}}{\partial r}(\beta, r, \tau; \varepsilon) = T(r) + \varepsilon \partial_r \mathscr{H}_1(\beta, r, \tau; \varepsilon), \\ \dfrac{\mathrm{d}r}{\mathrm{d}\tau} = -\dfrac{\partial \mathscr{H}}{\partial \beta}(\beta, r, \tau; \varepsilon) = -\varepsilon \partial_\beta \mathscr{H}_1(\beta, r, \tau; \varepsilon). \end{cases} \tag{3.16}$$

The functions $\partial_r \mathscr{H}_1$ and $\partial_\beta \mathscr{H}_1$ are $\mathcal{C}^6$ and $\mathcal{C}^5$ function, respectively, in β, smooth in r, and continuous in τ. Furthermore, Peano's Theorem (see Theorem 2.1.1) ensures that, for each initial condition $(\beta_0, r_0, \tau_0) \in \mathbb{S}_\sigma \times \mathbb{R} \times \mathbb{S}_{2\pi}$, with r_0 sufficiently large, there exists a solution of (3.16) passing through (β_0, r_0, τ_0), as follows.

Lemma 3.4.2. *There exists $r^* > 0$ and a constant $C^* \in (0,1)$ such that if $r_0 \geqslant r^*$, then the solution with initial condition $(\beta(0, \beta_0, r_0; \varepsilon), r(0, \beta_0, r_0; \varepsilon)) = (\beta_0, r_0)$ of (3.16) is defined on the interval $[0, 2\pi]$ and satisfies*

$$[1 - C^*]\, r_0 \leqslant r(\tau) \leqslant [C^* + 1]\, r_0.$$

Proof. We appeal to the quantitative character provided by Peano's theorem (Theorem 2.1.1) to prove this lemma. We define $G(\tau, \beta, r; \varepsilon) = (T(r) + \varepsilon \partial_r \mathscr{H}_1(\beta, r, \tau; \varepsilon), -\varepsilon \partial_\beta \mathscr{H}_1(\beta, r, \tau; \varepsilon))$ as the function associated to the non-autonomous differential equation (3.16). For sufficiently large r, it follows from Lemma 3.4.1 that

$$\|G\| \leqslant |T(r) + \varepsilon \partial_r \mathscr{H}_1| + |\varepsilon \partial_\beta \mathscr{H}_1| \leqslant \frac{\frac{4\pi}{\sqrt{\theta}} r^{1/2} + \varepsilon C_{0,1} + \varepsilon C_{1,0} r^{1/2}}{r^{1/2}}.$$

By taking $a = 2\pi$ and $b = C^* r_0$, with $C^* \in (0,1)$, we define $R = R(0, \beta_0, r_0, a, b)$ as in (2.3). We have that G is continuous on R and , for sufficiently small ε, it satisfies

$$M := \sup_{(\tau, \beta, r) \in R} \{|G(\tau, \beta, r; \varepsilon)|\} \leqslant \frac{\frac{4\pi}{\sqrt{\theta}} \left[(1 + C^*)\, r_0\right]^{1/2} + C_{0,1} + C_{1,0} \left[(1 + C^*)\, r_0\right]^{1/2}}{\left((1 - C^*)\, r_0\right)^{1/2}}.$$

Consequently, we have

$$\frac{b}{M} = \frac{C^* r_0}{M} \geqslant \frac{C^* (1 - C^*)\, r_0^{3/2}}{\frac{4\pi}{\sqrt{\theta}} \left[(1 + C^*)\, r_0\right]^{1/2} + C_{0,1} + C_{1,0} \left[(1 + C^*)\, r_0\right]^{1/2}}.$$

We notice that $b/M \to +\infty$ as $r_0 \to +\infty$. Thus, there exists $r^* > 0$ such that $\inf\{b/M, a\} = 2\pi$ if $r_0 > r^*$, and this concludes the proof. $\square$

Let C^* be the constant provided by Lemma 3.4.2. There exists $n_1^* \in \mathbb{N}$ such that $b_n^+/(1+C^*) - b_n^-/(1-C^*) > 0$, for every $n \geqslant n_1^*$. Hence, we can define the intervals

$$\mathcal{J}_n := \left[\frac{b_n^-}{1-C^*}, \frac{b_n^+}{1+C^*} \right] \subset (r^*, +\infty) \tag{3.17}$$

for $n \geqslant n_1^*$. Notably, n_1^* can be chosen so that the properties provided in Remark 3.2.3 still hold for the intervals $\mathcal{J}_n$, and, then, we construct a sequence of coordinate changes between $\mathbb{S}_\sigma \times \mathcal{J}_n$ and $\mathbb{S}_1 \times [1/2, 1]$, for every $n \geqslant n_1^*$, in the following way

$$\Phi_{3,n} : (\beta, r) \longmapsto (\bar{\beta}, \bar{\rho}) := \left(\frac{\beta}{\sigma}, \frac{T(r) - 2\pi/\sqrt{\theta}}{\lambda_n} \right), \tag{3.18}$$

with $\{\lambda_n\}_{n\in\mathbb{N}}$ being the sequence introduced in Remark 3.2.3. Therefore, upon applying the coordinate changes $\Phi_{3,n}$ to the system of differential equations (3.16), we obtain

$$\begin{cases} \dfrac{\mathrm{d}\bar{\beta}}{\mathrm{d}\tau} = \dfrac{\lambda_n}{\sigma}\bar{\rho} + \dfrac{2\pi}{\sigma\sqrt{\theta}} + \varepsilon \bar{f}_{1,n}(\bar{\beta}, \bar{\rho}, \tau; \varepsilon), \\ \dfrac{\mathrm{d}\bar{\rho}}{\mathrm{d}\tau} = \varepsilon \bar{f}_{2,n}(\bar{\beta}, \bar{\rho}, \tau; \varepsilon), \end{cases} \tag{3.19}$$

where we are introducing

$$\bar{f}_{1,n}(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) := \frac{1}{\sigma} \partial_r \mathscr{H}_1 \left(\sigma\bar{\beta}, T^{-1}\left(\lambda_n \bar{\rho} + \frac{2\pi}{\sqrt{\theta}} \right), \tau; \varepsilon \right),$$

and

$$\bar{f}_{2,n}(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) := -\lambda_n^{-1} T' \left(T^{-1}\left(\lambda_n \bar{\rho} + \frac{2\pi}{\sqrt{\theta}} \right) \right) \partial_\beta \mathscr{H}_1 \left(\sigma\bar{\beta}, T^{-1}\left(\lambda_n \bar{\rho} + \frac{2\pi}{\sqrt{\theta}} \right), \tau; \varepsilon \right).$$

We can deduce that both $\bar{f}_{1,n}$ and $\bar{f}_{2,n}$ are of class $\mathcal{C}^5$ with respect to $\bar{\beta}$, while they are smooth with respect to $\bar{\rho}$. Furthermore, they exhibit a 1-periodicity in terms of $\bar{\beta}$ and a 2π-periodicity in the variable τ. The subsequent lemma provide the bounds for such functions.

Lemma 3.4.3. *There exists a constant $C > 0$ depending only on $\|p\|_{C^6(\mathbb{S}_\sigma)}$ and θ, but not on n, and a natural $n_2^* \in \mathbb{N}$ such that, for sufficiently small ε,*

$$\left\| \bar{f}_{1,n}(\cdot, \cdot, \tau; \varepsilon) \right\|_{\mathcal{C}^5\left(\mathbb{R}\times\left[\frac{1}{2},1\right]\right)} + \left\| \bar{f}_{2,n}(\cdot, \cdot, \tau; \varepsilon) \right\|_{\mathcal{C}^5\left(\mathbb{R}\times\left[\frac{1}{2},1\right]\right)} \leqslant C\lambda_n, \tag{3.20}$$

for every $n \geqslant n_2^*$.

3.4.1 Time-2π-map

Let us consider $(\bar{\beta}(\tau;\varepsilon), \bar{\rho}(\tau;\varepsilon)) = (\bar{\beta}(\tau, \bar{\beta}_0, \bar{\rho}_0; \varepsilon), \bar{\rho}(\tau, \bar{\beta}_0, \bar{\rho}_0; \varepsilon))$ as the solution of (3.19) whose initial conditions is $(\bar{\beta}(0;\varepsilon), \bar{\rho}(0;\varepsilon)) = (\bar{\beta}_0, \bar{\rho}_0)$. For the sake of simplicity in the following notations, we will omit the dependence of λ_n, $\bar{f}_{1,n}$, and $\bar{f}_{2,n}$ on n until explicitly needed. The integral equation for the solutions of a differential equation provides us

$$\bar{\beta}(\tau;\varepsilon) = \bar{\beta}_0 + \int_0^\tau \left(\frac{\lambda}{\sigma}\bar{\rho}(s;\varepsilon) + \frac{2\pi}{\sigma\sqrt{\theta}} + \varepsilon \bar{f}_1(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\right) \mathrm{d}s,$$
$$\bar{\rho}(\tau;\varepsilon) = \bar{\rho}_0 + \int_0^\tau \left(\varepsilon \bar{f}_2(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\right) \mathrm{d}s.$$

Thus, for sufficiently small ε,

$$\begin{aligned}\bar{\beta}(\tau;\varepsilon) = \bar{\beta}_0 + \left(\frac{\lambda}{\sigma}\bar{\rho}_0 + \frac{2\pi}{\sigma\sqrt{\theta}}\right)\tau + \varepsilon \Bigg(\frac{\lambda}{\sigma}\int_0^\tau \int_0^s \bar{f}_2(\bar{\beta}(\xi;\varepsilon), \bar{\rho}(\xi;\varepsilon), \xi;\varepsilon)\mathrm{d}\xi \mathrm{d}s \\ + \int_0^\tau \bar{f}_1(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\mathrm{d}s\Bigg), \\ \bar{\rho}(\tau;\varepsilon) = \bar{\rho}_0 + \varepsilon \int_0^\tau \bar{f}_2(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\mathrm{d}s.\end{aligned} \tag{3.21}$$

As a consequence of Lemma 3.4.2, the solutions of (3.19) are defined in the interval $[0, 2\pi]$, which enables us to construct the Poincaré map (time-2π-map) associated with the system (3.19). This map consists in the function that takes $(\bar{\beta}_0, \bar{\rho}_0)$ as input and returns the point where the solution with $(\bar{\beta}_0, \bar{\rho}_0)$ as the initial condition will be at time $\tau = 2\pi$. Let us denote it by $\bar{P} : \mathbb{R} \times [1/2, 1] \to \mathbb{R} \times \mathbb{R}$ the Poincaré map associated with (3.19), then

$$\bar{P}(\bar{\beta}_0, \bar{\rho}_0;\varepsilon) = (\bar{\beta}(2\pi, \bar{\beta}_0, \bar{\rho}_0;\varepsilon), \bar{\rho}(2\pi, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)).$$

Based on the solutions expressed in (3.21), we can establish asymptotic expressions for $\bar{P}$, as follows

$$\bar{P}(\bar{\beta}_0, \bar{\rho}_0;\varepsilon) = (\bar{\beta}_0 + \bar{\alpha} + \bar{\lambda}\bar{\rho}_0 + \bar{\lambda}\varepsilon \bar{P}_1(\bar{\beta}_0, \bar{\rho}_0;\varepsilon), \bar{\rho}_0 + \bar{\lambda}\varepsilon \bar{P}_2(\bar{\beta}_0, \bar{\rho}_0;\varepsilon)), \tag{3.22}$$

with $\bar{\alpha} = 4\pi^2/(\sigma\sqrt{\theta})$ and $\bar{\lambda} = 2\pi\lambda/\sigma$, and $\bar{P}_1$ and $\bar{P}_2$ being expressed by

$$\bar{\lambda}\bar{P}_1(\bar{\beta}_0, \bar{\rho}_0;\varepsilon) = \frac{\bar{\lambda}^2}{2\pi}\int_0^{2\pi}\int_0^s \bar{f}_2(\bar{\beta}(\xi;\varepsilon), \bar{\rho}(\xi;\varepsilon), \xi;\varepsilon)\mathrm{d}\xi\mathrm{d}s + \int_0^{2\pi} \bar{f}_1(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\mathrm{d}s$$

and

$$\bar{\lambda}\bar{P}_2(\bar{\beta}_0, \bar{\rho}_0;\varepsilon) = \int_0^{2\pi} \bar{f}_2(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\mathrm{d}s,$$

respectively. We notice that $\bar{P}$ is 1-periodic with respect to $\bar{\beta}_0$ and it is of class $\mathcal{C}^5$ in the variables $(\bar{\beta}_0, \bar{\rho}_0)$. Additionally, we remind that $\bar{P}$, $\bar{P}_1$, $\bar{P}_2$, and $\bar{\lambda}$ depend on n. Next result provides a uniform bound for the functions $\bar{P}_{1,n}$ and $\bar{P}_{2,n}$, for sufficiently large n and sufficiently small ε.

Lemma 3.4.4. *There exists $n_3^* \in \mathbb{N}$ such that, for each $n \geqslant n_3^*$ and for sufficiently small ε, we have the bounds*

$$\left\|\bar{P}_{1,n}(\cdot;\varepsilon)\right\|_{\mathcal{C}^5(\mathbb{R}\times[1/2,1])} + \left\|\bar{P}_{2,n}(\cdot;\varepsilon)\right\|_{\mathcal{C}^5(\mathbb{R}\times[1/2,1])} \leqslant C,$$

with $C > 0$ depending on $p(\beta_0)$ and θ, but not on n.

We notice that (3.22) is not currently in the required format for the Invariant Curve Theorem, as the irrationality of $\bar{\alpha}$ as a number of constant type is unknown. In order to overcome this hindrance, we will conjugate (3.22) with a map that satisfies the conditions of the Invariant Curve Theorem, specially those related to irrational numbers of constant type. To this end, we provide a technical result that offers approximations of $\bar{\alpha}$ using irrational numbers of constant type along with their associated Markoff constants.

Lemma 3.4.5. *Let us consider $\bar{\alpha} = 4\pi^2/(\sigma\sqrt{\theta})$. There exists $n_4^* \in \mathbb{N}$ such that, for every natural $n \geqslant n_4^*$, there exist an irrational number of constant type, denoted as α_n, with Markoff constant Ω_n, satisfying*

$$4\bar{\lambda}_n \leqslant \alpha_n - \bar{\alpha} \leqslant \frac{13}{2}\bar{\lambda}_n, \tag{3.23}$$

and

$$\frac{\bar{\lambda}_n}{16} \leqslant \Omega_n \leqslant \frac{\bar{\lambda}_n}{4}. \tag{3.24}$$

Proof. Let us denote the fractional part of $\bar{\alpha} = 4\pi^2/(\sigma\sqrt{\theta})$ as $\alpha = \bar{\alpha} - \lfloor\bar{\alpha}\rfloor$. For $n \in \mathbb{N}$, we define the numbers $a_n := \alpha + 4\bar{\lambda}_n$ and $b_n := \alpha + \frac{13}{2}\bar{\lambda}_n$. We notice that there exists $n_4^* \in \mathbb{N}$ such that $[a_n, b_n] \subseteq [0,1]$ for every $n \geqslant n_4^*$. Then, from Proposition 2.3.5, it follows that there exists an irrational number $\alpha_n' \in [a_n, b_n]$ of constant-type with the corresponding Markoff constant Ω_n satisfying relation (3.24). As a consequence of Remark 2.3.2, we can deduce that for $n \in \mathbb{N}$ satisfying $n \geqslant n_4^*$, $\alpha_n = \alpha_n' + \lfloor\bar{\alpha}\rfloor$ is also an irrational number of constant type with the same Markoff constant Ω_n. Furthermore, from definition of α_n, a_n, and b_n, relationship (3.23) holds. □

Let us define $n^* := \max\{n_1^*, n_2^*, n_3^*, n_4^*\}$. Then, taking into account Lemma 3.4.5, we construct, for $n \geqslant n^*$, a sequence of transformations between $\mathbb{R} \times [1/2, 1]$ and $\mathbb{R} \times [-6, -3]$ in the following way

$$\Psi_n : (\bar{\beta}_0, \bar{\rho}_0) \longmapsto (u(\bar{\beta}_0, \bar{\rho}_0), v(\bar{\beta}_0, \bar{\rho}_0)) := \left(\bar{\beta}_0, \bar{\rho}_0 + \frac{\bar{\alpha} - \alpha_n}{\bar{\lambda}_n}\right). \tag{3.25}$$

By performing Ψ_n to $\bar{P}_n$ as given in (3.22), the conjugate map $\mathcal{P}_n = \Psi_n \circ \bar{P}_n \circ \Psi_n^{-1}$ is given by

$$\mathcal{P}_n(u, v; \varepsilon) = (u + \alpha_n + \bar{\lambda}_n v + \bar{\lambda}_n \varepsilon \mathcal{P}_{1,n}(u, v; \varepsilon), v + \bar{\lambda}_n \varepsilon \mathcal{P}_{2,n}(u, v; \varepsilon)), \tag{3.26}$$

where

$$\mathcal{P}_{j,n}(u, v; \varepsilon) = \bar{P}_{j,n}\left(u, v + \frac{\bar{\alpha}_n - \alpha}{\bar{\lambda}_n}; \varepsilon\right), \quad \text{for} \quad j = 1, 2.$$

Furthermore, under the same assumptions, the bounds presented in Lemma 3.4.4 naturally extend to the functions $\mathcal{P}_{1,n}$ and $\mathcal{P}_{2,n}$. In the following, we show that, for sufficiently large n, $\mathcal{P}_n$ satisfies the intersection property condition.

Proposition 3.4.6. *For each $n \geqslant n^* = \max\{n_1^*, n_2^*, n_3^*, n_4^*\}$, the map $\mathcal{P}_n$ has the intersection property.*

Proof. Let $\mathscr{P}$ be the Poincaré map corresponding to the Hamiltonian system (3.16). From Theorem 2.3.11, it follows that $\mathscr{P}$ has the intersection property. By considering the transformation $\tilde{\Psi}_n := \Psi_n \circ \Phi_{3,n}$, where $\Phi_{3,n}$ and Ψ_n are the transformations given in (3.18) and (3.25), respectively, we see that, for $n \geqslant n^*$, the maps $\mathscr{P}$ and $\mathcal{P}_n$ are conjugated. Then, according with Proposition 2.3.12, it follows that, for $n \geqslant n^*$, the map $\mathcal{P}_n$ also exhibits the intersection property. □

3.5 Proof of the main results

Proof of Theorem A. Let us consider the maps $\mathcal{P}_n$ obtained in (3.26). We are interested in applying the Invariant Curve Theorem to these maps, which are in the same form presented in (2.17).

Let us remind that $n^* = \max\{n_1^*, n_2^*, n_3^*, n_4^*\}$. Then, for each $n \geqslant n^*$, the irrational number α_n is of constant-type with corresponding Markoff constant Ω_n satisfying relationship (3.24). Moreover, if necessary, the natural n^* can be chosen sufficiently large such that $\bar{\lambda}_n = 2\pi\lambda_n/\sigma \in (0,2)$ for every $n \geqslant n^*$, satisfying the inequality

$$\Omega_n \leqslant \frac{\bar{\lambda}_n}{4} < \bar{\lambda}_n = 16\frac{\bar{\lambda}_n}{16} \leqslant 16\Omega_n.$$

Therefore, by taking $M = 16$, the relation (2.18) is satisfied. In addition, Proposition 3.4.6 guarantees that the maps $\mathcal{P}_n$ has the intersection property. Also, due to the uniqueness of solutions of (3.19), they are one-to-one maps having 1-periodicity in the variable u. This enables the application of the Invariant Curve Theorem to $\mathcal{P}_n$. Hence, the constant M^* in (2.19) can be selected independently of $n \geqslant n^*$. By defining $F_j = \varepsilon\mathcal{P}_{j,n}$ for $j = 1, 2$, and $\varepsilon^*_{(p,\theta)} = M^*/C$, where C is the constant depending on $\|p\|_{\mathcal{C}^5(\mathbb{S}_\sigma)}$ and θ as provided in Lemma 3.4.4, we can deduce that

$$\begin{aligned}\|F_1\|_{\mathcal{C}^5(\mathbb{R}\times[-6,-3])} + \|F_2\|_{\mathcal{C}^5(\mathbb{R}\times[-6,-3])} &= \varepsilon\left(\|\mathcal{P}_{1,n}(\cdot;\varepsilon)\|_{\mathcal{C}^5(\mathbb{R}\times[-6,-3])} + \|\mathcal{P}_{2,n}(\cdot;\varepsilon)\|_{\mathcal{C}^5(\mathbb{R}\times[-6,-3])}\right)\\ &\leqslant \varepsilon C.\end{aligned}$$

Thus, relationship (2.19) is satisfied for every $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$ and for every $n \geqslant n^*$. We can therefore conclude that, for every $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$ and for each $n \geqslant n^*$, the map $\mathcal{P}_n$ has an invariant closed curve, denoted by Γ_ε^n, carrying quasi-periodic motion, since the rotation number of $\mathcal{P}_n$ restricted to Γ_ε^n is the irrational α_n.

Upon transforming the system back to its original form, we notice that the curves corresponding to Γ_ε^n in the plane $(x,y) \in \mathbb{R}^2$ give rise to nested invariant tori of (3.1) in the extended phase space $\mathbb{S}_\sigma \times \mathbb{R}^2$. These tori, to be denoted $\bar{\mathcal{T}}_\varepsilon^i$, carry quasi-periodic solutions of (3.1). Finally, the sequence of nested invariant tori introduced in Theorem A is obtained by defining $\mathcal{T}_\varepsilon^i = \bar{\mathcal{T}}_\varepsilon^{i+n^*}$, for $i \in \mathbb{N}$. □

Proof of Theorem B. Let us consider the invariant curves Γ_ε^n of $\mathcal{P}_n$ provided in the proof of Theorem A along with their parametrizations $\gamma_\varepsilon^n \in \mathcal{C}^3(\mathbb{S}_1)$, for $n \geqslant n^*$ and $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$. By performing the inverse transformations given in (3.25) and (3.18) to $\mathcal{P}_n$, that is, $\tilde{\Psi}_n^{-1} = \Phi_{3,n}^{-1} \circ \Psi_n^{-1}$ (see Proof of Proposition 3.4.6) we get the Poincaré map $\mathscr{P}$ associated with (3.16) and defined on the sequence of disjoint annuli $\{\mathbb{S}_\sigma \times \mathcal{J}_n\}_{n \geqslant n^*} \subset (r^*, +\infty)$. Besides that, the invariant curve Γ_ε^n of $\mathcal{P}_n$ in $\mathbb{S}_1 \times [-6, -3]$ are transformed into an invariant curve $\tilde{\Gamma}_\varepsilon^n = \tilde{\Psi}_n^{-1}(\Gamma_\varepsilon^n)$ of $\mathscr{P}$ in the annulus $\mathbb{S}_\sigma \times \mathcal{J}_n$, for each $n \geqslant n^*$ and $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$. We denote the parametrizations of $\tilde{\Gamma}_\varepsilon^n$ by $\tilde{\gamma}_\varepsilon^n$, that is, $\tilde{\Gamma}_\varepsilon^n = \{(\beta, \tilde{\gamma}_\varepsilon^n(\beta)) : \beta \in \mathbb{S}_\sigma\}$.

Taking (3.26) into account, we notice that $\rho\left(\mathcal{P}_n\big|_{\Gamma_\varepsilon^n}\right) = \alpha_n$, with α_n being the irrational number of constant type provided in Lemma 3.4.5. This implies that $\rho\left(\mathscr{P}\big|_{\tilde{\Gamma}_\varepsilon^n}\right) = \sigma\alpha_n$. From the relation (3.23), it can be shown that, for each $i \geqslant n^*$ there exists $j > i$ such that $\alpha_i > \alpha_j$. These values are assumed to be fixed from now on.

Given that the annuli $\mathbb{S}_\sigma \times \mathcal{J}_i$ and $\mathbb{S}_\sigma \times \mathcal{J}_j$ are pairwise disjoint, we have that the curves $\tilde{\Gamma}_\varepsilon^i$ and $\tilde{\Gamma}_\varepsilon^j$ are also pairwise disjoint, implying that $\tilde{\gamma}_\varepsilon^i(\beta) \neq \tilde{\gamma}_\varepsilon^j(\beta)$, for every $\beta \in \mathbb{S}_\sigma$.

Now, let us denote by $\tilde{R}_{(i,j)}$ the invariant region of $\mathscr{P}$ comprised between the invariant curves $\tilde{\Gamma}_\varepsilon^i$ and $\tilde{\Gamma}_\varepsilon^j$. This region can be described by means of the parametrizations of $\tilde{\Gamma}_\varepsilon^i$ and $\tilde{\Gamma}_\varepsilon^j$ as follows

$$\tilde{R}_{(i,j)} = \{(\beta, r) \in \mathbb{S}_\sigma \times \mathbb{R} : \beta \in \mathbb{S}_\sigma \quad \text{and} \quad \tilde{\gamma}_\varepsilon^i(\beta) \leqslant r \leqslant \tilde{\gamma}_\varepsilon^j(\beta)\}.$$

By taking into account the diffeomorphism

$$\begin{array}{rccc} \tilde{\chi}: & \tilde{R}_{(i,j)} & \longrightarrow & \mathbb{S}_1 \times [0,1] \\ & (\beta, r) & \longmapsto & \left(\dfrac{\beta}{\sigma}, \dfrac{r - \tilde{\gamma}_i(\beta)}{\tilde{\gamma}_j(\beta) - \tilde{\gamma}_i(\beta)}\right), \end{array}$$

we notice that the conjugated map $G = \tilde{\chi} \circ \mathscr{P} \circ \tilde{\chi}^{-1} : \mathbb{S}_1 \times [0,1] \to \mathbb{S}_1 \times [0,1]$ matches all the conditions of Theorem 2.4.5. Indeed, since G is conjugated with $\mathscr{P}$, it has the intersection property. Besides that, it leaves the boundaries of $\mathbb{S}_1 \times [0,1]$ invariant and satisfies $\rho\left(G\big|_{\mathbb{S}_1 \times \{0\}}\right) = \alpha_i$ and $\rho\left(G\big|_{\mathbb{S}_1 \times \{1\}}\right) = \alpha_j$, with $\alpha_i, \alpha_j \in \mathbb{R}\backslash\mathbb{Q}$ satisfying $\alpha_i - \alpha_j > 0$. Thus, there exists $q^* \in \mathbb{N}$ such that for each $q \in \mathbb{N}$ satisfying $q \geqslant q^*$, the map G has at least one periodic point of period q. This implies the existence of periodic solutions for (3.19) and, consequently, for (3.1), whenever $0 \leqslant \varepsilon < \varepsilon^*_{(p,\theta)}$.

By recursively applying this procedure to the sequence of invariant curves $\{\Gamma_\varepsilon^n\}_{n \geqslant j}$, we conclude the proof of the corollary. □

3.6 Proofs of the technical results

In this section, we are concerned with the proofs of the technical results previously presented throughout the corresponding chapter, particularly those related to estimating functions. Before we proceed with those proofs, we introduce an auxiliary lemma concerning higher-order derivatives of composite functions (see, for instance, [1, 46] for more details).

Lemma 3.6.1. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ be sufficiently differentiable functions. Then, for $i + j \geqslant 1$, we have*

$$\partial_x^i \partial_y^j [F \circ (f, g)] = \sum_{\substack{(k,p) \in \mathbb{N}_0^2:\ 1 \leqslant k+p \leqslant i+j, \\ \vec{i}=(i_1,\cdots,i_{k+p}),\ |\vec{i}|=i, \\ \vec{j}=(j_1,\cdots,j_{k+p}),\ |\vec{j}|=j}} C_{k,p,\vec{i},\vec{j}} (\partial_1^k \partial_2^p F(f,g)) \Big[(\partial_x^{i_1} \partial_y^{j_1} f) \ldots (\partial_x^{i_k} \partial_y^{j_k} f)(\partial_x^{i_{k+1}} \partial_y^{j_{k+1}} g) \ldots (\partial_x^{i_{k+p}} \partial_y^{j_{k+p}} g) \Big],$$

where the coefficients $C_{k,p,\vec{i},\vec{j}}$ are integers and satisfy $C_{k,p,\vec{i},\vec{j}} = 0$ if $i_l = j_l = 0$ for $1 \leqslant l \leqslant k + p$, meaning that when the l-th entries of vectors $\vec{i}$ and $\vec{j}$ are both zero, the corresponding coefficients will also be zero.

In particular, if $F : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently differentiable functions, then

$$\partial_x^i \partial_y^j [F \circ f] = \sum_{\substack{1 \leqslant k \leqslant i+j, \\ \vec{i}=(i_1,\cdots,i_k),\ |\vec{i}|=i, \\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=j}} C_{k,\vec{i},\vec{j}} (D^k F(f)) \Big[(\partial_x^{i_1} \partial_y^{j_1} f) \ldots (\partial_x^{i_k} \partial_y^{j_k} f) \Big]. \tag{3.27}$$

Lemma 3.6.2 (General Leibniz rule [52])**.** *Let f ang g be real function n-times differentiable. Then the product $f \cdot g$ is n-times differentiable and its derivative is given by the formula*

$$D^n(f(x) \cdot g(x)) = \sum_{k=0}^{n} \binom{n}{k} D^k f(x) \cdot D^{n-k} g(x).$$

Proof of Lemma 3.2.2. In order to establish the relationship given in (3.7), we start by observing that

$$T'(h) = -\frac{2\sqrt{2}}{h^{1/2}(1 + 2\theta h)},$$

which also ensures the strictly decreasing character that T possess. Subsequently, by proceeding with induction over $i \geqslant 1$, we verify that $D^i T(h) = C_i P_i(h) h^{1/2 - i}(1 + 2\theta h)^{-i}$, where C_i is a positive constant dependent on θ, and P_i is a polynomial of degree $i - 1$. Thus, relationship (3.7) holds for every $i \geqslant 1$. The inverse of T is obtained by considering the relations $T(h) = \rho$, for $\rho \in (2\pi/\sqrt{\theta}, 4\pi/\sqrt{\theta})$, and $1/\cos^2(x) = 1 + \tan^2(x)$, for $x \in (-\pi/2, \pi/2)$. We achieve relationship (3.8) by noticing that

$$\lim_{\rho \to 2\pi/\sqrt{\theta}} \frac{T^{-1}(\rho)}{|2\pi - \sqrt{\theta}\rho|^{-2}} = \frac{8}{\theta}.$$

The bounds stated in (3.9) are obtained through an induction approach over i. For the base case $i = 1$, we derive the equation $T(T^{-1}(\rho)) = \rho$ to obtain

$$DT^{-1}(\rho) = (DT(T^{-1}(\rho)))^{-1}.$$

Therefore, for the case where $i = 1$, the bounds in (3.9) holds by considering those from (3.7) and (3.8). From the induction step, we assume that for $i = n - 1$ relationship (3.9) holds. From Lemma

3.6.1, we have

$$D^nT^{-1}(\rho) = \frac{1}{(DT(T^{-1}(\rho)))^n} \sum_{\substack{1\leqslant r<n \\ \vec{m}=(m_1,\dots,m_r) \\ |\vec{m}|=n}} C_{r,\vec{m}} \cdot D^rT^{-1}(\rho)\left(D^{m_1}T(T^{-1}(\rho))\cdots D^{m_r}T(T^{-1}(\rho))\right), \quad (3.28)$$

for $n \geqslant 2$. To conclude the proof, it is sufficient to show that each therm of the sum (3.28) satisfies the bounds in (3.9). By considering relationships (3.7) and (3.8), we find

$$c|2\pi - \sqrt{\theta}\rho|^{-3n} \leqslant |(DT(T^{-1}(\rho)))|^{-n} \leqslant C|2\pi - \sqrt{\theta}\rho|^{-3n},$$

and

$$c|2\pi - \sqrt{\theta}\rho|^{r+2n} \leqslant |D^{m_1}T(T^{-1}(\rho))\cdots D^{m_r}T(T^{-1}(\rho))| \leqslant C|2\pi - \sqrt{\theta}\rho|^{r+2n},$$

since $m_1 + \cdots + m_r = n$, for every $1 \leqslant r < n-1$. By the induction hypothesis, we deduce

$$c|2\pi - \sqrt{\theta}\rho|^{-(r+2)} \leqslant |D^rT^{-1}(\rho)| \leqslant C|2\pi - \sqrt{\theta}\rho|^{-(r+2)}.$$

Consequently, for $1 \leqslant r < n$,

$$c|2\pi - \sqrt{\theta}\rho|^{-(n+2)} \leqslant \left|\frac{D^rT^{-1}(\rho)|}{(DT(T^{-1}(\rho)))^n}\left(D^{m_1}T(T^{-1}(\rho))\cdots D^{m_r}T(T^{-1}(\rho))\right)\right| \leqslant C|2\pi - \sqrt{\theta}\rho|^{-(n+2)}.$$

This completes the induction step, as (3.28) consists in a finite sum, thereby completing the proof of the lemma. □

Proof of Lemma 3.3.1. It is not difficult to see that A is a smooth function, since it consists in a combination of elementary smooth functions. Direct computations and also Remark 2.2.5 provide us $A'(h) = T(h)$ for every $h \in (0, +\infty)$, and, since $T(h) > 0$ for any $h \in (0, +\infty)$, we conclude that A is invertible. Furthermore, this fact in combination with

$$\lim_{h\to+\infty} \frac{A(h)}{h} = \frac{2\pi}{\sqrt{\theta}}$$

and the bounds in (3.7) yield relationship (3.11).

In order to establish the estimates stated in (3.12), we start by noticing that h_0 is an increasing smooth function, and that the differentiation of the equality $A(h_0(I)) = I$ provides

$$Dh(I) = \frac{1}{DA(h_0(I))}.$$

Hence, for sufficiently large I,

$$c_0 I \leqslant h_0(I) \leqslant C_0 I \quad \text{and} \quad c_1 \leqslant Dh_0(I) \leqslant C_1. \quad (3.29)$$

In order to complete the proof of the lemma, it only remains to show that

$$|D^ih_0(I)| \leqslant C_i I^{1/2-i} \quad \text{for} \quad i \geqslant 2.$$

To this end, we proceed by induction over i. For the base step, $i = 2$, we notice that

$$D^2 h_0(I) = -\frac{Dh_0(I) D^2 A(h_0(I))}{[DA(h_0(I))]^2},$$

which, in conjunction with (3.11) and (3.29), yields

$$\left|D^2 h_0(I)\right| \leqslant C \left|D^2 A(h_0(I))\right| \leqslant C I^{-3/2}.$$

Therefore, relationship (3.12) holds for $i = 2$. Now, as induction step, we assume that relationship also holds for $i = n-1$. Then, by invoking Lemma 3.6.1, we can establish the following for $i = n$:

$$D^n h_0(I) = \frac{1}{(DA(h_0(I)))^n} \sum_{\substack{2 \leqslant r < n \\ \vec{m}=(m_1,\ldots,m_r) \\ |\vec{m}|=n}} C_{r,\vec{m}} \cdot D^r h_0(I) \left[D^{m_1} A(h_0(I)) \ldots D^{m_r} A(h_0(I))\right]. \tag{3.30}$$

Similar to what was done in Lemma 3.2.2, we demonstrate that each term from the sum (3.30) is bounded by $CI^{1/2-n}$. However, we must distinguish between the case where $\vec{m}$ has coordinates of the form $m_j = 1$, with $1 \leqslant j \leqslant r$, and the case where does not. Then, for $2 \leqslant r < n$, we consider the cases:

The vector $\vec{m}$ does not have coordinates of the form $m_j = 1$: The bounds in (3.11) and the induction hypothesis imply that

$$\begin{aligned}
\left|C_{r,\vec{m}} \cdot \frac{D^r h_0(I)}{(DA(h_0(I)))^n} \left[D^{m_1} A(h_0(I)) \ldots D^{m_r} A(h_0(I))\right]\right| &\leqslant C|D^r h_0(I)| \left|D^{m_1} A(h_0(I)) \ldots D^{m_r} A(h_0(I))\right| \\
&\leqslant C I^{1/2-r} I^{r/2-n} \\
&\leqslant C I^{1/2-r/2-n} \\
&\leqslant C I^{1/2-n},
\end{aligned}$$

for sufficiently large I.

The vector $\vec{m}$ has $1 \leqslant M \leqslant r$ coordinates of the form $m_j = 1$: Once again, from the relation (3.11) and the induction hypothesis, we can deduce that, for sufficiently large I,

$$\begin{aligned}
\left|C_{r,\vec{m}} \cdot \frac{D^r h_0(I)}{(DA(h_0(I)))^n} \left[D^{m_1} A(h_0(I)) \ldots D^{m_r} A(h_0(I))\right]\right| &\leqslant C I^{1/2-r} I^{(r-M)/2-(n-M)} \\
&\leqslant C I^{(1-r+M)/2-n} \\
&\leqslant C I^{1/2-n},
\end{aligned}$$

since $1 - r + M \leqslant 1$. Then we conclude that for $i = n$ relationship (3.12) holds, and this completes the proof of the lemma. □

Proof of Lemma 3.3.2. The transformation Φ_1 is onto by construction. Injectivity is shown using the level curves $\mathcal{C}_h$, for $h > 0$, of the Hamiltonian $\mathbf{H}$. Suppose that $(\phi_1, I_1) = (\phi_2, I_2) \in \mathbb{S}_{2\pi} \times$

$(2\pi/\theta^{3/2}, +\infty)$. Since A is a bijection between $(0, +\infty)$ and $(2\pi/\theta^{3/2}, +\infty)$, there exist $h_1 = h_2 \in (0, +\infty)$ such that $A(h_i) = I_i$, for $i = 1, 2$. For fixed values of h, we have that $\phi(x, h)$ is one-to-one in the interval $[0, a(h)]$, which implies that, if $\phi_1 = \phi(x_1, h_1) = \phi(x_2, h_2) = \phi_2$, then $x_1 = x_2$. We conclude that $y_1 = y_2$ by considering the definition of $\phi(x, h)$. The continuity of Φ_1 follows from the fact that each term of the function is continuous. In order to show that its inverse is continuous, we state the relation

$$x(\phi, I) = \tilde{x}(\phi, h_0(I)) = \tilde{x}(\phi, h). \tag{3.31}$$

Then, if $\phi \in (0, \pi/2)$, we have $\phi = \phi_1(\tilde{x}, h)$, that is,

$$\phi = \frac{2\pi}{\sqrt{\theta}T(h)}\left(\arcsin\left(\frac{1}{\sqrt{2h\theta+1}}\right) + \arcsin\left(\frac{\theta\tilde{x}(\phi,h)-1}{\sqrt{2h\theta+1}}\right)\right),$$

which implies that

$$\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi - \arcsin\left(\frac{1}{\sqrt{2h\theta+1}}\right)\right) = \frac{\theta\tilde{x}(\phi,h)-1}{\sqrt{2h\theta+1}}$$

Therefore, considering $|\sqrt{2h\theta+1}| > 1$ when $h > 0$, and using the relationship $\sin(z + \arcsin(y)) = \sqrt{1-y^2}\sin(z) + y\cos(z)$ for $y \in [0, 1]$ and $z \in \mathbb{R}$, we obtain

$$\sqrt{1-\left(\frac{1}{\sqrt{2h\theta+1}}\right)^2}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right) - \frac{1}{\sqrt{2h\theta+1}}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right) = \frac{\theta\tilde{x}(\phi,h)-1}{\sqrt{2h\theta+1}},$$

and consequently,

$$\tilde{x}(\phi,h) = \frac{1}{\theta} + \sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right) - \frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right).$$

Reapplying the previous argument to the remaining angles, we get

$$\tilde{x}(\phi,h) = \begin{cases} \frac{1}{\theta} + \sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right) - \frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right) & \text{if} \quad \phi \in [0, \frac{\pi}{2}), \\ \frac{1}{\theta} + \sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}(\pi-\phi)\right) - \frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}(\pi-\phi)\right) & \text{if} \quad \phi \in [\frac{\pi}{2}, \pi), \\ -\frac{1}{\theta} - \sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}(\phi-\pi)\right) + \frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}(\phi-\pi)\right) & \text{if} \quad \phi \in [\pi, \frac{3\pi}{2}), \\ -\frac{1}{\theta} - \sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}(2\pi-\phi)\right) + \frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}(2\pi-\phi)\right) & \text{if} \quad \phi \in [\frac{3\pi}{2}, 2\pi]. \end{cases} \tag{3.32}$$

We observe that, for all $h > 0$, $\lim_{\phi \to \phi_0} \tilde{x}(\phi, h) = \tilde{x}(\phi_0, h)$ holds for every $\phi_0 \in \{0, \pi/2, \pi, 3\pi/2, 2\pi\}$. Consequently, $\tilde{x}$ exhibits continuity over $\mathbb{S}_{2\pi} \times (0, +\infty)$, and this continuity extends to x (regarded as a function) over $\mathbb{S}_{2\pi} \times (2\pi/\theta^{3/2}, +\infty)$ through the relationship (3.31), and this concludes that Φ_1 is a homeomorphism.

The smoothness of $x(\phi, I)$ with respect to the variable I is also achieved from relationship (3.31) and the smoothness of the function h_0 provided in the Lemma 3.3.1. In order to verify the bounds for (3.13), let us begin by checking that, for a fixed ϕ and $i \geqslant 0$,

$$|\partial_h^i \tilde{x}(\phi, h)| \leqslant C_i h^{1/2-i}. \tag{3.33}$$

As seen in (3.32), $\tilde{x}(\phi, h)$ is a piecewise smooth function. However, within each quadrant of the plane $\mathbb{R}^2$, $\tilde{x}(\phi, h)$ is composed by functions of the same nature when considering their asymptotic behavior. For this reason, we will demonstrate that the relationship (3.33) holds for $\phi \in [0, \pi/2)$, and this result will be naturally extended to the remaining angles. Specifically, for $\phi \in [0, \pi/2)$ and $i = 0$, we have

$$\tilde{x}(\phi, h) = \frac{1}{\theta} + \sqrt{\frac{2h}{\theta}} \sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right) - \frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right).$$

Consequently, for sufficiently large h, it follows that

$$|\tilde{x}(\phi, h)| \leqslant C h^{1/2}.$$

For $i \geqslant 1$, the higher-order derivatives of $\tilde{x}(\phi, h)$ with respect to h are given by

$$\partial_h^i \tilde{x}(\phi, h) = \partial_h^i \left[\sqrt{\frac{2h}{\theta}} \sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right] - \partial_h^i \left[\frac{1}{\theta}\cos\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right]. \tag{3.34}$$

Claim 1. *For $i \geqslant 1$ and sufficiently large h, the following estimate holds*

$$\left|\partial_h^i \left[\mathcal{G}\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right]\right| \leqslant C h^{-1/2-i},$$

where $\mathcal{G}(x)$ is either $\cos(x)$ *or* $\sin(x)$.

Proof of Claim 1. By considering the formula provided in Lemma 3.6.1, we are led to

$$\partial_h^i \left[\mathcal{G}\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right] = \sum_{\substack{1 \leqslant r \leqslant i \\ \vec{m} = (m_1, \cdots, m_r) \\ |\vec{m}| = i}} C_{r,\vec{m}} (D^r \mathcal{G}(\tfrac{T(h)}{2\pi}\phi)) \left[D^{m_1} T(h) \dots D^{m_r} T(h)\right]. \tag{3.35}$$

We observe that in the sum (3.35), if the constant $C_{r,\vec{m}} \neq 0$,then $m_j \geqslant 1$ for every $1 \leqslant j \leqslant r$. Therefore, for $i \geqslant 1$ and sufficiently large h, any non-zero term in the sum (3.35) is bounded by

$$\left|C_{r,\vec{m}} D^r \mathcal{G}\left(\tfrac{T(h)}{2\pi}\phi\right) \left[D^{m_1} T(h) \dots D^{m_r} T(h)\right]\right| \leqslant C h^{-r/2-i} \leqslant C h^{-1/2-i},$$

where we have considered the bounds in (3.7) and the fact that $\left|D^r\mathcal{G}\left(\frac{T(h)}{2\pi}\phi\right)\right| \leqslant 1$. This completes the proof of Claim 1.

Claim 2. *For $i \geqslant 1$ and sufficiently large h, we have that*

$$\left|\partial_h^i\left[\sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right]\right| \leqslant Ch^{1/2-i}.$$

Proof of Claim 2. We first observe that, for sufficiently large h and $k \in \mathbb{N}$,

$$\left|D^k\left(\sqrt{\frac{2h}{\theta}}\right)\right| \leqslant Ch^{1/2-k}.$$

Then, after applying the General Leibniz rule to (3.34), we have

$$\begin{aligned}\partial_h^i\left[\sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right] &= \sum_{j=0}^{i}\binom{i}{j}D^j\left(\sqrt{\frac{2h}{\theta}}\right)D^{i-j}\left(\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right)\\ &= \sum_{j=0}^{i-1}\binom{i}{j}D^j\left(\sqrt{\frac{2h}{\theta}}\right)D^{i-j}\left(\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right) + D^i\left(\sqrt{\frac{2h}{\theta}}\right).\end{aligned}$$

By applying the bounds provided in Claim 1, it follows that

$$\left|\partial_h^i\left[\sqrt{\frac{2h}{\theta}}\sin\left(\frac{T(h)\sqrt{\theta}}{2\pi}\phi\right)\right]\right| \leqslant Ch^{-i} + Ch^{1/2-i} \leqslant Ch^{1/2-i},$$

for sufficiently large h, which concludes the proof of Claim 2.

Therefore, from Claims 1 and 2 we conclude that, for sufficiently large h and $i \geqslant 0$, relationship (3.33) holds. Back to our initial goal, we now consider the composite function $x(\phi, I) = \tilde{x}(\phi, h_0(I))$. For $i = 0$, taking into account the bounds (3.33), (3.31), and (3.12), we have that

$$|x(\phi, I)| = |\tilde{x}(\phi, h_0(I))| \leqslant C_0(h_0(I))^{1/2} \leqslant C_0 I^{1/2},$$

for sufficiently large I. For $i \geqslant 1$, it follows from Lemma 3.6.1 that

$$\partial_I^n x(\phi, I) = \partial_I^i[\tilde{x}(\phi, h_0(I))] = \sum_{\substack{1\leqslant k\leqslant i\\ (m_1,\ldots,m_k)=\vec{m}\\ |\vec{m}|=i}} C_{k,\vec{m}}\left[\partial_h^k\tilde{x}(\phi, h_0(I))\right]\left[D^{m_1}h_0(I)\ldots D^{m_k}h_0(I)\right],$$

with $C_{\vec{i},k} = 0$, if $i_l = 0$ for any $1 \leqslant l \leqslant k$. Thus, each term of the sum (3.36) is of the form

$$C_{k,\vec{m}}\left[\partial_h^k\tilde{x}(\phi, h_0(I))\right]\left[D^{m_1}h_0(I)\ldots D^{m_k}h_0(I)\right].$$

The bounds in (3.33) and (3.12) yields

$$|\partial_h^k\tilde{x}(\phi, h_0(I))| \leqslant C_k(h_0(I))^{1/2-k} \leqslant CI^{1/2-k},$$

for sufficiently large I. In order to provide estimations for the term $D^{m_1}h_0(I)\dots D^{m_k}h_0(I)$, we need to distinguish between two cases: the case where $\vec{m}$ does not have coordinates of the form $m_j = 1$, and the case where it does.

The vector $\vec{m}$ does not have coordinates of the form $m_j = 1$: In this case, we take into account (3.12), and then

$$|D^{i_1}h_0(I)\dots D^{i_k}h_0(I)| \leqslant CI^{1/2-i_1}\dots I^{1/2-i_k} = CI^{k/2-i},$$

hence

$$\begin{aligned}|\partial_h^k\tilde{x}(\phi,h_0(I))[D^{i_1}h_0(I)\dots D^{i_k}h_0(I)]| &\leqslant CA = I^{1/2-k}I^{k/2-i}\\ &= CI^{\frac{(1-k)}{2}-i}\\ &\leqslant CI^{1/2-i},\end{aligned}$$

since $1-k<1$, for any $1\leqslant k\leqslant i$.

The vector $\vec{m}$ has coordinates of the form $m_j = 1$: Suppose that $\vec{m}$ has $1\leqslant M\leqslant k$ coordinates of the type $m_j = 1$, then for such coordinates $|D^{i_l}h_0(A)|\leqslant C$. In this way,

$$|D^{m_1}h_0(I)\dots D^{m_k}h_0(I)| \leqslant CA^{\frac{(k-M)}{2}-i}I^M = CI^{\frac{(k+M)}{2}-i},$$

which implies that

$$\begin{aligned}|\partial_h^k\tilde{x}(\phi,h_0(I))[D^{m_1}h_0(I)\dots D^{m_k}h_0(I)]| &\leqslant CI^{1/2-k}I^{\frac{(k+M)}{2}-i}\\ &= CI^{\frac{(1-k+M)}{2}-i}\\ &\leqslant CI^{1/2-i},\end{aligned}$$

given that $1-k+M\leqslant 1$. Then, each term of the sum (3.36) is less or equal to $CI^{1/2-i}$ in absolute values for every $i\geqslant 1$ and sufficiently large I, which means that relationship (3.13) holds under the same conditions. This concludes the proof of the lemma. □

Proof of Lemma 3.4.1. For the sake of simplicity, we are going to omit the dependence of $\mathcal{H}_1$ on β, r, τ, and ε, unless necessary. It follows from (3.14) and (3.15) that

$$\begin{aligned}r &= \mathcal{H}(\tau,\mathscr{H}(\beta,r,\tau;\varepsilon),\beta;\varepsilon)\\ &= \mathcal{H}(\tau,A(r)+\varepsilon\mathscr{H}_1,\beta)\\ &= h_0(A(r)+\varepsilon\mathscr{H}_1)-\varepsilon x(\tau,A(r)+\varepsilon\mathscr{H}_1)p(\beta).\end{aligned}$$

Since $A(\cdot) = h_0^{-1}(\cdot)$, we obtain

$$\varepsilon\mathscr{H}_1 = A(r+\varepsilon x(\tau,A(r+\varepsilon\mathscr{H}_1))p(\beta)) - A(r).$$

Thus, by defining $\mathcal{F}(\rho) := A(r + \rho\varepsilon x(\tau, A(r + \varepsilon\mathscr{H}_1))p(\beta))$, we have $\mathcal{F}(1) = A(r + \varepsilon x(\tau, A(r + \varepsilon\mathscr{H}_1))p(\beta))$ and $\mathcal{F}(0) = A(r)$. Additionally,

$$\mathcal{F}'(\rho) = \varepsilon x(\tau, A(r) + \varepsilon\mathscr{H}_1)p(\beta)T(r + \rho\varepsilon x(\tau, A(r) + \varepsilon\mathscr{H}_1)p(\beta)),$$

since $A' = T$. Therefore, by the Fundamental Theorem of Calculus, we can rewrite $\varepsilon\mathscr{H}_1$ as

$$\mathscr{H}_1 = x(\tau, A(r) + \varepsilon\mathscr{H}_1)p(\beta)\int_0^1 T(r + \rho\varepsilon x(\tau, A(r) + \varepsilon\mathscr{H}_1)p(\beta))d\rho. \tag{3.36}$$

In order to avoid repetitive arguments, we will assume that r is sufficiently large and $\varepsilon \leqslant 1/2$ throughout this proof. The lemma is proved by induction over $i + j$.

For the base step, that is, for $i + j = 0$, we notice that (3.36), (3.13), and the boundedness of T imply that

$$|\mathscr{H}_1| \leqslant C|x(\tau, \mathscr{H})| \leqslant C|\mathscr{H}|^{1/2},$$

with C being a constant depending on p. Since $|A(r)| \leqslant Cr$, it follows that $|\mathscr{H}_1| \leqslant Cr^{1/2}$, as desired.

Before we proceed with the proof, we notice that, from (3.14) and (3.11), we have

$$cr \leqslant cr - Cr^{1/2} \leqslant |A(r)| - |\mathscr{H}_1| \leqslant |\mathscr{H}| \leqslant |A(r)| + |\mathscr{H}_1| \leqslant Cr + Cr^{1/2} \leqslant Cr, \tag{3.37}$$

and this together with (3.13) imply that

$$|\partial_I^k x(\tau, \mathscr{H})| \leqslant Cr^{1/2-k} \quad \text{for} \quad k \geqslant 0. \tag{3.38}$$

Now, let us assume that the bound in Lemma 3.4.1 holds for $i + j = n$. We will show that it also holds for $i + j = n + 1$. For this purpose, we define the auxiliary functions

$$\begin{aligned} &U(\beta, r, \tau; \varepsilon) := x(\tau, \mathscr{H}(\beta, r, \tau; \varepsilon))p(\beta), \\ &\tilde{T}(\rho) := T(r + \rho\varepsilon x(\tau, A(r) + \varepsilon\mathscr{H}_1)p(\beta)) = T(r + \rho\varepsilon U(\beta, r, \tau)), \end{aligned}$$

and we provide bounds for them in distinct claims.

Claim 1. For $0 \leqslant i + j \leqslant n + 1$,

$$|\partial_\beta^i\partial_r^j[x(\tau, \mathscr{H})]| \leqslant C(r^{-1/2}|\partial_\beta^i\partial_r^j\mathscr{H}| + r^{1/2-j}). \tag{3.39}$$

Proof of Claim 1. In order to prove the estimate (3.39), we will use (3.27) from Lemma 3.6.1, thus

$$\partial_\beta^i\partial_r^j[x(\tau, \mathscr{H})] = C\left(\partial_I x(\tau, \mathscr{H})\right)\left(\partial_\beta^i\partial_r^j\mathscr{H}\right) + \sum_{\substack{2\leqslant k\leqslant i+j, \\ \vec{i}=(i_1,\cdots,i_k),\ |\vec{i}|=i, \\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=j}} C_{k,\vec{i},\vec{j}}\left(\partial_I^k x(\tau, \mathscr{H})\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}\mathscr{H}\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}\mathscr{H}\right). \tag{3.40}$$

Let us observe that the absolute value of the first term in the equation (3.40) is smaller than $Cr^{-1/2}|\partial_\beta^i\partial_r^j\mathscr{H}|$, due to (3.38). Therefore, to conclude the proof of this first claim, it suffices to demonstrate that each non-zero term in the sum (3.40) is smaller in absolute value than $Cr^{1/2-j}$. To this end, we start by noticing that, from Lemma 3.6.1, each non-zero term in the expression

$$\sum_{\substack{2\leqslant k\leqslant i+j,\\ \vec{i}=(i_1,\cdots,i_k),\ |\vec{i}|=i,\\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=j}} C_{k,\vec{i},\vec{j}}\left(\partial_I^k x(\tau,\mathscr{H})\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}\mathscr{H}\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}\mathscr{H}\right) \tag{3.41}$$

should have $1\leqslant i_l+j_l\leqslant n$ for every $1\leqslant l\leqslant k+p$, otherwise if there exists $1\leqslant l\leqslant k$ such that $i_l+j_l=n+1$, then, since $\vec{i}$ and $\vec{j}$ must have at least two components, it would follow that $i_{l-1}+j_{l-1}=0$ or $i_{l+1}+j_{l+1}=0$, which contradicts Lemma 3.6.1. Thus, we can apply the induction hypothesis to $\partial_\beta^{i_l}\partial_r^{j_l}\mathscr{H}_1$, yielding $|\partial_\beta^{i_l}\partial_r^{j_l}\mathscr{H}_1|\leqslant Cr^{1/2-j_l}$ for such indices. Consequently, if $i_l\geqslant 1$, then from (3.11), (3.38), and the induction hypothesis, it follows that

$$\begin{aligned}|\partial_r^{j_l}\mathscr{H}| = |D^{j_l}A+\varepsilon\partial_r^{j_l}\mathscr{H}_1| &\leqslant \begin{cases} C+Cr^{-1/2} & \text{if}\quad j_l=1,\\ Cr^{1/2-j_l}+Cr^{1/2-j_l} & \text{if}\quad j_l\geqslant 1,\end{cases}\\ &\leqslant \begin{cases} C & \text{if}\quad j_l=1,\\ Cr^{1/2-j_l} & \text{if}\quad j_l\geqslant 1,\end{cases}\end{aligned}$$

which implies that

$$|\partial_\beta^{i_l}\partial_r^{j_l}\mathscr{H}|\leqslant \begin{cases} C & \text{if}\quad (i_l,j_l)=(0,1),\\ Cr^{1/2-j_l} & \text{if}\quad (i_l,j_l)\neq(0,1),\end{cases} \quad \text{for every}\ \ 1\leqslant l\leqslant k. \tag{3.42}$$

Therefore, if a non-zero term in the sum (3.40) does not have pairs of indices of the form $(i_l,j_l)=(0,1)$, it follows from (3.38) and (3.42) that

$$\begin{aligned}\left|\left(\partial_I^k x(\tau,\mathscr{H})\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}\mathscr{H}\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}\mathscr{H}\right)\right| &\leqslant Cr^{1/2-k}r^{1/2-j_1}\dots r^{1/2-j_k}\\ &= Cr^{(1-k)/2-j}.\end{aligned}$$

Now, let us consider a non-zero term from the sum (3.41) that has $1\leqslant M\leqslant k$ pairs of indices of the form $(i_l,j_l)=(0,1)$. In such a scenario,

$$\begin{aligned}\left|\left(\partial_I^k x(\tau,\mathscr{H})\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}\mathscr{H}\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}\mathscr{H}\right)\right| &\leqslant Cr^{1/2-k}r^{\frac{(k-M)}{2}-j}r^M = Cr^{\frac{(1-k+M)}{2}-j}\\ &\leqslant Cr^{1/2-j},\end{aligned}$$

where the last inequality follows from the fact that $1-k+M\leqslant 1$. This concludes the proof of the claim.

As a consequence of Claim 1 and the induction hypothesis, we have, for $0\leqslant i+j\leqslant n$,

$$|\partial_\beta^i\partial_r^j[x(\tau,\mathscr{H})]|\leqslant C(r^{-1/2}r^{1/2-j}+r^{1/2-j})\leqslant Cr^{1/2-j}. \tag{3.43}$$

Claim 2. For $0 \leqslant i + j \leqslant n + 1$,

$$|\partial_\beta^i \partial_r^j U| \leqslant C(r^{-1/2}|\partial_\beta^i \partial_r^j \mathscr{H}| + r^{1/2 - j}). \tag{3.44}$$

Proof of Claim 2. From the general Leibniz rule, we have

$$\partial_\beta^i \partial_r^j U = \partial_\beta^i(\partial_r^j[x(\tau, \mathscr{H})]p(\beta)) = \sum_{k=0}^{i-1} \binom{i}{k} D^{i-k}p(\beta)\partial_\beta^k \partial_r^j[x(\tau, \mathscr{H})] + \partial_\beta^i \partial_r^j[x(\tau, \mathscr{H})].$$

The above equality combined with the bounds (3.39) and (3.43), yield the claim.

Claim 3. For a fixed $\rho \in [0, 1]$ and for $0 \leqslant i + j \leqslant n + 1$, we have

$$|\partial_\beta^i \partial_r^j [T(r + \varepsilon\rho U)]| \leqslant C(r^{-2}|\partial_\beta^i \partial_r^j \mathcal{H}| + r^{-1/2 - j}).$$

Proof of Claim 3. We start the proof of the claim by noticing that

$$|U| \leqslant C(r^{-1/2}|\mathscr{H}| + r^{1/2}) \leqslant C(r^{-1/2}r + r^{1/2}) = Cr^{1/2},$$

where we have used (3.44) and (3.37). Thus, for sufficiently small ε, we have the bounds

$$|r + \varepsilon\rho U| \geqslant r - \varepsilon\rho|U| \geqslant r - Cr^{1/2} \geqslant cr. \tag{3.45}$$

Hence, relation (3.45) combined with the bounds for the derivatives of T provided in (3.7), gives us, for $k \geqslant 1$, the following:

$$|D^k T(r + \varepsilon\rho U)| \leqslant C|r + \varepsilon\rho U|^{-1/2 - k} \leqslant Cr^{-1/2 - k}. \tag{3.46}$$

Now, let us consider, for $i + j \geqslant 1$, the higher order derivatives of $r + \varepsilon\rho U$, which are given by:

$$\partial_\beta^i \partial_r^j [r + \varepsilon\rho U] = \begin{cases} 1 + \varepsilon\rho(\partial_r U) & \text{if} \quad (i, j) = (0, 1), \\ \varepsilon\rho(\partial_\beta^i \partial_r^j U) & \text{if} \quad (i, j) \neq (0, 1). \end{cases} \tag{3.47}$$

Thus, for $(i, j) = (0, 1)$, it follows from (3.44) and (3.46) that

$$\begin{aligned} |\partial_r [T(r + \varepsilon\rho U)]| &= |T'(r + \varepsilon\rho U)|(1 + \varepsilon\rho|\partial_r U|) \\ &\leqslant Cr^{-3/2}(1 + r^{-1/2}|\partial_r \mathscr{H}| + r^{-1/2}) \\ &\leqslant C(r^{-2}|\partial_r \mathscr{H}| + r^{-3/2}). \end{aligned}$$

On the other hand, if $(i, j) \neq (0, 1)$, we take into account the bounds (3.44) and (3.46) to show that

$$\left|T'(r + \varepsilon\rho U)\left(\partial_\beta^i \partial_r^j [r + \varepsilon\rho U]\right)\right| \leqslant Cr^{-3/2}(r^{-1/2}|\partial_\beta^i \partial_r^j \mathscr{H}| + r^{1/2 - j}) \leqslant C(r^{-2}|\partial_\beta^i \partial_r^j \mathscr{H}| + r^{-1/2 - j}).$$

We notice that in both cases, for $(i,j)=(0,1)$ and $(i,j)\neq(0,1)$, the first term of the sum

$$\partial_\beta^i\partial_r^j[T(r+\varepsilon\rho U)] = CT'(r+\varepsilon\rho U)\left(\partial_\beta^i\partial_r^j[r+\varepsilon\rho U]\right) + \sum_{\substack{2\leqslant k\leqslant i+j,\\ \vec{i}=(i_1,\cdots,i_k),\ |\vec{i}|=i,\\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=j}} C_{k,\vec{i},\vec{j}}\left(D^kT(r+\varepsilon\rho U)\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}[r+\varepsilon\rho U]\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}[r+\varepsilon\rho U]\right). \tag{3.48}$$

is smaller than $C(r^{-2}|\partial_\beta^i\partial_r^j\mathscr{H}|+r^{-1/2-j})$, for $j\geqslant 1$, in absolute values. Thus, it only remains to show that the remaining terms of the sum (3.48) are bounded by $Cr^{-1/2-j}$. In fact, it is known that each non-zero term of the sum (3.48) is of the form

$$C_{k,\vec{i},\vec{j}}\left(D^kT(r+\varepsilon\rho U)\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}[r+\varepsilon\rho U]\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}[r+\varepsilon\rho U]\right)$$

with $2\leqslant k\leqslant i+j$. We divide the analysis in two cases, namely

There are no pairs of indices of the form $(i_l,j_l)=(0,1)$ in the vectors $\vec{i}$ and $\vec{j}$: In this situation, for $2\leqslant k\leqslant i+j$, equation (3.47) and the induction hypothesis yield

$$\begin{aligned}\left|\left(D^kT(r+\varepsilon\rho U)\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}[r+\varepsilon\rho U]\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}[r+\varepsilon\rho U]\right)\right| &\leqslant Cr^{-1/2-k}r^{k/2-j}\\ &\leqslant Cr^{-1/2-j}.\end{aligned}$$

There exist M, with $1\leqslant M<k$, pairs of the form $(i_l,j_l)=(0,1)$ in the vectors $\vec{i}$ and $\vec{m}$: In such case, from equation (3.47) and the induction hypothesis, it follows that

$$\begin{aligned}\left|\left(D^kT(r+\varepsilon\rho U)\right)\left(\partial_\beta^{i_1}\partial_r^{j_1}[r+\varepsilon\rho U]\right)\dots\left(\partial_\beta^{i_k}\partial_r^{j_k}[r+\varepsilon\rho U]\right)\right| &\leqslant Cr^{-1/2-k}r^{1/2(k-M)-(j-M)}\\ &\leqslant Cr^{1/2(1-k+M)-j}\\ &\leqslant Cr^{-1/2-j},\end{aligned}$$

since $1-k+M\leqslant 1$. Thus, Claim 3 holds. We observe that the induction hypothesis implies, for $0\leqslant i+j\leqslant n$, that

$$|\partial_\beta^i\partial_r^j[T(r+\varepsilon\rho U)]|\leqslant Cr^{-1/2-j}.$$

We are now in position to complete the proof of Lemma 3.4.1, as we have established bounds for all the components involved in the higher-order derivatives of $\mathcal{H}_1=U\displaystyle\int_0^1\tilde{T}(\rho)d\rho$. From

Lemma (3.6.1), the Integral Leibniz Rule, and Claims 1, 2, and 3, we obtain

$$
\begin{aligned}
|\partial_\beta^i \partial_r^j \mathscr{H}_1(\beta, r, \tau)| &\leqslant |\partial_\beta^i \partial_r^j U| + \int_0^1 |\partial_\beta^i \partial_r^j [T(r+\varepsilon\rho U)]| d\rho \\
&\quad + C \sum_{\substack{p=0,\ldots,i \\ k=0,\ldots,j \\ (p,k)\neq(0,0),(p,k)\neq(i,j)}} |\partial_\beta^p \partial_r^k U| \left(\int_0^1 |\partial_\beta^{i-p} \partial_r^{j-k} [T(r+\varepsilon\rho U)]| d\rho \right) \\
&\leqslant C \left(r^{-1/2} |\partial_\beta^i \partial_r^j \mathscr{H}| + r^{1/2-j} + r^{-2} |\partial_\beta^i \partial_r^j \mathscr{H}| + r^{-1/2-j} + \sum_{(p,k)} r^{1/2-k} r^{-1/2-(j-k)} \right) \\
&\leqslant C(r^{-1/2} |\partial_\beta^i \partial_r^j \mathscr{H}| + r^{1/2-j}). \qquad (3.49)
\end{aligned}
$$

We observe that, when $(i,j) = (0,1)$, then

$$
\begin{aligned}
|\partial_r \mathscr{H}_1| &\leqslant C(r^{-1/2} |\partial_r \mathscr{H}| + r^{-1/2}) \\
&\leqslant C(r^{-1/2}(|A'(r)| + \varepsilon |\partial_r \mathscr{H}_1|) + r^{-1/2}) \\
&\leqslant C r^{-1/2} + C r^{-1/2} |\partial_r \mathscr{H}_1|,
\end{aligned}
$$

where we have used (3.11) to bound A. On the other hand, for $(i,j) \neq (0,1)$ and $i + j \geqslant 1$, it follows from (3.11) and the sum (3.49) that

$$
\begin{aligned}
|\partial_\beta^i \partial_r^j \mathscr{H}_1| &\leqslant C(r^{-1/2} |\partial_\beta^i \partial_r^j \mathscr{H}| + r^{1/2-j}) \\
&\leqslant C r^{-1/2} |\partial_\beta^i \partial_r^j A(r)| + \varepsilon C r^{-1/2} |\partial_\beta^i \partial_r^j \mathscr{H}_1| + C r^{1/2-j} \\
&\leqslant C r^{-1/2} |\partial_\beta^i \partial_r^j \mathscr{H}_1| + C r^{1/2-j}.
\end{aligned}
$$

Therefore, for sufficiently large r such that $Cr^{-1/2} \leqslant 1/2$, we have

$$
|\partial_\beta^i \partial_r^j \mathscr{H}_1| \leqslant C r^{1/2-j},
$$

and this completes the proof by induction. □

Proof of Lemma 3.4.3. In order to provide the bounds (3.20), we define the auxiliary function

$$
r_n(\bar{\rho}) := T^{-1}\left(\lambda_n \bar{\rho} + \frac{2\pi}{\sqrt{\theta}} \right),
$$

which allows us to rewrite $\bar{f}_{1,n}$ and $\bar{f}_{2,n}$ as

$$
\bar{f}_{1,n}(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) = \frac{1}{\sigma} \partial_r \mathscr{H}_1 \left(\sigma\bar{\beta}, r_n(\bar{\rho}), \tau; \varepsilon \right)
$$

and

$$
\bar{f}_{2,n}(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) := -\lambda_n^{-1} T' \left(r_n(\bar{\rho}) \right) \partial_\beta \mathscr{H}_1 \left(\sigma\bar{\beta}, r_n(\bar{\rho}), \tau; \varepsilon \right).
$$

Thus, the most significant task in this case is to determine the bounds for $r_n(\bar{\rho})$ and subsequently apply Lemma 3.4.1 to derive the bounds for $\bar{f}_{1,n}$ and $\bar{f}_{2,n}$. Since bounding the functions $\bar{f}_{1,n}$ and $\bar{f}_{2,n}$ meet a finite number of conditions, we can assume that there exists $n_2^* \in \mathbb{N}$, for which the bounds provided in (3.9) hold for every $n \geqslant n_2^*$.

We start by stating that, for $i \geqslant 0$,

$$|D^i[r_n(\bar{\rho})]| \leqslant C\lambda_n^{-2}.$$

Indeed, when $i = 0$, it follows that

$$c\lambda_n^{-2} \leqslant c|\lambda_n\bar{\rho}|^{-2} \leqslant r_n(\bar{\rho}) \leqslant C|\lambda_n\bar{\rho}|^{-2} \leqslant C\lambda_n^{-2},$$

since $\bar{\rho} \in [1/2, 2]$ and T^{-1} is bounded. For $i \geqslant 1$, we consider the bounds in (3.9) to obtain

$$|D^i[r_n(\bar{\rho})]| = \lambda_n^i|D^iT^{-1}(\lambda_n\bar{\rho} + 2\pi)| \leqslant C\lambda_n^i|\lambda_n\bar{\rho}|^{-(i+2)} \leqslant C\lambda_n^{-2}, \tag{3.50}$$

and it is a immediate consequence of (3.7) and (3.50) that

$$|D^iT(r_n(\bar{\rho}))| \leqslant C\lambda_n^{1+2i},$$

for every $i \geqslant 1$ and every $n \geqslant n_2^*$, which leads us to the following claim.

Claim 1. For every $i \geqslant 0$, the following bounds hold

$$|D^i[T'(r_n(\bar{\rho}))]| \leqslant C\lambda_n^3.$$

Proof of Claim 1. For $i = 0$, it is evident from the estimates above. Now, let us assume that $i \geqslant 1$. Then, from Lemma 3.6.1 and the bounds provided in (3.50), we have that

$$\begin{aligned}|D^i[T'(r_n(\bar{\rho}))]| &= \Big|\sum_{\substack{1\leqslant k\leqslant i,\\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=i}} C_{k,\vec{j}}(D^{k+1}T(r_n(\bar{\rho}))\left((D^{j_1}r_n(\bar{\rho})\dots D^{j_k}r_n(\bar{\rho}))\right)\Big| \\ &\leqslant C\lambda_n^{3+2k}\lambda_n^{-2}\dots\lambda_n^{-2} \leqslant C\lambda_n^3,\end{aligned}$$

which concludes the proof of the claim.

Taking into account Lemmas 3.6.1 and 3.4.1, we are in position to provide bounds for $\bar{f}_{1,n}$ and $\bar{f}_{2,n}$. We notice that, for $i + j \geqslant 0$,

$$\begin{aligned}|\partial_{\bar{\beta}}^i\partial_{\bar{\rho}}^j[\partial_\beta\mathscr{H}_1(\sigma\bar{\beta}, r_n(\bar{\rho}), \tau; \varepsilon)]| &= \sigma^i|\partial_{\bar{\rho}}^j[\partial_\beta^{i+1}\mathscr{H}_1(\sigma\bar{\beta}, r_n(\bar{\rho}), \tau)]| \\ &= \sigma^i\Big|\sum_{\substack{1\leqslant k\leqslant j,\\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=j}} C_{k,\vec{j}}(\partial_r^k\partial_\beta^{i+1}\mathscr{H}_1(\sigma\bar{\beta}, r_n(\bar{\rho}), \tau; \varepsilon))\left((D^{j_1}r_n(\bar{\rho})\dots D^{j_k}r_n(\bar{\rho}))\right)\Big| \\ &\leqslant C\sum(r_n(\bar{\rho}))^{1/2-k}\lambda_n^{-2}\dots\lambda_n^{-2} \leqslant C\sum\lambda_n^{-1+2k}\lambda_n^{-2k} \leqslant C\lambda_n^{-1},\end{aligned}$$

which allows us to conclude that

$$|\partial_{\bar\beta}^i\partial_{\bar\rho}^j[\partial_\beta \mathscr{H}_1(\sigma\bar\beta, r_n(\bar\rho),\tau;\varepsilon)]| \leqslant C\lambda_n^{-1}. \tag{3.51}$$

In an analogous way,

$$\begin{aligned}
|\partial_{\bar\beta}^i\partial_{\bar\rho}^j[\partial_r \mathscr{H}_1(\sigma\bar\beta, r_n(\bar\rho),\tau;\varepsilon)]| &= \sigma^i|\partial_{\bar\rho}^j[\partial_\beta^i\partial_r \mathscr{H}_1(\sigma\bar\beta, r_n(\bar\rho),\tau;\varepsilon)]| \\
&= \sigma^i\Big|\sum_{\substack{1\leqslant k\leqslant j,\\ \vec{j}=(j_1,\cdots,j_k),\ |\vec{j}|=j,\\ j_l\geqslant 1:\forall 1\leqslant l\leqslant k}} C_{k,\vec{j}}(\partial_r^{k+1}\partial_\beta^i \mathscr{H}_1(\sigma\bar\beta, r_n(\bar\rho),\tau;\varepsilon))\left((D^{j_1}r_n(\bar\rho)\dots D^{j_k}r_n(\bar\rho))\right)\Big| \\
&\leqslant C\sum (r_n(\bar\rho))^{1/2-(k+1)}\lambda_n^{-2}\dots\lambda_n^{-2} \leqslant C\sum \lambda_n^{1+2k}\lambda_n^{-2k} \leqslant C\lambda_n,
\end{aligned}$$

which implies that

$$|\bar f_{1,n}(\bar\beta,\bar\rho,\tau;\varepsilon)| \leqslant C\lambda_n. \tag{3.52}$$

Finally, by taking into account the general Leibniz rule, the bounds provided in (3.51), and Lemma 3.4.1, we have that

$$\begin{aligned}
|\partial_{\bar\beta}^i\partial_{\bar\rho}^j[T'(r_n(\bar\rho))\partial_\beta \mathscr{H}_1(\sigma\bar\beta, r_n(\bar\rho),\tau;\varepsilon)]| &= |\sum_{k=0}^{j} C_{k,\vec{j}}D^{j-k}[T'(r_n(\bar\rho))].\partial_{\bar\beta}^k[\partial_\beta \mathscr{H}_1(\sigma\bar\beta, r_n(\bar\rho),\tau;\varepsilon)]| \\
&\leqslant C\lambda_n^3\lambda_n^{-1} \\
&= C\lambda_n^2,
\end{aligned}$$

and consequently,

$$|\bar f_{2,n}(\bar\beta,\bar\rho,\tau;\varepsilon)| \leqslant C\lambda_n^2\lambda_n^{-1} = C\lambda_n,$$

which concludes the proof of the lemma. □

Proof of Lemma 3.4.4. Let us remind that $\bar\lambda = 2\pi\lambda/\sigma$ and $\bar\alpha = 4\pi^2/(\sigma\sqrt{\theta})$, which allows us to rewrite (3.21) as

$$(\bar\beta(\tau;\varepsilon),\bar\rho(\tau;\varepsilon)) = \left(\bar\beta_0 + \frac{\alpha}{2\pi}\tau + \frac{\bar\lambda}{2\pi}\bar\rho_0\tau + \bar\lambda\varepsilon A(\bar\beta_0,\bar\rho_0,\tau;\varepsilon), \bar\rho_0 + \bar\lambda\varepsilon B(\bar\beta_0,\bar\rho_0,\tau;\varepsilon)\right), \tag{3.53}$$

where

$$\bar\lambda A(\bar\beta_0,\bar\rho_0,\tau;\varepsilon) = \frac{\bar\lambda^2}{2\pi}\int_0^\tau B(\bar\beta_0,\bar\rho_0,s;\varepsilon)ds + \int_0^\tau \bar f_1(\bar\beta(s;\varepsilon),\bar\rho(s;\varepsilon),s;\varepsilon)\mathrm{d}s, \tag{3.54}$$

$$\bar\lambda B(\bar\beta_0,\bar\rho_0,\tau;\varepsilon) = \int_0^\tau \bar f_2(\bar\beta(s;\varepsilon),\bar\rho(s;\varepsilon),s;\varepsilon)\mathrm{d}s, \tag{3.55}$$

and $(\bar\beta_0,\bar\rho_0)$ is the initial condition of the solution $(\bar\beta(\tau;\varepsilon),\bar\rho(\tau;\varepsilon))$, which has the maximal interval of definition containing the interval $\tau\in[0,2\pi]$. We notice that $\bar P_1(\bar\beta_0,\bar\rho_0;\varepsilon) = A(\bar\beta_0,\bar\rho_0,2\pi;\varepsilon)$ and $\bar P_2(\bar\beta_0,\bar\rho_0;\varepsilon) = B(\bar\beta_0,\bar\rho_0,2\pi;\varepsilon)$. For the sake of simplicity, let us denote the norms of A and B by

$$\|A\|_N = \|A\|_{\mathcal{C}^N} \quad\text{and}\quad \|B\|_N = \|B\|_{\mathcal{C}^N},$$

where $\|\cdot\|_{C^N}$ is the norm introduced in (2.16). Thus, in order to proof the desired lemma, we provide the bounds for the auxiliary functions A and B using an inductive step over N. Subsequently, we extend this result to $\bar{P}_1$ and $\bar{P}_2$. We should mention that $\bar{\lambda}$ depends on n, and this proof specifically addresses cases where n is sufficiently large, that is, there exists $n_3^* \in \mathbb{N}$ such that all the bounds used in this proof are valid if $n \geqslant n_3^*$. This implies $\bar{\lambda}$ is sufficiently small, when n is sufficiently large. We also remind that $C > 0$ is recursively defined in order to meet a finite number of conditions.

For the base step $i + j = N = 0$, we first notice that $c\bar{\lambda} \leqslant \lambda \leqslant C\bar{\lambda}$. Then, taking into account (3.55) and Lemma 3.4.3, we obtain

$$\|B\|_0 = \frac{1}{\bar{\lambda}}\left\|\int_0^\tau \bar{f}_2(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\ \mathrm{d}s\right\|_\infty \leqslant \frac{1}{\bar{\lambda}}\int_0^\tau C\lambda\ \mathrm{d}s \leqslant C,$$

and consequently, by considering (3.54), we have

$$\begin{aligned}\|A\|_0 &= \frac{1}{\bar{\lambda}}\left\|\frac{\bar{\lambda}^2}{2\pi}\int_0^\tau B(\bar{\beta}_0, \bar{\rho}_0, s;\varepsilon)\ \mathrm{d}s + \int_0^\tau \bar{f}_1(\bar{\beta}(s;\varepsilon), \bar{\rho}(s;\varepsilon), s;\varepsilon)\ \mathrm{d}s\right\|_\infty \\ &\leqslant \frac{1}{\bar{\lambda}}(C\bar{\lambda}^2 + C\bar{\lambda}) \\ &\leqslant C\bar{\lambda} + C \\ &\leqslant C.\end{aligned}$$

Thus, for the base step $i + j = 0$, the bounds for A and B holds. Now, let us assume as induction step that

$$\|A\|_N + \|B\|_N \leqslant C$$

for some $N \in \mathbb{N}$. By differentiating equations (3.53), we get

$$\partial_{\bar{\beta}_0}[\bar{\beta}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)] = 1 + \bar{\lambda}\varepsilon(\partial_{\bar{\beta}_0}A), \quad \partial_{\bar{\beta}_0}[\bar{\rho}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)] = \bar{\lambda}\varepsilon(\partial_{\bar{\beta}_0}B), \tag{3.56}$$

$$\partial_{\bar{\rho}_0}[\bar{\beta}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)] = \frac{\bar{\lambda}}{2\pi}\tau + \bar{\lambda}\varepsilon(\partial_{\bar{\rho}_0}A), \quad \partial_{\bar{\rho}_0}[\bar{\rho}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)] = 1 + \bar{\lambda}\varepsilon(\partial_{\bar{\rho}_0}A) \tag{3.57}$$

$$\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\beta}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)] = \bar{\lambda}\varepsilon(\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}A), \quad \partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\rho}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)] = \bar{\lambda}\varepsilon(\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}B), \ \ i + j \geqslant 2.$$

The expressions above implies that, for $i + j \geqslant 1$,

$$|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\beta}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)]| + |\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\rho}(\tau, \bar{\beta}_0, \bar{\rho}_0;\varepsilon)]| \leqslant C(1 + \|A\|_{i+j} + \|B\|_{i+j}). \tag{3.58}$$

Then, by setting either $F = \bar{f}_1$ or $F = \bar{f}_2$, it follows from Lemma 3.6.1 that

$$\begin{aligned}
|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)]| &= \Bigg|\sum_{\substack{(k,p)\in\mathbb{N}_0^2:\ 1\leqslant k+p\leqslant i+j,\\ \vec{i}=(i_1,\cdots,i_{k+p}),\ |\vec{i}|=i,\\ \vec{j}=(j_1,\cdots,j_{k+p}),\ |\vec{j}|=j}} C_{k,p,\vec{i},\vec{j}}(\partial_1^k\partial_2^p F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)) \\
&\qquad \times \Big[(\partial^{i_1}_{\bar{\beta}_0}\partial^{j_1}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)])\dots(\partial^{i_k}_{\bar{\beta}_0}\partial^{j_k}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)]) \\
&\qquad \times (\partial^{i_{k+1}}_{\bar{\beta}_0}\partial^{j_{j+1}}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)])\dots(\partial^{i_{k+p}}_{\bar{\beta}_0}\partial^{j_{k+p}}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)])\Big]\Bigg| \\
&\leqslant C|\partial_{\bar{\beta}_0}F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)]| \\
&\qquad + C|\partial_{\bar{\rho}_0}F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)]| \\
&\qquad + \Bigg|\sum_{\substack{(k,p)\in\mathbb{N}_0^2:\ 2\leqslant k+p\leqslant i+j,\\ \vec{i}=(i_1,\cdots,i_{k+p}),\ |\vec{i}|=i,\\ \vec{j}=(j_1,\cdots,j_{k+p}),\ |\vec{j}|=j}} C_{k,p,\vec{i},\vec{j}}(\partial_1^k\partial_2^p F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)) \\
&\qquad \times \Big[(\partial^{i_1}_{\bar{\beta}_0}\partial^{j_1}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)])\dots(\partial^{i_k}_{\bar{\beta}_0}\partial^{j_k}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)]) \\
&\qquad \times (\partial^{i_{k+1}}_{\bar{\beta}_0}\partial^{j_{j+1}}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)])\dots(\partial^{i_{k+p}}_{\bar{\beta}_0}\partial^{j_{k+p}}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)])\Big]\Bigg|.
\end{aligned} \tag{3.59}$$

We recall that if $C_{k,p,\vec{i},\vec{j}} \neq 0$ in (3.59), then $1 \leqslant i_l + j_l \leqslant N$ for every $1 \leqslant l \leqslant k+p$. Then, taking into account the induction hypothesis, it follows that each non-zero term in the sum (3.59) must have, for $1 \leqslant l \leqslant k$,

$$|\partial^{i_l}_{\bar{\beta}_0}\partial^{j_l}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)]| \leqslant C(1+\|A\|_{i_l+j_l}+\|B\|_{i_l+j_l}) \leqslant C(1+\|A\|_N+\|B\|_N) \leqslant C,$$

and, similarly, for $k+1 \leqslant l \leqslant k+p$,

$$|\partial^{i_l}_{\bar{\beta}_0}\partial^{j_l}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)]| \leqslant C.$$

Consequently, Lemma 3.20 yields

$$\begin{aligned}
&\Bigg|\sum_{\substack{(k,p)\in\mathbb{N}_0^2:\ 2\leqslant k+p\leqslant i+j,\\ \vec{i}=(i_1,\cdots,i_{k+p}),\ |\vec{i}|=i,\\ \vec{j}=(j_1,\cdots,j_{k+p}),\ |\vec{j}|=j}} C_{k,p,\vec{i},\vec{j}}(\partial_1^k\partial_2^p F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)) \times \Big[(\partial^{i_1}_{\bar{\beta}_0}\partial^{j_1}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)])\dots(\partial^{i_k}_{\bar{\beta}_0}\partial^{j_k}_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)]) \\
&\qquad \times (\partial^{i_{k+1}}_{\bar{\beta}_0}\partial^{j_{j+1}}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)])\dots(\partial^{i_{k+p}}_{\bar{\beta}_0}\partial^{j_{k+p}}_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)])\Big]\Bigg| \leqslant C\bar{\lambda},
\end{aligned}$$

which implies that (3.59) is bounded as follows

$$|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)]| \leqslant C\bar{\lambda}(1+|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\beta}(\tau;\varepsilon)]|+|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[\bar{\rho}(\tau;\varepsilon)]|),$$

for either $F = \bar{f}_1$ or $F = \bar{f}_2$. By considering (3.56), (3.57), (3.58), and the fact that $\tau \in [0, 2\pi]$, we have

$$|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}[F(\bar{\beta}(\tau;\varepsilon),\bar{\rho}(\tau;\varepsilon),\tau;\varepsilon)]| \leqslant C\bar{\lambda}\left(1+\bar{\lambda}\left[\left\|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}A\right\|_\infty+\left\|\partial^i_{\bar{\beta}_0}\partial^j_{\bar{\rho}_0}B\right\|_\infty\right]\right). \tag{3.60}$$

Thus, from (3.55), (3.60), and the integral Leibniz rule, it follows that

$$\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}B\|_\infty \leqslant C\left(1+\bar\lambda\left[\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}A\right\|_\infty+\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}B\right\|_\infty\right]\right). \tag{3.61}$$

Hence, for sufficiently small $\bar\lambda$, we obtain $\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}B\right\|_\infty \leqslant C$. Taking into account (3.54), (3.60), and (3.61), we obtain

$$\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}A\right\|_\infty \leqslant C + C\bar\lambda\left(1+\bar\lambda\left[\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}A\right\|_\infty+\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}B\right\|_\infty\right]\right) \leqslant C,$$

and for sufficiently small $\bar\lambda$,

$$\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}A\right\|_\infty+\left\|\partial^i_{\bar\beta_0}\partial^j_{\bar\rho_0}B\right\|_\infty \leqslant C.$$

This concludes the proof by induction over $i+j$, since $\|A\|_{N+1}+\|B\|_{N+1}\leqslant C$, and consequently the proof of the lemma. □

4 Analysis of the Case $\mathcal{A}_2$

This chapter is dedicated to investigate the existence of invariant tori, boundedness of solutions, and periodic orbits of the differential equation

$$\ddot{x} + \operatorname{sgn}(x) = \theta x + \varepsilon\, p(t), \tag{4.1}$$

with $\theta > 0$, corresponding to case $\mathcal{A}_2$ in (1.3). This case presents a very rich dynamics, featuring phenomena such as the existence of heteroclinic connections, periodic orbits, unbounded solutions, and chaos (see Figure 13).

The persistence of heteroclinic connections and periodic orbits is explored in [23], where the Melnikov method is employed, and bifurcating functions are provided to verify the existence of such objects in a family of impact systems that comprises the differential equation (4.1). The authors in [23] also suggested the possibility of investigating the existence of two-dimensional invariant tori of (4.1) by means of the techniques employed in [34]. This suggestion was one of the motivations of the study presented in this chapter.

This chapter is organized as follows: Section 4.1 provides the main results of this chapter (Theorems C and D). Section 4.2 is devoted to present preliminary results concerning the unperturbed differential equation (4.1), which is very useful in constructing the coordinates changes, discussed in Section 4.3, that fit our problem into the Moser's Twist Map Theorem conditions. Finally, Section 4.5 is dedicated to the proofs of the main results.

4.1 Main results

In order to present the main results of this chapter, we recall some aspects concerning the differential equation (4.1). We start by considering the additional variable $y = \dot{x}$ and writing the corresponding differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \operatorname{sgn}(x) + \varepsilon\, p(t). \end{cases} \tag{4.2}$$

The differential system (4.2) is Hamiltonian with

$$H(x, y, t; \varepsilon) = H_{\mathcal{A}_2}(x, y, t; \varepsilon) = \frac{y^2}{2} + G(x) - \varepsilon x p(t), \tag{4.3}$$

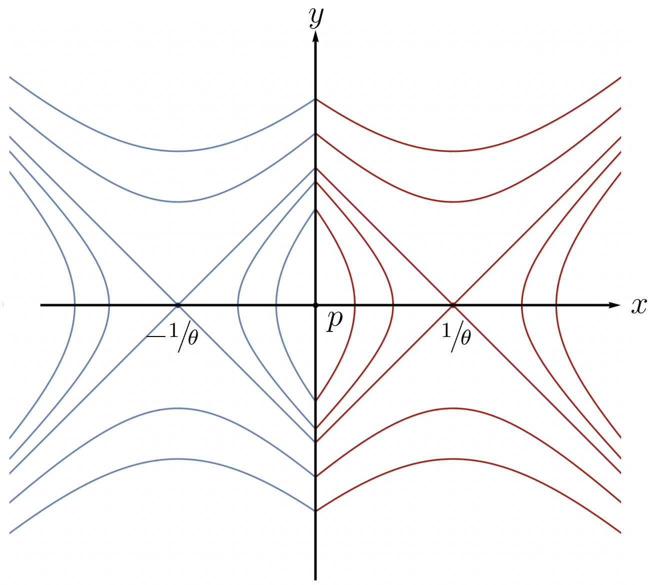


Figure 13 – Trajectories of (4.1) for $\varepsilon = 0$.

where $G(x) := G_{\mathcal{A}_2}(x) = -\theta\dfrac{x^2}{2} + |x|$. We stress that H is sufficiently smooth on y, t, and ε, but only continuous on x. When $\varepsilon = 0$, the differential system (4.2) simplifies to

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \operatorname{sgn}(x), \end{cases} \tag{4.4}$$

while the Hamiltonian H simplifies to $\mathbf{H}(x,y) := \frac{y^2}{2} + G(x) = H(x,y,t;0)$. Taking into account the set L_θ, whose definition is reminded below

$$L_\theta := \left\{(x,y) \in \mathbb{R}^2 : 0 \leqslant \mathbf{H}(x,y) < \frac{1}{2\theta} \text{ and } |x| < \frac{1}{\theta}\right\},$$

we state our first main result of this chapter concerning the existence of an invariant torus of (4.2). The proof is postponed to Section 4.5.

Theorem C. *Given a compact $K \subset L_\theta$ and a function $p \in \mathcal{C}^5(\mathbb{S}_\sigma)$, there exist $\varepsilon^*_{(K,p)} > 0$ such that, for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(K,p)}$, there exists an invariant torus $\mathcal{T}_\varepsilon$ of (4.2) whose intersection with the time section $t = 0$, to be denoted Λ_ε, satisfies*

$$K \subset \operatorname{int}(\Lambda_\varepsilon) \subset L_\theta,$$

where $\operatorname{int}(\Lambda_\varepsilon)$ denotes the open region in $\mathbb{R}^2$ enclosed by Λ_ε.

Since $\mathcal{T}_\varepsilon$ is an invariant torus of (4.2) it follows that any solution initiating in $\operatorname{int}(\Lambda_\varepsilon)$ must perpetually remain in $\operatorname{int}(\mathcal{T}_\varepsilon)$. This property leads us to following result concerning boundedness of solutions whose initial conditions lie within a compact set $K \subset L_\theta$.

Corollary 4.1.1. *For each compact $K \subset L_\theta$ and for each $p \in \mathcal{C}^5(\mathbb{S}_\sigma)$, there exist $\varepsilon^*_{(K,p)} > 0$ such that, for every $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(K,p)}$, all solutions of* (4.2) *with initial conditions $(x_0, y_0) \in K$ are bounded.*

Theorem C is very useful in constructing a family of nested invariant tori of (4.2), as given in the next result.

Theorem D. *Given $n \in \mathbb{N}$ and $p \in \mathcal{C}^5(\mathbb{S}_\sigma)$, there exists $\varepsilon^*_{(n,p)} > 0$ such that, for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(n,p)}$, the differential system* (4.2) *admits a family of nested invariant tori $\{\mathcal{T}^i_\varepsilon\}_{i=1}^n$. In addition, $\Lambda^n \to \partial L_\theta$ when $n \to +\infty$, with $\Lambda^n = \mathcal{T}^n_{\varepsilon^*_{(n,p)}} \cap (\{t = 0\} \times \mathbb{R}^2)$.*

The construction of a family of nested invariant tori for (4.2) implies the existence of periodic orbits for (4.2), as observed in the following result. The proof is entirely analogous to Theorem B.

Corollary 4.1.2. *Given $n \in \mathbb{N}$ and $p \in \mathcal{C}^5(\mathbb{S}_\sigma)$, there exists $\varepsilon^*_{(n,p)} > 0$ such that, for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(n,p)}$, the differential system* (4.2) *admits at least $n - 1$ periodic solutions.*

In order to illustrate the main result of this chapter (Theorem C), we present some numerical simulations concerning the solutions of the differential equation (4.1). Specifically, assuming $\theta = 1$ and $p(t) = \sin(2\pi t)$, and choosing a specific value for ε in (4.1), we consider several initial conditions for the differential equation (4.1). Subsequently, we plot 1000 points for each of them on the time section $(\{t = 1\} \times \mathbb{R}^2)$, as $p(t)$ is 1-periodic in this case.

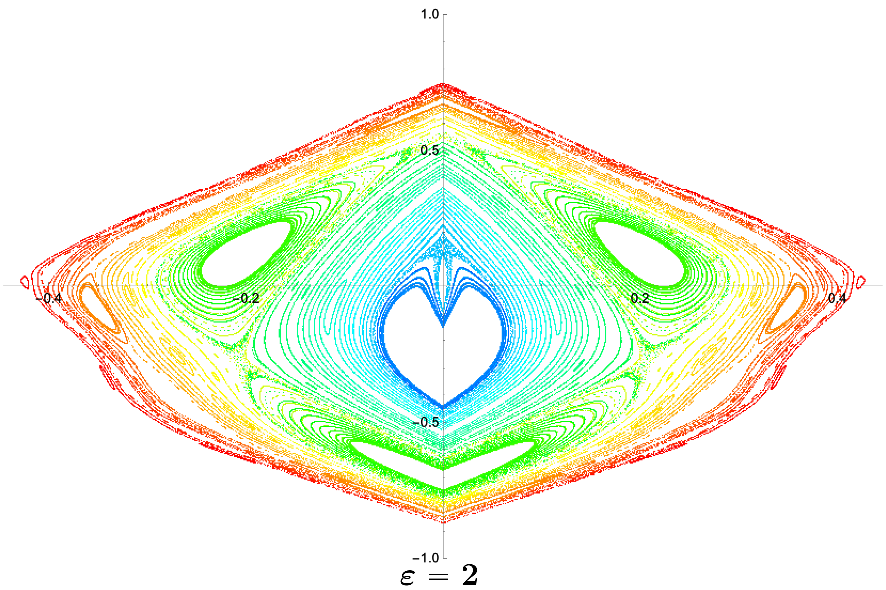
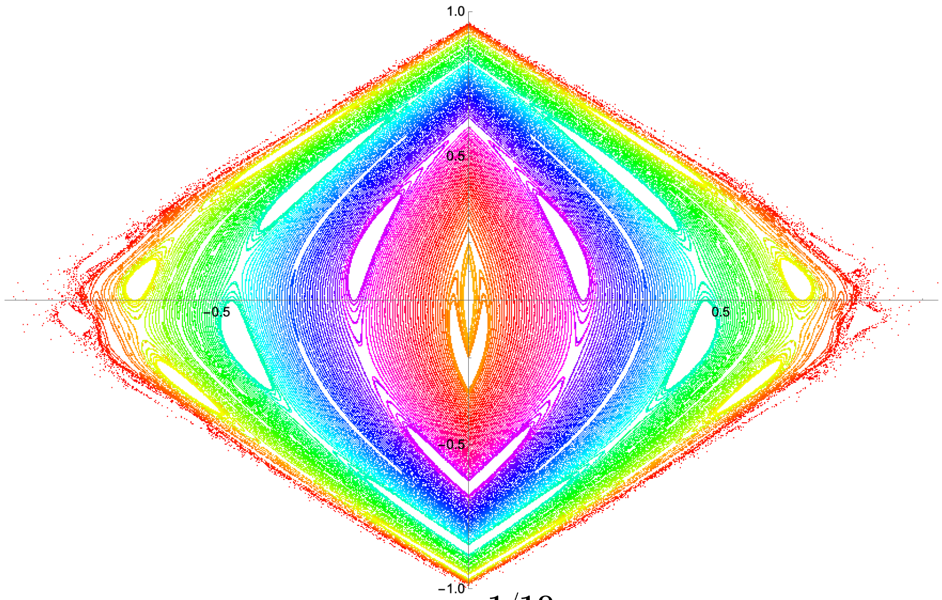
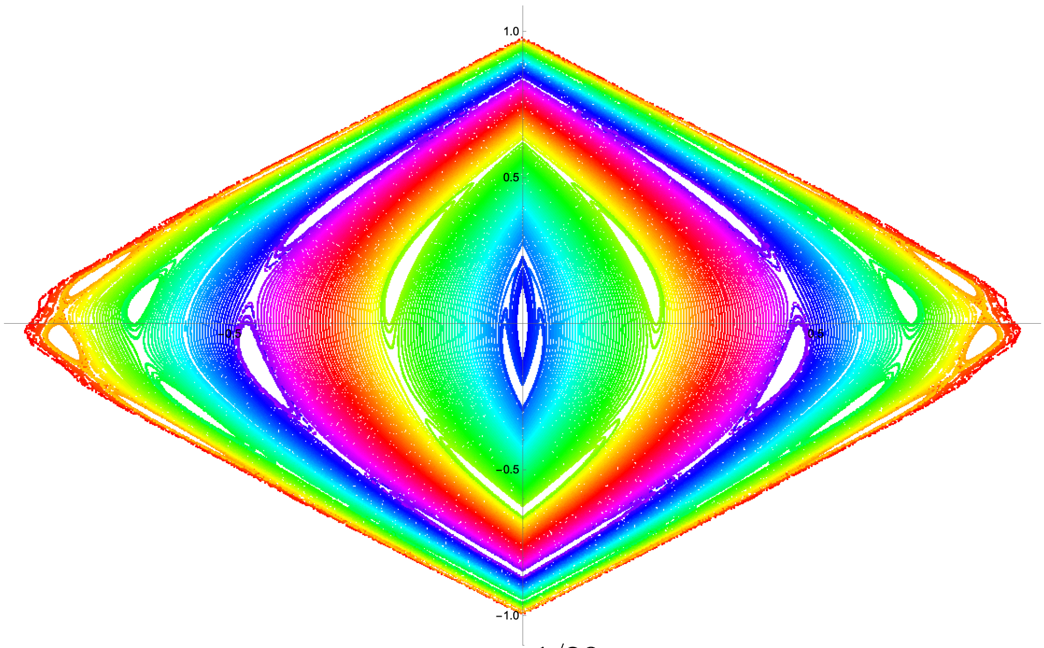


Figure 14 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (4.1), is indicated by the concentration of colors in the figure.



$\varepsilon = \mathbf{1/10}$

Figure 15 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (4.1), is indicated by the concentration of colors in the figure.



$\varepsilon = \mathbf{1/20}$

Figure 16 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (4.1), is indicated by the concentration of colors in the figure.

4.2 Preliminary results

This section provides important results concerning the unperturbed differential system (4.4), which is very useful in constructing coordinate changes that allow the application of Moser's Twist Map Theorem to a differential system equivalent to (4.2).

As previously remarked, the unperturbed Hamiltonian satisfies $\mathbf{H}(-x, y) = \mathbf{H}(x, y)$, which implies that the level curves $\mathcal{C}_h := \{(x, y) \in \mathbb{R}^2 : \mathbf{H}(x, y) = h\}$, with $h \in \mathbb{R}$, exhibit symmetry with respect to y-axis. We study these level curves by considering $x \geqslant 0$ in the Hamiltonian $\mathbf{H}$, and using the aforementioned symmetry to derive geometrical properties of $\mathcal{C}_h$ across the whole plane. For $x \geqslant 0$, the Hamiltonian $\mathbf{H}(x, y)$ can be expressed as

$$\mathbf{H}^r(x, y) = \frac{y^2}{2} - \frac{1}{2\theta}\left((1 - \theta x)^2 - 1\right),$$

with r being used to denote the Hamiltonian on the right side of the y-axis. We notice that, for $h \neq {}^1\!/_{2\theta}$, the energy level $\mathcal{C}^r_h$ is a hyperbola having $\mathcal{C}^r_{1/2\theta}$ as its asymptotes and the point $({}^1\!/_\theta, 0)$ as its focus. For $h > {}^1\!/_{2\theta}$, the vertices of $\mathcal{C}^r_h$ lie on the line $\{x = {}^1\!/_\theta\}$, while, for $h < {}^1\!/_{2\theta}$ the vertices of $\mathcal{C}^r_h$ lie on the x-axis. In the former case, the connected components of $\mathcal{C}^r_h$ are given by the sets

$$\mathcal{R}^r_h := \{(x, y) \in \mathcal{C}^r_h : |x| > {}^1\!/_\theta\} \quad \text{and} \quad \mathcal{L}^r_h := \{(x, y) \in \mathcal{C}^r_h : |x| < {}^1\!/_\theta\}.$$

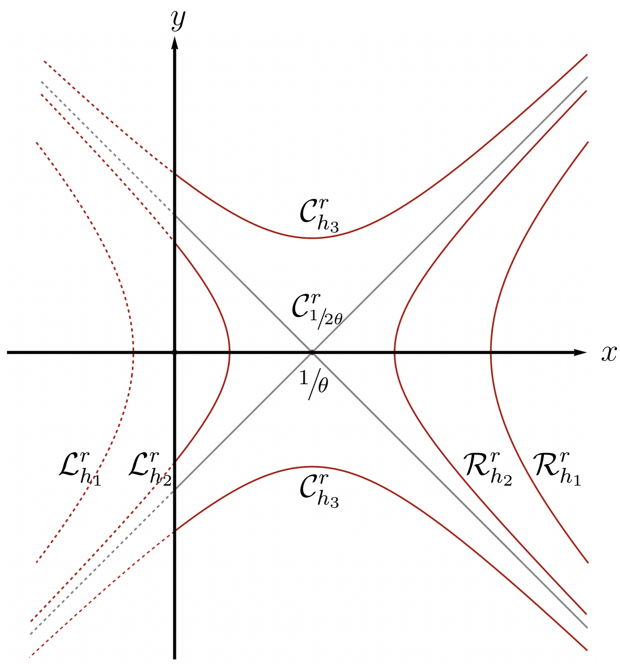


Figure 17 – Example of some level curves of $\mathbf{H}^r(x, y)$, with $h_1 < 0$, $0 < h_2 < {}^1\!/_{2\theta}$, and $h_3 > {}^1\!/_{2\theta}$.

When $h < 0$, the subsets $\mathcal{R}^r_h$ and $\mathcal{L}^r_h$ are entirely contained in the half-planes $\{x > 0\}$ and $\{x < 0\}$, respectively. On the other hand, if $0 \leqslant h < {}^1\!/_\theta$, then $\mathcal{L}^r_h \cap \{x > 0\} \neq \varnothing$ and $\mathcal{R}^r_h \cap \{x > 0\} = \mathcal{R}^r_h$. Moreover, $\mathcal{L}^r_h \cap \{x > 0\}$ is bounded, while $\mathcal{R}^r_h$ is unbounded (see Figure 17).

Back to the Hamiltonian $\mathbf{H}$, we notice that, for $0 < h < 1/2\theta$, the level curve $\mathcal{C}_h$ restricted to the band $|x| < 1/\theta$ is a closed curve carrying a periodic solution of (4.4). The set L_θ can, then, be described as follows

$$L_\theta = \dot{\bigcup}_{0 \leqslant h < 1/2\theta} \mathcal{C}_h.$$

Remark 4.2.1. *Given a compact $K \subset L_\theta$, there exists $h_K \in (0, 1/2\theta)$ such that $\mathbf{H}(x,y) \leqslant h_K$ for every $(x,y) \in K$. In other words, the level curve $\mathcal{C}_{h_K}$ encloses the set K.*

The points of intersection between $\mathcal{C}_h$ and the x-axis, when $0 \leqslant h \leqslant 1/2\theta$ and $|x| < 1/\theta$, are given by $\{(-a(h), 0), (a(h), 0)\}$, where

$$a(h) = \tfrac{1-\sqrt{1-2h\theta}}{\theta}.$$

Period function: Let us consider $G(x) = -\frac{1}{2\theta}F(x) = x - \theta\frac{x^2}{2}$. Given that G is an even function, the period of the solution in $(4.2)_{\varepsilon=0}$ on the level curve $\mathcal{C}_h$, for $0 \leqslant h < 1/2\theta$, is determined by the function

$$\begin{aligned} T(h) &= 4\int_0^{a(h)} \frac{\mathrm{d}u}{\sqrt{2(h-G(u))}} = \frac{4}{\sqrt{\theta}} \int_{-1/\sqrt{\theta}}^{\sqrt{\theta}a(h)-1/\sqrt{\theta}} \frac{\mathrm{d}u}{\sqrt{\frac{2h\theta-1}{\theta}+u^2}} \\ &= \frac{4}{\sqrt{\theta}}\left[\log\left|u + \sqrt{\frac{2h\theta-1}{\theta}+u^2}\right|\right]_{-1/\sqrt{\theta}}^{\sqrt{\theta}a(h)-1/\sqrt{\theta}} \\ &= \frac{2}{\sqrt{\theta}}\log\left(\frac{1+\sqrt{2h\theta}}{1-\sqrt{2h\theta}}\right). \end{aligned}$$

We notice that T is a strictly increasing function over $[0, 1/2\theta)$ and smooth in the interval $(0, 1/2\theta)$.

4.3 Action-angle transformation

In this Section we follow the procedure outlined in Section 3.3 to provide a transformation that puts the Hamiltonian (4.3) into action-angle variables. We recall that the **angle-function** is given by

$$\phi(x,h) = \begin{cases} \phi_1(x,h), & \text{if} \quad x, y \geqslant 0, \\ \pi - \phi_1(x,h), & \text{if} \quad x \geqslant 0,\ y \leqslant 0, \\ \pi + \phi_1(-x,h), & \text{if} \quad x, y \leqslant 0, \\ 2\pi - \phi_1(-x,h), & \text{if} \quad x \leqslant 0, y \geqslant 0, \end{cases}$$

for $0 < h < 1/2\theta$ and $|x| < 1/2\theta$. For the case $\mathcal{A}_2$, with $x \geqslant 0$, we have that $G(x) = -\theta\dfrac{x^2}{2} + x$, and the function $\phi_1(x,h)$ is given as follows

$$\begin{aligned}
\phi_1(x,h) &= \frac{2\pi}{T(h)}\int_0^x \frac{du}{\sqrt{2(h-G(u))}} \\
&= \frac{2\pi}{\sqrt{\theta}T(h)}\left[\log\left(u + \sqrt{\frac{2h\theta-1}{\theta} + u^2}\right)\right]_{-1/\sqrt{\theta}}^{\sqrt{\theta}x - 1/\sqrt{\theta}} \\
&= \frac{2\pi}{\sqrt{\theta}T(h)}\log\left(\frac{\theta x - 1 + \sqrt{2h\theta - 1 + (\theta x - 1)^2}}{1-\sqrt{2h\theta}}\right) \\
&= \pi\left(\frac{\log(\theta x - 1 + \sqrt{2h\theta - 1 + (\theta x-1)^2}) - \log(1-\sqrt{2h\theta})}{\log(1+\sqrt{2h\theta}) - \log(1-\sqrt{2h\theta})}\right),
\end{aligned}$$

whereas the **action-function** is given by

$$\begin{aligned}
A(h) &= 4\int_0^{a_-(h)} \sqrt{2(h-G(x))}\mathrm{d}x \\
&= \frac{4}{\sqrt{\theta}}\int_{-1/\sqrt{\theta}}^{\sqrt{\theta}a_-(h) - 1/\sqrt{\theta}} \sqrt{\frac{2h\theta-1}{\theta} + u^2\mathrm{d}u} \\
&= \frac{4}{\sqrt{\theta}}\left[\frac{u}{2}\sqrt{\frac{2h\theta-1}{\theta} + u^2} + \frac{1}{2}\frac{2h\theta-1}{\theta}\log\left|u + \sqrt{\frac{2h\theta-1}{\theta}+u^2}\right|\right]_{-1/\sqrt{\theta}}^{\sqrt{\theta}a_-(h)-1/\sqrt{\theta}} \\
&= \frac{2\sqrt{2h\theta}}{\theta^{3/2}} + \frac{(1-2h\theta)}{\theta^{3/2}}\log\left(\frac{1+\sqrt{2h\theta}}{1-\sqrt{2h\theta}}\right).
\end{aligned}$$

We remind that $A(h)$ corresponds to the area enclosed by the level curve $\mathcal{C}_h$ when $0 \leqslant h < 1/2\theta$ and $|x| < 1/\theta$.

Remark 4.3.1. *The functions $T : (0, 1/2\theta) \to (0,\infty)$ and $A : (0, 1/2\theta) \to (0, 2/\theta^{3/2})$ are smooth, and $A'(h) = T(h)$ for every $h \in (0, 1/2\theta)$. Since $T(h) > 0$ for every $h \in (0, 1/2\theta)$, it follows that A has a smooth inverse function, to be denoted by h_0.*

Lemma 4.3.2. *The transformation*

$$\begin{array}{rccc}
\Phi_1 : & L_\theta\backslash\{(0,0)\} & \longrightarrow & \mathbb{S}_{2\pi} \times (0, 2/\theta^{3/2}) \\
& (x,y) & \longmapsto & \left(\phi\left(x, \mathbf{H}(x,y)\right), A\left(\mathbf{H}(x,y)\right)\right) = \left(\phi(x,h), A(h)\right)
\end{array}$$

is a homeomorphism. Furthermore, if $x(\phi, I)$ is the one satisfying $\Phi_1^{-1}(\phi, I) = (x(\phi,I), y(\phi,I))$, then $x(\phi,I)$ is smooth on I.

Proof. In order to show that the transformation Φ_1 is onto, let us consider $(\phi^*, I^*) \in \mathbb{S}_{2\pi} \times \left(0, 2/\theta^{3/2}\right)$. Since A is a bijection between $(0, 1/(2\theta))$ and $\left(0, 2/\theta^{3/2}\right)$, there exists $h^* \in (0, 1/(2\theta))$ such that

$A(h^*) = I^*$. The value for x^* is given by $x^* = \tilde{x}(\phi^*, h^*)$ (see equation (4.6)), while y^* is obtained through the relation $h^* = \mathbf{H}(x^*, y^*)$. The injectivity of Φ_1 is demonstrated using the same procedure as outlined in Lemma 3.3.2. Continuity is clear from the explicit expressions of $\phi(x, h)$ and $A(h)$. In order to show that its inverse is continuous, we state the relation

$$x(\phi, I) = \tilde{x}(\phi, h_0(I)) = \tilde{x}(\phi, h). \tag{4.5}$$

Then, if $\phi \in (0, \pi/2)$, we have $\phi = \phi_1(\tilde{x}, h)$, that is,

$$\phi = \pi \left(\frac{\log(\theta\tilde{x}(\phi, h) - 1 + \sqrt{2h\theta - 1 + (\theta\tilde{x}(\phi, h) - 1)^2}) - \log(1 - \sqrt{2h\theta})}{\log(1 + \sqrt{2h\theta}) - \log(1 - \sqrt{2h\theta})} \right).$$

From the properties of the Logarithm function, we have the relation

$$\mathcal{K}(\phi, h) = (\theta\tilde{x}(\phi, h) - 1) - \sqrt{2h\theta - 1 + (\theta\tilde{x}(\phi, h) - 1)^2},$$

where $\mathcal{K}(\phi, h) := (1 + \sqrt{2h\theta})^{\phi/\pi}(1 - \sqrt{2h\theta})^{1-\phi/\pi}$. This implies that

$$-(\theta\tilde{x}(\phi, h) - 1) + \mathcal{K}(\phi, h) = -\sqrt{2h\theta - 1 + (\theta\tilde{x}(\phi, h) - 1)^2},$$

and consequently

$$-2(\theta\tilde{x}(\phi, h) - 1)\mathcal{K}(\phi, h) + (\mathcal{K}(\phi, h))^2 = 2h\theta - 1.$$

The relation above leads us to

$$\tilde{x}(\phi, h) = \frac{1}{\theta} - \frac{1}{2\theta}\left(\left(\frac{1 - \sqrt{2h\theta}}{1 + \sqrt{2h\theta}} \right)^{\phi/\pi} (1 + \sqrt{2h\theta}) + \left(\frac{1 + \sqrt{2h\theta}}{1 - \sqrt{2h\theta}} \right)^{\phi/\pi} (1 - \sqrt{2h\theta}) \right). \tag{4.6}$$

In analogous way, we obtain the expression for $\tilde{x}(\phi, h)$ when $\phi \in [\pi, 2\pi]$. Thus, as a complete expression for $\tilde{x}(\phi, h)$, we have

$$\tilde{x}(\phi, h) = \begin{cases} \dfrac{1}{\theta} - \dfrac{1}{2\theta}\left(\left(\dfrac{1 - \sqrt{2h\theta}}{1 + \sqrt{2h\theta}} \right)^{\phi/\pi} (1 + \sqrt{2h\theta}) + \left(\dfrac{1 + \sqrt{2h\theta}}{1 - \sqrt{2h\theta}} \right)^{\phi/\pi} (1 - \sqrt{2h\theta}) \right) & \text{if} \quad \phi \in [0, \pi), \\[2ex] -\dfrac{1}{\theta} + \dfrac{1}{2\theta}\left(\left(\dfrac{1 - \sqrt{2h\theta}}{1 + \sqrt{2h\theta}} \right)^{\phi/\pi - 1} (1 + \sqrt{2h\theta}) + \left(\dfrac{1 + \sqrt{2h\theta}}{1 - \sqrt{2h\theta}} \right)^{\phi/\pi - 1} (1 - \sqrt{2h\theta}) \right) & \text{if} \quad \phi \in [\pi, 2\pi]. \end{cases}$$

Notice that, for each $0 < h < 1/2\theta$, $\lim_{\phi \to \phi_0} \tilde{x}(\phi, h) = \tilde{x}(\phi_0, h)$ holds for every $\phi_0 \in \{0, \pi, 2\pi\}$. Consequently, $\tilde{x}$ exhibits continuity over $\mathbb{S}_{2\pi} \times (0, 1/2\theta)$, and this continuity extends to x (regarded as a function) over $\mathbb{S}_{2\pi} \times (0, 2/\theta^{3/2})$ through the relationship (4.5), and therefore Φ_1 is a homeomorphism. In addition, for fixed values of ϕ, we observe that $x(\phi, I)$ is composed only by smooth functions, which concludes the proof of the lemma.

□

By performing Φ_1 to the Hamiltonian (4.3), we have

$$\mathcal{H}(\phi, I, t; \varepsilon) = h_0(I) - \varepsilon x(\phi, I)p(t),$$

which is defined on $\mathbb{S}_{2\pi} \times (0, 2/\theta^{3/2}) \times \mathbb{S}_\sigma$, for every $\varepsilon \in \mathbb{R}$. Notice that the Hamiltonian $\mathcal{H}$ is continuous in ϕ, smooth in I, and of class $\mathcal{C}^5$ in t. Besides that, taking $I_K = A(h_K)$, where h_K is the energy provided in Remark 4.2.1 for a given compact $K \subset L_\theta$, the following lemma holds.

Lemma 4.3.3. *Given a compact set $K \subset L_\theta$, there exists $\varepsilon_{(K,p)} > 0$ and $I_1, I_2 \in (I_K, 2/\theta^{3/2})$, with $I_1 < I_2$, such that $\partial_I \mathcal{H}(\phi, I, t; \varepsilon) > 0$ for every $I \in [I_1, I_2]$ and $0 \leqslant \varepsilon \leqslant \varepsilon_{(K,p)}$.*

Proof. From Remark 4.2.1, it follows that there exists $h_K \in (0, 1/2\theta)$ such that $\mathbf{H}(x, y) \leqslant h_K$. As the sequence $\mathcal{C}_h$ converges to ∂L_θ as h tends to $1/2\theta^-$, there exist distinct values h_1 and h_2 in $(h_K, 1/2\theta)$, with $h_1 < h_2$, satisfying

$$\mathcal{D}(h_1, h_2) := \{(x, y) \in L_\theta : h_1 \leqslant \mathbf{H}(x, y) \leqslant h_2\} \subsetneq L_\theta, \tag{4.7}$$

with each level curve $\mathcal{C}_h \subset \mathcal{D}(h_1, h_2)$ enclosing the compact set K (see Figure 18). The action-angle transformation Φ_1 provided in Lemma 4.3.2 transforms the annulus $\mathcal{D}(h_1, h_2)$ into the annulus $\mathbb{S}_{2\pi} \times [I_1, I_2]$, where $I_i = A(h_i)$, for $i = 1, 2$. For fixed values of ϕ and t in the Hamiltonian $\mathcal{H}(\phi, I, t; \varepsilon)$, and taking into account Lemma 4.3.2, we have that

$$\partial_I \mathcal{H}(\phi, I, t; \varepsilon) = h_0'(I)(1 - \varepsilon \partial_h \tilde{x}(\phi, h_0(I))p(t)),$$

where we have used relationship (4.5). Since $h_0'(I) > 0$ for every $I \in [I_1, I_2]$ and $\partial_h \tilde{x}(\phi, h_0(I))p(t)$ is continuous on $\mathbb{S}_{2\pi} \times [I_1, I_2] \times \mathbb{S}_\sigma$, there exists $\varepsilon_{(K,p)} > 0$ such that

$$1 - \varepsilon \partial_h \tilde{x}(\phi, h_0(I))p(t) > 0,$$

for every $(\phi, I, t) \in \mathbb{S}_{2\pi} \times [I_1, I_2] \times \mathbb{S}_\sigma$ and for each $\varepsilon \in [0, \varepsilon_{(K,p)}]$. This completes the proof of the lemma. □

4.4 Angle and energy as new time and position

In order to overcome the lack of regularity of the Hamiltonian $\mathcal{H}$ in the angle variable, we follow the method presented in Section 2.2.3 to construct another coordinate change that turns $\mathcal{H}$ into a sufficiently smooth Hamiltonian in both angle and action variables. From Lemma 4.3.3, we have that, for fixed $\phi \in \mathbb{S}_{2\pi}$ and $t \in \mathbb{S}_\sigma$, and for each $\varepsilon \in [0, \varepsilon_{(K,p)})$, the function $\mathcal{H}(\phi, \cdot, t; \varepsilon)$ is invertible in $[I_1, I_2]$. Thus, for every $\varepsilon \in [0, \varepsilon_{(K,p)}]$, the transformation

$$\begin{array}{rccc} \Phi_2 : & \mathbb{S}_{2\pi} \times [I_1, I_2] \times \mathbb{S}_\sigma & \longrightarrow & \mathbb{S}_\sigma \times [r_1, r_2] \times \mathbb{S}_{2\pi} \\ & (\phi, I, t) & \longmapsto & (t, \mathcal{H}(\phi, I, t; \varepsilon), \phi), \end{array}$$

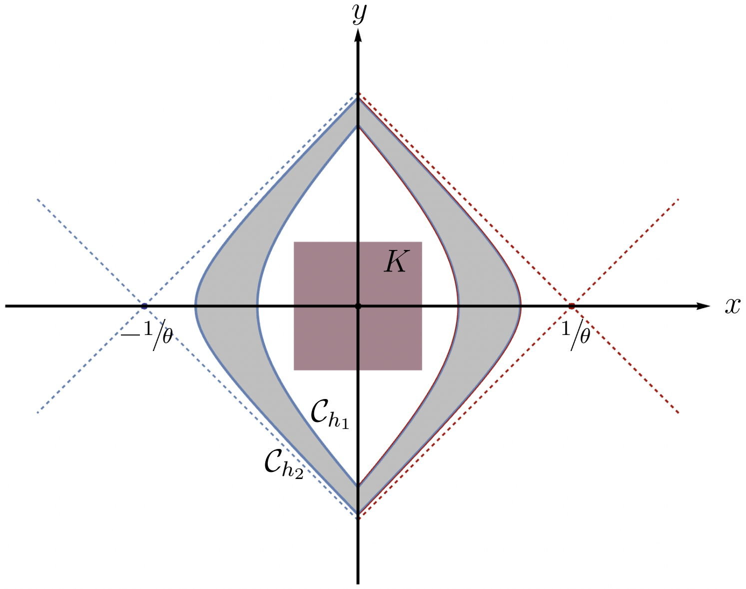


Figure 18 – The shaded gray area corresponds to the annulus $\mathcal{D}(h_1, h_2)$, where each level curve $\mathcal{C}_h \subset \mathcal{D}(h_1, h_2)$ encloses the compact set K.

with $r_i = h_0(I_i)$, for $i = 1, 2$, defines a diffeomorphism on the extended phase space that preserves the Hamiltonian character of $\mathcal{H}$, as mentioned in Section 2.2.3. Taking into account that $\mathcal{H}(\phi, I, t; 0) = h_0(I)$ and denoting the new variables as $(\beta, r, \tau) \in \mathbb{S}_\sigma \times [r_1, r_2] \times \mathbb{S}_{2\pi}$, we can express the new Hamiltonian as follows:

$$\mathscr{H}(\beta, r, \tau; \varepsilon) = A(r) + \varepsilon \mathscr{H}_1(\beta, r, \tau; \varepsilon), \tag{4.8}$$

where $\mathscr{H}_1$ is defined by the relation above.

Remark 4.4.1. *The Hamiltonian $\mathscr{H}(\beta, r, \tau; \varepsilon)$ is of class $\mathcal{C}^5$ in β, smooth in r, and continuous on τ. In addition, the compactness of $\mathbb{S}_\sigma \times [r_1, r_2] \times \mathbb{S}_{2\pi}$ ensures the boundedness of $\mathscr{H}_1(\beta, r, \tau; \varepsilon)$ and all its partial derivatives with respect to β and r, which means that there exists C depending on K and $p(\beta)$, such that*

$$\left|\partial_\beta^i \partial_r^j \mathscr{H}_1(\beta, r, \tau; \varepsilon)\right| \leqslant C \quad \textit{for} \quad 0 \leqslant i + j \leqslant 5.$$

Since $A'(r) = T(r)$, it follows that, for every $\varepsilon \in [0, \varepsilon_{(K,p)})$, the Hamiltonian differential system associated with (4.8) is given by

$$\begin{cases} \dfrac{\mathrm{d}\beta}{\mathrm{d}\tau} = \dfrac{\partial \mathscr{H}}{\partial r}(\beta, r, \tau; \varepsilon) = T(r) + \varepsilon \partial_r \mathscr{H}_1(\beta, r, \tau; \varepsilon), \\ \dfrac{\mathrm{d}r}{\mathrm{d}\tau} = -\dfrac{\partial \mathscr{H}}{\partial \beta}(\beta, r, \tau; \varepsilon) = -\varepsilon \partial_\beta \mathscr{H}_1(\beta, r, \tau; \varepsilon). \end{cases} \tag{4.9}$$

Moreover, the existence and uniqueness of solutions for (4.9) are established due to their smoothness in the respective components. We denote by $(\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon), r(\tau,\tau_0,\beta_0,r_0;\varepsilon))$ the solutions of (4.9) satisfying $(\beta(\tau_0,\tau_0,\beta_0,r_0;\varepsilon), r(\tau_0,\tau_0,\beta_0,r_0;\varepsilon)) = (\beta_0,r_0) \in \mathbb{S}_{2\pi} \times [r_1,r_2]$, and we state that such solutions are defined for every $\tau \in \mathbb{R}$, as noticed in the following result.

Lemma 4.4.2. *For every $\varepsilon \in [0,\varepsilon_{(K,p)}]$ and for each $(\beta_0,r_0) \in \mathbb{S}_{2\pi} \times [r_1,r_2]$, the maximal solution of* (4.9) *having (β_0,r_0) as initial condition has $\mathbb{R}$ as its maximal interval of definition.*

Proof. We first notice that the differential system (4.9) is defined for every $(\beta,r,\tau) \in \mathbb{S}_\sigma \times [r_1,r_2] \times \mathbb{S}_{2\pi}$ and for every $\varepsilon \in [0,\varepsilon_{(K,p)}]$. Thus, proceeding by reduction to absurdity, let us assume that there exists $\varepsilon_* \in [0,\varepsilon_{(K,p)}]$ and $(\beta_0,r_0) \in \mathbb{S}_\sigma \times [r_1,r_2]$ for which the corresponding solution $(\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon_*), r(\tau,\tau_0,\beta_0,r_0;\varepsilon_*))$ of (4.9) is not defined for every $\tau \in \mathbb{R}$. In other words, there exists $\tau_0 < \tau_1 < \infty$ such that $(\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon), r(\tau,\tau_0,\beta_0,r_0;\varepsilon))$ cannot be continuously extended to an interval that properly contains $[\tau_0,\tau_1]$.

Since $(\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon), r(\tau,\tau_0,\beta_0,r_0;\varepsilon_*))$ is a solution of (4.9), it follows that $r(\tau,\tau_0,\beta_0,r_0;\varepsilon_*) \in [r_1,r_2]$ for every $\tau \in [\tau_0,\tau_1]$, which implies that $T(r(\tau,\tau_0,\beta_0,r_0;\varepsilon_*)) \in [T(r_1),T(r_2)]$ for every $\tau \in [\tau_0,\tau_1]$. In addition, $\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon_*)$ must satisfy the integral equation

$$\begin{aligned}\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon_*) = \beta_0 + \int_{\tau_0}^{\tau} & T(r(s,\tau_0,\beta_0,r_0;\varepsilon_*)) \\ & + \varepsilon_* \partial_r \mathscr{H}_1(\beta(s,\tau_0,\beta_0,r_0;\varepsilon_*), r(s,\tau_0,\beta_0,r_0;\varepsilon), s;\varepsilon_*) \mathrm{d}s.\end{aligned}$$

From the comments above and Remark 4.4.1, it follows that, there exists a constant $K_0 > 0$ depending on $T(r_1)$, $T(r_2)$, and $\partial_r \mathscr{H}_1$ such that

$$|\beta(\tau,\tau_0,\beta_0,r_0;\varepsilon_*)| \leqslant \beta_0 + K_0(\tau_1 - \tau_0).$$

This implies that the curve described by the function $g(\tau) = (\tau, \beta(\tau,\tau_0,\beta_0,r_0;\varepsilon_*), r(\tau,\tau_0,\beta_0,r_0;\varepsilon_*))$ is entirely contained in the compact $[\tau_0,\tau_1] \times [-\beta_0 - K_0(\tau_1-\tau_0), \beta_0 + K_0(\tau_1-\tau_0)] \times [r_1,r_2]$, which contradicts Theorem 2.1.2. This completes the proof of the lemma. □

In order to simplify the expressions of the differential system (4.9), we propose the following change of variables:

$$\begin{aligned}\Phi_3 \quad : \mathbb{S}_\sigma \times [r_1,r_2] \quad &\longrightarrow \quad \mathbb{S}_{2\pi} \times [\bar{\rho}_1,\bar{\rho}_2] \\ (\beta,r) \qquad &\longmapsto \quad \left(\frac{2\pi}{\sigma}\beta, \frac{2\pi}{\sigma}T(r)\right),\end{aligned}$$

where $\bar{\rho}_i = \dfrac{2\pi}{\sigma}T(r_i)$, for $i = 1, 2$. Thus, by denoting $(\bar{\beta}, \bar{\rho}) = \Phi_3(\beta, r)$, the differential system (4.9) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}\bar{\beta}}{\mathrm{d}\tau} = \bar{\rho} + \varepsilon \overline{f}_1(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) \\ \dfrac{\mathrm{d}\bar{\rho}}{\mathrm{d}\tau} = \varepsilon \overline{f}_2(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) \end{cases} \tag{4.10}$$

with

$$\overline{f}_1(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) = \partial_r \mathscr{H}_1(\tfrac{\sigma}{2\pi}\bar{\beta}, \bar{r}(\bar{\rho}), \tau; \varepsilon)$$

and

$$\overline{f}_2(\bar{\beta}, \bar{\rho}, \tau; \varepsilon) = -\tfrac{2\pi}{\sigma} T'(\bar{r}(\bar{\rho}))\partial_\beta \mathscr{H}_1(\tfrac{\sigma}{2\pi}\bar{\beta}, \bar{r}(\bar{\rho}), \tau; \varepsilon),$$

where $\bar{r}(\bar{\rho}) = T^{-1}\left(\frac{\sigma}{2\pi}\bar{\rho}\right)$. In this setup, it follows that $\overline{f}_1$ and $\overline{f}_2$ are $\mathcal{C}^4$ and 2π-periodic in $\bar{\beta}$, and smooth in $\bar{\rho}$. Furthermore, the bounds provided in the Remark 4.4.1 extend to $\overline{f}_1$ and $\overline{f}_2$, yielding the lemma.

Lemma 4.4.3. *There exists a positive constant $\overline{C}$ depending on K, p and θ such that*

$$\left|\partial_{\bar{\beta}}^i \partial_{\bar{\rho}}^j \overline{f}_1(\bar{\beta}, \bar{\rho}, \tau; \varepsilon)\right| + \left|\partial_{\bar{\beta}}^i \partial_{\bar{\rho}}^j \overline{f}_2(\bar{\beta}, \bar{\rho}, \tau; \varepsilon)\right| \leqslant \overline{C} \quad \text{for} \quad 0 \leqslant i + j \leqslant 4,$$

for every $(\bar{\beta}, \bar{\rho}, \tau) \in \mathbb{S}_{2\pi} \times [\bar{\rho}_1, \bar{\rho}_2] \times [0, 2\pi]$, and for each $\varepsilon \in [0, \varepsilon_{(K,p)}]$.

Let us denote by $(\bar{\beta}(\tau; \varepsilon), \bar{\rho}(\tau; \varepsilon)) = (\bar{\beta}(\bar{\beta}_0, \bar{\rho}_0, \tau; \varepsilon), \bar{\rho}(\bar{\beta}_0, \bar{\rho}_0, \tau; \varepsilon))$ the solution of (4.10) whose initial condition is $(\bar{\beta}(0; \varepsilon), \bar{\rho}(0; \varepsilon)) = (\bar{\beta}_0, \bar{\rho}_0)$. We see that for $\varepsilon = 0$, the solutions of (4.10) are explicitly given by

$$(\bar{\beta}(\tau; 0), \bar{\rho}(\tau; 0) = (\bar{\beta}_0 + \bar{\rho}_0\tau, \bar{\rho}_0).$$

Then, for $0 \leqslant \varepsilon \leqslant \varepsilon_{(K,p)}$, where we redefine $\varepsilon_{(K,p)}$ to be the one given in Lemma 4.3.3 and also allows the Taylor expansion of $(\bar{\beta}(\tau; \varepsilon), \bar{\rho}(\tau; \varepsilon))$, it follows that

$$\bar{\beta}(\tau; \varepsilon) = \bar{\beta}_0 + \tau\bar{\rho}_0 + \varepsilon A(\bar{\beta}_0, \bar{\rho}_0, \tau; \varepsilon) \quad \text{and} \quad \bar{\rho}(\tau; \varepsilon) = \bar{\rho}_0 + \varepsilon B(\bar{\beta}_0, \bar{\rho}_0, \tau; \varepsilon), \tag{4.11}$$

where

$$B(\bar{\beta}_0, \bar{\rho}_0, \tau; \varepsilon) = \int_0^\tau \overline{f}_2(\bar{\beta}(\tau; \varepsilon), \bar{\rho}(s; \varepsilon), s; \varepsilon)\mathrm{d}s$$

and

$$A(\bar{\beta}_0, \bar{\rho}_0, \tau; \varepsilon) = \int_0^\tau B(\bar{\beta}_0, \bar{\rho}_0, s; \varepsilon) + \overline{f}_1(\bar{\beta}(s; \varepsilon), \bar{\rho}(s; \varepsilon), s; \varepsilon)\mathrm{d}s$$

For $\varepsilon \in [0, \varepsilon_{(K,p)}]$, we define the Poincaré-map $P : \mathbb{S}_{2\pi} \times [\bar{\rho}_1, \bar{\rho}_2] \to \mathbb{R}^2$ associated with (4.10) in the following way:

$$P(\bar{\beta}_0, \bar{\rho}_0; \varepsilon) = (\bar{\beta}(\beta_0, \bar{\rho}_0, 2\pi; \varepsilon), \bar{\rho}(\beta_0, \bar{\rho}_0, 2\pi; \varepsilon)),$$

which, by considering relationship (4.11), can be expressed as

$$P(\bar{\beta}_0, \bar{\rho}_0; \varepsilon) = (\bar{\beta}_0 + 2\pi\bar{\rho}_0 + \varepsilon P_1(\bar{\beta}_0, \bar{\rho}_0; \varepsilon),\ \bar{\rho}_0 + \varepsilon P_2(\bar{\beta}_0, \bar{\rho}_0; \varepsilon)),$$

with $P_1(\bar{\beta}_0, \bar{\rho}_0; \varepsilon) = A(\bar{\beta}_0, \bar{\rho}_0, 2\pi; \varepsilon)$ and $P_2(\bar{\beta}_0, \bar{\rho}_0; \varepsilon) = B(\bar{\beta}_0, \bar{\rho}_0, 2\pi; \varepsilon)$. One can see that P is 2π-periodic in $\bar{\beta}_0$ and of class $\mathcal{C}^4$ in $\mathbb{S}_{2\pi} \times [\bar{\rho}_1, \bar{\rho}_2]$. Besides that, the bounds in Lemma 4.4.3 naturally extend to P_1 and P_2, yielding the following.

Lemma 4.4.4. *There exists a positive constant $\tilde{C} > 0$ depending on $p(\bar{\beta}_0)$ and K such that*

$$\|P_1(\cdot; \varepsilon)\|_{\mathcal{C}^4(\mathbb{S}_{2\pi}\times[\bar{\rho}_1,\bar{\rho}_2])} + \|P_2(\cdot; \varepsilon)\|_{\mathcal{C}^4(\mathbb{S}_{2\pi}\times[\bar{\rho}_1,\bar{\rho}_2])} \leqslant \tilde{C},$$

for every $\varepsilon \in [0, \varepsilon_{(K,p)}]$.

4.5 Proofs of the main results

We start by proving Theorem C.

Proof of Theorem C. Let us begin by defining $\alpha(\bar{\rho}) = 2\pi\bar{\rho}$. We notice that $\alpha \in \mathcal{C}^4([\bar{\rho}_1, \bar{\rho}_2])$ and $\alpha'(\bar{\rho}) = 2\pi > 0$, for every $\bar{\rho} \in [\bar{\rho}_1, \bar{\rho}_2]$. The map P has the intersection property since it is conjugated to a Poincaré-map originating from a Hamiltonian system (Proposition 2.3.11), and its lift can be written in the same form presented in *Moser's Twist Map Theorem* due to its 1-periodicity in $\bar{\beta}_0$. Hence, there exists $\kappa > 0$, depending on $\bar{\rho}_2 - \bar{\rho}_1$ and α, such that if

$$\|\varepsilon P_1\|_{\mathcal{C}^4(\mathbb{S}_{2\pi}\times[\bar{\rho}_1,\bar{\rho}_2])} + \|\varepsilon P_2\|_{\mathcal{C}^4(\mathbb{S}_{2\pi}\times[\bar{\rho}_1,\bar{\rho}_2])} < \kappa,$$

then P has invariant curves.

By taking $\varepsilon^*_{(K,p)} = \min\{\kappa/2\tilde{C}, \varepsilon_{(K,p)}\}$, where $\tilde{C} > 0$ is the constant provided in Lemma 4.4.4 and $\varepsilon_{(K,p)} > 0$ is provided in Lemma 4.3.3, it follows that for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(K,p)}$, there exists an invariant closed curve Γ_ε of P. This implies that the saturation of Γ_ε by the flow of (4.10) correspond to an invariant torus of (4.10), which is going to be denoted $\bar{\mathcal{T}}_\varepsilon$. By reversing all the transformations applied so far, we have that $\bar{\mathcal{T}}_\varepsilon$ is transformed into an invariant torus of (4.2), to be denoted $\mathcal{T}_\varepsilon$, whose intersection with the time section $t = 0$ is a Jordan curve Λ_ε entirely contained in the annulus $\mathcal{D}(h_1, h_2)$ (see (4.7)). Since every closed curve contained in $\mathcal{D}(h_1, h_2)$ encloses the compact set K, we conclude this proof. □

As previously mentioned, Corollary 4.1.1 follows as a direct consequence of Theorem C, while Theorem D is proved by recursively applying Theorem C, as we shall see in the sequel.

Proof of Theorem D. Let $n \in \mathbb{N}$ and $p \in \mathcal{C}^5(\mathbb{S}_\sigma)$ be fixed. We define $K_1 = \mathcal{C}_{h_1}$, with $h_1 = 0$, and we notice that K_1 is compact set contained in L_θ. Then, from Theorem C, there exists $\varepsilon^*_{(K_1,p)} > 0$, such that for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(K_1,p)}$, there exists an invariant torus $\mathcal{T}^1_\varepsilon$ of (4.2) such that $\Lambda^1_\varepsilon := \mathcal{T}^1_\varepsilon \cap (\{t = 0\} \times \mathbb{R}^2)$ satisfies

$$K_1 \subset \mathrm{int}(\Lambda^1_\varepsilon) \subsetneq L_\theta.$$

From Remark 4.2.1, there exists $h_2 \in (0, 1/2\theta)$ such that the level curve $\mathcal{C}_{h_2}$ encloses Λ^1_ε. By defining $K_2 := \mathcal{C}_{h_2}$, we can apply Theorem C and, then, obtain an $0 < \varepsilon^*_{(K_2,p)} \leqslant \varepsilon^*_{(K_1,p)}$, such that for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(K_2,p)}$, there exists an invariant torus $\mathcal{T}^2_\varepsilon$ of (4.2) such that $\Lambda^2_\varepsilon := \mathcal{T}^2_\varepsilon \cap (\{t = 0\} \times \mathbb{R}^2)$ satisfies

$$\mathrm{int}(\Lambda^1_\varepsilon) \subsetneq \mathrm{int}(\mathcal{C}_{h_2}) \subsetneq \mathrm{int}(\Lambda^2_\varepsilon) \subsetneq L_\theta.$$

By taking Remark 4.2.1 into account and recursively applying Theorem C, we obtain an increasing sequence of energies $0 = h_1 < h_2 < \cdots < h_n < 1/2\theta$ and $0 < \varepsilon^*_{(K_n,p)} = \varepsilon^*_{(n,p)} < \cdots < \varepsilon^*_{(K_1,p)}$, such that for each $0 \leqslant \varepsilon \leqslant \varepsilon^*_{(n,p)}$ there exists a sequence $\{\mathcal{T}^i_\varepsilon\}_{i=1}^n$ of invariant tori of (4.2). Furthermore, by defining $\Lambda^n := \mathcal{T}^n_{\varepsilon^*_{(n,p)}} \cap (\{t = 0\} \times \mathbb{R}^2)$, we have that, for $n \in \mathbb{N}$,

$$\mathcal{C}_{h_1} \subsetneq \mathrm{int}(\Lambda^1) \subsetneq \mathrm{int}(\mathcal{C}_{h_2}) \subsetneq \mathrm{int}(\Lambda^2) \subsetneq \cdots \mathrm{int}(\mathcal{C}_{h_n}) \subsetneq \mathrm{int}(\Lambda^n) \subsetneq L_\theta.$$

Since $\mathcal{C}_h \to \partial L_\theta$ when $h \to 1/(2\theta)$, it follows from the construction above that $\Lambda^n \to \partial L_\theta$ when $n \to +\infty$, and this completes the proof of the theorem. $\square$

5 Analysis of the Case $\mathcal{A}_3$

The findings presented in this chapter are based on the paper [51].

In this chapter, we investigate the existence of invariant tori and the boundedness of solutions for the differential equation

$$\ddot{x} + \operatorname{sgn}(x) = \varepsilon p(t),$$

which corresponds to case $\mathcal{A}_3$. For such a case, our analysis is non-perturbative, meaning that the results presented in this chapter holds for every value of ε. Consequently, the perturbative parameter ε in (1.3) is absorbed by the function $p(t)$, simplifying the differential equation above to

$$\ddot{x} + \operatorname{sgn}(x) = p(t). \tag{5.1}$$

Furthermore, by assuming $p(t)$ to be a periodic function with a vanishing average, we employ a constructive method to obtain results similar to those in Chapters 3 and 4. In addition to not using KAM theory for case $\mathcal{A}_3$, a notable difference between the approach used in this chapter and the former ones lies in the fact that the regularity of $p(t)$ can be significantly reduced to Lebesgue-integrable.

As mentioned in [37], Ortega in a talk [54] at Academia Sinica in 1998 suggested the question of whether all solutions of (1.1), when $g(x) = \arctan(x)$ and $p(t)$ is periodic, are bounded or not. In this case, the saturation function is bounded and generates a small twist at infinity, which makes it difficult to apply the standard versions of the *Moser's Twist Map Theorem*. It fell to Li [37] to first answered this question, in the case that $p(t)$ is a $\mathcal{C}^\infty$ periodic function with vanishing average. In [64], Wang improved the result of Li by considering $p(t)$ as a $\mathcal{C}^5$ periodic function with some smallness condition on its average. The non-smooth forced oscillator (5.1) represents a limit scenario to the case introduced by Ortega in [54]. Particularly, the differential equation (5.1) provides models of electronic circuit in the presence of a relay as noticed by [30].

The goal of this chapter is to provide a simple proof for the boundedness of every solutions of the differential equation (5.1) in the case that $p(t)$ is a Lebesgue-integrable periodic function with vanishing average. Despite of the vanishing average restriction, equation (5.1) still presents a rich dynamics, for instance, periodic solutions [30] and chaotic behaviour [11, 60]. Our reasoning is based on a simple constructive approach that allows us to prove the existence of a sequence of invariant tori such that the union of their interiors covers all the $(t, x, \dot{x})$-space, $(t, x, \dot{x}) \in \mathbb{S}_\sigma \times \mathbb{R}^2$. In addition, we will see that these tori are foliated by periodic solutions, representing a highly exceptional phenomenon.

This chapter is structured as follows. In Section 5.1, we define some objects to be used throughout this chapter and we state our main result (Theorem E) concerning the existence of infinitely many invariant tori. Section 5.2 is dedicated to provide a sufficient condition for the existence of an invariant torus of (5.2), while Section 5.3 is devoted to the proof the main result of this chapter. In Section 5.4, we explore further directions for this study, particularly addressing an approach to handle cases where $p(t)$ has a non-zero average.

5.1 Main result

In order to address the properties of the solutions of the differential equation (5.1), it is convenient to consider it written as the following first-order autonomous differential system in the extended phase space, by taking $y = x'$, $t = \phi$ and $\phi' = 1$

$$\begin{cases} \phi' = 1, \\ x' = y, \\ y' = -\operatorname{sgn}(x) + p(\phi), \end{cases} \tag{5.2}$$

$(\phi, x, y) \in \mathbb{R} \times \mathbb{R}^2$.

5.1.1 Existence and uniqueness of solutions

The differential system (5.2) has two kinds of discontinuities, namely, the ones generated by the sgn function and the ones possibly generated by the Lebesgue-integrable function $p(t)$. We recall that, by taking $y = x'$, the differential equation (5.1) can be written as the following first-order differential system:

$$\begin{cases} x' = y, \\ y' = -\operatorname{sgn}(x) + p(t). \end{cases} \tag{5.3}$$

As previously mentioned, Filippov convention will be assumed for solutions of the differential system (5.3) (see Definition 2.1.14), which exist for every initial conditions (see Example 2.1.21). As usual, solutions of the differential system (5.2) are given in terms of solutions of the Filippov system (5.3) and, therefore, also exist for every initial conditions.

The solutions of (5.3) can be investigated by considering the following differential systems:

$$\begin{cases} x' = y, \\ y' = -1 + p(t), \end{cases} \quad x \geqslant 0, \quad \text{and} \quad \begin{cases} x' = y, \\ y' = 1 + p(t), \end{cases} \quad x \leqslant 0, \tag{5.4}$$

which match (5.3) restricted to $x \geqslant 0$ and $x \leqslant 0$, respectively. Since $p(t)$ is a Lebesgue integrable function, the differential systems in (5.4) correspond to Carathéodory differential systems for which all the conditions for the existence and uniqueness of solutions are satisfied (see Example 2.1.12). In the extended phase space, such differential systems become:

$$\begin{cases} \phi' = 1, \\ x' = y, \\ y' = -1 + p(\phi), \end{cases} \quad x \geqslant 0, \quad \text{and} \quad \begin{cases} \phi' = 1, \\ x' = y, \\ y' = 1 + p(\phi). \end{cases} \quad x \leqslant 0. \tag{5.5}$$

We remind that, except for $y_0 = 0$, the switching plane $\Sigma' = \{(\phi, x, y) \in \mathbb{R} \times \mathbb{R}^2 : x = 0\}$ is a crossing region of (5.2) (see Example 2.1.23). In addition, we have seen in Example 2.1.12 that each one of the system in (5.5) satisfy the conditions for local uniqueness of solutions. These facts allow to conclude that, for each initial condition $(\phi_0, 0, y_0) \in \Sigma'$ with $y_0 \neq 0$, the unique maximal solutions of the differential systems in (5.5) are transversal to Σ' at $(\phi_0, 0, y_0)$ and concatenate in order to form a (local) solution of the differential system (5.2) that is unique around $(\phi_0, 0, y_0)$. The explicit expressions of the solutions of the differential systems in (5.5) will be provided below in Section 5.2.

5.1.2 Main result

Let us define

$$P_1(t) := \int_0^t p(s)\mathrm{d}s \quad \text{and} \quad P_2(t) := \int_0^t P_1(s)\mathrm{d}s,$$

and, as usual, let $\overline{p}$ denote the average of $p(t)$, i.e.

$$\overline{p} := \frac{1}{\sigma} \int_0^\sigma p(s)\mathrm{d}s = \frac{P_1(\sigma)}{\sigma}.$$

Notice that the function $P_1(t)$ is continuous and the function $P_2(t)$ is continuously differentiable.

Our main result provides the existence of a sequence of invariant tori for (5.2) under the assumption $\overline{p} = 0$.

Theorem E. *Suppose that $p(t)$ is a Lebesgue integrable σ-periodic function satisfying $\overline{p} = 0$. Then, there exists a sequence $\{\mathbb{T}_n\}_{n \in \mathbb{N}} \subset \mathbb{S}_\sigma \times \mathbb{R}^2$ of nested invariant tori of the differential system (5.2) foliated by periodic solutions and satisfying:*

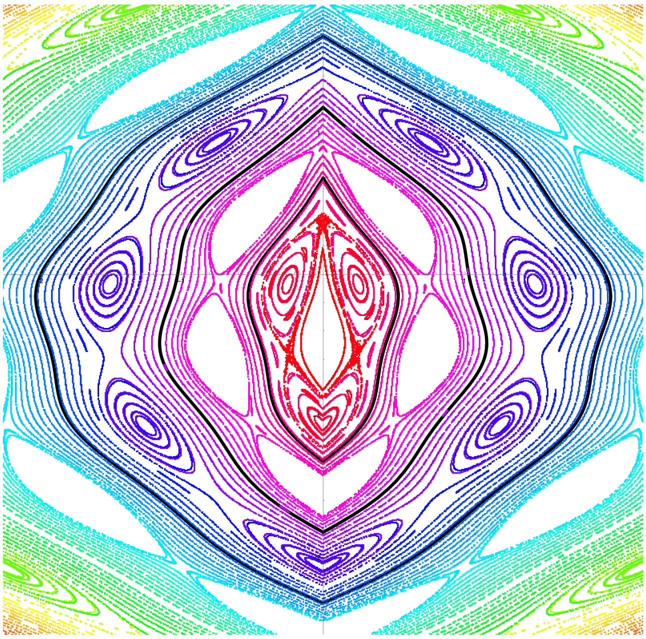
$$\mathbb{S}_\sigma \times \mathbb{R}^2 = \bigcup_{n \in \mathbb{N}} \mathrm{int}(\mathbb{T}_n),$$

where $\mathrm{int}(\mathbb{T}_n)$ denotes the open region enclosed by $\mathbb{T}_n$. In addition, all the maximal solutions of (5.2) are defined for every $t \in \mathbb{R}$ and the ones starting at $(\mathbb{S}_\sigma \times \mathbb{R}^2) \backslash \mathrm{int}(\mathbb{T}_1)$ are unique and transversal to Σ'.

Since $\mathbb{T}_n$ is invariant for each $n \in \mathbb{N}$, the uniqueness property provided by Theorem E implies that a solution starting at $\overline{\mathrm{int}(\mathbb{T}_n)}$ cannot leave it for all $t \in \mathbb{R}$. This leads us to the following corollary.

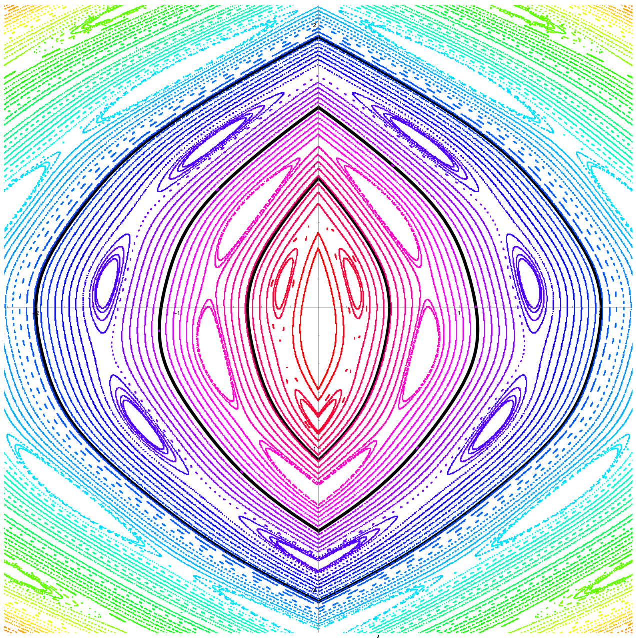
Corollary 5.1.1. *Suppose that $p(t)$ is a Lebesgue integrable σ-periodic function satisfying $\overline{p} = 0$. Then, all the solutions of the differential system* (5.3) *are bounded.*

In order to illustrate the main result of this chapter (Theorem E), we present some numerical simulations concerning the solutions of the differential equation (5.1). Specifically, $p(t) = \varepsilon \sin(2\pi t)$, and choosing a specific value for ε, we consider several initial conditions for the differential equation (5.1). Subsequently, we plot 2500 points for each of them on the time section $(\{t = 1\} \times \mathbb{R}^2)$, as $p(t)$ is 1-periodic in this case.



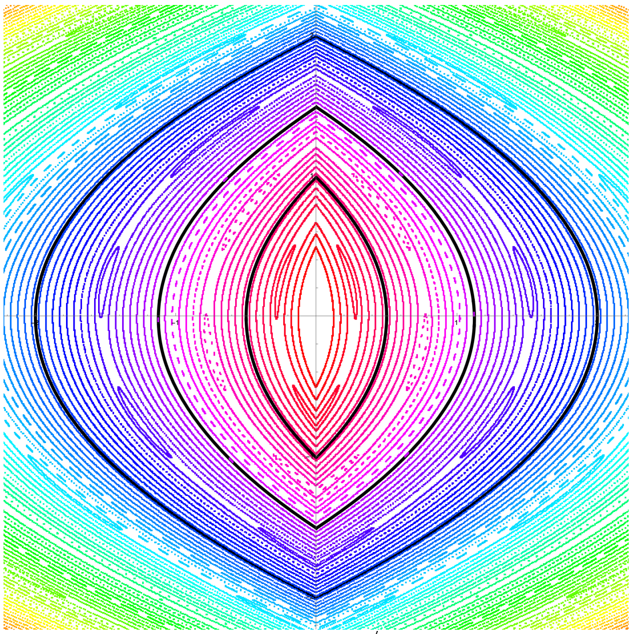
$\boldsymbol{\varepsilon = 2}$

Figure 19 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (4.1), is indicated by the concentration of colors in the figure. On the other hand, the black curves represent the intersection between the tori $\mathbb{T}_n$ provided by Theorem E and the time section $(\{t = 1\} \times \mathbb{R}^2)$.



$\varepsilon = 1/2$

Figure 20 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (4.1), is indicated by the concentration of colors in the figure. On the other hand, the black curves represent the intersection between the tori $\mathbb{T}_n$ provided by Theorem E and the time section $(\{t = 1\} \times \mathbb{R}^2)$.



$\varepsilon = 1/20$

Figure 21 – The existence of invariant curves for the time-1-map, and consequently invariant tori for the differential equation (4.1), is indicated by the concentration of colors in the figure. On the other hand, the black curves represent the intersection between the tori $\mathbb{T}_n$ provided by Theorem E and the time section $(\{t = 1\} \times \mathbb{R}^2)$.

5.2 Fundamental Lemma

This section is devoted to provide a sufficient condition for the existence of invariant tori of (5.2).

For each $n \in \mathbb{N}$, define the functions $y_n^+ : [0,\sigma] \to \mathbb{R}$ and $y_n^- : [0,\sigma] \to \mathbb{R}$ by

$$y_n^{\pm}(\phi_0) = \pm\frac{n\sigma}{2} + P_1(\phi_0) - \frac{P_2(\sigma)}{\sigma} \tag{5.6}$$

and, for each $n \in \mathbb{N}$, such that $y_n^-(\phi_0) < y_n^+(\phi_0)$ for every $\phi_0 \in [0,\sigma]$, define the surface

$$\begin{aligned} &\mathcal{T}_n := \mathcal{T}_n^+ \cup \mathcal{T}_n^-, \quad \text{where} \\ &\mathcal{T}_n^{\pm} := \{(\phi_0, \Psi_n^{\pm}(\phi_0,y_0), y_0) : \phi_0 \in \mathbb{R},\ y_0 \in [y_n^-(\phi_0), y_n^+(\phi_0)]\}, \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \Psi_n^{\pm}(\phi_0,y_0) := \frac{1}{8}\Bigg(&\pm n^2\sigma^2 \mp 4y_0^2 - 8P_2\left(\frac{n\sigma}{2} \pm y_0 \mp P_1(\phi_0) \pm \frac{P_2(\sigma)}{\sigma} + \phi_0\right) \\ &+ 4P_2(\sigma)\left(n \pm \frac{P_2(\sigma)}{\sigma^2}\right) - 4P_1(\phi_0)(\pm P_1(\phi_0) \mp 2y_0) + 8P_2(\phi_0)\Bigg). \end{aligned}$$

The following result provides sufficient conditions for which the surface $\mathcal{T}_n$, for some $n \in \mathbb{N}$, corresponds to an invariant torus of (5.2).

Lemma 5.2.1 (Fundamental Lemma). *Let $n \in \mathbb{N}$ be fixed and suppose that $\overline{p} = 0$. Assume that, for every $\phi_0 \in [0,\sigma]$,*

$$\left|\sigma P_1(\phi_0) - P_2(\sigma)\right| < \frac{n\sigma^2}{2} \tag{5.8}$$

and

$$\left|tP_2(\sigma) + \sigma P_2(\phi_0) - \sigma P_2(t+\phi_0)\right| < \frac{\sigma}{2}t(n\sigma - t),\ \ t \in (0, n\sigma). \tag{5.9}$$

Then, $\mathcal{T}_n$ is an invariant torus of the differential system (5.2). Moreover, $\mathcal{T}_n$ is foliated by $2n\sigma$-periodic solutions.

Before we proceed with the proof of Lemma 5.2.1, we define some important objects to help us along the process. Consider the following sequence of curves in the plane Σ':

$$\gamma_n^+ = \{(\phi_0, 0, y_n^+(\phi_0)) : \phi_0 \in \mathbb{S}_\sigma\} \text{ and } \gamma_n^- = \{(\phi_0, 0, y_n^-(\phi_0)) : \phi_0 \in \mathbb{S}_\sigma\}, \tag{5.10}$$

where y_n^+ and y_n^- are the continuous functions defined in (5.6).

We observe that the solutions of the differential systems in (5.5) are given, respectively, by

$$\begin{aligned} \varphi^+(t,\phi_0,x_0,y_0) = \Big(&t+\phi_0, -\frac{t^2}{2} + x_0 + ty_0 - tP_1(\phi_0) - P_2(\phi_0) + P_2(t+\phi_0), \\ &-t + y_0 - P_1(\phi_0) + P_1(t+\phi_0)\Big) \end{aligned}$$

and

$$\varphi^-(t,\phi_0,x_0,y_0) = \Big(t+\phi_0, \frac{t^2}{2}+x_0+ty_0-tP_1(\phi_0)-P_2(\phi_0)+P_2(t+\phi_0), \\ t+y_0-P_1(\phi_0)+P_1(t+\phi_0)\Big),$$

which are obtained via direct integration. As mentioned in Section 5.1.1, if a solution $\varphi(t)$ of the differential system (5.2) is transversal to Σ', then it writes as concatenations of φ^+ and φ^-.

Remark 5.2.2. *Expressions for y_n^+ and y_n^- are obtained by forcing a solution initiating in Σ' to firstly return to Σ' after a time $n\sigma$.*

Also, the next lemma concerning the primitives P_1 and P_2 plays an important role throughout this work.

Lemma 5.2.3. *The following identities hold for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$:*

$$P_1(t+n\sigma) = P_1(t) + nP_1(\sigma) \tag{5.11}$$

and

$$P_2(t+n\sigma) = P_2(t) + nP_1(\sigma)t + \frac{n^2-n}{2}\sigma P_1(\sigma) + nP_2(\sigma). \tag{5.12}$$

In particular, if $p(t)$ has vanishing average, then P_1 is σ-periodic.

Proof. The identity (5.11) is obtained through elementary properties of the Lebesgue integral, as follows:

$$\begin{aligned} P_1(t+n\sigma) &= \int_0^{t+n\sigma} p(s)\mathrm{d}s = \int_0^t p(s)\mathrm{d}s + \int_t^{t+n\sigma} p(s)\mathrm{d}s \\ &= P_1(t) + \int_0^{n\sigma} p(s)\mathrm{d}s \\ &= P_1(t) + \underbrace{\int_0^{\sigma} p(s)\mathrm{d}s + \cdots + \int_0^{\sigma} p(s)\mathrm{d}s}_{n} \\ &= P_1(t) + nP_1(\sigma). \end{aligned}$$

In order to derive the identity (5.12), we define the auxiliary function

$$\delta_n(t) := P_2(t+n\sigma) - \left(P_2(t) + nP_1(\sigma)t + \frac{n^2-n}{2}\sigma P_1(\sigma) + nP_2(\sigma)\right),$$

which is continuously differentiable for every $n \in \mathbb{N}$ and every $t \in \mathbb{R}$. Considering (5.11), we notice that

$$\delta_n'(t) = P_1(t+n\sigma) - P_1(t) - nP_1(\sigma) = 0,$$

for every $n \in \mathbb{N}$. This means that $\delta_n(t)$ is constant over $\mathbb{R}$. Proceeding by induction over n, we are going to show that $\delta_n(0) = 0$ for every $n \in \mathbb{N}$, and, then, conclude that relationship (5.12) holds for every $t \in \mathbb{R}$ and $n \in \mathbb{N}$.

For the base step $n = 1$, we have that

$$\delta_1(0) = P_2(\sigma) - P_2(\sigma) = 0,$$

which means that the identity (5.12) holds for $n = 1$. Now, as induction step, let us assume that $\delta_k(0) = 0$. Then, for $n = k + 1$, we have

$$\begin{aligned}
\delta_{k+1}(0) &= P_2((k+1)\sigma) - \left(\frac{(k+1)^2 - (k+1)}{2}\sigma P_1(\sigma) + (k+1)P_2(\sigma)\right) \\
&= \int_0^{(k+1)\sigma} P_1(s)\mathrm{d}s - \left(\frac{k^2+k}{2}\sigma P_1(\sigma) + (k+1)P_2(\sigma)\right) \\
&= \int_0^{k\sigma} P_1(s)\mathrm{d}s + \int_{k\sigma}^{(k+1)\sigma} P_1(s)\mathrm{d}s - \left(\frac{k^2+k}{2}\sigma P_1(\sigma) + (k+1)P_2(\sigma)\right) \\
&= P_2(k\sigma) + \int_0^{\sigma} P_1(s + k\sigma)\mathrm{d}s - \left(\frac{k^2+k}{2}\sigma P_1(\sigma) + (k+1)P_2(\sigma)\right) \\
&= P_2(k\sigma) + P_2(\sigma) + k\sigma P_1(\sigma) - \left(\frac{k^2+k}{2}\sigma P_1(\sigma) + (k+1)P_2(\sigma)\right) \\
&= P_2(k\sigma) - \left(\frac{k^2-k}{2}\sigma P_1(\sigma) + kP_2(\sigma)\right) + P_2(\sigma) - P_2(\sigma) \\
&= \delta_k(0) \\
&= 0.
\end{aligned}$$

This completes the proof of the lemma. □

5.2.1 Proof of the Fundamental Lemma

Let us first prove that $\varphi^{\pm}$ takes the curves $\gamma_n^{\pm}$ into $\gamma_n^{\mp}$ (see Figure 22), that is

$$\varphi^{\pm}(t, \phi_0, 0, y_n^{\pm}(\phi_0)) \notin \Sigma' \quad \text{for every} \quad t \in (0, n\sigma) \tag{5.13}$$

and

$$\varphi^{\pm}(n\sigma, \phi_0, 0, y_n^{\pm}(\phi_0)) = (n\sigma + \phi_0, 0, y_n^{\mp}(\phi_0)). \tag{5.14}$$

Indeed, condition (5.8) implies that

$$y_n^{+}(\phi_0) > 0 \quad \text{and} \quad y_n^{-}(\phi_0) < 0,$$

which means that the points in γ_n^+ and γ_n^- follows the forward flows φ^+ and φ^-, respectively. From now on, $\varphi_i^{\pm}$, $i = 1, 2, 3$, denotes the i-th coordinate of the function $\varphi^{\pm}$, thus

$$\begin{aligned}
\varphi_2^{\pm}(t, \phi_0, 0, y_n^{\pm}(\phi_0)) &= -\frac{t^2}{2} + t\left(\pm\frac{n\sigma}{2} + P_1(\phi_0) - \frac{P_2(\sigma)}{\sigma}\right) - tP_1(\phi_0) - P_2(\phi_0) + P_2(t + \phi_0) \\
&= \mp\frac{t^2}{2} \pm \frac{n\sigma}{2}t - \frac{P_2(\sigma)}{\sigma}t - P_2(\phi_0) + P_2(t + \phi_0).
\end{aligned}$$

The condition (5.9) implies that

$$\varphi_2^+(t, \phi_0, 0, y_n^+(\phi_0)) > 0 \quad \text{and} \quad \varphi_2^-(t, \phi_0, 0, y_n^-(\phi_0)) < 0,$$

for every $t \in (0, n\sigma)$, which provides that relationship (5.13) holds. Moreover, taking Lemma 5.2.3 into account, we have

$$\begin{aligned}
\varphi_2^\pm(n\sigma, \phi_0, 0, y_n^\pm(\phi_0)) &= \mp\frac{(n\sigma)^2}{2} + n\sigma\ y_n^\pm(\phi_0) - n\sigma\ P_1(\phi_0) - P_2(\phi_0) + P_2(n\sigma + \phi_0) \\
&= P_2(n\sigma + \phi_0) - P_2(\phi_0) - nP_2(\sigma) \\
&= P_1(\sigma)(n(\phi_0 + \frac{n-1}{2})) \\
&= 0,
\end{aligned}$$

because $P_1(\sigma) = 0$. Also,

$$\begin{aligned}
\varphi_3^\pm(n\sigma, \phi_0, 0, y_n^\pm(\phi_0)) &= \mp n\sigma + y_n^\pm(\phi_0) - P_1(\phi_0) + P_1(n\sigma + \phi_0) \\
&= \mp n\sigma \pm \frac{n\sigma}{2} + P_1(\phi_0) - \frac{P_2(\sigma)}{\sigma} - P_1(\phi_0) + P_1(n\sigma + \phi_0) \\
&= \mp\frac{n\sigma}{2} + P_1(\phi_0) - \frac{P_2(\sigma)}{\sigma} \\
&= y_n^\mp(\phi_0).
\end{aligned}$$

Hence, relationship (5.14) holds.

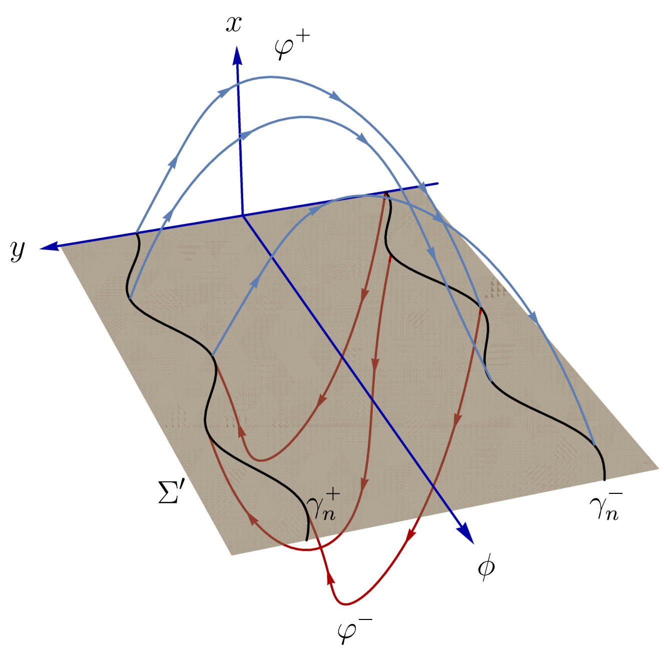


Figure 22 – For $n \in \mathbb{N}$ satisfying (5.8) and (5.9) , the flow $\varphi^\pm$ takes $\gamma_n^\pm$ into $\gamma_n^\mp$.

Accordingly, we notice that

$$\begin{aligned}&\mathcal{S}_n := \mathcal{S}_n^+ \cup \mathcal{S}_n^-, \quad \text{where}\\ &\mathcal{S}_n^\pm := \{\varphi^\pm(t, \phi_0, 0, y_n^\pm(\phi_0)) : t \in [0, n\sigma],\ \phi_0 \in \mathbb{R}\},\end{aligned}$$

is an invariant surface of (5.2) whose intersections with Σ' correspond to the curves γ_n^+ and γ_n^- given by (5.10) (see Figure 22). Now, by solving the system of equations

$$\begin{aligned}\varphi_1^\pm(t, \phi, 0, y_n^\pm(\phi)) &= \phi_0,\\ \varphi_3^\pm(t, \phi, 0, y_n^\pm(\phi)) &= y_0,\end{aligned}$$

in the variables (t, ϕ) we have

$$\begin{aligned}t &= \mp y_0 \pm \frac{n\sigma}{2} \mp \frac{P_2(\sigma)}{\sigma} \pm P_1(\phi_0),\\ \phi &= \phi_0 \pm y_0 \mp \frac{n\sigma}{2} \pm \frac{P_2(\sigma)}{\sigma} \mp P_1(\phi_0),\end{aligned}$$

which, after substituting the solutions (t, ϕ) into $\varphi_2^\pm(t; \phi, 0, y_n^\pm(\phi))$, provides us $\mathcal{S}_n = \mathcal{T}_n$. Furthermore, $\mathcal{T}_n^+ = \mathcal{T}_n \cap \{x \geqslant 0\}$ and $\mathcal{T}_n^- = \mathcal{T}_n \cap \{x \leqslant 0\}$ are homeomorphic to

$$\mathcal{D}_n := \{(\phi_0, y_0) : \phi_0 \in \mathbb{R},\ y_0 \in [y_n^-(\phi_0), y_n^+(\phi_0)]\},$$

because they are graphs of Ψ_n^+ and Ψ_n^-, respectively. This in turn implies that $\mathcal{T}_n^+$ and $\mathcal{T}_n^-$ are simply connected surfaces and, consequently, $\mathcal{T}_n$ is an invariant cylinder of (5.2).

Now, let $U \subset \Lambda_0$ be the set of initial conditions in Λ_0 for which the corresponding maximal solutions of (5.2) are transversal to Σ'. As discussed in Section 5.1.1, such solutions are unique and defined for every $t \in \mathbb{R}$. Thus, consider the time-σ-map $\mathcal{P}_\sigma$ defined on U into Λ_σ:

$$\begin{array}{cccc}\mathcal{P}_\sigma : & U & \longrightarrow & \Lambda_\sigma\\ & (0, x_0, y_0) & \longmapsto & \varphi(\sigma; 0, x_0, y_0).\end{array}$$

Since Λ_0 and Λ_σ coincide in the quotient space $\mathbb{S}_\sigma \times \mathbb{R}^2$ and taking into account that $\mathcal{P}_\sigma(U) \subset \Lambda_\sigma$ corresponds to the set of initial conditions in Λ_σ for which the maximal solutions of (5.2) are transversal to Σ', it follows that $\mathcal{P}_\sigma$ can be seen as an automorphism on U.

Let $\mathcal{C}_n^0$ and $\mathcal{C}_n^\sigma$ denote the intersections between the invariant cylinder $\mathcal{T}_n$ with the time sections Λ_0 and Λ_σ, respectively. Notice that, from the considerations above, $\mathcal{C}_n^0 \subset U$. In addition,

$$\begin{aligned}\mathcal{C}_n^0 :=& \mathcal{T}_n \cap \Lambda_0 = \mathcal{C}_{n,0}^+ \cup \mathcal{C}_{n,0}^-, \quad \text{where}\\ \mathcal{C}_{n,0}^\pm :=& \{(0, \Psi_n^\pm(0, y_0), y_0) : y_0 \in [y_n^-(0), y_n^+(0)]\}\end{aligned} \tag{5.15}$$

and

$$\begin{aligned}\mathcal{C}_n^\sigma :=& \mathcal{T}_n \cap \Lambda_\sigma = \mathcal{C}_{n,\sigma}^+ \cup \mathcal{C}_{n,\sigma}^-, \quad \text{where}\\ \mathcal{C}_{n,\sigma}^\pm :=& \{(\sigma, \Psi_n^\pm(\sigma, y_0), y_0) : y_0 \in [y_n^-(\sigma), y_n^+(\sigma)]\}.\end{aligned} \tag{5.16}$$

In what follows, we show that $\mathcal{C}_n^0$ is invariant under the map $\mathcal{P}_\sigma$. For this purpose, we will examine the parametrizations of $\mathcal{C}_n^0$ and $\mathcal{C}_n^\sigma$ given by (5.15) and (5.16), respectively. We first observe that the functions y_n^+ and y_n^- are σ-periodic, thus $\Psi_n^\pm(0,\cdot)$ and $\Psi_n^\pm(\sigma,\cdot)$ have the same domain, namely, $\mathcal{I}_n = [y_n^-(0), y_n^+(0)]$. Then, it is sufficient to show that

$$\Psi_n^\pm(0,y_0) - \Psi_n^\pm(\sigma,y_0) = 0 \quad \text{for all} \quad y_0 \in \mathcal{I}_n.$$

In fact,

$$\begin{aligned}\Psi_n^\pm(0,y_0) - \Psi_n^\pm(\sigma,y_0) =& n\frac{P_2(\sigma)}{2} - P_2\left(\frac{n\sigma}{2} \pm \frac{P_2(\sigma)}{\sigma} \pm y_0\right)\\ &+ \left(P_2\left(\sigma + \frac{n\sigma}{2} \pm \frac{P_2(\sigma)}{\sigma} \pm y_0\right) - (n+2)\frac{P_2(\sigma)}{2}\right)\\ =& n\frac{P_2(\sigma)}{2} + P_2(\sigma) - (n+2)\frac{P_2(\sigma)}{2}\\ =& 0.\end{aligned}$$

The second equality above was obtained by the identity (5.12). Hence, it follows that the invariant cylinder $\mathcal{T}_n$ corresponds to an invariant torus of (5.2) in the quotient space $\mathbb{S}_\sigma \times \mathbb{R}^2$ (see Figure 23), which concludes this proof.

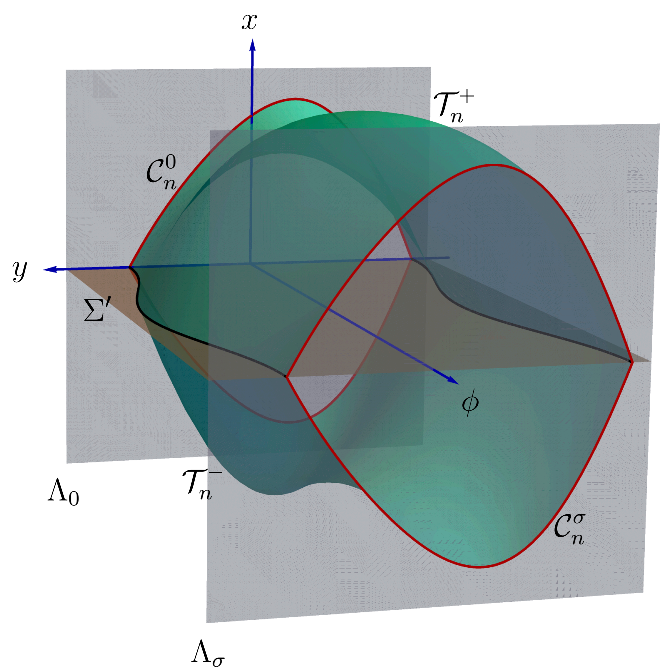


Figure 23 – Invariant torus $\mathcal{T}_n = \mathcal{T}_n^+ \cup \mathcal{T}_n^-$ provided by the Fundamental Lemma.

5.3 Proof of the main result

The proof of Theorem E will follow as a consequence of the next results. The first one, Proposition 5.3.1, will provide the existence of $n^* \in \mathbb{N}$ such that the conditions (5.8) and (5.9) of the

Fundamental Lemma 5.2.1 are satisfied for every $n \geqslant n^*$. Accordingly, the sequence of invariant tori stated by Theorem E will be given by $\mathbb{T}_n := \mathcal{T}_{n+n^*}$, $n \in \mathbb{N}$. Finally, Corollary 5.3.2 will provide that each maximal solution of the differential system (5.2) is defined for every $t \in \mathbb{R}$ and the ones starting in $(\mathbb{S}_\sigma \times \mathbb{R}^2)\backslash\mathrm{int}(\mathbb{T}_1)$ are transversal to Σ' and, consequently, unique.

Proposition 5.3.1. *Let $p(t)$ be a Lebesgue integrable σ-periodic function such that $\overline{p} = 0$. Then, there exists $n^* \in \mathbb{N}$ such that $\mathcal{T}_n$ is an invariant torus of (5.2) for every $n \geqslant n^*$.*

Proof. From lemma 5.2.3 we have that $\sigma P_1(\phi_0) - P_2(\sigma)$ is continuous σ- periodic in ϕ_0 and consequently bounded, thus there exists $n_0 \in \mathbb{N}$ such that the relationship (5.8) holds for every $n \geqslant n_0$.

In order to obtain (5.9), we define the functions

$$\begin{aligned} f(t) :=& tP_2(\sigma) + \sigma P_2(\phi_0) - \sigma P_2(t + \phi_0) \text{ and} \\ h_n(t) :=& \frac{\sigma}{2} t(n\sigma - t), \text{ for } n \in \mathbb{N}. \end{aligned} \tag{5.17}$$

By Remark 5.2.3, we notice that f is continuously differentiable and σ-periodic. Besides that, $h_n(t) > 0$ for every $t \in (0, n\sigma)$. We start by proving the following claim:

Claim 1. *There exists $n^* \geqslant n_0$ such that*

$$|f(t)| < h_{n^*}(t) \quad \textit{for every} \;\; t \in (0, n^*\sigma). \tag{5.18}$$

Consider the functions

$$q_n^+(t) := f(t) - h_n(t) \quad \text{and} \quad q_n^-(t) := f(t) + h_n(t).$$

Since, $q_{n_0}^+(n\sigma) = q_{n_0}^-(n\sigma) = 0$, $(q_{n_0}^+)'(n_0\sigma) > 0$ and $(q_{n_0}^-)'(n_0\sigma) < 0$, then there exists $\varepsilon_0 > 0$ such that $q_{n_0}^+(t) > 0$ and $q_{n_0}^-(t) < 0$ for every $t \in (n_0\sigma - \varepsilon_0, n_0\sigma)$. This implies that

$$|f(t)| < h_{n_0}(t) \quad \text{for every} \;\; t \in (n_0\sigma - \varepsilon_0, n_0\sigma). \tag{5.19}$$

Now, for each $k \in \mathbb{N}$, consider the following $(2^k n_0\sigma)$-periodic function:

$$\overline{h}_k(t) := \sum_{m\in\mathbb{N}} \chi_{I_m}(t) h_{2^k n_0}(t - 2^k(m-1)\sigma), \tag{5.20}$$

where $I_m := \big[2^k(m-1)\sigma, 2^k m\sigma\big)$ and χ_{I_m} is the characteristic function of I_m. Notice that $\overline{h}_0(t)$ is a $n_0\sigma$-periodic extension of h_{n_0} and that $\overline{h}_{k+1}(t) \geqslant \overline{h}_k(t)$ for every $t \geqslant 0$ and $k \in \mathbb{N}$ (see Figure 24). Taking (5.19) into account, this implies that

$$|f(t)| < \overline{h}_k(t) \quad \text{for every} \;\; t \in (2^k n_0\sigma - \varepsilon_0, 2^k n_0\sigma), \tag{5.21}$$

for every $k \in \mathbb{N}$.

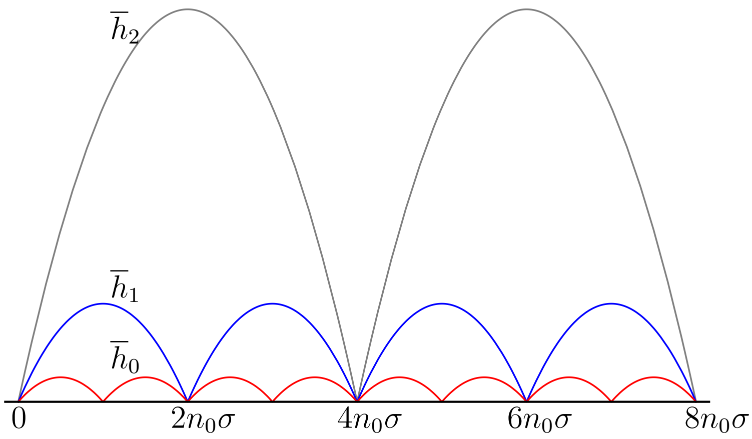


Figure 24 – Graphs of the functions $\overline{h}_k$ constructed in (5.20) for $k = 0, 1, 2$.

On the other hand, from Lemma 5.2.3, it follows that P_1 is bounded, so let $M > 0$ satisfy $\|P_1\|_\infty = M$. Then,

$$|f(t)| \leqslant t\int_0^\sigma |P_1(s)|\mathrm{d}s + \sigma\int_0^t |P_1(s+\phi_0)|\mathrm{d}s \leqslant 2\sigma M t, \ \forall\, t \in \mathbb{R}. \tag{5.22}$$

Consequently,

$$|f(t)| < \overline{h}_k(t) \quad \text{for all} \quad t \in (0, 2^k n_0\sigma - 4M). \tag{5.23}$$

In addition, since $f(t)$ is σ-periodic, (5.22) also implies that

$$|f(t)| \leqslant 2\sigma^2 M \;\text{ for every }\; t \in \mathbb{R}. \tag{5.24}$$

Assume, by reduction to absurdity, that (5.18) does not hold. In particular, for each $k \in \mathbb{N}$, there exists $t_k \in (0, 2^k n_0\sigma)$ such that $|f(t_k)| = \overline{h}_k(t_k)$. From (5.23), it follows that $t_k \in [2^k n_0\sigma - 4M, 2^k n_0 T)$, which means that $t_k \to \infty$ when $k \to \infty$. Moreover, from (5.24), one has

$$|t_k - 2^k n_0\sigma| = \frac{\overline{h}_k(t_k)}{|t_k|} = \frac{|f(t_k)|}{|t_k|} \leqslant \frac{2\sigma^2 M}{|t_k|} \to 0, \quad \text{when} \quad k \to \infty.$$

Then, there exists $k_0 \in \mathbb{N}$ such that $|t_k - 2^k n_0\sigma| < \varepsilon_0$ for every $k \geqslant k_0$, with $\varepsilon_0 > 0$ being the one satisfying (5.19). This is contradiction with (5.21).

Thus, there must exists $k^* \in \mathbb{N}$, for which $|f(t)| < \overline{h}_{k^*}(t)$ for all $t \in (0, 2^{k^*} n_0\sigma)$. Hence, Claim 1 follows by defining $n^* = 2^{k^*} n_0$.

Finally, the proof of the proposition will follow by proving the next claim:

Claim 2. *Condition* (5.9) *holds for every* $n \geqslant n^*$.

We notice that

$$|f(t)| < h_{n^*}(t-\sigma) \quad \text{for all} \quad t \in (\sigma, (n^*+1)\sigma).$$

Taking into account that $h_{n^*}(t)$ and $h_{n^*}(t-\sigma)$ coincide for $t = (n^*+1)\sigma/2$, we can define

$$\tilde{h}_{n^*}(t) := \begin{cases} h_{n^*}(t) & \text{if } t \in [0, \frac{(n^*+1)\sigma}{2}], \\ h_{n^*}(t-\sigma) & \text{if } t \in (\frac{(n^*+1)\sigma}{2}, (n^*+1)\sigma]. \end{cases}$$

Now, since $h_{n^*}(t) \leqslant h_{n^*+1}(t)$ for all $t \in (0, n^*\sigma)$ and $h_{n^*}(t-\sigma) \leqslant h_{n^*+1}(t)$ for all $t \in (\sigma, (n^*+1)\sigma)$, we have that $|f(t)| < \tilde{h}_{n^*}(t) \leqslant h_{n^*+1}(t)$ for all $t \in (0, (n^*+1)\sigma)$. The proof of the claim follows, then, by repeating this procedure recursively.

Hence, Lemma 5.2.1 ensures that, for each $n \geqslant n^*$, $\mathcal{T}_n$ is an invariant torus of (5.2). □

Corollary 5.3.2. *Let $p(t)$ be a Lebesgue integrable function with vanishing average. Then, all the maximal solutions of (5.2) are defined for every $t \in \mathbb{R}$ and the ones whose initial conditions lie on $(\mathbb{S}_\sigma \times \mathbb{R}^2)\backslash\mathrm{int}(\mathbb{T}_1)$ are unique and transversal to Σ'.*

Proof. We start by proving that, for each $(\phi_0, x_0, y_0) \in (\mathbb{S}_\sigma \times \mathbb{R}^2)\backslash\mathrm{int}(\mathbb{T}_1)$, there exists a unique maximal solution passing through (ϕ_0, x_0, y_0) which is transversal to Σ' and it is defined for every $t \in \mathbb{R}$.

Notice that a maximal solution with initial condition (ϕ_0, x_0, y_0) may intersect the plane Σ' or not. If such an intersection does not occur, then such a solution is unique as a maximal solution of one the differential systems in (5.5). On the other hand, if $\varphi(t, \phi_0, x_0, y_0)$ intersects Σ', then such an intersection must be transversal. Otherwise, there would exist a time $t^* > 0$ such that

$$\varphi_2(t^*, \phi_0, x_0, y_0) = 0 \quad \text{and} \quad \varphi_3(t^*, \phi_0, x_0, y_0) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_2(t^*, \phi_0, x_0, y_0) = 0,$$

which implies that such a solution would cross $\mathbb{T}_1$ (from outside to inside) contradicting the fact that $\mathbb{T}_1$ is invariant and that the solutions there defined are unique. Thus, as noticed in Section 5.1.1, this transversality implies the uniqueness of all solutions starting in $(\mathbb{S}_\sigma \times \mathbb{R}^2)\backslash\mathrm{int}(\mathbb{T}_1)$ which, therefore, are defined for every $t \in \mathbb{R}$.

For the remaining initial conditions, as a consequence of the invariance of $\mathbb{T}_1$, the maximal solutions starting in the set $\mathrm{int}(\mathbb{T}_1)$ are confined in the compact set $\overline{\mathrm{int}(\mathbb{T}_1)}$. Therefore, they must be defined for all $t \in \mathbb{R}$ because of Theorem 2.1.22. This concludes the proof of the corollary. □

5.3.1 A simpler approach for L^∞-forcing term

In the next result, we shall see that the proof of Proposition 5.3.1 simplifies a lot by assuming p to be an L^∞-function on $[0, \sigma]$, instead of just Lebesgue integrable. In this case, we will show that the surface $\mathcal{T}_n$ provided in (5.7) is an invariant torus of (5.2) for every $n \in \mathbb{N}$ bigger than $2\|p\|_{L^\infty}$.

Proposition 5.3.3. *Let p be a σ-periodic function with vanishing average and suppose that there exists $M > 0$ such that $\|p\|_{L^\infty} < M$. Then, the surface $\mathcal{T}_n$ is an invariant torus of (5.2) for all $n \in \mathbb{N}$ satisfying $n \geqslant 2M$.*

Proof. We recall that in order to obtain this result, it is sufficient to show that conditions (5.8) and (5.9) hold for all $n \in \mathbb{N}$ such that $n \geqslant 2M$.

Define $\alpha(\phi_0) := \sigma P_1(\phi_0) - P_2(\sigma)$. Notice that α is σ-periodic by Remark 5.2.3, which restricts our analysis to $\phi_0 \in [0, \sigma]$. Then, taking into account that

$$|P_1(\phi_0) - P_1(t)| = \left| \int_0^{\phi_0} p(s)\mathrm{d}s - \int_0^t p(s)\mathrm{d}s \right| = \left| \int_t^{\phi_0} p(s)\mathrm{d}s \right| \leqslant \|p\|_{L^\infty} |\phi_0 - t|$$

and assuming that $\|p\|_{L^\infty} < M$, we see that

$$\begin{aligned} |\alpha(\phi_0)| &= \left| \int_0^\sigma P_1(\phi_0) - P_1(t)\mathrm{d}t \right| \\ &\leqslant \int_0^\sigma \|p\|_{L^\infty} |\phi_0 - t|\mathrm{d}t \\ &< M \int_0^\sigma |\phi_0 - t|\mathrm{d}t = M\left(\frac{\sigma^2}{2} - \sigma\phi_0 + \phi_0^2\right) \leqslant M\frac{\sigma^2}{2} \leqslant \frac{n\sigma^2}{4}, \end{aligned} \tag{5.25}$$

whenever $n \in \mathbb{N}$ satisfies $n \geqslant 2M$. Therefore, condition (5.8) holds for every $n \in \mathbb{N}$ satisfying $n \geqslant 2M$.

In order to obtain (5.9), we define the functions

$$d_n(t) := \frac{n\sigma^2}{4}t \quad \text{and} \quad e_n(t) := -\frac{n\sigma^2}{4}(t - n\sigma).$$

We notice that $d_n(t) > 0$ for $t > 0$; $e_n(t) > 0$ for $t < n\sigma$; and $d_n(t) = e_n(t)$ if, and only if $t = \frac{n\sigma}{2}$. Consider the functions

$$r_n^+(t) := f(t) - d_n(t) \quad \text{and} \quad s_n^+(t) := f(t) - e_n(t),$$

where $f(t)$ is the function defined in (5.17). We notice that $r_n^+(0) = 0$ and $s_n^+(n\sigma) = 0$, since f is σ-periodic. In addition, given that $\alpha(t + \phi_0) = -f'(t)$ for every $t \in \mathbb{R}$, and taking (5.25) into account, it follows that

$$(r_n^+)'(t) < 0 \quad \text{and} \quad (s_n^+)'(t) > 0 \quad \text{for all} \quad t \in \mathbb{R},$$

and $n \geqslant 2M$. This means that $f(t) < d_n(t)$ for all $t > 0$, and $f(t) < e_n(t)$ for all $t < n\sigma$. Thus, by defining the function

$$g_n^+(t) := \begin{cases} d_n(t) & \text{if} \ \ t \in (0, \frac{n\sigma}{2}], \\ e_n(t) & \text{if} \ \ t \in (\frac{n\sigma}{2}, n\sigma), \end{cases}$$

and taking into account that $g_n^+(t) \leqslant h_n(t)$[1] for all $t \in (0, n\sigma)$, where h_n is the function defined in (5.17), it follows that $f(t) < h_n(t)$ for all $t \in (0, n\sigma)$ and $n \geqslant 2M$. In an analogous way, we can show that $-h_n(t) < f(t)$ for all $t \in (0, n\sigma)$ and $n \geqslant 2M$. It concludes this proof. □

5.4 Further directions: A glimpse of the non-vanishing average case

The main results presented in this chapter are contingent upon the assumption that $p(t)$ has a vanishing average. A natural question that emerges from this assumption is: What if $p(t)$ has a non-vanishing average?

Enguiça and Ortega in [20] showed that (5.1) has infinitely many bounded solutions provided the limit

$$\lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} p(s)\mathrm{d}s,$$

uniformly exists with respect to $t \in \mathbb{R}$. Unlike Theorem E, this outcome is not global, in the sense that not all solutions are necessarily bounded.

In this section, we present a work in progress to address the existence of invariant tori and global stability of solutions of (5.2) in the cases where $p(t)$ has a non-vanishing average. Since it consists in a perturbative approach, we reintroduce the perturbative parameter ε, transforming the equation (5.1) into

$$\ddot{x} + \operatorname{sgn}(x) = \varepsilon\, p(t). \tag{5.26}$$

Here we are assuming that $p(t)$ is analytic and σ-periodic.

Notice that in the unperturbed scenario, every solution is periodic and the extended phase space $(t, x, \dot{x}) \in \mathbb{S}_\sigma \times \mathbb{R}^2$, except for the line $\{(t, 0, 0) : t \in \mathbb{R}\}$, are foliated by invariant tori. The intersection between these invariant tori and the switching manifold Σ' results in the curves $\{(t, 0, y_0) : t \in \mathbb{R}\}$, with $y_0 \neq 0$ (see Figure 25). This implies that, if we consider the impact map (to be elaborated) associated with the unperturbed equation defined on Σ', every curve $\gamma_{y_0} := \{\psi_{y_0}(t) = (t, 0, y_0) : t \in \mathbb{R}\}$, with $y_0 > 0$, is invariant under such map.

We briefly explain the impact map associated with the differential equation (5.26). We remind that (5.26) can be seen as the vector field (1.7).

We denote the solutions of (1.7) with initial condition (ϕ_0, x_0, y_0) and (ϕ_1, x_1, y_1), if $x > 0$ and $x < 0$, respectively, by

$$\varphi^+(\tau, \phi_0, x_0, y_0; \varepsilon) = (\phi_+^\tau(\phi_0, x_0, y_0; \varepsilon), x_+^\tau(\phi_0, x_0, y_0; \varepsilon), y_+^\tau(\phi_0, x_0, y_0; \varepsilon)), \tag{5.27}$$

[1] There is a typo in the definition of the function $g_n^+(t)$ in the published paper [51]. Such typo does not compromise the accuracy of the result.

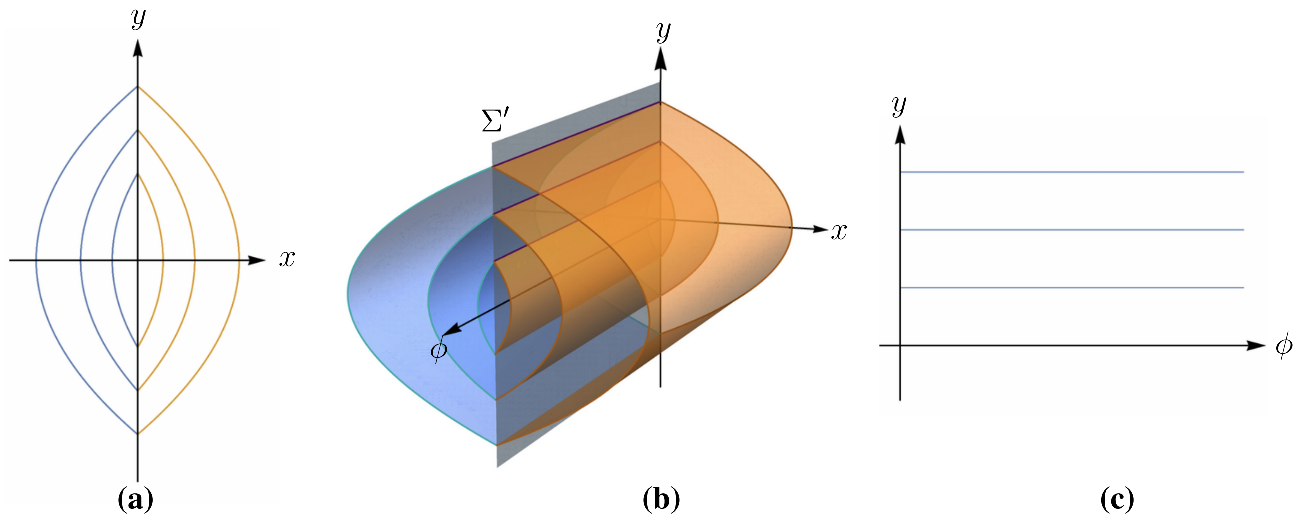


Figure 25 – **(a)**-Intersection between the invariant tori and the time section $\{\phi_0\} \times \mathbb{R}^2$ **(b)**- Invariant tori in $\mathbb{S}_\sigma \times \mathbb{R}^2$ **(c)**- Intersection between the invariant tori and the plane Σ'

and

$$\varphi^-(\tau, \phi_1, x_1, y_1; \varepsilon) = (\phi_-^\tau(\phi_1, x_1, y_1; \varepsilon), x_-^\tau(\phi_1, x_1, y_1; \varepsilon), y_-^\tau(\phi_1, x_1, y_1; \varepsilon)),$$

respectively. The subsets

$$\Sigma^+ = \{(\phi_0, 0, y_0) \in \Sigma' : y_0 > 0\} \quad \text{and} \quad \Sigma^- = \{(\phi_1, 0, y_1) \in \Sigma' : y_1 < 0\}$$

play an important role in defining the domain of the half impact maps. For $(\phi_0, 0, y_0) \in \Sigma^+$, we denote by $\tau^+(\phi_0, y_0; \varepsilon)$ the smallest positive time such that

$$x_+^{\tau^+(\phi_0, y_0; \varepsilon)}(\phi_0, 0, y_0; \varepsilon) = 0.$$

We note that $\tau^+(\phi_0, y_0; \varepsilon)$ is σ-periodic in ϕ_0. The half positive impact map is given by

$$\mathcal{P}_\varepsilon^+ : (\phi_0, 0, y_0) \in \Sigma^+ \mapsto (\phi_+^{\tau^+(\phi_0, y_0; \varepsilon)}(\phi_0, 0, y_0; \varepsilon), 0, y_+^{\tau^+(\phi_0, y_0; \varepsilon)}(\phi_0, 0, y_0; \varepsilon) \in \Sigma^-.$$

The initial condition $(\phi_1, 0, y_1)$, with $(\phi_1, 0, y_1) \in \Sigma^-$, follows the negative flow given by (5.2). We consider $\tau^-(\phi_1, y_1; \varepsilon)$ as the smallest positive time such that $x_-^{\tau^-(\phi_1, y_1; \varepsilon)}(\phi_1, 0, y_1; \varepsilon) = 0$, with $\tau^-(\phi_1, y_1; \varepsilon)$ being σ periodic in ϕ_1. Then, the half negative impact map is given by

$$\mathcal{P}_\varepsilon^- : (\phi_1, 0, y_1) \in \Sigma^- \mapsto (\phi_-^{\tau^-(\phi_1, y_1; \varepsilon)}(\phi_1, 0, y_1; \varepsilon), y_-^{\tau^-(\phi_1, y_1; \varepsilon)}(\phi_1, 0, y_1; \varepsilon)) \in \Sigma^+.$$

Thus, the complete impact map for equation (5.26) is provided by the composition of the negative with the positive half impact map, respectively, i.e.,

$$\mathcal{P}_\varepsilon : (\phi_0, 0, y_0) \in \Sigma^+ \mapsto (\bar{t}_0, \bar{y}_0) = \mathcal{P}_\varepsilon^- \circ \mathcal{P}_\varepsilon^+(\phi_0, 0, y_0) \in \Sigma^+,$$

with σ-periodic dependence in ϕ_0, so that it can be read as a map of the annulus $\mathbb{S}_\sigma \times \mathbb{R}^+$. When $\varepsilon = 0$, it is easy to check that

$$\mathcal{P}_0(\phi_0, 0, y_0) = (\phi_0 + \alpha(y_0), 0, y_0),$$

with $\alpha(y_0) = 4y_0$ representing the period of the solution of the unperturbed autonomous system (5.26) whose initial condition is $(\phi_0, 0, y_0)$. This implies that the curves γ_{y_0} are invariant under $\mathcal{P}_0$, for every $y_0 > 0$. Moreover, for sufficiently small $\varepsilon \geqslant 0$, the trajectories of (5.26) starting in Σ^+ cross Σ^+ again, then, $\mathcal{P}_\varepsilon$ is well defined and also analytic as the solutions (5.27) restrict to Σ^+. Since $\alpha'(y_0) = 4 > 0$, for all $y_0 > 0$, it follows that $\mathcal{P}_0$ is an integrable twist map of the annulus, which makes $\mathcal{P}_\varepsilon$ a close to integrable twist map. Then, the parametrization method turns out to be very useful to detect analytic periodic curves of $\mathcal{P}_\varepsilon$ in Σ^+.

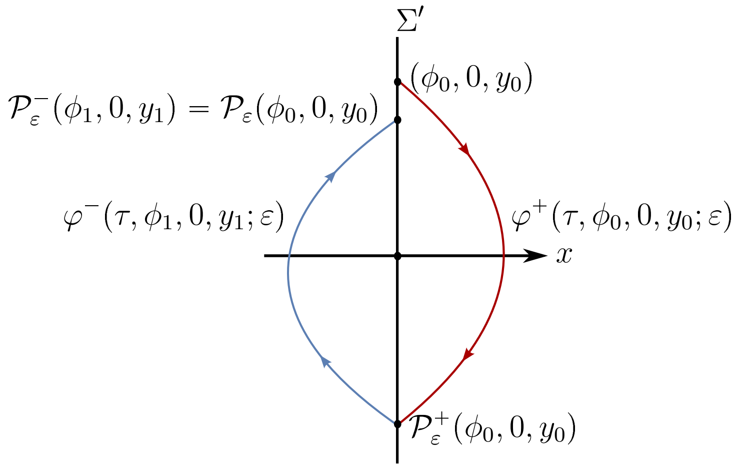


Figure 26 – Impact map $\mathcal{P}_\varepsilon$.

Parametrization method: The understanding of the parametrization method comprehends the introduction of some objects.

Let $\omega \in \mathbb{R}$ be a fixed Diophantine number (see Definition 2.3.1) and let F be a real analytic and exact sympletic map defined on a cylinder $\mathbb{S}_\sigma \times \mathbb{R}$ endowed by the $2-$form $\mathrm{d}\phi \wedge \mathrm{d}I$. We suppose that ψ is a real analytic parametrization for a given torus $\mathcal{T} \subset \mathbb{S}_\sigma \times \mathbb{R}$, and we define the invariance error associated with ψ

$$e(\theta) := F(\psi(\theta)) - \psi(\theta + \omega), \quad \theta \in \mathbb{S}_\sigma.$$

Hence, if for a given parametrization ψ^*, the invariance error e^* vanishes, it implies that ψ^* corresponds to an invariant torus of F.

Definition 5.4.1. *Let f be a complex valued analytic function defined in an open set $\mathcal{U} \subset \mathcal{C}^2$, which is bounded in the closure of $\mathcal{U}$. We introduce the norm*

$$\|f\|_{\mathcal{U}} = \sup_{z \in \mathcal{U}} |f(z)|,$$

where $|\cdot|$ is the sup-norm. In the case where f is defined on a torus $\mathbb{S}_\sigma$ and can be analytically extended to the complex strip $\Delta(\rho) := \{\phi \in \mathbb{C} : |Im(\phi)| < \rho\}$, then we refer to the size of this function with the notation $\|f\|_\rho = \|f\|_{\Delta(\rho)}$.

The parametrization method consists in finding a sequence of parametrizations $\{\psi^{(n)}\}_n$ that converges uniformly to a function ψ^* that parametrizes an invariant torus $\mathcal{T}^*$ under F. The idea behind the construction of such sequence is start with a parametrization for a quasi- torus, $\psi^{(0)}$, in a way that, for some $0 < \rho < 1$, the associated invariance error is sufficiently small, i.e., $\left\|e^{(0)}\right\|_\rho \ll 1$. Next parametrization, $\psi^{(1)}$, can be obtained by means of the quasi-Newton method, where we compute the linear approximation of $e^{(1)}$ around $\psi^{(0)}$ and we try to cancel out the resulting expression. This process is interactively applied in order to improve the previous parametrization, in the sense that the new associated error have quadratic size with respect to the previous one. The advantage in work with this method lies in the fact that no change on the original system is needed, but only on the parametrizations. This method was first introduced in [16] and was consistently applied in several works as we can see in [27, 62, 63] and references therein. Besides that, the parametrization method is very useful in computer assisted proofs, since we can obtain a better threshold for the size of the initial error than methods based in transformation theory (see, for instance, [21]).

A strategy for establishing the existence of an infinite collection of invariant tori, as outlined in Theorems A and E, involves considering an invariant curve γ_{y_0}, where $y_0 > 0$ satisfies $\alpha(y_0) = \omega$. Here, ω is a Diophantine number of type (ζ, ν). The approach then applies the parametrization method to γ_{y_0} to generate a sequence $\{\psi_{y_0}^{(n)}\}_{n\in\mathbb{N}}$ of parametrizations, for which the associated invariant errors satisfy $\left\|e^{(n+1)}\right\|_{\rho^{(n+1)}} = \mathcal{O}(\left\|e^{(n)}\right\|^2_{\rho^{(n)}})$, for every $n \in \mathbb{N}$. Under these conditions, the sequence $\{\psi_{y_0}^{(n)}\}_{n\in\mathbb{N}}$ converges to a parametrization of an invariant torus of $\mathcal{P}_\varepsilon$. Moreover, the upper bound for ε ensuring the aforementioned convergence must be independent on the amplitude y_0 initially chosen. This fact together with Remark 2.3.2 enable the construction of a sequence of invariant curves of $\mathcal{P}_\varepsilon$, and consequently a sequence of invariant tori of (5.26) (see Figure 27).

Certainly, there is still much work to be done, including the demonstration of the exact symplectic nature of $\mathcal{P}_\varepsilon$. We believe that this is likely true given the piecewise Hamiltonian structure of (5.26) noticed in (1.6). Additionally, it is imperative to precisely establish the bounds of $\mathcal{P}_\varepsilon$ while ensuring their independence on the initially chosen amplitude y_0.

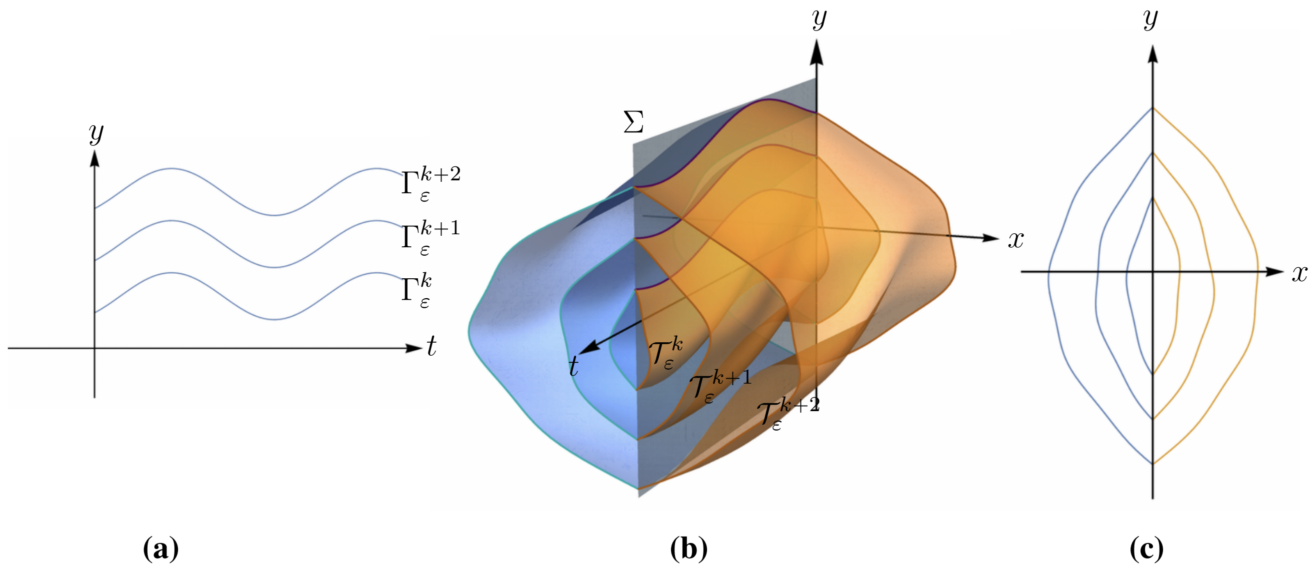


Figure 27 – **(a)**-Invariant curves of $\mathcal{P}_\varepsilon$ **(b)**- Invariant tori of (1.7) obtained after the invariant curves Γ_k **(c)**- Intersection between the invariant tori $\mathcal{T}_{k,\varepsilon}$ and the time section $\{0\} \times \mathbb{R}^2$

6 Melnikov analysis for detecting periodic orbits

The results presented in this chapter are based on the manuscript [50].

This chapter is dedicated to exploring the existence of periodic solutions through a Melnikov analysis for the more general class of second-order discontinuous differential equations (1.2), which, recalling its definition, is given by

$$\ddot{x} + \mu \operatorname{sgn}(x) = \theta x + \varepsilon f(t, x, \dot{x}), \tag{6.1}$$

where $\mu \in \{-1, 1\}$, and θ and ε are real parameters. The function f is assumed to be $\mathcal{C}^1$ and σ-periodic in the variable t.

The Melnikov method [41] is one of the main tools for determining the persistence of periodic solutions in planar smooth differential systems when subjected to non-autonomous periodic perturbations. It basically consists in providing a bifurcation function, called *Melnikov function*, whose simple zeros are associated with periodic solutions bifurcating from a period annulus of the corresponding differential system. The Melnikov function is derived by expanding a Poincaré map, typically the time-T-stroboscopic map, into Taylor series, since, in the smooth context, this map inherits the regularity of the flow.

The Melnikov analysis has also been applied in the investigation of existence of crossing periodic solutions in non-smooth differential systems in [8, 2, 23, 48, 49]. Therefore, in the same direction of the former references, our main goal is to provide an explicit expression, through a Melnikov procedure, for a function that controls the existence of periodic solutions in (6.1) for the cases where the unperturbed equation admits a period annulus.

Since the discontinuous nature of (6.1) imposes challenges in verifying the regularity of the time-σ stroboscopic map associated with it, we proceed by introducing the time as a variable and using the discontinuous set generated by the sign function as a Poincaré section. This approach allows the construction of a smooth displacement function, previously explored by J. Sotomayor in his thesis [58] for autonomous differential equations. This function quantifies the distance between the positive forward flow and the negative backward flow where both intersect the discontinuous set. A Melnikov-like function will, then, be obtained by expanding this displacement function into Taylor series.

This chapter is structured as follows: In section 6.1, we mainly discuss some additional properties for the unperturbed Filippov system (6.2). By taking into account the classification done in Chapter 1, we provide the statement of our main result (Theorem F) in Section 6.2. An application of our main result is also provided (see section 6.2.1). Finally, Section 6.3 is devoted for the proof of our main result.

6.1 Additional properties on the unperturbed Filippov system

Taking $y = \dot{x}$, we recall that (6.1) can be written as the differential system

$$X_{\theta,\mu}(t,x,y;\varepsilon) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu\,\mathrm{sgn}(x) + \varepsilon f(t,x,y). \end{cases} \tag{6.2}$$

for which we are adopting the Filippov convention for its solutions (see Section 2.1.2). Consequently, the solutions of (6.1) are derived from the solutions of the Filippov system (6.2).

It is important to emphasize that the switching plane Σ' corresponds to a crossing region of (6.2) (see Example 2.1.23), and only maximal solutions of (6.2) intersecting Σ' transversely are considered in the subsequent analysis. Furthermore, we highlight that the existence and local uniqueness of solutions, as discussed in Chapter 1 for the differential system (1.4), remain unaffected even when considering the $\mathcal{C}^1$-function $f(t,x,\dot{x})$ instead of $p(t)$.

Before presenting our main results, some additional comments on the unperturbed Filippov system are necessary. We remind that, for $\varepsilon = 0$, the differential system (6.2) becomes

$$X_{\theta,\mu}(t,x,y;0) = \mathbf{X}_{\theta,\mu}(x,y) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu\mathrm{sgn}(x), \end{cases}$$

which matches

$$\mathbf{X}^{+}_{\theta,\mu}(x,y) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x - \mu, \end{cases} \quad \text{and} \quad \mathbf{X}^{-}_{\theta,\mu}(x,y) : \begin{cases} \dot{x} = y, \\ \dot{y} = \theta x + \mu, \end{cases}$$

when restricted to $x \geqslant 0$ and $x \leqslant 0$, respectively. We notice that the line $\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$ represents the set of discontinuity of $\mathbf{X}_{\theta,\mu}$. Additionally, if we consider the involution $R(x,y) = (-x,y)$, we notice that $\mathbf{X}^{-}_{\theta,\mu}(x,y) = -R\mathbf{X}^{+}_{\theta,\mu}(R(x,y))$ and Fix$(R) \subset \Sigma$, which means that $\mathbf{X}_{\theta,\mu}$ is R-*reversible*. In the classical sense, a planar vector field X is said to be R-reversible if it satisfies $X(x,y) = -R(X(R(x,y)))$. The geometric meaning of this property is that the phase portrait of $\mathbf{X}_{\theta,\mu}$ is symmetric with respect to Fix(R). Furthermore, by considering the involution $S(x,y) = (x,-y)$, we verify that both $\mathbf{X}^{+}_{\theta,\mu}$ and $\mathbf{X}^{-}_{\theta,\mu}$ are S-reversible. This indicates that the trajectories of $\mathbf{X}^{+}_{\theta,\mu}$ and $\mathbf{X}^{-}_{\theta,\mu}$ exhibit symmetry with respect to Fix$(S) = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ (see Figures 2, 3, and 4).

Remark 6.1.1. *Let us denote* $\mathbf{x} = (x,y)$ *and* $\mathbf{a} = (0,\mu)$. *Then, taking*

$$A = \begin{pmatrix} 0 & 1 \\ \theta & 0 \end{pmatrix},$$

the vector fields $\mathbf{X}^+_{\theta,\mu}$ *and* $\mathbf{X}^-_{\theta,\mu}$ *can be rewritten as follows*

$$\mathbf{X}^+_{\theta,\mu}(\mathbf{x}) = A\mathbf{x} - \mathbf{a} \quad \textit{and} \quad \mathbf{X}^-_{\theta,\mu}(\mathbf{x}) = A\mathbf{x} + \mathbf{a},$$

for $\mathbf{x} \in \Sigma^+ = \{x > 0\}$ *and* $\mathbf{x} \in \Sigma^- = \{x < 0\}$*, respectively. By denoting the solutions of* $\mathbf{X}^+_{\theta,\mu}$ *and* $\mathbf{X}^-_{\theta,\mu}$*, as* $\Gamma^+(t, \mathbf{z}_0)$ *and* $\Gamma^-(t, \mathbf{z}_0)$*, respectively, with initial conditions* $\mathbf{z}_0 \in \Sigma^+$ *and* $\mathbf{z}_0 \in \Sigma^-$*, respectively, we notice that both solutions can be explicitly expressed by means of the variation of parameters as follows*

$$\Gamma^+(t, \mathbf{z}_0) = e^{At}(\mathbf{z}_0 - \int_0^t e^{-As} \cdot \mathbf{a}\, \mathrm{d}s) \quad \textit{and} \quad \Gamma^-(t, \mathbf{z}_0) = e^{At}(\mathbf{z}_0 + \int_0^t e^{-As} \cdot \mathbf{a}\, \mathrm{d}s),$$

where e^{At} *is the exponential matrix of* At *given by*

$$e^{At} = \begin{cases} \begin{pmatrix} \cosh(t\sqrt{\theta}) & \dfrac{\sinh(t\sqrt{\theta})}{\sqrt{\theta}} \\ \sqrt{\theta}\sinh(t\sqrt{\theta}) & \cosh(t\sqrt{\theta}) \end{pmatrix} & \textit{if} \quad \theta > 0, \\ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} & \textit{if} \quad \theta = 0, \\ \begin{pmatrix} \cos(t\sqrt{-\theta}) & \dfrac{\sin(t\sqrt{-\theta})}{\sqrt{-\theta}} \\ -\sqrt{-\theta}\sin(t\sqrt{-\theta}) & \cos(t\sqrt{-\theta}) \end{pmatrix} & \textit{if} \quad \theta < 0. \end{cases} \tag{6.3}$$

The R*-reversibility of* $\mathbf{X}_{\theta,\mu}$ *implies that* $\Gamma^-(t, \mathbf{z}_0) = R(\Gamma^+(-t, R \cdot \mathbf{z}_0))$ *and*

$$e^{At} = Re^{-At}R. \tag{6.4}$$

Now let us consider $\mathbf{x}_0 = (0, y_0)$, with $y_0 > 0$. For the sake of simplicity, we denote by $\Gamma^+(t, y_0) = (\Gamma^+_1(t, y_0), \Gamma^+_2(t, y_0))$ and $\Gamma^-(t, y_0) = (\Gamma^-_1(t, y_0), \Gamma^-_2(t, y_0))$ the solutions of $\mathbf{X}^+_{\theta,\mu}$ and $\mathbf{X}^-_{\theta,\mu}$, respectively, having $\mathbf{x}_0$ as the initial condition. Taking Remark 6.1.1 and the expression for e^{At} given in (6.3), the solutions of $\mathbf{X}^+_{\theta,\mu}$ having $\mathbf{x}_0$ as initial condition are given by

$$\Gamma(t, y_0) = \begin{cases} \left(\dfrac{\mu(1 - \cosh\left(\sqrt{\theta}t\right)) + \sqrt{\theta}y_0 \sinh\left(\sqrt{\theta}t\right)}{\theta}, y_0 \cosh\left(\sqrt{\theta}t\right) - \dfrac{\mu \sinh\left(\sqrt{\theta}t\right)}{\sqrt{\theta}} \right), & \theta > 0, \\ \left(ty_0 - \dfrac{\mu t^2}{2}, y_0 - \mu t \right), & \theta = 0, \\ \left(\dfrac{\mu(\cos(\zeta t) - 1) + \zeta y_0 \sin(\zeta t)}{\zeta^2}, y_0 \cos(\zeta t) - \dfrac{\mu \sin(\zeta t)}{\zeta} \right), & \theta < 0, \end{cases} \tag{6.5}$$

where $\zeta = \sqrt{-\theta}$. As previously mentioned, the R-reversibility of $\mathbf{X}_{\theta,\mu}$ allows us to easily obtain the expressions of $\Gamma^-(t, y_0)$ just by considering the relations

$$\Gamma^+(t, y_0) = \Gamma(t, y_0) \quad \text{and} \quad \Gamma^-(t, y_0) = R(\Gamma^+(-t, y_0)). \tag{6.6}$$

The following presents a reminder of the cases in which the unperturbed Filippov system $\mathbf{X}_{\theta,\mu}$ features a period annulus (see Figure 5):

- $\mathcal{A}_0$: $\theta < 0$ and $\mu = 1$;
- $\mathcal{A}_1$: $\theta < 0$ and $\mu = -1$;
- $\mathcal{A}_2$: $\theta > 0$ and $\mu = 1$;
- $\mathcal{A}_3$: $\theta = 0$ and $\mu = 1$.

For each one of these cases, there exist a half–period function, $\tau^+(y_0)$ (resp. $\tau^-(y_0)$), providing the smallest (resp. greatest) time for which the solution $\Gamma^+(t, y_0)$ (resp. $\Gamma^-(t, y_0)$), $(0, y_0) \in \mathcal{A}_i$, reaches the discontinuous line Σ again. In each of these cases, the function is given by $\tau^+(y_0) = \tau_0(y_0)$ and $\tau^-(y_0) = -\tau_0(y_0)$, where, using the S-reversibility of $\mathbf{X}^+_{\theta,\mu}$, the function $\tau_0(y_0)$ is the solution of the boundary problem

$$\begin{cases} \Gamma(0, y_0) = (0, y_0), \\ \Gamma_2\left(\dfrac{\tau_0(y_0)}{2}, y_0\right) = 0. \end{cases}$$

The expression for τ_0 is given by

$$\tau_0 : \begin{cases} (0,\infty) \to \left(0, \dfrac{\pi}{\zeta}\right) & \text{for } \mathcal{A}_0, \\ (0,\infty) \to \left(\dfrac{\pi}{\zeta}, \dfrac{2\pi}{\zeta}\right) & \text{for } \mathcal{A}_1, \\ \left(0, \dfrac{1}{\sqrt{\theta}}\right) \to (0,\infty) & \text{for } \mathcal{A}_2, \\ (0,\infty) \to (0,\infty) & \text{for } \mathcal{A}_3, \end{cases} \quad \text{and} \quad \tau_0(y_0) = \begin{cases} \dfrac{2}{\zeta} \arctan(\zeta y_0) & \text{for } \mathcal{A}_0, \\ \dfrac{2}{\zeta} (\pi - \arctan(\zeta y_0)) & \text{for } \mathcal{A}_1, \\ \dfrac{1}{\sqrt{\theta}} \log\left(\dfrac{1 + y_0\sqrt{\theta}}{1 - y_0\sqrt{\theta}}\right) & \text{for } \mathcal{A}_2, \\ 2y_0 & \text{for } \mathcal{A}_3. \end{cases} \tag{6.7}$$

It is noteworthy that in cases $\mathcal{A}_0$ and $\mathcal{A}_1$ the boundary problem above provides infinitely many solutions. However, due to the dynamics of the unperturbed system near $y_0 = 0$, the ones to be considered must satisfy the conditions $\lim_{y_0 \to 0^+} \tau_0(y_0) = 0$ in case $\mathcal{A}_0$ and $\lim_{y_0 \to 0^+} \tau_0(y_0) = 2\pi/\zeta$ in case $\mathcal{A}_1$.

Taking into account the the S-reversibility of $\mathbf{X}^+_{\theta,\mu}$ and $\mathbf{X}^-_{\theta,\mu}$, and equation (6.6), it follows that

$$\Gamma^-(-\tau_0(y_0), y_0) = \Gamma^+(\tau_0(y_0), y_0) = (0, -y_0). \tag{6.8}$$

Remark 6.1.2. *We notice that the points $p^+ = (-1/\eta, 0)$ and $p^- = (1/\eta, 0)$ in case $\mathcal{A}_1$ correspond to linear centers, which have been consistently studied in the literature (see, for instance, [47]). For this reason, we are not considering these centers in our study.*

For $i \in \{0,1,2,3\}$, we denote by $\mathcal{D}_i$, the interval of definition of τ_0, and by $\mathcal{I}_i$ the image of τ_0. From the possible expressions of τ_0 in (6.7), it can be observed that $\tau_0'(y_0) > 0$ in $\mathcal{D}_i$, for $i \in \{0,2,3\}$, and $\tau_0'(y_0) < 0$ in $\mathcal{D}_1$. Consequently, τ_0 is a bijection between $\mathcal{D}_i$ and $\mathcal{I}_i$ for $i \in \{0,1,2,3\}$. Its inverse is given by

$$v : \begin{cases} \left(0, \dfrac{\pi}{\zeta}\right) \to (0,\infty) & \text{for } \mathcal{A}_0, \\ \left(\dfrac{\pi}{\zeta}, \dfrac{2\pi}{\zeta}\right) \to (0,\infty) & \text{for } \mathcal{A}_1, \\ (0,\infty) \to \left(0, \dfrac{1}{\sqrt{\theta}}\right) & \text{for } \mathcal{A}_2, \\ (0,\infty) \to (0,\infty) & \text{for } \mathcal{A}_3, \end{cases} \quad \text{and} \quad v(\sigma) = \begin{cases} \dfrac{1}{\zeta}\tan\left(\dfrac{\zeta\,\sigma}{2}\right) & \text{for } \mathcal{A}_0 \\ -\dfrac{1}{\zeta}\tan\left(\dfrac{\zeta\,\sigma}{2}\right) & \text{for } \mathcal{A}_1, \\ \dfrac{1}{\sqrt{\theta}}\tanh\left(\dfrac{\sqrt{\theta}\,\sigma}{2}\right) & \text{for } \mathcal{A}_2, \\ \dfrac{\sigma}{2} & \text{for } \mathcal{A}_3. \end{cases} \tag{6.9}$$

6.2 Main result

As usual, the Melnikov method applied for determining the persistence of periodic solutions provides a bifurcation function, whose simple zeros are associated with periodic solutions bifurcating from a period annulus. In what follows we are going to introduce this function for system (6.1).

Let $f(t,x,\dot{x})$ be the $\sigma-$periodic function in t constituting the differential equation (6.1). Consider $\Gamma(t,y_0)$ and $v(\sigma)$ the functions defined in (6.5) and (6.9), respectively. We define the Melnikov–like function $M:\mathbb{R}\to\mathbb{R}$ as

$$M(\phi) = \int_0^{\frac{\sigma}{2}} U\left(t, \frac{\sigma}{2}\right)\left(f\left(\phi+t, \Gamma\left(t, v\left(\frac{\sigma}{2}\right)\right)\right) + f\left(\phi-t, R\Gamma\left(t, v\left(\frac{\sigma}{2}\right)\right)\right)\right) dt, \tag{6.10}$$

where

$$U(t,\sigma) = \begin{cases} -\dfrac{1}{\sqrt{\theta}}\operatorname{sech}\left(\dfrac{\sqrt{\theta}\,\sigma}{2}\right)\sinh\left(\dfrac{\sqrt{\theta}\,(2t-\sigma)}{2}\right) & \theta > 0, \\ -\dfrac{2t-\sigma}{2} & \theta = 0, \\ -\dfrac{\mu}{\zeta}\sec\left(\dfrac{\zeta\,\sigma}{2}\right)\sin\left(\dfrac{\zeta\,(2t-\sigma)}{2}\right) & \theta < 0. \end{cases} \tag{6.11}$$

with $\zeta = \sqrt{-\theta}$. Notice that, since $f(t,x,\dot{x})$ is σ-periodic in t, the Melnikov–like function M is σ-periodic. We derive the expression of M in the proof of our main result, which concerns periodic solutions of the differential equation (6.1). Its proof is postponed to Section 6.3.

Theorem F. *Suppose that for some $i \in \{0,1,2,3\}$ the parameters α and θ of the differential equation (6.1) satisfy the condition $\mathcal{A}_i$ and that $\sigma/2 \in \mathcal{I}_i$. Then, for each $\phi^* \in [0,\sigma]$, such that $M(\phi^*) = 0$ and $M'(\phi^*) \neq 0$, there exists $\overline{\varepsilon} > 0$ and a unique smooth branch $x_\varepsilon(t)$, $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$, of isolated σ-periodic solutions of the differential equation (6.1) satisfying $(x_0(\phi^*), \dot{x}_0(\phi^*)) = (0, v(\sigma/2))$.*

Remark 6.2.1. *It is important to emphasize that periodic solutions obtained in Theorem F bifurcate from the interior of the period annulus $\mathcal{A}_i$ for $i \in \{0, 1, 2, 3\}$. In case $\mathcal{A}_1$, the two homoclinic orbits joining $p = (0, 0)$ constitute the boundary of $\mathcal{A}_1$. Therefore, their persistence are not being considered in Theorem F, which is elucidated by the conclusion $v(\sigma/2) \in \mathcal{D}_1 = (0, \infty)$.*

6.2.1 Example

In order to illustrate the application of Theorem F, we examine the following differential equation

$$\ddot{x} + \operatorname{sgn}(x) = \theta x + \varepsilon \sin(\beta\, t), \quad \text{with} \quad \theta > 0. \tag{6.12}$$

This equation was previously studied in [15], where the authors provided conditions on θ and β in order to determine the existence of a discrete family of simple periodic solutions of (6.12). By assuming $f(t, x, \dot{x}) = \sin(\beta t)$ in (6.1), we reproduce a similar result as in [15, Theorem 2.1.1], as follows.

Proposition 6.2.2. *Given $n \in \mathbb{N}$, there exists $\overline{\varepsilon}_n > 0$ such that, for every $\varepsilon \in (-\overline{\varepsilon}_n, \overline{\varepsilon}_n)$, the equation (6.12) has n isolated $\frac{2\pi}{\beta}(2k-1)$-periodic solutions, for $k \in \{1, \ldots, n\}$, whose initial conditions are* $(x_\varepsilon^k(0), \dot{x}_\varepsilon^k(0)) = \left(0, \dfrac{1}{\sqrt{\theta}} \tanh\left(\dfrac{\pi\sqrt{\theta}\,(2k-1)}{\beta}\right)\right) + \mathcal{O}(\varepsilon)$.

The main difference between both results is that Proposition 6.2.2 is based on perturbation theory, while Theorem 2.1.1 in [15] provides a precise upper bound for ε by means of direct computations.

Proof. Since $\theta > 0$, the unperturbed equation (6.12) represents the case $\mathcal{A}_2$. Let us define $\sigma_i = 2\pi\beta^{-1} i$, for $i \in \mathbb{N}$. Since $f(t, x, \dot{x}) = \sin(\beta t)$ is σ_1-periodic in t, it is natural that $f(t, x, \dot{x})$ is also σ_i-periodic in t. Besides that, for every $i \in \mathbb{N}$, we have $\sigma_i/2 \in \mathcal{I}_1$. Then, for $i \in \mathbb{N}$, we compute the Melnikov function, defined as (6.10), corresponding to the period σ_i. This function takes the form

$$M_i(\phi) = \frac{2(1 + (-1)^{i-1}) \sin(\beta\, \phi)}{\beta^2 + \theta}.$$

Notice that if i is odd, then

$$M_i(\phi) = \frac{4 \sin(\beta\, \phi)}{\beta^2 + \theta},$$

while for even values of i, $M_i(\phi) = 0$. Given $n \in \mathbb{N}$, we observe that $M_{2k-1}(0) = 0$ and $M'_{2k-1}(0) \neq 0$ for each $k \in \{1, \ldots, n\}$. Applying Theorem F, it follows that, for each $k \in \{1, \ldots, n\}$, there exists $\varepsilon_k > 0$ and a unique branch $x_\varepsilon^k(t)$, $\varepsilon \in (-\varepsilon_k, \varepsilon_k)$, of isolated $\overline{\sigma}_k$-periodic solutions of the differential equation (6.12) satisfying $(x_\varepsilon^k(0), \dot{x}_\varepsilon^k(0)) = (0, \nu(\overline{\sigma}_k/2)) + \mathcal{O}(\varepsilon)$, where $\overline{\sigma}_k = \sigma_{2k-1}$. We see that, for each $k \in \{1, \ldots, n\}$, the periods $\overline{\sigma}_k$ are pairwise distinct, indicating that $\nu(\overline{\sigma}_{k_1}/2) \neq \nu(\overline{\sigma}_{k_2}/2)$ whenever $k_1 \neq k_2$. Therefore, by considering $\overline{\varepsilon}_n = \min\limits_{1 \leqslant k \leqslant n} \{\varepsilon_k\}$, we conclude the proof of the proposition. $\square$

6.3 Proof of the main result

In order to prove Theorem F, we consider the vector field associated to the differential system (6.2), given by

$$\begin{cases} \dot{\phi} = 1, \\ \dot{x} = y, \\ \dot{y} = \theta x - \mu \operatorname{sgn}(x) + \varepsilon f(\phi, x, y). \end{cases} \tag{6.13}$$

The differential system (6.13) matches

$$\begin{cases} \dot{\phi} = 1, \\ \dot{x} = y, \\ \dot{y} = \theta x - \mu + \varepsilon f(\phi, x, y), \end{cases} \quad \text{and} \quad \begin{cases} \dot{\phi} = 1, \\ \dot{x} = y, \\ \dot{y} = \theta x + \mu + \varepsilon f(\phi, x, y), \end{cases} \tag{6.14}$$

when it restricted to $x \geqslant 0$ and $x \leqslant 0$, respectively. The solutions for the differential systems in (6.14) with initial condition $(\phi_0, 0, y_0)$, for $y_0 > 0$, are given by the functions

$$\Phi^+(\tau, \phi_0, y_0; \varepsilon) = (\tau + \phi_0, \varphi^+(\tau, \phi_0, y_0; \varepsilon)), \tag{6.15}$$

and

$$\Phi^-(\tau, \phi_0, y_0; \varepsilon) = (\tau + \phi_0, \varphi^-(\tau, \phi_0, y_0; \varepsilon)), \tag{6.16}$$

respectively, where $\varphi^\pm(\tau, \phi_0, y_0; \varepsilon) = (\varphi_1^\pm(\tau, \phi_0, y_0; \varepsilon), \varphi_2^\pm(\tau, \phi_0, y_0; \varepsilon))$ is the solution for the Cauchy problem

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \mp \mathbf{a} + \varepsilon F(t + \phi_0, \mathbf{x}), \\ \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{6.17}$$

with $\mathbf{x}_0 = (0, y_0)$ and $F(t, \mathbf{x}) = (0, f(t, \mathbf{x}))$. Since the discontinuous plane $\Sigma' = \{(\phi, x, y) \in \mathbb{R}^3 : x = 0\}$, with $y \neq 0$, is a crossing region for the differential system (6.13), it follows that solutions of (6.13) arise from the concatenation of Φ^+ and Φ^- along Σ' when these solutions intersect Σ' transversely, as previously mentioned. We denote this concatenated solution by $\Phi(\tau, \phi_0, y_0; \varepsilon)$.

6.3.1 Construction of the displacement function

We notice that, for $y_0 > 0$,

$$\varphi_1^\pm(\tau_0^\pm(y_0), \phi_0, y_0; 0) = \Gamma_1^\pm(\tau_0^\pm(y_0), y_0) = 0,$$

and

$$\partial_t \varphi_1^\pm(\tau_0^\pm(y_0), \phi_0, y_0; 0) = \varphi_2^\pm(\tau_0^\pm(y_0), \phi_0, y_0; 0) = \Gamma_2^\pm(\tau_0^\pm(y_0), y_0) = -y_0 \neq 0, \tag{6.18}$$

with the last equality being given in equation (6.8). According to the Implicit Function Theorem, there exist smooth functions $\tau^+ : \mathcal{U}_{(\phi_0,y_0;0)} \to \mathcal{V}^+_{\tau_0^+(y_0)}$ and $\tau^- : \mathcal{U}_{(\phi_0,y_0;0)} \to \mathcal{V}^-_{\tau_0^-(y_0)}$, where $\mathcal{U}_{(\phi_0,y_0;0)}$ and $\mathcal{V}^\pm_{\tau_0^\pm(y_0)}$ are small neighborhoods of $(\phi_0, y_0; 0)$ and $\tau_0^\pm(y_0)$, respectively. These functions satisfy

$$\tau^\pm(\phi_0, y_0; 0) = \tau_0^\pm(y_0) \quad \text{and} \quad \varphi_1^\pm(\tau^\pm(\phi, y; \varepsilon), \phi, y; \varepsilon) = 0, \tag{6.19}$$

for every $(\phi, y; \varepsilon) \in \mathcal{U}_{(\phi_0,y_0;0)}$.

Thus, the functions $\tau^+(\phi_0, y_0; \varepsilon)$ and $\tau^-(\phi_0, y_0; \varepsilon)$, combined with the solutions (6.15) and (6.16) of the vector field (6.13), allow us to construct a displacement function, $\Delta(\phi_0, y_0; \varepsilon)$, that "controls" the existence of periodic solutions of (6.1) as follows

$$\begin{aligned}\Delta(\phi_0, y_0; \varepsilon) &= \Phi^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon) - \Phi^-(\tau^-(\phi_0 + \sigma, y_0; \varepsilon), \phi_0 + 2\sigma, y_0; \varepsilon)\\ &= \Phi^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon) - \Phi^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon),\end{aligned}$$

since $\tau^-(\phi_0, y_0; \varepsilon)$ is σ-periodic in ϕ_0. The displacement function $\Delta(\phi_0, y_0; \varepsilon)$ computes the difference in Σ' between the points $\Phi^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon)$ and $\Phi^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)$ (see Fig. 28). Thus it is straightforward that if $\Delta(\phi_0^*, y_0^*; \varepsilon^*) = 0$, for some $(\phi^*, y_0^*; \varepsilon^*) \in [0, \sigma] \times \mathbb{R}^+ \times \mathbb{R}$, then the solution $\Phi(\tau, \phi_0^*, y_0^*; \varepsilon^*)$ is σ-periodic in τ, meaning that $\Phi(\tau, \theta_0^*, y_0^*; \varepsilon^*)$ and $\Phi(\tau + \sigma, \theta_0^*, y_0^*; \varepsilon^*)$ are identified in the quotient space $\mathbb{S}_\sigma \times \mathbb{R}^2$. Furthermore, from the definition of Φ^+ and Φ^- in (6.15) and (6.16), respectively, we have that

$$\begin{aligned}\Delta(\phi_0, y_0; \varepsilon) &= (\Delta_1(\phi_0, y_0; \varepsilon), 0, \Delta_3(\phi_0, y_0; \varepsilon))\\ &:= (\tau^+(\phi_0, y_0; \varepsilon) - \tau^-(\phi_0, y_0; \varepsilon) - \sigma, 0,\\ &\qquad \varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon) - \varphi_2^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)).\end{aligned}$$

In what follows, we provide preliminary results concerning the main ingredients constituting the displacement function $\Delta(\phi_0, y_0; \varepsilon)$.

6.3.2 Preliminary results

This section is dedicated to presenting preliminary results regarding the solutions of (6.17) and the time functions $\tau^+(\phi_0, y_0; \varepsilon)$ and $\tau^+(\phi_0, y_0; \varepsilon)$ mentioned earlier. We begin by providing a result concerning the behavior of the solutions of (6.17) as ε approaches to zero.

Proposition 6.3.1. *For sufficiently small $|\varepsilon|$, the function $\varphi^\pm(t, \phi_0, y_0; \varepsilon)$ writes as*

$$\varphi^\pm(t, \phi_0, y_0; \varepsilon) = \Gamma^\pm(t, y_0) + \varepsilon\psi^\pm(t, \phi_0, y_0) + \mathcal{O}(\varepsilon^2), \tag{6.20}$$

where $\Gamma^+(t, y_0)$ and $\Gamma^-(t, y_0)$ are the functions given in (6.5) and (6.6), respectively, and

$$\psi^\pm(t, \phi_0, y_0) = e^{At}\int_0^t e^{-As}F(s + \phi_0, \Gamma^\pm(s, y_0))\mathrm{d}s. \tag{6.21}$$

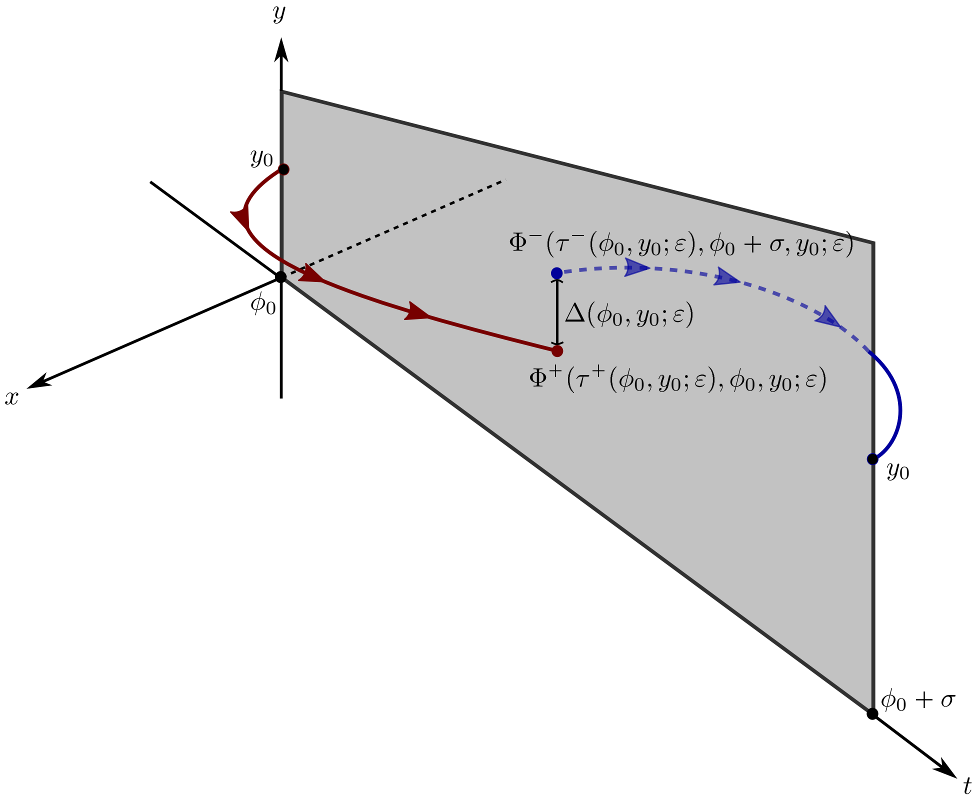


Figure 28 – Representation of the points $\Phi^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)$ and $\Phi^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon)$, which originate the displacement function Δ.

Proof. Since $\varphi^{\pm}(t, \phi_0, y_0; \varepsilon)$ is the solution to the Cauchy problem (6.17), then it must satisfy the integral equation

$$\varphi^{\pm}(t, \phi_0, y_0; \varepsilon) = \mathbf{x}_0 + \int_0^t \big[A\varphi^{\pm}(s, \phi_0, y_0; \varepsilon) \mp \mathbf{a} + \varepsilon F(s + \phi_0, \varphi^{\pm}(s, \phi_0, y_0; \varepsilon)\big]\,\mathrm{d}s,$$

which, by expanding in Taylor series around $\varepsilon = 0$, gives us

$$\begin{aligned}\varphi^{\pm}(t, \phi_0, y_0; \varepsilon) = (0, y_0) + \int_0^t \big[A\Gamma^{\pm}(s, y_0) \mp \mathbf{a}\big]\,\mathrm{d}s + \varepsilon \int_0^t \big[A\psi^{\pm}(s, \phi_0, y_0) \\ + F(s + \phi_0, \Gamma^{\pm}(s, y_0))\big]\,\mathrm{d}s + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Then, taking into account the expression for $\varphi^{\pm}(t, \phi_0, y_0; \varepsilon)$ in (6.20) and the computations above, we have that

$$\psi^{\pm}(t, \phi_0, y_0) = \int_0^t \big[A\psi^{\pm}(s, \phi_0, y_0) + F(s + \phi_0, \Gamma^{\pm}(s, y_0))\big]\,\mathrm{d}s,$$

which implies that $\psi^{\pm}$ is the solution to the Cauchy problem

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + F(t + \phi_0, \Gamma^{\pm}(s, y_0)), \\ \mathbf{x}(0) = (0, 0). \end{cases}$$

Then, the general formula for solutions of linear differential equations yields relationship (6.21). □

In what follows, we describe the behavior of the time functions $\tau^+(\phi, y; \varepsilon)$ and $\tau^-(\phi, y; \varepsilon)$ satisfying (6.19).

Proposition 6.3.2. *Let $\tau^\pm(\phi_0, y_0; \varepsilon)$ be the time satisfying equation (6.19). Then, for sufficiently small $|\varepsilon|$ and $y_0 > 0$ we have*

$$\tau^\pm(\phi_0, y_0; \varepsilon) = \tau_0^\pm(y_0) + \varepsilon\tau_1^\pm(\phi_0, y_0) + \mathcal{O}(\varepsilon^2),$$

with

$$\tau_1^\pm(\phi_0, y_0) = \frac{\psi_1^\pm(\tau_0^\pm(y_0), \phi_0, y_0)}{y_0}.$$

Proof. By expanding (6.19) in Taylor series around $\varepsilon = 0$, we have

$$\varepsilon\left(\partial_t\Gamma_1^\pm(\tau_0^\pm(y_0), y_0)\tau_1^\pm(\phi_0, y_0) + \psi_1^\pm(\tau_0^\pm(y_0), \phi_0, y_0)\right) + \mathcal{O}(\varepsilon^2) = 0,$$

which implies that

$$\partial_t\Gamma_1^\pm(\tau_0^\pm(y_0), y_0)\tau_1^\pm(\phi_0, y_0) + \psi_1^\pm(\tau_0^\pm(y_0), \phi_0, y_0) = 0. \tag{6.22}$$

Then equation (6.22) together with (6.18) conclude the proof of the proposition. □

The following result plays an important role in describing the behaviour of $\varphi_2^\pm$ around $\varepsilon = 0$. It is important to mention that, in our context, we identify vectors with column matrix.

Proposition 6.3.3. *Let us consider $v_1(t)$ and $v_2(t)$ as the lines of the matrix e^{At}. Then for every $y_0 > 0$, the following identity holds*

$$\mu v_1(\tau_0(y_0)) - y_0 v_2(\tau_0(y_0)) = (\mu \quad y_0),$$

where $\tau_0(y_0)$ is the half–period function defined in (6.7).

Proof. We define the auxiliary matrix-valued function

$$\beta(t) = \left(\mu - \theta\Gamma_1^+(t, y_0) \quad \Gamma_2^+(t, y_0)\right) \cdot e^{At},$$

which is continuously differentiable for every $t \in \mathbb{R}$. By differentiating $\beta(t)$, we have that

$$\begin{aligned}
\beta'(t) &= \left(-\theta\Gamma_2^+(t, y_0) \quad \theta\Gamma_1^+(t, y_0) - \mu\right) \cdot e^{At} + \left(\mu - \theta\Gamma_1^+(t, y_0) \quad \Gamma_2^+(t, y_0)\right) \cdot Ae^{At} \\
&= \left(-\theta\Gamma_2^+(t, y_0) \quad \theta\Gamma_1^+(t, y_0) - \mu\right) \cdot e^{At} + \left(\theta\Gamma_2^+(t, y_0) \quad \alpha - \theta\Gamma_1^+(t, y_0)\right) \cdot e^{At} \\
&= (0 \quad 0) \cdot e^{At} \\
&= (0 \quad 0).
\end{aligned}$$

Computations above imply that $\beta(t)$ is a constant function in each one of its entries, that is,

$$\beta(t) = \beta(0) = (\mu \quad y_0) \cdot \mathrm{Id} = (\mu \quad y_0)$$

for every $t \in \mathbb{R}$. In particular,

$$\mu v_1(\tau_0(y_0)) - y_0 v_2(\tau_0(y_0)) = \beta(\tau_0(y_0)) = (\mu \quad y_0),$$

and this concludes the proof of the proposition. □

In the following discussion, we provide key relationships for the fundamental components that appear in the expressions of $\varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon)$ and $\varphi_2^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)$ for sufficiently small values of $|\varepsilon|$. We start by formulating a more detailed expression for $\psi^+(t, \phi_0, y_0)$. By taking relation (6.21) into account, we have

$$\begin{aligned}\psi^+(t, \phi_0, y_0) &= \begin{pmatrix} \psi_1^+(t, \phi_0, y_0) \\ \psi_2^+(t, \phi_0, y_0) \end{pmatrix} = e^{At} \int_0^t e^{-As} F(s + \phi_0, \Gamma^+(s, y_0)) \mathrm{d}s \\ &= \begin{pmatrix} \langle v_1(t)^\top, I^+(t, \phi_0, y_0) \rangle \\ \langle v_2(t)^\top, I^+(t, \phi_0, y_0) \rangle \end{pmatrix},\end{aligned} \tag{6.23}$$

where

$$I^+(t, \phi_0, y_0) := \int_0^t e^{-As} F(s + \phi_0, \Gamma^+(s, y_0)) \mathrm{d}s. \tag{6.24}$$

This remark leads us to the following result.

Lemma 6.3.4. *For sufficiently small $|\varepsilon|$, the function $\varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon)$ writes as*

$$\varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon) = -y_0 - \frac{\varepsilon}{y_0} \langle (\mu, y_0), I^+(\tau_0(y_0), \phi_0, y_0) \rangle + \mathcal{O}(\varepsilon^2).$$

Proof. By expanding $\varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon)$ in Taylor series around $\varepsilon = 0$, we have that

$$\varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon) = \Gamma_2^+(\tau_0(y_0), y_0) + \varepsilon \xi^+(\phi_0, y_0) + \mathcal{O}(\varepsilon^2),$$

where

$$\begin{aligned}\xi^+(\phi_0, y_0) &= \partial_t \Gamma_2^+(\tau_0(y_0), y_0) \tau_1^+(\phi_0, y_0) + \psi_2^+(\tau_0(y_0), \phi_0, y_0) \\ &= \left(\theta \Gamma_1^+(\tau_0(y_0), y_0) - \mu\right) \frac{\psi_1^+(\tau_0(y_0), \phi_0, y_0)}{y_0} + \psi_2^+(\tau_0(y_0), \phi_0, y_0) \\ &= -\frac{1}{y_0} \left(\mu \psi_1^+(\tau_0(y_0), \phi_0, y_0) - y_0 \psi_2^+(\tau_0(y_0), \phi_0, y_0)\right).\end{aligned}$$

Thus, by considering the relationship (6.23), the function $\xi^+(\phi_0, y_0)$ can be rewritten as follows

$$\begin{aligned}\xi^+(\phi_0, y_0) &= -\frac{1}{y_0} \left(\mu \langle v_1(\tau_0(y_0))^\top, I^+(\tau_0(y_0), \phi_0, y_0) \rangle - y_0 \langle v_2(\tau_0(y_0))^\top, I^+(\tau_0(y_0), \phi_0, y_0) \rangle\right) \\ &= -\frac{1}{y_0} \left(\langle \mu v_1(\tau_0(y_0))^\top - y_0 v_2(\tau_0(y_0))^\top, I^+(\tau_0(y_0), \phi_0, y_0) \rangle\right) \\ &= -\frac{1}{y_0} \left(\langle (\mu, y_0), I^+(\tau_0(y_0), \phi_0, y_0) \rangle\right),\end{aligned}$$

where the last equality above is obtained after Proposition 6.3.3. Then, taking into account (6.8), the proof of the lemma is completed. □

In order to obtain analogous results as those achieved for $\varphi_2^+(\tau^+(\phi_0, y_0; \varepsilon), \phi_0, y_0; \varepsilon)$ in the context of the function $\varphi_2^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)$, we follow the previously outlined procedure. Then, by taking into account that F is a σ-periodic function and relationship (6.4), we notice that

$$\begin{aligned}
\psi^-(-t, \phi_0 + \sigma, y_0) &= \begin{pmatrix} \psi_1^-(-t, \phi_0 + \sigma, y_0) \\ \psi_2^-(-t, \phi_0 + \sigma, y_0) \end{pmatrix} \\
&= e^{-At} \int_0^{-t} e^{-As} F(s + \phi_0 + \sigma, \Gamma^-(s, y_0)) \mathrm{d}s \\
&= -Re^{At} R \int_0^{t} e^{As} RRF(-s + \phi_0, R\Gamma^+(s, y_0)) \mathrm{d}s \\
&= -R \begin{pmatrix} \langle v_1(t)^\top, \int_0^t e^{-As} RF(-s + \phi_0, R\Gamma^+(s, y_0)) \mathrm{d}s \rangle \\ \langle v_2(t)^\top, \int_0^t e^{-As} RF(-s + \phi_0, R\Gamma^+(s, y_0)) \mathrm{d}s \rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle v_1(t)^\top, I^-(t, \phi_0, y_0) \rangle \\ \langle -v_2(t)^\top, I^-(t, \phi_0, y_0) \rangle \end{pmatrix},
\end{aligned}$$

where

$$I^-(t, \phi_0, y_0) := \int_0^t e^{-As} RF(-s + \phi_0, R\Gamma^+(s, y_0)) \mathrm{d}s. \tag{6.25}$$

Similarly proceeding as in Lemma 6.3.4, we provide the following result concerning the behavior of $\varphi_2^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)$ around $\varepsilon = 0$.

Lemma 6.3.5. *For sufficiently small $|\varepsilon|$, the function $\varphi_2^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon)$ writes as*

$$\varphi_2^-(\tau^-(\phi_0, y_0; \varepsilon), \phi_0 + \sigma, y_0; \varepsilon) = -y_0 + \frac{\varepsilon}{y_0} \left(\langle (\mu, y_0), I^-(\tau_0(y_0), \phi_0, y_0) \rangle \right) + \mathcal{O}(\varepsilon^2).$$

Before we proceed with the proof of Theorem F, let us perform some essential computations which will play an important role in deriving the desired Melnikov-like function. Let $I^+(t, \phi_0, y_0)$ and $I^-(t, \phi_0, y_0)$ be the integrals defined in (6.24) and (6.25), respectively. We remind that $F(t, x, y) = (0, f(t, x, y))$. Taking $u(t) = (u_1(t), u_2(t))$ to be the second column of e^{-At}, that is,

$$u(t) = \begin{cases} \left(-\dfrac{\sinh(t\sqrt{\theta})}{\sqrt{\theta}}, \cosh(t\sqrt{\theta}) \right) & \text{if} \quad \theta > 0, \\ (-t, 1) & \text{if} \quad \theta = 0, \\ \left(-\dfrac{\sin(t\zeta)}{\zeta}, \cos(t\zeta) \right) & \text{if} \quad \theta < 0, \end{cases}$$

with $\zeta = \sqrt{-\theta}$, it follows that

$$I^{+}(t, \phi_0, y_0) + I^{-}(t, \phi_0, y_0) = \int_0^t e^{-As} \begin{pmatrix} 0 \\ g(s, \phi_0, y_0) \end{pmatrix} \mathrm{d}s \tag{6.26}$$
$$= \begin{pmatrix} \displaystyle\int_0^t u_1(s) g(s, \phi_0, y_0) \mathrm{d}s \\ \displaystyle\int_0^t u_2(s) g(s, \phi_0, y_0) \mathrm{d}s \end{pmatrix},$$

where we are defining $g(s, \phi_0, y_0) := f(\phi_0 + s, \Gamma^{+}(s, y_0)) + f(\phi_0 - s, R\Gamma^{+}(s, y_0))$. Notice that $g(s, \phi_0, y_0)$ is σ-periodic in ϕ_0.

6.3.3 Conclusion of the proof

The task of obtaining a point $(\phi_0, y_0; \varepsilon)$ that directly makes the function Δ vanish is quite challenging. Thus, in our approach, we proceed with a Melnikov–like method, which basically consists in computing the Taylor expansion of $\Delta(\cdot, \cdot; \varepsilon) = 0$ around $\varepsilon = 0$ up to order 1 and solving the resulting expression. In this direction, we start by examining the first component of the function Δ, which, after its Taylor expansion around $\varepsilon = 0$, is given by

$$\Delta_1(\phi_0, y_0; \varepsilon) = \tau^{+}(\phi_0, y_0; \varepsilon) - \tau^{-}(\phi_0, y_0; \varepsilon) - \sigma = 2\tau_0(y_0) - \sigma + \mathcal{O}(\varepsilon).$$

Let $i \in \{0, 1, 2, 3\}$ be fixed such that the parameters μ and θ satisfy condition $\mathcal{A}_i$ and $\sigma/2 \in \mathcal{I}_i$. Since τ_0 is a bijection between $\mathcal{D}_i$ and $\mathcal{I}_i$, there exist $y_0^* \in \mathcal{D}_i$ such that $\tau_0(y_0^*) = \sigma/2$. Additionally, as discussed in Section 6.1, $\partial_{y_0}\Delta_1(y_0^*; 0) = 2\tau_0'(y_0^*) \neq 0$, for every $\phi_0 \in \mathbb{S}_\sigma$. Therefore, from the Implicit Function Theorem and the compactness of $\mathbb{S}_\sigma$, there exist $\varepsilon_1 > 0$, $\delta_1 > 0$, and a unique $\mathcal{C}^1$-function $\overline{y} : \mathbb{S}_\sigma \times (-\varepsilon_1, \varepsilon_1) \to (y_0^* - \delta_1, y_0^* + \delta_1)$ such that $\overline{y}(\phi_0, 0) = y_0^*$ and $\Delta_1(\overline{y}(\phi_0, \varepsilon); \varepsilon) = 0$, for every $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and every $\phi_0 \in \mathbb{S}_\sigma$.

By substituting $\overline{y}(\phi_0, \varepsilon)$ into $\Delta_3(\phi_0, y_0; \varepsilon)$, and taking into account Lemmas 6.3.4 and 6.3.5, we have, for sufficiently small $|\varepsilon|$,

$$\Delta_3(\phi_0, \overline{y}(\phi_0, \varepsilon); \varepsilon) = -\frac{2\varepsilon}{y_0^*} \left(\left\langle (\mu, y_0^*), I^{+}\left(\frac{\sigma}{2}, \phi_0, y_0^*\right) + I^{-}\left(\frac{\sigma}{2}, \phi_0, y_0^*\right) \right\rangle \right) + \mathcal{O}(\varepsilon^2),$$

where $I^{+}(\sigma, \phi_0, y_0^*)$ and $I^{-}(\sigma, \phi_0, y_0^*)$ are the integrals defined in (6.24) and (6.25), respectively. We can then define the function, for $|\varepsilon|$ sufficiently small,

$$\tilde{\Delta}_3(\phi_0; \varepsilon) := -\frac{y_0^*}{2\varepsilon} \Delta_3(\phi_0, \overline{y}(\phi_0, \varepsilon); \varepsilon),$$

which, after being expanded in Taylor series around $\varepsilon = 0$, gives us

$$\tilde{\Delta}_3(\phi_0; \varepsilon) = M(\phi_0) + \mathcal{O}(\varepsilon),$$

with

$$M(\phi_0) = \left\langle (\mu, y_0^*), I^+\left(\frac{\sigma}{2}, \phi_0, y_0^*\right) + I^-\left(\frac{\sigma}{2}, \phi_0, y_0^*\right)\right\rangle.$$

The identity (6.26) and the fact that $y_0^* = v(\sigma/2)$, allow us to rewritten $M(\phi_0)$ as follows

$$M(\theta_0) = \int_0^{\frac{\sigma}{2}} \left\langle \left(\mu, v\left(\frac{\sigma}{2}\right)\right), u(s)\right\rangle g\left(s, \phi_0, v\left(\frac{\sigma}{2}\right)\right) \mathrm{d}s,$$

and this lead us to the expression stated in (6.10), with the auxiliary function $U(t,\sigma) = \langle (\mu, v(\sigma)), u(t)\rangle$ expressed in (6.11).

Notice that the σ-periodicity of $g(s, \phi_0, v(\sigma/2))$ in ϕ_0 implies that M is σ-periodic, which enables us to restrict our analysis to the interval $[0,\sigma]$. Now suppose that $\phi^* \in [0,\sigma]$ is such that $M(\phi^*) = 0$ and $M'(\phi^*) \neq 0$. Then, by the Implicit Function Theorem, there exist $0 < \overline{\varepsilon} < \varepsilon_1$ and a branch $\overline{\phi}(\varepsilon)$ of simple zeros of M satisfying $\overline{\phi}(0) = \phi^*$ and $M(\overline{\phi}(\varepsilon)) = \tilde{\Delta}_3(\overline{\phi}(\varepsilon);\varepsilon) = 0$, for every $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$. Back to the solution of the differential system (6.13), we have that $\Phi(\tau, \overline{\phi}(\varepsilon), \overline{y}(\overline{\phi}(\varepsilon), \varepsilon); \varepsilon)$ is a σ-periodic solution of (6.13), whenever $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$.

Notice that, by defining

$$x_\varepsilon(t) := \Phi_2(t - \overline{\phi}(\varepsilon), \overline{\phi}(\varepsilon), \overline{y}(\overline{\phi}(\varepsilon), \varepsilon); \varepsilon),$$

and taking into account that $\dot{x}_\varepsilon(t) = \Phi_3(t - \overline{\phi}(\varepsilon), \overline{\phi}(\varepsilon), \overline{y}(\varepsilon); \varepsilon)$, where Φ_2 and Φ_3 are the second and third entries of Φ, respectively, we have that

$$x_0(\phi^*) = \Phi_2(0, \phi^*, y_0^*; 0) = 0 \quad \text{and} \quad \dot{x}_0(\phi^*) = \Phi_3(0, \phi^*, y_0^*; 0) = y_0^* = v\left(\frac{\sigma}{2}\right).$$

It concludes the proof of Theorem F.

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