UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

HANNAH DE LÁZARI DA COSTA E SILVA

The extended differential for approximate solutions of the Heterotic G_2 system

A diferencial estendida para soluções aproximadas do sistema heterótico G_2

Campinas

2023

Hannah De Lázari da Costa e Silva

The extended differential for approximate solutions of the Heterotic G_2 system

A diferencial estendida para soluções aproximadas do sistema heterótico G_2

Dissertação apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestra em Matemática.

Dissertation presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Master in Mathematics.

Supervisor: Henrique Nogueira de Sá Earp

Este trabalho corresponde à versão final da Dissertação defendida pela aluna Hannah De Lázari da Costa e Silva e orientada pelo Prof. Dr. Henrique Nogueira de Sá Earp.

Campinas

2023

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

C823e
Costa e Silva, Hannah de Lázari da, 1999-
The extended differential for approximate solutions of the heterotic G_2 system / Hannah De Lázari da Costa e Silva. – Campinas, SP : [s.n.], 2023.

Orientador: Henrique Nogueira de Sá Earp.
Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. G-estruturas. 2. Variedades sasakianas. 3. Geometria diferencial. 4. Teoria das supercordas. I. Sá Earp, Henrique Nogueira de, 1981-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações Complementares

Título em outro idioma: A diferencial estendida para soluções aproximadas do sistema heterótico G_2
Palavras-chave em inglês:
G-structures
Sasakian manifolds
Differential geometry
Superstring theory
Área de concentração: Matemática
Titulação: Mestra em Matemática
Banca examinadora:
Henrique Nogueira de Sá Earp [Orientador]
Viviana Jorgelina Del Barco
Eirik Eik Svanes
Data de defesa: 03-08-2023
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
- ORCID do autor: https://orcid.org/0009-0008-3587-9234
- Currículo Lattes do autor: http://lattes.cnpq.br/7631874017462532

Dissertação de Mestrado defendida em 03 de agosto de 2023 e aprovada pela banca examinadora composta pelos Profs. Drs.

Prof(a). Dr(a). HENRIQUE NOGUEIRA DE SÁ EARP

Prof(a). Dr(a). VIVIANA JORGELINA DEL BARCO

Prof(a). Dr(a). EIRIK EIK SVANES

A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

Acknowledgements

I would like to thank my supervisor, Henrique Sá Earp, for these many years of guidance, since my first steps into geometry back in 2018. Thanks for introducing me to the wonders of this rich subject, for the help in understanding the sometimes esoteric language, for the patience during the harder times, and the confidence that I would eventually get here. I would also like to thank Jason Lotay and Eirik Svanes, for the constructive discussions and for hosting my stays at the University of Oxford and the University of Stavanger, respectively. The months I stayed abroad were fundamental in this project's development and in my own, due to interactions with Jason and Eirik, and their respective research groups.

It is impossible to overestimate the importance of my friends and family during this project, even when most of them don't understand a word of what I was studying. Thanks for all the laughs, vents, beers, support and encouragement. In particular I need to deeply thank my closest friends, Julia and Carol, as I would not have gotten this far without them.

I'm also thankful to Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) for funding this project through the grant number 2020/15525-5, and funding my stays abroad through grants 2021/11442-0 and 2021/11467-3. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

Resumo

Nesse trabalho, revisamos os conceitos necessários para entender resultados recentes relacionados ao sistema heterótico G_2. Lembramos alguns fatos sobre geometria G_2, em particular relacionados à torção de uma G_2-estrutura e a G_2 instantons. Seguimos com a exploração de uma classe de 7-variedades Sasakianas chamadas de Calabi-Yau de contato, que são realizadas por fibrações por círculos sobre variedades Calabi-Yau, e admitem uma G_2-estrutura cocalibrada natural.

Em sequência, mostramos alguns aspectos de teoria das cordas heteróticas, e como sua compactificação em 7d leva ao sistema heterotico G_2. Então, revisamos um método recente de calcular o espaço de moduli infinitesimal do sistema heterotico G_2 desenvolvido por de la Ossa, Larfors and Svanes (2017), e soluções aproximadas deste sistema encontradas por Lotay and Sá Earp (2021) em variedades Calabi-Yau de contato.

Finalmente, mostramos dois resultados originais relacionados a propriedades destas soluções aproximadas: que a diferencial extendida definida por tais soluções não define a cohomologia necessária para as contas da de la Ossa, e que estas soluções são incompativeis com a estrutura de fibrado Sasakiano holomorfo do espaço tangente da variedade base.

Palavras-chave: Geometria G_2, systema Heterótico G_2, Calabi-Yau de contato

Abstract

In this work, we give an overview of the necessary concepts to understand recent results related to the heterotic G_2 system. We recall some facts of G_2 geometry, in particular related to the torsion of G_2-structures and to G_2 instantons. Then, we survey a class of Sasakian 7-manifolds called contact Calabi-Yau manifolds, which are realized as circle fibrations over Calabi-Yau threefolds, and admit a natural cocalibrated G_2-structure.

Following this, we show some aspects of heterotic string theory, and how its 7d compactification leads to the heterotic G_2 system. We then review a recent method of calculating the infinitesimal moduli space of the heterotic G_2 system developed by de la Ossa, Larfors and Svanes (2017), and approximate solutions found by Lotay and Sá Earp (2021) on contact Calabi-Yau manifolds.

Finally, we show two original results related to the properties of these approximate solutions: that the extended differential defined by these solutions don't define the cohomology necessary for de la Ossa's calculations, and that these solutions aren't compatible with the Saskian holomorphic bundle structure for the tangent space of the base manifold.

Keywords: G_2 Geometry, Heterotic G_2 system, contact Calabi-Yau.

Contents

Introduction 9
I Gauge theory on Special Geometries 11
A) G_2 geometry 11
B) Contact Calabi-Yau and Sasakian holomorphic bundles 20
II Heterotic string theory 24
A) Equations of motion and Supersymmetry 24
B) G_2 compactification 26
III Heterotic G_2 system 27
A) The infinitesimal moduli space 28
B) Approximate solutions 29
C) Extended differential for approximate solutions 31
IV Conclusions and further work 36

BIBLIOGRAPHY 37

Appendix 42
I Spinors and Supersymmetry 43
A) Clifford Algebra 43
B) Spinor representations and Gamma Matrices 44
C) Supersymmetry 49
II 7d compactification 50

Introduction

We give a general overview of G_2 geometry, in particular a class of 7-manifolds known as contact Calabi-Yau, and its application to Heterotic string theory. This fits into a larger context of gauge theories in higher dimensions and special geometries. We assume knowledge of manifolds, bundles and connections (LEE, 2013).

Gauge theory is, in the most general terms, the study of connections on bundles, and equations involving them. Probably the most famous gauge-theoretic equation is the Yang-Mills equation

$$d * F = 0, \tag{.1}$$

where F is the curvature of a connection A, $*$ is the Hodge star, and d is the covariant exterior derivative associated to A. This equation determines the critical points of the Yang-Mills functional

$$S_{YM}(A) = \int_X \mathrm{tr}(F \wedge *F), \tag{.2}$$

and is of great interest to physics. Furthermore, it has shown to be very fruitful to mathematics as well. In particular, Donaldson showed (DONALDSON, 1983) that, for 4-manifolds, we can construct a topological invariant using the moduli of anti-self-dual connections, i.e., connections that satisfy

$$-F = *F. \tag{.3}$$

These anti-self-dual connections are known as instantons, and are solutions to the Yang-Mills equation.

The success of Donalson's theory inspired the search for analogous constructions on higher dimensions, as proposed by Donalson and Thomas (DONALDSON; THOMAS, 1998). These higher dimensional instantons require special geometric structure, and so are related to special geometries and holonomy. For example, in dimension 7 we can define instantons when the manifold has G_2-holonomy through the equation

$$*F = \varphi \wedge F. \tag{.4}$$

where φ is the associative form of the G_2 structure. Even still, the respective moduli space has been only recently and scarcely studied (SÁ EARP, 2009; de la OSSA; LARFORS; SVANES, 2016), which is due to the difficulty of constructing G_2 holonomy manifolds. An abundance of examples and applications in physics leads to the study of instantons on manifolds with G_2-structures with torsion.

A class of torsional G_2-structure manifolds that has garnered recent attention is contact Calabi-Yau manifolds (Definition 15), first introduced in (TOMASSINI; VEZZONI, 2007) as manifolds with a transverse Calabi-Yau structure, and so transverse $SU(n)$ holonomy. Soon it was shown that these manifolds admit a natural co-calibrated G_2-structure in dimension 7, and so G_2 instantons were studied over them (CALVO-ANDRADE; DíAZ; Sá EARP, 2020).

Gauge theory has its origins in physics, where it was noted that the potentials in Maxwell's equations for electromagnetism had a certain internal (not spacial or temporal) symmetry, which came to be known as a gauge symmetry. It was eventually noticed that these potentials could be described as a $U(1)$-connection over spacetime, then the gauge symmetries were simply a change of coordinates and Maxwell's equations were the critical points of the Yang-Mills functional. This allowed the development of theories using other, non-Abelian structure groups, and today the standard model of particle physics uses a $SU(3) \times SU(2) \times U(1)$-connection, and describes the strong, weak and electromagnetic forces. For a more thorough review on the history of gauge theory in physics, see (O'RAIFEARTAIGH; STRAUMANN, 2000).

Note that the standard model does not describe gravity, which leads to many proposals to unify general relativity and particle physics (KIEFER, 2006). One such attempt, called string theory, substitutes the basic point-like particles from the standard model with strings, and requires extra, compact dimensions. Since this still has gauge symmetries, string theory is closely related to gauge theory in higher dimensions and special geometries. In fact, many of the studies of G_2-structure manifolds, with and without torsion, has been motivated by string and M-theory, e.g. (HARVEY; MOORE, 1999; FRIEDRICH; IVANOV, 2003b; de la OSSA; LARFORS; SVANES, 2016). This forms a plentiful source of gauge-theoretic problems, which can be inspiring for mathematics.

One particular type of string theory is heterotic string theory, which has both bosonic and fermionic chiral coordinates, was developed Gross *et al.* in the 1980's (GROSS et al., 1985a) and gives rise to a $\mathcal{N} = 1$ effective supergravity theory in dimension 10. The compactification of this theory to $\mathbb{R}^{1,9-d} \times M^d$ requires special structure on M. The case of $d = 6$ has been extensively studied (STROMINGER, 1986; BECKER et al., 2003; BECKER et al., 2004; GOLDSTEIN; PROKUSHKIN, 2004; FU; YAU, 2006), and gives rise to Kähler with torsion[1] manifolds (GILLARD; PAPADOPOULOS; TSIMPIS, 2003). Our interest is in $d = 7$, which gives rise to integrable G_2-structures (FRIEDRICH; IVANOV, 2003a; GAUNTLETT; MARTELLI; WALDRAM, 2004).

The moduli space of such compactifications have been shown to describe the massless state spectrum (de la OSSA; HARDY; SVANES, 2016) in dimension 6, and

[1] Kähler with torsion is not a commonly used term, but we are using it here to adhere to the notation in the reference (GILLARD; PAPADOPOULOS; TSIMPIS, 2003), where it is defined in the appendix.

this moduli space can be studied from the point of view of G_2 geometry (de la OSSA; LARFORS; SVANES, 2014). This lead to the recent interest of the moduli space of the G_2 compactification by de la Ossa *et al.* (de la OSSA; LARFORS; SVANES, 2016; de la OSSA; LARFORS; SVANES, 2017; de la OSSA et al., 2020), in which a method to compute the infinitesimal moduli space through G_2 Dolbeault cohomology (Theorem 9) of an extended differential has been constructed.

There also has been a construction of approximate solutions to the heterotic G_2 system (Definition 25) over contact Calabi-Yau manifolds (LOTAY; SÁ EARP, 2021). Given the recent advances on the study of instantons over contact Calabi-Yau manifolds, and on the construction of a method of computing the infinitesimal moduli of a G_2 compactification, it seems natural to try to calculate the moduli for these new solutions.

The main purpose of this work is to show the original theorems 28 and 30. Theorem 28 shows that the extended differential (III.4) defined by these approximate solutions fails to be a G_2 instanton, and thus we cannot define the G_2 Dolbeault cohomology and compute the infinitesimal moduli through de la Ossa *et. al.*'s method. Theorem 30 shows that these solutions are also not integrable in the Sasakian holomorphic sense (Definition 20), so this avenue is also not available.

In section 1, we recall the basic facts about G_2 geometry, since its definitions from vector cross products, the 16 classes of torsion and G_2 instantons. Then we define contact Calabi-Yau manifolds and Sasakian holomorphic bundles, showing their relation to G_2 geometry.

In section 2, we look at heterotic string theory, its supersymmetric conditions and equations of motion, and how the G_2 compactification restricts the geometry of the manifold.

Finally, in section 3 we show the results related to the heterotic G_2 system. First we go over the infinitesimal moduli space, then we show the construction of the approximate solutions. Lastly, we prove the original results of this work, theorems 28 and 30.

We gathered facts about spinors and supersymmetry in appendix I, as they aren't the focus of the work, but are necessary for certain calculations.

I Gauge theory on Special Geometries

A) G_2 geometry

The name G_2 can refer to three related simple Lie Groups, though we will generally only use it to talk about the compact real group. This group has the smallest exceptional simple Lie algebra, and is one of the two sporadic special holonomy groups in Berger's list along with Spin(7). These facts make this group particularly interesting both to mathematics and theoretical physics. The group G_2 is also related to vector cross products in dimension 7, which is a good way to introduce and define this group.

Vector products

We define a binary vector cross product as follows:

Definition 1. *A* ***vector cross product*** *is a non-trivial bilinear operator in* $\mathbb{R}^d$ *such that for any* $v_1, v_2 \in \mathbb{R}^d$,

$$\langle v_1 \times v_2, v_i \rangle = 0, \;\; i = 1, 2; \tag{I.1}$$
$$||v_1 \times v_2||^2 = ||v_1||^2 ||v_2||^2 - \langle v_1, v_2 \rangle^2 . \tag{I.2}$$

It is easy to check that this definition is consistent with the standard vector cross product in $\mathbb{R}^3$. In fact, any vector cross product in $\mathbb{R}^3$ differs from the standard one by at most a a factor of -1: Let $\{e_1, e_2, e_3\}$ be the coordinate orthonormal basis for $\mathbb{R}^3$, and $\times$ be any vector cross product. Then, from (I.1), we know $e_1 \times e_2$ must be orthogonal to both of these, so it must be a multiple of e_3, so $e_1 \times e_2 = \lambda_3 e_3$. From (I.2) we can deduce that $\lambda_3 = \pm 1$. Similarly, $e_2 \times e_3 = \lambda_1 e_1$ and $e_3 \times e_1 = \lambda_2 e_2$. Take $v_1 = e_1 + e_2 + e_3$, and $v_2 = v_2^i e_i$ be generic. Then, from (I.1), we have

$$\begin{aligned} 0 = \langle v_1 \times v_2, v_1 \rangle = & \sum_{i,k=1}^{3} v_2^j \langle e_i \times e_j, e_k \rangle \\ =& v_2^1(\langle e_2 \times e_1, e_3 \rangle + \langle e_3 \times e_1, e_2 \rangle) + v_2^2(\langle e_1 \times e_2, e_3 \rangle + \langle e_3 \times e_2, e_1 \rangle) + \\ & + v_2^3(\langle e_1 \times e_3, e_2 \rangle + \langle e_2 \times e_3, e_1 \rangle) \\ =& v_2^1(-\lambda_3 + \lambda_2) + v_2^2(\lambda_3 - \lambda_1) + v_2^3(-\lambda_2 + \lambda_1). \end{aligned}$$

Since v_2 is generic, we have that $\lambda_1 = \lambda_2 = \lambda_3$, so either they're all $+1$, which is the standard cross product, or they are all -1.

A reasonable question is in what other dimensions does such a cross product exist.

Theorem 2. *(GRAY; BROWN, 1967) Let* $\times$ *be a vector cross product in* $\mathbb{R}^d$*, then,* $d = 3$ *or* $d = 7$*.*

Proof. Assume $\mathbb{R}^d$ has a vector cross product. Define a multiplication on $\mathbb{R}^{d+1} = \mathbb{R}^d \oplus \{e\}$ by

$$(a + \alpha e)(b + \beta e) = a \times b - \langle a, b \rangle e + \beta a + \alpha b + \alpha\beta e, \qquad a, b \in \mathbb{R}^7, \alpha, \beta \in \mathbb{R}. \tag{I.3}$$

This gives $\mathbb{R}^{d+1}$ the structure of a composition algebra. However, composition algebras only exist in dimension 1, 2, 4 and 8 (JACOBSON, 1989). That would mean d is 0, 1, 3 or 7. Clearly, the 0 and 1 dimensional case are identically null, so we are left with $d = 3$ and $d = 7$. ■

An example of such a 7d cross product can be found using octonions.

Definition 3. *Let $\{e_0, ..., e_7\}$ be the coordinate orthonormal basis for $\mathbb{R}^8$. We add a division algebra structure to it by the following multiplication rule:*

$$e_i e_j = \begin{cases} e_i, & \text{if } j = 0; \\ e_j, & \text{if } i = 0; \\ -\delta_{ij} e_0 + \epsilon_{ijk} e_k, & \text{else} \end{cases} \tag{I.4}$$

where δ_{ij} is the Kronecker delta, and ϵ_{ijk} is a totally antisymmetric tensor that is 1 when $ijk = 123, 145, 176, 246, 257, 347, 365$.

The division algebra defined by this process is called the Octonions and is noted $\mathbb{O}$.

Define a cross product on $\mathbb{R}^7 = \operatorname{Im} \mathbb{O}$ by $a \times b = ab + \langle a, b \rangle e_0$

We know that the 3 dimensional cross product is unique up to sign, so what about the 7 dimensional case? To answer this question, we will show again that the 3d case is unique through a 3-form: Given a cross product $\times$, define a 3-form $\phi_\times$ by

$$\phi_\times(v, u, w) = \langle v \times u, w \rangle . \tag{I.5}$$

It is easy to check that $\phi_\times$ is multilinear, and that it is alternating, therefore it is in fact a 3-form. In 3d, $\phi_\times$ is the volume form, so if the transformations that preserve it are all those in SO(3). Since we are using the inner product to define this form, it only makes sense to deform it by elements in SO(3), so we conclude that the cross product is unique.

However, in 7d this is not the case. The 3-form defined by the cross product above is

$$\phi_0 = e^{123} + e^{145} - e^{176} + e^{246} - e^{257} - e^{347} + e^{365}, \tag{I.6}$$

and we have that

$$G_2 = \{A \in \mathrm{SO}(7); A^* \phi_0 = \phi_0\}. \tag{I.7}$$

Since G_2 is a proper subgroup of SO(7), the cross product is not unique in dimension 7.

Decomposition of forms into G_2 representations

The group G_2 — as we defined here — is the 14 dimensional compact Lie group with Lie algebra $\mathfrak{g}_2$, the smallest exceptional simple Lie algebra (FULTON; HARRIS, 1991, Section 21.2). A historical overview on the study of this group, see (AGRICOLA, 2008). Its fundamental representations have dimensions 7 and 14.

As a subset of $SO(7)$, G_2 has a natural action on $\mathbb{R}^7$, which extends to an action on k-forms. This defines different representations of G_2, which can be decomposed into its irreducible components, giving rise to additional structure on each $\Lambda^k\mathbb{R}^7$:

Proposition 4. *Each Λ^k can be decomposed into irreducible G_2-modules Λ^k_d of dimension d as follows:*

$$\begin{aligned} \Lambda^0 =& \Lambda^0_1; & \Lambda^7 =& \Lambda^7_1; \\ \Lambda^1 =& \Lambda^1_7; & \Lambda^6 =& \Lambda^6_7; \\ \Lambda^2 =& \Lambda^2_7 \oplus \Lambda^2_{14}; & \Lambda^5 =& \Lambda^5_7 \oplus \Lambda^5_{14}; \\ \Lambda^3 =& \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}; & \Lambda^4 =& \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}; \end{aligned}$$

Notice that the decomposition of Λ^k and Λ^{7-k} are identical. This is because the Hodge star is an isomorphism and preserves this decomposition. With this, we only need to describe the decomposition of 2 and 3-forms, as the rest are trivial or dual to these:

$$\begin{aligned} \Lambda^2_7 =& \{\alpha \in \Lambda^2;\ *(\alpha \wedge \phi_0) = -2\alpha\}; & \Lambda^3_1 =& \{f\phi_0;\ f \in \Lambda^0\} \\ \Lambda^2_{14} =& \{\alpha \in \Lambda^2;\ *(\alpha \wedge \phi_0) = \alpha\}; & \Lambda^3_7 =& \{X \lrcorner * \phi_0; X \in T\mathbb{R}^7\} \\ & & \Lambda^3_{27} =& \{\gamma \in \Lambda^3;\ \gamma \wedge \phi_0 = 0 = \gamma \wedge *\phi_0\} \end{aligned}$$

Other than the Hodge star, there are other natural isomorphisms between these spaces given by the 3-form ϕ_0 defined in (I.6) and its dual $*\phi_0$:

Proposition 5. *(KARIGIANNIS, 2005, Proposition 2.2.1) The map $\alpha \mapsto \phi_0 \wedge \alpha$ realizes the following isomorphisms:*

$$\begin{aligned} \Lambda^0_1 &\simeq \Lambda^3_1 & \Lambda^1_7 &\simeq \Lambda^4_7 \\ \Lambda^2_7 &\simeq \Lambda^5_7 & \Lambda^2_{14} &\simeq \Lambda^5_{14} \\ \Lambda^3_7 &\simeq \Lambda^6_7 & \Lambda^4_1 &\simeq \Lambda^7_1. \end{aligned}$$

The map $\alpha \mapsto (*\phi_0) \wedge \alpha$ realizes the following isomorphisms:

$$\begin{aligned} \Lambda^0_1 &\simeq \Lambda^4_1 & \Lambda^1_7 &\simeq \Lambda^5_7 \\ \Lambda^2_7 &\simeq \Lambda^6_7 & \Lambda^3_1 &\simeq \Lambda^7_1. \end{aligned}$$

G_2-structures on manifolds

Recall that the frame bundle $P_{GL}(M)$ of a manifold M^n is a $GL(n,\mathbb{R})$-bundle whose fibers are the linear isomorphisms from T_xM to $\mathbb{R}^n$, i.e. the local frames of the tangent bundle. Given a Lie subgroup $H \subset GL(n)$, a H-structure on M is reduction to a H-subbundle $P_H(M) \subset P_{GL}(M)$. For most cases, and in particular for the ones of interest to this work, a H structure can be equivalently defined by a collection of tensors $(\xi_1, ..., \xi_n)$ such that H is the stabilizer of all ξ_i, e.g., an orientable Riemannian manifold is a manifold with a metric g and a compatible volume form M, or equivalently a $SO(n)$ structure, since locally $Stab(g) \cap Stab(vol) = O(n) \cap SL(n) = SO(n)$. For a review on general geometric structures, see (JOYCE, 2000; FADEL et al., 2023).

Following this, a G_2-structure on M is a G_2-reduction of the frame bundle. Since we know G_2 is the stabilizer of ϕ_0 (I.6), this is equivalent to

Definition 6. *A G_2-structure on a 7-manifold M is a 3-form $\varphi \in \Omega^3(M)$, called the* ***associative form****, such that for each $p \in M$, there is a frame $f_p : T_pM \to \mathbb{R}^7$ such that $\varphi_p = f_p^*\phi_0$.*

Since $G_2 \subset \mathrm{SO}(7)$, we find that if M admits a G_2-structure, then it is orientable and admits a Riemannian metric. The metric induced by φ is given by

$$6g_\varphi(a,b)\,\mathrm{vol}_\varphi = (a \lrcorner \varphi) \wedge (b \lrcorner \varphi) \wedge \varphi, \tag{I.8}$$

where $\mathrm{vol}_\varphi \overset{loc}{=} dx^{1234567}$ in the frames f_p from the definition above.

The existence of a metric and a volume form permits us to define the Hodge operator $*_\varphi$ in these manifolds, which in turn is used to define the **coassociative 4-form** $\psi = *_\varphi\varphi$. We will usually omit the $_\varphi$ subscript for a cleaner notation, though it is worth keeping in mind this dependence.

It can be shown that a 7-manifold admits a G_2-structure if and only if it is orientable and spinnable (Theorem 43). In the appendix, we explore a bit more the relation between G_2-structures and spinors.

Everything discussed for k-forms on $\mathbb{R}^7$ extends to differential k-forms on M, with Ω^k_d denoting the sections of Λ^k_d.

Any $H \subset SO(n)$-structure has an intrinsic torsion, which lives in the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{so}(n)$. For our purposes, we can consider the torsion of a G_2-structure to be $\nabla^g\varphi$, where ∇^g is the Levi-Civita connection induced by g_φ.

Proposition 7. *(FERNáNDEZ; GRAY, 1982) The tensor $\nabla^g\varphi$ lives in the space*

$$W = \{\alpha \in T^*M \otimes \Omega^3; \alpha(X, Y \wedge Z \wedge (Y \wedge Z)) = 0,\ X, Y, Z \in \Gamma(TM)\},$$

that can be decomposed as $W = W_1 \oplus W_{14} \oplus W_{27} \oplus W_7$*, where* W_d *is a d-dimensional irreducible* G_2*-module.*

These are the torsion components of the G_2-structure. A more tractable description of the torsion is in terms of the torsion forms:

Proposition 8. *Let* φ *be the* G_2 *associative form of a 7 manifold, and* $\psi = *\varphi$*. The following decomposition exists:*

$$\begin{aligned} d\varphi =& \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3 \\ d\psi =& 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi, \end{aligned}$$

where $\tau_0 \in \Omega^0_1,\ \tau_1 \in \Omega^1_7,\ \tau_2 \in \Omega^2_{14},\ \tau_3 \in \Omega^3_{27}$ *are the torsion forms of* φ*.*

This result is simply the decomposition of $d\varphi$ and $d\psi$ into the irreducible G_2-modules, with the only surprise being the agreement of τ_1 in both. This is due to there only being one 7 component in the torsion of φ.

Given a G_2 structure φ, we can calculate the torsion forms by

$$\begin{aligned} \tau_0 =& \frac{1}{7} * (\varphi \wedge d\varphi); & \tau_1 =& \frac{1}{12} * (\varphi \wedge *d\varphi); \\ \tau_2 =& - *(d\psi) + 4 * (\tau_1 \wedge \psi); & \tau_3 =& * d\varphi - \tau_0\varphi - 3 * (\tau_1 \wedge \varphi). \end{aligned}$$

We can classify any G_2-structure into one of 16 types depending on which torsion forms vanish, such as:

(i) Torsion free: $\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$ (this is equivalent to $\nabla^g\varphi = 0$);

(ii) Nearly parallel: $\tau_1 = \tau_2 = \tau_3 = 0$;

(iii) Closed: $\tau_0 = \tau_1 = \tau_3 = 0$;

(iv) Coclosed: $\tau_1 = \tau_2 = 0$;

(v) Integrable: $\tau_2 = 0$.

Dolbeault cohomology

We have a particular interest in the **integrable** case. This is due to the possibility of defining a Dolbeault like differential complex as follows:

Theorem 9. *(FERNÁNDEZ; UGARTE, 1998) Let* (M, φ) *be a manifold with a* G_2*-structure and consider the following sequence:*

$$0 \longrightarrow \Omega^0 \xrightarrow{\check{d}_0} \Omega^1 \xrightarrow{\check{d}_1} \Omega^2_7 \xrightarrow{\check{d}_2} \Omega^3_1 \longrightarrow 0, \tag{I.9}$$

where $\check{d}_0 = d$, $\check{d}_1 = \pi_7 \circ d$, $\check{d}_2 = \pi_1 \circ d$, *and* $\pi_7 : \Omega^2 \to \Omega^2_7$, $\pi_1 : \Omega^3 \to \Omega^3_1$ *are the natural projections.*

Then, (I.9) is a differential complex (i.e. $\operatorname{im} \check{d}_i \subset \ker \check{d}_{i+1}$*) if and only if* φ *is integrable, that is,* $\tau_2 = 0$.

To prove this we will need the following Lemma:

Lemma 10. *If* M *is an integrable* G_2*-manifold and* $\beta \in \Omega^2_{14}$*, then* $d\beta \in \Omega^3_7 \oplus \Omega^3_{27}$.

Proof. Notice that since $\cdot \wedge \psi$ is an isomorphism from Ω^2_7 to Ω^6_7, $\beta \wedge \psi = 0$.

$$\begin{aligned} d\beta \wedge \psi =& d(\beta \wedge \psi) - \beta \wedge d\psi \\ =& - 4\beta \wedge \tau_1 \wedge \psi \\ =& 0. \end{aligned}$$

Similarly, the isomorphism $\Omega^3_1 \simeq \Omega^7_1$ implies $d\beta \in \Omega^3_7 \oplus \Omega^3_{27}$. ∎

Proof. (Theorem 9) Note first that $d_1 \circ d_0 = \pi_7 \circ d \circ d = 0$, since $d^2 = 0$, so we only need to check that $d_2 \circ d_1 = 0$.

Given $\alpha \in \Omega^1$, we can write $\check{d}_1\alpha = \pi_7 \circ d\alpha = d\alpha + \beta$, for some $\beta \in \Omega^2_{14}$, then we have

$$\begin{aligned} \check{d}_2 \circ \check{d}_1\alpha =& \check{d}_2(d\alpha + \beta) \\ =& \pi_1(d^2\alpha + d\beta) \\ =& \pi_1(d\beta) \\ =& 0. \end{aligned}$$

The last equality is due to Lemma 10. ∎

As a differential complex, we can define a cohomology with classes $H^k_{\check{d}} = \dfrac{\ker \check{d}_k}{\operatorname{im} \check{d}_{k-1}}$. This complex has also been shown to be elliptic.

In fact, if $E \to M$ is a vector bundle with a connection A such that $F_A \in \Omega^2_{14}(\operatorname{End}(E))$ (as we will see, this is to say A is a G_2 instanton: Definition 12), then this complex extends to bundle valued forms as well (de la OSSA; LARFORS; SVANES, 2017, Theorem 2).

Compatible connections

In the torsion-free case, we have that $\nabla^g \phi = 0$. This is equivalent to saying that the holonomy of g is contained in G_2. We will not discuss holonomy in depth here, but it is worth noting that this is one of the seven possible special holonomy groups in Berger's list (BERGER, 1955):

Theorem 11. *(JOYCE, 2000, Theorem 3.4.1) Suppose (M, g) is a simply-connected Riemannian manifold of dimension n, that is irreducible and non-symmetric. Then the holonomy of g is exactly one of these:*

i) $SO(n)$;

ii) $U(m)$, *with* $n = 2m$;

iii) $SU(m)$, *with* $n = 2m$;

iv) $Sp(m)$, *with* $n = 4m$;

v) $Sp(m)Sp(1)$, *with* $n = 4m$;

vi) G_2, *with* $n = 7$;

vii) $Spin(7)$, *with* $n = 8$.

In the general case, the Levi-Civita connection isn't compatible with the G_2-structure, but other connections are. In fact, in (BRYANT, 2005, Remark 8) Bryant shows there is a two parameter family of connections that preserve φ, and is only unique if the G_2-structure is closed or nearly parallel.

For integrable G_2-structures, one important G_2-compatible connection in heterotic string theory is the Bismut connection, which is the unique metric connection which makes φ parallel and has totally skew-symmetric torsion (LOTAY; SÁ EARP, 2021, Remark 3.14). A thorough exploration on the existence of connections with totally skew-symmetric torsion on general G-structures is done in (FRIEDRICH; IVANOV, 2003b). Though not G_2-compatible, the connection with opposite sign torsion to the Bismut connection is called the Hull connection, and is also useful for physics. We will denote the Bismut connection as ∇^+, and the Hull connection as ∇^-.

G_2 instantons

Another class of special connections are instantons

Definition 12. *A connection A on a G_2 manifold is called a G_2 instanton when its curvature F_A is in the 14 dimensional submodule of 2-forms, i.e., $F_A \in \Omega^2_{14} \simeq \mathfrak{g}_2$.*

This concept originates from the critical points to the Yang-Mills functional

$$S_{YM}(A) = \int_M \operatorname{tr}(F_A \wedge *F_A), \tag{I.10}$$

where A is a connection on a bundle over M, and F_A is its curvature. These are the focus of study in Gauge Theory. Using variational principles, we can show that the critical points of this functional are those that satisfy

$$d_A * F_A = 0, \tag{I.11}$$

known as the Yang-Mills equation. For compact Riemannian 4-manifolds, there is a class of solutions known as *instantons*, which are solutions to the equation

$$*F_A = \pm F_A, \tag{I.12}$$

and have been used to understand the geometry of such manifolds (DONALDSON; KRONHEIMER, 1997).

Since the curvature is a 2-form, this equation doesn't make sense in the context of higher dimensions. However, there is a generalization of the instanton equation for d dimensional manifolds, which depends on the choice of a $(d-4)$-form σ. This is the eigenvalue equation

$$\lambda F_A = *(\sigma \wedge F_A). \tag{I.13}$$

For an overview on gauge theory in higher dimensions, see (FADEL; Sá EARP, 2019). Thus, we want to look at connections A that satisfy the equation

$$\lambda F_A = *(\varphi \wedge F_A). \tag{I.14}$$

It is clear that if φ is closed, or indeed torsion-free, then these connections will always satisfy the Yang-mills equation (I.11), since

$$\begin{aligned} d_A * F_A =& d_A \frac{1}{\lambda}(\varphi \wedge F_A) \\ =& \frac{1}{\lambda}(d\varphi \wedge F_A - \varphi \wedge d_A F_A) \\ =& 0, \end{aligned}$$

by $d\varphi = 0$ and the Bianchi Identity. We will see, however, that there are non closed G_2-structures that also have this property.

Recall that $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$ (Proposition 4), with

$$\begin{aligned} \Lambda^2_7 =& \{\alpha; *(\varphi \wedge \alpha) = -2\alpha\} \\ \Lambda^2_{14} =& \{\alpha; *(\varphi \wedge \alpha) = \alpha\} \simeq \mathfrak{g}_2, \end{aligned}$$

so the only possible λ are -2 and 1. We call A a G_2 instanton if it curvature lies in the Lie algebra of G_2, i.e, when $\lambda = 1$.

Theorem 13. *Let A be a connection on a vector bundle E over a G_2-manifold. Then the following are equivalent:*

i) A *is a* G_2 *instanton;*

ii) $*(\varphi \wedge F_A) = F_A$*;*

iii) $F_A \wedge \psi = 0$*;*

iv) $F_A \in \mathfrak{g}_2$.

B) Contact Calabi-Yau and Sasakian holomorphic bundles

First studied in (TOMASSINI; VEZZONI, 2007), contact Calabi-Yau (cCY) manifolds are a generalization of Calabi-Yau structures for odd dimensions. The study of these manifolds from the point of view of Calabi-Yau foliations was done in (HABIB; VEZZONI, 2013) where it was shown that such manifolds have transverse $SU(n)$ holonomy. Habib and Vezzoni also showed that certain links are examples of cCY manifolds; subsequently, (CALVO-ANDRADE; DíAZ; Sá EARP, 2020) studied gauge theory over these links.

Recall the definition of a Sasakian manifold:

Definition 14. *A Sasakian manifold is a quintuple* (M, η, ξ, g, Φ) *where* (M, g) *is a Riemannian manifold,* (M, η) *is a contact manifold with Reeb field* ξ*, and* $\Phi \in \operatorname{End} TM$ *is a transverse almost complex structure, satisfying:*

(i) $g(\xi, \xi) = 1$*;*

(ii) $\Phi^2 = -\mathbb{I}_{TM} + \eta \otimes \xi$*;*

(iii) $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$*;*

(iv) $\nabla^g_X \xi = -\Phi X$*;*

(v) $(\nabla^g_X \Phi)Y = g(X, Y)\xi - \eta(Y)X$.

On Sasakian manifolds, we may use the transverse complex structure Φ to decompose the complexified tangent bundle $T_{\mathbb{C}}M$ similarly to how it's done on complex manifolds in even dimensions:

$$T_{\mathbb{C}}M = T_H^{(1,0)} \oplus T_H^{(0,1)} \oplus T_V, \tag{I.15}$$

where $T_V = \ker \Phi = \operatorname{span} \xi$ is the vertical space, and $T_H^{(1,0)}$ and $T_H^{(0,1)}$ are the i and $-i$ eigenspaces of Φ respectively, and together form the horizontal or transverse space T_H. This decomposition can, of course, be extended to differential forms. On T_H, we have that Φ is a complex structure, $\omega = d\eta$ is a symplectic 2-form, g is a metric, and they are compatible in the Kähler sense. This can be understood as M having a transverse $U(n)$ holonomy, or transverse Kähler structure. Naturally, we may ask when is there a transverse Calabi-Yau structure:

Definition 15. *A contact Calabi-Yau manifold* $(M, \eta, \xi, g, \Phi, \Omega)$ *is a Sasakian manifold* (M, η, ξ, g, Φ) *with a nowhere vanishing transverse form* Ω *of type* $(n, 0)$ *such that*

$$\begin{aligned} \Omega \wedge \bar{\Omega} =& (-1)^{n(n+1)/2} i^n \omega^n, \\ d\Omega =& 0. \end{aligned}$$

It is known (HABIB; VEZZONI, 2013) that cCY manifolds have transverse $SU(n)$ holonomy, and in dimension 7 ($n = 3$) they admit a co-calibrated G_2-structure

$$\varphi = \eta \wedge d\eta + \mathrm{Re}(\Omega), \tag{I.16}$$

$$\psi = \frac{1}{2} d\eta \wedge d\eta + \eta \wedge \mathrm{Im}\,\Omega \tag{I.17}$$

This makes cCY 7-manifolds appropriate for interpolations of Calabi-Yau 3-fold and G_2 geometries.

cCY example: $Y \times \mathbf{S}^1$

The trivial example of a cCY manifold is to take a Calabi-Yau manifold (Y, J, h), and consider the product $M = Y \times \mathbf{S}^1$. Using t as the coordinate on $\mathbf{S}^1$, we can construct a metric $g = dt^2 + h$, define $\eta = dt$ and $\xi = \partial_t$, extend the complex structure to $\Phi|_Y = J$ and $\Phi(\xi) = 0$, and a similar extension of Ω. Thus, $(M, \eta, \xi, g, \Phi, \Omega)$ is a contact Calabi-Yau manifold. If $\dim_{\mathbb{R}} Y = 6$, the G_2-structure on M defined as in (I.16) is torsion-free (JOYCE, 2004).

Calabi-Yau links

Another class of examples of cCY manifold can be found in the theory of singular hypersurface links:

Definition 16. *(CALVO-ANDRADE; DíAZ; Sá EARP, 2020, Definition 5) Let* $f : \mathbb{C}^{n+1} \to \mathbb{C}$ *be weighted homogeneous polynomial with degree* d *and weights* $w = (w_0, ..., w_n)$ *with an isolated critical point at 0, so that each sphere* $\mathbf{S}^{n+1} = \partial B_\epsilon(0)$ *intersects* $\mathcal{V} := f^{-1}(0)$ *transversely. Then* $K_f = \mathcal{V} \cap \mathbf{S}^{n+1}$ *is called a* ***weighted link of degree*** d ***and weight*** w*.*

These links admit a Sasakian structure (BOYER; GALICKI, 2008), and under certain circumstances, they are cCY.

Proposition 17. *(HABIB; VEZZONI, 2013, Proposition 6.7) Let* K_f *be a weighted link of degree* d *and weight* w*. Assume*

$$\sum_{i=0}^{n} w_i = d, \tag{I.18}$$

then K_f *admits a contact Calabi-Yau structure.*

We call these **Calabi-Yau links**. These examples can be understood as generalizations of our $Y \times \mathbf{S}^1$, as every Calabi-Yau link is a non-trivial $\mathbf{S}^1$ fibration over a Calabi-Yau orbifold.

Sasakian holomorphic bundle

The existence of a transverse complex structure allows us to define an analogue of holomorphic bundles over Sasakian manifolds, first introduced in (BISWAS; SCHUMACHER, 2010). We will follow the construction made on (PORTILLA; Sá EARP, 2019).

Definition 18. *Let $S \subset T_{\mathbb{C}}M$ be an integrable sub-bundle, and $E \to M$ a complex vector bundle. Then, a* ***partial connection*** *in the direction of S is an operator $D : \Gamma(E) \to \Gamma(S^*) \otimes \Gamma(E)$ satisfying the Leibniz rule*

$$D(fs) = fD(s) + q_S(df) \otimes s,$$

where $q_S : T^*_{\mathbb{C}}M \to S^*$ is the dual of the inclusion map.

By extending a partial connection D to act on bundle valued differential forms in the usual way, we can define the curvature of D as D^2. We know that both T_V and $T_H^{0,1} \oplus T_V$ are integrable sub-bundles, so we can define a complex Sasakian structure as follows:

Definition 19. *A* ***complex Sasakian*** *vector bundle on a Sasakian manifold M is a pair $\mathbf{E} = (E, D_0)$ where $E \to M$ is a complex vector bundle and D_0 is a partial connection in the direction of T_V.*

And a Sasakian holomorphic structure:

Definition 20. *A* ***Sasakian holomorphic*** *vector bundle on M is a pair $\mathcal{E} = (\mathbf{E}, \bar{\partial})$, where $\mathbf{E} = (E, D_0)$ is a complex Sasakian bundle and $\bar{\partial}$ is a flat partial connection in the direction of $T_H^{0,1} \oplus T_V$ whose restriction to T_V is D_0.*

With this, we can do gauge theory analogously to how it's done in the complex holomorphic case.

Gauge theory on Sasakian holomorphic bundles

A connection A on E is called **integrable** if its restriction to $T_H^{0,1} \oplus T_V$ is the Sasakian holomorphic structure partial connection. Since the partial connection needs to be flat, we find that F_A cannot have components in $\Omega_H^{(0,2)} \oplus \Omega_V^2$. A hermitian structure on $\mathbf{E}$ is a hermitian metric h that is compatible with D_0. If A is compatible with h, it is called **unitary**. This leads naturally to the notion of a Chern connection:

Proposition 21. *(PORTILLA; Sá EARP, 2019, Proposition A.6) Let $\mathcal{E}$ be a holomorphic Sasakian bundle with Hermitian structure h. Then there is a unique unitary and integrable connection $A_h \in \mathcal{A}(E)$ called the* ***Chern connection****.*

The notion of Hermitian Yang-Mills (HYM) is also adapted to the Sasakian case by satisfying

$$(F, \omega) = 0, \qquad F^{0,2} = 0, \tag{I.19}$$

with $\omega = d\eta$.

Theorem 22. *(PORTILLA; Sá EARP, 2019) Let A be the Chern connection of a Sasakian holomorphic bundle $\mathcal{E}$ over a 7-dimensional closed cCY manifold. Then A is a G_2 instanton if and only if it is HYM.*

Proof. Assume A is HYM.

$$\begin{aligned} F_A \wedge \psi =& F_A \wedge (\frac{1}{2}\omega \wedge \omega + \eta \wedge \mathrm{Im}(\Omega)) \\ =& \frac{1}{2}(F_A, \omega) + \eta \wedge F_A \wedge \mathrm{Im}\,\Omega \end{aligned}$$

Since A is HYM, $(F_A, \omega) = 0$, and since A is Chern, $F_A \in \Omega^{1,1}$ and so $F_A \wedge \mathrm{Im}\,\Omega = 0$. We conclude that $F_A \wedge \psi = 0$, i.e., A is a G_2 instanton.

Assume now that A is a G_2 instanton. Since A is Chern, $F_A^{0,2} = 0$. But $(F_A, \omega) = 0$ for the equality of the last case, so we conclude A is HYM.

■

Theorem 23. *(CALVO-ANDRADE; DíAZ; Sá EARP, 2020) Over a contact Calabi-Yau manifold with the natural cocalibrated G_2-structure, G_2 instantons are absolute minimizers of the Yang-Mills functional.*

Proof. First, define a charge

$$\kappa(A) = \int_M Tr F_A^2 \wedge \varphi. \tag{I.20}$$

It is shown in (CALVO-ANDRADE; DíAZ; Sá EARP, 2020) that $\kappa(A_h)$ for the chern connection does not depend on the choice of Hermitian structure h. This then defines a topological charge $\kappa(\mathcal{E})$.

We can decompose $F_A = F_7 + F_{14}$. Since this decomposition is orthogonal, we get

$$S_{YM}(A) = ||F_A||^2 = ||F_7||^2 + ||F_{14}||^2. \tag{I.21}$$

Similarly, we have

$$
\begin{aligned}
\kappa(\mathcal{E}) &= \int_M Tr(F_A \wedge (F_7 \wedge \varphi + F_{14} \wedge \varphi)) \\
&= \int (Tr(F_A(-2 * F_7 + *F_{14})) \\
&= -2||F_7||^2 + ||F_{14}||^2
\end{aligned}
$$

Therefore, $S_{YM}|_{\mathcal{A}(\mathcal{E})}(A) = \kappa(\mathcal{E}) + 3||F_7||^2$, and so, if the charge is positive, S_{YM} is minimum at $F_7 = 0$, which is the G_2 instanton condition. If the charge is negative, instantons cannot exist, since $S_{YM} \geq 0$. ∎

II Heterotic string theory

A fundamental issue with modern physics is the incompatibility of General Relativity and the Standard Model of particle physics. There have been many attempts to try to create a unified field theory that could describe all of physics. One of the earliest and most notorious of these unified theories is String Theory (ST), which describes the fundamental particles of nature as vibrating strings instead of being point-like. The first version of this theory only included bosonic strings, and required 26 dimensions. With the invention of supersymmetry, fermionic strings became possible and the number of dimensions required fell to 10. Consistency conditions for the cancelation of fermionic anomalies restricted known possibilities to three types of superstring theories (sST): Type I, with open and closed strings, and Types IIA and IIB, with only closed strings.

Every sST is required to reduce to a theory of supergravity. This leads to the possibility of gravitational anomalies, which Gaumé and Witten believed they showed obstructed 10-dimensional $\mathcal{N} = 1$ supergravity, so Type I sST was inconsistent. However, Green and Schwarz showed that there is an anomaly cancelation mechanism when the gauge group is $SO(32)$ or $E_8 \times E_8$. Only $SO(32)$ is consistent with Type I sST, but this also led to two new types of sST.

Heterotic string theory is a sST developed in the 1980's by Gross *et al.* (GROSS et al., 1985a; GROSS et al., 1985b; GROSS et al., 1986) with gauge group $SO(32)$ or $E_8 \times E_8$. The heterotic theory contains a combination of the 10-dimensional superstring and the 26-dimensional bosonic string, which leads to a $\mathcal{N} = 1$ supergravity theory in $D = 10$.

A) Equations of motion and Supersymmetry

Heterotic string theory has four bosonic fields, the metric g, a scalar field $\Phi \in \Omega^0$ called the dilaton, a 3-form $H \in \Omega^3$ called the NS field strength, and the connection A on a gauge bundle E over a manifold X. The effective bosonic action up to quartic terms is deduced in (BERGSHOEFF; DE ROO, 1989), and can be written as (IVANOV, 2010)

$$S = \frac{1}{2k^2}\int_X d^{10}x\sqrt{-g}e^{-2\Phi}\left[Scal^g + 4(\nabla^g\Phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\left(Tr|F|^2 - Tr|R|^2\right)\right], \quad \text{(II.1)}$$

Where F is the curvature of A on E and R is the curvature of a connection θ on TX (Remark 1).

The equations of motion up to two-loops are (GILLARD; PAPADOPOULOS; TSIMPIS, 2003):

$$Ric^g_{ij} - \frac{1}{4}H_{imn}H^{mn}_j + 2\nabla^g_i\nabla^g_j\Phi - \frac{\alpha'}{4}\left[F_{imab}F^{mab}_j - R_{imnq}R^{mnq}_j\right] + O(\alpha'^2) = 0 \quad \text{(II.2a)}$$

$$\nabla^g_i(e^{-2\Phi}H^i_{jk}) + O(\alpha'^2) = 0 \quad \text{(II.2b)}$$

$$\nabla^+_i(e^{-2\Phi}F^i{}_j) + O(\alpha'^2) = 0 \quad \text{(II.2c)}$$

Notice that the first equation reduces to the Einstein equation for supergravity when $\alpha' = 0$.

The Green-Schwarz anomaly cancelation mechanism requires that the 3-form H is globally defined and satisfies the heterotic Bianchi Identity (BI)

$$dH = \frac{\alpha'}{4}(Tr(R\wedge R) - Tr(F\wedge F)) + O(\alpha'^2). \quad \text{(II.3)}$$

Remark 1. *There is a controversy in the choice of connection θ on the tangent bundle. In (HULL, 1986), Hull showed that any connection that differs from the Levi-Civita by a covariant contorsion tensor would cancel the anomalies, so it might seem that ∇^+ would be the natural connection to consider, as it has reduced holonomy, however Hull argues that the connection with opposite sign torsion ∇^- can have useful symmetries in the $H \neq 0$ case. On the other hand, since the Hull connection ∇^- doesn't satisfy the equations of motion exactly, some, like de la Ossa et al. (de la OSSA; LARFORS; SVANES, 2017), choose an instanton connection, which will be the choice we follow in this work.*

In order for the theory to preserve supersymmetry, certain variations must vanish. This condition is equivalent to the existence of at least one Majorana-Weyl (appendix B)) spinor ϵ such that the following equations hold:

$$\left(\nabla^g_M + \frac{1}{4}H_{MNP}\Gamma^{NP}\right)\epsilon = \nabla^+\epsilon + O(\alpha'^2) = 0 \quad \text{(II.4a)}$$

$$(d\Phi - \frac{1}{2}H)\cdot\epsilon + O(\alpha'^2) = 0 \quad \text{(II.4b)}$$

$$F\cdot\epsilon + O(\alpha'^2) = 0, \quad \text{(II.4c)}$$

where $\Gamma^1, ..., \Gamma^{10}$ are the (1,9) gamma matrices for the metric g_{10}, i.e. $\Gamma^P\Gamma^N + \Gamma^N\Gamma^P = 2g^{NP}$, and $\Gamma^{NP} = [\Gamma^N, \Gamma^P]$.

The first equation (II.4a) restricts the holonomy of the torsion connection ∇^+, while (II.4c) restricts F to the Lie algebra of the stabilizer of ϵ, and thus restricts the connection A.

From now on, we will only consider the terms up to first order in α'. Notice that this makes it so that the SUSY conditions (II.4) are of zeroth order, while the BI (II.3) and the equations of motion (II.2) have first order terms.

When we compactify the theory to $X = \mathbb{R}^{1,9-d} \times M^d$, we get the following relation between the SUSY conditions and the equations of motion

Theorem 24 ((IVANOV, 2010))**.** *The heterotic supersymmetry equations (II.4) together with the anomaly cancellation (II.3) imply the heterotic equations of motion (II.2) on a manifold in dimensions five, six, seven and eight if and only if the connection θ on TM with curvature R is an SU(2), SU(3), G_2 and Spin(7) instanton in dimension five, six, seven and eight, respectively.*

B) G_2 compactification

At a macroscopic level, we clearly do not experience 10 spacetime dimensions. The solution to this, inspired by Kaluza-Klein theory (KLEIN, 1926), is that the extra dimensions are compact with a small radius. Usually, we take the ansatz that this compactification is of the form $X^{10} = \mathbb{R}^{1,9-d} \times M^d$, for a compact manifold M of dimension d. Since we usually want to consider a 4 dimensional spacetime, the most commonly studied case is $d = 6$. However, there has been some interest in other dimensions as well, which might give broader insights into the theory, and are of mathematical interest in their own right. Here we are interested in the case of $d = 7$, which can give insights into $d = 7$ compactifications in the 11 dimensional M-theory, but is also closely related to the 6d case.

Following the technique used in (STROMINGER, 1986), we can take the metric of X to decompose as

$$g_{10} = \begin{pmatrix} g_3 & 0 \\ 0 & g_7 \end{pmatrix} \tag{II.5}$$

into the metric of $\mathbb{R}^{1,2}$ and M^7. Assuming all fields vanish in the directions of $\mathbb{R}^{1,2}$, we can follow the 7d compactification done in (POLYDOROU; ROCéN; ZABZINE, 2017) and conclude that the SUSY conditions on M^7 have the same form as for the full space, i.e. substitute ϵ by η in equations (II.4) (see appendix II for details).

The existence of a ∇^+-parallel spinor (II.4a) in dimension 7 implies $Hol(\nabla^+) \subset G_2$, so M admits a G_2-structure. More than that, since the torsion of ∇^+ is skew-symmetric, (FRIEDRICH; IVANOV, 2003a) shows the G_2-structure must be integrable, and the torsion of ∇^+ is

$$T = H = - * d\varphi + \frac{1}{6}(d\varphi, \psi)\psi + *(4\tau_1 \wedge \varphi). \tag{II.6}$$

In this same paper, it shows that (II.4b) further implies $\tau_1 = -\frac{1}{2}d\Phi$, and the torsion simplifies to

$$H = - * d\varphi - 2(d\Phi \wedge \varphi). \tag{II.7}$$

Finally, the last SUSY equation implies F is a stabilizer of ϵ, i.e., $F \in \mathfrak{g}_2$, so A is a G_2-instanton.

III Heterotic G_2 system

As we see from the last section, the heterotic equations of motion for a G_2 compactification is determined by a set of geometric conditions. This means that we can rephrase them in terms of geometric constraints, analogously to the Hull-Strominger system in dimension 6 (GARCIA-FERNANDEZ, 2018), which we call the Heterotic G_2 system, and is defined as follows:

Definition 25. *(de la OSSA; LARFORS; SVANES, 2017) A heterotic G_2 system is a quadruple* $([Y, \varphi], [V, A], [TY, \tilde{\theta}], H)$*, where*

(i) Y is a 7-manifold with an integrable G_2-structure $\varphi \in \Omega^3(Y)$;

(ii) $\tilde{\theta}$ is a G_2 instanton connection on the tangent bundle TY, i.e., $\tilde{R}_\theta \wedge \psi = 0$;

(iii) A is a G_2 instanton connection on a vector bundle V over Y, i.e. $F_A \wedge \psi = 0$;

(iv) H is a 3-form on Y uniquely determined by φ through[2]

$$H = \frac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3, \tag{III.1}$$

where τ_i are the torsion forms of φ (Proposition 8), and satisfies the heterotic Bianchi identity, also known as the anomaly-free condition

$$dH = \frac{\alpha'}{4}(\mathrm{tr}(F_A \wedge F_A) - \mathrm{tr}(\tilde{R}_\theta \wedge \tilde{R}_\theta)). \tag{III.2}$$

Since this system implies the heterotic equations of motion (Theorem 24), it is of physical interest, but it is also a interesting geometrical condition in its own right.

[2] The 1/6 term differs from the previous result due to a difference in normalization of the volume form

A) The infinitesimal moduli space

The study of the moduli space of solutions to the heterotic system is motivated by the fact that it describes the massless spectrum of the theory (de la OSSA; HARDY; SVANES, 2016). A method of computing the infinitesimal moduli space (i.e. the tangent space of the moduli space) of the heterotic G_2 system was developed in (de la OSSA; LARFORS; SVANES, 2017), which we will recall here.

Given a heterotic G_2 system, we want to construct a bundle $\mathcal{Q}$ over Y, and a connection $\mathcal{D}$ such that

$$0 \longrightarrow \Omega^0_1(\mathcal{Q}) \xrightarrow{\check{\mathcal{D}}} \Omega^1_7(\mathcal{Q}) \xrightarrow{\check{\mathcal{D}}} \Omega^2_7(\mathcal{Q}) \xrightarrow{\check{\mathcal{D}}} \Omega^3_1(\mathcal{Q}) \longrightarrow 0 \tag{III.3}$$

is a differential complex, i.e $\check{\mathcal{D}}^2 = 0$ where $\check{\mathcal{D}}$ is the projection of $\mathcal{D}$, as is done in theorem 9, and the infinitesimal deformations of the system is contained in the first cohomology group $\check{H}^1_{\mathcal{D}}(\mathcal{Q})$.

Following (de la OSSA; LARFORS; SVANES, 2017), we take $\mathcal{Q} = T^*Y \oplus \mathrm{End}(TY) \oplus \mathrm{End}(V)$, and

$$\mathcal{D} = \begin{pmatrix} d_\theta & \mathcal{R} & -\mathcal{F} \\ \mathcal{R} & d_{\tilde{\theta}} & 0 \\ \mathcal{F} & 0 & d_A \end{pmatrix}, \tag{III.4}$$

where θ is the Hull connection on T^*Y, d_θ and $d_{\tilde{\theta}}$ are the covariant exterior derivatives associated to these connections, $\mathcal{F}$ is defined as

$$\mathcal{F} : \Omega^p(T^*Y) \oplus \Omega^p(\mathrm{End}(V)) \to \Omega^{p+1}(\mathrm{End}(V)) \oplus \Omega^{p+1}(T^*Y)$$
$$\begin{pmatrix} y \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{F}(y) \\ \mathcal{F}(\alpha) \end{pmatrix} = \begin{pmatrix} (-1)^p g^{ab} y_a \wedge F_{bc} e^c \\ (-1)^p \frac{\alpha'}{4} \operatorname{tr}(\alpha \wedge F_{ab} e^b) dx^a \end{pmatrix}$$

where $e^1, ..., e^n$ is a local frame for V. $\mathcal{R}$ is defined similarly, switching $\mathrm{End}(V)$ for $\mathrm{End}(TY)$, and F for R:

$$\mathcal{R}(y) = (-1)^p g^{ab} y_a \wedge R_{bc} dx^c,$$
$$\mathcal{R}(\kappa)_a = (-1)^p \frac{\alpha'}{4} \operatorname{tr}(\kappa \wedge R_{ab} dx^b)$$

This choice in fact defines a differential complex:

Theorem 26. *(de la OSSA; LARFORS; SVANES, 2017, Theorem 6) For the differential defined above, we have* $\check{\mathcal{D}}^2 = 0$.

The proof of this theorem is a specific case of theorem 28 when $\tilde{R} \wedge \psi = 0$.

We may note that, using theorem 9, we have that if $([Y,\varphi],[V,A],[TY,\tilde{\theta}],H)$ is a heterotic G_2 solution, then $\mathcal{D}$ is a G_2 instanton. The reciprocal is true up to an expansion in orders of α'.

Having a differential complex allows us to use the cohomology to compute moduli spaces, and in fact, as is shown in (de la OSSA; LARFORS; SVANES, 2017, Section 5.4), the infinitesimal moduli of heterotic solutions is given by the first cohomology $\check{H}^1_{\mathcal{D}}(\mathcal{Q})$.

B) Approximate solutions

For physical theories, we normally can get away with approximate (up to terms in higher order of α') solutions to the equations of motion. With this in mind Lotay and Sá Earp constructed a family of heterotic G_2 systems on contact Calabi-Yau manifolds (LOTAY; SÁ EARP, 2021) in which $\tilde{\theta}$ is only a G_2 instanton up to higher orders of α', i.e., $\tilde{R}\wedge\psi = O((\alpha')^N)$, for any choice of $N\in\mathbb{N}$. In this section, we describe these solutions.

Let (Y,η,ξ,J,Ω) be a contact Calabi-Yau 7-manifold, then we know Y is a $\mathbf{S}^1$ bundle over a Calabi-Yau 3-orbifold (X,ω,Ω), with connection 1-form η and curvature $d\eta=\omega$. For any $\epsilon>0$, we can define a S^1-invariant cocalibrated G_2-structure on Y by

$$\begin{aligned}\varphi_\epsilon =&\epsilon\eta\wedge\omega+\mathrm{Re}(\Omega),\\ \psi_\epsilon =&\frac{1}{2}\omega^2-\epsilon\eta\wedge\mathrm{Im}(\Omega).\end{aligned}$$

Lemma 27. *For each $\epsilon>0$, the torsion forms of the G_2-structure φ_ϵ are*

$$\begin{aligned}\tau_0 = \frac{6}{7}\epsilon, &\qquad \tau_1 = 0\\ \tau_2 = 0 &\qquad \tau_3 = \frac{8}{7}\epsilon^2\eta\wedge\omega - \frac{6}{7}\epsilon\,\mathrm{Re}\,\Omega.\end{aligned}$$

The flux is then

$$H_\epsilon = -\epsilon^2\eta\wedge\omega+\epsilon\,\mathrm{Re}\,\Omega.$$

Proof. First, remember that $d\psi = 4\tau_1\wedge\psi + *\tau_2$, and

$$d\psi_\epsilon = \frac{1}{2}(d\omega\wedge\omega+\omega\wedge d\omega)-\epsilon\omega\wedge\mathrm{Im}\,\Omega+\epsilon\eta\wedge\mathrm{Im}\,d\Omega.$$

Since $d\omega=d^2\eta=0$, the first term is zero; the second term is also zero, since ω is of transverse type $(1,1)$, and $\mathrm{Im}\,\Omega$ is $(3,0)+(0,3)$, so their product is of type $(4,1)+(1,4)$, which is a null space; finally, we know that $\mathrm{Im}\,d\Omega=0$, so we have $d\psi_\epsilon=0$, and $\tau_1=0=\tau_2$.

On the other hand, $d\varphi=\tau_0\psi+*\tau_3$, and

$$d\varphi_\epsilon = \epsilon\omega^2+\mathrm{Re}\,d\Omega=\epsilon\omega^2;$$

this means $d\varphi \wedge \varphi = \tau_0 \psi \wedge \varphi + *\tau_3 \wedge \varphi = \tau_0 7 \,\mathrm{vol}_\epsilon$, since $\psi \wedge \varphi = 7\,\mathrm{vol}$ and $*\tau_3 \in \Omega^4_{27}$ so it's product with φ is zero. So

$$\begin{aligned} 7\tau_0 \,\mathrm{vol}_\epsilon =& \epsilon\omega^2 \wedge (\epsilon\eta \wedge \omega + \mathrm{Re}(\Omega)) \\ =& 6\epsilon(\epsilon\eta \wedge \frac{\omega^3}{3!}) \\ =& 6\epsilon \,\mathrm{vol}_\epsilon, \end{aligned}$$

therefore, $\tau_0 = \dfrac{6}{7}\epsilon$. Now,

$$*\tau_3 = d\varphi - \tau_0\psi,$$

which gives us the result.

Since $H = \dfrac{1}{6}\tau_0\varphi - \tau_1 \lrcorner \psi - \tau_3$, we get the expected result.

■

Let $\pi : Y \to X$ be the circle fibration over the Calabi-Yau base. Then taking the vector bundle $V = \pi^* TX$ over Y with connection $A = \pi^*\nabla^{LC}$ the pullback of the Levi-Civita connection on X, we know by (CALVO-ANDRADE; DíAZ; Sá EARP, 2020, Section 4.3) that A is a G_2-instanton since it is the pullback of a Hermitian Yang-Mills connection.

To have a solution to the heterotic G_2 system, we only need an instanton connection on TX such that the heterotic Bianchi Identity (II.3) is satisfied.

Taking an orthonormal coframe compatible with the Sasakian structure $\{e^0 = \epsilon\eta, e^1, e^2, e^3, Je^1, Je^2, Je^3\}$, and consider the connection on TY

$$\theta^{\delta,k}_{\epsilon,m} = A + \frac{k\epsilon}{2}B + \frac{k\epsilon\delta}{2}C + \frac{km\epsilon}{2}e^0\mathcal{I}, \tag{III.5}$$

where

$$B = \begin{pmatrix} 0 & Je^T & -e^T \\ -Je & 0 & -e^0\mathbb{I}_3 \\ e & e^0\mathbb{I}_3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & Je^T & -e^T \\ -Je & -[e] & [Je] \\ e & [Je] & [e] \end{pmatrix} - e^0\mathcal{I}, \quad \mathcal{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathbb{I}_3 \\ 0 & \mathbb{I}_3 & 0 \end{pmatrix}, \tag{III.6}$$

and

$$[e] = \begin{pmatrix} e^1 \\ e^2 \\ e^3 \end{pmatrix} = \begin{pmatrix} 0 & e^3 & -e^2 \\ -e^3 & 0 & e^1 \\ e^2 & -e^1 & 0 \end{pmatrix}. \tag{III.7}$$

Then $\theta^{\delta,k}_{\epsilon,m}$ can be an approximate instanton on TY up to higher orders of α', but never exactly (LOTAY; SÁ EARP, 2021, Proposition 3.24), i.e., we can choose parameters such that $R^{\delta,k}_{\epsilon,m} \wedge \psi_\epsilon = O(\alpha'^N)$ for any $N > 1$.

As for the anomaly cancelation condition (II.3), it can be shown (LOTAY; SÁ EARP, 2021, Section 4.4) that it is satisfied if

$$-\epsilon^2\omega^2 = -\frac{\alpha'}{8}\lambda_0\epsilon^2\omega^2 \tag{III.8}$$

for

$$\lambda_0 = k^2\epsilon^2(k^2\delta^2(1+\delta)^2 + (1-\delta+m)(k^2(4\delta^2-(1+\delta)^2)-3)), \tag{III.9}$$

so we get an approximate solution to the heterotic G_2 system for appropriate choices of ϵ, δ, k and m.

C) Extended differential for approximate solutions

The approximate nature of these solutions should not be an issue for the underlying physics, however, this poses a difficulty for the infinitesimal moduli space calculation, as the extended differential $\mathcal{D}$ defined no longer satisfies $\check{\mathcal{D}}^2 = 0$, so we can't define $\check{H}^1_{\mathcal{D}}(\mathcal{Q})$.

Theorem 28. *Let (Y, φ) be a manifold with an integrable G_2 structure, $V \to Y$ a vector bundle, $\tilde{\theta}$ be a generic connection on* $\mathrm{End}(TY)$ *with curvature $\tilde{R}$, θ the Hull connection on T^*Y with curvature R, and A a G_2-instanton on* $\mathrm{End}(V)$ *with curvature F.*

Then, the covariant derivative defined in (III.4) satisfies

$$\mathcal{D}^2 \wedge \psi = \begin{pmatrix} (11) & (12) & 0 \\ (21) & (22) & (23) \\ 0 & (32) & 0 \end{pmatrix} \tag{III.10}$$

where

$$\begin{aligned}
((11)y)_a =& -\frac{\alpha'}{4}y^b \wedge \mathrm{tr}(\tilde{R}_{ba}\tilde{R}\wedge\psi);\\
((22)\kappa) =& \tilde{R}\wedge\psi\wedge\kappa - \kappa\wedge\tilde{R}\wedge\psi - \frac{\alpha'}{4}g^{ac}\,\mathrm{tr}(\kappa\wedge\tilde{R}_{ab}e^b)\wedge\tilde{R}_{cd}e^d\wedge\psi;\\
((12)\kappa)_a =& \frac{\alpha'}{4}\,\mathrm{tr}(\kappa\wedge\nabla^{\tilde{\theta}}_a(\tilde{R}\wedge\psi));\\
((21)y) =& y^a\wedge\nabla^{\tilde{\theta}}_a(\tilde{R}\wedge\psi);\\
((32)\kappa) =& -g^{ac}\frac{\alpha'}{4}\,\mathrm{tr}(\kappa\wedge\tilde{R}_{ab}e^b)\wedge F_{cd}e^d\wedge\psi;\\
((23)\alpha) =& g^{ac}\frac{\alpha'}{4}\,\mathrm{tr}(\alpha\wedge F_{ab}e^b)\wedge\tilde{R}_{cd}e^d\wedge\psi.
\end{aligned} \tag{III.11}$$

Proof. We will consider the action of

$$\mathcal{D}^2 = \begin{pmatrix} d_\theta + \mathcal{R}^2 - \mathcal{F}^2 & d_\theta\mathcal{R} + \mathcal{R}d_{\tilde{\theta}} & -d_\theta\mathcal{F} - \mathcal{F}d_A \\ \mathcal{R}d_\theta + d_{\tilde{\theta}}\mathcal{R} & \mathcal{R}^2 + d_{\tilde{\theta}}^2 & -\mathcal{R}\mathcal{F} \\ \mathcal{F}d_\theta + d_A\mathcal{F} & \mathcal{F}\mathcal{R} & -\mathcal{F}^2 + d_A^2 \end{pmatrix} \tag{III.12}$$

on a section

$$s = \begin{pmatrix} y \\ \kappa \\ \alpha \end{pmatrix} \in \Omega^p(\mathcal{Q}) \tag{III.13}$$

term by term. Many of the identities used in these calculations were proven in section 2.2 and appendix A of (de la OSSA; LARFORS; SVANES, 2017). We will begin by the terms that we know should be 0:

(13): $-(d_\theta\mathcal{F} + \mathcal{F}d_A)$ acting on $\alpha \in \Omega^p(\mathrm{End}(V))$.

$$\begin{aligned} \mathcal{F}(d_A\alpha)_a =& (-1)^{p+1}\frac{\alpha'}{4}\,\mathrm{tr}(d_A\alpha \wedge F_{ab}e^b); \\ d_\theta\mathcal{F}(\alpha)_a =& (-1)^p\frac{\alpha'}{4}(d\,\mathrm{tr}(\alpha \wedge F_{ab}e^b) - \theta_a^b \wedge \mathrm{tr}(\alpha \wedge F_{bc}e^c)) \\ =& (-1)^p\frac{\alpha'}{4}\,\mathrm{tr}(d_A(\alpha \wedge F_{ab}e^b) - \theta_a^b \wedge \alpha \wedge F_{bc}e^c) \\ =& (-1)^p\frac{\alpha'}{4}\,\mathrm{tr}(d_A\alpha \wedge F_{ab}e^b + (-1)^p\alpha \wedge (d_AF_{ab}e^b - \theta_a^b \wedge F_{bc}e^c)); \\ \mathcal{F}(d_A\alpha)_a + d_\theta\mathcal{F}(\alpha)_a =& \frac{\alpha'}{4}\,\mathrm{tr}(\alpha \wedge (d_AF_{ab}e^b - \theta_a^b \wedge F_{bc}e^c)) \end{aligned}$$

Using $d_AF_{ab}e^b = -(d_AF)_a + \partial_{Aa}F$ and $\partial_{Aa}F - \theta_a^b \wedge F_{bc}e^c = \nabla_a^A F$, we have

$$\begin{aligned} ((12)\alpha)_a =& \frac{\alpha'}{4}\,\mathrm{tr}(\alpha \wedge \nabla_a^A F \wedge \psi) \\ =& \frac{\alpha'}{4}\,\mathrm{tr}(\alpha \wedge (\nabla_a^A(F \wedge \psi) - F \wedge \nabla_a\psi)) \\ =& 0. \end{aligned}$$

(31): $\mathcal{F}d_\theta + d_A\mathcal{F}$ acting on $y \in \Omega^p(T^*Y)$.

$$\begin{aligned} d_A\mathcal{F}(y) =& (-1)^p d_A i_y(F) \\ =& (-1)^p\left([d_A, i_y]F + i_y d_AF\right) \\ =& (-1)^p\left((-1)^p y^a \wedge \nabla_a^A F + i_{d_\theta y}F\right) \\ =& y^a \wedge \nabla_a^A F - \mathcal{F}(d_\theta y) \end{aligned}$$

Therefore, we have

$$(d_A\mathcal{F}(y) + \mathcal{F}(d_\theta y)) \wedge \psi = y^a \wedge \nabla_a^A(F \wedge \psi) = 0$$

(33): $-\mathcal{F}^2 + d_A^2$ acting on $\alpha \in \Omega^p(\mathrm{End}(V))$. A is a G_2 instanton, so $d_A^2 \wedge \psi = 0$. This leaves us with

$$\mathcal{F}^2(\alpha) = -\frac{\alpha'}{4} g^{ac}\,\mathrm{tr}(\alpha \wedge F_{ab}e^b) \wedge F_{cd}e^d.$$

Since F_{ab} is in the 14 representation of G_2, $\mathcal{F}^2$ is in the $14 \times 14 = 1+14+27+77+77'$ (FEGER; KEPHART; SASKOWSKI, 2020, Table A.121), so it cannot have a component on 7. Since $\mathcal{F}^2 \in \Omega^2 = \Omega^2_7 \oplus \Omega^2_{14}$, we conclude that $\mathcal{F}^2 \in \Omega^2_{14}$.

(11): $d_\theta^2 + \tilde{R}^2 - \mathcal{F}^2$ acts on $y \in \Omega^p(T^*Y)$.

$$\begin{aligned} d_\theta^2 y_c =& - R^a_c \wedge y_a = -y^a \wedge R_{ca}; \\ \mathcal{F}^2(y)_c =& - \frac{\alpha'}{4} y^a \wedge \mathrm{tr}(F_{ab}e^b \wedge F_{cd}e^d); \\ \tilde{R}^2(y)_c =& - \frac{\alpha'}{4} y^a \wedge \mathrm{tr}(\tilde{R}_{ab}e^b \wedge \tilde{R}_{cd}e^d); \end{aligned}$$

The Heterotic Bianchi Identity can be written as

$$(dH)_{abcd}e^{bd} = \alpha'\,\mathrm{tr}(F_{ab}e^b \wedge F_{cd}e^d - F_{ac}F - \tilde{R}_{ab}e^b \wedge \tilde{R}_{cd}e^d + \tilde{R}_{ac}\tilde{R})$$

So the (11) term is

$$\begin{aligned} ((11)y)_c =& y^a \wedge \left(-R_{ca} + \frac{1}{4}(dH)_{abcd}e^{bd} + \frac{\alpha'}{4}\left[\mathrm{tr}(F_{ab}F) - \mathrm{tr}(\tilde{R}_{ab}\tilde{R})\right] \right) \wedge \psi \\ =& y^a \wedge \mathrm{tr}(\tilde{R}_{ab}\tilde{R} \wedge \psi), \end{aligned}$$

where we used $F \wedge \psi = 0$ and $(-R_{ca} + \frac{1}{4}(dH)_{abcd}e^{bd}) \wedge \psi = 0$. This second condition can be shown using (de la OSSA; LARFORS; SVANES, 2017, Proposition 6), which shows $R_{ab} \lrcorner \varphi = -\frac{1}{4}(dH)_{cdab}\varphi^{cd}_e e^e$, so

$$\begin{aligned} -R_{ca} \lrcorner \varphi + \frac{1}{4}(dH)_{abcd}e^{bd} \lrcorner \varphi =& \frac{1}{4}(dH)_{bdca}\phi^{bd}_e e^e + \frac{1}{4}(dH)_{abcd}\phi^{bd}_e e^e \\ =& \frac{1}{4}(-(dH)_{abcd} + (dH)_{abcd})\phi^{bd}_e e^e \\ =& 0 \end{aligned}$$

(22): $\mathcal{R}^2 + d_{\tilde{\theta}}^2$ acts on $\kappa \in \Omega^p(\mathrm{End}(TY))$.

$$\begin{aligned} \mathcal{R}^2(\kappa) =& - \frac{\alpha'}{4} g^{ac}\,\mathrm{tr}(\kappa \wedge \tilde{R}_{ab}e^b) \wedge \tilde{R}_{cd}e^d \\ (d_{\tilde{\theta}}^2 \kappa)^\mu_\nu =& \tilde{R}^\mu_\lambda \wedge \kappa^\lambda_\nu - \tilde{R}^\lambda_\nu \wedge \kappa^\mu_\lambda \\ (22)\kappa =& \tilde{R} \wedge \psi \wedge \kappa - \kappa \wedge \tilde{R} \wedge \psi - \frac{\alpha'}{4} g^{ac}\,\mathrm{tr}(\kappa \wedge \tilde{R}_{ab}e^b) \wedge \tilde{R}_{cd}e^d \wedge \psi \end{aligned}$$

(12): $d_\theta \mathcal{R} + \mathcal{R} d_{\tilde{\theta}}$ acts on $\kappa \in \Omega^p(\mathrm{End}(TY))$.

$$\begin{aligned}
\mathcal{R}(d_{\tilde{\theta}}\kappa)_a =& (-1)^{p+1}\frac{\alpha'}{4}\operatorname{tr}(d_{\tilde{\theta}}\kappa \wedge \tilde{R}_{ab}e^b);\\
d_\theta \mathcal{R}(\kappa)_a =& (-1)^p \frac{\alpha'}{4}(d\operatorname{tr}(\kappa \wedge \tilde{R}_{ab}e^b) - \theta_a^b \wedge \operatorname{tr}(\kappa \wedge \tilde{R}_{bc}e^c))\\
=& (-1)^p \frac{\alpha'}{4}\operatorname{tr}(d_{\tilde{\theta}}(\kappa \wedge \tilde{R}_{ab}e^b) - \theta_a^b \wedge \kappa \wedge \tilde{R}_{bc}e^c)\\
=& (-1)^p \frac{\alpha'}{4}\operatorname{tr}(d_{\tilde{\theta}}\kappa \wedge \tilde{R}_{ab}e^b + (-1)^p \kappa \wedge (d_{\tilde{\theta}}\tilde{R}_{ab}e^b - \theta_a^b \wedge \tilde{R}_{bc}e^c));\\
\mathcal{R}(d_{\tilde{\theta}}\kappa)_a + d_\theta \mathcal{R}(\kappa)_a =& \frac{\alpha'}{4}\operatorname{tr}(\kappa \wedge (d_{\tilde{\theta}}\tilde{R}_{ab}e^b - \theta_a^b \wedge \tilde{R}_{bc}e^c))
\end{aligned}$$

Using $d_{\tilde{\theta}}\tilde{R}_{ab}e^b = -(d_{\tilde{\theta}}\tilde{R}) + \partial_{\tilde{\theta}a}\tilde{R}$ and $\partial_{\tilde{\theta}a}\tilde{R} - \theta_a^b \wedge \tilde{R}_{bc}e^c = \nabla_a^{\tilde{\theta}}\tilde{R}$, we have

$$\begin{aligned}
((12)\kappa)_a =& \frac{\alpha'}{4}\operatorname{tr}(\kappa \wedge \nabla_a^{\tilde{\theta}}\tilde{R} \wedge \psi)\\
=& \frac{\alpha'}{4}\operatorname{tr}(\kappa \wedge (\nabla_a^{\tilde{\theta}}(\tilde{R} \wedge \psi) - \tilde{R} \wedge \nabla_a \psi))
\end{aligned}$$

Since $\nabla\psi = 0$, we get what we wanted.

(21): $\mathcal{R} d_\theta + d_{\tilde{\theta}}\mathcal{R}$ acts on $y \in \Omega^p(T^*Y)$.

$$\begin{aligned}
d_{\tilde{\theta}}\mathcal{R}(y) =& (-1)^p d_{\tilde{\theta}} i_y(\tilde{R})\\
=& (-1)^p \left([d_{\tilde{\theta}}, i_y]\tilde{R} + i_y d_{\tilde{\theta}}\tilde{R}\right)\\
=& (-1)^p \left((-1)^p y^a \wedge \nabla_a^{\tilde{\theta}}\tilde{R} + i_{d_\theta y}\tilde{R}\right)\\
=& y^a \wedge \nabla_a^{\tilde{\theta}}\tilde{R} - \mathcal{R}(d_\theta y)
\end{aligned}$$

Therefore, we have

$$(d_{\tilde{\theta}}\mathcal{R}(y) + \mathcal{R}(d_\theta y)) \wedge \psi = y^a \wedge \nabla_a^{\tilde{\theta}}(\tilde{R} \wedge \psi)$$

(32): $\mathcal{F}\mathcal{R}$ acts on $\kappa \in \Omega^p(\mathrm{End}(TY))$.

$$\mathcal{F}(\mathcal{R}(\kappa)) = -\frac{\alpha'}{4}g^{ac}\operatorname{tr}(\kappa \wedge \tilde{R}_{ab}e^b) \wedge F_{cd}e^d$$

(23): $-\mathcal{R}\mathcal{F}$ acts on $\alpha \in \Omega^p(\mathrm{End}(V))$.

$$-\mathcal{R}(\mathcal{F}(\alpha)) = \frac{\alpha'}{4}g^{ac}\operatorname{tr}(\alpha \wedge F_{ab}e^b) \wedge \tilde{R}_{cd}e^d$$

■

This means that, up to an expansion in orders of α', $\mathcal{D}$ is a G_2 instanton if and only if $\tilde{\theta}$ is one as well. This is evident in the term (22) in which the only zeroth order α' term is $\tilde{R} \wedge \psi \wedge \kappa - \kappa \wedge \tilde{R} \wedge \psi$, which is only zero for all κ if $\tilde{R} \wedge \psi = 0$. We know the solutions from (LOTAY; SÁ EARP, 2021) are not instantons, so we have

Corolary 29. *Let $\tilde{\theta}$ be the connection induced by $\theta^{\delta,k}_{\epsilon,m}$, and A induced from the HYM connection. Then, since $\tilde{\theta}$ is not an exact G_2-instanton, the constructed D isn't either.*

Note however that if $\tilde{R} \wedge \psi = O(\alpha'^N)$, then $\mathcal{D}^2 \wedge \psi = O(\alpha'^N)$.

With this, we find this solution isn't appropriate for computing the moduli using the methods described in (de la OSSA; LARFORS; SVANES, 2017). Even still, we can try to replicate the results in $d = 6$ that use integrable connections to construct the differential on the extended bundle (de la OSSA; SVANES, 2014). This could be possible due to the Sasakian holomorphic bundle structure that is possible for contact Calabi-Yau manifolds. Alas, the connections we have are not appropriate for this either

Theorem 30. *If $\theta^{\delta,k}_{\epsilon,m}$ is an integrable connection in the Sasakian holomorphic sense, then it cannot satisfy the Bianchi Identity.*

Proof. As we know, $\tilde{\theta} = \theta^{\delta,k}_{\epsilon,m}$ and (LOTAY; SÁ EARP, 2021, Proposition 3.21)

$$R^{\delta,k}_{\epsilon,m} = F_A + \frac{k\epsilon^2(1-\delta+m)}{2}\omega\mathcal{I} + \frac{k^2\epsilon^2}{4}Q^{\delta}_{m},$$

where

$$Q^{\delta}_{m} = (1-\delta+m)Q^{\delta}_{-} + (1+\delta)Q^{\delta}_{+} + \delta^2 Q_0.$$

If we want $\tilde{\theta}$ to be integrable, we want its curvature to be in the $\Omega^{(1,1)}_{H}$. F_A and ω are of type $(1,1)$, so we only need to adjust Q^{δ}_{m}.

$$Q_0 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -[e\times e + Je\times Je] & -2([e]\wedge[Je]-[Je]\wedge[e]) \\ 0 & 2([e]\wedge[Je]-[Je]\wedge[e]) & -[e\times e+Je\times Je] \end{pmatrix}$$

Lemma 2.8 of the same paper shows $-[e\times e + Je\times Je]$ and $[e]\wedge[Je]-[Je]\wedge[e]$ are of type $(1,1)$ so this term isn't a problem.

$$Q^{\delta}_{+} = \begin{pmatrix} 0 & 2\delta(e\times Je)^T & \delta(e\times e - Je\times Je)^T \\ -2\delta(e\times Je) & -(1+\delta)(Je\wedge Je^T) & (1+\delta)(Je\wedge e^T) \\ -\delta(e\times e-Je\times Je)^T & (1+\delta)(e\wedge Je^T) & -(1+\delta)(e\wedge e^T) \end{pmatrix}$$

The same lemma tells us $e\times Je$ and $e\times e - Je\times Je$ are of type $(2,0)+(0,2)$. Since Q^{δ}_{-} is a vertical form, we need to eliminate both these terms separately, which can be done by setting $m = \delta - 1$ and $\delta = 0$ or $\delta = -1$.

However, the condition for the heterotic bianchi identity is

$$-\epsilon^2\omega^2 = -\frac{\alpha'\lambda_0}{8}\omega^2,$$

where $\lambda_0 = k^2\epsilon^2(k^2\delta^2(1+\delta)^2 + (1-\delta+m)(k^2(4\delta^2-(1+\delta)^2)-3))$. Notice that for $\tilde{\theta}$ to be integrable, $\lambda_0 = 0$. Since $\epsilon > 0$ and ω is non zero, this means we cannot solve the Bianchi identity in this condition.

■

From physics, perhaps this is not surprising, as the solutions of (LOTAY; SÁ EARP, 2021) preserve $\mathcal{N} = 1$ supersymmetry ($\tau_0 \neq 0$), but imposing an integrable holomorphic structure on the full solution would indicate extended supersymmetry ($\mathcal{N} = 2$).

IV Conclusions and further work

The objective of this dissertation was to summarize the necessary concepts to understand the heterotic G_2 system, in particular the results on the infinitesimal moduli described in (de la OSSA; LARFORS; SVANES, 2017) and the approximate solutions constructed in (LOTAY; SÁ EARP, 2021). For this, we recalled some concepts on G_2 geometry, with a focus on the effects of torsion of G_2-structures on connections, in particular for instanton connections. Then, we introduced the notion of contact Calabi-Yau manifolds, a geometric structure studied only relatively recently, and that is the basis on which we build the approximate solutions to the heterotic system. Following this, we give an introduction to heterotic string theory, and describe the conditions the compactification to 7d imposes on the manifold.

In the last section, we describe in detail the main results, such as the construction of the extended differential $\mathcal{D}$ used to compute the infinitesimal moduli and the connections $\theta^{\delta,k}_{\epsilon,m}$ used to construct the approximate solutions. Finally, the author showed, as an original contribution, that the solutions of (LOTAY; SÁ EARP, 2021) aren't compatible with the methods used in (de la OSSA; LARFORS; SVANES, 2017) to compute the moduli, nor are they compatible with a Sasakian holomorphic generalization of the methods used in (de la OSSA; SVANES, 2014).

This work leads well to further research. One route is to try this computation for actual solutions of the system, such as the ones described in (CLARKE; GARCIA-FERNANDEZ; TIPLER, 2020) for T^3-bundles over K3 surfaces. Another route is to construct a new differential $\mathcal{D}$ that doesn't count the θ degrees of freedom, as is done in (MCORIST; SVANES, 2022) for dimension 6. One could also use the tools developed here to study other dimensions of compactifications, or other base theories.

Bibliography

AGRICOLA, I. Old and new on the exceptional group g 2. *Notices of the American Mathematical Society*, v. 55, p. 922–929, 09 2008. cited on page 14.

AGRICOLA, I.; CHIOSSI, S. G.; FRIEDRICH, T.; HöLL, J. Spinorial description of su(3)- and g2-manifolds. *Journal of Geometry and Physics*, v. 98, p. 535–555, 2015. ISSN 0393-0440. Disponível em: https://www.sciencedirect.com/science/article/pii/S0393044015002144. cited on page 49.

BECKER, K.; BECKER, M.; DASGUPTA, K.; GREEN, P. S. Compactifications of heterotic theory on non-kähler complex manifolds, i. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2003, n. 04, p. 007–007, apr 2003. Disponível em: https://doi.org/10.1088%2F1126-6708%2F2003%2F04%2F007. cited on page 10.

BECKER, K.; BECKER, M.; DASGUPTA, K.; GREEN, P. S.; SHARPE, E. Compactifications of heterotic strings on non-kähler complex manifolds II. *Nuclear Physics B*, Elsevier BV, v. 678, n. 1-2, p. 19–100, feb 2004. Disponível em: https://doi.org/10.1016%2Fj.nuclphysb.2003.11.029. cited on page 10.

BERGER, M. Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes. *Bulletin de la Société Mathématique de France*, Société mathématique de France, v. 83, p. 279–330, 1955. Disponível em: http://eudml.org/doc/86895. cited on page 18.

BERGSHOEFF, E. A.; DE ROO, M. The Quartic Effective Action of the Heterotic String and Supersymmetry. *Nucl. Phys. B*, v. 328, p. 439–468, 1989. cited on page 25.

BISWAS, I.; SCHUMACHER, G. Vector bundles on Sasakian manifolds. *Advances in Theoretical and Mathematical Physics*, International Press of Boston, v. 14, n. 2, p. 541 – 562, 2010. Disponível em: https://doi.org/. cited on page 22.

BOYER, C.; GALICKI, K. *Sasakian Geometry*. OUP Oxford, 2008. (Oxford Mathematical Monographs). ISBN 9780198564959. Disponível em: https://books.google.no/books?id=ERYZAQAAIAAJ. cited on page 21.

BRYANT, R. L. *Some remarks on G 2-structures*. 2005. cited on page 18.

CALVO-ANDRADE, O.; DíAZ, L. O. R.; Sá EARP, H. N. Gauge theory and G2-geometry on Calabi-Yau links. *Revista Matemática Iberoamericana*, European Mathematical Society Publishing House, v. 2020, Feb 2020. ISSN 0213-2230. Disponível em: http://dx.doi.org/10.4171/rmi/1182. cited on pages 10, 20, 21, 23, and 30.

CLARKE, A.; GARCIA-FERNANDEZ, M.; TIPLER, C. *T-Dual solutions and infinitesimal moduli of the G_2-Strominger system*. 2020. cited on page 36.

de la OSSA, X.; HARDY, E.; SVANES, E. E. The massless spectrum of heterotic compactifications. *Fortschritte der Physik*, Wiley, v. 64, n. 4-5, p. 365–366, mar 2016. Disponível em: https://doi.org/10.1002%2Fprop.201500066. cited on pages 10 and 28.

de la OSSA, X.; LARFORS, M.; MAGILL, M.; SVANES, E. E. Superpotential of three dimensional $$ \mathcal{n} $$ = 1 heterotic supergravity. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2020, n. 1, jan 2020. Disponível em: https://doi.org/10.1007%2Fjhep01%282020%29195. cited on page 11.

de la OSSA, X.; LARFORS, M.; SVANES, E. E. *Exploring SU(3) Structure Moduli Spaces with Integrable G2 Structures*. 2014. cited on page 11.

______. Infinitesimal moduli of g2 holonomy manifolds with instanton bundles. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2016, n. 11, nov 2016. Disponível em: https://doi.org/10.1007%2Fjhep11%282016%29016. cited on pages 9, 10, and 11.

______. The infinitesimal moduli space of heterotic g 2 systems. *Communications in Mathematical Physics*, Springer Science and Business Media LLC, v. 360, n. 2, p. 727–775, nov 2017. Disponível em: https://doi.org/10.1007%2Fs00220-017-3013-8. cited on pages 6, 7, 11, 17, 25, 27, 28, 29, 32, 33, 35, and 36.

de la OSSA, X.; SVANES, E. E. Holomorphic bundles and the moduli space of n=1 supersymmetric heterotic compactifications. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2014, n. 10, oct 2014. Disponível em: https://doi.org/10.1007%2Fjhep10%282014%29123. cited on pages 35 and 36.

DONALDSON, S.; KRONHEIMER, P. *The Geometry of Four-manifolds*. Clarendon Press, 1997. (Oxford mathematical monographs). ISBN 9780198502692. Disponível em: https://books.google.com.br/books?id=LbHmMtrebi4C. cited on page 19.

DONALDSON, S.; THOMAS, R. Gauge theory in higher dimensions. *The Geometric Universe*, 12 1998. cited on page 9.

DONALDSON, S. K. An application of gauge theory to four-dimensional topology. *Journal of Differential Geometry*, Lehigh University, v. 18, n. 2, p. 279 – 315, 1983. Disponível em: https://doi.org/10.4310/jdg/1214437665. cited on page 9.

FADEL, D.; LOUBEAU, E.; MORENO, A. J.; Sá EARP, H. N. *Flows of geometric structures*. 2023. cited on page 15.

FADEL, D. G.; Sá EARP, H. N. *Gauge Theory in Higher Dimensions*. [S.l.]: IMPA, 2019. (32º Colóquio Brasileiro de Matemática). ISBN 978-85-244-0428-3. cited on page 19.

FEGER, R.; KEPHART, T. W.; SASKOWSKI, R. J. LieART 2.0 – a mathematica application for lie algebras and representation theory. *Computer Physics Communications*, Elsevier BV, v. 257, p. 107490, dec 2020. Disponível em: https://doi.org/10.1016%2Fj.cpc.2020.107490. cited on page 33.

FERNÁNDEZ, M.; UGARTE, L. Dolbeault cohomology for g2-manifolds. *Geometriae Dedicata*, v. 70, p. 57–86, 1998. cited on page 16.

FERNáNDEZ, M.; GRAY, A. Riemannian manifolds with structure group g2. *Annali di Matematica Pura ed Applicata*, v. 132, p. 19–45, 12 1982. cited on page 15.

FRIEDRICH, T.; IVANOV, S. Killing spinor equations in dimension 7 and geometry of integrable g2-manifolds. *Journal of Geometry and Physics*, Elsevier BV, v. 48, n. 1, p. 1–11, oct 2003. Disponível em: https://doi.org/10.1016%2Fs0393-0440%2803%2900005-6. cited on pages 10 and 27.

______. *Parallel spinors and connections with skew-symmetric torsion in string theory*. 2003. cited on pages 10 and 18.

FU, J.-X.; YAU, S.-T. *The theory of superstring with flux on non-Kahler manifolds and the complex Monge-Ampere equation*. 2006. cited on page 10.

FULTON, W.; HARRIS, J. *Representation Theory: A First Course*. Springer New York, 1991. (Graduate Texts in Mathematics). ISBN 9783540974956. Disponível em: https://books.google.com.br/books?id=TuQZAQAAIAAJ. cited on page 13.

GARCIA-FERNANDEZ, M. *Lectures on the Strominger system*. 2018. cited on page 27.

GAUNTLETT, J. P.; MARTELLI, D.; WALDRAM, D. Superstrings with intrinsic torsion. *Physical Review D*, American Physical Society (APS), v. 69, n. 8, apr 2004. Disponível em: https://doi.org/10.1103%2Fphysrevd.69.086002. cited on page 10.

GILLARD, J.; PAPADOPOULOS, G.; TSIMPIS, D. Anomaly, fluxes and (2,0) heterotic-string compactifications. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2003, n. 06, p. 035–035, jun 2003. Disponível em: https://doi.org/10.1088%2F1126-6708%2F2003%2F06%2F035. cited on pages 10 and 25.

GOLDSTEIN, E.; PROKUSHKIN, S. Geometric model for complex non-kähler manifolds with SU (3) structure. *Communications in Mathematical Physics*, Springer Science and Business Media LLC, v. 251, n. 1, p. 65–78, sep 2004. Disponível em: https://doi.org/10.1007%2Fs00220-004-1167-7. cited on page 10.

GRAY, A.; BROWN, R. B. Vector cross products. *Commentarii mathematici Helvetici*, v. 42, p. 222–236, 1967. Disponível em: http://eudml.org/doc/139340. cited on page 12.

GROSS, D. J.; HARVEY, J. A.; MARTINEC, E.; ROHM, R. Heterotic string. *Phys. Rev. Lett.*, American Physical Society, v. 54, p. 502–505, Feb 1985. Disponível em: https://link.aps.org/doi/10.1103/PhysRevLett.54.502. cited on pages 10 and 24.

GROSS, D. J.; HARVEY, J. A.; MARTINEC, E. J.; ROHM, R. Heterotic String Theory. 1. The Free Heterotic String. *Nucl. Phys. B*, v. 256, p. 253, 1985. cited on page 24.

______. Heterotic String Theory. 2. The Interacting Heterotic String. *Nucl. Phys. B*, v. 267, p. 75–124, 1986. cited on page 24.

HABIB, G.; VEZZONI, L. *Some remarks on Calabi-Yau and hyper-Kähler foliations*. 2013. cited on pages 20 and 21.

HARVEY, F. *Spinors and Calibrations*. Elsevier Science, 1990. (Perspectives in mathematics). ISBN 9780123296504. Disponível em: https://books.google.no/books?id=3lTvAAAAMAAJ. cited on page 43.

HARVEY, J. A.; MOORE, G. *Superpotentials and Membrane Instantons*. 1999. cited on page 10.

HULL, C. Anomalies, ambiguities and superstrings. *Physics Letters B*, v. 167, n. 1, p. 51–55, 1986. ISSN 0370-2693. Disponível em: https://www.sciencedirect.com/science/article/pii/0370269386905447. cited on page 25.

IVANOV, S. Heterotic supersymmetry, anomaly cancellation and equations of motion. *Physics Letters B*, Elsevier BV, v. 685, n. 2-3, p. 190–196, mar 2010. Disponível em: https://doi.org/10.1016%2Fj.physletb.2010.01.050. cited on pages 25 and 26.

JACOBSON, N. Composition algebras and their automorphisms. In: ______. *Nathan Jacobson Collected Mathematical Papers: Volume 2 (1947–1965)*. Boston, MA: Birkhäuser Boston, 1989. p. 341–366. ISBN 978-1-4612-3694-8. Disponível em: https://doi.org/10.1007/978-1-4612-3694-8_24. cited on page 13.

JOYCE, D. *Compact Manifolds with Special Holonomy*. Oxford University Press, 2000. (Oxford mathematical monographs). ISBN 9780198506010. Disponível em: https://books.google.com.br/books?id=c3P-YUD8GZQC. cited on pages 15 and 18.

______. *The exceptional holonomy groups and calibrated geometry*. 2004. cited on page 21.

KARIGIANNIS, S. Deformations of g2 and spin(7) structures. *Canadian Journal of Mathematics*, Canadian Mathematical Society, v. 57, n. 5, p. 1012–1055, oct 2005. Disponível em: https://doi.org/10.4153%2Fcjm-2005-039-x. cited on page 14.

KIEFER, C. Quantum gravity: general introduction and recent developments. *Annalen der Physik*, Wiley Online Library, v. 15, n. 1-2, p. 129–148, 2006. cited on page 10.

KLEIN, O. The Atomicity of Electricity as a Quantum Theory Law. *Nature*, v. 118, p. 516, 1926. cited on page 26.

LAWSON, H.; MICHELSOHN, M. *Spin Geometry (PMS-38)*. Princeton University Press, 1990. (Princeton Mathematical Series, v. 38). ISBN 9780691085425. Disponível em: https://books.google.no/books?id=pZx_wAEACAAJ. cited on page 43.

LEE, J. *Introduction to Smooth Manifolds*. Springer New York, 2013. (Graduate Texts in Mathematics). ISBN 9780387217529. Disponível em: https://books.google.com.br/books?id=w4bhBwAAQBAJ. cited on page 9.

LOTAY, J. D.; SÁ EARP, H. N. *The heterotic G_2 system on contact Calabi–Yau 7-manifolds*. arXiv, 2021. Disponível em: https://arxiv.org/abs/2101.06767. cited on pages 6, 7, 11, 18, 29, 30, 31, 34, 35, and 36.

MCORIST, J.; SVANES, E. E. Heterotic quantum cohomology. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2022, n. 11, nov 2022. Disponível em: https://doi.org/10.1007%2Fjhep11%282022%29096. cited on page 36.

MICHELSOHN, M.-L. Spinors, twistors, and reduced holonomy(geometry of moduli spaces and 4-dimensional manifolds). v. 616, 3 1987. Disponível em: http://hdl.handle.net/2433/99835. cited on page 48.

O'RAIFEARTAIGH, L.; STRAUMANN, N. Gauge theory: Historical origins and some modern developments. *Reviews of Modern Physics*, APS, v. 72, n. 1, p. 1, 2000. cited on page 10.

POLYDOROU, K.; ROCéN, A.; ZABZINE, M. 7D supersymmetric Yang-Mills on curved manifolds. *Journal of High Energy Physics*, Springer Science and Business Media LLC, v. 2017, n. 12, Dec 2017. ISSN 1029-8479. Disponível em: http://dx.doi.org/10.1007/JHEP12(2017)152. cited on page 26.

PORTILLA, L. E.; Sá EARP, H. N. *Instantons on Sasakian 7-manifolds*. 2019. Https://arxiv.org/abs/1906.11334. cited on pages 22 and 23.

SÁ EARP, H. N. *Instantons on G2- manifolds (with Addendum)*. Tese (Doutorado), 2009. cited on page 9.

STROMINGER, A. Superstrings with torsion. *Nuclear Physics B*, v. 274, n. 2, p. 253–284, 1986. ISSN 0550-3213. Disponível em: https://www.sciencedirect.com/science/article/pii/0550321386902865. cited on pages 10 and 26.

TOMASSINI, A.; VEZZONI, L. *Contact Calabi-Yau manifolds and Special Legendrian submanifolds*. 2007. cited on pages 10 and 20.

VAN PROEYEN, A. *Tools for supersymmetry*. arXiv, 1999. Disponível em: https://arxiv.org/abs/hep-th/9910030. cited on pages 43 and 46.

WEINBERG, S. *The Quantum Theory of Fields: Volume 3, Supersymmetry*. Cambridge University Press, 2005. ISBN 9781139643436. Disponível em: https://books.google.no/books?id=QMkgAwAAQBAJ. cited on page 43.

Appendix

I Spinors and Supersymmetry

Our main references for this section are (LAWSON; MICHELSOHN, 1990; HARVEY, 1990) and (VAN PROEYEN, 1999; WEINBERG, 2005).

A) Clifford Algebra

Definition 31. *Let V be a vector space over a field $\mathbb{K}$, and q is a quadratic form on V. Then the **Clifford algebra** $Cl(V)$ is defined by*

$$Cl(V) = \bigotimes V/I(V),$$

where $\bigotimes V$ is the tensor algebra of V and $I(V)$ is the ideal generated by the elements of the form $v \otimes v + q(v)$ for $v \in V$.

From here on out, we will always use q as the norm squared induced by an inner product on V, and the field $\mathbb{K}$ as $\mathbb{R}$ or $\mathbb{C}$.

The first thing we may notice is, since for any $v \in V$, $v \otimes v + \langle v, v\rangle \in I(V)$, for we have that, in the Clifford algebra, $v \cdot v + \langle v, v\rangle = 0$, i.e.,

$$v \cdot v = -\langle v, v\rangle = -||v||^2, \tag{I.1}$$

which in turn, means for any $v, w \in V$,

$$v \cdot w + w \cdot v = -2\langle v, w\rangle. \tag{I.2}$$

This property actually characterizes the Clifford algebra, in the sense of the following Lemma:

Lemma 32. *[Fundamental Lemma of Clifford Algebras] Let A be an associative algebra with unit and $\phi : V \to A$ a linear map satisfying*

$$\phi(v)\phi(v) = -||v||^2 1, \ \forall v \in V.$$

Then, ϕ admits a unique extension to an algebra homomorphism $Cl(V) \to A$.

We use this lemma to define the canonical automorphism $u \mapsto \tilde{u}$ for $u \in Cl(V)$ as the unique extension of the linear map $v \mapsto -v$ for $v \in V$. Clearly, this operator squares to the Identity, so we can decompose $Cl(V)$ into the ± 1 eigenspaces of the canonical automorphism, $Cl^0(V)$, $Cl^1(V)$ called the even and odd subspaces. This provides a $\mathbb{Z}_2$ grading to $Cl(V)$.

As an algebra, we can look at the multiplicative group of units $Cl^*(V)$, given by all elements of $Cl(V)$ that have a multiplicative inverse. First, notice that $v \in V$ with $||v||^2 \neq 0$ is in $Cl^*(V)$, since $v^{-1} = -v/||v||^2$. The group generated by all $v \in V$ such that $||v||^2 = \pm 1$ is called the Pin group $Pin(V)$.

Lemma 33. *The center* $\operatorname{cen} Cl(V) = \{x \in Cl(V);\ xv = vx\ \forall v \in V\}$ *and the twisted center* $\operatorname{twcen} Cl(V) = \{x \in Cl(V); x\tilde{v} = vx\ \forall v \in V\}$ *of* $Cl(V)$ *are given by:*

- *if* $\dim V = n$ *is even:*

$$\operatorname{cen} Cl(V) = \Lambda^0 V, \qquad \operatorname{twcen} Cl(V) = \Lambda^n V;$$

- *if* $\dim V = n$ *is odd:*

$$\operatorname{cen} Cl(V) = \Lambda^0 V \oplus \Lambda^n V, \qquad \operatorname{twcen} Cl(V) = \{0\}.$$

Proposition 34. *The map* $\tilde{Ad} : Pin(V) \to O(V)$ *defined by* $\tilde{Ad}_y(v) = \tilde{y}vy^{-1},\ \forall y \in Pin(V), v \in V$ *is a double cover.*

Proof. For $v, w \in V$, with v non-null, we have $\tilde{Ad}_v(w) = -vwv^{-1} = vwv/||v||^2 = (-v^2w - v2\langle v, w\rangle)/||v||^2 = w - \dfrac{2\langle v, w\rangle}{||v||^2}v$, which is the reflection in the direction of v. Since $\tilde{Ad}_{vu} = \tilde{Ad}_v\tilde{Ad}_u$, $Pin(V)$ acts though composition of reflections, that is, it acts by orthogonal transformations on V. Since all orthogonal transformations can be achieved as a composition of reflections, we have that this mapping is onto. Now, if we show that $\ker(\tilde{Ad}) = \mathbb{Z}_2$, we will have shown this is a double cover.

Suppose that $x \in Pin(V)$ and $\tilde{Ad}_x = \mathbb{I}$, then we have that $\tilde{x}vx^{-1} = v,\ \forall v \in V$. If x is odd, we have $-xa = ax \iff x\tilde{a} = ax$, that is, x is in the twisted center of $Cl(V)$, which can't have odd elements by the previous lemma. If x is even, then $xa = ax$, and so $x \in \operatorname{cen} Cl(V)$, which implies $x \in \Lambda^0 V = \mathbb{R}$. Since $x \in Pin(V) \implies ||x|| = \pm 1$, we have $x \in \{1, -1\} = \mathbb{Z}_2$.

■

Similarly, we can define $Spin(V) = Pin(V) \cap Cl^0(V)$, and we will have that $Spin(V)$ is a double cover of $SO(V)$. This covering is, in fact, the universal cover of $SO(V)$ when $\dim V \geq 3$.

B) Spinor representations and Gamma Matrices

Since $Cl(V)$ only depends on the dimension and signature of V, we will denote $Cl_{p,q} = Cl(\mathbb{R}^{p+q})$, where $\mathbb{R}^{p+q}$ is the $(p+q)$-dimensional real vector space with inner product signature (p, q). For ease, $Cl_n = Cl_{n,0}$ as well.

We are interested in the representations of the spin group. We remind the general case of a representation of the Clifford algebra

Definition 35. *A* $\mathbb{K}$*-representation of* $Cl_{p,q}$ *is an algebra homomorphism*

$$\rho : Cl_{p,q} \to Hom(W),$$

where W is a vector space over $\mathbb{K}$.

We often simplify notation by writing $\rho(\phi)w = \phi \cdot w$, and we call this operation the **Clifford product**. We are interested in real and complex representations of $Cl_{p,q}$, and remark that any complex representation extends to a representation of $Cl_{p,q} \otimes \mathbb{C} = \mathbb{C}l_{p+q}$. Representations are said to be equivalent if they are related by a linear isomorphism, and they are said to be irreducible if W has no proper invariant subspaces.

Proposition 36. *For n even, there is a single inequivalent complex representation of $\mathbb{C}l_n$, with dimension $2^{n/2}$.*

For n odd, there are two inequivalent complex representations of $\mathbb{C}l_n$, with dimension $2^{(n-1)/2}$.

Since $Spin_{p,q} \subset \mathbb{C}l^0_{p+q} \equiv \mathbb{C}l_{p+q-1}$, we'll find that the parity relations of the representations of the spin group are flipped.

Definition 37. *The complex spinor representation of the spin group $Spin_n$,*

$$\Delta_n : Spin_n \to GL_{\mathbb{C}}(S),$$

is given by the restriction of a irreducible representation $\mathbb{C}l_n \to Hom_{\mathbb{C}}(S)$ to $Spin_n \subset \mathbb{C}l^0_n \subset \mathbb{C}l_n$;. S is then called the space of spinors.

Proposition 38. *For n odd, this representation is irreducible and independent of the choice of irreducible representation of $\mathbb{C}l_n$.*

For n even, this representation is reducible, with $S = S_+ \oplus S_-$ leading to two irreducible representations.

The fact that, in even dimensions, the spinor space is reducible leads to the concept of **Weyl spinors**, sometimes known as *chiral spinors*, which is when a spinor is in one of $S_\pm$. These representations can be realized with gamma matrices, a very important tool for doing calculations involving spinors.

We say a set of $d \times d$ matrices $\{\gamma^1, \gamma^2, ..., \gamma^n\}$ are gamma matrices if they satisfy the following relations:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} = \begin{cases} \pm 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases},$$

where the sign of ± 1 is determined by the signature of the inner product used.

From lemma 32, it is easy to see that the map $e_\mu \mapsto \gamma^\mu$, where e_μ is a basis for $\mathbb{C}^n$, extends to a representation of $\mathbb{C}l_n$. When d is the appropriate dimension, this is the irreducible representation. For $n = 3$, the gamma matrices are the well known Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{I.3}$$

In fact, for any n, there is a way to explicit construct gamma matrices using Pauli matrices:

$$\begin{aligned}
\gamma^1 &= \sigma^1 \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots \\
\gamma^2 &= \sigma^2 \otimes \mathbb{I} \otimes \mathbb{I} \otimes \cdots \\
\gamma^3 &= \sigma^3 \otimes \sigma^1 \otimes \mathbb{I} \otimes \cdots \\
\gamma^4 &= \sigma^3 \otimes \sigma^2 \otimes \mathbb{I} \otimes \cdots \\
\gamma^5 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \cdots \\
&\vdots
\end{aligned}$$

For even dimensions, this representation has dimension $d = 2^{n/2}$, and for odd dimensions, we can ignore the final σ^1, and so we have $d = 2^{n-1/2}$, so this is a irreducible representation. For other signatures, multiply the first q matrices by i and we get the $(n-q, q)$ signature matrices. There are other possibilities of gamma matrices, and these might not be the most appropriate for a specific application.

For even dimensions, we can also define the chirality element $\gamma^{n+1} = \lambda\gamma^1\gamma^2\cdots\gamma^n$, where λ is a normalization so that $\gamma^{n+1}\gamma^{n+1} = 1$. Since γ^{n+1} squares to 1, we can decompose the spinor space into the ± 1 eigenspaces with $S = S_+ \oplus S_-$. As the notation suggests, this is in fact the Weyl decomposition into irreducible spinor representations described earlier.

Another decomposition we can look at is the **Majorana** condition. In any dimension, there will be at least one matrix C satisfying $C^t = -\epsilon C$ and $\gamma^{\mu t} = -\eta C\gamma^\mu C^{-1}$, for $\epsilon, \eta = \pm 1$ called the charge conjugation matrix. For euclidean metrics, the Majorana condition is expressed as $s^\dagger = s^t C$, for other signatures, we must modify the left hand side a bit (VAN PROEYEN, 1999). However, if the gamma matrices are all real or purely imaginary (known as a Majorana representation), the Majorana condition becomes that the spinor s must be real.

The existence of spinors that are both Majorana and Weyl only happens when $p - q \equiv 0 \pmod 4$.

Spinor Bundles

Let M be a manifold with an orientable vector bundle E of dimension n. Consider then the $SO(n)$-bundle of orthonormal frames $P_{SO}(E)$.

Definition 39. *Suppose $n \geq 3$[3], then a* ***spin structure*** *on E is a $Spin(n)$-bundle $P_{Spin}(E)$ with a 2-covering*

$$\xi : P_{Spin}(E) \to P_{SO}(E) \tag{I.4}$$

such that $\xi(pg) = \xi(p)\tilde{Ad}, p \in P_{Spin}(E),\ g \in Spin(n)$ and $\tilde{Ad}$ is the universal covering defined in proposition 34.

The condition for the existence of a spin structure on E is topological, as it exists if and only if the second Stiefel-Whitney class of E is zero.

When TM admits a spin structure, we say M is a spin manifold. Similar to the construction of spinors from the representation of the spin group, we can make a spinor bundle from a bundle with spin structure.

Definition 40. *A* ***real spinor bundle*** *of E is the associated bundle*

$$S(E) = P_{Spin}(E) \times_{\mu} L, \tag{I.5}$$

where L is a left module of Cl_n and $\mu : Spin(n) \to SO(L)$ is the left multiplication of $Spin(n) \subset Cl^0$.

Analogously, a ***complex spinor bundle*** *is constructed using $\mathbb{C}l_n$.*

Spin connection

With spinor bundles, we can now talk about spinor fields over a manifold as sections of such bundles, and with this comes the question of how what is a constant spinor field, or more generally, how do we differentiate spinor fields. In particular, we are interested in a covariant derivative on $S(E)$ induced by a covariant derivative on E.

Definition 41. *Let A be a $SO(n)$ connection on E. Then, the induced spin covariant derivative ∇^s on $S(E)$ is defined by*

$$\nabla^s \sigma = d\sigma + A \cdot \sigma, \tag{I.6}$$

where $\cdot$ is the Clifford multiplication.

This definition is sound since $A \in \Omega^1(\mathfrak{so}(n))$, so the 1-form acts through the Clifford multiplication, and $\mathfrak{so} = \mathfrak{spin}$ are the standard structure preserving endomorphisms.

In coordinates, this can be expressed as:

[3] We can define a spin structure for $n = 1$ and 2 with a slight modification.

Proposition 42. *Let $\omega \in \Omega^1(\mathfrak{so}(n))$ be the connection 1-form on $P_{SO}(E)$. Then, the spin covariant derivative in coordinates is given by*

$$\nabla^s \sigma_N = \frac{1}{2} \sum_{i<j} \tilde{\omega}_{ji} e_i e_j \cdot \sigma_N, \tag{I.7}$$

for $\mathcal{E} = \{e_1, ..., e_n\}$ a section of $P_{SO}(E)$, $\tilde{\omega} = \mathcal{E}^* \omega$, and $\{\sigma_1, ..., \sigma_N\}$ a local section of $P_{SO}(S(E)$ determined by $\mathcal{E}$.

The existence of parallel ($\nabla^s \sigma = 0$) and Killing ($\nabla^s_X \sigma = \lambda X \cdot \sigma$) spinors is strongly tied to the geometry of the underlying manifold.

G-Structures and Holonomy

The existence of sections of particular spinor bundles is directly associated to the topological reduction of the structure group of a manifold, and in particular, the existence of parallel spinors is associated to the reduction of the holonomy group (MICHELSOHN, 1987). Here we will show the case of G_2-structures, but a similar construction can be done for other G-structures and holonomy reductions.

Take S the irreducible real spinor space over $\mathbb{R}^7$, we know that $\dim S = 8$. For each non-trivial spinor $\sigma \in S$, we can define a 3-form ϕ_σ by the relation

$$\phi_\sigma(X, Y, Z) = (X \cdot Y \cdot Z \cdot \sigma, \sigma), \tag{I.8}$$

where $\cdot$ is the Clifford multiplication and $(\cdot, \cdot)$ is a inner product on S. We can also define a vector cross product on $\mathbb{R}^7$ by

$$\phi_\sigma(X, Y, Z) = \langle X \times Y, Z \rangle \,. \tag{I.9}$$

This means ϕ_σ is in fact a G_2-structure on $\mathbb{R}^7$.

If we consider the spinor bundle $S(M)$ over a 7-manifold M, the existence of a global nowhere vanishing section $\sigma \in \Gamma(S(E))$ defines a 3-form φ_σ pointwise by (I.8). It is clear then that φ_σ is a G_2-structure for M.

Theorem 43. *A 7-manifold M admits a G_2-structure if and only if it is orientable and spinnable.*

Proof. Let M be a G_2 manifold. Since its structure group is G_2, which is connected, it is orientable. Since G_2 is simply connected, M is admits a spin structure.

Now assume M admits a spin structure. Then it's spinor bundle $S(M)$ is 8-dimensional, and $\dim S(M) > \dim M$ guarantees the existence of a global non-vanishing section of $S(M)$, which we just saw defines a G_2-structure on M. ■

The torsion $\nabla\sigma$ is directly related to the torsion of φ_σ. A complete spinorial description of the G_2 torsion classes is done in (AGRICOLA et al., 2015). We will only summarize the results for the classes we are interested in. Let $T \in \mathrm{End}(TM)$ be the endomorphism such that

$$\nabla_X\sigma = T(X)\cdot\sigma, \tag{I.10}$$

then φ_σ is

i) torsion-free iff $T \equiv 0$;

ii) nearly parallel iff $T = \lambda\mathbb{I},\ \lambda \in C^\infty(M)$;

iii) closed iff $T \in \mathfrak{g}_2$;

iv) coclosed iff T is symmetric;

v) integrable iff $T \in \mathfrak{g}_2^\perp$.

C) Supersymmetry

It is useful to remember a few facts about regular symmetries, henceforth known as *bosonic symmetries*. In dimension 4, bosonic symmetries are generated by the spacetime transformations P_μ (translations) and $M_{\mu\nu}$ (Lorentz boosts), forming the Poincare algebra, and internal symmetries generated by T_μ. This algebra can be described by its commutation relations

$$[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, M^{\rho\sigma}] = -2\delta^{[\rho}_{[\mu}M^{\sigma]}_{\nu]}, \qquad [P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu}P_{\rho]},$$

and the commutation of T_μ depends on the internal symmetry algebra. The Colemann-Mandula theorem guarantees that, for certain types of quantum field theories, these are all the symmetries possible and that T commutes with the spacetime symmetries trivially.

Supersymmetry adds a new type of symmetry, called fermionic symmetries, one whose coefficients anticommute. The most important fact for us is that these fermionic symmetries are generated by spinors, which will allow us to do the necessary calculations in the next section.

A more formal description of supersymmetries can be done using Lie superalgebras instead of Lie algebras.

Definition 44. *A Lie superalgebra A is a $\mathbb{Z}_2$ graded vector space equipped with a Lie superbracket $[,\}$ satisfying*

$$[x, y\} = (-1)^{|x||y|}[y, x\},$$
$$(-1)^{|x||z|}[x, [y, z\}\} + (-1)^{|y||x|}[y, [z, x\}\} + (-1)^{|z||y|}[z, [x, y\}\} = 0,$$

where $|x| = 0, 1$ *is the degree of* x.

In this language, bosonic symmetries are those generated by degree 0 elements, and fermionic symmetries are those generated by degree 1 elements. These are sometimes called even and odd respectively. The possibilities for the bracket of bosonic symmetries are described by the Coleman-Mandula theorem, and the possible brackets between fermionic elements, and mixed brackets are described by the Haag–Lopuszański–Sohnius theorem. Although the details of these relations are not relevant for us, these are important results, and so are worth mentioning.

II 7d compactification

Let $X^{10} = M^7 \times \mathbb{R}^{1,2}$ be 10-manifold with Lorentzian metric $g_{10} = g_7 \oplus g_3$, and $\{\Gamma^1, ..., \Gamma^{10}\}$ the 10d gamma matrices with respect to the metric g_{10}. We know the spinor space S has dimension 32, but can be decomposed into Cl_{10} modules as $S = S_+ \oplus S_-$, with dimension 16 each. This induces a block matrix structure for Γ^M as

$$\Gamma_M = \begin{pmatrix} 0 & \Gamma^M_- \\ \Gamma^M_+ & 0 \end{pmatrix}, \tag{II.1}$$

where $\Gamma^M_+\Gamma^N_- + \Gamma^N_+\Gamma^M_- = 2g^{MN}_{10}$ and $\Gamma^M_-\Gamma^N_+ + \Gamma^N_-\Gamma^M_+ = 2g^{MN}_{10}$. Similarly, a spinor $\sigma \in S$ can be decomposed as

$$\sigma = \begin{pmatrix} \epsilon + \\ \epsilon_- \end{pmatrix}. \tag{II.2}$$

Now, let $\gamma^1, ..., \gamma^7$ be a set gamma matrices for g_7 on M. Then we can construct $\Gamma^M_\pm$ by

$$\Gamma^m_+ = \begin{pmatrix} 0 & i\gamma^m \\ -i\gamma^m & 0 \end{pmatrix}, \ \Gamma^8_+ = \begin{pmatrix} 0 & \mathbb{I}_8 \\ \mathbb{I}_8 & 0 \end{pmatrix}, \ \Gamma^9_+ = \begin{pmatrix} \mathbb{I}_8 & 0 \\ 0 & -\mathbb{I}_8 \end{pmatrix}, \ \Gamma^{10}_+ = \mathbb{I}_{16}, \tag{II.3}$$

and $\Gamma^M_+ = \Gamma^M_-$ for $M = 1, ..., 9$ and $\Gamma^{10}_+ = -\Gamma^{10}_-$. Then, given we can write ϵ_+ with two 7d spinors η_1, η_2

$$\epsilon_+ = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{II.4}$$

Now, a Weyl spinor is one that only lives in S_+ or S_-, and, if Γ^M are all real or all imaginary, a Majorana spinor is one that is real. If γ^m are purely imaginary, we get that Γ^M are purely real, so a Majorana spinor η in M^7 defines a Majorana-Weyl spinor in X^{10} by

$$\sigma = \begin{pmatrix} \eta \\ \eta \\ 0 \\ 0 \end{pmatrix}, \tag{II.5}$$

which we will just call ϵ as a 16 component spinor. Then $\Gamma^M \sigma = \Gamma_+^M \epsilon$.

Notice that the Levi-Civita spinor connection acts on ϵ by

$$\nabla^{g_{10}} \epsilon = \begin{pmatrix} \nabla^{g_7}\eta \\ \nabla^{g_7}\eta \end{pmatrix}, \tag{II.6}$$

since the connection symbols $\tilde{\omega}_{MN}^K = \frac{1}{2} g^{KL}(\partial_M g_{LN} + \partial_N g_{LM} - \partial_L g_{MN})$. But since g_3 is Minkowski, $\tilde{\omega}_{\mu N} = 0$ for $\mu = 8, 9, 10$. Then the derivative is

$$\begin{aligned}
\nabla^g \epsilon_K =& \frac{1}{2} \sum_{M<N} \tilde{\omega}_{NM} \Gamma^M \Gamma^N \epsilon_K \\
=& \frac{1}{2} \sum_{m<n} \tilde{\omega}_{nm} \begin{pmatrix} 0 & i\gamma^m \\ -i\gamma^m & 0 \end{pmatrix} \begin{pmatrix} 0 & i\gamma^n \\ -i\gamma^n & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \eta \end{pmatrix} \\
=& \begin{pmatrix} \frac{1}{2} \sum_{m<n} \tilde{\omega}_{nm} \gamma^n \gamma^m \eta \\ \frac{1}{2} \sum_{m<n} \tilde{\omega}_{nm} \gamma^n \gamma^m \eta \end{pmatrix} \\
=& \begin{pmatrix} \nabla^{g_7}\eta \\ \nabla^{g_7}\eta \end{pmatrix}
\end{aligned}$$

since $\tilde{\omega}_{nm}$ only depends on g_7, and so is the connection symbols for the 7d metric. Now, consider that the heterotic fields $\Phi,\ H,\ A$ vanish in the directions of $\mathbb{R}^{1,2}$, then

$$\begin{aligned}
H \cdot \epsilon &= H_{MNP} \Gamma_+^{MNP} \epsilon \\
&= H_{mnp} \Gamma_+^{mnp} \epsilon \\
&= \begin{pmatrix} 0 & iH_{mnp}\gamma^{mnp} \\ -iH_{mnp}\gamma^{mnp} & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \eta \end{pmatrix} \\
&= \begin{pmatrix} -iH \cdot \eta \\ iH \cdot \eta \end{pmatrix},
\end{aligned}$$

and similar results for the other fields. With this, is is clear that the SUSY conditions (II.4a,II.4b,II.4c) are written as

$$\nabla^{+}\eta = 0 \tag{II.7}$$

$$(d\Phi - H) \cdot \eta = 0 \tag{II.8}$$

$$F \cdot \eta = 0, \tag{II.9}$$

on M^7.