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Grothendieck Ring of Varieties

Anel de Grothendieck de Variedades

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Resumo

O anel de Grothendieck de variedades é uma construção essencial na Geometria Algébrica. É formado tomando o conjunto de todas as classes de isomorfismo de variedades algébricas e definindo uma estrutura de anel sobre esse conjunto. Essa estrutura tem implicações significativas em vários ramos da Geometria Algébrica, em particular, o Anel de Grothendieck pode ser utilizado para provar teoremas sobre variedades, estudar suas propriedades algébricas e geométricas e classificar variedades com base em suas propriedades no anel. Além disso, está intimamente relacionado com a Geometria Birracional e a Teoria das Singularidades, e desempenha um papel central na Teoria de Integração Motívica. Tendo em vista sua importância, este trabalho tem como objetivo principal apresentar o anel de Grothendieck de variedades, estudar propriedades e explorar resultados que permitem o cálculo de classes de variedades neste anel. Mais ainda, a fim de explicitar a relação entre a Geometria Birracional e o anel de Grothendieck, será mostrado que a racionalidade estável de uma variedade suave e própria sobre um corpo de característica zero pode ser detectada na classe da variedade no anel de Grothendieck. Entretanto, para realizar este estudo, faremos uso de ferramentas básicas de Geometria Algébrica e, por isso, também serão apresentados conceitos, resultados e exemplos de variedades algébricas e esquemas.

Palavras-chave: Anel de Grothendieck de Variedades. Geometria Algébrica. Geometria Birracional.

Abstract

The Grothendieck Ring of Varieties is a fundamental construction in Algebraic Geometry. It is formed by taking the set of all isomorphism classes of varieties and defining a ring structure on this set. This structure has significant implications in various branches of Algebraic Geometry. In particular, the Grothendieck Ring can be used to prove theorems about varieties, study their algebraic and geometric properties, and classify varieties based on their properties in the ring. Furthermore, it is closely related to Birational Geometry and the Theory of Singularities, and it also plays a central role in the Theory of Motivic Integration. Given its importance, the main objective of this work is to present the Grothendieck Ring of Varieties, study its properties, and explore results that allow for the computation of classes of varieties in this ring. Moreover, to elucidate the relationship between Birational Geometry and the Grothendieck Ring, it will be shown that the stable rationality of a smooth and proper variety over a field of characteristic zero can be detected in the class of the variety in the Grothendieck Ring. However, to achieve this study, we will employ basic tools of Algebraic Geometry, therefore, we will also introduce concepts, results, and examples of algebraic varieties and schemes.

Keywords: Grothendieck Ring of Varieties. Algebraic Geometry. Birational Geometry.

Contents

Introduction

Since Ancient Greece, around 400 BC, Geometry and Algebra have been used together to solve mathematical problems. The challenges faced by Greek mathematicians led to the development of "Geometric Algebra", an approach where the most prevalent technique of resolving algebraic problems was through the intersection of auxiliary curves. In this form of expression, writing equations was more geometric. A clear example of this is the use of simple geometric constructions to find roots of equations of the form "$x^2 = ab$". Later on, the connection between these two areas became even more evident with the publication of Book II of Euclid's "Elements" in the 3rd century BC, which presented ten propositions dealing with algebraic identities described through geometric representations.

With the introduction of Analytical Geometry in the 16th century and the works of René Descartes, the relationship between Algebra and Geometry went beyond assigning geometric meanings to algebraic expressions. In Descartes' treatise "La Géométrie", he sought to liberate geometry from diagrams through algebraic processes. Descartes used both geometry and algebra as tools to solve geometric and algebraic problems. This change marked the beginning of contemporary Algebraic Geometry.

One of the fundamental results in algebraic geometry is the equivalence between the category of affine varieties over an algebraically closed field k and the dual category of commutative reduced k-algebras of finite type. This result was significantly generalized by Alexander Grothendieck around 1957 when he undertook a gigantic program aimed at a vast generalization of algebraic geometry. His objective was to encompass all the previous developments and move beyond the framework of reduced algebras of finite type over an algebraically closed field, instead working with the category of all commutative rings. Grothendieck's groundbreaking contribution to algebraic geometry came through the development of the theory of schemes. This powerful framework allowed mathematicians to work with objects at a more abstract level, providing deeper insights and leading to new discoveries in the field.

Another accomplishment of Grothendieck was the definition of the Grothendieck ring of varieties, a ring that is additively generated by isomorphism classes of varieties over k, modulo a "cut and paste" relation with respect to closed subschemes. This is one of the central objects in the Kontsevich's theory of motivic integration and is highly useful in various branches of Algebraic Geometry, especially in Birational Geometry and Singularity Theory. A class of varieties in the Grothendieck ring hold a wealth of geometric data about the variety, such as the Hodge polynomials, Euler characteristic, the number of points of the variety in the case where k is a finite field, among others.

Thus, the objective of this work is to provide an introduction to algebraic geometry by presenting fundamental concepts and results from the field. Part of this study has as the main focus the Chapters 1 and 2 of “Algebraic Geometry” by Robin Hartshorne, which serve as the foundational material for this study. In addition to covering the initial concepts of algebraic geometry, this work explore into the study of the Grothendieck ring of varieties, since it is a fundamental structure that captures important information about algebraic varieties and their classes. To achieve these objectives, the study is divided into three chapters.

In Chapter 1, we present the initial objects of this theory and prerequisites that will be used throughout the work. In this chapter, we review and study concepts from category theory and commutative algebra. Section 1 provides a brief introduction to category theory, presenting the initial concepts and some examples. Section 2 is dedicated to a review of commutative algebra, covering concepts such as modules, monoids, localizations, exact sequences, Noetherian rings, and important results like the Snake lemma and Hilbert’s Basis theorem.

In Chapter 2, we lay the groundwork for our study of Algebraic Geometry. We begin by introducing the fundamental concepts of affine and projective algebraic varieties and delve into the language of schemes. We also explore the properties of abelian varieties and complex tori. This chapter is divided into three sections, each focusing on different aspects of the subject. The first section is dedicated to the study of varieties. We cover topics such as affine varieties, projective varieties, morphisms, and rational maps and lastly we look into important techniques like blow-ups and showcase the grassmannians as significant examples of varieties. The second section shifts our attention to the study of schemes. We introduce the notions of pre-sheaves, sheaves, affine schemes, and schemes themselves. We explore various properties of schemes and conclude the section with insights into solving singularities. Finally, in the third section, our focus turns to abelian varieties, complex tori, and specifically elliptic curves. These examples of schemes serve as illustrative instances that will be utilized throughout our work. The construction of this chapter is crucial for the development of the subsequent chapter, as it provides the necessary foundation and tools for further investigations, proofs, and examples.

Finally, Chapter 3 is dedicated to the study of the Grothendieck ring of varieties. Section 1 focuses on presenting classical properties of the Grothendieck ring of varieties and some examples that illustrate how to calculate classes, we also provide an exposition of the generators of the Grothendieck ring and introduce Bittner’s alternative presentations, which offer alternative perspectives on this ring. In Section 2, we delve into the fascinating connection between birational geometry and the Grothendieck ring of varieties. We explore the significant role played by the class $[\mathbb{L}]$ and its implications in understanding the relationship between birational geometry and the algebraic structure of the Grothendieck ring. The chapter concludes with Section 3, which is dedicated to give two ideas on how to prove that the Grothendieck ring of varieties is not a domain. The historical information and contexts presented in this introduction and throughout the paper are sourced from the MacTutor History of Mathematics archive and the reference (DIEUDONNÉ, 1985).

1 Preliminaries

1.1 Categories

The theory of Categories, despite its generality, provides very useful concepts in the study of Algebraic Geometry, simplifying the notation and language of the objects we work with. With that in mind, the purpose of this section is to present the initial definitions and some examples. The content presented in this section is based on the reference (VAKIL, 2017).

Definition 1.1.1. *A category $\mathcal{C}$ consists of:*

i) A collection of objects, denoted by $Obj\ \mathcal{C}$;

ii) For every pair of objects $X, Y \in Obj\ \mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, whose elements are called morphisms or arrows from X to Y;

iii) For every triple of objects $X, Y, Z \in Obj\ \mathcal{C}$, a composition law for morphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \quad \to \quad \mathrm{Hom}_{\mathcal{C}}(X, Z)\ (f, g) \quad \mapsto \quad g \circ f.$$

Item (iii) must satisfy the following axioms:

(a) (Identity) For every object $X \in Obj\ \mathcal{C}$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, called the identity morphism of X, such that

$$f \circ \mathrm{id}_X = f \quad \textit{and} \quad \mathrm{id}_X \circ g = g,$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Z, X)$.

(b) (Associativity) For any $f \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Here are some examples of categories:

Example 1.1.2. *The category $Vect_k$ of vector spaces over a fixed field k. The objects are vector spaces over k, and the morphisms are linear transformations between such spaces.*

Example 1.1.3. *The category $Groups$ of all groups. The objects are groups, and the morphisms are group homomorphisms.*

Example 1.1.4. *The category Top of all topological spaces. The objects are topological spaces, and the morphisms are continuous maps between spaces.*

Example 1.1.5. *The dual category $\mathcal{C}^{\circ}$. The objects are the same as in the category $\mathcal{C}$, and we define the set of morphisms as* $\mathrm{Hom}_{\mathcal{C}^{\circ}}(X^{\circ}, Y^{\circ}) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

From now on, we will write $X \in \mathcal{C}$ instead of $X \in \mathrm{Obj}, \mathcal{C}$. Let's also see the definition of isomorphism between objects.

Definition 1.1.6. *Given $X, Y \in \mathcal{C}$, we say that X and Y are isomorphic if there exist $\varphi \colon X \to Y$ and $\psi \colon Y \to X$ such that* $\mathrm{id}_X = \psi \circ \varphi$ *and* $\mathrm{id}_Y = \varphi \circ \psi$.

Now that we have defined what a category is, let's see how categories relate to each other.

Definition 1.1.7. *A (covariant) functor F between categories $\mathcal{C}$ and $\mathcal{D}$ is a rule that associates to each object $M \in \mathcal{C}$ an object $FM \in \mathcal{D}$ and to each morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ a morphism $F\varphi \in \mathrm{Hom}_{\mathcal{D}}(FM, FN)$, satisfying the following conditions:*

i) $F(\mathrm{id}\, M) = \mathrm{id}\, FM$;

ii) $F(\varphi \circ \psi) = F\varphi \circ F\psi$ *for* $\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ *and* $\psi \in \mathrm{Hom}_{\mathcal{C}}(N, P)$.

We say that a functor F is contravariant if it satisfies the above conditions, except that F reverses the direction between morphisms, i.e.,

$$\varphi \in \mathrm{Hom}_{\mathcal{C}}(M, N) \Rightarrow F\varphi \in \mathrm{Hom}_{\mathcal{D}}(FN, FM).$$

The following diagram provides a more visual way to understand the distinction between a covariant functor and a contravariant functor.

$$\begin{tikzcd}
F\colon & \mathcal{C} \arrow[r] & \mathcal{D} \\
& M \arrow[r, mapsto] & FM \\
& M \arrow[d, "\varphi"'] \arrow[dr, phantom, "\longmapsto"] & FM \\
& N & FN \arrow[u, "F_\varphi"']
\end{tikzcd}$$

An example of a functor is the mapping $F\colon Top(X) \to Rings$, where $Top(X)$ is the category of open sets in a topological space X, with morphisms being continuous mappings between the open sets. And *Rings* is the category of commutative rings, with morphisms being ring homomorphisms. The functor F associates each open set of the topological space with the ring of continuous functions on it, and for each continuous function $\varphi \in \mathrm{Hom}(U, V)$, it associates the homomorphism $F\varphi \in \mathrm{Hom}(FV, FU)$, defined as $F\varphi(f) = f \circ \varphi$. Note that F is a contravariant functor.

Now let's explore a way to generalize the notions of injectivity and surjectivity to functors.

Definition 1.1.8. *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be faithful if, for any two objects X and Y in $\mathcal{C}$, the mapping $F\colon \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ is injective. We say that F is full if this mapping is surjective.*

An interesting and useful example of a category is the category of abelian categories. Let's present the definition and some examples.

Definition 1.1.9. *An abelian category is a category $\mathfrak{A}$ such that: for every $A, B \in \mathfrak{A}$, $\mathrm{Hom}(A,B)$ has the structure of an abelian group; the composition law is linear; finite direct sums exist; every morphism has a kernel and cokernel; every monomorphism is the kernel of its cokernel; every epimorphism is the cokernel of its kernel, and every morphism can be factored into an epimorphism followed by a monomorphism.*

As examples of abelian categories, we have:

i) $\mathfrak{Ab}$, the category of abelian groups.

ii) $\mathfrak{Mod}(A)$, the category of modules over a commutative ring A with unity.

1.2 Commutative algebra

In this section, we present a review of commutative algebra, discussing various concepts like modules, monoids, localizations, exact sequences, Noetherian rings, and significant results such as the Snake lemma and Hilbert's Basis theorem. This study is based on the reference (ATIYAH; MACDONALD, 1969).

The concept of a module over a ring is a generalization of the notion of a vector space, where instead of a field, we have a ring as the set of scalars. Thus, a module, like vector spaces, is the product of elements from an abelian group with a ring. In this section, we introduce the definition and elementary properties of modules. Additionally, we briefly present the notion of sequences of A-modules, localization, and graded rings (and modules).

Definition 1.2.1. *Let A be a ring. An A-module is a pair (M, μ), where M is an abelian group and $\mu : A \times M \to M$ maps (a, m) to am, satisfying the following properties:*

i) *$a(m + n) = am + an$;*

ii) *$(a + b)m = am + bm$;*

iii) $(ab)m = a(bm)$;

iv) $1m = m$;

for all $a, b \in A$ *and* $m, n \in M$.

Definition 1.2.2. *Let* A *be a ring and* M *an* A*-module. A submodule* M' *of* M *is a subgroup of* M *that is closed under multiplication by elements of* A.

The abelian group M/M' inherits an A-module structure from M, defined by $a(m + M') = am + M'$. Thus, M/M' is defined as the quotient A-module of M by M'.

The sum $\sum M_i$ is a submodule of M and is the smallest submodule of M containing all M_i. The intersection $\cap M_i$ and the product IM are also submodules of M.

If a module $M = \sum Am_i$, we say that the m_i's form a set of generators for M. This means that every element of M can be expressed (not necessarily uniquely) as a finite linear combination of the m_i's with coefficients in A. An A-module is said to be finitely generated if it has a finite set of generators.

Just as we defined homomorphisms between rings, we now define homomorphisms between A-modules.

Definition 1.2.3. *Let* M *and* N *be* A*-modules. A function* $f : M \to N$ *is called an* A*-homomorphism if for any* $m_1, m_2 \in M$ *and* $a \in A$*, the following conditions hold:*

i) $f(m_1 + m_2) = f(m_1) + f(m_2)$;

ii) $f(am_1) = af(m_1)$.

Next, we present a way to represent a relationship between homomorphisms of A-modules using diagrams: exact sequences.

Definition 1.2.4. *Let* $\{\dots, M_{i-1}, M_i, M_{i+1}, \dots\}$ *be a family of* A*-modules, and* $\{\dots, f_{i+1} : M_i \to M_{i+1}, \dots\}$ *be a family of* A*-homomorphisms. The sequence*

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is said to be exact at M_i *if* $Im(f_i) = Ker(f_{i+1})$*. The sequence is exact if it is exact at each* M_i.

Definition 1.2.5. *An exact sequence of* A*-modules and* A*-homomorphisms of the form*

$$0 \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow 0$$

is called a short exact sequence.

Next, we present an important result known as the Snake Lemma.

Proposition 1.2.6. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \downarrow{f'} & & \downarrow{f} & & \downarrow{f''} & & \\ 0 & \longrightarrow & N' & \underset{u'}{\longrightarrow} & N & \underset{v'}{\longrightarrow} & N'' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of A-modules and homomorphisms, with the rows exact. Then there exists an exact sequence

$$0 \to \operatorname{Ker}\left(f'\right) \xrightarrow{\bar{u}} \operatorname{Ker}(f) \xrightarrow{\bar{v}} \operatorname{Ker}\left(f''\right) \xrightarrow{d}$$
$$\operatorname{Coker}\left(f'\right) \xrightarrow{\bar{u}'} \operatorname{Coker}(f) \xrightarrow{\bar{v}'} \operatorname{Coker}\left(f''\right) \to 0$$

in which $\bar{u}, \bar{v}$ are restrictions of u, v, and $\bar{u}', \bar{v}'$ are induced by u', v'.

Next, we define some more algebraic structures.

Definition 1.2.7. *Let k be a field and A be a vector space over k, equipped with a binary operation $\cdot\ :\ A \times A \to A$ called multiplication. Then A is a k-algebra if the following properties hold for all $x, y, z \in A$ and $a, b \in k$:*

i) $(x + y) \cdot z = x \cdot z + y \cdot z$;

ii) $z \cdot (x + y) = z \cdot x + z \cdot y$;

iii) $(ax) \cdot (by) = (ab)(x \cdot y)$.

Definition 1.2.8. *Let M be a set with a binary operation $\cdot\ :\ M \times M \to M$. Then M is a monoid if the following properties hold:*

i) $a \cdot b \in M$, $\forall a, b \in M$;

ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c$, $\forall a, b, c \in M$;

iii) *There exists a unique element e such that $a \cdot e = e \cdot a = a$, $\forall a \in M$.*

Definition 1.2.9. *A graded ring A is a ring with a family $(A_n)_{n\in\mathbb{Z}}$ of additive subgroups of A such that $A = \bigoplus_{n\in\mathbb{Z}} A_n$ and $A_m A_n \subseteq A_{n+m}$ for all $n, m \in \mathbb{Z}$.*

Definition 1.2.10. *Let A be a graded ring and M an A-module. Then M is said to be graded if there exists a family of additive subgroups $(M_n)_{n\geqslant 0}$ of M such that*

$$M = \oplus_{n=0}^{\infty} M_n \quad \textit{and} \quad A_m M_n \subseteq M_{n+m}\ \forall\, m, n \geqslant 0.$$

A non-zero element $x \in M$ is called homogeneous of degree n if $x \in M_n$.

Definition 1.2.11. *Let A be a graded ring and I an ideal of A. The ideal I is said to be homogeneous if it is generated by a set of homogeneous elements.*

A graded module that is also a graded ring is called a graded algebra.

Example 1.2.12. *The ring of polynomials $\mathbb{C}[x] = \oplus A_i$ is a graded ring where $A_i = \{ax^i\}$. The ideal $I = \langle x^2 \rangle$ is a homogeneous ideal in $\mathbb{C}[x]$. Similarly, $J = \langle x^2 + y^2 + z^2, xy, yz + zx, z^5 \rangle$ is a homogeneous ideal in $\mathbb{C}[x, y, z]$.*

As a generalization of how we construct the field of fractions of a domain, we can construct localization in a multiplicative subset of a ring by formally inverting the elements in this subset.

Definition 1.2.13. *Let A be a ring. A multiplicative set $S \subseteq A$ is a subset that is closed under multiplication, meaning that if $s, t \in S$, then $st \in S$, and such that $1 \in S$. Define a relation $\equiv$ in $A \times S$ as follows:*

$$(a, s) \equiv (a', s') \Leftrightarrow (as' - a's)u = 0 \text{ for some } u \in S. \tag{1.1}$$

The relation $\equiv$ is an equivalence relation.

Let us denote by $\dfrac{a}{s}$ the equivalence class of (a, s), and let $S^{-1}A = (A \times S)/\equiv$ be the set of equivalence classes. We define addition and multiplication in $S^{-1}A$ in the usual way:

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2} \quad \text{and} \quad \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}. \tag{1.2}$$

With these addition and multiplication operations, we define the following:

Definition 1.2.14. *Let $S^{-1}A$ be a commutative ring with zero element $\dfrac{0}{1}$ and identity element $\dfrac{1}{1}$. This ring is called the localization of A with respect to S. Associated with $S^{-1}A$, there is a ring homomorphism $\rho : A \to S^{-1}A$ given by $a \mapsto \dfrac{a}{1}$, called the localization map.*

If A is a graded ring and $\mathfrak{p}$ is a homogeneous prime ideal in A, then we denote by $A_{(\mathfrak{p})}$ the subring of degree 0 elements in the localization $T^{-1}A$ with respect to the multiplicative set T formed by the homogeneous elements of A that are not in $\mathfrak{p}$.

The set $T^{-1}A$ has a natural grading given by

$$degree(f/g) = degree(f) - degree(g), \tag{1.3}$$

for f homogeneous in A and $g \in T$. Thus, $A_{(\mathfrak{p})}$ is a local ring, and the only maximal ideal is the intersection $(\mathfrak{p} \cdot T^{-1}A) \cap A_{(\mathfrak{p})}$. In particular, if A is a domain, then for $\mathfrak{p} = \langle 0 \rangle$, we obtain the field $A_{(\langle 0 \rangle)}$. Similarly, if $f \in A$ is a homogeneous element, we denote by $A_{(f)}$ the subring of degree 0 in the localized ring S.

Next, we present the universal property of localization.

Proposition 1.2.15. *Let $g : A \to B$ be a ring homomorphism such that all elements $g(s) \in B, s \in S$ are invertible. Then there exists a unique homomorphism $h : S^{-1}A \to B$ such that $g = h \circ \rho$.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\rho"'] & B \\ S^{-1}A \arrow[ur, "h"'] & \end{tikzcd}$$

Emmy Noether was a German mathematician who made significant contributions to the fields of theoretical physics and abstract algebra. She is considered one of the most important women in the history of mathematics and revolutionized theories related to rings, fields and algebras. In 1921, in her article "Idealtheorie in Ringbereichen," she elegantly employed the Ascending Chain Condition, and as a tribute to Emmy Noether, objects that satisfy this condition are called Noetherian. We will now present some of these concepts.

Definition 1.2.16. *We say that a ring A is a Noetherian ring if every ideal I of A is finitely generated, i.e., there exist $r_1, \ldots, r_k \in A$ such that $I = \langle r_1, \ldots, r_k \rangle$.*

Next, we present results that characterize the above definition.

Theorem 1.2.17. *Let A be a ring. A is a Noetherian ring if, and only if, A satisfies the ascending chain condition, i.e., for any chain of nested ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$, there exists $k \in \mathbb{N}$ such that $I_k = I_j$ for all $j \geqslant k$.*

In the following, we present Hilbert's Basis Theorem.

Theorem 1.2.18. *Let A be a Noetherian ring. Then the ring of polynomials $A[x]$ is also Noetherian.*

Theorem 1.2.18 admits a natural generalization as follows.

Corollary 1.2.19. *Let A be a Noetherian ring and $x_1, \ldots, x_n$ variables. Then the ring of polynomials $A[x_1, \ldots, x_n]$ is Noetherian.*

2 Algebraic geometry

In this chapter, we establish the foundation for our exploration of Algebraic Geometry. We start by introducing the fundamental concepts of affine and projective algebraic varieties, and then delve into the language of schemes. Next we present some results about resolution of singularities. Additionally, we explore the properties of abelian varieties and complex tori. The chapter is structured into four sections, each dedicated to exploring different aspects of this fascinating subject.

2.1 Algebraic varieties

This section is based on chapter I of (HARTSHORNE, 1977) and as supplementary material, we relied on the reference (MUMFORD, 1999).

2.1.1 Affine varieties

In this chapter, we will work with a fixed algebraically closed field denoted by k, until we introduce the language of schemes in section 2. Rings are assumed to be commutative with a multiplicative unit 1. The polynomial ring in n variables, namely $x_1, \dots, x_n$, over k will be denoted by $k[x_1, \dots, x_n]$. Additionally, we use the notation $f(P)$ to represent the value of a polynomial $f \in k[x_1, \dots, x_n]$ at a point $P = (a_1, \dots, a_n) \in k^n$.

Definition 2.1.1. *The set $\mathbb{A}^n_k := \{(a_1, \dots, a_n) | a_i \in k \text{ for } i = 1, \dots, n\}$ of all n-tuples of elements of a field k is called the affine n-space over k. When the field k is clear, sometimes we simplify the notation and use just $\mathbb{A}^n$.*

Note that $\mathbb{A}^n_k$ is just k^n as a set. However, it is customary to use two different notations here since k_n is also a k-vector space and a ring. The notation $\mathbb{A}^n_k$ is typically used when we want to disregard these additional structures. For instance, while addition and scalar multiplication are defined on k^n, they are not defined on $\mathbb{A}^n_k$. We consider the affine space $\mathbb{A}^n_k$ as the ambient space for our zero loci of polynomials, which will be defined subsequently.

Definition 2.1.2. *If $f \in k[x_1, \dots, x_n]$, we define the set of zeros of f as*

$$V(f) = \{P \in \mathbb{A}^n | f(P) = 0\}.$$

More generally, if T is a subset of $k[x_1, \dots, x_n]$, we define the zero locus of T to be the commom zeros of all the elements of T:

$$V(T) = \{P \in \mathbb{A}^n | f(P) = 0 \text{ for all } f \in T\}.$$

If T is a finite set $T = \{[f_1, \dots, f_l\}$ we will write $V(T) = V(\{f_1, \dots, f_l\})$ also as $V(f_1, \dots, f_l)$.

Definition 2.1.3. *A subset X of $\mathbb{A}^n$ is an algebraic set if there exists a subset $T \subseteq k[x_1, \ldots, x_n]$ such that $X = V(T)$.*

We will begin by defining the algebraic set determined by an ideal, and subsequently, we will demonstrate that if I is an ideal generated by T in $k[x_1, \ldots, x_n]$, then $V(T) = V(I)$. In other words, algebraic sets correspond to zero loci of ideals.

Definition 2.1.4. *For an ideal I in a polynomial ring $k[x_1, \ldots, x_n]$, we define*

$$V(I) = \{(b_1, \ldots, b_n) \in k^n \mid \mathscr{F}orall\, g \in I,\ g(b_1, \ldots, b_n) = 0\}. \tag{2.1}$$

$V(I)$ is referred to as the algebraic set determined by the ideal I.

Lemma 2.1.5. *If $I = \langle f_1, \ldots, f_l \rangle$ is an ideal in $k[x_1, \ldots, x_n]$ then $V(I) = V(f_1, \ldots, f_l)$.*

Proof. For an arbitrary element f in the ideal I generated by $f_1, \ldots, f_l$ in the polynomial ring $k[x_1, \ldots, x_n]$ of n variables, there exist $g_\alpha \in k[x_1, \ldots, x_n]$, $\alpha = 1, \ldots, l$, such that

$$f(x_1, \ldots, x_n) = \sum_{\alpha=1}^{l} g_\alpha(x_1, \ldots, x_n) f_\alpha(x_1, \ldots, x_n). \tag{2.2}$$

Therefore, if $(a_1, \ldots, a_n) \in V(f_1, \ldots, f_l)$, it follows that $f_1(a_1, \ldots, a_n) = \ldots = f_l(a_1, \ldots, a_n) = 0$, which implies $f(a_1, \ldots, a_n) = 0$. Hence, $V(f_1, \ldots, f_l) \subset V(I)$. We will now show that $V(I) \subset V(f_1, \ldots, f_l)$. Let $(b_1, \ldots, b_n) \in V(I)$, and since $f_\alpha \in I$, we have $f_\alpha(b_1, \ldots, b_n) = 0$ for every $\alpha = 1, \ldots, l$. Therefore, $V(I) \subset V(f_1, \ldots, f_l)$, which completes the proof. □

The Hilbert's Basis Theorem guarantees that every ideal of the polynomial ring $k[x_1, \ldots, x_n]$ is of the form $I = \langle f_1, \ldots, f_l \rangle$. Therefore, by Lemma 2.1.5, the algebraic set $V(I)$ determined by an ideal is precisely the algebraic set $V(f_1, \ldots, f_l)$ determined by the polynomials $f_1, \ldots, f_l$.

Next, we present some properties that will be essential for establishing the correspondence between ideals and algebraic sets.

Proposition 2.1.6. *For ideals I, J, and $I_{\lambda\lambda\in\Lambda}$ in the polynomial ring $k[x_1, \ldots, x_n]$ over a field k, where $\lambda \in \Lambda$ and Λ can be an infinite set, we have:*

i) *$V(I) \cup V(J) = V(I \cap J)$;*

ii) *$\cap_{\lambda\in\Lambda} V(I_\lambda) = V\left(\sum_{\lambda\in\Lambda} I_\lambda\right)$;*

iii) *$V(I) \supset V(J)$ when $\sqrt{I} \subset \sqrt{J}$,*

where $\sum_{\lambda\in\Lambda} I_\lambda$ denotes the ideal in $k[x_1, \ldots, x_n]$ generated by I_λ, $\lambda \in \Lambda$ and $\sqrt{I} = \{f \in k[x_1, \ldots, x_n] | f^m \in I$ for some integer $m\}$ is the radical of I.

Proof. **i)** From the definition of an algebraic set, we can deduce that $V(I) \supset V(J)$ for $I \subset J$. In fact, if $(a_1, \ldots, a_n) \in V(J)$, a zero of all polynomials in J is certainly a zero of all polynomials in I. Therefore, we have $V(I \cap J) \supset V(I)$ and $V(I \cap J) \supset V(J)$. Thus, $V(I) \cup V(J) \subset V(I \cap J)$.

On the other hand, let $(a_1, \ldots, a_n) \in V(I \cap J)$. If $(a_1, \ldots, a_n) \notin V(I)$, then there exists a polynomial $f \in I$ such that $f(a_1, \ldots, a_n) \neq 0$. Now, for an arbitrary element $g(x_1, \ldots, x_n) \in J$, we have $h = f \cdot g \in I \cap J$. Consequently,

$$h(a_1, \ldots, a_n) = f(a_1, \ldots, a_n)g(a_1, \ldots, a_n) = 0. \tag{2.3}$$

Therefore, $g(a_1, \ldots, a_n) = 0$, which implies that $(a_1, \ldots, a_n) \in V(J)$. Thus, $V(I \cap J) \subset V(I) \cup V(J)$. Hence, $V(I \cap J) = V(I) \cup V(J)$.

ii) Since $I_\mu \subset \sum_{\lambda \in \Lambda} I_\lambda$ for every $\mu \in \Lambda$, we have

$$V(I_\mu) \supset V(\sum_{\lambda \in \Lambda} I_\lambda) \ \Rightarrow \ \bigcap_{\mu \in \Lambda} V(I_\mu) \supset V(\sum_{\lambda \in \Lambda} I_\lambda).$$

For each λ, let us express I_λ in terms of generators $I_\lambda = \langle h_{\lambda 1}, \ldots, h_{\lambda m_\lambda} \rangle$.

For $(a_1, \ldots, a_n) \in \cap_{\lambda \in \Lambda} V(I_\lambda)$, we have $h_{\lambda j}(a_1, \ldots, a_n) = 0$ for $j = 1, \ldots, m_\lambda$. On the other hand, $h_{\lambda j} \lambda \in \Lambda, : 1 \leqslant j \leqslant m\lambda$ generates the ideal $\sum_{\lambda \in \Lambda} I_\lambda$. Thus, $(a_1, \ldots, a_n) \in V(\sum_{\lambda \in \Lambda} I_\lambda)$. Therefore, $\cap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$.

iii) We just need to show that $V(\sqrt{I}) = V(I)$. Let $f \in \sqrt{I}$, which means that $f^m \in I$ for some integer m. For $(a_1, \ldots, a_n) \in V(I)$, we have $f^m(a_1, \ldots, a_n) = 0$. Consequently, $f(a_1, \ldots, a_n) = 0$, i.e., $(a_1, \ldots, a_n) \in V(\sqrt{I})$, which shows that $V(I) \subset V(\sqrt{I})$. The inclusion $V(\sqrt{I}) \subset V(I)$ follows from the fact that $I \subset \sqrt{I}$. Therefore, $V(I) = V(\sqrt{I})$, which completes the proof.

□

Corollary 2.1.7. *For a finite number of ideals $I_1, \ldots, I_s$ in $k[x_1, \ldots, x_n]$, we have*

$$\bigcup_{j=1}^{s} V(I_j) = V\left(\bigcap_{j=1}^{s} I_j\right).$$

Proof. The proof follows by induction on s and by Proposition 2.1.6, item (i). □

Note that $V(\varnothing) = \mathbb{A}^n$ and $V(1) = \varnothing$. Those equalities along with the properties presented in Lemma 2.1.6 allow us to define:

Definition 2.1.8. *The Zariski topology in $\mathbb{A}^n$ is defined by considering the complements of the algebraic sets as the open subsets.*

The Zariski topology has unique properties that distinguish it from other topologies. It is non-Hausdorff, meaning that two distinct points may not be separated by disjoint open sets. This non-separation reflects the inherent algebraic interplay between equations and points. Next we present an example:

Example 2.1.9. *Let's consider the Zariski topology on the affine line $\mathbf{A}^1$. In the ring of polynomials $A = k[x]$, every ideal is principal, which means that every algebraic set can be described as the set of zeros of a single polynomial. Since the field k is algebraically closed, any nonzero polynomial $f(x)$ can be factored as $f(x) = c(x - a_1)\cdots(x - a_n)$, where $c, a_1, \ldots, a_n \in k$. Consequently, the algebraic set $Z(f)$ corresponds to the set $\{a_1, \ldots, a_n\}$. Therefore, the algebraic sets in $\mathbf{A}^1$ consist of finite subsets (including the empty set) and the entire space (corresponding to $f = 0$). As a result, the open sets are the empty set and the complements of finite subsets.*

We introduce some additional notations.

Definition 2.1.10. *Given a set V in the n-dimensional affine space k^n over an algebraically closed field k, we define the ideal $I(V)$ determined by V as follows:*

$$I(V) = \{f \in k[x_1, \ldots, x_n] \mid f(b_1, \ldots, b_n) = 0 \textit{ for all } (b_1, \ldots, b_n) \in V\}.$$

The following properties demonstrate some of the relationships between ideals and algebraic sets.

Proposition 2.1.11. *For subsets X, Y of k^n and a subset S of $k[x_1, \ldots, x_n]$, we have:*

i) *If $X \subset Y$, then $I(X) \supset I(Y)$.*

ii) *$I(X \cup Y) = I(X) \cap I(Y)$.*

iii) *$I(V(S)) \supset S$ and $V(I(X)) \supset X$.*

iv) *$I(V(I(X))) = I(X)$. Hence, if I is the ideal of an algebraic set W, then $I = I(V(I))$.*

Proof. **i)** Let $f \in I(Y)$. We have $f(P) = 0$ for all $P \in Y$. In particular, since $X \subset Y$, $f(P) = 0$ holds for all $P \in X$. Therefore, $f \in I(X)$, and we conclude that $I(Y) \subset I(X)$.

ii) Since $X \cup Y \supset X$ and $X \cup Y \supset Y$, by item (i) we have $I(X \cup Y) \subset I(X)$ and $I(X \cup Y) \subset I(Y)$. Consequently, $I(X \cup Y) \subset I(X) \cap I(Y)$.

On the other hand, let $f \in I(X) \cap I(Y)$. We have $f \in I(X)$ and $f \in I(Y)$. Hence, $f(P) = 0$ for all $P \in X$ and $f(P) = 0$ for all $P \in Y$. Thus, $f(P) = 0$ for all $P \in X \cup Y$. Therefore, $f \in I(X \cup Y)$, and we have $I(X \cup Y) \supset I(X) \cap I(Y)$. Hence, $I(X \cup Y) = I(X) \cap I(Y)$.

iii) If $f \in S$, then $f(P) = 0$ for all $P \in V(S)$, by the definition of $V(S)$. Hence, $f \in I(V(S))$. Similarly, if $P \in X$, then $g(P) = 0$ for all $g \in I(X)$. Thus, $P \in V(I(X))$.

iv) By the second assertion of item (iii), we have $V(I(X)) \supset X$. Therefore, by item (ii), we obtain $I(V(I(X))) \subset I(X)$. Moreover, by the first assertion of item (iii) with $S = I(X)$, we have $I(V(I(X))) \supset I(X)$. This completes the proof. □

It is necessary for $V(I)$ to be non-empty for an algebraic set $V(I)$ in k^n to have geometric meaning. The Weak Nullstellensatz guarantees this condition.

Lemma 2.1.12. *A maximal ideal in the polynomial ring $k[x_1, \dots, x_n]$ over an algebraically closed field k has the following form:*

$$\langle x_1 - a_1, \dots, x_n - a_n \rangle, \quad a_j \in k, \ j = 1, \dots, n.$$

Proof. Let m be a maximal ideal of $k[x_1, \dots, x_n]$. We define $J = k[x_1, \dots, x_n]/m$. So J is a field. We also know that J is finitely generated over k, i.e., $J = k[\overline{x_1}, \dots, \overline{x_n}]$. Therefore J is algebraic over k, and since k is algebraically closed, we conclude that $J = k$.

Thus, for each x_i, there

exists $a_i \in k$ such that a_i is the residue of x_i, and hence $x_i - a_i \in m$ for all $i = 1, \dots, n$. Consequently, $\langle x_1 - a_1, \dots, x_n - a_n \rangle \subseteq m$. Then $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is a maximal ideal, and therefore, $m = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. □

The next theorem, known as the Nullstellensatz, explains the relationship between J and $I(V(J))$. The Nullstellensatz establishes a fundamental connection between algebra and geometry and is considered one of the foundations of algebraic geometry. It was discovered and proved by David Hilbert in his work "Ueber die Theorie der algebraischen Formen" (1893).

Theorem 2.1.13 (Hilbert's Nullstellensatz)**.** *For an ideal J in the polynomial ring $k[x_1, \dots, x_n]$ over an algebraically closed field k, we have*

$$I(V(J)) = \sqrt{J}.$$

Proof. According to Definition 2.1.10, we have $\sqrt{J} \subset I(V(J))$. Indeed, if $g \in \sqrt{J}$, then $g^m \in J$ for some positive integer m. Therefore, $g(P) = 0$ for all $P \in V(J)$, which implies $g \in I(V(J))$. This shows that $\sqrt{J} \subset I(V(J))$.

It suffices to prove that for $f \in I(V(J))$, we have $f \in \sqrt{J}$.

Let x_0 be a new variable, and let $\tilde{J}$ be the ideal generated by $1 - x_0 f(x_1, \dots, x_n)$ and J, where J is considered as an ideal in the polynomial ring $k[x_0, \dots, x_n]$ with $n + 1$ variables.

Firstly, assume $V(\tilde{J}) \neq \emptyset$. Let $(a_0, \dots, a_n) \in V(\tilde{J})$. Then $(a_1, \dots, a_n) \in V(J)$ since $J \subset \tilde{J}$. Consequently, $f(a_1, \dots, a_n) = 0$ because $(a_1, \dots, a_n) \in V(J)$, and by assumption, $f \in I(V(J))$. However, as $1 - x_0 f \in \tilde{J}$ and $(a_0, \dots, a_n) \in V(\tilde{J})$, we obtain the contradiction

$$0 = 1 - a_0 f(a_1, \dots, a_n) = 1. \tag{2.4}$$

Therefore, we must have $V(\tilde{J}) = \varnothing$. Hence, $\tilde{J} = k[x_0, \ldots, x_n]$, which means that $\tilde{J}$ contains the identity element 1. We can write

$$1 = h(x_0, \ldots, x_n)(1 - x_0 f(x_1, \ldots, x_n)) + \sum_{j=1}^{l} g_j(x_0, \ldots, x_n) f_j(x_1, \ldots, x_n), \tag{2.5}$$

where $h, g_j \in k[x_0, \ldots, x_n]$ and $f_j \in J$. Substituting x_0 with $1/f$ in the equation and multiplying both sides by a suitable power m of f, we obtain

$$f^m = \sum_{j=1}^{l} \tilde{g_j}(x_1, \ldots, x_n) f_j(x_1, \ldots, x_n), \tag{2.6}$$

where $\tilde{g_j} \in k[x_1, \ldots, x_n]$. Consequently, $f^m \in J$, which concludes the proof. □

Due to this theorem, to study algebraic sets $V(J)$, we can focus only on ideals that satisfy $J = \sqrt{J}$. Ideals with this property are called reduced ideals.

We now introduce the concept of irreducible algebraic sets and present some additional notations and results. In particular, we will prove that every algebraic set can be decomposed into irreducible components.

Definition 2.1.14 (Algebraic Variety)**.** *An algebraic set V in the affine space k^n over a field k is said to be reducible if V can be expressed as the union of algebraic sets V_1 and V_2,*

$$V = V_1 \cup V_2, \quad V \neq V_1 \text{ and } V \neq V_2.$$

When an algebraic set is not reducible, it is called irreducible. An irreducible algebraic set is called an algebraic variety.

Proposition 2.1.15. *An algebraic set V is irreducible if and only if the associated ideal $I(V)$ is a prime ideal.*

Proof. The proofs will be by contradiction. Let V be an irreducible algebraic set and assume that $I(V)$ is not a prime ideal, i.e., there exist $f_1, f_2 \notin I(V)$ such that $f_1 f_2 \in I(V)$. Then $V(f_1 f_2) \supset V(I(V)) = V$, so we have

$$V = V \cap V(f_1 f_2) = V \cap (V(f_1) \cup V(f_2)) = (V \cap V(f_1)) \cup (V \cap V(f_2)). \tag{2.7}$$

We have $V \cap V(f_i) \subsetneq V$ because, otherwise, $V \subset V(f_i)$ and $I(V) \supset I(V(f_i))$, which would imply $f_i \in I(V)$ for $i = 1, 2$. Thus, V is reducible, which is a contradiction.

Conversely, if $I(V)$ is a prime ideal, suppose that $V = V_1 \cup V_2$ with $V_i \subsetneq V$. Then $I(V_i) \supsetneq I(V)$. Let $f_i \in I(V_i)$ with $f_i \notin I(V)$ for $i = 1, 2$. Since $I(V) = I(V_1 \cup V_2)$, we have that for every $P \in V$, $P \in V_1$ or $P \in V_2$. Therefore, $f_1 f_2(P) = f_1(P) f_2(P) = 0$, and thus $f_1 f_2 \in I(V)$. Consequently, $I(V)$ is not a prime ideal, which is a contradiction. □

As a consequence of Theorem 2.1.13 and Proposition 2.1.15, we have the following result.

Corollary 2.1.16. *If I is a prime ideal, then $V(I)$ is irreducible. There exists a bijection between prime ideals and irreducible algebraic sets. Maximal ideals correspond to points.*

The next lemma will be used in the proof of Theorem 2.1.18. We have that any non-empty collection of ideals in a Noetherian ring has a maximal element. As a consequence of this result, we have the following lemma:

Lemma 2.1.17. *Any non-empty collection of algebraic sets in k^n has a minimal element.*

Proof. Let $\{V_i\}$ be a non-empty collection of algebraic sets in k^n. Consider the collection of ideals $\{I(V_i)\}$. The collection $\{I(V_i)\}$ has a maximal element $I(V_m)$. Since $I(V_m) \supset I(V_i)$ for all $i \in \mathbb{N}$, we have $V_m = V(I(V_m)) \subset V(I(V_i)) = V_i$. Therefore, V_m is a minimal element of the collection $\{V_i\}$. □

The decomposition of an algebraic set into irreducible components is unique and finite, as stated in the following result.

Theorem 2.1.18. *Let V be an algebraic set in k^n. Then there exist $V_1, \ldots, V_n$ irreducible algebraic sets such that $V = V_1 \cup \cdots \cup V_n$, and $V_i \not\subset V_j$ for $i \neq j$. The algebraic sets $V_1, \ldots, V_n$ are unique.*

Proof. Let S be the set of all algebraic sets V in k^n that are not the union of irreducible algebraic sets. We want to show that S is empty. Suppose S is non-empty and let V be a minimal element of S. Since $V \in S$, there exist $V_1, V_2 \in k^n$ such that $V = V_1 \cup V_2$ with $V_i \subsetneq V$ for $i = 1$ or 2. As $V_i \subset V$ and V is a minimal element of S, we have $V_i \notin S$. Hence, V_i is reducible, i.e., $V_i = V_{i1} \cup \cdots \cup V_{im}$ with V_{ij} irreducible for all j. Thus, we conclude that $V = \cup_{ij} V_{ij}$, which contradicts the assumption that $V \in S$. Therefore, any algebraic set V can be written as $V = V_1 \cup \cdots \cup V_n$, V_i irreducible to $i = 1, \ldots, n$. To obtain $V_i \not\subset V_j$ for $i \neq j$, we discard V_i such that $V_i \subset V_j$ for $i \neq j$.

We will now show the uniqueness of $V_1, \ldots, V_n$. Let $V = W_1 \cup \cdots \cup W_m$ another decomposition, then

$$V_i = V \cap V_i = \left(\bigcup_{j=1}^{m} W_j \right) \cap V_i = \bigcup_{j=1}^{m} (W_j \cap V_i). \tag{2.8}$$

So $V_i \subset W_{j(i)}$ for some $j(i)$ and $W_{j(i)} \subset V_l$ for some l. But $V_i \subset V_l$ implies $i = l$ and therefore $V_i = W_{j(i)}$. Analogously, each W_i is equal to some $V_{i(j)}$. □

The sets V_i are called irreducible components of V and $V = V_1 \cup \cdots \cup V_n$ is the decomposition of V into irreducible components.

Finally, we present another notation.

Definition 2.1.19 (Coordinate ring). *For an algebraic variety V in the n-dimensional affine space k^n, the quotient ring*

$$A(V) := k[x_1, \dots, x_n]/I(V)$$

is called the coordinate ring of V.

Thus, the Proposition 2.1.15 can be rewritten as follows:

Proposition 2.1.20. *An algebraic set V in k^n is irreducible if and only if its coordinate ring $A(V)$ is a domain.*

The coordinate ring $A(V)$ can be seen as the set of polynomial functions defined in V.

2.1.2 Projective varieties

To define projective varieties, we follow a similar approach to the definition of affine varieties, but working in a projective space.

We define the projective space $\mathbb{P}^n$ over an algebraically closed field k as the set of equivalence classes of $(n+1)$-tuples $(a_0, \dots, a_n)$, where the elements $a_i \in k$ are not all zero, given by the equivalence relation

$$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n),$$

for every non-zero $\lambda \in k$. An element P of $\mathbb{P}^n$ is called a point, and any $(n+1)$-tuple in the equivalence class of P is called homogeneous coordinates of P.

To establish a graded ring, we transform the polynomial ring $S = k[x_0, \dots, x_n]$ by defining S_d as the collection of all linear combinations of monomials with a total degree d in $x_0, \dots, x_n$. Due to the non-uniqueness of homogeneous coordinates, it is not possible to directly employ a polynomial $f \in S$ to define a function on $\mathbb{P}^n$. Nevertheless, if f represents a homogeneous polynomial with degree d, we observe that $f(ia_0, \dots, ia_n) = i^d f(a_0, \dots, a_n)$. Consequently, the determination of whether f is zero or non-zero relies solely on the equivalence class of $(a_0, \dots, a_n)$. Thus, f gives a function from $\mathbb{P}^n$ to $\{0, 1\}$, where $f(P) = 0$ if $f(a_0, \dots, a_n) = 0$, and $f(P) = 1$ if $f(a_0, \dots, a_n) \neq 0$.

We can now define the zeros of a homogeneous polynomial, denoted $V(f) = \{P \in \mathbb{P}^n \mid f(P) = 0\}$. The zero set of a given set T consist of homogeneous elements of S is defined as follows:

$$V(T) = \{P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

When considering a homogeneous ideal $\mathfrak{a}$ of S, we define $V(\mathfrak{a})$ as $V(T)$, where T represents the collection of all homogeneous elements in $\mathfrak{a}$. As S is a Noetherian ring, it follows that for any set T of homogeneous elements, there exists a finite subset $f_1, \dots, f_r$ of T such that $V(T) = V(f_1, \dots, f_r)$.

Definition 2.1.21. *If there exists a set T consisting of homogeneous elements of S such that $Y = V(T)$, then the subset Y of $\mathbb{P}^n$ is considered an algebraic set.*

Note that the empty set $V(1) = \varnothing$ and the whole space $V(0) = \mathbb{P}^n$ are algebraic sets. And we have the following properties.

Proposition 2.1.22. *The intersection of any family of algebraic sets is an algebraic set and the union of two algebraic sets is an algebraic set.*

So, similarly to the affine case we can establish a topology.

Definition 2.1.23. *The Zariski topology on $\mathbb{P}^n$ is defined by considering the open sets as the complements of algebraic sets.*

Next we present an exemple that show us the difference between the affine and the projective algebraic set.

Example 2.1.24. *Consider the homogeneous polynomial $f = x_1^2 - x_2^2 - x_0^2 \in \mathbb{C}[x_0, x_1, x_2]$. This polynomial's real component in the affine zero locus $V_a(f) \subset \mathbb{A}^3$ takes the form of a 2-dimensional cone (see Figure 1). Simultaneously, its projective zero locus $V_p(f) \subset \mathbb{P}^2$ is the set of all 1-dimensional linear subspaces contained in this cone.*

By considering $\mathbb{P}^2$ as an affine plane $\mathbb{A}^2$ with additional points at infinity (embedded in $\mathbb{A}^3$ at $x_0 = 1$), the application of $x_0 = 1$ in f yields an insightful portrayal of $V_p(f)$. In this context, $V_p(f)$ consists of the hyperbola $x_1^2 - x_2^2 - 1 = 0$ together with two points a and b situated at infinity (see Figure 1).

In the figure of the cone, the points a and b correspond to the two 1-dimensional linear subspaces parallel to the plane at $x_0 = 1$. In the figure of the hyperbola contained in $\mathbb{A}^2$, the points a and b can be thought of as points at infinity in the corresponding directions. Notably, under this latter interpretation, "opposite" points at infinity are the same, as they correspond to identical 1-dimensional linear subspaces in $\mathbb{C}^3$.

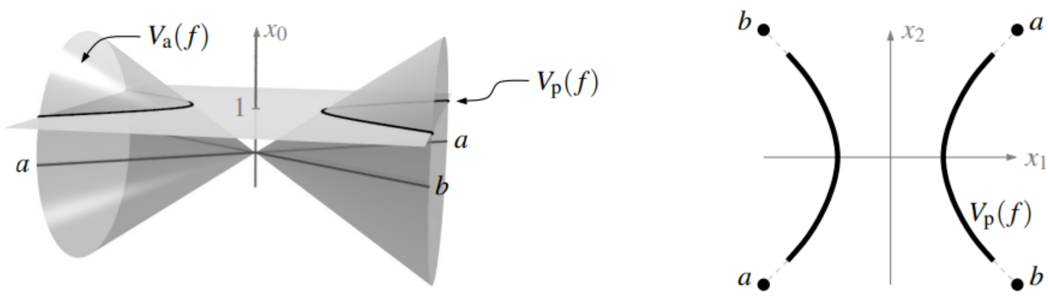

Figure 1 – Affine and projective zero locus of f. From (GATHMANN, 2022).

Note that both $V(f) \subset \mathbb{A}^3$ and $V(f) \subset \mathbb{P}^2$ hold fundamentally identical geometric information. The distinction lies solely in how we interpret the cone: either as a collection of individual points or as a union of 1-dimensional linear subspaces in $\mathbb{A}^3$.

Now that we have a topological space, we can define the notions of irreducibility and dimension of a subset in a similar way to the previous section.

Definition 2.1.25. *An irreducible algebraic set in $\mathbb{P}^n$ with the induced Zariski topology is called a projective algebraic variety or a projective variety. A quasi-projective variety is an open subset of a projective variety.*

Definition 2.1.26. *The dimension of a projective or quasi-projective variety is the dimension of the variety as a topogical space.*

We introduce some additional notations.

Definition 2.1.27. *We define the homogeneous ideal of subset $Y \subset \mathbb{P}^n$ in S as*

$$I(Y) = \langle \{ f \in S \mid f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y \} \rangle.$$

Definition 2.1.28. *We define the homogeneous coordinate ring of an algebraic set Y as $S(Y) = S/I(Y)$.*

The zero set of a linear homogeneous polynomial $f \in S$ is called a hyperplane. Specifically, we represent the zero set of x_i by H_i, for $i = 0, \ldots, n$.

We next aim to demonstrate that a projective space of dimension n can be covered by open affine spaces of dimension n. Which means that every (quasi-)projective variety has an open covering by (quasi-)affine varieties.

Proposition 2.1.29. *Let U_i be the open set $\mathbb{P}^n - H_i$. Then $\mathbb{P}^n$ is covered by the open sets U_i.*

Proof. If $P = (a_0, \ldots, a_n)$ is a point, then at least one of the a_i's is nonzero, hence $P \in U_i$. We define a mapping $\varphi_i : U_i \to \mathbb{A}^n$ as follows: if $P = (a_0, \ldots, a_n) \in U_i$, then $\varphi_i(P) = Q$, where the point Q has the affine coordinates

$$\left(\frac{a_0}{a_i}, \ldots, \frac{a_n}{a_i} \right),$$

disregarding the coordinate $1 = a_i/a_i$. The map φ_i is well-defined because the quotients a_j/a_i do not depend on the choice of homogeneous coordinates. □

2.1.3 Morphisms

Up to this point, we have introduced the concepts of affine and projective varieties; however, we have not talked about the mappings allowed between them or the notion of isomorphism. So in this section, we define regular functions on a variety and then present the concept of morphisms between varieties. This will provide us with a well-defined category to work with.

Definition 2.1.30. *Let $Y \subset \mathbb{A}^n$ be a quasi-affine variety. A function $f : Y \to k$ is said to be regular at a point $P \in Y$ if there exists an open neighborhood U with $P \in U \subseteq Y$ and polynomials $g, h \in A = k[x_1, \ldots, x_n]$ such that h is nonzero on U and $f = g/h$ on U. We say that f is regular on Y if it is regular at every point of Y.*

Lemma 2.1.31. *A regular function is continuous when k is identified with $\mathbb{A}^1_h$ in its Zariski topology.*

Proof. It suffices to show that the inverse image of a closed set is closed. In the case of $\mathbb{A}^1_k$, where a closed set consists of a finite number of points, it is enough to prove the closedness of $f^{-1}(a) = \{P \in Y \mid f(P) = a\}$ for any $a \in k$. We can determine whether Z is closed by verifying that Y can be covered by open sets U such that $Z \cap U$ is closed in U for each U. Let U be an open set where f can be expressed as the ratio g/h, with both g and h belonging to A and $h \neq 0$ on U. In this case, we find that $f^{-1}(a) \cap U = \{P \in U \mid g(P)/h(P) = a\}$. However, $g(P)/h(P) = a$ is equivalent to $(g - ah)(P) = 0$. Consequently, we have $f^{-1}(a) \cap U = Z(g - ah) \cap U$, which is closed. □

Now, analogous to Definition 2.1.30, we present the notion of regular function defined on a quasi-projective variety.

Definition 2.1.32. *Let $Y \subseteq \mathbb{P}^n$ be a quasi-projective variety. A function $f : Y \to k$ is said to be regular at a point $P \in Y$ if there exists an open neighborhood U with $P \in U \subseteq Y$ and homogeneous polynomials $g, h \in S = k[x_0, \ldots, x_n]$ of the same degree, such that $f = g/h$ and $h \neq 0$ on U. We called f a regular function on Y if it is regular function at every point $P \in Y$.*

Although g and h are not functions on the projective space $\mathbb{P}^n$, they are both homogeneous polynomials of the same degree, therefore their quotient is a well-defined function whenever h is nonzero.

Furthermore, we have that a regular function is continuous, then if f and g are regular functions on a variety X and they coincide on an open subset $U \subseteq X$ where U is non-empty, then it follows that f and g are equal everywhere. In fact, the set of points where $f - g = 0$ is both closed and dense, so it must be equal to the whole space X.

With this preparation we can now introduce the category of varieties.

Definition 2.1.33. *The set X is called a variety over an algebraically closed field k if it is any of the following: affine, quasi-affine, projective, or quasi-projective variety.*

Definition 2.1.34. *Let X and Y be varieties, a morphism is a continuous map $\varphi : X \to Y$ such that for every open set $V \subseteq Y$ and every regular function $f : V \to k$, the function $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.*

If $\varphi : X \to Y$ and $\psi : Y \to Z$ are morphisms of varieties, then the composition $\psi \circ \varphi$ is a morphism, which establishes the category structure. Moreover, we define an isomorphism $\varphi : X \to Y$ of varieties as a morphism that admits an inverse morphism $\psi : Y \to X$ with $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

We informally say that a variety is affine if it is isomorphic to an affine variety. Next we present some rings of functions associated with a variety.

Definition 2.1.35. *Let Y be a variety. The ring of all regular functions on Y is denoted by $\mathcal{O}(Y)$. The local ring of a point P on Y, denoted as $\mathcal{O}_{P,Y}$ or $\mathcal{O}_P$, is the ring of germs of regular functions on Y that are defined near P. That is, an element of $\mathcal{O}_P$ is a pair $\langle U, f\rangle$, where U is an open subset of Y containing P and f is a regular function on U. Two such pairs $\langle U, f\rangle$ and $\langle V, g\rangle$ are considered to be the same if $f = g$ on $U \cap V$.*

Observe that $\mathcal{O}_P$ is really a local ring with its maximal ideal m defined as the collection of function germs that vanish at the point P. In the case where $f(P)$ is non-zero, we have that $1/f$ is a regular function in a neighborhood of P. Furthermore, the quotient field $\mathcal{O}_P/m$ is isomorphic to k.

Definition 2.1.36. *Let Y be a variety. We denote the function field of Y by $k(Y)$, and it consists of equivalence classes of pairs $\langle U, f\rangle$, where U represents a nonempty open subset of Y, and f is a regular function defined on U. Two pairs, $\langle U', f\rangle$ and $\langle V, g\rangle$, are identified if $f = g$ on $U \cap V$. The elements of $k(Y)$ are rational functions on Y.*

Note that $k(Y)$ is a field. Indeed, we have that Y is irreducible, so the intersection of two nonempty open sets is nonempty. Therefore, we can define addition and multiplication in $k(Y)$, giving it the structure of a ring. If $\langle U, f\rangle \in k(Y)$ with $f \neq 0$, we can consider the restriction of f to the open set $V = U - U \cap Z(f)$ where it never vanishes. Hence $1/f$ is regular on V and $\langle V, 1/f\rangle$ is an inverse for $\langle U, f\rangle$.

Up to this point, we have described the ring of global functions $\mathcal{O}(Y)$, the local ring $\mathcal{O}_P$ at a point P and the function field $k(Y)$ for a variety Y. Considering the restriction of functions, we have natural injective maps $\mathcal{O}(Y) \to \mathcal{O}_P \to k(Y)$. Thus, we usually consider $\mathcal{O}(Y)$ and $\mathcal{O}_P$ as subrings of $k(Y)$.

These rings, $\mathcal{O}(Y)$, $\mathcal{O}_P$, and $k(Y)$, are invariants of the variety Y (and the point P) up to isomorphism since they remain unchanged when we replace Y with an isomorphic variety.

Now, we aim to establish the connections among $\mathcal{O}(Y)$, $\mathcal{O}_P$, $k(Y)$, the affine coordinate ring $A(Y)$ of an affine variety and the homogeneous coordinate ring $S(Y)$ of a projective variety. On further study, we will discover that for an affine variety Y we have $A(Y) = \mathcal{O}(Y)$, therefore the affine coordinate ring is invariant up to isomorphism. On the other hand, the coordinate ring $S(Y)$ of a projective variety Y is not an invariant, since it relies on the embedding of Y in the projective space.

Theorem 2.1.37. *Let $V \subseteq k^n$ be an algebraic variety, $A(V)$ be the coordinate ring of V, $\mathcal{O}(V)$ be the ring of regular functions on V, and $k(V)$ be the field of regular functions on V. Then:*

(1) $\mathcal{O}(V) \simeq A(V)$;

(2) *For each point $P \in V$, let $m_P \subseteq A(V)$ be the ideal of functions vanishing at P. Then the correspondence $P \leftrightarrow m_P$ is a one-to-one correspondence between the points of V and the maximal ideals of $A(V)$;*

(3) *For each point $P \in V$, there exists a ring isomorphism $A(V)_{m_P} \simeq \mathcal{O}_P$ and $\dim \mathcal{O}_P = \dim V$;*

(4) *The fraction field of $A(V)$ is isomorphic to $k(V)$, and consequently, $k(V)$ is a finitely generated extension of k with $tr.deg_k k(V) = \dim V$.*

Let's introduce some notation before stating the next result. Let S be a graded ring and $\mathfrak{p}$ be a homogeneous prime ideal in S. We denote by $S_{(\mathfrak{p})}$ the subring of elements of degree 0 in the localization of S with respect to the multiplicative subset T, which consists of the homogeneous elements of S that are not in $\mathfrak{p}$. $S_{(\mathfrak{p})}$ is a local ring and $(\mathfrak{p} \cdot T^{-1}S) \cap S_{(\mathfrak{p})}$ is its maximal ideal, furthermore $\deg(f/g) = \deg(f) - \deg(g)$ gives a natural grade to $T^{-1}S$, where $f \in S$ homogeneous and $g \in T$. We express the subring consisting of elements with degree 0 in the localized ring S_f as $S_{(f)}$, when $f \in S$ is a homogeneous element.

Theorem 2.1.38. *Consider a projective variety Y contained in $\mathbb{P}^n$ with a homogeneous coordinate ring $S(Y)$. Then:*

(1) $\mathcal{O}(Y) = k$;

(2) *For each any point $P \in Y$, let $m_P \subseteq S(Y)$ be the ideal generated by the set of homogeneous $f \in S(Y)$ vanishing at P. Then $\mathcal{O}_P = S(Y)_{(m_P)}$;*

(3) $K(Y) \cong S(Y)_{((0))}$.

Proposition 2.1.39. *Let X be a variety and Y be an affine variety. Then there exists a natural bijective mapping of sets*

$$\alpha : \mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(A(Y), \mathcal{O}(X)),$$

where $\mathrm{Hom}(X,Y)$ *is the set of morphisms between varieties and* $\mathrm{Hom}(A(Y), \mathcal{O}(X))$ *the set of homomorphisms of k-algebras.*

From this proposition, we immediately obtain the following corollary:

Corollary 2.1.40. *Given two affine varieties X and Y, we have the following equivalence: X and Y are isomorphic if and only if their respective coordinate rings $A(X)$ and $A(Y)$ are isomorphic as k-algebras.*

In categorical language, we can rephrase this result as:

Corollary 2.1.41. *The functor $X \mapsto A(X)$ induces a contravariant equivalence of categories between the category of affine varieties over k and the category of finitely generated integral domains over k.*

2.1.4 Rational maps

In this section, we define the concepts of rational maps and birational equivalence, which play a crucial role in the classification of varieties. The notion of a rational map captures the idea of a well-defined mapping between varieties even when it may not be defined on the entire domain. It allows us to study partial correspondences between varieties and is particularly useful in situations where a regular map may not be defined everywhere or may not be well-behaved.

Lemma 2.1.42. *Let φ and ψ be morphisms between varieties X and Y. If there exists a nonempty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$, then $\varphi = \psi$.*

Definition 2.1.43. *Let X and Y be varieties. A rational map $\varphi : X \to Y$ is an equivalence class of pairs $\langle U', \varphi_{U'} \rangle$, where U' is a nonempty open subset of X and $\varphi_{U'}$ is a morphism from U' to Y. Two pairs $\langle U, \varphi_U \rangle$ and $\langle V, \varphi_V \rangle$ are considered equivalent if φ_U and φ_V agree on $U \cap V$. The rational map φ is said to be dominant if, for some (and hence every) pair $\langle U', \varphi_{U'} \rangle$, the image of $\varphi_{U'}$ is dense in Y.*

Note that if the image of φ_U is dense in Y for some pair $\langle U, \varphi_U \rangle$, then it is dense for every pair. Also observe that a rational map $\varphi : X \to Y$ is not generally a map from the set X to Y. Furthermore, it is possible to compose dominant rational maps, allowing us to consider the category of varieties and dominant rational maps. A birational map is an "isomorphism" in this category:

Definition 2.1.44. *A birational map $\varphi : X \to Y$ is a rational map that admits an inverse, namely a rational map $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$ as rational maps. We say that X and Y are birationally equivalent or birational if there exists a birational map from X to Y,.*

In this section, one of the main conclusions is that the category of varieties and dominant rational maps is in bijective correspondence with the category of finitely generated field extensions of k, with the arrows reversed. Before stating this result, we present some results.

Lemma 2.1.45. *Let Y be a hypersurface in $\mathbf{A}^n$ given by the equation $f(x_1, \ldots, x_n) = 0$. Then $\mathbf{A}^n - Y$ is isomorphic to the hypersurface H in $\mathbf{A}^{n+1}$ defined by $x_{n+1} f = 1$. In particular, $\mathbf{A}^n - Y$ is affine and its affine ring is $k[x_1, \ldots, x_n]_f$.*

Proposition 2.1.46. *Every variety Y has a base for its topology consisting of open affine subsets.*

Now we arrive at the key result of this section. Consider a dominant rational map $\varphi : X \to Y$, represented by the pair $\langle U, \varphi_U \rangle$. Let $f \in k(Y)$ be a rational function represented by $\langle V, f \rangle$, where V is an open set in Y and f is a regular function on V. Since $\varphi_U(U)$ is dense in Y, the preimage $\varphi_U^{-1}(V)$ is a nonempty open subset of X. Thus, $f \circ \varphi_U$ is a regular function on $\varphi_U^{-1}(V)$. This defines a rational function on X and establishes a homomorphism of k-algebras from $k(Y)$ to $k(X)$.

Theorem 2.1.47. *Let X and Y be varieties, then there exists a bijection between the set of dominant rational maps from X to Y and the set of k-algebra homomorphisms from $k(Y)$ to $k(X)$. Moreover, this correspondence yields a reverse arrow equivalence between the category of varieties and dominant rational maps, and the category of finitely generated field extensions of k.*

From this theorem we get the following corollary:

Corollary 2.1.48. *Let X and Y be varieties, the following statements are equivalent:*

(1) X and Y are birationally equivalent;

(2) There exist open subsets $U \subseteq X$ and $V \subseteq Y$ such that U is isomorphic to V;

(3) The k-algebras $k(X)$ and $k(Y)$ are isomorphic.

Proposition 2.1.49. *Every variety X with dimension r is birational to a hypersurface $Y \in \mathbb{P}^{r+1}$.*

We end the study of rational maps by presenting the concept of blowing-up, which is a fundamental tool in algebraic geometry, playing a crucial role in the study of singularities and birational geometry. It provide a way to resolve singularities and refine the geometry of a variety by introducing new geometric objects called exceptional divisors. The main idea behind blowing-ups is to replace a singular point or a subvariety with a new space that captures the local behavior of the variety. This new space is constructed in such a way that it retains the essential information about the original variety while resolving its singularities.

Proposition 2.1.50. *The blowing-up of the point $O = (0, \dots, 0)$ in $\mathbb{A}^n$ can be constructed as a closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = x_j y_i \mid i, j = 1, \dots, n\}$. The blowing-up is a birational map from X to $\mathbb{A}^n$.*

The construction of the blowing-up involves considering the product space $\mathbb{A}^n \times \mathbb{P}^{n-1}$, where the closed subsets are defined by homogeneous polynomials in the affine coordinates x_i of $\mathbb{A}^n$ and the homogeneous coordinates y_j of $\mathbb{P}^{n-1}$. By restricting the projection map from $\mathbb{A}^n \times \mathbb{P}^{n-1}$ to $\mathbb{A}^n$, we obtain the blowing-up map $\varphi : X \to \mathbb{A}^n$.

The following diagram represents those maps:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{A}^n \times \mathbb{P}^{n-1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

Proposition 2.1.51. *The properties of the blowing-up are as follows:*

(1) If $P \in \mathbb{A}^n$ and $P \neq O$, then $\varphi^{-1}(P)$ consists of a single point. In fact, φ gives an isomorphism between $X - \varphi^{-1}(O)$ and $\mathbb{A}^n - O$. Moreover, for $P \in \mathbb{A}^n - O$, there exists an inverse morphism $\psi(P) = (a_1, \ldots, a_n) \times (a_1, \ldots, a_n)$ that shows the isomorphism;

(2) The preimage $\varphi^{-1}(O)$ is isomorphic to $\mathbb{P}^{n-1}$;

(3) The points of $\varphi^{-1}(O)$ correspond to the lines passing through O in $\mathbb{A}^n$;

(4) X is an irreducible variety.

The blowing-up can also be performed on a closed subvariety Y of $\mathbb{A}^n$ by considering the inverse image $\tilde{Y}$ of Y under the blowing-up map $\varphi : X \to \mathbb{A}^n$. The blowing-up process "pulls apart" Y near O according to the different directions of lines through O.

An example is the blowing-up of a plane cubic curve Y in $\mathbb{A}^2$ defined by the equation $y^2 = x^2(x+1)$:

Example 2.1.52. *Consider the example of the plane cubic curve Y given by the equation $y^2 = x^2(x+1)$. We will perform the blowing-up of Y at the point O (see Figure 2 below).*

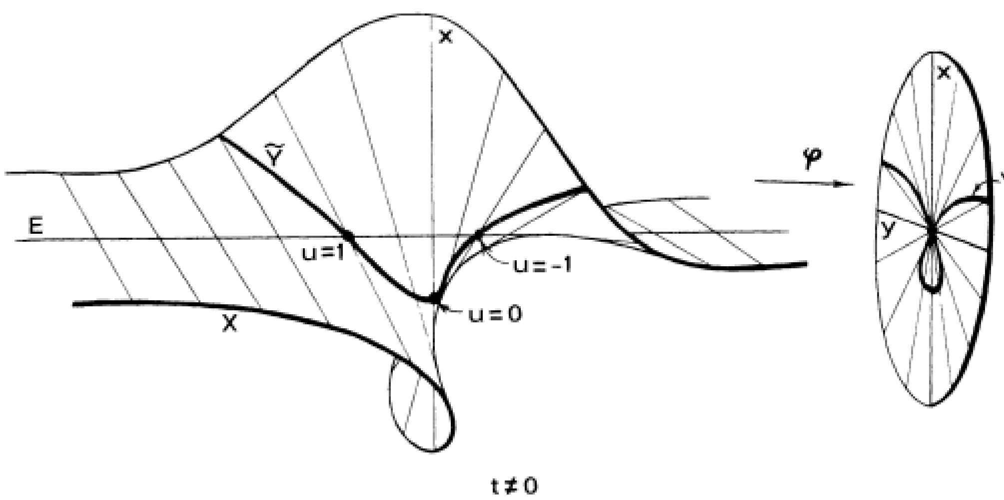

Figure 2 – Blow-up of a cubic curve. From (HARTSHORNE, 1977).

To do this, we introduce homogeneous coordinates t and u for $\mathbb{P}^1$. The blowing-up X of $\mathbb{A}^2$ at O is defined by the equation $xu = ty$ inside $\mathbb{A}^2 \times \mathbb{P}^1$. Geometrically, X appears similar to $\mathbb{A}^2$, but with the point O replaced by a $\mathbb{P}^1$ that represents the slopes of lines passing through O. We will refer to this $\mathbb{P}^1$ as the exceptional curve and denote it as E.

To obtain the complete inverse image of Y in X, we consider the equations $y^2 = x^2(x+1)$ and $xu = ty$ in $\mathbb{A}^2 \times \mathbb{P}^1$. The $\mathbb{P}^1$ space is divided into open sets $t \neq 0$ and $u \neq 0$, which we analyze separately. Assuming $t \neq 0$, we set $t = 1$ and treat u as an affine parameter. This leads us to the following equations:

$$y^2 = x^2(x+1)$$
$$y = xu$$

in $\mathbb{A}^3$ with coordinates x, y, u. By substituting, we obtain $x^2u^2 - x^2(x+1) = 0$, which can be factored. Hence, we have two irreducible components: one defined by $x = 0$, $y = 0$, and any value of u, denoted as E, and the other defined by $u^2 = x + 1$ and $y = xu$, representing $\tilde{Y}$. Notably, $\tilde{Y}$ intersects E at the points $u = \pm 1$, which correspond to the slopes of the two branches of Y at O.

Analogously, we can verify that the complete inverse image of the x-axis consists of E and another irreducible curve known as the strict transform of the x-axis (previously described as $\bar{L}'$ corresponding to the line $L = x$-axis). The strict transform intersects E at $u = 0$. By considering the complementary open set $u \neq 0$ in $\mathbb{A}^2 \times \mathbb{P}^1$, we observe that the strict transform of the y-axis intersects E at the point $t = 0$ and $u = 1$.

These findings are summarized in Figure 2. The blowing-up operation effectively separates the branches of curves passing through O based on their slopes. If the slopes differ, their strict transforms no longer intersect in X. Instead, they meet E at distinct points corresponding to their respective slopes.

2.1.5 Grassmanians

Let us now shift our focus from the general theory to the construction of a fascinating and valuable class of examples: Grassmannians. For this study we use the references (GATHMANN, 2022) and (GORODENTSEV, 2017).

The motivation behind this construction is quite straightforward. We have already seen the definition of projective spaces as the collection of 1-dimensional linear subspaces of K^n, and this concept has proven to be incredibly useful. Now, we aim to generalize this idea further by considering sets of k-dimensional linear subspaces of K^n, where k can range from 0 to n.

Definition 2.1.53. *Let $n \in \mathbb{N}_{>0}$ and $k \in \mathbb{N}$ with $0 \leqslant k \leqslant n$. We denote by $G(k, n)$ the set of all k-dimensional linear subspaces of K^n. This set is called the Grassmannian of k-planes in K^n.*

It is worth noting that k-dimensional linear subspaces of K^n (for $k > 0$) are naturally in bijection with $(k-1)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$. Consequently, we can alternatively view $G(k,n)$ as the set of such projective linear subspaces. As a result, our Grassmannian $G(k,n)$ from Definition 2.1.67 is sometimes denoted as $\mathbb{G}(k-1,n-1)$ in the literature, as the dimensions k and n are reduced by 1 in this alternative notation.

Our goal is to turn the Grassmannian $G(k,n)$ into a variety. In fact, we will show that it can be naturally equipped with a projective variety structure. To achieve this, we will introduce the algebraic concept of alternating tensor products, which is a slight variation of the ordinary tensor products commonly used in commutative algebra. Let us provide a brief introduction to alternating tensor products.

Definition 2.1.54. *Let V be a vector space over K, and let $k \in \mathbb{N}$. A k-fold multilinear map $f : V^k \to W$ to another vector space W is called alternating if $f(v_1, \ldots, v_k) = 0$ for all $v_1, \ldots, v_k \in V$ such that $v_i = v_j$ for some $i \neq j$.*

Remark 2.1.55. *It is important to note that for an alternating multilinear map $f : V^k \to W$ and $v_1, \ldots, v_k \in V$, if we exchange two arguments of f, the result is multiplied by -1. This property can be extended to any permutation of the arguments, and we obtain the following result:*

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sign}(\sigma) \cdot f(v_1, \ldots, v_k),$$

where σ is a permutation in the symmetric group S_k and $\operatorname{sign}(\sigma)$ denotes the sign of the permutation.

This concept of alternating multilinear maps will play a crucial role in the construction of Grassmannians.

Definition 2.1.56. *Let V be a vector space over K and $k \in \mathbb{N}$. An alternating tensor product of V is a vector space T equipped with an alternating k-fold multilinear map $\tau : V^k \to T$ that satisfies the following universal property: for any k-fold alternating multilinear map $f : V^k \to W$ to another vector space W, there exists a unique linear map $g : T \to W$ such that $f = g \circ \tau$, i. e., such that the diagram below commutes.*

$$\begin{tikzcd}
V^k \arrow[r] \arrow[d] & W \\
T \arrow[ur, dashed] &
\end{tikzcd}$$

Proposition 2.1.57. *For any vector space V and any $k \in \mathbb{N}$, there is a k-fold alternating tensor product $\tau : V^k \to T$ as in Definition 2.1.56, and it is unique up to unique isomorphism. We will write T as $\Lambda^k V$ and $\tau(v_1, \ldots, v_k)$ as $v_1 \wedge \cdots \wedge v_k \in \Lambda^k V$ for all $v_1, \ldots, v_k \in V$.*

Alternating tensor products provide an elegant framework for capturing the linear dependence and linear spans of vectors.

Lemma 2.1.58. *Let $v_1, \ldots, v_k \in K^n$ for some $k \leqslant n$. Then $v_1 \wedge \cdots \wedge v_k = 0$ if and only if $v_1, \ldots, v_k$ are linearly dependent.*

Lemma 2.1.59. *Let $v_1, \ldots, v_k \in K^n$ and $w_1, \ldots, w_k \in K^n$ both be linearly independent. Then the alternating tensor products $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are linearly dependent in $\Lambda^k K^n$ if and only if* $\operatorname{Lin}(v_1, \ldots, v_k) = \operatorname{Lin}(w_1, \ldots, w_k)$

We can now use our results to realize the Grassmannian $G(k, n)$ as a subset of a projective space.

Construction 2.1.60 (Plücker embedding)**.** . *Consider the map $f : G(k, n) \to \mathbb{P}^{\binom{n}{k}-1}$ defined as follows: for a linear subspace* $\operatorname{Lin}(v_1, \ldots, v_k) \in G(k, n)$*, we send it to the class of the alternating tensor $v_1 \wedge \cdots \wedge v_k \in \Lambda^k K^n \cong K^{\binom{n}{k}}$ in $\mathbb{P}^{\binom{n}{k}-1}$.*

This map is well-defined: $v_1 \wedge \cdots \wedge v_k$ is non-zero (Lemma 2.1.58), and changing the basis of the same subspace does not alter the resulting point in $\mathbb{P}^{\binom{n}{k}-1}$ (Lemma 2.1.59). Furthermore, the map f is injective (Lemma 2.1.59). We refer to this map as the Plücker embedding of $G(k, n)$. For a k-dimensional linear subspace $L \in G(k, n)$, the homogeneous coordinates of $f(L)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are known as the Plücker coordinates of L. These coordinates correspond to the maximal minors of the matrix whose rows are $v_1, \ldots, v_k$.

We have successfully embedded the Grassmannian $G(k, n)$ into a projective space. However, we still need to show that it is a closed subset, or in other words, a projective variety. By construction, $G(k, n)$ consists of the classes in $\mathbb{P}^{\binom{n}{k}-1}$ that correspond to non-zero alternating tensors in $\Lambda^k K^n$ and can be expressed as "pure tensors," i.e., of the form $v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in K^n$—not just as linear combinations of such expressions. Therefore, we need to find suitable equations that describe these pure tensors in $\Lambda^k K^n$. The key lemma that will help us achieve this is the following.

Lemma 2.1.61. *For a fixed non-zero $\omega \in \Lambda^k K^n$ with $k < n$ consider the K-linear map*

$$f : K^n \to \Lambda^{k+1} K^n, v \mapsto v \wedge \omega.$$

Then $\operatorname{rk} f \geqslant n - k$*, with equality holding if and only if $\omega = v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in K^n$.*

Example 2.1.62. *Let $k = 2$ and $n = 4$.*

(1) For $\omega = e_1 \wedge e_2$ the map f of Lemma 2.1.61 is given by

$$\begin{aligned} f(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) &= (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge e_1 \wedge e_2 \\ &= a_3e_1 \wedge e_2 \wedge e_3 + a_4e_1 \wedge e_2 \wedge e_4 \end{aligned}$$

for $a_1, a_2, a_3, a_4 \in K$, and thus has rank $\operatorname{rk} f = 2 = n - k$ *in accordance with the statement of the lemma.*

(2) For $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$ we get

$$
\begin{aligned}
&f(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4)\\
&\quad = (a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)\\
&\quad = a_1e_1 \wedge e_3 \wedge e_4 + a_2e_2 \wedge e_3 \wedge e_4 + a_3e_1 \wedge e_2 \wedge e_3 + a_4e_1 \wedge e_2 \wedge e_4
\end{aligned}
$$

instead, so that $\operatorname{rk} f = 4$. *Hence Lemma 2.1.61 tells us that there is no way to write ω as a pure tensor $v_1 \wedge v_2$ for some vectors $v_1, v_2 \in K^4$.*

The following corollary presents $G(k, n)$ as a projective variety.

Corollary 2.1.63. *The Plücker embedding ensures that the Grassmannian $G(k, n)$ is a closed subset of $\mathbb{P}^{\binom{n}{k}-1}$. Therefore, it can be regarded as a projective variety.*

Proof. Considering $G(n, n)$, which is a single point and therefore a variety, we can focus on the case where $k < n$. In this case, a point $\omega \in \mathbb{P}^{\binom{n}{k}-1}$ belongs to $G(k, n)$ if and only if it represents a pure tensor $v_1 \wedge \cdots \wedge v_k$. Lemma 2.1.61 establishes that this condition holds if and only if the rank of the linear map $f : K^n \to \Lambda^{k+1}K^n, v \mapsto v \wedge \omega$ is $n - k$. Since we know that the rank of this map is always at least $n - k$, we can verify this condition by checking the vanishing of all $(n - k + 1) \times (n - k + 1)$ minors of the matrix corresponding to f. These minors are polynomials in the matrix entries and, consequently, in the coordinates of ω. Thus, we conclude that the condition for ω to belong to $G(k, n)$ defines a closed subset. □

Example 2.1.64. *The proof of Corollary 2.1.63 shows that the Grassmannian $G(2, 4)$ is defined as the set of points in $\mathbb{P}^5$ where all sixteen 3×3 minors of a 4×4 matrix corresponding to a linear map $K^4 \to \Lambda^3K^4$ vanish. In other words, $G(2, 4)$ is represented by a system of sixteen cubic equations in $\mathbb{P}^5$.*

Surprisingly, a single quadratic equation is sufficient to define the Grassmannian $G(2, 4)$ in $\mathbb{P}^5$. The condition $\omega \wedge \omega = 0$ for $\omega = \sum_{i,j} a_{ij}e_ie_j$ is expressed by the quadratic equation $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$. More details about the construction of this quadratic equation can be found in Chapter 4 of (GORODENTSEV, 2017).

Our proof of Corollary 2.1.63 is just the easiest way to show that $G(k, n)$ is a variety, however this is not the most efficient way to describe $G(2, 4)$, as it does not provide a nice representation of the variety.

Fortunately, there is an alternative approach to describe the Grassmannian using affine patches, which offers a more convenient perspective. This description allows us to easily determine the dimension of $G(k, n)$, which is a challenging task when relying solely on the equations as shown in Corollary 2.1.63.

Construction 2.1.65. *[Affine cover of the Grassmannian]. Consider the affine open subset* $U_0 \subset G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ *where the* $e_1 \wedge \cdots \wedge e_k$*-coordinate is non-zero. We have that a linear subspace* $L \in G(k,n)$ *lies in* U_0 *if and only if it can be expressed as the row span of a* $k \times n$ *matrix of the form* $(A \mid B)$*, where* A *is an invertible* $k \times k$ *matrix and* B *is an arbitrary* $k \times (n-k)$ *matrix.*

By multiplying such a matrix by A^{-1} *from the left, which preserves the row span, we can see that* U_0 *is the image of the map*

$$f : \mathbb{A}^{k(n-k)} = \mathrm{Mat}(k \times (n-k), K) \to U_0,$$
$$C \mapsto \text{the row span of } (E_k \mid C),$$

where $C = A^{-1}B$ *in the above notation. Since different matrices* C *lead to distinct row spans of* $(E_k \mid C)$*, we can conclude that* f *is a bijective map. Furthermore, since the maximal minors of* $(E_k \mid C)$ *are polynomial functions in the entries of* C*, we can see that* f *is actually a morphism.*

Conversely, we can reconstruct the (i,j)*-entry of* C *up to sign from* $f(C)$ *by considering the maximal minor of* $(E_k \mid C)$ *formed by all columns of* E_k *except the* i*-th column, along with the* j*-th column of* C*. This implies that* f^{-1} *is also a morphism, demonstrating that* f *is an isomorphism.*

In other words, we have established that $U_0 \cong \mathbb{A}^{k(n-k)}$ is isomorphic to an affine space, not just an affine variety. By a similar argument, the same holds for all other affine patches where one of the Plücker coordinates is non-zero. Therefore, we can conclude that $G(k,n)$ can be covered by such affine spaces. As a result, we have the following consequence:

Corollary 2.1.66. $G(k,n)$ *is an irreducible variety of dimension* $k(n-k)$*.*

In the next example, we will make use of the concept of open cells. To provide a brief reminder of the definition: An n-dimensional open cell, typically denoted as "e^n," is a topological space that is homeomorphic to the open n-dimensional ball in Euclidean space. In simpler terms, it is a space that looks like an open ball in n-dimensional Euclidean space.

Example 2.1.67. *Let* $p \in P$ *be a point in the Plücker quadric* $P = \{\omega \in \Lambda^2 V \mid \omega \wedge \omega = 0\} \subset \mathbb{P}^5$ *and* $H \simeq \mathbb{P}^3$ *a hyperplane complementary to* p *in the tangent plane* $T_pP \simeq \mathbb{P}^4$*, i.e.,* $p \notin H$ *and* p *and the hyperplane* H *together provide a complete set of basis elements for the tangent space at that point . Then the intersection* $C = P \cap T_pP$ *is a simple cone with vertex at* p *over a smooth quadric* $G = H \cap P$ *that can be regarded as the Segre quadric in* $\mathbb{P}^3 = H$*.*

Now if we choose a point $p' \in G$ *and write* π_α*,* π_β *for planes spanned by* p *and two lines on* G *passing through* p'*. Linked with this information, we can establish a stratification of the Plücker quadric represented by closed subvarieties, as illustrated in the Figure 3.*

First, we have $\{p\} \simeq \mathbb{A}^0$*, then we have the projective line (intersection of* π_α *and* π_β*) without* p*:* $(\pi_\alpha \cap \pi_\beta) \backslash p \simeq \mathbb{A}^1$*. Next, we consider the pair of projective spaces without this*

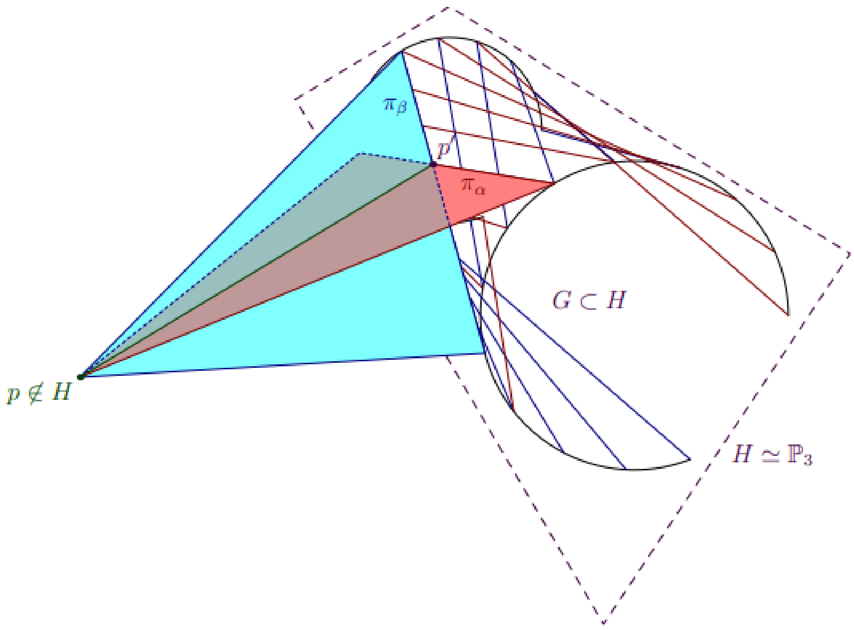

Figure 3 – The cone $C = P \cap T_pP \subset \mathbb{P}^4 = T_pP$. From (GORODENTSEV, 2017).

line: $\pi_\alpha \backslash (\pi_\alpha \cap \pi_\beta) \simeq \pi_\beta \backslash (\pi_\alpha \cap \pi_\beta) \simeq \mathbb{A}^2$. *Also, we have the cone* C *over* G *without the planes* π_α *and* π_β*:* $C \backslash (\pi_\alpha \cup \pi_\beta) \simeq \mathbb{A}^1 \times (G \backslash (G \cap T_{p'}G))$. *Finally, there are natural identifications* $G \backslash (G \cap T_{p'}G) \simeq \mathbb{A}^2$ *and* $P \backslash C \simeq \mathbb{A}^4$.

So we can write $\mathrm{Gr}(2,4)$ *as a disjoint union of open cells isomorphic to affine spaces:*

$$\mathrm{Gr}(2,4) = \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \cup \mathbb{A}^3 \sqcup \mathbb{A}^4.$$

More details about this example can be founded in Chapter 5 of (GORODENTSEV, 2017).

2.2 Schemes

The theory of schemes is a fundamental framework in modern algebraic geometry that provides a powerful and flexible language for studying geometric objects defined by polynomial equations. Unlike classical algebraic varieties, which are restricted to being defined by sets of zeros of polynomials over an algebraically closed field, schemes allow for more general geometric objects that can be defined over any commutative ring. The key idea behind schemes is to associate a commutative ring to each open subset of a given space, such that these rings encode geometric information about the space. Instead of working solely with the points of a variety, schemes incorporate the structure of the underlying ring into the geometric study. This allows for a deeper understanding of geometric phenomena, such as singularities and intersections, by examining the algebraic properties of the associated rings. The main reference used to construct this section was chapter II of (HARTSHORNE, 1977) and as supplementary material, we relied on the references (VAKIL, 2017) and (SHAFAREVICH; REID, 1994).

2.2.1 Presheaves and sheaves

We will define the notion of a presheaf and a sheaf on a topological space. The concept of sheaves allows us to study local information in a topological space, particularly in an algebraic variety. Let's start with the definition of a presheaf.

Definition 2.2.1. *Let X be a topological space, and $Top(X)$ be the category whose objects are the open sets of X and morphisms are the inclusion maps. A presheaf $\mathscr{F}$ of abelian groups on X is a contravariant functor from the category $Top(X)$ to the category of abelian groups.*

Similarly, we can define presheaves of commutative rings or any other category.

Definition 2.2.2. *A sheaf of abelian groups (or rings, or valued in objects of a certain category) is a presheaf that satisfies the following additional conditions:*

(i) If $U \subset X$ is an open set, V_i is an open covering of U, and there exists $s \in \mathscr{F}(U)$ such that $s\big|_{V_i} = 0$ for all i, then $s = 0$ (the locality property).

(ii) Let $U \subset X$ be an open set, V_i be an open covering of U, and suppose we have elements $s_i \in \mathscr{F}(V_i)$ such that for all i and j, $s_i\big|_{V_i \cap V_j} = s_j\big|_{V_i \cap V_j}$. Then there exists an element $s \in \mathscr{F}(U)$ such that $s\big|_{V_i} = s_i$ for each i (the gluing property).

To fix the terminology, if $\mathscr{F}$ is a presheaf (or a sheaf) of abelian groups on X, the elements of $\mathscr{F}(U)$ are called local sections over U. If $V \subset U$, then the induced morphism from $\mathscr{F}(U)$ to $\mathscr{F}(V)$ is called the restriction map, and we denote the image of a section s from $\mathscr{F}(U)$ to $\mathscr{F}(V)$ as $s\big|_V$.

Example 2.2.3. *The canonical example of a sheaf is when X is an algebraic variety over an algebraically closed field k, and for each open set $U \subset X$, we associate the ring of regular functions on U. Additionally, we can define sheaves of continuous, differentiable, or holomorphic functions over a topological, smooth, or complex manifold, respectively.*

Example 2.2.4. *Let's consider an example of a presheaf that is not a sheaf. Let $X = \mathbb{R}$, and consider the open cover $U_i \ : \ |x| < i$. Let $\mathscr{F}$ be the presheaf of continuous and bounded functions on X.*

Suppose that $\mathscr{F}$ is a sheaf. Since the section $f_i(x) = x$ is bounded on each U_i, the function $f(x) = x$ should be bounded on $\mathbb{R}$, which is not the case.

The following result was proved by Grothendieck and gives us a criterion for when a presheaf is a sheaf.

Theorem 2.2.5. *Every representable presheaf on an algebraic variety is a sheaf.*

The proof of the above theorem can be found in (VISTOLI, 2005).

When defining a new structure, some natural questions arise. For example, what are the smaller structures found within these objects? Can we relate two different elements of our structure? We will explore the answers to these questions.

Definition 2.2.6. *A sub-sheaf of a sheaf $\mathscr{F}$ is a sheaf $\mathscr{F}'$ such that for every open subset $U \subset X$, $\mathscr{F}'(U)$ is a subgroup (subring) of $\mathscr{F}(U)$ and the restriction maps of the sheaf $\mathscr{F}'$ are induced by those of $\mathscr{F}$.*

Definition 2.2.7. *Given two sheaves $\mathscr{F}$ and $\mathscr{G}$ over the same topological space, a morphism of sheaves between $\mathscr{F}$ and $\mathscr{G}$ is a functor morphism between the presheaves.*

Given a morphism of sheaves $\varphi \colon \mathscr{F} \to \mathscr{G}$ over X, we can associate a presheaf related to the kernel of the morphism, where for each open subset $U \subset X$, we associate the group $\ker \varphi(U)$. Similarly, we can construct the presheaves that associate each open subset with the groups $\operatorname{coker} \varphi(U)$ and $\operatorname{im} \varphi(U)$. The presheaf ker is actually a sheaf, but the presheaves image and cokernel are sheaves.

Since not every presheaf is a sheaf, a natural question is how we can associate a presheaf with a sheaf. For this, we need the notion of the stalk of a sheaf at a point in the topological space where the sheaf is defined, which will give us algebraic information in the neighborhoods of the point.

Definition 2.2.8. *If $\mathscr{F}$ is a presheaf on X and P is a point of X, the stalk of $\mathscr{F}$ at P, denoted by $\mathscr{F}_P$, is the set of all ordered pairs (U, s), equipped with an equivalence relation, where $(U, s) \sim (V, t)$ if and only if there exists an open subset W with $P \in W \subset U \cap V$ such that $s\big|_W = t\big|_W$.*

The following theorem tells us about the local nature of the stalk concept and is an important technical result. The proof can be found in II.1.1 of the reference (HARTSHORNE, 1977).

Theorem 2.2.9. *A morphism of sheaves $\varphi \colon \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if, for every $P \in X$, the induced map $\varphi_P \colon \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism.*

Now we have the necessary tools to associate a sheaf with a presheaf.

Theorem 2.2.10. *Given a presheaf $\mathscr{F}$, there exists a sheaf $\mathscr{F}^+$ with the property that for any sheaf $\mathscr{G}$ and any morphism $\varphi \colon \mathscr{F} \to \mathscr{G}$, there exists a unique morphism $\psi \colon \mathscr{F}^+ \to \mathscr{G}$ such that $\varphi = \psi \circ \theta$. Moreover, $(\mathscr{F}^+, \theta)$ is unique up to isomorphisms and is called the sheaf associated to the presheaf $\mathscr{F}$.*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d, "\theta"'] & \mathscr{G} \\ \mathscr{F}^+ \arrow[ur, "\psi"'] \arrow[ur, phantom, shift left=3, "\circlearrowleft"] & \end{tikzcd}$$

The proof of the above theorem can be found in (HARTSHORNE, 1977). Therefore, we can define the sheaf image and cokernel as the sheafification of the presheaf image and cokernel, respectively.

2.2.2 Ringed spaces and schemes

We will introduce the concept of schemes, beginning with the study of affine schemes and then extending to projective schemes. This construction is inspired by the idea of algebraic and projective varieties, which are defined as the solution sets of polynomials and regular functions on them, respectively. Ultimately, we will discover that a scheme provides a natural generalization of the notion of an algebraic variety.

Let A be a commutative ring with unity. We define the set $Spec(A)$ as the set of all prime ideals of A. For an ideal $I \subset A$, we define the subset $V(I) \subset Spec(A)$ as the set of prime ideals of A that contain the ideal I.

Lemma 2.2.11. *(i) If I and J are ideals of A, then $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.*

(ii) If I_α with $\alpha \in \Lambda$ are ideals of A, then $V\left(\sum I_\alpha\right) = \cap V(I_\alpha)$.

(iii) Let I, J be ideals of A. Then $V(I) \subset V(J)$ if and only if $\sqrt{J} \subset \sqrt{I}$.

Proof. The proof of this lemma can be found in (HARTSHORNE, 1977). □

Note that $V(A) = \varnothing$ and $V(0) = Spec(A)$. Therefore, we can define a topology on $Spec(A)$ by taking the sets $V(I)$ as the closed sets that define this topology, known as the Zariski Topology.

A basis of open sets for this topology can be given by

$$D(f) := \{p \in Spec(A) \ : \ f \notin p\} = Spec(A_f).$$

With that in mind, let's explore a way to define a sheaf on $Spec(A)$. Our goal is for the stalks of this sheaf to satisfy the properties of regular functions on varieties, as we defined them in the classical sense.

Definition 2.2.12. *Let $U \subset Spec(A)$ be an open set. We define $\mathcal{O}(U)$ as the set of functions*

$$s : U \to \bigsqcup_{p \in U} A_p$$

such that for every p in U, $s(p)$ is in A_p. Furthermore, s is locally a quotient of elements of A, i.e., for each $p \in U$, there exists an open neighborhood V contained in U and elements $a, f \in A$ such that for every $q \in V$, $f \notin q$ and $s(q) = a/f$ in A_p.

This definition is similar to the definition of the sheafification of a presheaf, and therefore, the proof that $\mathcal{O}$ is a sheaf is similar to the proof that the sheafification of a presheaf is a sheaf.

Definition 2.2.13. *A germ of a regular function at $p \in Spec(A)$ is an equivalence class of pairs (U, s) such that if U and U' are open sets in $Spec(A)$ and $s \in \mathcal{O}(U)$ and $s' \in \mathcal{O}(U')$, then $(U, s) \sim (U', s')$ if and only if there exists $V \subset U \cap U'$ such that s and s' coincide on V.*

From this, it follows that the stalk $\mathcal{O}_p \simeq A_p$.

We observe that the sheaf $\mathcal{O}$ satisfies the following properties:

Proposition 2.2.14. *Let A be a ring, and $(Spec(A), \mathcal{O})$ be its spectrum. Then,*

(i) For every $p \in Spec(A)$, we have $\mathcal{O}_p \simeq A_p$;

(ii) If $f \in A$, then $\mathcal{O}(D(f)) \simeq A_f$, and in particular, $\mathcal{O}(Spec(A)) \simeq A$.

The proof of this proposition can be found in (HARTSHORNE, 1977).

With these definitions, we associate each commutative ring with unity with a pair $(Spec(A), \mathcal{O})$, consisting of a topological space and a sheaf, which we call the *Spectrum*.

We would like this correspondence to be functorial, but for that, we need a category whose objects are topological spaces with sheaves of rings on them.

Definition 2.2.15. *A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf $\mathcal{O}_X$ on X.*

Now we desire a morphism between ringed spaces, for which we need the notion of direct image sheaf, defined as follows.

Definition 2.2.16. *Let X, Y be topological spaces, and let $f : X \to Y$ be a continuous function. For any sheaf $\mathcal{F}$ on X, we define the direct image sheaf $f_{\mathcal{F}}$ on Y as $f_{\mathcal{F}}(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subset Y$.*

Definition 2.2.17. *A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ consisting of a continuous map $f : X \to Y$ and $f^{\#} : \mathcal{O}Y \to f_*\mathcal{O}_X$ a morphism of sheaves of rings over Y.*

However, for our studies, the category of ringed spaces is too broad, so we need the following notion:

Definition 2.2.18. *A ringed space is a locally ringed space if $\mathcal{O}_{X,p}$ is a local ring for every $p \in X$.*

Note that from the previous proposition, if A is a local ring, then $(\mathrm{Spec}(A), \mathcal{O})$ is a locally ringed space. Furthermore, let's recall that if $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings and $\varphi : A \to B$ is a ring homomorphism, we say that φ is a local homomorphism when $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

A morphism of locally ringed spaces is a morphism $(f, f^{\#})$ of ringed spaces such that, for each point $p \in X$, the induced map of local rings $f_p^{\#} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism. In other words, for each point $p \in X$, the morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f\mathcal{O}_X$ induces a ring homomorphism $\mathcal{O}_Y(V) \to f\mathcal{O}_X(V)$ for every $V \subset Y$. Let $\{V\}$ be the collection of open neighborhoods of $f(p)$, then $f^{-1}(\{V\})$ forms a collection of open neighborhoods of p. Taking the direct limit, we obtain $\mathcal{O}_{Y,f(p)} = \varinjlim_V \mathcal{O}_Y(V) \to \varinjlim_V \mathcal{O}_X(f^{-1}(V))$. Note that the latter limit applies to the stalk $\mathcal{O}_{X,p}$. This gives us an induced map $f_p^{\#} : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$.

An interesting point to note is that we use the pair $(f, f^{\#})$ to denote a morphism between ringed spaces, as the definition does not impose a relationship between f and $f^{\#}$. However, given a ring homomorphism $\varphi : A \to B$, there is a natural way to induce a morphism of locally ringed spaces. For this, we define

$$\begin{array}{rccc} f\colon & Spec(B) & \to & Spec(A) \\ & p & \mapsto & \varphi^{-1}(p) \end{array}$$

The continuity of f follows from the fact that $f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a}))$. If $V \subset \mathrm{Spec}(A)$, we have a ring homomorphism $f^{\#} : \mathcal{O}\mathrm{Spec}(A)(V) \to \mathcal{O}\mathrm{Spec}(B)(f^{-1}(V))$. By the definition of $\mathcal{O}$ and composing the maps f and φ_p, we obtain a sheaf morphism $f^{\#} : \mathcal{O}\mathrm{Spec}(A) \to f_*(\mathcal{O}_{\mathrm{Spec}(B)})$. Notice that the induced maps on the stalks are local homomorphisms, and therefore $(f, f^{\#})$ is a morphism of locally ringed spaces.

So in the first section of this chapter, we established an equivalence duality between affine varieties and regular maps to finitely generated k-algebras with no nilpotents. Interestingly, the restriction of working only with finitely generated algebras over a field without nilpotents can be quite problematic and inconvenient. Therefore, it is more convenient to extend our scope to include all commutative rings, without imposing these three conditions. This extension leads us to the concept of an affine scheme, which bears the same relationship to commutative rings as affine varieties have to those special types of commutative rings. Thus, when we study the theory of affine schemes, it will be similar to the theory of affine varieties, but with the flexibility of allowing our rings to be more general. Now we are ready to introduce the notion of schemes.

Definition 2.2.19. *An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic as a locally ringed space to the spectrum of a ring. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that for every point there exists an open neighborhood U of the point such that $(U, \mathcal{O}_{X|_U})$ is an* *affine scheme. In other words, a scheme is a locally ringed space that can be covered by affine schemes.*

Note that, just like algebraic varieties can be covered by affine algebraic varieties, a scheme can be covered by affine schemes. We say that X is the underlying topological space of the scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ is its structure sheaf. In situations where the structure sheaf is clear, we denote the scheme simply by X. A morphism of schemes is a morphism of locally ringed spaces.

The underlying topological space of an affine scheme will be $X = V(I)$, where I is an ideal of the ring of polynomials in n variables, and the structure sheaf, which consists of regular functions on X, is precisely $k[x_1, \ldots, x_n]/I$, where k is a field. Let's see some examples.

If k is a field, then $\operatorname{Spec}(k)$ is a topological space with a single element, and its structure sheaf is k. We have seen that if k is algebraically closed, then by the Nullstellensatz, the points of $\mathbb{A}^1_k$ are in bijective correspondence with the maximal ideals of $k[x]$ and they are contained in $\operatorname{Spec}(k[x])$ and are closed, as they correspond to irreducible non-constant polynomials in x.

However, $(0) \in \operatorname{Spec}(k[x])$ is a prime ideal that is not maximal. Thus, consider now the scheme $(Spec(k[x]), \mathscr{O}_{Spec(k[x])}) := (\mathbb{A}^1, \mathscr{O}_{\mathbb{A}^1})$. We see that it has an additional point compared to $\mathbb{A}^1_k$. This point, which is not closed, is called the generic point.

This suggests that an algebraic variety is a scheme and that there can be more points and, therefore, more information in algebraic varieties if they are thought of as schemes.

An interesting example of a scheme that is not affine is as follows:

Let $X_1 = X_2 = \mathbb{A}^1$ and $U_1 = U_2 = \mathbb{A}^1 \backslash (x)$. Let $\varphi : U_1 \to U_2$ be the identity map. Let X be the scheme obtained by gluing U_1 and U_2 together via the isomorphism φ. X is the affine line with two origins and is not an affine scheme, but it can be covered by affine opens.

Analogously to the case of algebraic varieties in the classical sense, the next step is to define a projective scheme. For this purpose, we need to define a topology on a graded ring. In the case of the spectrum of a ring, we use the Zariski Topology, where the closed sets are sets of prime ideals containing a certain ideal I. For a graded ring S, we consider the topological space $Proj(S)$ as the set of homogeneous prime ideals, recalling that a homogeneous ideal is an ideal generated by homogeneous elements. The closed sets of this topological space are given by $V(I) := \mathfrak{p} \ : \ I \subset \mathfrak{p}$ and $\mathfrak{p} \in Proj(S)$.

To define the structure sheaf on $Proj(S)$, let's recall the localization of degree zero for each $\mathfrak{p} \in Proj(S)$, $S_{(\mathfrak{p})} := \dfrac{a}{b} \ : \ a, b \in S$ homogeneous elements, $b \notin \mathfrak{p}, \ \deg(a) = \deg(b)$.

Definition 2.2.20. *Let $U \subset Proj(S)$ be an open set. We define $\mathcal{O}(U)$ as the set of functions*

$$s \colon U \to \bigsqcup_{\mathfrak{p} \in U} S_{\mathfrak{p}}$$

such that for every $\mathfrak{p}$ in U, $s(\mathfrak{p})$ is in $S_{\mathfrak{p}}$. And s is locally a quotient of homogeneous elements of S, i.e., for each $\mathfrak{p} \in U$, there exists an open neighborhood V contained in U and homogeneous elements $a, b \in S$ such that for every $\mathfrak{q} \in V$, $b \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \dfrac{a}{b}$ in $S_{\mathfrak{p}}$.

Analogously to the previous case, we obtain the following result.

Proposition 2.2.21. *Let S be a graded ring. Then:*

(i) For every $\mathfrak{p} \in Proj(S)$, the stalk $\mathcal{O}_{\mathfrak{p}} \simeq S_{\mathfrak{p}}$.

(ii) For every element $f \in S_+ = \bigoplus_{d>0} S_d$, let $D_+(f) = \mathfrak{p} \in Proj(S) : f \notin \mathfrak{p}$. Then, $D_+(f)$ forms an open category for $Proj(S)$ such that $\left(D_+(f), \mathcal{O}\big|_{D_+(f)}\right) \simeq Spec(S_{(f)})$.

(iii) $\left(Proj(S), \mathcal{O}_{Proj(S)}\right)$ is a scheme.

The proof of the above proposition can be found in (HARTSHORNE, 1977).

If A is a ring, we define the n-projective space over A as $Proj(A[x_0, \ldots, x_n])$. In particular, if $A = k$ is an algebraically closed field, $Proj(k[x_0, \ldots, x_n])$ is isomorphic to the variety $\mathbb{P}^n_k$ in the classical sense.

Now let's see that it is possible to define a scheme over another scheme.

Definition 2.2.22. *Let S be a scheme. A scheme over S is a scheme X along with a morphism $X \to S$. If X and Y are schemes over S, a morphism of schemes over S is a morphism $f : X \to Y$ that is compatible with the given morphisms.*

We denote by $\mathfrak{Sch}(S)$ the category of schemes over S. If A is a ring, then by abuse of notation, we write $\mathfrak{Sch}(A)$ for schemes over the ring A.

We can now turn our attention to a theorem that demonstrates how every variety defined as the zeros of polynomials can be viewed as a scheme. This result highlights the fact that the notion of scheme generalizes the classical definition of an algebraic variety.

Theorem 2.2.23. *If k is an algebraically closed field, then there exists a full and faithful functor $t : \mathfrak{Var}(k) \to \mathfrak{Sch}(k)$, where $\mathfrak{Var}(k)$ is the category of varieties over k.*

The proof of this theorem can be found in (HARTSHORNE, 1977).

2.2.3 Properties of schemes

After defining a mathematical object, a natural step is to characterize it based on its properties. Therefore, we will now classify different types of schemes and explore some morphisms between them. As schemes are based on ringed spaces, many properties of schemes are related to the underlying topological space. Let's begin with two definitions in this context.

Definition 2.2.24. *A scheme is said to be connected if its underlying topological space is connected. Similarly, we say that a scheme is irreducible if its underlying topological space is irreducible.*

In addition to associating properties of the topological space with the scheme, we can also associate properties of the rings. Let's look at two definitions.

Definition 2.2.25. *A scheme X is reduced if for every open subset $U \subset X$, $\mathcal{O}_X(U)$ does not have any nilpotent elements. We also say that a scheme X is integral if for every $U \subset X$, the ring $\mathcal{O}_X(U)$ is an integral domain.*

We can observe that in the case where X is an affine scheme, there exists a ring A such that $X \simeq \operatorname{Spec}(A)$. Therefore, if X is an affine scheme, we have that X is irreducible when $nil(A)$, where $nil(A)$ denotes the nilradical of A. Moreover, X is integral if and only if A is an integral domain. We present a result that tells us that two properties of schemes, related to the topological space, are sufficient and necessary for a result about the rings associated with the space.

Proposition 2.2.26. *A scheme is integral if and only if it is reduced and irreducible.*

Proof. The proof can be found in II.3.1 of the reference (HARTSHORNE, 1977). □

Let us recall that a topological space is said to be Noetherian if it satisfies the ascending chain condition for open subsets, i.e., if X is a topological space and $U_1 \subset U_2 \subset \cdots$ is a sequence of open subsets U_i of X, then there exists an integer n_0 such that $U_{n_0} = U_{n_0+1} = \cdots$. We will use the definitions of compact and quasi-compact given by Bourbaki in (BOURBAKI, 1966). Recall that according to Bourbaki, a topological space X is said to be quasi-compact if for every open cover of X, we can find a finite subcover. And X will be compact if it is quasi-compact and Hausdorff.

Now, to define a Noetherian scheme, we can relate the property to either the topological space or the associated ring. We will define it based on the ring, and then we will see that this definition implies that the topological space is Noetherian. However, there is an example where the converse implication is not always valid.

Definition 2.2.27. *A scheme is locally Noetherian if it can be covered by affine opens, $Spec(A_i)$, where each A_i is a Noetherian ring. X is Noetherian if it is locally Noetherian and quasi-compact.*

Let's see that if X is Noetherian, then its topological space is also Noetherian. Notice that X is Noetherian if, and only if, every open subset is quasi-compact. Therefore, let's assume that X is a Noetherian scheme, and consider the chain $U_1 \subset U_2 \subset \cdots$. We define $U = \bigcup_{i=1} U_i$. Since U is open, there exists a finite subcover of U consisting of open sets U_i, that is, $U = \bigcup_{i=1}^{n} U_i$ for some $n \in \mathbb{N}$. Consequently, the chain stabilizes, and we conclude that the underlying topological space of X is Noetherian.

Let's see an example where the reverse is not true.

Example 2.2.28. *Consider $A = \mathbb{C}[x_1, x_2, ...]/(x_1^2, x_2^2, ...)$. Observe that every x_i is a nilpotent element and therefore belongs to every prime ideal of A. Thus, $(x_1, x_2, ...)$ is contained in every prime ideal. However, $(x_1, x_2, ...)$ is a maximal ideal, so A has only one prime ideal, meaning that $Spec(A)$ is a one-point topological space and therefore Noetherian. However, $(x_1) \subset (x_1, x_2) \subset \cdots$ is a chain that does not stabilize, so A is not Noetherian.*

Note that in the definition of a Noetherian scheme, we do not require that every affine open set is isomorphic to the spectrum of a Noetherian ring. Now let's consider a result that tells us that being a Noetherian scheme is a local property.

Proposition 2.2.29. *A scheme X is locally Noetherian if and only if for every affine open set $U = Spec(A)$, A is a Noetherian ring.*

The proof of the above proposition can be found in (HARTSHORNE, 1977).

Now let's look at a way to characterize morphisms between schemes. The following two definitions describe properties of a morphism $f : X \to Y$ that relate to a cover by affine opens of Y satisfying certain properties.

Definition 2.2.30. *A morphism of schemes $f : X \to Y$ is locally of finite type if there exists a cover of Y by affine open sets $V_i = Spec(B_i)$ such that for each i, $f^{-1}(V_i)$ can be covered by affine open sets $U_{ij} = Spec(A_{ij})$, where each A_{ij} is a finitely generated B_i-algebra. The morphism is of finite type if each $f^{-1}(V_i)$ can be covered by a finite number of U_{ij}.*

Definition 2.2.31. *A morphism of schemes $f : X \to Y$ is finite if there exists a cover of Y by affine open sets $V_i = Spec(B_i)$ such that for each i, $f^{-1}(V_i) = Spec(A_i)$, where each A_i is a finitely generated B_i-algebra.*

Now let's see how to define substructures of a scheme. Since we have a topological space, we are interested in open and closed subschemes, as well as the concept of open and closed immersions.

Definition 2.2.32. *An open subscheme of a scheme X is a scheme U such that $sp(U)$ is open in $sp(X)$ and the structural sheaf $\mathcal{O}_U$ is isomorphic to the sheaf $\mathcal{O}_X\big|_U$.*

Definition 2.2.33. *A closed immersion is a morphism of schemes $f: Y \to X$ such that f induces a homeomorphism from $sp(Y)$ to a closed subset of $sp(X)$. The induced map of sheaves $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$ on X is surjective. A closed subscheme of a scheme X is an equivalence class of closed immersions, where $f: Y \to X$ and $f': Y' \to X$ are equivalent if there exists an isomorphism $i: Y' \to Y$ such that $f' = f \circ i$.*

Now let's see an example. Let A be a Noetherian ring and I an ideal of A. Let $X = \operatorname{Spec} A$ and $Y = A/I$. Then, the ring homomorphism $A \to A/I$ induces a morphism of schemes $f: Y \to X$ that is a closed immersion. The map f is a homeomorphism from Y to the closed subspace $V(I) \subset X$, and the map of structural sheaves $f^\#: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective because it is surjective on the stalks, which are the localizations of A and A/I.

Another important definition is the dimension of a scheme and the codimension of a subscheme.

Definition 2.2.34. *The dimension of a scheme X, denoted by* $\dim X$*, is its dimension as a topological space. And if Z is an irreducible closed subset of X, the codimension of Z in X, denoted by* $\operatorname{codim}(Z, X)$*, is the supremum of integers n such that there exists a chain*

$$Z = Z_0 < Z_1 < \cdots < Z_m$$

of distinct irreducible closed sets of X, starting with Z. If Y is an arbitrary closed set, we define

$$\operatorname{codim}(Y, X) = \inf_{Z \leqslant Y} \operatorname{codim}(Z, X),$$

where the infimum is taken over the closed irreducible sets of Y.

Note that in the case where $X = \operatorname{Spec}(A)$ is an affine scheme, the dimension of the topological space X coincides with the Krull dimension of the ring.

Remember that a regular local ring is a Noetherian local ring having the property that the minimal number of generators of its maximal ideal is equal to its Krull dimension. So we can define a regular scheme.

Definition 2.2.35. *A scheme X is regular if it is a locally Noetherian scheme whose local rings are regular everywhere.*

Every smooth scheme is regular, and every regular scheme of finite type over a perfect field is smooth.

Now, a natural step would be to define the product of schemes. However, when defining the product in the usual way, we can encounter some problems. Let's consider an example.

Let $X = Y = \mathbb{A}^1_k = \operatorname{Spec}(k[x])$. If we define the cartesian product of sets, we would obtain $X \times Y = \operatorname{Spec}(k[x]) \times \operatorname{Spec}(k[x])$, which is a topological space different from $\mathbb{A}^2_k = \operatorname{Spec}(k[x, y])$. In $X \times Y$, the closed sets are finite unions of vertical and horizontal lines together with points. However, in $\mathbb{A}^2_k$, the line $x = y$ is a closed set. Therefore, we need a different way to construct the product of schemes, and for that, we need the definition of fibered product.

Definition 2.2.36. *Let S be a scheme, and let X and Y be schemes over S. The fibered product of X and Y over S is the scheme $X \times_S Y$ together with two morphisms $p_1 \colon X \times_S Y \to X$ and $p_2 \colon X \times_S Y \to Y$ that form a commutative diagram with X and Y with the morphisms $X \to S$ and $Y \to S$. For any scheme Z over S and for any morphisms $f \colon Z \to X$ and $g \colon Z \to Y$ that make the diagram commute with the morphisms $X \to S$ and $Y \to S$, there exists a unique morphism $\theta \colon Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$.*

The following diagram provides a visual representation of the definition, aiding in a better understanding of its concept:

$$\begin{tikzcd}
Z \arrow[rr, dashed] \arrow[dr] \arrow[drrr] & & X \times_S Y \arrow[dl] \arrow[dr] & \\
& X \arrow[dr] & & Y \arrow[dl] \\
& & S &
\end{tikzcd}$$

If two schemes X and Y are given without specifying the base scheme, we take $S = \operatorname{Spec} \mathbb{Z}$ and define the product of X by Y, denoted $X \times Y$, as $X \times_{\operatorname{Spec} \mathbb{Z}} Y$. The following theorem states the universal property of the fibered product, i.e., its existence and uniqueness up to isomorphism.

Theorem 2.2.37. *Let X and Y be two schemes over S. The fibered product $X \times_S Y$ exists and is unique up to isomorphism.*

The proof of the Theorem 2.2.37 can be found in (HARTSHORNE, 1977).

We know that the Zariski topology gives us either a trivial topological space or a non-Hausdorff space. However, we would like to find an analogous property that tells us whether the diagonal is closed or not. Thus, we say that a scheme that possesses this property is a separated scheme. In the first edition of "Éléments de géométrie algébrique" the term used for what we now call schemes is "préschéma," while "schéma" is reserved for separated schemes. Although there are several non-separated schemes, they are somewhat peculiar and not commonly encountered in practice.

The motivation for classifying separated schemes is that several properties are found only in schemes that possess this property, such as the fact that the intersection of two affine subsets is also an affine subset. With that said, let's now formally define a separated scheme, starting with the definition of the diagonal.

Definition 2.2.38. *Let $f\colon X \to Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $\delta\colon X \to X \times_Y X$ whose composition with the projection maps is the identity map.*

Definition 2.2.39. *We say that the map f is separated when the diagonal morphism δ is a closed immersion. In this case, we say that X is separated over Y.*

Note that when we say that a scheme X is separated without specifying the base scheme, we mean that it is separated over $\mathrm{Spec}(\mathbb{Z})$. Let's consider an example of a non-separated scheme X over k. Consider the affine line over k with two origins. In this case, $X \times_k X$ is the plane with 4 origins. The image of the diagonal morphism is the usual diagonal with two origins, but its closure is the diagonal with four origins. Therefore, it is not a closed immersion, which implies that the scheme is not separated.

A classic example of separated schemes is the affine schemes.

Proposition 2.2.40. *If $f\colon X \to Y$ is a morphism of affine schemes, then f is separated.*

Proof. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$. Then, A is a B-algebra, and we have that $X \times_Y X = \mathrm{Spec}(A) \otimes_{\mathrm{Spec}(B)} \mathrm{Spec}(A)$ is also affine. By construction, the diagonal morphism is induced by the morphism $A \otimes A = A$ such that $a \otimes a' = aa'$, which gives us a surjective map, and therefore, the diagonal morphism is a closed immersion. □

Now, let's recall that in general topology, a function between topological spaces is said to be proper if the preimage of a compact set is compact. Just as in the case of Hausdorff spaces, this definition does not fit well in the context of algebraic geometry. Thus, let's see how to define an analogous property. But first, we need the concept of closed and universally closed morphisms.

Definition 2.2.41. *A morphism $f\colon X \to Y$ is closed if the image of any closed set is closed. And we say that f is universally closed if it is closed and for every morphism $Z \to Y$, the morphism $X \times_Y Z \to Z$ is closed.*

An important theorem about closed morphisms is the following.

Theorem 2.2.42. *For each $n, m \in \mathbb{N}$, the projection $\mathbb{P}^n_k \times \mathbb{A}^m_k \to \mathbb{A}^m_k$ is a closed morphism.*

As a direct consequence of the theorem, we obtain the following corollary.

Corollary 2.2.43. *If X is a projective set, then for every algebraic variety Y, the projection $X \times Y \to Y$ is closed.*

Definition 2.2.44. *A morphism $f\colon X \to Y$ is proper if it is separated, of finite type, and universally closed.*

After classifying schemes according to their properties, we can generalize the notion of algebraic variety, and from this point on, the word "variety" refers to an abstract variety as defined below.

Definition 2.2.45. *An abstract variety is a separated, integral, finite-type scheme over an algebraically closed field k. If the scheme is proper, then we say it is a complete abstract variety.*

In commutative algebra, a quasi-excellent ring is a Noetherian commutative ring that behaves well under completion, and an excellent ring is one that is also universally catenary, meaning that for any pair of prime ideals p and q, any two strictly increasing chains $p = p_0 \subset p_1 \subset \cdots \subset p_n = q$ of prime ideals are contained in maximal strictly increasing chains from p to q of the same finite length.

The concept of excellent rings was introduced by Alexander Grothendieck in 1965 as a class of rings that are well-behaved. It is conjectured that quasi-excellent rings serve as the appropriate base rings for which the problem of resolution of singularities can be solved. This motivates the definition of an excellent scheme.

Definition 2.2.46. *A scheme is called excellent if it has a cover by open affine subschemes that are excellent, which implies that every open affine subscheme has this property.*

We conclude this section by providing the definition of scheme stratification and presenting a result that proves the existence of a refinement when given two stratifications.

Definition 2.2.47. *A stratification is a decomposition of a scheme into locally closed regular subschemes, typically defined based on the constancy of certain local invariants such as dimension.*

Proposition 2.2.48. *If X is a separated scheme of finite type and has two stratifications $X = \bigsqcup_{i\in I} X_i$ and $X = \bigsqcup_{j\in J} Y_j$, then there exists a stratification $X = \bigsqcup_{k\in K} Z_k$ that refines both, meaning that each X_i and Y_j can be expressed as a union of some subsets Z_k.*

Proof. We begin with some preliminary observations. First, if $X_1 \subset X$ is an irreducible component and we have a locally closed subset L that contains the generic point of X_1, then L contains an open subset of X_1. This is because a locally closed subset can be written as the intersection of a closed subset and an open subset, and any closed subset containing the generic point must contain all of X_1. Next, we note that our scheme X is Noetherian since it is of finite type over a Noetherian ring. Therefore, X has finitely many irreducible components and any locally finite collection of sets is actually globally finite.

Now, let's address the problem at hand. We can write $X = X_1 \cup \ldots \cup X_n$ for the decomposition of X into irreducible components. For any irreducible component X_i, the collection of points X_i° that belong only to that irreducible component and no others is open.

We find Y_j and Z_k that contain the generic point of X_i. By our observation above, each of Y_j and Z_k contains an open subset of X_i. By taking the intersection of these open sets and the smooth locus of X_i, we can find a smooth open subset $W \subset X$ that is contained in X_i, Y_j and Z_k, and its complement inside X_i consists of a finite number of closed subschemes of strictly lower dimension. Here, we use the fact that any locally closed subset of a Noetherian scheme is again locally Noetherian to conclude the finiteness and irreducibility of X_i and assert that the dimension must drop.

Observe that in order to provide a stratification of X that is a common refinement of $\bigsqcup Y_j$ and $\bigsqcup Z_k$, it is sufficient to provide a stratification of $X \backslash W$ that refines the stratifications $\bigsqcup (Y_j \cap W^c)$ and $\bigsqcup (Z_k \cap W^c)$. This suggests an inductive approach: we consider the collection of maximal-dimensional irreducible components of X and apply the above procedure for each of them. At each stage, when we go from X to $X \backslash W$, we eliminate one top-dimensional irreducible component at the cost of potentially introducing finitely many more irreducible components of strictly lower dimension in $X \backslash W$.

By repeating this procedure a finite number of times, we reduce the problem to a scheme of finite type over C with strictly smaller dimension. Eventually, we reach a finite discrete union of points, and we can simply take the disjoint union of all these points. Therefore, the proposition holds. □

2.3 Resolution of singularities

The original problem of resolution of singularities aimed to find a nonsingular model for the function field of a variety X, in other words, a complete non-singular variety X' with the same function field. However, in practice, it is more convenient to consider a slightly different condition, which can be stated as follows:

Definition 2.3.1. *Let X be a locally noetherian reduced scheme. A proper birational morphism $\pi : Z \to X$ with Z regular is called a desingularization of X. If π is an isomorphism at every regular point of X, we say it is a strong desingularization.*

The condition that the map is proper is necessary to exclude trivial solutions, such as taking X' to be the subvariety of non-singular points of X. Without the requirement of properness, one could simply consider the non-singular locus of X as the resolution, which does not provide a substantial improvement. By imposing properness, we ensure that the resolution is a significant modification of X that captures its singularities in a more meaningful way.

Hironaka showed that there exists a strong desingularization satisfying certain conditions when X is defined over a field of characteristic 0. This result is presented in the following theorem.

Theorem 2.3.2 (Hironaka)**.** *Let X be a reduced algebraic variety over a field of characteristic zero, or more generally, a reduced scheme that is locally of finite type over an excellent scheme of characteristic zero (i.e.,* $\mathrm{char}(k(x)) = 0$ *for every $x \in X$), and locally noetherian. Then X admits a strong desingularization.*

The main idea of Hironaka's theorem proof is to perform a sequence of blow-ups and modifications on the given variety X in order to resolve its singularities. This is achieved by carefully constructing a sequence of smooth varieties that approximate X while eliminating the singularities at each step. The proof involves several steps and can be found on reference (HIRONAKA, 1964).

Next, we present Chow's Lemma, an important tool in algebraic geometry that finds applications in various areas, such as birational geometry and resolution of singularities. This lemma establishes a connection between proper morphisms and projective morphisms, providing a way to "compactify" varieties by embedding them into projective spaces.

Lemma 2.3.3 (Chow's Lemma)**.** *For any complete irreducible variety X, there exists a projective variety $\overline{X}$ and a surjective birational morphism $f : \overline{X} \to X$.*

Now, we present Nagata's Theorem, which provides a method to "complete" a given variety by embedding it as a dense open subset of a larger, complete variety.

Theorem 2.3.4 (Nagata)**.** *Every variety can be embedded as a dense open subset of a complete variety.*

The proof of Nagata's Theorem involves constructing the complete variety by taking the closure of the given variety in a suitable projective space. A detailed proof can be found in Chapter 6 of Shafarevich's book (SHAFAREVICH; REID, 1994) or in Section 7 of Chapter II of Hartshorne's book (HARTSHORNE, 1977).

Finally, the Weak Factorization Theorem is a fundamental result in algebraic geometry that provides insights into the structure of algebraic varieties. We state it as follows:

Theorem 2.3.5. *(1) Let $f : X \to Y$ be a birational map between smooth complete varieties over a characteristic zero field K. Suppose f is an isomorphism over an open set $U \subset X$. Then, we can factorize f as follows:*

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n = Y,$$

where each X_i is a smooth complete variety and f_i is a blow-up or blow-down at a smooth center that is an isomorphism over U.

(2) Furthermore, if $X\backslash U$ and $Y\backslash U$ are divisors with simple normal crossings, then each $D_i := X_i\backslash U$ also becomes a divisor with simple normal crossings and the map f_i corresponds to a blow-up or blow-down at a smooth center that has normal crossings with the components of D_i.

(3) There exists an index $1 \leqslant r \leqslant n$ such that for all $i \leqslant r$, the induced birational map $X_i \xrightarrow{f_0} X$ is a projective morphism. Moreover, for all $r \leqslant i \leqslant n$, the map $X_i \xrightarrow{f_0} Y$ is a projective morphism.

2.4 Abelian varieties

Abelian varieties, complex tori, elliptic curves, and Albanese varieties are fascinating objects in algebraic geometry and complex analysis that have deep connections and rich structures. In this section, we will introduce and discuss these four concepts.

Definition 2.4.1. *A complex torus of dimension n is obtained by quotienting a complex vector space V of dimension n by a complete lattice Λ.*

A complex torus can be seen as a particular type of complex manifold M whose underlying smooth manifold is a torus in the usual sense (i.e. the cartesian product of some number n circles). Furthermore, complex tori can be regarded as a higher-dimensional analogue of elliptic curves and are closely related to abelian varieties. They provide a fruitful framework for studying the interplay between algebraic geometry and complex analysis. Particularly, next we present the one-dimensional complex torus, which is an elliptic curve.

Definition 2.4.2. *An elliptic curve is a smooth, projective, algebraic curve of genus one, on which there is a specified point O.*

In the following, we explore the abelian varieties, which generalize the notion of elliptic curves to higher dimensions.

Definition 2.4.3. *An abelian variety is a complex projective variety that is also an algebraic group. In other words, it is a variety equipped with a group structure that behaves well with respect to algebraic operations.*

Abelian varieties can be characterized by equations with coefficients in any field, and such varieties are referred to as being defined over that specific field. The initial focus of study for abelian varieties was on those defined over the field of complex numbers. Interestingly, these abelian varieties are precisely the complex tori that can be embedded into a complex projective space.

A morphism of abelian varieties is a morphism of the underlying algebraic varieties that preserves the identity element for the group structure. An isogeny is a morphism that is surjective and has a finite kernel. For abelian varieties, such as elliptic curves, this notion can also be formulated as follows:

Definition 2.4.4. *An isogeny between abelian varieties E_1 and E_2 of the same dimension over a field k is a dense morphism $f : E_1 \to E_2$ of varieties that preserves basepoints. In other words, the isogeny f maps the identity point on E_1 to the identity point on E_2.*

Abelian varieties arise naturally as Jacobian varieties, which are the connected components of zero in Picard varieties, as well as Albanese varieties of other algebraic varieties.

Definition 2.4.5. *The Jacobian variety $J(X)$ of a ringed space X is the group of isomorphism classes of invertible sheaves L (or line bundles) on X such that* $\deg(L) = 0$*, i.e.,* $J(X) = \mathrm{Pic}^0(X)$.

Albanese varieties represent a distinct type of algebraic variety associated with a given variety X. In the following, we will explore the concept of Albanese variety.

Definition 2.4.6. *The Albanese variety of a variety X is defined as the universal object that maps X to all other abelian varieties. The Albanese variety is dual to the Picard variety.*

It captures the geometric and algebraic properties of X and plays a fundamental role in the study of the geometry of X. The Albanese variety provides a powerful tool for understanding the structure and properties of arbitrary varieties.

3 Grothendieck ring of varieties

The Grothendieck ring of varieties over a field k is a ring that is additively generated by isomorphism classes of varieties over k, modulo a “cut and paste” relation with respect to closed subvarieties, and where the product is cartesian product. This concept was initially introduced in a letter from Grothendieck in the Serre-Grothendieck correspondence (dated 16/8/64) and it is a fascinating object that lies at the core of the theory of motivic integration.

In this chapter, the main goal is to present the Grothendieck ring, exploring some properties and alternative presentations of this structure. Furthermore, we will calculate some classes of varieties in this ring, present the demonstration given by Larsen and Lunts (LARSEN; LUNTS, 2003) that stable rationality of a smooth and proper variety over a field of characteristic zero can be detected on the class of the variety in the Grothendieck ring and finally show that this ring is not an integrity domain. The primary source utilized for constructing this chapter was (SAHASRABUDHE, 2007) and as supplementary material, we relied on the references (POONEN, 2002), (KIEFNER, 2016), (NICAISE; SHINDER, 2019) and (LARSEN; LUNTS, 2003).

Definition 3.0.1. *The Grothendieck Ring $K_0(\mathrm{Var}_k)$ is the quotient of the free abelian group generated by isomorphism classes of k-varieties by the relation $[X\backslash Y] = [X] - [Y]$, where Y is a closed subscheme of X; the fiber product over k induces a ring structure defined by $[X] \cdot [X'] = [(X \times_k X')_{red}]$.*

The relation $[X\backslash Y] = [X] - [Y]$ is often called scissor relation as we think of cutting a variety into disjoint subschemes.

Remark 3.0.2. *Indeed, $K_0(\mathrm{Var}_k)$ is a commutative ring. We observe that the multiplication is well-defined. In particular, the fiber product $X \times_k Y$ of any two varieties X and Y is again a reduced, separated scheme of finite type over k. It is important to note that this is not always true over a field that is not algebraically closed, as the product of reduced schemes over a non-perfect field k may not be reduced. However, in such cases, the multiplication can be defined by taking the reduction of the product over k.*

Furthermore, the fiber product induces a bilinear map $K_0(\mathrm{Var}_k) \times K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$ that is both commutative and associative and the zero is given by the class of the empty set and the unit is $[\mathrm{Spec}\, k]$.

Lemma 3.0.3. *Let k be a field. Then in the Grothendieck ring of varieties, we have:*

(1) $[\varnothing] = 0$.

(2) $[\mathrm{Spec}(k)] = 1$.

(3) $\left[\mathbb{P}_k^n\right] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n$, *where* $\mathbb{L} := \left[\mathbb{A}_k^1\right]$.

Proof. Let $[X] \in K_0(\mathrm{Var}_k)$. To demonstrate item (1), let us consider $X \subseteq X$ as a closed subvariety. We have:

$$[X \backslash X] = [X] - [X]$$
$$\text{i.e., } [\varnothing] = 0.$$

For part (2), observe that $\mathrm{Spec}(k) \times_k Y = Y \times_k \mathrm{Spec}(k) = Y$ for all $Y \in \mathrm{Var}_k$. Therefore, $\mathrm{Spec}(k) = 1$.

To prove (3), we use induction. Note that $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$ is an open subvariety with ∞ as its complement. Therefore, by definition, we have:

$$\left[\mathbb{P}_k^1 \backslash \mathbb{A}_k^1\right] = \left[\mathbb{P}_k^1\right] - \left[\mathbb{A}_k^1\right] \Rightarrow \left[\mathbb{P}_k^1\right] = 1 + \left[\mathbb{A}_k^1\right] = 1 + \mathbb{L}.$$

Since $\mathbb{P}_k^{n+1} \backslash \mathbb{A}_k^{n+1} \simeq \mathbb{P}_k^n$, we have $[\mathbb{P}_k^{n+1}] = [\mathbb{P}_k^n] + [\mathbb{A}_k^{n+1}]$. Applying the induction hypothesis $[\mathbb{P}_k^n] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n$, we conclude that $[\mathbb{P}_k^{n+1}] = 1 + \mathbb{L} + \cdots + \mathbb{L}^n + \mathbb{L}^{n+1}$.

□

3.1 Classical properties

Here are a few important properties of the Grothendieck Ring of Varieties:

Lemma 3.1.1. *(1) If X is a variety and U and V are two locally closed subvarieties in X then,*

$$[U \cup V] + [U \cap V] = [U] + [V].$$

(2) If a variety X is a disjoint union of locally closed subvarieties $X_1, X_2, \ldots X_n$ for some $n \in \mathbb{N}$, then

$$[X] = \Sigma_{i=1}^n [X_i].$$

(3) Let C be a constructible subset of a variety X, then C has a class in $K_0(\mathrm{Var}_k)$.

Proof. (1) Note that $V - U \cap V = U \cup V - U$. If U and V are both open or closed then from the definition we get that $[V] - [U \cap V] = [V - U \cap V] = [U \cup V - U] = [U \cup V] - [U]$. Thus,

$$[U \cup V] + [U \cap V] = [U] + [V], \tag{3.1}$$

if both U and V are either open or closed.

Now, in general, since U and V are locally closed, we can express them as the intersection of an open set and a closed set. Let $U = U_1 \cap F_1$ and $V = U_2 \cap F_2$, where U_1 and U_2 are open subsets of X, and F_1 and F_2 are closed subsets of X. Then, by equation 3.1,

$$[U_1 \cup U_2] + [U_1 \cap U_2] = [U_1] + [U_2]\,. \tag{3.2}$$

Given that $U_1 \cap F_1$ is a closed subset of U_1 (likewise, $U_2 \cap F_2 \subset U_2$ is closed), we have

$$\begin{aligned} [U_1] &= [U_1 \cap F_1] + \left[U_1 \backslash U_1 \cap F_1\right] \\ \text{then } [U_1] &= [U] + \left[U_1 \backslash U_1 \cap F_1\right]. \end{aligned} \tag{3.3}$$

Similarly, we get

$$\begin{aligned} [U_2] &= [U_2 \cap F_2] + \left[U_2 \backslash U_2 \cap F_2\right] \\ \text{then } [U_2] &= [V] + \left[U_2 \backslash U_2 \cap F_2\right]. \end{aligned} \tag{3.4}$$

Adding equations 3.2, 3.3 and 3.4 we obtain

$$[U_1 \cup U_2] + [U_1 \cap U_2] = [U] + [V] + \left[U_1 \backslash U_1 \cap F_1\right] + \left[U_2 \backslash U_2 \cap F_2\right]. \tag{3.5}$$

Again, note that since $U_1 \cap F_1$ is closed in U_1 (resp. $U_2 \cap F_2$ is closed in U_2), therefore, $U_1 \backslash U_1 \cap F_1 \subset U_1$ is an open subset (resp. $U_2 \backslash U_2 \cap F_2 \subset U_2$ is an open subset). Moreover, since U_1 and U_2 are open in X, it follows that $U_1 \backslash (U_1 \cap F_1)$ and $U_2 \backslash (U_2 \cap F_2)$ are also open subsets of X. So by equation 3.1, we get

$$\begin{aligned} \left[U_1 \backslash U_1 \cap F_1\right] + \left[U_2 \backslash U_2 \cap F_2\right] = {} & \left[\left(U_1 \backslash U_1 \cap F_1\right) \cup \left(U_2 \backslash U_2 \cap F_2\right)\right] \\ & + \left[\left(U_1 \backslash U_1 \cap F_1\right) \cap \left(U_2 \backslash U_2 \cap F_2\right)\right] \end{aligned} \tag{3.6}$$

So by substituting the given equality in 3.6 into equation 3.5:

$$\begin{aligned} [U_1 \cup U_2] + [U_1 \cap U_2] = [U] + [V] + {} & \left[\left(U_1 \backslash U_1 \cap F_1\right) \cup \left(U_2 \backslash U_2 \cap F_2\right)\right] \\ & + \left[\left(U_1 \backslash U_1 \cap F_1\right) \cap \left(U_2 \backslash U_2 \cap F_2\right)\right] \end{aligned}$$

then

$$\begin{aligned} [U] + [V] = {} & \left([U_1 \cup U_2] - \left[\left(U_1 \backslash U_1 \cap F_1\right) \cup \left(U_2 \backslash U_2 \cap F_2\right)\right]\right) \\ & + \left([U_1 \cap U_2] - \left[\left(U_1 \backslash U_1 \cap F_1\right) \cap \left(U_2 \backslash U_2 \cap F_2\right)\right]\right) \end{aligned}$$

or

$$[U] + [V] = [U_1 \cap F_1 \cup U_2 \cap F_2] + [U_1 \cap F_1 \cap U_2 \cap F_2]$$

therefore,

$$[U] + [V] = [U \cup V] + [U \cap V].$$

Hence proved.

(2) By induction, it suffices to prove that if X is a disjoint union of two locally closed subvarieties X_1 and X_2, that is, $X = X_1 \cup X_2$, then $[X] = [X_1] + [X_2]$. Using property (1), we have

$$[X_1 \cup X_2] + [X_1 \cap X_2] = [X_1] + [X_2],$$

but since X is a disjoint union of X_1 and X_2, we have

$$[X_1 \cap X_2] = [\varnothing] = 0, \text{ therefore } [X] = [X_1] + [X_2].$$

(3) Since C is a constructible set in X, we know that it can be expressed as a finite disjoint union of locally closed subsets. Thus, we can write $C = \cup_{i=1}^{n} C_i$. This construction does not depend on choices and therefore, using property (2), we obtain the class $[C]$ in $K_0(\mathrm{Var}_k)$. □

One of the main tools for computations in the Grothendieck ring of varieties is the concept of Zariski locally trivial fibrations. The following results and examples demonstrate its usefulness.

Definition 3.1.2. *Let X and Y be k-varieties, and let $\pi : Y \to X$ be a morphism of varieties. The morphism π is called a Zariski locally trivial fibration with fiber $F \in \mathrm{Var}_k$ if for every closed point $x \in X$, there exists an open set (in the Zariski topology) $U \subseteq X$ containing x, and an isomorphism $\pi^{-1}(U) \simeq U \times F$.*

Zariski locally trivial fibrations behave like products in the Grothendieck ring. More precisely, the following property holds.

Proposition 3.1.3. *If the morphism of k-varieties $\pi : Y \to X$ is a Zariski locally trivial fibration with fiber F, then*

$$[Y] = [F] \cdot [X]$$

in $K_0(\mathrm{Var}_k)$.

Proof. By considering an open neighborhood U for any $x \in X$, as described in Definition 3.1.2, we obtain a covering of X. Since X is a variety, it is quasi-compact. Therefore, we can find a finite covering $X = \bigcup_{i=1}^{n} X_i$ consisting of such open subvarieties. This implies that $Y_i := \varphi^{-1}(X_i) \cong X_i \times F$, and $\varphi : Y_i \to X_i$ corresponds to the projection onto X_i for all i. We have $Y = \bigcup_{i=1}^{n} Y_i$ and $[Y_i] = [X_i]\,[F]$ for all i.

Let $Y^r := \bigcup_{i=1}^{r} Y_i$ and $X^r := \bigcup_{i=1}^{r} X_i$. Since $X = X^n$ and $Y = Y^n$, it suffices to show that

$$[Y^r] = [X^r]\,[F],$$

for all $r \in \{1, \ldots, n\}$. We already know this holds for $r = 1$. Let's assume it is true for some r. Since $\varphi : Y_i \cong X_i \times F \to X_i$ is the projection for all i, we have $Y_{r+1} \cap Y^r = \varphi^{-1}(X_{r+1} \cap X^r) \cong (X_{r+1} \cap X^r) \times F$. Therefore,

$$\begin{aligned} \left[Y^{r+1}\right] &= [Y^r] + [Y_{r+1}] - [Y_{r+1} \cap Y^r] \\ &= [X^r][F] + [X_{r+1}][F] - [X_{r+1} \cap X^r][F] \\ &= \left[X^{r+1}\right][F]. \end{aligned}$$

By induction, this proves the proposition. □

Corollary 3.1.4. *(1) Consider a proper morphism $f : X \to Y$ between smooth varieties, where f is a blow-up with a smooth center $Z \subset Y$ of codimension d. Then*

$$[f^{-1}(Z)] = [Z] \cdot [\mathbb{P}^{d-1}]$$

in $K_0(\mathrm{Var}_k)$.

(2) Consider a homogeneous polynomial $f \in k[Z_0, \ldots, Z_n]$, where $Y := V(f)$ is the corresponding projective variety in $\mathbb{P}^n$, and $X \subseteq \mathbb{A}^{n+1}$ is the affine cone over Y. In the Grothendieck ring $K_0(\mathrm{Var}_k)$, we have the following relation:

$$[X] = (\mathbb{L} - 1)[Y] + 1.$$

Note that the second item of Corollary 3.1.4 confirms our intuition: the affine cone over Y can be thought of as the origin together with an affine line (excluding the origin) for each point of Y.

Proof. Item (1) follows directly from Proposition 3.1.3. To prove item (2) consider the morphism $\pi : X \backslash \{0\} \to Y$, which maps a closed point $(a_0, \ldots, a_n) \neq 0$ to the line $[a_0 : \ldots : a_n] \in Y$ that contains this point. It is worth noting that all fibers of this morphism are isomorphic to $\mathbb{A}^1 \backslash \{0\}$.

To simplify the discussion, let us assume that $y \in Y$ is contained in the open subset $U := Y \cap \{Z_0 \neq 0\}$. We observe that

$$\begin{aligned} \pi^{-1}(U) &= \left\{a \in X \backslash \{0\} \mid a_0 \neq 0, f(a) = 0\right\} \\ &\cong \left(\mathbb{A}^1 \backslash \{0\}\right) \times \{[a] = [1 : a_1 : \ldots : a_n] \mid f(a) = 0\} \\ &= \left(\mathbb{A}^1 \backslash \{0\}\right) \times U, \end{aligned}$$

where $\pi_{|\pi^{-1}(U)}$ corresponds to the projection onto U. Thus, π is a Zariski locally trivial fibration. According to Proposition 3.1.3, we conclude that $[X \backslash \{0\}] = (\mathbb{L} - 1)[Y]$, and therefore $[X] = (\mathbb{L} - 1)[Y] + 1$.

□

With the properties established in the previous results, we are now able to compute some classes. Let k be a field. In the Grothendieck Ring of varieties $K_0(Var_k)$ we have the following examples.

Example 3.1.5. *Consider the grassmanian $Gr(2,4)$, let's compute its class in the Grothendieck ring $K_0(\mathrm{Var}_k)$. By Construction 2.1.65 and Example 2.1.67 we can write $Gr(2,4)$ as the disjoint union of open cells that are isomorphic to affine spaces, which are locally closed subvarieties, so we have:*

$$\mathrm{Gr}(2,4) = \mathbb{A}^0 \sqcup \mathbb{A}^1 \sqcup \begin{pmatrix} \mathbb{A}^2 \\ \sqcup \\ \mathbb{A}^2 \end{pmatrix} \sqcup \mathbb{A}^3 \sqcup \mathbb{A}^4.$$

So by item (2) *of Lemma 3.1.1, we get*

$$[Gr(2,4)] = \mathbb{L}^4 + \mathbb{L}^3 + 2\mathbb{L}^2 + \mathbb{L} + 1.$$

Example 3.1.6. *We claim that*

$$[Bl(\mathbb{P}^2,\ pt)] = (\mathbb{L}+1)^2 = [\mathbb{P}^1 \times \mathbb{P}^1].$$

To prove this, consider the blow-up of $\mathbb{P}^2$ at a point $p \in \mathbb{P}^2$, denoted by $\tilde{\mathbb{P}^2}$. We can write $\mathbb{P}^2$ as the union of the affine plane with a line at infinity, i.e., $\mathbb{P}^2 = \mathbb{A}^2 \cup l_\infty$. The blow-up $\tilde{\mathbb{P}^2}$ can then be expressed as the union $\tilde{\mathbb{P}^2} = X \cup l_\infty$, where X is the blow-up of $\mathbb{A}^2$ at a point p, and the exceptional divisor of the blow-up is denoted by D and is isomorphic to $\mathbb{P}^1$, since the exceptional divisor D corresponds to the projectivization of the tangent plane of $\mathbb{P}^2$ at the point p.

Using Proposition 2.1.51, we have $X \backslash D = \mathbb{A}^2 \backslash \{p\}$, which implies $[X\backslash D] = [\mathbb{A}^2] - [\{p\}] = \mathbb{L}^2 - 1$. By the definition of classes in the Grothendieck ring, we have $[X] = [X\backslash D] + [D] = \mathbb{L}^2 - 1 + \mathbb{L} + 1 = \mathbb{L}^2 + \mathbb{L}$. And since $\tilde{\mathbb{P}^2} = X \cup l_{infty}$, which is a disjoint union, we get $[\tilde{\mathbb{P}^2}] = [X] + \mathbb{L} + 1 = \mathbb{L}^2 + 2\mathbb{L} + 1 = (\mathbb{L}+1)^2$. It is worth noting that if we were to calculate the class of the blow-up of $\mathbb{P}^2$ at a point using the result on Zariski locally trivial fibrations, the result would be the same.

Example 3.1.7. *Consider Gl_n the general linear group of degree n, i.e., the set of $n \times n$ invertible matrices. We have*

$$[Gl_n] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \dots (\mathbb{L}^n - \mathbb{L}^{n-1}).$$

Indeed, to compute the class of the algebraic group Gl_n, we can intuitively consider matrices in GL_n corresponding to bases of k^n (via the columns of the matrix, for example). For the first column, we can choose any nonzero vector in k^n, which has class $(\mathbb{L}^n - 1)$. For the second column, we can choose any vector in k^n that is not in the span of the first column. This has class $(\mathbb{L}^n - \mathbb{L})$. By iterating this process for each subsequent column, we obtain the result. However, it is important to note that this approach does not give a global product of varieties.

To formalize the reasoning in this example, we can use Proposition 3.1.3:

Let $V := k^n$ with the standard basis $e_1, \ldots, e_n$. We define $V_r \subset \mathbb{A}^{n\times r}$ to be the set of r-tuples of nonzero linearly independent vectors in V. Specifically, $V_1 = \mathbb{A}^n\backslash\{0\}$ and $V_n = \mathrm{GL}_n$. We want to compute the classes of V_r in the Grothendieck ring $K_0(\mathrm{Var}_k)$.

We proceed by induction. For $r = 1$, we have $[V_1] = [\mathbb{A}^n\backslash\{0\}] = \mathbb{L}^n - 1$, which serves as the base case.

Now, consider the projection $\pi_r : V_r \to V_{r-1}$ that forgets the last vector. We want to show that π_r is a Zariski locally trivial fibration with fiber $\mathbb{A}^n\backslash\mathbb{A}^{r-1}$. To do this, we define rational functions β_i as solutions to the $r-1$ equations $v_{rj} = \sum_{i=1}^{r-1} \beta_i v_{ij}$, where v_{rj} and v_{ij} are the components of the vectors in V_r and V_{r-1}, respectively. It can be shown that the β_i are regular on $\pi_r^{-1}(U)$, where U is an open set in V_{r-1}.

For any $(v_1, \ldots, v_{r-1}) \in U$ and $v_r \in V$, we define a vector $v'_r \in V$ with components given by

$$v'_{rj} := \begin{cases} v_{rj} & \text{if } j \leqslant r-1 \\ v_{rj} - \sum_{i=1}^{r-1} \beta_i v_{ij} & \text{if } r \leqslant j. \end{cases}$$

It can be shown that v'_r is contained in the span of $v_1, \ldots, v_{r-1}$ if and only if $v_r \in \langle v_1, \ldots, v_{r-1}\rangle$. Therefore, we obtain an isomorphism $\pi_r^{-1}(U) \xrightarrow{\sim} U \times \left(\mathbb{A}^n\backslash\mathbb{A}^{r-1}\right)$ given by $(v_1, \ldots, v_r) \mapsto (v_1, \ldots, v_{r-1}, v'_r)$.

Similarly, the fiber over any $U_\sigma := \{(v_1, \ldots, v_{r-1}) \in V_{r-1} \mid v_{i\sigma(i)} \neq 0, \forall i\}$, where $\sigma \in S_n$, is isomorphic to $U_\sigma \times \left(\mathbb{A}^n\backslash\mathbb{A}^{r-1}\right)$. These sets form an open cover of V_{r-1}, establishing that π_r is a Zariski locally trivial fibration with fiber $\mathbb{A}^n\backslash\mathbb{A}^{r-1}$.

By using Proposition 3.1.3, we can conclude that $[V_r] = \prod_{i=0}^{r-1} \left(\mathbb{L}^n - \mathbb{L}^i\right)$. This holds trivially for $r = 1$ and proves the claim for $r = n$.

Example 3.1.8. *Let $Fl(k_1, k_2, V) = \{L_1 \subset L_2 \subset V \mid \dim(L_1) = k_1, \dim(L_2) = k_2\}$ be the flag variety. Consider the morphism $\tau : Fl(k_1, k_2, V) \to Gr(k_2, V)$, which is a Zariski locally trivial fibration. In particular, when $k_1 = 1$, we have that τ is a Zariski locally trivial fibration with fiber $\mathbb{P}^{n-1}$, where $n = k_2$. So by Proposition 3.1.3, we obtain*

$$[Fl(1, n, V)] = [Gr(n, V)][\mathbb{P}^{n-1}].$$

Example 3.1.9. *By setting $f = 0$ in the second part of Corollary 3.1.4, we obtain $\left[\mathbb{A}^{n+1}\backslash\{0\}\right] = \left[\mathbb{A}^1\backslash\{0\}\right][\mathbb{P}^n]$, even though $\mathbb{A}^{n+1}\backslash\{0\}$ and $\left(\mathbb{A}^1\backslash\{0\}\right) \times \mathbb{P}^n$ are not isomorphic. This explains why Zariski locally trivial fibrations are highly valuable for calculations in $K_0(\mathrm{Var}_k)$: they often allow us to represent the class of a variety as a product, even if the variety itself is not a product.*

The following result shows us the generators of the Grothendieck ring.

Proposition 3.1.10. *Let k be a field of characteristic zero. The Grothendieck ring of varieties is generated by:*

(1) the classes of smooth varieties;

(2) the classes of projective smooth varieties;

(3) the classes of projective smooth connected varieties.

Proof. (1) Let $d = \dim(X)$. By Nagata's theorem (Theorem 2.3.4), X can be embedded as an open dense subset of a complete variety X'. Therefore, we have $[X] = \left[X'\right] - [Z]$, where $\dim(Z) \leqslant d - 1$. Next, using Hironaka's theorem (Theorem 2.3.2), we resolve the singularities of $\overline{X'} \longrightarrow X'$. This results in an expression of the form $\left[\overline{X'}\right] = \left[X'\right] - ([C] - [E])$, where E is the exceptional divisor and C is the center of the blow-up of X', and C is smooth and both $\dim(C)$ and $\dim(E)$ are at most $d - 1$. Thus, we can express X as a finite union of smooth varieties using induction on the dimension.

(2) This result can also be proven by employing a similar induction argument based on the dimension d of X. By using Nagata's theorem and Chow's lemma (Lemma 2.3.3), we can embed X into a complete variety that is birational to a projective variety. Then, by resolving the singularities and applying the same proof as described earlier, we obtain the desired result.

(3) By using property (2), we can express X as a finite union of smooth projective varieties. Let $X = \cup_{i=1}^{n} X_i$, where each X_i is smooth and projective. In this case, the connected components and irreducible components coincide, allowing us to represent each X_i as a finite union of its connected components. Therefore, we can express X as a finite union of smooth projective connected varieties.

□

The structure of the Grothendieck ring of varieties remains largely enigmatic. However, when the base field k has characteristic zero, two significant results emerge from Hironaka's resolution of singularities and the Weak Factorization theorem. The first result, contributed by Franziska Bittner (BITTNER, 2004), offers an alternative presentation of the group $K_0(\mathrm{Var}_k)$ in terms of more convenient generators and relations. This alternative presentation facilitates the construction of motivic invariants and finds applications in various contexts. In this paper, we will explore some of these applications. The second notable result, formulated by Larsen and Lunts, establishes a close connection between the Grothendieck ring of varieties and birational geometry. This result will be further examined in the subsequent section. Thus, we present Bittner's definition:

Proposition 3.1.11. . *The Grothendieck ring of k-varieties can be alternatively characterized by the following presentations:*

(1) ***(sm)*** *As the abelian group generated by the isomorphism classes of smooth varieties over k subject to the relations $[X] = [X - Y] + [Y]$, where X is smooth and $Y \subset X$ is a smooth closed subvariety.*

(2) ***(bl)*** *As the abelian group generated by the isomorphism classes of smooth complete k-varieties subject to the relation $[\emptyset] = 0$ and $[\mathrm{Bl}_Y X] = [X] - [Y] + [E]$, where X is smooth and complete, $Y \subset X$ is a closed smooth subvariety, $\mathrm{Bl}_Y X$ is the blow-up of X along Y and E is the exceptional divisor of this blow-up.*

Remark 3.1.12. *Restricting to quasi-projective varieties in case (sm) or projective varieties in case (bl) yields the same group. Additionally, we can consider only connected varieties in both presentations.*

Proof. We define the group mentioned in (1) of Proposition 3.1.11 as $K_0^{sm}(\mathrm{Var}_k)$. We will now show that the ring homomorphism

$$\varphi : K_0^{sm}(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Var}_k)$$

defined on the generators by

$$[X]_{sm} \longmapsto [X]$$

is an isomorphism.

To construct an inverse map, let $[X] \in K_0(\mathrm{Var}\, k)$. We stratify $X = \bigcup_{N \in \mathcal{N}} N$, where N is smooth and equidimensional, and $\bar{N}$ is a union of strata for all $N \in \mathcal{N}$. Consider the expression $\Sigma_{N\in\mathcal{N}}[N]_{sm}$ in $K0^{sm}(\mathrm{Var}\, k)$. If X is smooth, $\Sigma_{N\in\mathcal{N}}[N]_{sm} = [X]_{sm}$, as can be shown by induction on the number m of elements in $\mathcal{N}$:

Let $N \in \mathcal{N}$ be an element of minimal dimension. For the case $m = 1$, we have $[X]_{sm} = [X - N]_{sm} + [N]_{sm}$. By the induction hypothesis for $m = n - 1$, we have $[X]_{sm} = \Sigma_{i=1}^{n-1}[N_i]_{sm}$. Then, by definition, we have $[X - N_n]_{sm} = [X]_{sm} - [N_n]_{sm}$, and the induction hypothesis implies that $\Sigma_{i=1}^{n-1}[N_i]_{sm} = [X]_{sm} - [N_n]_{sm}$.

For two stratifications $\mathcal{N}$ and $\mathcal{N}'$ of X, we can always find a common refinement $\mathcal{L}$ (see Proposition 2.2.48). The above argument shows that for $N \in \mathcal{N}$, we get $\Sigma_{N\supset L\in\mathcal{L}}[L]_{sm}$. Hence, $\Sigma_{L\in\mathcal{L}}[L]_{sm}$ is equal to $\Sigma_{N\in\mathcal{N}}[N]_{sm}$, and analogously, it is equal to $\Sigma_{N\in\mathcal{N}'}[N]_{sm}$. Therefore, $\Sigma_{N\in\mathcal{N}}[N]_{sm}$ is independent of the choice of stratification. Thus, we can set $e(X) := \Sigma_{N\in\mathcal{N}}[N]_{sm}$.

If $Y \subset X$ is a closed subvariety, we can find a stratification in which Y is a union of strata, which yields $e(X) = e(X - Y) + e(Y)$. Thus, we have the following diagram:

$$\begin{tikzcd} \mathbb{Z}[Var_k] \arrow[r] \arrow[d, two heads] & K_0^{sm}(\mathrm{Var}_k) \\ K_0(\mathrm{Var}_k) \arrow[ur] & \end{tikzcd}$$

The induced map on $K_0^{sm}(\mathrm{Var}_k)$ is obviously an inverse for $K_0^{sm}(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Var}_k)$, which proves the isomorphism between the two groups: $K_0^{sm}(\mathrm{Var}_k)$ and $K_0(\mathrm{Var}_k)$.

We call the group defined in (2) of Definition 3.1.11 as $K_0^{bl}(\mathrm{Var}_k)$. The proof of the equivalence between this definition and the others can be found in (SAHASRABUDHE, 2007).

□

3.2 Stable birational geometry

The main result of this section establishes a unique homomorphism between $K_0(\mathrm{Var}_k)$ and $\mathbb{Z}[SB]$, where SB represents the multiplicative monoid of classes of stable birational equivalence of varieties. This result was originally proven by Larsen and Lunts in their paper (LARSEN; LUNTS, 2003). Their proof involves induction, where the induction step requires verification of various constructions and conditions on the image of the map. The proof is lengthy and complex. So, we present a new proof of the same result using Bittner's result as stated in Definition 3.1.11. This proof was developed by Neeraja Sahasrabudhe in her thesis (SAHASRABUDHE, 2007).

From this point onwards, we assume that k is an algebraically closed field of characteristic zero.

Definition 3.2.1. . *We say that (irreducible) varieties X and Y are stably birational if $X \times \mathbb{P}^k$ is birational to $Y \times \mathbb{P}^l$ for some $k, l \geqslant 0$.*

Example 3.2.2. *$\mathbb{P}_k^1$ and $\mathbb{P}_k^2$ are stably birational.*

Theorem 3.2.3. *Consider an abelian monoid $\mathbf{G}$ and its corresponding monoid ring $\mathbb{Z}[\mathbf{G}]$. Let $\mathfrak{M}$ be the multiplicative monoid consisting of isomorphism classes of smooth complete irreducible varieties. We have a monoid homomorphism $\Psi : \mathfrak{M} \longrightarrow \mathbf{G}$ satisfying the following conditions:*

(1) $\Psi([X]) = \Psi([Y])$ if X and Y are birational.

(2) $\Psi\,[\mathbb{P}^n] = 1$ for all $n \geqslant 0$.

Under these conditions, there exists a unique homomorphism

$$\Phi : K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}[\mathbf{G}]$$

such that $\Phi([X]) = \Psi([X])$ for all $[X] \in \mathfrak{M}$.

Proof. It is worth noting that according to Bittner's definition, we have $K_0(\mathrm{Var}_k) \cong \mathbb{Z}[\mathfrak{M}]/\sim$, where the equivalence relation $\sim$ is given by $[\mathrm{Bl}_Y X] - [E] = [X] - [Y]$. Let ρ denote the quotient map $\rho : \mathbb{Z}[\mathfrak{M}] \longrightarrow \mathbb{Z}[\mathfrak{M}]/\sim$.

Then we write $Z = Bl_Y X$ and let X be a smooth projective variety. Consider the blow-up map $\pi : Z \longrightarrow X$ with Y as its center, and let $E = \pi^{-1}(Y)$ be the exceptional divisor. With this notation, the equivalence relation $\sim$ can be written as $[Z] - [\pi^{-1}(Y)] = [X] - [Y]$.

Now, let $\Psi : \mathfrak{M} \longrightarrow \mathbf{G}$ be a map. From this, we obtain a natural extension $\widehat{\Psi} : \mathbb{Z}[\mathfrak{M}] \longrightarrow \mathbb{Z}[\mathbf{G}]$. Therefore, we have the following diagram:

$$\begin{tikzcd}
\ker p \arrow[r, equal] \arrow[d, hook] & \ker p \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d] \\
\ker \widehat{\psi} \arrow[r, hook] \arrow[d] & \mathbb{Z}[\mathfrak{M}] \arrow[r] \arrow[d, two heads] & \mathbb{Z}[G] \arrow[d, equal] \\
\dfrac{\ker \widehat{\psi}}{\ker p} \arrow[r, hook] & K_0\left(\mathrm{Var}_k\right) \arrow[r, dashed] & \mathbb{Z}[G]
\end{tikzcd}$$

By applying the Snake Lemma (Proposition 1.2.6), in order to obtain a map Φ : $K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}[\mathbf{G}]$, we need to show that $\mathrm{Ker}(\rho) \subseteq \mathrm{Ker}(\widehat{\Psi})$, which means that $\widehat{\Psi}([Z] - [\pi^{-1}(Y)]) = \widehat{\Psi}([X] - [Y])$. To demonstrate this, it is sufficient to show the following:

(1) $\Psi([Z]) = \Psi([X])$,

(2) $\Psi\left(\pi^{-1}([Y])\right) = \Psi([Y])$.

Item (1) follows from the fact that blowing up map is a birational map. For item (2), we recall Corollary 3.1.4, which states that $\Psi\left(\pi^{-1}([Y])\right) = \Psi([Y])\Psi\left([\mathbb{P}^r]\right)$ for some r determined by the codimension of the center Y in X.

□

Remark 3.2.4. *Consider SB, the multiplicative monoid of stable birational equivalence classes of varieties. There is a canonical surjective homomorphism $\Psi_{SB} : \mathfrak{M} \longrightarrow SB$ which satisfies conditions (1) and (2) of the Theorem 3.2.3 (with $\Psi_{SB} = \Psi, \mathbf{G} = SB$)). By definition, any homomorphism Ψ as in the theorem factors through Ψ_{SB}. Denote by Φ_{SB} the ring homomorphism from $K_0\left(\mathrm{Var}_k\right)$ to $\mathbb{Z}[SB]$, corresponding to Ψ_{SB} by the theorem.*

Note that the multiplication is well-defined: let $[X] = [X']$ in SB, i.e. $X \times \mathbb{P}^n$ and $X' \times \mathbb{P}^m$ are birational for some $n, m \geqslant 0$. Then for any variety Y also $Y \times X \times \mathbb{P}^n$ and $Y \times X' \times \mathbb{P}^m$

are birational, and hence one has $[Y \times X] = [Y \times X']$ in SB if Y is smooth, projective and irreducible.

Corollary 3.2.5. *Let $X_1, \ldots, X_k, Y_1, \ldots, Y_m$ be smooth complete connected varieties. Let $m_i, n_j \in \mathbb{Z}$ be such that*

$$\Sigma m_i [X_i] = \Sigma n_j \left[Y_j\right],$$

in $K_0(\mathrm{Var}_{\mathbb{C}})$. Then $k = m$ and after renumbering the varieties X_i and Y_i are stably birational and $m_i = n_i$.

Proof. Applying the ring homomorphism Φ_{SB} to the above equality, we obtain the equality in the monoid ring $\mathbb{Z}[SB]$:

$$\Sigma m_i \Psi_{SB}(X_i) = \Sigma n_j \Psi_{SB}\left(Y_j\right)$$

and the corollary follows. □

Thus, any variety can be written uniquely (upto stable birational equivalence) as linear combination (in $K_0(\mathrm{Var}_{\mathbb{C}})$) of smooth complete varieties.

Proposition 3.2.6. *The kernel of the (surjective)homomorphism $\Phi_{SB} : K_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}[SB]$ is the principal ideal generated by the class $\left[\mathbb{A}^1\right]$ of the affine line $\mathbb{A}^1$.*

Proof. Note that the uniqueness of Φ_{SB} follows directly from the above theorem applied to Ψ_{SB}. Consequently, we have $\Phi_{SB}([\mathbb{P}^n]) = 1$ for all n. In particular, $\Phi_{SB}\left(\left[\mathbb{P}^1\right]\right) = 1$. Thus, we obtain $\Phi_{SB}\left([1] + \left[\mathbb{A}^1\right]\right) = 1$ and $\Phi_{SB}\left(\mathbb{A}^1\right) = 0$.

Now, let $a \in \mathrm{Ker}(\Phi_{SB})$. We can express a as a linear combination

$$a = [X_1] + \ldots + [X_k] - [Y_1] - \ldots - [Y_l]$$

where X_i and Y_j are smooth, complete, and connected varieties. Since $\Phi_{SB}(a) = \sum \Psi_{SB}(X_i) - \sum \Psi_{SB}(Y_j) = 0$, we deduce that $k = l$. After renumbering, we can assume that X_i is stably birational to Y_i. Thus, it suffices to show that if X and Y are smooth, complete, and stably birational, then $[X] - [Y] \in K_0(\mathrm{Var}_k) \cdot \left[\mathbb{A}^1\right]$.

We observe that $\left[X \times \mathbb{P}^k\right] - [X] = [X] \cdot \left[\mathbb{A}^1 + \mathbb{A}^2 + \ldots + \mathbb{A}^k\right]$. Therefore, we can assume that $[X]$ and $[Y]$ are birational. Moreover, by the weak factorization theorem, we can further assume that X is a blow-up of Y with a smooth center $Z \subset Y$ and exceptional divisor $E \subset X$. Then, $[E] = \left[\mathbb{P}^t\right] \cdot [Z]$ for some t, and

$$[X] - [Y] = [E] - [Z] = \left(\left[\mathbb{A}^1\right] + \left[\mathbb{A}^2\right] + \ldots + \left[\mathbb{A}^t\right]\right)[Z].$$

□

Corollary 3.2.7. *Let X and X' be smooth and proper schemes over a field F of characteristic zero. Then X and X' are stably birational if and only if their classes $[X]$ and $[X']$ in $K(Var_F)$ are congruent modulo $\mathbb{L}$. In particular, $[X]$ is congruent to an integer c modulo $\mathbb{L}$ if and only if each of its connected components is stably rational; in that case, c is the number of connected components of X.*

3.3 The Grothendieck ring is not a domain

In this section we give two ideas to prove the fact that Grothendieck Ring is not a domain. The first one was proved by Poonen (POONEN, 2002) and consists essencially in using a result about abelian varieties. The second one consists in using a result on elliptic curves that was published by Tetsuji Shioda (SHIODA, 1977).

Remark 3.3.1. *It is worth noting that for any field extension $k \subseteq k'$, there exists a natural ring homomorphism $K_0(\mathrm{Var}_k) \longrightarrow K_0(\mathrm{Var}_{k'})$ that maps $[X]$ to $[X_{k'}]$ for any variety X. We have already established in Theorem 3.2.3 and Proposition 3.2.6 that when $k = \mathbb{C}$, there exists a unique ring homomorphism $K_0(\mathrm{Var}_k) \longrightarrow \mathbb{Z}[SB_k]$ that maps the class of any smooth projective integral variety to its stable birational class. Moreover, this homomorphism is surjective, and its kernel is generated by $\mathbb{L} = [\mathbb{A}^1]$.*

Consider the set AV_k of isomorphism classes of abelian varieties over k, which forms a monoid. The Albanese functor, which assigns to each smooth, projective, geometrically integral variety its Albanese variety, induces a homomorphism of monoids $SB_k \longrightarrow AV_k$. This follows from the fact that the Albanese variety is a birational invariant, the formation of the Albanese variety commutes with products, and the Albanese variety of $\mathbb{P}^n$ is trivial. Consequently, we obtain a ring homomorphism $Z[SB_k] \longrightarrow Z[AV_k]$.

The proof uses the above fact and the following lemma:

Lemma 3.3.2. *Let k be an algebraically closed field of characteristic zero. There exist abelian varieties A and B over k such that $A \times A \cong B \times B$ but $A_k \ncong B_k$.*

Theorem 3.3.3. *Let k be an algebraically closed field of characteristic zero. The Grothendieck ring of varieties $K_0\,(\mathrm{Var}_k)$ is not a domain.*

Proof. Consider the maps:

$$K_0\,(\mathrm{Var}_k) \longrightarrow \mathbb{Z}\,[SB_k] \longrightarrow \mathbb{Z}\,[AV_k]$$

Let A and B be as in Lemma 3.3.2. Then, the images of $[A] + [B]$ and $[A] - [B]$ are non-zero under the composition above. In fact, if we call this composition α and assume $[A] - [B] = 0$, we would have $\alpha([A]) = \alpha([B])$, and since A and B are abelian varieties, this implies $[A] = [B]$

in $\mathbb{Z}[AV_k]$, which only happens if $A \cong B$. Therefore, we have that $[A] + [B]$ and $[A] - [B]$ are non-zero elements, but $([A] + [B])([A] - [B]) = 0$.

$\square$

An alternative proof of Theorem 3.3.3 can be based on the following example:

Example 3.3.4. *Consider the elliptic curve $C(\tau) = C/\mathbb{Z} + \tau\mathbb{Z}$, and let $E = C(i)$, $E' = C(3i)$, and $E'' = C\left(\frac{-1+3i}{2}\right)$, where $i = \sqrt{-1}$. We have the following isomorphisms:*

$$E \times E' \cong E \times E'',$$

but $E' \ncong E''$.

This example, which is a result from the study of abelian varieties and can be found in (SHIODA, 1977), provides a counterexample to the claim that there are no zero divisors in $K_0(\mathrm{Var}_k)$. Now we can proceed with the proof of Theorem 3.3.3:

Proof. Note that $[E] \neq 0$, since E is non-empty. We also have $[E' - E''] \neq 0$, because otherwise, following a similar argument as the previous proof, we would have $E' \cong E''$. Now observe that $[E(E' - E'')] = [EE' - EE''] = 0$, which concludes the proof.

$\square$

Remark 3.3.5. *We have seen that $K_0(Var_k)$ is not an integral domain, this fact was proved by Poonen in 2002. So a natural question arises: is $\mathbb{L}$ a zero-divisor in $K_0(Var_k)$?*

Although $K_0(Var_k)$ is not a domain, it was possible that $\mathbb{L}$ is a non-zerodivisor. This was believed to be true for several mathematicians. However, in 2014, Lev Borisov proved that $\mathbb{L}$ is in fact a zero divisor, he shows that for two smooth derived equivalent non-birational Calabi-Yau 3-folds X and Y (pfaffian-grassmannian double mirror correspondence) we have

$$[X] - [Y](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = 0.$$

For further details and a complete exposition of this result, refer to reference (BORISOV, 2018).

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