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Can KS-contextuality hide in a crowd? : investigating state-independent contextuality in systems with multiple observers

A contextualidade-KS pode se esconder na multidão? : investigando a contextualidade independente de estado em sistema com múltiplos observadores

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O correr da vida embrulha tudo,
a vida é assim: esquenta e esfria,
aperta e daí afrouxa, sossega e depois desinquieta.
O que ela quer da gente é coragem.
O que Deus quer é ver a gente
aprendendo a ser capaz
de ficar alegre a mais,
no meio da alegria,
e inda mais alegre
ainda no meio da tristeza!
A vida inventa!
A gente principia as coisas,
no não saber por que,
e desde aí perde o poder de continuação
porque a vida é mutirão de todos,
por todos remexida e temperada.
O mais importante e bonito, do mundo, é isto:
que as pessoas não estão sempre iguais,
ainda não foram terminadas,
mas que elas vão sempre mudando.
Afinam ou desafinam. Verdade maior.
Viver é muito perigoso; e não é não.
Nem sei explicar estas coisas.
Um sentir é o do sentente, mas outro é do sentidor.

João Guimarães Rosa, Grande Sertão: Veredas.

Resumo

Nesta tese, investigamos como um sistema compartilhado por múltiplos observadores, o que chamamos de um Sistema Público, afeta a contextualidade-KS. A contextualidade-KS é uma característica fascinante da teoria quântica, que não faz parte da descrição clássica da realidade. A literatura indica que Sistemas Públicos conseguem suprimir contextualidade dependente de estado, na forma de violações das desigualdades de N-ciclo. Uma possível explicação para este limite clássico é que os observadores adicionais estariam degradando o estado e esgotando o recurso quântico. Esta justificativa, todavia, não se estende à contextualidade independente de estado, indicando que ela pudesse sobreviver em um Sistema Público. Nesta tese, mostramos que este não é o caso, analisando um observador tentando violar a desigualdade de Peres-Mermin em um Sistema Público. A partir de uma simulação numérica do sistema, mostramos que um único observador adicional é suficiente para impedir violações. Também oferecemos uma descrição analítica do sistema, esclarecendo o desaparecimento da contextualidade. Em síntese, nossos resultados mostram que mesmo a contextualidade independente de estado não é independente do que ocorre entre medições de um contexto.

Palavras-chave: Teoria quântica; Contextualidade KS; Correlações quânticas; Limite clássico.

Abstract

In this thesis, we investigate what happens to KS-contextuality in a system shared with multiple observers, which we call a Public System. KS-contextuality is a fascinating feature of quantum theory, though it is not present in the classical description of reality. Previous research indicated that Public Systems could suppress state-dependent KS-contextuality, expressed as violations of N-cycle inequalities. One possible explanation for this classical limit is the additional observers' measurements degrading the state and depleting the quantum resource. However, this justification does not apply to state-independent contextuality, indicating it might survive in a Public System. This thesis shows this is not the case by exploring an observer trying to violate the Peres-Mermin noncontextuality inequality in a Public System. Through a numerical simulation of the system, we show that a single additional observer is enough to prevent violations. We also give an analytical description of the setup, elucidating the loss of contextuality. Ultimately, our results show that even state-independent contextuality is not independent of what takes place between measurements of a context.

Keywords: Quantum theory; KS-contextuality; Quantum correlations; Classical limit.

List of Figures

List of Tables

Contents

1 Introduction

Quantum theory is very successful and provides accurate predictions of various physical phenomena. Although it is now a very established field, it retains a sort of uncanny aura, perhaps best encapsulated by the famous quote attributed to Niels Bohr: "[...] those who are not shocked when they first come across quantum theory cannot possibly have understood it."(HEISENBERG, 1971). Many foundational quantum properties have no classical analogs and are difficult to understand from a classical outlook.

Classically, we expect every measurement to reveal a predetermined value, irrespective of other measurements one may implement. Quantum theory is inconsistent with this perspective; it is what we call a KS-contextual theory. This result, known as the Kochen-Specker theorem (KOCHEN; SPECKER, 1967), stands out as one of the most distinctive nonclassical features of the quantum world.

Since KS-contextuality is a nonclassical phenomenon, what processes can suppress KS-contextuality from a quantum world? Studying these classical limits is an essential step toward understanding the natural world and might help prevent classicality from manifesting itself in applications that rely on this quantum resource.

A convenient way to detect KS-contextuality is to look for violations of a noncontextuality inequality (KLYACHKO et al., 2008). If any state can produce a violation, the inequality reveals state-*independent* contextuality (SIC), but if only some states do so, it reveals state-*dependent* contextuality.

Ref. (BALDIJÃO; TERRA CUNHA, 2020) showed that if multiple independent observers have full access to a system, which we call a Public System (PS), they cannot detect state-dependent contextuality. A natural suspect for the disappearance of KS-contextuality is state degradation: after other observers have interacted with the system, the state might no longer violate the inequality. Such an explanation would be unable to suppress state-independent contextuality.

In this thesis, we expand on Baldijão and Terra Cunha's work by showing that, perhaps surprisingly, a Public System can also wash away state-independent contextuality. We achieve this by simulating a public system where an observer tries to violate the Peres-Mermin inequality. We found that, even though KS-contextuality can be independent of the state the system is in before measurements, it is sensitive to what happens between compatible measurements of an observer.

We also examine what causes this phenomenon by modeling our system as an open quantum system. We can understand the emergence of classicality in two ways: In the

Schrödinger picture, additional observers are equivalent to a depolarization channel that breaks the correlations among measurements. In the Heisenberg picture, where the state does not change due to the additional observers, the measurements are altered, impacting their compatibility structure and the observer's ability to witness violations.

In Chapter 2, we set the grounds properly defining KS-contextuality and how to detect it, Chapter 3 discusses experimental measurement rounds and defines the Public System, Chapter 4 explains how we simulated our Public System, Chapter 5 showcases our results from the simulation and an analytical description and Chapter 6 delineates our conclusions.

Appendices supplement our main text: Appendix A presents results from an alternative implementation of the Public System. In Appendix C, we analyze the quantum channel influencing our Public System. Lastly, Appendix B explores the expected values of sequential measurements.

This thesis assumes previous knowledge on topics of linear algebra (LANG, 1987; DEBNATH; MIKUSINSKI, 2005), and quantum mechanics, especially in finite dimensions (NIELSEN; CHUANG, 2010; AMARAL; BARAVIERA; TERRA CUNHA, 2011) that are not fully contained within the text. The reader is encouraged to look at the suggested references for a thorough exposition.

2 Background

The classical world admits a determined answer to any question, and a probabilistic description is only necessary when we lack knowledge about the system. For example, in a well shuffled deck of cards, there is roughly a 2% chance to draw any specific card. This probabilistic description does not mean that it is impossible to know what card will be drawn; rather, it is a consequence of not having enough information about how it was shuffled. In other words, classical probability pertains to knowledge about events, making it epistemological. It does not reveal the fundamental nature of reality itself (ontology).

In quantum theory, probabilities are ubiquitous due to the nature of quantum states, which can exist in superpositions. Even if we prepare the same state and perform the same measurement multiple times, the results may differ each time. However, it could be the case that the current description of a quantum state does not completely describe the system. It is not unreasonable to think that when we prepare various systems at the same state, even though we think we are preparing the same state over and over, we are actually preparing an ensemble of different states, each of which could have a different deterministic outcome for our measurement, and it is this incomplete information that necessitates the probabilistic description. Historically, the idea that quantum theory could be "completed" is most famously associated with Einstein, Podolsky, and Rosen due to the paper (EINSTEIN; PODOLSKY; ROSEN, 1935).

Kochen-Specker contextuality (KS-contextuality) is a property that helps us assess the possibility of completing a model in a way that allows a noncontextual value assignment to all measurement outcomes. In this chapter, we will formalize this idea, drawing inspiration from other works in the field, particularly Refs. (AMARAL, 2014; SANTOS, 2018; BALDIJÃO, 2021; BARBOSA; KARVONEN; MANSFIELD, 2021).

In Section 2.1 we establish the foundational concepts by defining a scenario and its behavior. In Section 2.2 we formalize the process of completing our description of the system and investigate when a behavior is consistent with noncontextual completion. Section 2.3 introduces the nondisturbance property. Section 2.4 defines KS-contextuality and how it relates to the possibility of completion. Finally, and Section 2.5 explores how we can detect KS-contextuality using noncontextuality inequalities.

2.1 Behaviors

In this chapter, we will adopt an operational theory-independent perspective. Our approach involves treating the system as a black box, originating from a preparation process, which is subjected to a single measurement and subsequently discarded, a process called a prepare-and-measure round.

This framework does not ascribe an intrinsic state to the system; rather, we define it's state as an equivalence class of preparation processes that have the same outcome probability distributions for all available measurements. This underscores the versatility of our approach to KS-contextuality, as it serves as it can be applied not only to quantum theory but also in the exploration of other, and perhaps future, theories.

Throughout this thesis, will use the following notation: a measurement will be represented by a capital letter with an index, such as A_i, while an outcome of the i-th measurement will be represented by the corresponding lowercase letter, a_i. An outcome probability distribution, $P_{\mathbb{p}}(a_i|A_i)$, is the probability of obtaining each measurement outcome a_i from measuring A_i on a system subject to preparation $\mathbb{p}$.

The general problem we are interested in is whether the randomness from measurements can be can be attributed to an incomplete description of the state. In other words, we investigate whether we could think of the preparation process as generating an ensemble of latent states $\{(\mathbb{p}, \lambda)\}_{\lambda\in\Lambda}$, each represented by a pair $(\mathbb{p}, \lambda)$, having predetermined outcomes for all available measurements.

For this hypothesis to even be reasonable, repeated measurements on the same system should yield consistent results. So, in this chapter, and when working with KS-contextuality in general, we exclusively focus on repeatable measurements:

Definition 1 (Repeatable measurement)**.** *A measurement A_i is said to be repeatable if, after a preparation $\mathbb{p}$, measuring A_i and obtaining the result a_i, implies that any subsequent measurement of A_i on the system will always yield the same result a_i with probability* 1*.*

In quantum theory, repeatability limits us to projective measurements, since more general measurements, are not fully repeatable (HEINOSAARI; ZIMAN, 2011). To study contextuality with more general measurements, readers may be interested in exploring a different formulation called Spekkens (or generalized) contextuality, which is discussed in Ref. (SPEKKENS, 2005).

Suppose that we would like to implement a set of measurements $\{A_i\}_i$ in a system, after a preparation $\mathbb{p}$. Ideally, we would like to have a single measurement that simultaneously provides the outcomes for all measurements in our set. This concept is known as a joint measurement. In the classical world, joint measurability is always possible, however, in a general operational theory, various factors can prevent joint measurability.

To illustrate this limitation, consider that security concerns may necessitate restricting simultaneous access to certain attributes in a database. Nonetheless, the main interest in this limitation lies in the fact that quantum theory does not allow joint implementation of arbitrary measurements.

Each theory will have its own rules for when joint measurability is allowed, which defines a compatibility structure. The notion of compatibility can be defined in several ways (GÜHNE et al., 2023; GUERINI; TERRA CUNHA, 2018), depending on the property one wishes to emphasize. Here, since our focus is joint measurability, we provide the following definition:

Definition 2 (Compatibility)**.** *A set of measurements* $\mathcal{A} = \{A_1, \dots, A_n\}$*, each* $A_k \in \mathcal{A}$ *with measurement outcomes* $a_k \in \mathcal{O}_k$*, is compatible if it is possible to define a single measurement* $A_{(1,\dots,n)}$*, called a joint measurement, with outcomes* $(a_1, \dots, a_n) \in \mathcal{O}_1 \times \dots \times \mathcal{O}_n$ *such that, for any preparation* $\mathbb{p}$*,* $A_{(1,\dots,n)}$ *produces the outcome probability distribution of each measurement* $A_k \in \mathcal{A}$ *through marginalization*

$$P_{\mathbb{p}}(a_k|A_k) = \sum_{a_i : i \neq k} P_{\mathbb{p}}((a_1, \dots, a_k, \dots, a_n)|A_{(1,\dots,k,\dots,n)}), \tag{2.1}$$

$\forall A_k \in \mathcal{A}$.

To further highlight joint measurability, we will write the distribution $P_{\mathbb{p}}((a_1, \dots, a_n)|A_{(1,\dots,n)})$ *as*

$$P_{\mathbb{p}}(a_1, \dots, a_n|A_1, \dots, A_n). \tag{2.2}$$

In practice, joint measurability means that, in a prepare-and-measure round, we can only discover the outcomes of a set of compatible measurements. The fact that it is impossible to discover the values of incompatible measurements simultaneously does not imply that they do not simultaneously exist. This highlights again the difference between epistemology, which is concerned with our knowledge and understanding of the world, and ontology, which deals with the actual existence of entities and properties. Therefore, incompatibility does not rule out global deterministic outcomes.

We call a set of compatible measurements a context.

Definition 3 ((Maximal) Context)**.** *A context is a set of compatible measurements. We say that a context is maximal if it is not properly contained by any other context.*

In order to simplify notation, we will also use the contexts to index compatible measurements. Given a context $C = \{A_1, A_3, A_5\}$ we will refer to the probability distribution associated with its joint measurement as $P_{\mathbb{p}}(a_{C^{(1)}}, a_{C^{(2)}}, a_{C^{(3)}}|A_{C^{(1)}}, A_{C^{(2)}}, A_{C^{(3)}})$.

We can now define a compatibility scenario, which condenses all the information of how the observers are allowed to interact with the system.

Definition 4 (Compatibility scenarios)**.** *A compatibility scenario* $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ *is defined by three sets: a set of measurements* $\mathcal{A}$*, their outcomes set* $\mathcal{O}$*, and the set of maximal contexts* $\mathcal{C}$*.*

We will restrict ourselves to finite scenarios, where there is a finite number of measurements that all have a finite number of outcomes. For convenience, we will consider only homogeneous scenarios, where all measurements have the same number of outcomes[1] and all maximal contexts have the same number of measurements. We will represent the total number of measurements in a scenario with the letter n and the number of measurements in each maximal context with the letter m.

Compatibility scenarios provide a very general framework. A scenario is not a single physical experiment that can be realized in a laboratory. Instead, a compatibility scenario is an abstraction of a whole class of "questions", with a set of possible "answers", that can be jointly known according to the compatibility structure.

We are interested in the outcome probability distribution to the joint measurements of all contexts from the scenario. We call this set of outcome probability distributions the behavior.

Definition 5 (Behavior)**.** *The behavior* $\mathcal{B}^{\Gamma}_{\mathbb{p}}$ *of a scenario* $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ *is the family of probability distributions over* $\mathcal{O}^m$*:*

$$\mathcal{B}^{\Gamma}_{\mathbb{p}} := \{P_{\mathbb{p}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}})\}_{C\in\mathcal{C}}. \tag{2.3}$$

We can visualize behaviors using tables. Each line displays the results for one of the maximal contexts and each row corresponds to one possible output from $\mathcal{O}^m$, as illustrated in Ex. 1.

Example 1. *A generic behavior for the scenario*

$$\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}} := \begin{cases} \mathcal{A} = \{A_1, A_2, A_3\}; \\ \mathcal{O} = \{-1, 1\}; \\ \mathcal{C} = \{\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_1\}\}; \end{cases} \tag{2.4}$$

can be displayed in table, as shown with the generic behavior in Table 1.

Table 1 – Tabular representation of a behavior

Maximal contexts	**Outcomes** $(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\{A_1, A_2\}$	$P_{\mathbb{p}}(-1,-1\|A_1,A_2)$	$P_{\mathbb{p}}(-1,1\|A_1,A_2)$	$P_{\mathbb{p}}(1,-1\|A_1,A_2)$	$P_{\mathbb{p}}(1,1\|A_1,A_2)$
$\{A_2, A_3\}$	$P_{\mathbb{p}}(-1,-1\|A_2,A_3)$	$P_{\mathbb{p}}(-1,1\|A_2,A_3)$	$P_{\mathbb{p}}(1,-1\|A_2,A_3)$	$P_{\mathbb{p}}(1,1\|A_2,A_3)$
$\{A_3, A_1\}$	$P_{\mathbb{p}}(-1,-1\|A_3,A_1)$	$P_{\mathbb{p}}(-1,1\|A_3,A_1)$	$P_{\mathbb{p}}(1,-1\|A_3,A_1)$	$P_{\mathbb{p}}(1,1\|A_3,A_1)$

[1] It's worth noting that the labels given to the outcomes are arbitrary; what holds significance is the number elements in $\mathcal{O}$.

Each line of the table corresponds to an outcome probability distribution $P_{\mathbb{p}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}})$, so all values in a line sum to 1.

Through an analysis of various behaviors of a scenario, we can identify attributes that indicate if it would be possible to complete the description of the system and achieve determinism. This allows us to investigate the underlying model just by looking at the behaviors it produces.

2.2 Noncontextual completion

This section will formalize the process of completing a theory and explore when it can be done. In a classical description, measurements have predetermined values, which the measurement process reveals. Any non-deterministic behavior in a classical world can thus be attributed to an incomplete description of the system's state.

A completion introduces an extra-variables space Λ. Under this new description, the preparation outputs an ensemble of "latent" states $\{(\mathbb{p}, \lambda)\}_\lambda$, and it is possible to assign a predetermined value $V_{\mathbb{p},\lambda}(A_k)$ for all measurements on each latent state. The outcome probability distributions on these states are deterministic

$$D_{\mathbb{p},\lambda}(a_k, A_k) : \mathcal{O} \to \{0, 1\}, \tag{2.5}$$

with $D_{\mathbb{p},\lambda}(V_{\mathbb{p},\lambda}(A_k)|A_k) = 1$, and the probabilistic description arises from the observer's lack of knowledge of the extra variables:

$$P_{\mathbb{p}}(a_k|A_k) = \sum_\lambda p(\lambda) D_{\mathbb{p},\lambda}(a_k|A_k), \tag{2.6}$$

where $p(\lambda)$ is the probability that the state preparation for $\mathbb{p}$ outputs the latent state $(\mathbb{p}, \lambda)$.

A deterministic completion is noncontextual if the distributions $D_{\mathbb{p},\lambda}(a_k|A_k)$ are independent of other jointly performed measurements. In these cases, the joint outcome probability distributions for contexts can be obtained by multiplying the individual measurements' outcome distributions:

$$D_{\mathbb{p},\lambda}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) = \prod_{k: A_k \in C} D_{\mathbb{p},\lambda}(a_k|A_k). \tag{2.7}$$

Classical reasoning presumes a noncontextual completion since if the outcomes $V_{\mathbb{p},\lambda}(A_k)$ exist before the measurement, which simply reveals them, they cannot depend on choices made in the moment of the measurement. In this chapter, we are exclusively considering noncontextual completions.

Every behavior in a classical theory emerges from noncontextual predetermined values, meaning that every behavior would be consistent with a noncontextual completion.

Definition 6. *A behavior $\mathcal{B}_{\mathbb{P}}^{\Gamma}$ of a scenario $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ is consistent with a noncontextual completion if there exists measurable space Λ, a probability distribution $p(\lambda)$ on Λ, and family of context-independent deterministic outcome probability distributions*

$$\{D_{\mathbb{P},\lambda}(a_k|A_k)\}_{\lambda\in\Lambda}, \tag{2.8}$$

such that

$$P_{\mathbb{P}}(a_{C^{(1)}},...,a_{C^{(m)}}|A_{C^{(1)}},...,A_{C^{(m)}}) = \sum_{\lambda} p(\lambda) \prod_{k:A_k\in C} D_{\mathbb{P},\lambda}(a_k|A_k) \tag{2.9}$$

$\forall C \in \mathcal{C}$.

Labeling a behavior as *consistent* with a noncontextual completion is not an assessment of whether the predetermined outcomes or extra variables physically exist. Instead, it means that a noncontextual deterministic model could also generate the same behavior. In contrast, when a noncontextual completion is deemed impossible, we can dismiss the possibility of noncontextual predetermined outcomes.

We will now investigate a couple of properties from behaviors and how they relate to noncontextual outcome determinism.

2.3 Nondisturbance

In a scenario, measurements can be part of multiple contexts. If two contexts C and $\tilde{C}$ overlap, $C \cap \tilde{C} = \{A_1,...,A_k\}$, one could perform a joint measurement of either context and forget the results of all measurements not in the intersection. These measurements would yield two marginal distributions for the measurements in $C \cap \tilde{C}$. If the marginals from all contexts coincide, we say the behavior is nondisturbing.

Definition 7 (Nondisturbance)**.** *A behavior is nondisturbing if, given a set of compatible measurements $A_1,...,A_k \subset \mathcal{A}$,*

$$\begin{aligned} &P_{\mathbb{P}}(a_1,...,a_k|A_1,...,A_k,A_{C^{(k+1)}},...,A_{C^{(m)}}) := \\ &\sum_{a_{C^{(i)}}:C^{(i)}\notin\{1,...,k\}} P_{\mathbb{P}}(a_1,...,a_k,a_{C^{(k+1)}},...,a_{C^{(m)}}|A_1,...,A_k,A_{C^{(k+1)}},...,A_{C^{(m)}}) = \\ &P_{\mathbb{P}}(a_1,...,a_k|A_1,...,A_k). \end{aligned} \tag{2.10}$$

for every $C \in \mathcal{C}$ such that $\{A_1,...,A_k\} \subset C$.

We can think of each outcome probability distribution in the behavior as a glimpse into part of the information in the system; due to incompatibility constraints, we can never see the whole picture. It is as if we were looking at a painting but were only allowed to see a fragment each time we look. In this case, we would expect to be able to stitch together two overlapping patches into a coherent image, as illustrated in Figure 1.

To offer a more practical sense of nondisturbance, we provide an example with behaviors of a simple scenario.

Example 2 ((Non)Disturbing behaviors)**.** *Tables 2 and 3 display behaviors for the scenario*

$$\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}} := \begin{cases} \mathcal{A} = \{A_1, A_2, A_3\}; \\ \mathcal{O} = \{-1, 1\}; \\ \mathcal{C} = \{\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_1\}\}. \end{cases} \tag{2.11}$$

Table 2 – Nondisturbing behavior

Maximal contexts	**Outcomes** $(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\{A_1, A_2\}$	0	1/2	1/2	0
$\{A_2, A_3\}$	0	1/2	1/2	0
$\{A_3, A_1\}$	0	1/2	1/2	0

Table 3 – Disturbing behavior

Maximal contexts	**Outcomes** $(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\{A_1, A_2\}$	1/2	0	1/2	0
$\{A_2, A_3\}$	0	0	1/2	1/2
$\{A_3, A_1\}$	1/4	1/4	1/4	1/4

Let us focus on measurement A_2 in both tables. Notice that, in Table 2, A_2 will output -1 with probability $1/2$ and outputs 1 with probability $1/2$ regardless of whether one jointly measures it with A_1 or A_3. In other words, measuring A_1 or A_3 does not disturb A_2's outcome distribution.

On the other hand, at Table 3 measurement A_2 outputs -1 with probability 1 if measured alongside A_1, and outputs 1 with probability 1 if measured alongside A_2. In this extreme case of disturbance, what one measures alongside A_2 essentially determines its result.

One important distinction we would like to stress is that, in Table 2, the outputs of measurement A_2 are strongly ***anti-correlated*** *with those of the other measurement. This phenomenon is different from disturbance. If one were to measure A_1 alongside A_2 but looked only at the result of A_1, they would automatically* ***know*** *the result of A_2. However, since they cannot control the output of A_1, there is no way to* ***influence*** *the result of A_2*

Figure 1 – In this figure, we can see fragments of an image depicting flowing water, which represent different contexts. Whenever we look across two overlapping fragments, the water flow makes sense, representing a nondisturbing behavior. Source: Adapted from (ESCHER, 1961 apud ART INSTITUTE OF CHICAGO, 2014)

with their measurement choice. The behavior from Table 2 is an example of overprotective seer correlations from Specker's parable of the overprotective seer (LIANG; SPEKKENS; WISEMAN, 2017).

In disturbing behaviors, measurements are not consistent across contexts, so labeling them independently of context is somewhat misleading. While the classical world is nondisturbing, as measurements that reveal preexisting deterministic values cannot disturbed by other compatible measurements, nondisturbance is not a sufficient condition for completing a behavior, as we will explore in the upcoming section.

2.4 KS-contextuality

We are now ready to define Kochen-Specker contextuality (KS-contextuality), a concept introduced in the authors' seminal paper (KOCHEN; SPECKER, 1967). Here we will present a more modern approach, due to the contributions of Abramsky and Brandenburger (ABRAMSKY; BRANDENBURGER, 2011).

If nondisturbance pertains to local coherence, where the contexts overlap, KS-noncontextuality requires global coherence of the behavior. Specifically, it necessitates that the outcome probability distribution of any context can be marginalized from a single global probability distribution.

Definition 8 (KS-contextuality)**.** *A behavior $\mathcal{B}^{\Gamma}_{\mathbb{P}}$ is KS-noncontextual if there exists a global probability distribution $P(a_1, ..., a_n|A_1, ..., A_n)$ over $\mathcal{O}^n$, called a global section, such that any outcome probability distribution $P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) \in \mathcal{B}^{\Gamma}_{\mathbb{P}}$ can be marginalized from the global section*

$$P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) = \sum_{a_i : A_i \notin C} P_{\mathbb{P}}(a_1, ..., a_n|A_1, ..., A_n). \tag{2.12}$$

If a global section does not exist, the behavior is KS-contextual.

It is important to emphasize that the existence of the global section does not imply that a joint implementation of all measurements in a scenario is possible. Instead, it indicates that it is mathematically possible to define the global probability distribution over $\mathcal{O}^n$.

Extending on the metaphor used in Figure 1, a nondisturbing KS-contextual behavior would be a paradoxical image, where you can stitch together any two overlapping patches to create a coherent image, but trying to paste all of them to recreate the whole picture gives rise to a paradox. This is the case with the lithographs of M. C. Escher, as we can see in Figure 2.

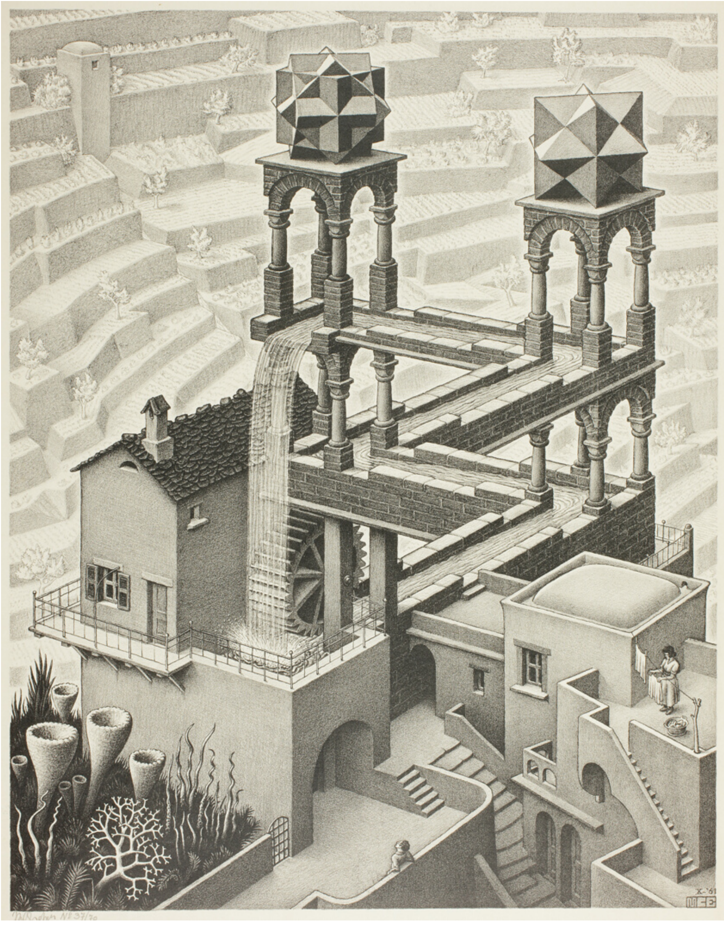


Figure 2 – Here we can fully view the image from Fig. 1. Even though the water flow made sense when we looked across any two overlapping fragments, taken as a whole, the water flow is absurd, representing a nondisturbing KS-contextual behavior. Source: (ESCHER, 1961 apud ART INSTITUTE OF CHICAGO, 2014)

We will now exemplify KS-contextuality with a few behaviors of a simple scenario.

Example 3. *Let us again consider behavior of the scenario*

$$\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}} := \begin{cases} \mathcal{A} = \{A_1, A_2, A_3\}; \\ \mathcal{O} = \{-1, 1\}; \\ \mathcal{C} = \{\{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_1\}\}. \end{cases} \tag{2.13}$$

The behavior in Table 4 is KS-noncontextual because it is a marginalization of the global section in Table 5.

Table 4 – Example of KS-noncontextual behavior.

Maximal contexts	**Outcomes** $(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\{A_1, A_2\}$	3/8	1/2	1/8	0
$\{A_2, A_3\}$	1/2	0	1/4	1/4
$\{A_3, A_1\}$	1/2	1/4	1/4	0

In this case, marginalization means that each element from Table 4 is the sum of the two corresponding elements in Table 5.

Table 5 – Global section of Table 4.

Maximal contexts	**Outcomes** $(-1,-1,-1)$	$(-1,-1,1)$	$(-1,1,-1)$	$(-1,1,1)$	$(1,-1,-1)$	$(1,-1,1)$	$(1,1,-1)$	$(1,1,1)$
$\{A_1, A_2, A_3\}$	3/8	0	1/8	1/4	1/8	0	1/8	0

Not every behavior has a global section, though. Let us look again at the behavior in Table 2, which we will repeat in Table 6 for convenience.

Table 6 – Nondisturbing behavior.

Maximal contexts	**Outcomes** $(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\{A_1, A_2\}$	0	1/2	1/2	0
$\{A_2, A_3\}$	0	1/2	1/2	0
$\{A_3, A_1\}$	0	1/2	1/2	0

Suppose this behavior had a global section, which we will represent generically in Table 7.

Table 7 – Attempt at a global section for Table 6.

Maximal contexts	**Outcomes** $(-1,-1,-1)$	$(-1,-1,1)$	$(-1,1,-1)$	$(-1,1,1)$	$(1,-1,-1)$	$(1,-1,1)$	$(1,1,-1)$	$(1,1,1)$
$\{A_1, A_2, A_3\}$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8

Looking at Table 8, we see that the marginalization requirements imposed by the probabilities in the $(-1,-1)$ *and* $(1,1)$ *columns from Table 6 would force all the probabilities from Table 7 to be set be set to zero.*

Table 8 – Behavior marginalized from generic global section.

Maximal contexts	**Outcomes** $(-1,-1)$	$(-1,1)$	$(1,-1)$	$(1,1)$
$\{A_1, A_2\}$	$p_1 + p_2$	$p_3 + p_4$	$p_5 + p_6$	$p_7 + p_8$
$\{A_2, A_3\}$	$p_3 + p_7$	$p_1 + p_5$	$p_2 + p_6$	$p_4 + p_8$
$\{A_3, A_1\}$	$p_1 + p_3$	$p_5 + p_7$	$p_2 + p_4$	$p_6 + p_8$

An identically zero global section contradicts the probabilities in the other columns of Table 6. This inconsistency reveals that the overprotective seer behavior is KS-contextual, although it cannot generated through quantum measurements (LIANG; SPEKKENS; WISEMAN, 2017).

Since Definition 8 does not explicitly involve deterministic probability distributions, KS-contextuality may, at first glance, appear unrelated to the possibility of achieving a noncontextual deterministic completion, as outlined in Definition 6. Perhaps surprisingly, these two definitions are equivalent, a result known as the Fine-Abramsky-Branderburger theorem, which was first shown by Ref. (FINE, 1982) specifically for the CHSH scenario (CLAUSER et al., 1969), and then generalized by Ref. (ABRAMSKY; BRANDENBURGER, 2011). Here we present a proof based on Ref. (KUNJWAL, 2015).

Theorem 1 (Fine-Abramsky-Branderburger)**.** *Given a nondisturbing behaviour $\mathcal{B}_{\mathbb{P}}^{\Gamma}$ for the compatibility scenario $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$, the behavior is KS-noncontextual if and only if it is consistent with a noncontextual deterministic assignment.*

Proof. KS-noncontextual → Noncontextual deterministic assignment:

Assuming $\mathcal{B}_{\mathbb{P}}^{\Gamma}$ has a global section $P_{\mathbb{P}}(a_1, ..., a_n|A_1, ..., A_n)$, let us define a latent variable $\lambda = (a_1, ..., a_n) \in \Lambda = \mathcal{O}^n$, the family of deterministic outcome probability distributions

$$D_{\mathbb{P},\lambda}(a_k|A_k) := \begin{cases} 1 & \text{if } a_k = \lambda_k; \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

with λ_k representing the k-th coordinate of λ, and the probability distribution over Λ

$$p(\lambda) = P_{\mathbb{P}}(\lambda|A_1, ..., A_n), \tag{2.15}$$

we can rewrite the global section of the behavior as

$$P_{\mathbb{P}}(a_1, ..., a_n|A_1, ..., A_n) = \sum_{\lambda} p(\lambda) \prod_{k} D_{\mathbb{P},\lambda}(a_k|A_k). \tag{2.16}$$

We can interpret the deterministic distributions $D_{\mathbb{P},\lambda}(a_k|A_k)$ like delta functions, that selects λ to be equal to the global outcome $(a_1, ..., a_n)$.

Now given $P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) \in \mathcal{B}_{\mathbb{P}}^{\Gamma}$, since $\mathcal{B}_{\mathbb{P}}^{\Gamma}$ is KS-noncontextual,

$$\begin{aligned} P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) &= \sum_{a_i : A_i \notin C} P_{\mathbb{P}}(a_1, ..., a_n|A_1, ..., A_n) \\ &= \sum_{a_i : A_i \notin C} \sum_{\lambda} p(\lambda) \prod_{k} D_{\mathbb{P},\lambda}(a_k|A_k) \\ &= \sum_{\lambda} p(\lambda) \prod_{k : A_k \in C} D_{\mathbb{P},\lambda}(a_k|A_k). \end{aligned} \tag{2.17}$$

Noncontextual deterministic assignment → KS-noncontextual:

If $\mathcal{B}_{\mathbb{P}}^{\Gamma}$ is consistent with a noncontextual deterministic deterministic assignment, any probability distribution $P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) \in \mathcal{B}_{\mathbb{P}}^{\Gamma}$ can be written as

$$P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) = \sum_{\lambda} p(\lambda) \prod_{k:A_k \in C} D_{\mathbb{P},\lambda}(a_k|A_k), \tag{2.18}$$

where the family of deterministic outcome distributions $\{D_{\mathbb{P},\lambda}(a_k|A_k)\}_{\lambda,k}$ is defined independently of context. It is easy to see that

$$P_{\mathbb{P}}(a_1, ..., a_n|A_1, ..., A_n) := \sum_{\lambda} p(\lambda) \prod_{k} D_{\mathbb{P},\lambda}(a_k|A_k), \tag{2.19}$$

is a global section that marginalizes to the outcome probability distributions of $\mathcal{B}_{\mathbb{P}}^{\Gamma}$. □

Quantum theory allows for KS-contextual behaviors. This result, known as the Kochen–Specker theorem (KOCHEN; SPECKER, 1967), reveals that quantum theory cannot be completed. In this sense, the Kochen–Specker theorem is a generalization of Bell's theorem (BELL, 1964), in fact, Bell-nonlocality–see Ref. (BRUNNER et al., 2014)–can be seen as a special case of KS-contextuality, where the contexts are composed of space-like separated measurements.

Nonetheless, uncovering whether a specific set of physical measurements exhibits KS-contextuality is an important question with broad implications for our understanding of quantum theory and its practical applications, since this nonclassical resource can be an asset in quantum computation (HOWARD et al., 2014), random number generation (ABBOTT et al., 2012), and error correction (DIVINCENZO; PERES, 1997).

2.5 Noncontextuality inequalities

Now that we have established and thoroughly explored the concept of KS-contextuality, we seek a more practical method to determine if a nondisturbing behavior is KS-contextual, without proving it has no global section. That is where noncontextuality inequalities come in.

Definition 9 (Noncontextuality inequalities)**.** *A noncontextuality inequality for a scenario* $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ *is a linear inequality in the form*

$$\sum_{\substack{C \in \mathcal{C} \\ a_{C^{(i)}} \in \mathcal{O}}} \gamma^{C}_{a_{C^{(1)}}, ..., a_{C^{(m)}}} P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) \overset{NC}{\leqslant} \beta \tag{2.20}$$

or

$$\sum_{\substack{C \in \mathcal{C} \\ a_{C^{(i)}} \in \mathcal{O}}} \gamma^{C}_{a_{C^{(1)}}, ..., a_{C^{(m)}}} P_{\mathbb{P}}(a_{C^{(1)}}, ..., a_{C^{(m)}}|A_{C^{(1)}}, ..., A_{C^{(m)}}) \overset{NC}{\geqslant} \beta, \tag{2.21}$$

with $\gamma^{C}_{a_{C(1)},...,a_{C(m)}}, \beta \in \mathbb{R}$.

It is respected by every KS-noncontextual behavior of the scenario, as alluded by the superscript "NC". If a nondisturbing behavior violates the inequality, then it is KS-contextual. We say an inequality is tight if there exists a KS-noncontextual behavior $\mathcal{B}^{\Gamma}_{\mathbb{P}}$ *that reaches the bound* β.

Interestingly, the idea that probability distributions of logically related events must satisfy inequalities in order to be, in a sense, globally consistent, dates back to Boole (BOOLE, 1862), as discussed in Refs. (PITOWSKY, 1994; ABRAMSKY, 2020). This formalism was extended to KS-contextuality by Ref. (ABRAMSKY; HARDY, 2012), after modern noncontextuality inequalities had already been independently developed (KLYACHKO et al., 2008; CABELLO, 2008), and long after the first Bell inequalities (BELL, 1964; CLAUSER et al., 1969).

While verifying if a nondisturbing behavior satisfies a given inequality may be straightforward, determining all tight inequalities of a scenario is an immensely challenging computational task. The space of noncontextual behaviors forms a polytope inside the space of nondisturbing behaviors, with each face corresponding to a tight noncontextuality inequality (PITOWSKY, 1989) and finding all faces is an NP-hard problem (BARAHONA; MAHJOUB, 1986).

A simple approach to generating a tight noncontextual inequality for finite scenarios with real outcomes involves looking at the expected value of measurements.

Remark 1 (Expected value)**.** *The expected value of a set of compatible measurements* $\{A_{C(1)}, ..., A_{C(m)}\}$ *is*

$$\langle A_{C(1)}...A_{C(m)}\rangle_{\mathbb{P}} = \sum_{a_{C(i)}\in\mathcal{O}} a_{C(1)} \cdot ... \cdot a_{C(m)} P_{\mathbb{P}}(a_{C(1)}, ..., a_{C(m)}|A_{C(1)}, ..., A_{C(m)}). \tag{2.22}$$

We start with a weighted sum of the expected values of a joint measurement of the contexts and search computationally for the maximum value of this sum while assigning deterministic values to the measurements. This maximum value will bound the result of any noncontextual behavior.

Corollary 1. *Given a scenario* $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ *with* $\mathcal{O} \subset \mathbb{R}$*, the inequality*

$$\sum_{C\in\mathcal{C}} \gamma^{C} \langle A_{C(1)}...A_{C(m)}\rangle_{\mathbb{P}} \overset{NC}{\leqslant} \beta = \max_{a_{C(j)}\in\mathcal{O}} \sum_{C\in\mathcal{C}} \gamma_{C} \prod_{j} a_{C(j)}, \tag{2.23}$$

with $\gamma^{C}, \beta \in \mathbb{R}$*, is a tight KS-noncontextuality inequality.*

Proof. The expected value of a context is

$$\langle A_{C(1)}...A_{C(m)}\rangle_{\mathbb{P}} = \sum_{a_{C(i)}\in\mathcal{O}} a_{C(1)} \cdot ... \cdot a_{C(m)} P_{\mathbb{P}}(a_{C(1)}, ..., a_{C(m)}|A_{C(1)}, ..., A_{C(m)}). \tag{2.24}$$

Theorem 1 tells us that noncontextual outcome probability distribution can generated by a convex sum of deterministic values

$$\langle A_{C(1)} \ldots A_{C(m)} \rangle_{\mathbb{P}} = \sum_{\lambda} p(\lambda) V_{\mathbb{P},\lambda}(A_{C(1)}) \cdot \ldots \cdot V_{\mathbb{P},\lambda}(A_{C(m)}). \tag{2.25}$$

Substituting Eq. (2.25) on the left side of Ineq. (2.23), we get

$$\begin{aligned} \sum_{C\in\mathcal{C}} \gamma^C \langle A_{C(1)} \ldots A_{C(m)} \rangle_{\mathbb{P}} &= \sum_{\lambda} p(\lambda) \sum_{C\in\mathcal{C}} \gamma^C \prod_{i=1}^{m} V_{\mathbb{P},\lambda}(A_{C(i)}) \\ &\leqslant \max_{\lambda} \sum_{C\in\mathcal{C}} \gamma^C \prod_{i=1}^{m} V_{\mathbb{P},\lambda}(A_{C(i)}) \\ &\leqslant \max_{a_{C(j)}\in\mathcal{O}} \sum_{C\in\mathcal{C}} \gamma^C \prod_{j} a_{C(j)}. \end{aligned} \tag{2.26}$$

□

To experimentally detect violations of a KS-noncontextuality inequality, we must look for quantum measurements that reflect the compatibility relations of a scenario, which we call a quantum realization. Of course, since KS-contextuality requires repeatable measurements, we are limited to quantum projective measurements.

Definition 10 (Quantum realization)**.** *A quantum realization $\mathcal{R}$ of a compatibility scenario $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ assigns to each measurement $A_i \in \mathcal{A}$ a self-adjoint linear operator M_i over a Hilbert space $\mathcal{H}$, such that $M_i = \sum_{a_i\in\mathcal{O}} a_i \Pi_i^{a_i}$ and $[M_j, M_k] = 0$ if A_j and A_k are compatible.*

When testing a quantum realization of a scenario, the preparations are quantum states ρ, which are positive-semidefinite operators over $\mathcal{H}$ with $\mathrm{tr}(\rho) = 1$. Quantum theory predicts a different behavior $\mathcal{B}_\rho^{\mathcal{R}}$ for different states, some of which may violate the inequality, while others adhere to it. This gives rise to the following distinction:

Definition 11 (State-(in)dependent contextuality)**.** *A realization $\mathcal{R}$ of a scenario $\Gamma^{\mathcal{A},\mathcal{O},\mathcal{C}}$ exhibits state independent contextuality in relation to a KS-noncontextuality inequality*

$$\sum_{\substack{C\in\mathcal{C} \\ a_{C(i)}\in\mathcal{O}}} \gamma^C_{a_{C(1)},\ldots,a_{C(m)}} P_\rho(a_{C(1)}, \ldots, a_{C(m)} | A_{C(1)}, \ldots, A_{C(m)}) \overset{NC}{\leqslant} \beta. \tag{2.27}$$

if, for every possible state ρ, the behavior $\mathcal{B}_\rho^{\mathcal{R}}$ violates said inequality. On the other hand, if the behaviors of only a subset of the possible preparations violate the inequality we say that the realization exhibits state-dependent contextuality.

Quantum theory allows for experimental realizations where we can witness both these of contextuality. Let us now look at two examples, both of which will be referenced later in this thesis.

Example 4 (State-dependent KCBS inequality). *The KCBS scenario, first proposed by (KLYACHKO et al., 2008), is defined as*

$$\Gamma^{KCBS} := \begin{cases} \mathcal{A} = \{A_0, A_1, A_2, A_3, A_4\}; \\ \mathcal{O} = \{-1, 1\}; \\ \mathcal{C} = \{\{A_0, A_1\}, \{A_1, A_2\}, \{A_2, A_3\}, \{A_3, A_4\}, \{A_4, A_0\}\}. \end{cases} \tag{2.28}$$

This scenario has the following noncontextuality inequality (KLYACHKO et al., 2008), obtained from the method describe in Corollary 1:

$$\langle A_0 A_1 \rangle_{\mathbb{P}} + \langle A_1 A_2 \rangle_{\mathbb{P}} + \langle A_2 A_3 \rangle_{\mathbb{P}} + \langle A_3 A_4 \rangle_{\mathbb{P}} + \langle A_4 A_0 \rangle_{\mathbb{P}} \overset{NC}{\geqslant} -3. \tag{2.29}$$

A quantum realization of Γ^{KCBS} *is possible in a three dimensional quantum system, with the measurements*

$$A_i = 2\,|v_i\rangle\langle v_i| - I, \tag{2.30}$$

where I *is the identity matrix and*

$$|v_i\rangle = \left(\frac{1}{\sqrt[4]{5}}, \sqrt{1 - \frac{1}{\sqrt{5}}} \cos\left(\frac{i\pi 4}{5}\right), \sqrt{1 - \frac{1}{\sqrt{5}}} \sin\left(\frac{i\pi 4}{5}\right) \right)^T. \tag{2.31}$$

This realization exhibits state-dependent contextuality because, by preparing the state $|\psi\rangle\langle\psi|$*, with*

$$|\Psi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2.32}$$

we can get a maximum violation of

$$\sum_{i=0}^{n-1} \langle A_i, A_{i+1} \rangle_{|\psi\rangle\langle\psi|} \overset{Q}{=} 5 - 4\sqrt{5} < -3, \tag{2.33}$$

where "Q" stands for "quantum", as this equality emerges from the quantum description. However, if the state prepared is different from $|\psi\rangle\langle\psi|$*, its behavior may no longer produce a violation.*

Experimental detections of KS-contextuality with a quantum realization of the KCBS scenario were made made using single photons (LAPKIEWICZ et al., 2011) and ions (UM et al., 2013; MALINOWSKI et al., 2018).

Example 5 (State-independent PM inequality)**.** *The Peres-Mermin (PM) scenario, presented in the works of (PERES, 1990; MERMIN, 1990; PERES, 1991), is defined as*

$$\Gamma^{PM} := \begin{cases} \mathcal{A} = \{A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}\}; \\ \mathcal{O} = \{-1, 1\}; \\ \mathcal{C} = \{\{A_{i1}, A_{i2}, A_{i3}\}, \{A_{1i}, A_{2i}, A_{3i}\}\}_{i\in\{1,2,3\}}. \end{cases} \tag{2.34}$$

These nine dichotomic measurements can be placed in a square grid

$$\begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}, \end{matrix} \tag{2.35}$$

such that each row and column of the grid forms one of the contexts:

$$R_1 := \{A_{11}, A_{12}, A_{13}\}; \tag{2.36a}$$
$$R_2 := \{A_{21}, A_{22}, A_{23}\}; \tag{2.36b}$$
$$R_3 := \{A_{31}, A_{32}, A_{33}\}; \tag{2.36c}$$
$$C_1 := \{A_{11}, A_{21}, A_{31}\}; \tag{2.37a}$$
$$C_2 := \{A_{12}, A_{22}, A_{32}\}; \tag{2.37b}$$
$$C_3 := \{A_{13}, A_{23}, A_{33}\}. \tag{2.37c}$$

This scenario has the following KS-noncontextuality inequality (CABELLO, 2008)[2]*, again obtained from the method described in Corollary 1:*

$$\begin{aligned} \Sigma :=&\langle A_{11}A_{12}A_{13}\rangle_{\mathbb{P}} + \langle A_{21}A_{22}A_{23}\rangle_{\mathbb{P}} + \langle A_{31}A_{32}A_{33}\rangle_{\mathbb{P}} \\ &-\langle A_{11}A_{21}A_{31}\rangle_{\mathbb{P}} - \langle A_{12}A_{22}A_{32}\rangle_{\mathbb{P}} - \langle A_{13}A_{23}A_{33}\rangle_{\mathbb{P}} \overset{NC}{\leqslant} 4, \end{aligned} \tag{2.38}$$

There are many possible quantum realizations of the Peres-Mermin square in a two-qubit system with Pauli measurements–see Ref. (SANIGA et al., 2007) and Ref. (HOLWECK, 2019, Fig. 13)–each one can violate a slightly different noncontextuality inequality. The one that violates Eq. (2.38), which is also the most symmetric implementation, is:

$$\begin{matrix} A_{11} = \sigma_y \otimes \sigma_z & A_{12} = \sigma_z \otimes \sigma_y & A_{13} = \sigma_x \otimes \sigma_x \\ A_{21} = \sigma_z \otimes \sigma_x & A_{22} = \sigma_x \otimes \sigma_z & A_{23} = \sigma_y \otimes \sigma_y \\ A_{31} = \sigma_x \otimes \sigma_y & A_{32} = \sigma_y \otimes \sigma_x & A_{33} = \sigma_z \otimes \sigma_z, \end{matrix} \tag{2.39}$$

Through quantum theory we can calculate the expected value of each term of inequality (2.38) by multiplying the observables of each context. The ones from each row (2.36) multiply to I while those from the columns (2.37) multiply to $-I$. This means that, for any quantum state ρ, quantum theory predicts that

$$\Sigma \overset{Q}{=} 6\langle I\rangle_\rho = 6, \tag{2.40}$$

[2] The inequality presented in (CABELLO, 2005) is slightly different from Ineq. (2.38). One can be obtained from the other through relabeling $A_{11} \to -A_{11}$ and $A_{12} \to -A_{12}$.

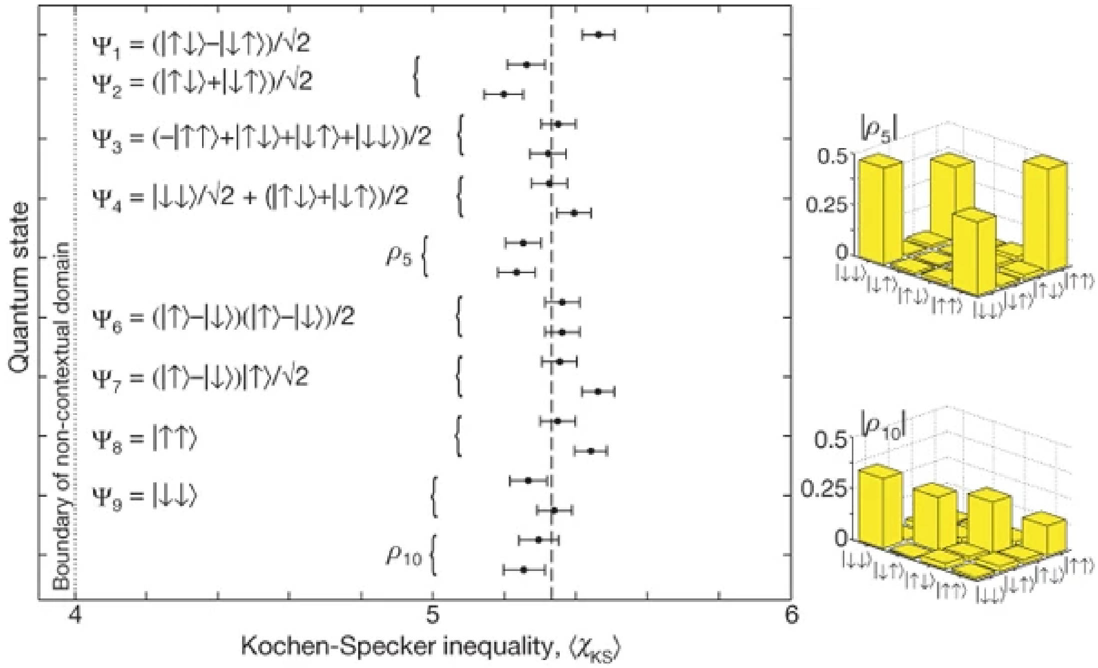


Figure 3 – Experimental results of violations of the PM noncontextuality inequality with different quantum states. Source: (KIRCHMAIR et al., 2009).

which shows that this realization exhibits state-independent contextuality.

Researchers have experimentally detected KS-contextuality in quantum realizations of the PM scenario using ions (KIRCHMAIR et al., 2009), nuclear magnetic resonance (MOUSSA et al., 2010), and photons (AMSELEM et al., 2009). We present the results from (KIRCHMAIR et al., 2009) in Figure 3.

3 Public systems

In Chapter 2, we discussed how violations of a noncontextuality inequality reveal KS-contextuality in a quantum realization of a scenario. To experimentally detect violations, it is necessary to estimate the expected values of each context on a state. Following the argumentation from the previous chapter, one might do so by implementing a series of prepare-and-measure rounds, with randomly chosen contexts.

This joint-measurement approach, however, has a flaw. It assumes that the joint measurement of a context produces the outcomes of the individual, context-independent measurements it contains. However, the execution of a joint measurement involves a distinct operational procedure compared to implementing each individual measurement. In practice, there is no assurance that the same measurement is implemented across the different contexts[1].

This possible discrepancy introduces a loophole in the experiment, which could allow deterministic outcomes to produce a violation. One way to bypass this problem is to obtain the expected values by sequentially implementing the individual measurements on what we call a sequential-measurement round[2].

Figure 4 showcases our representation of a sequential-measurement round where an observer implements the compatible measurements $\{A_1, A_2, A_3\}$ on state $\rho^{(0)}$. In this illustration, the arrows symbolize measurements and connect the states before and after each measurement. Additionally, the paired labels accompanying each arrow indicate the specific measurement carried out and the corresponding outcome obtained from it. The superscript of $\rho^{(i)}$ labels the state after i measurements, and $\rho^{(i)}$ can be calculated using the state-update rule

$$\rho^{(i+1)} = \frac{\Pi_k^{a_k} \rho^{(i)} \Pi_k^{a_k}}{\mathrm{tr}[\Pi_k^{a_k} \rho^{(i)} \Pi_k^{a_k}]}, \tag{3.1}$$

where $a_k \in \mathcal{O}$ is the outcome obtained in the measurement and $\Pi_k^{a_k}$ is the projection in to the k-th measurements' a_k eigenspace.

The sequential-measurement approach is not without its drawbacks. Since physical measurements cannot be perfectly compatible, there is always some disturbance, a problem known as the compatibility loophole. While efforts have been made to mitigate

[1] Note that verifying the behavior is nondisturbing is not sufficient to guarantee measurements are the same.

[2] Ref. (WAJS et al., 2016) proposed a comparable protocol, experimentally implemented by (LEUPOLD et al., 2018). Here, an observer performs a random sequence of measurements, and subsequently, a post-selection process is employed to discard those sub-sequences that do not form a context.

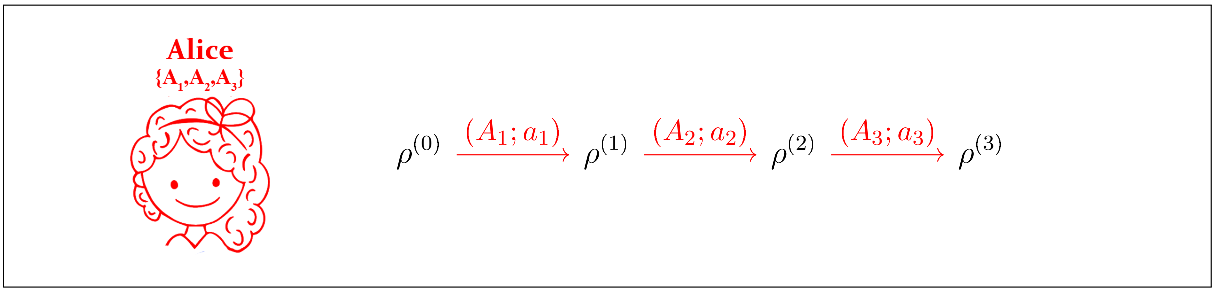


Figure 4 – Example of a sequential-measurement round with a single observer. Each arrow represents a measurement - it starts on the state before the measurement and points to the output state. The pair $(A_i; a_i)$ above the arrows indicates the observable measured and the measurement result.

this issue (SZANGOLIES; KLEINMANN; GÜHNE, 2013; GÜHNE et al., 2010; CABELLO; CUNHA, 2011), there is currently no loophole-free test of KS-contextuality.

Nevertheless, we think that the sequential setup is closer to the essence of KS-noncontextuality: that there exist individual context-less measurement devices yielding results identical to those obtained through joint measurements. Many experiments utilize sequential measurements to verify KS-contextuality, and an illustrative example can be found in Ref. (AHRENS et al., 2013). For the remainder of this thesis, we will explore KS-contextuality using sequential-measurement rounds.

Of course, isolating a single observer within a system, as depicted in Figure 4, is impossible. The system will inevitably interact with its surrounding environment and could be influenced by other, perhaps uninvited, observers. One way to generalize our model is to consider additional independent observers having full access to the system and concurrently implementing measurements. We refer to these systems as Public Systems (PS).

In a sequential-measurement round on a PS, observer Alice cannot perform her sequential measurements uninterruptedly, as is the case in Figure 4. Between her measurements, different observers will implement other, potentially incompatible, measurements. This situation is illustrated in Figure 5, where Alice aims to measure $\{A_1, A_2, A_3\}$, while Bob intends to measure $\{B_1, B_2, B_3\}$ on a system prepared in state $\rho^{(0)}$. Measurements are color-coded according to the respective observers.

With different nomenclatures, PSs have been already explored in the literature. One interesting avenue of research involves the exploration of quantum properties within such systems. Notably, prior studies have examined steering (SASMAL et al., 2018), Bell nonlocality (SILVA et al., 2015; DAS et al., 2019; SASMAL; KANJILAL; PAN, 2023), preparation contextuality (ANWER et al., 2021) and KS-contextuality (BALDIJÃO; TERRA CUNHA, 2020).

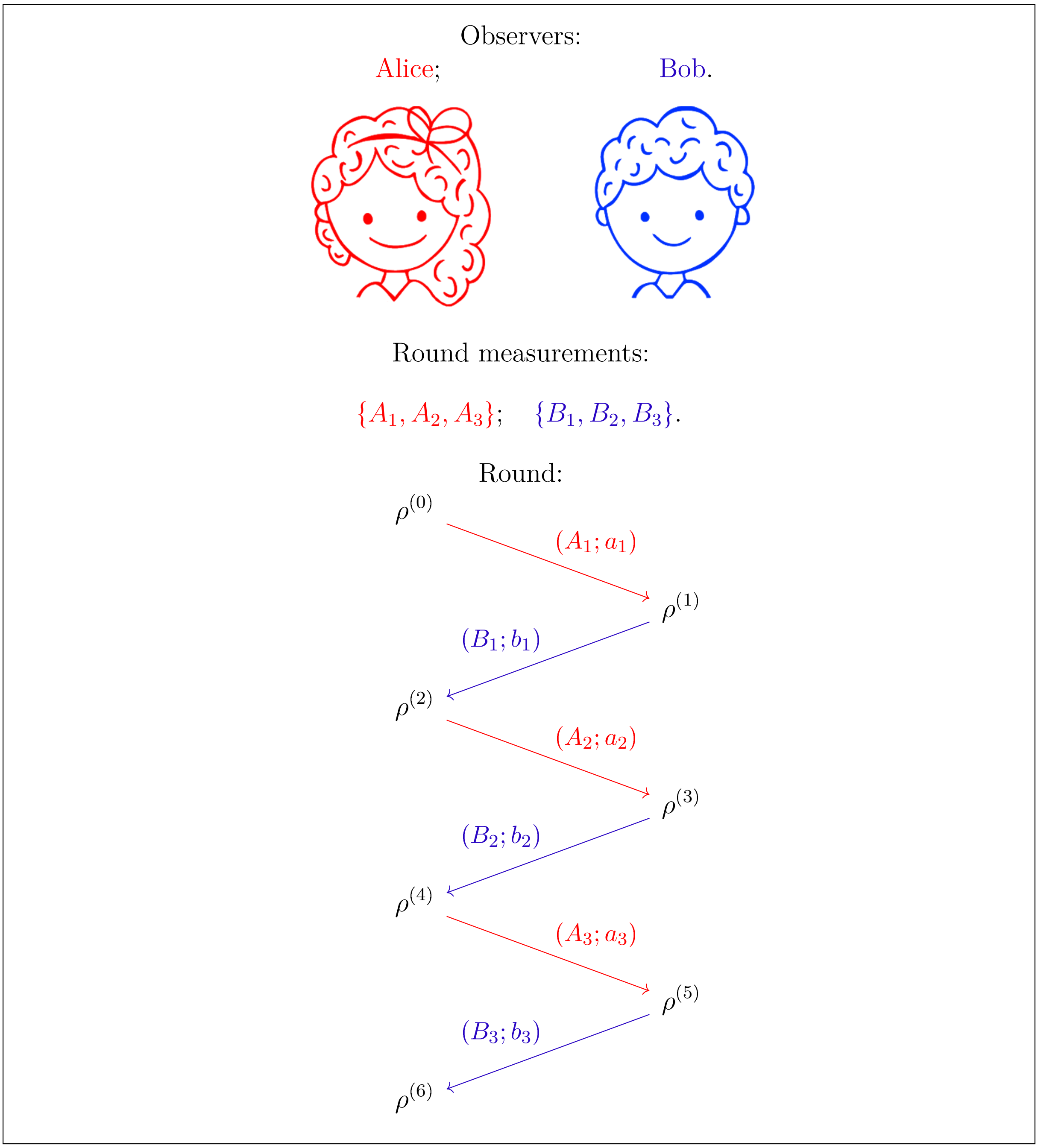


Figure 5 – Example of sequential-measurement round in a public system with Alice and Bob. Each arrow represents a measurement, and the pair $(A_i; a_i)$ or $(B_i; b_i)$ above it displays the observable measured and the measurement result. Every measurement is color-coded to the observer who implemented it.

Ref. (BALDIJÃO; TERRA CUNHA, 2020) investigated what happens when observers try to detect state-dependent contextuality in a PS. They tested odd N-cycle inequalities (ARAÚJO et al., 2013)–such as the KCBS inequality (KLYACHKO et al., 2008)– with different implementations of PSs and found that only a limited number of observers could detect violations.

Their findings point to state degradation as the reason for the emergence of KS-noncontextuality. As the N-cycle inequalities are pretty sensitive to the state used for the violation, it is reasonable to attribute this phenomenon to the multiple observers degrading the state, thereby depleting the quantum resource. This hypothesis is illustrated in Figure 6, where Bob needed to measure B_1 on state $\rho^{(0)}$ to get a violation, represented by the dashed blue line. However, Alice's measurement leads Bob to measure on $\rho^{(1)}$, which might not violate the inequality.

If state degradation is what erodes KS-contextuality, SIC should survive in a PS since any state could produce a violation. As we will show, our work falsifies this hypothesis.

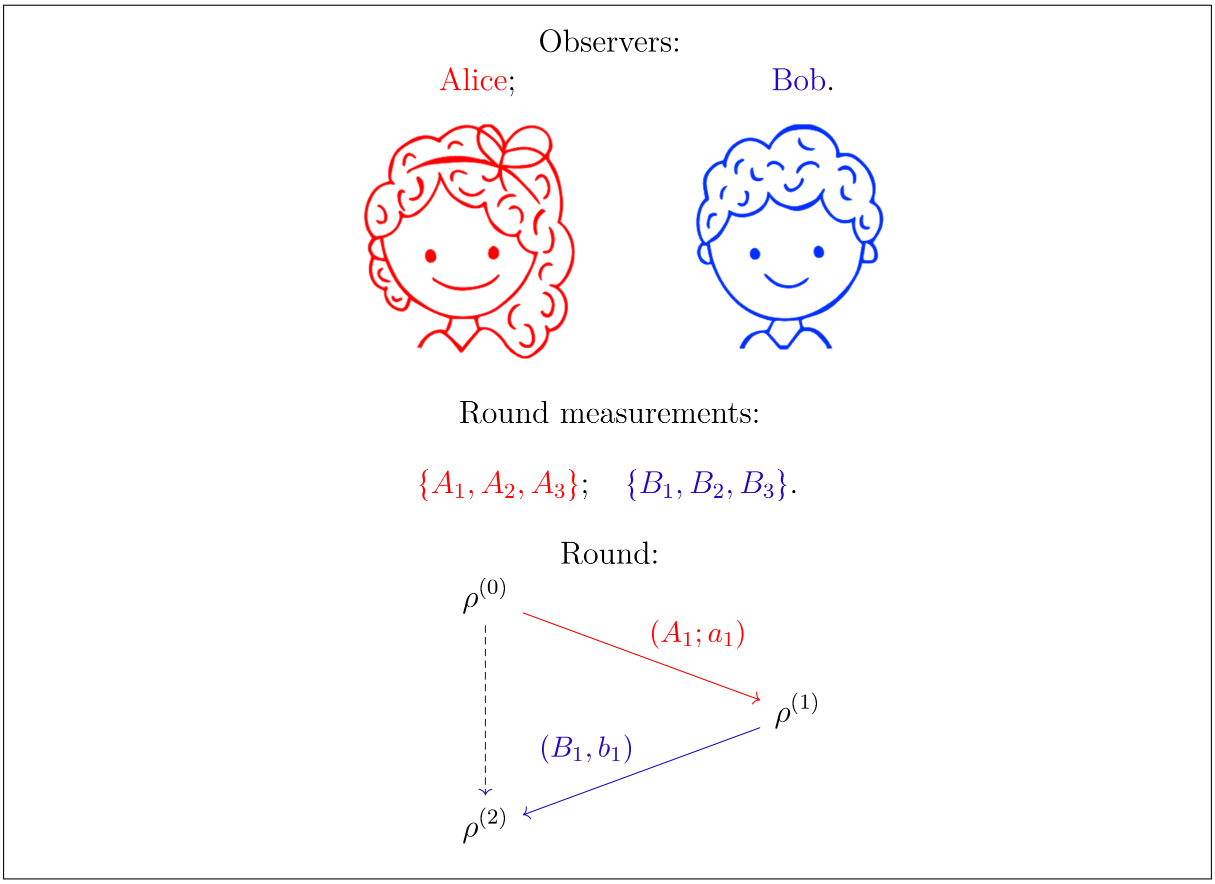


Figure 6 – Beginning of a sequential-measurement round in a public system with Alice and Bob. Alice's measure degrades the state before it gets to Bob, causing him to measure on a state that does not produce a violation (full blue arrow) instead of on the one that does (dashed blue arrow).

4 Simulation

To investigate how SIC responds to a PS, we conducted a computer simulation of a two-qubit system. In our simulation a single observer is trying to violate the PM state-independent noncontextuality inequality, while other independent observers also interact with the system. In this section, we will explain the parameters of this simulation.

4.1 The scenario

To verify if a Public System can suppress state-independent contextuality, we executed a computer simulation of two qubit Public System to try and violate the Peres-Mermin (PM) inequality, described in detail in Example 5. To recall, we are implementing the set of nine measurements:

$$\begin{aligned} A_{11} &= \sigma_y \otimes \sigma_z & A_{12} &= \sigma_z \otimes \sigma_y & A_{13} &= \sigma_x \otimes \sigma_x \\ A_{21} &= \sigma_z \otimes \sigma_x & A_{22} &= \sigma_x \otimes \sigma_z & A_{23} &= \sigma_y \otimes \sigma_y \\ A_{31} &= \sigma_x \otimes \sigma_y & A_{32} &= \sigma_y \otimes \sigma_x & A_{33} &= \sigma_z \otimes \sigma_z, \end{aligned} \tag{4.1}$$

organized in six contexts

$$R_1 := \{A_{11}, A_{12}, A_{13}\}; \tag{4.2a}$$
$$R_2 := \{A_{21}, A_{22}, A_{23}\}; \tag{4.2b}$$
$$R_3 := \{A_{31}, A_{32}, A_{33}\}; \tag{4.2c}$$
$$C_1 := \{A_{11}, A_{21}, A_{31}\}; \tag{4.3a}$$
$$C_2 := \{A_{12}, A_{22}, A_{32}\}; \tag{4.3b}$$
$$C_3 := \{A_{13}, A_{23}, A_{33}\}; \tag{4.3c}$$

to try and violate the KS-noncontextuality inequality

$$\begin{aligned} \Sigma :=& \langle A_{11}A_{12}A_{13}\rangle_\rho + \langle A_{21}A_{22}A_{23}\rangle_\rho + \langle A_{31}A_{32}A_{33}\rangle_\rho \\ &-\langle A_{11}A_{21}A_{31}\rangle_\rho - \langle A_{12}A_{22}A_{32}\rangle_\rho - \langle A_{13}A_{23}A_{33}\rangle_\rho \overset{NC}{\leqslant} 4. \end{aligned} \tag{4.4}$$

4.2 The protocol

In our simulation, one observer is trying to witness a violation of the noncontextuality inequality (the Main Observer) while N other observers are simply measuring aimlessly (Passersby Observers). The observers perform several sequential-measurement rounds adhering to the following script:

1. The system is prepared in state $\rho^{(0)}$;

2. An order of access to the system for the round is randomly decided among the $N+1$ observers;

3. Observers randomly pick their measurements for the round - one PM context (4.2-4.3) for the Main Observer and three PM observables (4.1) for each Passerby Observer;

4. The Main Observer randomly selects an order of implementation for the selected context;

5. Observers take turns accessing the system, performing a single measurement every access, on the state coming from the last observer's measurement, until all observers have implemented their three measurements;

6. The Main Observer records the outcomes of their three measurements to estimate the expected value of the context at the end of the simulation.

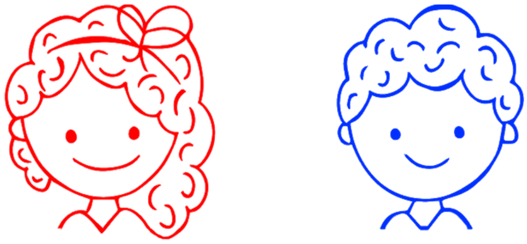
To illustrate the protocol, Figure 7 shows an example of a prepare an measure round of a simulation where Main Observer Alice is trying to violate the PM inequality with state $|00\rangle$ in a Public System shared with Passerby Observer Bob.

Remark 2. *There are many different ways we could have implemented the additional observers of the Public System. At one end of the spectrum, all observers could be independently trying to violate the PM noncontextuality inequality. At the opposite end, the Main Observer could be trying to obtain violations while the others implement any random observable, not limited to the PM square. The former scenario resembles an adversarial game, where observers compete to detect violations, while in the latter, the additional observers act more like noise, impeding the Main Observer's efforts.*

Our Passerby Observers are somewhere in the middle; they don't actively seek to violate the inequality but are constrained to PM measurements. This framework draws inspiration from Ref. (WAJS et al., 2016), which proposed an approach to violate noncontextuality inequalities by sequentially implementing random measurements from the scenario, and post-selecting the results arising from measurement sequences that form a context. Our PS is equivalent to a generalization of their approach, wherein the post-selection process is adapted to look for sequences of measurements forming a context but with a specified interspersing of N measurements between each. In essence, instead of requiring three consecutive measurements to form a context, as in the original method, it would look for a context with N in-between measurements.

We choose this setting due to its simpler analytical description, which allows for a deeper understanding of how such systems can affect KS-contextuality. Specifically, having each Passerby Observer behave stochastically, but with a limited number of possible measurements allows each Passerby to be modeled by a simple quantum channel, as we will showcase in Section 5.2.

Main observer: Alice; Passerby observer: Bob.



Chosen measurements:

$$R_2 = \{A_{21}, A_{22}, A_{23}\}, \quad \{A_{13}, A_{33}, A_{32}\}.$$

Round:

$$|00\rangle$$

$(A_{22}; 1)$

$$\frac{1}{\sqrt{2}}[|00\rangle + |10\rangle]$$

$(A_{13}; -1)$

$$\frac{1}{2}[|00\rangle - |01\rangle + |10\rangle - |11\rangle]$$

$(A_{21}; -1)$

$$\frac{1}{\sqrt{2}}[|10\rangle - |11\rangle]$$

$(A_{33}; -1)$

$$|10\rangle$$

$(A_{23}; 1)$

$$\frac{1}{\sqrt{2}}[|10\rangle + |01\rangle]$$

$(A_{32}; -1)$

$$\frac{1}{2}[(1+i)\,|01\rangle + (1-i)\,|10\rangle]$$

Figure 7 – Example of a prepare-and-measure round in a Public System following the described protocol with Main Observer Alice and Passerby Observer Bob. Each arrow represents a measurement - it starts on the state before the measurement and points to the output state. The pair $(A_{ij}; \pm 1)$ above the arrow indicates the observable measured and the measurement result, color-coded to the observer.

Moreover, we also simulated the adversarial version of the system, with all observers chasing violations, to show that overall conclusions are the same. Those results are showcased in Appendix A.

4.3 The code

We have developed a Python code designed to simulate sequential measurement rounds in a PS and test for violations of a noncontextuality inequality. Our code was created not only to simulate our implementation of the PM square, but as a versatile tool that allows the user to input the measurements, compatibility structure, and the noncontextuality inequality that they are working on.

The user can also define parameters such as the number of Passerby Observers N, the initial state $\rho^{(0)}$ and the number of measurement rounds. Such customization empowers researchers to adapt the simulation to their experimental setups, ensuring the code's broad applicability and relevance across various contexts. The code is available in the Github repository at the link https://github.com/turdutra/PS_simulation/.

5 Results

In this chapter we present our results regarding the possibility to violate the PM SIC inequality in a Public System. We obtained numerical results, through a simulation of the system, and analytical results, by modeling the system as an open quantum system. The results presented here are strongly correlated with the ones we published on Ref. (DUTRA; BALDIJÃO; TERRA CUNHA, 2023).

5.1 Numerical results

We conducted the computer simulation of a Public System following the protocol outlined in Chapter 4, examining how the value of Σ (4.4), obtained by the Main Observer, varies with the number of Passerby Observers N. For each $N \in \{1, 2, 3\}$, we choose 10 random pure states as the initial state $\rho^{(0)}$ and ran simulations lasting 1 million rounds. We present the outcomes of these simulations in Figure 8.

We found that a PS is capable of suppressing KS-contextuality, even SIC. Furthermore, for the PM scenario, one Passerby Observer is enough to prevent violations of the noncontextual quota, *i.e.* $\Sigma \leqslant 4$. We also see that Σ maintains its independence from the initial state $\rho^{(0)}$, as its value depends only on the number of Passerby Observers (N).

5.2 Analytical results

We will now give an analytical description of our PS. In the protocol described in Chapter 4, Passerby Observers behave stochastically: they choose each measurement independently from the last. Their Markovian conduct allows us to model the influence of each of their measurements on the system as a simple quantum channel.

We will analyze the system from the Main Observer's perspective and model it as an open quantum system. To begin, we will consider a PS with only one Passerby Observer. Recall that every PM observable (4.1) can be decomposed as a difference of projectors into the subspace of ± 1 eigenvalues

$$A_{ij} = \Pi_{ij}^{+} - \Pi_{ij}^{-}. \tag{5.1}$$

We denote by Γ_{ij} the map induced by implementing measurement A_{ij} when the outcome

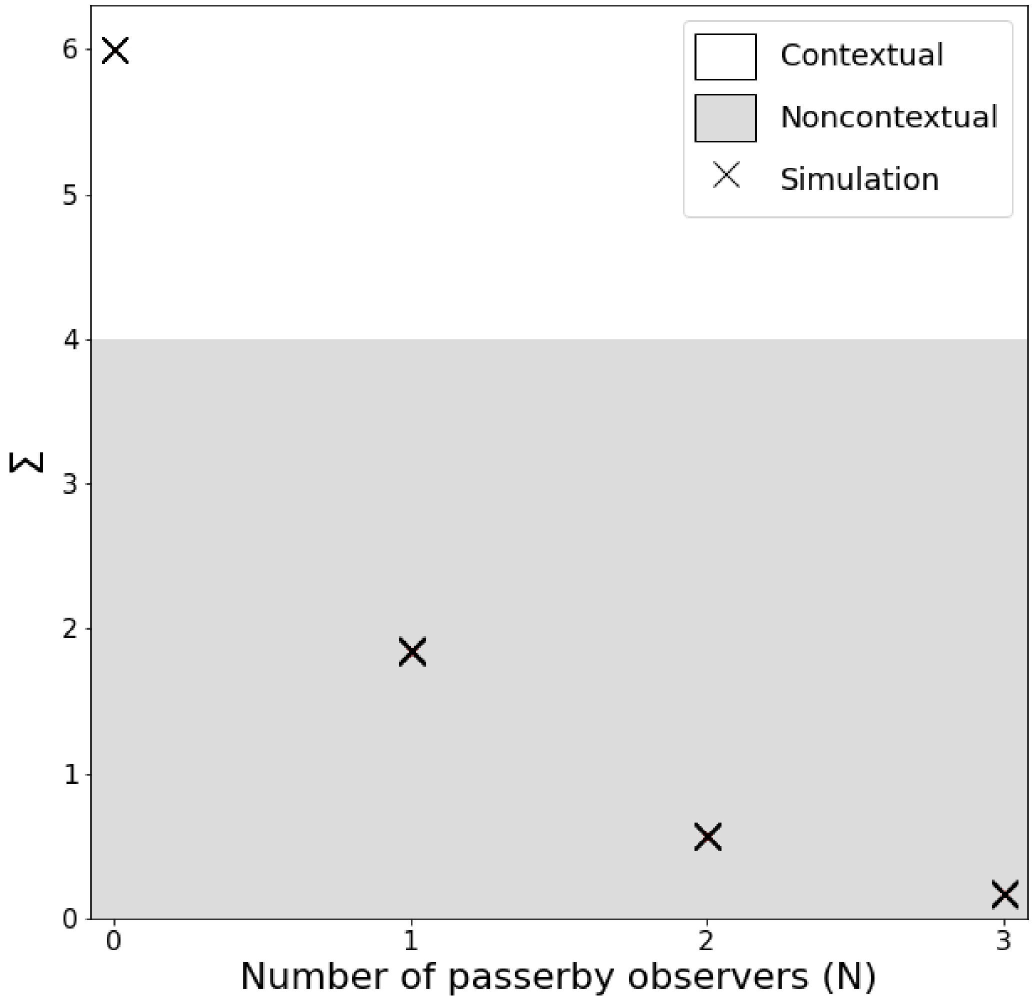


Figure 8 – Σ as a function of the number of Passerby Observers after a million prepare-and-measure rounds in a Public System.

is unknown. From the state-update rule (3.1), we ca write Γ_{ij} as

$$\begin{aligned}\Gamma_{ij}(\rho) &= \mathrm{tr}\big(\Pi^+_{ij}\rho\big)\frac{\Pi^+_{ij}\rho\Pi^+_{ij}}{\mathrm{tr}\big(\Pi^+_{ij}\rho\big)} + \mathrm{tr}\big(\Pi^-_{ij}\rho\big)\frac{\Pi^-_{ij}\rho\Pi^-_{ij}}{\mathrm{tr}\big(\Pi^-_{ij}\rho\big)} = \\ &= \Pi^+_{ij}\rho\Pi^+_{ij} + \Pi^-_{ij}\rho\Pi^-_{ij},\end{aligned} \tag{5.2}$$

which can be expressed as

$$\begin{aligned}\Gamma_{ij}(\rho) &= \Pi^+_{ij}\rho\Pi^+_{ij} + \Pi^-_{ij}\rho\Pi^-_{ij} = \\ &= \frac{1}{2}(\Pi^+_{ij} + \Pi^-_{ij})\rho(\Pi^+_{ij} + \Pi^-_{ij}) + \\ &+ \frac{1}{2}(\Pi^+_{ij} - \Pi^-_{ij})\rho(\Pi^+_{ij} - \Pi^-_{ij}) = \\ &= \frac{1}{2}\rho + \frac{1}{2}A_{ij}\rho A_{ij}.\end{aligned} \tag{5.3}$$

Since on each round, the Passerby Observer can implement any of the PM measurements with equal probability, and their outcome is unknown to the Main Observer, their measurement induces the channel

$$\Gamma(\rho) = \frac{1}{2}\rho + \frac{1}{18}\sum_{ij} A_{ij}\rho A_{ij}, \tag{5.4}$$

which is just an average of all the possible Γ_{ij} (5.3). Figure 9 illustrates this open quantum system, where Main Observer Alice implements their first PM measurement on state $\rho^{(0)}$, changing the system to state $\rho^{(1)}$, but her subsequent measurements are implemented on states disturbed by channel Γ, *e.g.*,

$$\rho^{(2)} = \Gamma(\rho^{(1)}) = \frac{1}{2}\rho^{(1)} + \frac{1}{18}\sum_{ij} A_{ij}\rho^{(1)} A_{ij}. \tag{5.5}$$

After numerous measurement rounds, when the Main Observer calculates the expected value of a context, for example, $R_1 = \{A_{11}, A_{12}, A_{13}\}$, channel Γ will impact the result. While without any Passerby Observers, they would get

$$\langle A_{11}A_{12}A_{13}\rangle_{\rho^{(0)}} = \langle A_{11}[\rho^{(0)}] \cdot A_{12}[\rho^{(1)}] \cdot A_{13}[\rho^{(2)}]\rangle, \tag{5.6}$$

where $\rho^{(1)}$ is the updated state after measurement A_{11} (and similarly for $\rho^{(2)}$), and the brackets indicate a mean value estimation of the three-point correlation function. If we consider the Passerby's presence and actions, the expected value becomes

$$\langle A_{11}A_{12}A_{13}\rangle_{\rho^{(0)}} \xrightarrow{\Gamma} \langle A_{11}[\rho^{(0)}] \cdot A_{12}[\Gamma(\rho^{(1)})] \cdot A_{13}[\Gamma(\rho^{(3)})]\rangle, \tag{5.7}$$

where $\rho^{(1)}$ and $\rho^{(3)}$ are again the post-measurement states of A_{11} and A_{12}, as depicted in Figure 9. Here, the multi-time aspect of these correlations becomes more consequential.

Eq. (5.7) shows that the Main Observer's best effort to estimate the mean value of a context, under the same strategy they have always used, will now be a good estimator

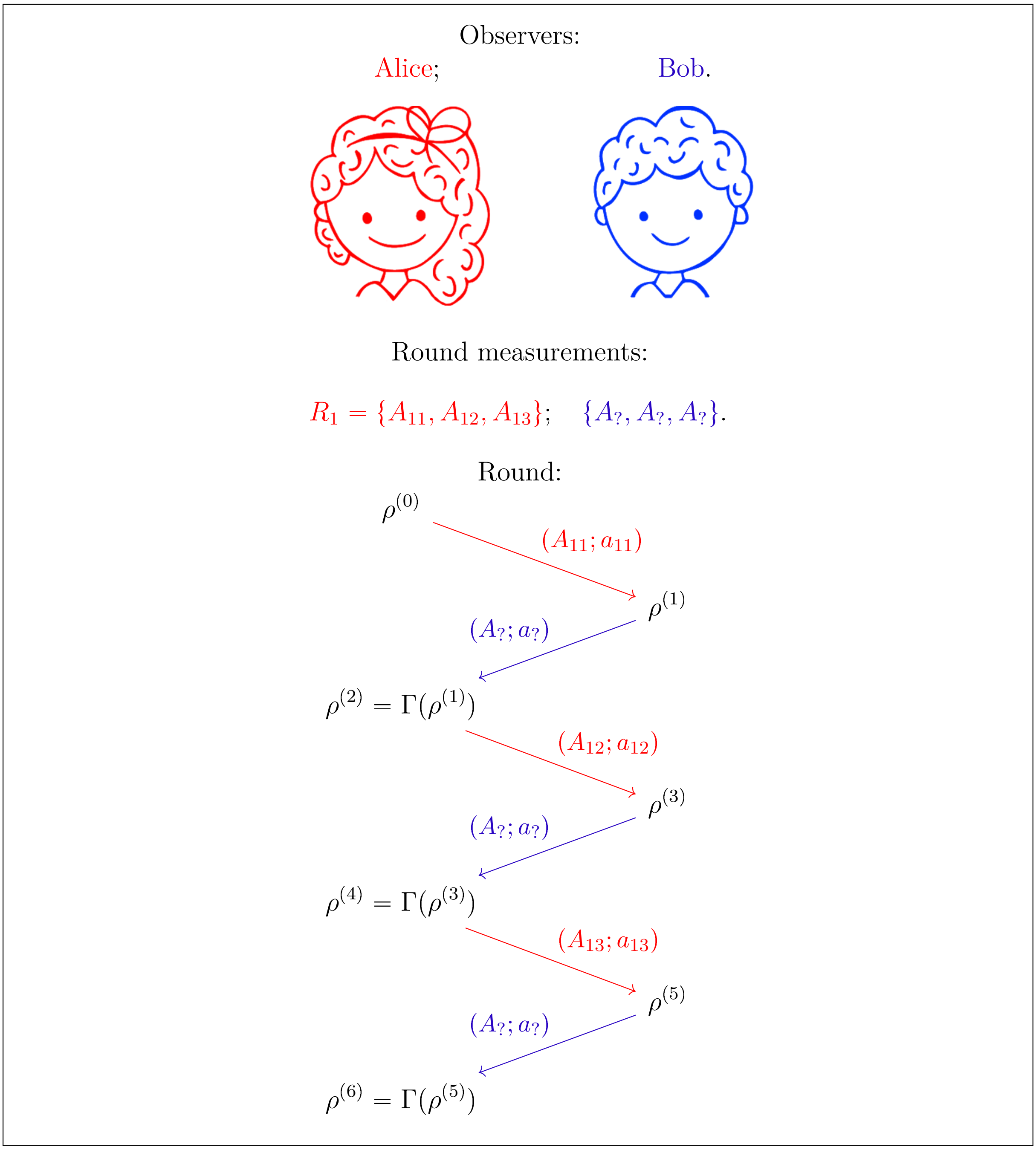


Figure 9 – Beginning of a prepare-and-measure round from the perspective of Main Observer Alice. She shares the system with Bob but does not know what he measures of his results, so he is modeled as a quantum channel.

for a different three-point correlation function. Operationally it is still the average product of the outcomes of three compatible measurements over many rounds; however, each is now acting on a different state. Since Γ alters the post-measurement state, this object generally differs from the usual expected value (see Appendix B).

To more clearly see how channel Γ affects the state, first recall that we can write any two-qubit density operator in the Pauli basis

$$\rho = \frac{1}{4}\left(I + \sum_{ij} t_{ij} A_{ij} + \sum_i l_i L_i + \sum_i r_i R_i\right), \tag{5.8}$$

where

$$L_i = \sigma_i \otimes I, \quad R_i = I \otimes \sigma_i. \tag{5.9}$$

This expression allows us to see that, for any two-qubit state ρ

$$\frac{\sum_{ij} A_{ij}\rho A_{ij} + 4[\rho_L \otimes \frac{1}{2}I + \frac{1}{2}I \otimes \rho_R] - \rho}{8} = 2\rho^*, \tag{5.10}$$

where $\rho_L = \text{tr}_R(\rho)$ is the reduced density matrix obtained by tracing over the right qubit, and analogously, $\rho_R = \text{tr}_L(\rho)$ is obtained by tracing over the right qubit.

Substituting Eq. 5.10 in the Eq. 5.4, we reveal the depolarizing effect exerted by a Passerby Observer on the state, which is responsible for dampening correlations between the Main Observer's measurements:

$$\Gamma(\rho) = \frac{5}{9}\rho + \frac{4}{9}\left(2\rho^* - \frac{1}{2}\left[\rho_L \otimes \frac{1}{2}I + \frac{1}{2}I \otimes \rho_R\right]\right), \tag{5.11}$$

with ρ^* denoting the maximally mixed state.

We can see this interference in full effect with a large number of Passerby Observers. To simplify the calculations, we will leverage the fact that, from the perspective of the Main Observer, who is restricted to implementing PM measurements, channel Γ is equivalent to the simpler depolarization channel (see Appendix C)

$$\tilde{\Gamma}(\rho) := \frac{5}{9}\rho + \frac{4}{9}\rho^*. \tag{5.12}$$

Generalizing this channel to N Passerby Observers yields

$$\tilde{\Gamma}^N(\rho) = \left(\frac{5}{9}\right)^N \rho + \left[1 - \left(\frac{5}{9}\right)^N\right]\rho^*. \tag{5.13}$$

By substituting channel $\tilde{\Gamma}^N$ in Eq. (5.7) and taking limit $N \to \infty$ we see that its depolarizing effect completely decorrelates the outcomes of the Main Observer's measurements:

$$\begin{aligned}
&\lim_{N\to\infty} \langle A_{11}[\rho^{(0)}] \cdot A_{12}[\Gamma^N(\rho^{(1)})] \cdot A_{13}[\Gamma^N(\rho^{(3)})]\rangle \\
&= \lim_{N\to\infty} \langle A_{11}[\rho^{(0)}] \cdot A_{12}[\tilde{\Gamma}^N(\rho^{(1)})] \cdot A_{13}[\tilde{\Gamma}^N(\rho^{(3)})]\rangle \\
&= \langle A_{11}[\rho^{(0)}] \cdot A_{12}[\rho^*] \cdot A_{13}[\rho^*]\rangle = \langle A_{11}\rangle_{\rho^{(0)}}\langle A_{12}\rangle_{\rho^*}\langle A_{13}\rangle_{\rho^*} = 0.
\end{aligned} \tag{5.14}$$

Notice that, even if the initial state $\rho^{(0)}$ is already the maximally mixed state, a fixed point of Γ^N, the channel still destroys the correlations, since

$$1 = \langle A_{11}A_{12}A_{13}\rangle_{\rho^*} \neq \langle A_{11}\rangle_{\rho^*}\langle A_{12}\rangle_{\rho^*}\langle A_{13}\rangle_{\rho^*} = 0. \tag{5.15}$$

In such extreme case, the Passerby Observers essentially acts like a t-design (AMBAINIS; EMERSON, 2007), preparing a new state between each measurement and turning the measurement round into three independent measurements. We can also see this in Eq. (B.11), since the depolarizing effect of Γ^N suppresses the state update channel, which is responsible for outcome correlations.

While we have shown how non-contextuality can emerge within a PS, particularly in the limit $N \to \infty$, this description does not encapsulate all of our numerical results. Notably, we cannot yet predict the simulation results depicted in Figure 8. We can get a different perspective on the effect of the Passerby Observers by moving our focus from the state to the measurements. For that, we need to analyze the system through the Heisenberg picture.

In this alternative approach, the Passerby Observers now disturb the Main Observer's *measurements*, and their effect is given by the adjoint of channel Γ (5.4):

$$\Gamma^\dagger(\cdot) = \frac{1}{2}(\cdot) + \frac{1}{18}\sum_{ij} A_{ij}(\cdot)A_{ij}. \tag{5.16}$$

From the commutation relation between the PM observables, we know that

$$A_{ij}A_{kl}A_{ij} = \begin{cases} A_{kl} & \text{if } k = i \text{ or } l = j, \\ -A_{kl} & \text{otherwise.} \end{cases} \tag{5.17}$$

Combining Eq. (5.17) and Eq. (5.16), we see that a single Passerby Observer has a depolarizing effect on the Main Observer's measurements:

$$\Gamma^\dagger(A_{kl}) = \frac{5}{9}A_{kl}. \tag{5.18}$$

This Heisenberg channel provides an alternative explanation for the loss of SIC: In the Heisenberg picture, we eliminate the in-between-measurement dynamics that can destroy the correlations, but the measurements themselves change. Since SIC is sensitive to the measurements, one may not witness KS-contextuality[1].

We can finally predict what the Main Observer obtains when calculating the expected value of a PM context in a PS with N Passerby Observers. The N-Passerby Heisenberg channel is

$$\Gamma^{\dagger N}(A_{kl}) = \left(\frac{5}{9}\right)^N A_{kl}. \tag{5.19}$$

[1] This results seem even more plausible when one remembers that depolarized Pauli measurements can become jointly measurable (BUSCH, 1986; GUERINI et al., 2017), and collective joint measurability brings forth KS-noncontextuality.

Consequently, the expected value of any "row" context, for example $R_1 = \{A_{11}, A_{12}, A_{13}\}$, becomes

$$\langle A_{11}\Gamma^{\dagger N}(A_{12})\Gamma^{\dagger N}(A_{13})\rangle_{\rho^{(0)}} = \left(\frac{5}{9}\right)^{2N} \langle A_{11}A_{12}A_{13}\rangle_{\rho^{(0)}} = \left(\frac{5}{9}\right)^{2N}, \tag{5.20}$$

while for any "column" context, for example $C_1 = \{A_{11}, A_{21}, A_{31}\}$, they obtain

$$\langle A_{11}\Gamma^{\dagger N}(A_{21})\Gamma^{\dagger N}(A_{31})\rangle_{\rho^{(0)}} = \left(\frac{5}{9}\right)^{2N} \langle A_{11}A_{21}A_{31}\rangle_{\rho^{(0)}} = -\left(\frac{5}{9}\right)^{2N}. \tag{5.21}$$

Up to now, we have considered that the Main Observer is the first to access the system, measuring on the initial state $\rho^{(0)}$. We now see that this does not result in any loss of generality, as the right side of Eqs. (5.20) and (5.21) would be the same regardless of the state on which the Main Observer implements their first measurement. This also explains why the value of Σ retains its state-independence.

Substituting Eqs. (5.20) and (5.21) in the PM inequality (4.4) we obtain

$$\Sigma = 6\left(\frac{5}{9}\right)^{2N}, \tag{5.22}$$

which matches the values from the numerical simulation for all $N \in \{1, 2, 3\}$, as shown in Figure 10.

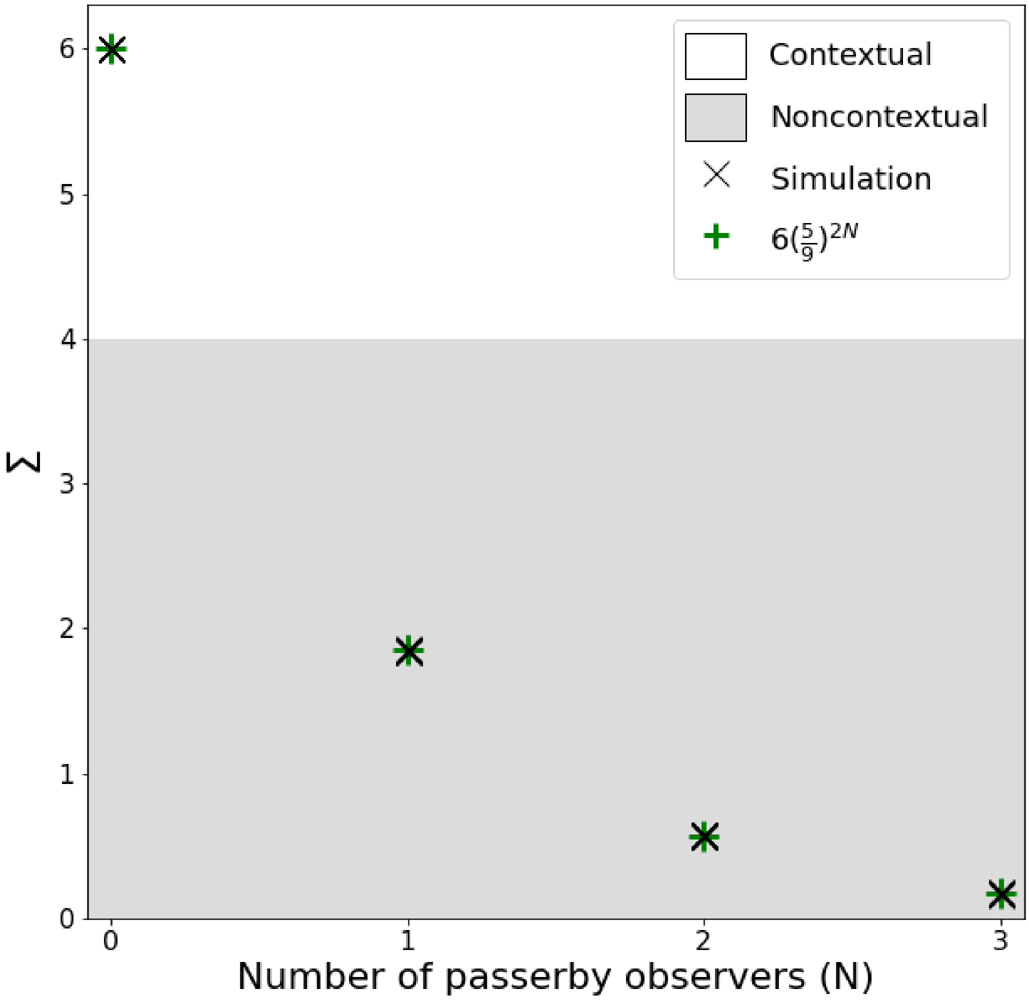


Figure 10 – Numerical and theoretical values of Σ as a function of the number of Passerby Observers in a Public System.

6 Conclusions

This thesis explores if observers can witness state-independent contextuality (SIC) within a Public System (PS).

The first chapter lays the foundations of our work. We define a scenario and its behaviors, introduce the proprieties of nondisturbance and KS-contextuality, and discuss how they relate to noncontextual determinism. We also present non-contextuality inequalities, which enable the detection of KS-contextuality and explore the distinction between its state-dependent and state-independent forms.

In Chapter 3, we present our definition of systems with multiple observers, which we call Public Systems, and discuss its effect on KS-contextuality. Ref. (BALDIJÃO; TERRA CUNHA, 2020) showed that state-dependent contextuality, manifest through the odd N-cycle inequalities, did not survive a multiple observers setup. Due to other observers' measurements, state degradation could be responsible for the phenomenon. This hypothesis immediately suggests that state-*in*dependent contextuality should be immune to such a process.

Perhaps surprisingly, we have shown that even SIC cannot resist a Public System. We obtained this result via a numerical simulation of numerous rounds of PM measurements in a Public System, as described in Section 4.2. Chapter 5 presents our numerical findings and also offers an analytical description that explains our results and enhances our comprehension of the system.

Firstly, in the Schrödinger picture, we showed that the in-between measurements act like a depolarizing channel. As outlined in Appendix B, the transformation of the state between sequential measurements is responsible for establishing the correlations among their outcomes. Introducing an additional channel between the measurements disrupts the correlations, ultimately rendering the Main Observer's measurements completely independent as the number of Passerby Observers increases. This result points to an important distinction: SIC is not sensitive to the state (or whatever happens to it) before the system enters the first measurement or after the last measurement of a context. SIC is, however, extremely sensitive to what happens between the measurements of a context.

When we look at the Heisenberg picture, we can show how the in-between measurements have a shrinking effect on the Main Observer's measurements. This effect provides an alternative explanation for why violations might disappear: the measurements effectively implemented on the state are not those the Main Observer expected to implement. The Heisenberg picture also allowed us to straightforwardly predict the value of Σ obtained by the Main Observer as a function of the number of Passerby Observers, which perfectly matched the results obtained from the simulation.

Finally, the analysis within the Heisenberg picture also emphasizes that KS-contextuality is not solely a resource residing in the state; to comprehend it fully, we should also consider the measurements and their compatibility structure. This work then highlights the necessity of a KS-contextuality resource theory that emphasizes channels and multi-time measurements rather than exclusively on the state (LIU; WINTER, 2019; BERK et al., 2021).

The phenomenon we describe here is also present, even if subtly concealed, in Ref. (WAJS et al., 2016), which discusses the possibility of recycling the state in SIC experiments. They sequentially implement measurements of a SIC scenario chosen uniformly at random. This way of measuring also leads to interleaving incompatible measurements. However, post-processing discards all outcomes that arise from sequences that do not form a context. This data analysis procedure, as they show, leads to violations. Therefore, our work also highlights the importance of post-selection in this process; otherwise, they would not obtain violations. In our PS, however, where passerby observers are independent, post-processing is unavailable to the Main Observer, which blocks the possibility of violations.

An interesting byproduct of our results is that Eq. (5.22) can be considered in the case of a continuous parameter $\Sigma \approx 6(5/9)^t$, which could be a way to interpret why experiments with sequential measurements of the PM square (KIRCHMAIR et al., 2009) never reach the quantum value $\Sigma = 6$: besides the error associated with the implementation of each measurement, noise introduced to the state between the measurements (which, as we have seen, is equivalent to changing the measurements themselves), scales down the violations seen in the laboratory. They show that experimental implementations of SIC tests – or protocols where SIC is the underlying resource– should be careful about the interval between the measurements and the dynamics that may affect this in-between state.

There is also more to explore in the future regarding SIC in PS setups. For instance, Public Systems could provide a model to analyze the disappearance of SIC due to the interaction of many quantum systems, such as shown to happen to generalized contextuality in quantum Darwinist processes (BALDIJÃO et al., 2021). Further studies could also incorporate CP maps or varying reference frames in our analysis in Chapter 5 to get a continuous model that can quantify the loss of contextuality in experimental realizations of the Peres-Mermin square, such as in Ref. (KIRCHMAIR et al., 2009). From a more practical perspective, the disappearance of violations that depend on the access of third parties to the system could have exciting applications, such as the certification of private channels.

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Appendix

APPENDIX A – Adversarial Setup

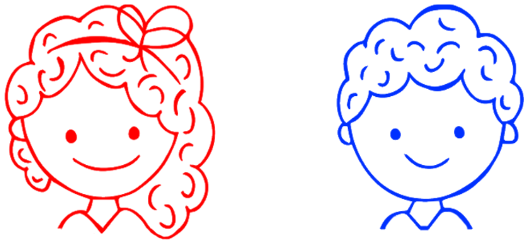
In our Public System, we choose to have the additional observers measure aimlessly. Another reasonable implementation, which we will call the Adversarial Setup, would be to have additional observers also trying to violate the noncontextuality inequality. This was the setup utilized on the principal reference for this work, Ref. (BALDIJÃO; TERRA CUNHA, 2020), which showed that a limited number of observers could detect violations of the N-Cycle state-dependent inequalities in a PS.

The difference between the Adversarial Setup and the one described in Section 4.2, which we will now call the Passerby Setup, is that all observers behave in the same way, measuring one context each round and gathering statistics to try and violate the inequality. A typical round in adversarial setup, with adversaries Alice and Bob, is depicted in Figure 11.

We also simulated a PS under the Adversarial Setup to see if this implementation would yield different results. For each number total observer $N \in \{1, 2, 3, 4\}$, we chose ten random initial states ρ and simulated one million measurement rounds. Figure 12 displays the results for the Adversarial Setup, compared to the ones obtained through the Passerby Setup.

We see that the values for Σ obtained in the Passerby Setup are slightly lower than those in the Adversarial Setup. However, qualitatively, the conclusion is the same: if there is more than one observer, none of them can witness KS-contextuality.

Main observer: Alice; Passerby observer: Bob.



Chosen measurements:

$$\{A_{21}, A_{22}, A_{23}\}, \quad \{A_{13}, A_{33}, A_{32}\}.$$

Round:

$$|00\rangle \xrightarrow{(A_{22};\,1)} \frac{1}{\sqrt{2}}[|00\rangle + |10\rangle]$$

$$\xrightarrow{(A_{13};\,-1)} \frac{1}{2}[|00\rangle - |01\rangle + |10\rangle - |11\rangle]$$

$$\xrightarrow{(A_{21};\,-1)} \frac{1}{\sqrt{2}}[|10\rangle - |11\rangle]$$

$$\xrightarrow{(A_{33};\,-1)} |10\rangle$$

$$\xrightarrow{(A_{23};\,1)} \frac{1}{\sqrt{2}}[|10\rangle + |01\rangle]$$

$$\xrightarrow{(A_{32};\,-1)} \frac{1}{2}[(1+i)\,|01\rangle + (1-i)\,|10\rangle]$$

Figure 11 – Example of a measurement round in a Public System following the adversarial protocol with main observer Alice and passerby observer Bob. Each arrow represents a measurement - it starts on the state before the measurement and points to the output state. The pair $(A_{ij}; \pm 1)$ above the arrow indicates the observable measured and the measurement result, color-coded to the observer.

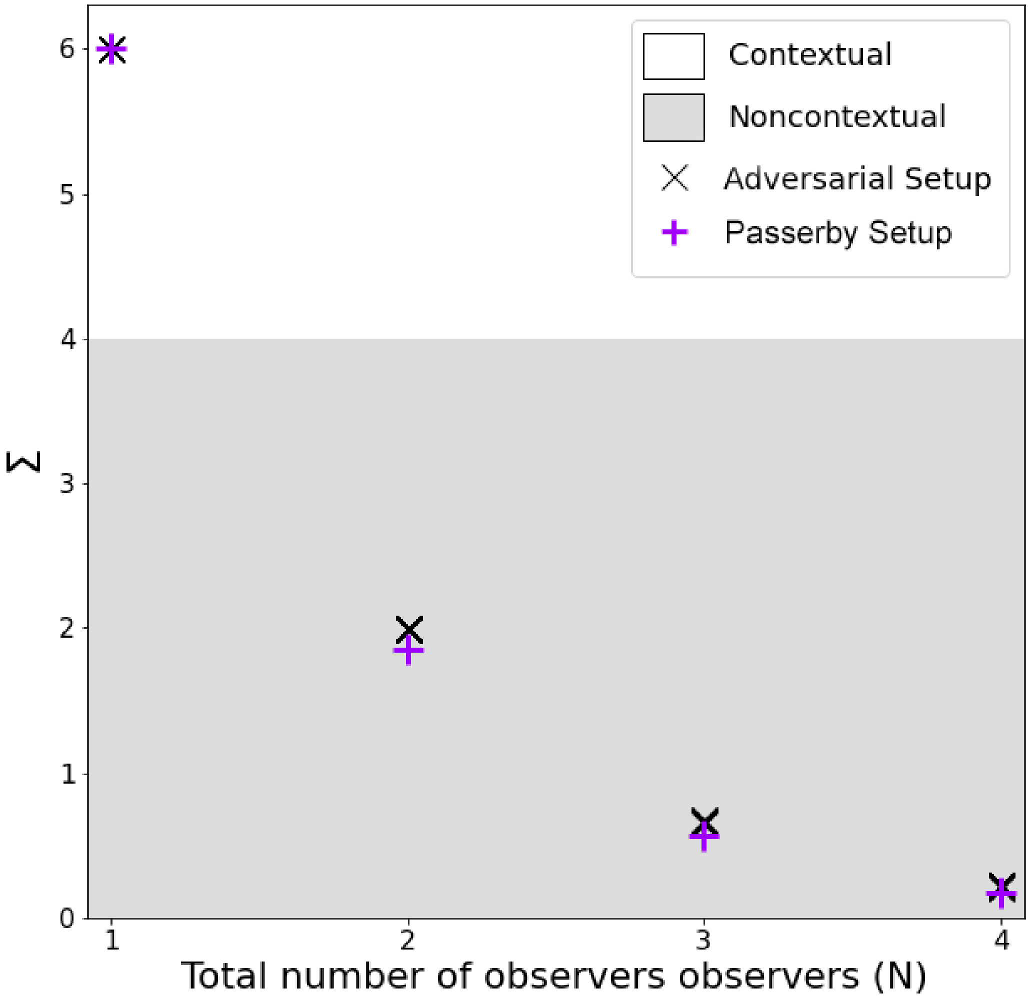


Figure 12 – Σ as a function of the total number of observers after a million rounds in the Adversarial and Passerby Setups.

APPENDIX B – Expected value in sequential measurements

The expected value of a measurement A_i on a state ρ is:

$$\langle A_i \rangle_\rho := \sum_{a_i \in \sigma(A_i)} p_\rho(a_i) a_i, \tag{B.1}$$

with

$$p_\rho(a_i) = \mathrm{tr}[\Pi_i^{a_i}(\rho)], \tag{B.2}$$

where $\sigma(A_i)$ is the spectrum of A_i and $\Pi_i^{a_i}$ is the projection operator on the a_i subspace.

Using the state-update rule (Eq. 3.1), we can define the expected value of sequentially measuring A_i followed by A_j on state ρ

$$\langle A_i \to A_j \rangle_\rho := \sum_{\substack{a_i \in \sigma(A_i) \\ a_j \in \sigma(A_j)}} p_\rho(a_i \to a_j) a_i a_j, \tag{B.3}$$

with

$$p_\rho(a_i \to a_j) := p_\rho(a_i) p_\rho(a_j | a_i), \tag{B.4}$$

where $p_\rho(a_j|a_i)$ is the probability of obtaining a_j from measuring A_j given that a_i was obtained from measuring A_i. This distribution is obtained from

$$p_\rho(a_j|a_i) = \mathrm{tr}\big[\Pi_j^{a_j} \Phi_i^{a_i}(\rho)\big], \tag{B.5}$$

where $\Phi_i^{a_i}$ is a channel that implements the state update rule (3.1). Notice how the dependency on a_i in Eq. (B.5) is what allows for correlations between the measurements.

We can understand the expected value of Eq. (B.5) as an average, over many sequential-measurement rounds, of the product of two outcomes: the outcome of A_i implemented on the initial state (*i.e.* $A_i[\rho]$) and the outcome A_j implemented on the post-measurement state (*i.e.* $A_j[\Phi_i^{a_i}(\rho)]$).These values can be interpreted as two-point correlation functions, so we can write Eq. (B.5) as:

$$\langle A_i \to A_j \rangle_\rho = \langle A_i[\rho] \cdot A_j[\Phi_i^{a_i}(\rho)] \rangle. \tag{B.6}$$

These two-point statistics are fairly common in many areas of physics, particularly in statistical mechanics (see Ref. (SETHNA, 2021, ch. 10)).

Since the sequential measurements are generally compatible and there are no disturbances between measurements, the order of implementation and state-update process is commonly omitted,

$$\langle A_i \to A_j \rangle_\rho = \langle A_i A_j \rangle_\rho = \langle A_j A_i \rangle_\rho = \langle A_j \to A_i \rangle_\rho, \tag{B.7}$$

and all these values can be identified as the expected value of the observable $A_i \cdot A_j$ implemented on state ρ. However, if we introduce a disturbance between the measurements, given by a channel Γ, the expected value of the sequential measurements becomes

$$\langle A_i \xrightarrow{\Gamma} A_j \rangle_\rho := \sum_{\substack{a_i \in \sigma(A_i) \\ a_j \in \sigma(A_j)}} p(a_i \xrightarrow{\Gamma} a_j) a_i a_j, \tag{B.8}$$

with

$$p(a_i \xrightarrow{\Gamma} a_j) = \mathrm{tr}[\Pi_i^{a_i}(\rho)]\,\mathrm{tr}\big[\Pi_j^{a_j}\Gamma\Phi_i^{a_i}(\rho)\big]. \tag{B.9}$$

Since Γ acts on $\Phi_i^{a_i}(\rho)$, it can modulate the correlations between the measurements.

This can be straightforwardly generalized to three sequential measurements

$$\langle A_i \xrightarrow{\Gamma} A_j \xrightarrow{\Gamma} A_k \rangle_\rho := \sum_{\substack{a_i \in \sigma(A_i) \\ a_j \in \sigma(A_j) \\ a_k \in \sigma(A_k)}} p(a_i \xrightarrow{\Gamma} a_j \xrightarrow{\Gamma} a_k) a_i a_j a_k, \tag{B.10}$$

where

$$p(a_i \xrightarrow{\Gamma} a_j \xrightarrow{\Gamma} a_k) = \mathrm{tr}[\Pi_i^{a_i}(\rho)]\,\mathrm{tr}\big[\Pi_j^{a_j}\Gamma\Phi_i^{a_i}(\rho)\big]\,\mathrm{tr}\big[\Pi_k^{a_k}\Gamma\Phi_j^{a_j}\Gamma\Phi_i^{a_i}(\rho)\big], \tag{B.11}$$

allowing us to identify Eq. B.10 with the three-point correlation function

$$\langle A_i \xrightarrow{\Gamma} A_j \xrightarrow{\Gamma} A_k \rangle_\rho = \langle A_i[\rho] \cdot A_j[\Gamma\Phi_i^{a_i}(\rho)] \cdot A_k[\Gamma\Phi_j^{a_j}\Gamma\Phi_i^{a_i}(\rho)] \rangle. \tag{B.12}$$

APPENDIX C – Analyzing the Passerby channel Γ

In this appendix, we will show that the Main Observer, being limited to implementing PM measurements, cannot differentiate between the application of the channels

$$\Gamma(\rho) = \frac{5}{9}\rho + \frac{4}{9}\left(2\rho^* - \frac{1}{2}\left[\rho_L \otimes \frac{1}{2}I + \frac{1}{2}I \otimes \rho_R\right]\right), \tag{C.1}$$

and

$$\tilde{\Gamma}(\rho) = \frac{5}{9}\rho + \frac{4}{9}\rho^*. \tag{C.2}$$

If we write the state in the Pauli basis

$$\rho = \frac{1}{4}\left(I + \sum_{ij} t_{ij}A_{ij} + \sum_i l_i L_i + \sum_i r_i R_i\right), \tag{C.3}$$

where

$$L_i = \sigma_i \otimes I, \quad R_i = I \otimes \sigma_i, \tag{C.4}$$

we can see that $\Gamma(\rho)$ and $\tilde{\Gamma}(\rho)$ differ only on the l_i and r_i coefficients.

When the Main Observer implements a PM measurement

$$A_{ij} = \Pi_{ij}^+ - \Pi_{ij}^- \tag{C.5}$$

on a state ρ, the probability of obtaining output ± 1 is

$$p_\rho(\pm 1) = \mathrm{tr}\big(\Pi_{ij}^\pm \rho\big) = \mathrm{tr}\left(\frac{1}{2}\left(I \pm A_{ij}\right)\rho\right) = \frac{1}{2} \pm \frac{1}{2}t_{ij}, \tag{C.6}$$

which depends only on the t_{ij} coefficient of ρ, so the outcome of a single PM measurement does not distinguish between the channels. But the Main Observer implements more than one PM measurement, so we still need to show that the state-update rule

$$\rho^{(i+1)} = \frac{\Pi_{ij}^\pm \rho^{(i)} \Pi_{ij}^\pm}{\mathrm{tr}\big[\Pi_{ij}^\pm \rho^{(i)} \Pi_{ij}^\pm\big]}, \tag{C.7}$$

does not propagate the disturbances on l_i, r_i introduced by channel Γ to the t_{ij} components.

Let us first analyze the L_k observables. Considering that

$$\Pi_{ij}^\pm = \frac{1}{2}\left(I \pm A_{ij}\right), \tag{C.8}$$

we can write

$$\Pi_{ij}^{\pm}(L_k)\Pi_{ij}^{\pm} = \frac{1}{4}\left(L_k \pm A_{ij}L_k \pm L_k A_{ij} + A_{ij}L_k A_{ij}\right)$$
$$= \begin{cases} \frac{1}{2}(L_k \pm A_{ij}L_k) \text{ if } A_{ij}, L_k \text{ commute;} \\ 0 \text{ if } A_{ij}, L_k \text{ anticommute.} \end{cases} \tag{C.9}$$

If L_k and A_{ij} commute, $A_{ij}L_k$ is the third element from a context, which is also not in the PM square–see Ref. (HOLWECK, 2021, Fig. 2)–so in either case of Eq. (C.9)

$$\Pi_{ij}^{\pm}(L_i)\Pi_{ij}^{\pm} \in \text{span}(L, R), \tag{C.10}$$

and analogously

$$\Pi_{ij}^{\pm}(R_i)\Pi_{ij}^{\pm} \in \text{span}(L, R). \tag{C.11}$$

Since the the disturbance introduced by the factor $P_{L+R}(\rho)$ does not propagate to the t_{ij} coefficients of ρ, the Main Observer cannot differentiate the effects of channels Γ and $\tilde{\Gamma}$.