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Exploring N-Freeness and Numerical Invariants of Logarithmic Tangent Sheaves in Reduced Hypersurfaces: A Stratified Approach with Algorithmic Implementation

Explorando a N-Liberdade e os Invariantes Numéricos de Feixes de Tangentes Logarítmicos em Hipersuperfícies Reduzidas: Uma Abordagem Estratificada com Implementação Algorítmica

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Resumo

Esta tese investiga o espaço de módulos de feixes tangentes logarítmicos $\mathcal{T}_f$ de hipersuperfícies reduzidas $V(f)$ em $\mathbb{P}^2$ e sua "n-liberdade", um conceito introduzido com base no comprimento de um esquema de dimensão zero Z, que é definido como o sub-esquema de zeros de uma seção global de $\mathcal{T}_f$ de grau mínimo. Construímos uma estratificação do espaço projetivo. Além disso, exploramos invariantes numéricos associados a curvas planas para vários graus e implementamos um algoritmo de "força bruta"para gerar polinômios de grau superior.

Palavras-chave: Feixe de diferenciais. Esquema Quot e famílias de feixes. Feixes sem torção.

Abstract

This thesis investigates the moduli space of logarithmic tangent sheaves $\mathcal{T}_f$ of reduced hypersurfaces $V(f)$ in $\mathbb{P}^2$ and their "n-freeness", a concept introduced based on the length of a zero-dimensional scheme Z which is defined as the zero-subscheme of a global section of $\mathcal{T}_f$ of minimal degree. We construct a stratification of the projective space. Additionally, we explore numerical invariants associated with plane curves for various degrees and implement a "brute-force" algorithm to generate higher-degree polynomials.

Keywords: Sheaf of differentials. Quot schemes and families of sheaves. Torsion-free sheaves.

Contents

1 Introduction

Let $f \in k[x, y, z]$ be a homogeneous polynomial over some algebraically closed field k defining a curve C and denote $d + 1 = deg(f)$.

1.1 Introduction

The partial derivatives $\partial_x f, \partial_y f, \partial_z f$ are homogeneous polynomials of degree d and they form a function of locally free sheaves $\nabla(f)$ defined by $(\partial_x f, \partial_y f, \partial_z f) \in Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}, \mathcal{O}_{\mathbb{P}^2}(d))$, the gradient map of f. Because we are in the projective plane, the kernel of $\nabla(f)$ is locally free $\mathcal{T}_C$. So

$$0 \to \mathcal{T}_C \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(d) \tag{1.1}$$

and if we denote $\mathcal{I}_J$ the ideal sheaf associated to the ideal $(\partial_x f, \partial_y f, \partial_z f)$ then

$$0 \to \mathcal{T}_C \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{I}_J(d) \to 0 \tag{1.2}$$

is exact, $\mathcal{T}_C$ is called the logarithmic tangent sheaf of C.

Such sheaves have been extensively studied e.g. (SAITO, 1980), (DIMCA; SERNESI, 2014). We also cite (MARCHESI; VALLÈS, 2019) which heavily inspired this thesis, we are interested in particular on the zero dimensional scheme Z of length n (Z is the zero-subscheme given by a nonzero minimal degree global section of $\mathcal{T}_f$, definition 36).

1.1.1 Moduli

One of our main objectives is to construct a moduli space of logarithmic tangent sheaves of reduced hypersurfaces in $\mathbb{P}^2$. We describe the idea:

Fix the degree d, we have that $\nabla(f) \in Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^0\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(d)$, then $\nabla(f)$ is an element of the k-vector space generated by monomials in a direct sum.

Now projectivize the vector space $W_d := H^0\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(d)$ over k forming a projective space $\mathbb{P}W_d$ of dimension $3dim_k(H^0\mathcal{O}_{\mathbb{P}^2}(d)) - 1$, we will show that any such $\nabla(f)$ is inside a linear subspace of W_d. Furthermore it defines a projective variety (irreducible) of dimension $dim_k(H^0\mathcal{O}_{\mathbb{P}^2}(d+1)) - 1$ given by the image of an injective linear map $\nabla'_d : H^0\mathcal{O}_{\mathbb{P}^2}(d+1) \to W_d$ (Lemma 16). Now we have a projective morphism $\nabla_d := \mathbb{P}(\nabla'_d) : \mathbb{P}H^0\mathcal{O}_{\mathbb{P}^2}(d+1) \to \mathbb{P}W_d$ and its image $Y_d := im(\nabla_d)$ is irreducible and closed.

We will then define (Theorem 21) a stratification of $\mathbb{P}W_d$ with respect to the Hilbert polynomial of $im(h)$ where we now view $h \in W_d$ as a morphism. Each stratum Z_i

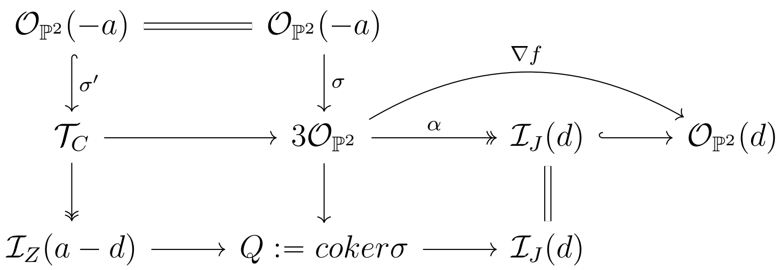
is locally closed with the property that the Hilbert polynomial of $im(h)$ is the same P_i for any $[h] \in Z_i \subset \mathbb{P}W_d$. We then consider the intersections $\nu_{d,j} := Z_j \cap Y_d$ and ask for the strata of Z_j with Hilbert polynomial $P_j = \binom{d+2}{2} - j$ of degree zero (a constant). This implies that $im(h) = \mathcal{I}_h(d)$ define a zero-dimensional scheme of degree j. In particular $\mathcal{I}_h(d)$ is of the form $\mathcal{I}_J(d)$ where $\mathcal{I}_J$ is the Jacobian ideal of some f homogeneous of degree $d+1$.

Each stratum Z_j induces a flat family of quotient sheaves, then the representability of the Quot functor implies that we have morphisms $Z_j \to Quot^{P_j}\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ which we then restrict to $\nu_{d,j}$.

Now each stratum $\nu_{d,j}$ is inside the image of ∇_d, this implies that we have a stratification of square-free global sections in $H^0\mathcal{O}_{\mathbb{P}^2}(d+1)$ corresponding to reduced curves with $h^0\mathcal{O}_J = j$.

1.1.2 n-freeness

Consider the following diagram where f is homogeneous of degree $d+1$:



Another objective is to study and understand the role of the zero-dimensional subscheme Z. If Z has length 1 then $\mathcal{T}_f$ is said to be nearly-free. A theorem given by (MARCHESI; VALLÈS, 2019) states that a nearly-free bundle is uniquely determined by the point Z and the minimal degrees $a, d-a$ corresponding to generators of $\mathcal{T}_f$.

We consider the cases where Z has length n in (36) and we call $\mathcal{T}_f$ n-free. There is then an equation (4.7) involving n, j, d, a:

$$n + j = d^2 - a.d + a^2. \tag{1.3}$$

Where $n = 0, ..., d^2$ ($n = d^2$ if $j = 0$, that is, f has no singularities) and j is the number of singular points.

We pose the question whether every value of n is realizable. For cubics and quartics the values $n = 2$ and $n = 5$ (respectively) seem to be missing.

1.1.3 Numerical invariant associated to plane curves

We compute explicitly the values of n, j, a for each normal form of plane curves of degree $d = 3, 4$. We also implement a "brute-force" algorithm and show it can recover a polynomial representing each case. This algorithm is used to generate polynomials of higher degrees (d=5,6) where we also recover interesting cases from the literature (irreducible free curves with one singular point of multiplicity d-1).

2 Preliminaries

2.1 Basic definitions

Fix the following notations:

- k is an algebraically closed field of characteristic 0
- $h^i(\mathcal{F}) := dim_k(H^i(\mathcal{F}))$
- $ext^i(\mathcal{F}, \mathcal{G}) = dim_k Ext^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$

2.1.1 Sheaf of differentials

We begin this section with a general definition of the Kähler differential of a morphism of schemes:

Definition 1. *Let $\pi : X \to S$ be a separated morphism of schemes and consider the fiber product, its natural projections and the diagonal morphism (which is a closed immersion)*

$$p_1, p_2 : X \times_S X \to X \tag{2.1}$$

$$\Delta : X \to X \times_S X \tag{2.2}$$

The closed immersion Δ induces a closed subscheme defined by an ideal sheaf $\mathcal{I}_D$. Locally on affine patches $\mathcal{I}_D$ is just the kernel of the multiplication map $\sum_i s_i \otimes t_i \mapsto \sum_i s_i t_i$ and $\mathcal{I}_D = (x_0 - a_0, x_1 - a_1, ..., x_n - a_n)$ where x_i, a_i are local coordinates for each copy of X.

Then the Kähler module of differentials is defined to be the $\mathcal{O}_X$-module $\Omega_{X/S} = \Delta^* \mathcal{I}_D / {\mathcal{I}_D}^2$.

Remark 2. *(Stacks project authors, 2021, Tag 07HX) The Kähler module of differentials comes with a morphism $\mathcal{O}_X \to \Omega_{X/S}$ taking a local section $s \in \mathcal{O}_X(U)$ to $ds := p_2^* s - p_1^* s = (1 \otimes s - s \otimes 1) mod(\mathcal{I}_D^2)$. We denote $\Omega^i_{X/S} := \bigwedge^i \Omega_{X/S}$ and $\Omega^0_{X/S} := \mathcal{O}_X$. Then there is the algebraic de Rham complex defined by the differential d above: it is the unique complex of the form*

$$0 \to \mathcal{O}_X \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to ... \tag{2.3}$$

such that in degree 0 the morphism is the differential d.

Let $S = Spec(k)$ and $X \to Spec(k)$ a k-scheme. We may omit "Spec" and just write k instead if no confusion arises.

Definition 3. *Let*

$$(-)^* = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$$

be the dual functor and

$$(-)^{**} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X), \mathcal{O}_X)$$

be the double dual functor taking a $\mathcal{O}_X$-module $\mathcal{F}$ to the $\mathcal{O}_X$-module $\mathcal{F}^*$ and $\mathcal{F}^{**}$ respectively.

Definition 4. *The sheaf is $\mathcal{T}_X = \mathcal{T}_{X/k} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \Omega^*_{X/k}$.*

Let $X = \mathbb{P}^n$ be the projective space over k, there is a sequence called the Euler sequence:

$$0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0 \tag{2.4}$$

and its dual version using the tangent sheaf

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{\oplus n+1} \to \mathcal{T}_X \to 0 \tag{2.5}$$

The next theorem is a classical theorem which will be used a few times along this thesis.

Theorem 5. *(HARTSHORNE, 2013, Serre Duality, Thm 3.7.1) Let $X = \mathbb{P}^n$ over k as before, let $\omega_{X/k} := \Omega^n_{X/k} \cong \mathcal{O}_X(-n-1)$ be the dualizing sheaf of $\mathbb{P}^n$ and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for any $i = 0, ..., n$*

$$Ext^i(\mathcal{F}, \omega_{X/k}) \cong H^{n-i}(\mathcal{F})^* \tag{2.6}$$

are naturally isomorphic k-vector spaces.

2.1.2 Hilbert polynomial

Definition 6 (Hilbert polynomial)**.** *Let X be a projective scheme over a field k with fixed ample line bundle $\mathcal{O}_X(1)$, $\mathcal{F}$ a coherent sheaf on X and $h^i(X, \mathcal{F}) := dim_k H^i(X, \mathcal{F})$ (each $H^i(X, \mathcal{F})$ is a finite k-vector space). Under those conditions we define the* ***Euler characteristic of*** *$\mathcal{F}$ to be $\chi(\mathcal{F}) = \sum_{i\geqslant 0}(-1)^i h^i(X, \mathcal{F})$ and the* ***Hilbert polynomial of*** *$\mathcal{F}$ by $P_\mathcal{F} : m \mapsto \chi(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(m))$. Both are additive over exact sequences and $P_\mathcal{F}$ is a polynomial in $\mathbb{Q}[m]$.*

We now list some observations that will be useful for the main result:

Remark 7. *Let $Y \to X$ be a closed immersion with X projective over k and $\mathcal{I}_Y$ be the quasi-coherent ideal sheaf on X. Then we have the defining exact sequence of $\mathcal{O}_X$-modules*

$$0 \to I_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \tag{2.7}$$

X is Noetherian because it is a projective scheme thus $\mathcal{I}_Y$ is coherent and $P_{\mathcal{I}_Y}$ is well defined. Since the Hilbert polynomial is additive for exact sequences of coherent sheaves we have:

$$P_{\mathcal{I}_Y} = P_{\mathcal{O}_X} - P_{\mathcal{O}_Y} \tag{2.8}$$

If $dim(Y) = 0$ we have that $\mathcal{O}_Y(m) = \mathcal{O}_Y$ and $P_{\mathcal{O}_Y} = h^0(X, \mathcal{O}_Y)$. Then:

$$P_{\mathcal{I}_Y(d)} = P_{\mathcal{O}_X(d)} - h^0(X, \mathcal{O}_Y) \tag{2.9}$$

$h^0(X, \mathcal{O}_Y)$ is the ***length*** *of the zero-dimensional projective subscheme Y. Furthermore, if Y is reduced, $length(Y)$ represents the set-theoretical number of points of Y.*

Example 1. *Let $X = \mathbb{P}^2$ and $\mathcal{I}_Y = (\partial_x f, \partial_y f, \partial_z f)$ be the ideal sheaf defined by the partial derivatives of $f = a_1 z^4 + y^4 + 2xy^2 z + x^2 z^2$ for some $a_1 \in k$. For $a_1 \neq 0$ we have that Y is zero dimensional and non-reduced consisting of the point $(1; 0; 0)$ with length 7.*

Definition 8. *(HUYBRECHTS; LEHN, 2010, Definition 6.1.1) Let $(\mathcal{F}, \mathcal{G})$ be a pair of coherent $\mathcal{O}_X$-modules. Then the Euler characteristic of the pair $(\mathcal{F}, \mathcal{G})$ is*

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_i (-1)^i ext^i(\mathcal{F}, \mathcal{G}) \tag{2.10}$$

Suppose that $\mathcal{F}$ is locally free, then we may rewrite the Hirzebruch-Riemann-Roch Theorem using the Euler characteristic of a pair of sheaves:

Theorem 9. *(HUYBRECHTS; LEHN, 2010, Lemma 6.1.1)*

$$\chi(\mathcal{F}, \mathcal{G}) = \int_X ch(\mathcal{F}^*) ch(\mathcal{G}) td(X) \tag{2.11}$$

We will use this equation in the next chapter.

2.1.3 Quot-scheme and families of sheaves

Definition 10. *(HUYBRECHTS; LEHN, 2010, Definition 2.1.1) A flat family of sheaves on the fibers of a morphism of schemes $f : X \to S$ is a coherent sheaf $\mathcal{F}$ on X which is flat over S, that is: for any $x \in X$ we have that $\mathcal{F}_x$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.*

We restrict our attention to the case where X and S are k-schemes:

Let S be a Noetherian scheme over an algebraically closed field k and $f : X \to S$ a projective morphism with fixed $\mathcal{O}_X(1)$ an f-ample line bundle on X. Let $\mathcal{H}$ be a coherent

$\mathcal{O}_X$-module and, for any S-scheme T, let $\mathcal{H}_T := \pi^*_{X/T}\mathcal{H}$ denote the pullback of $\mathcal{H}$ by the natural projection $\pi_{X/T} : X \times_S T \to X$. Consider the functor

$$\underline{Quot}^P_{\mathcal{H}/X/S} : (Sch/S)^\circ \to Sets$$

defined on objects by $\underline{Quot}^P_{\mathcal{H}/X/S}(T) := \{q : \mathcal{H}_T \twoheadrightarrow \mathcal{F}\}$, the set of all T-flat families of quotient sheaves with Hilbert polynomial $P \in \mathbb{Q}[\lambda]$, and on morphisms $\underline{Quot}_{\mathcal{H}/X/S}(g : T' \to T)$ it sends $\mathcal{H}_T \to \mathcal{F}$ to $\mathcal{H}_{T'} \to (1 \times g)^*\mathcal{F}$. Then it follows that $\underline{Quot}^P_{\mathcal{H}/X/S}$ is a representable functor, in other words there is a S-scheme (namely $Quot^P_{\mathcal{H}/X/S}$) and a natural isomorphism of functors $Mor_{Sch/S}(-, Quot^P_{\mathcal{H}/X/S}) \cong \underline{Quot}^P_{\mathcal{H}/X/S}$.

Remark 11. *Let $f : Y \to Quot^P_{\mathcal{H}/X/S}$ be a morphism of schemes. Then*

$$Mor_{Sch/S}(Y, Quot^P_{\mathcal{H}/X/S}) \cong \underline{Quot}^P_{\mathcal{H}/X/S}(Y)$$

hence any f is naturally mapped to a family of quotients parameterized by Y.

The tautological family is defined by applying the $\underline{Quot}$ functor to the projective scheme representing itself and taking the identity morphism Id:

$$Id \in Mor_{Sch/S}(Quot^P_{\mathcal{H}/X/S}, Quot^P_{\mathcal{H}/X/S}) \cong \underline{Quot}^P_{\mathcal{H}/X/S}(Quot^P_{\mathcal{H}/X/S}). \tag{2.12}$$

It is a coherent sheaf $\mathcal{G}$ on $Quot^P_{\mathcal{H}/X/S} \times_S X$ such that any family of quotients of $\mathcal{H}$ can be defined as a pullback from $\mathcal{G}$ (see (HUYBRECHTS; LEHN, 2010, p.41 Tautological Family)).

The construction of the Quot scheme was introduced by Grothendieck. We cite references (GROTHENDIECK, 1961) or (NITSURE, 2005) for the existence:

Theorem 12. *(NITSURE, 2005) Let S be a Noetherian scheme, $\pi : X \to S$ a projective morphism, and L a relatively very ample line bundle on X. Then for any coherent $\mathcal{O}_X$-module $\mathcal{H}$ and any polynomial $\Phi \in \mathbb{Q}[\lambda]$, the functor $\underline{Quot}^{\Phi,L}_{\mathcal{H}/X/S}$ is representable by a projective S-scheme $\mathrm{Quot}^{\Phi,L}_{\mathcal{H}/X/S}$.*

We also have that torsion-freeness is an open condition:

Theorem 13. *(MARUYAMA, 1976, Section 2) Let F be a family of sheaves on X parameterized by S, suppose that $s \in S$ such that $F_s := F|_{X\times\{s\}}$ is torsion-free. Then there is $S' \subset S$ open such that F_s is torsion-free for any $s \in S'$.*

Last theorem guarantees us that the torsion-free quotients form an open set of the Quot scheme. We denote this open set $Quot^P_{tf}$ (with appropriate sub-indexes $X, \mathcal{H}, S$).

3 Construction of Moduli

3.1 Definitions and computations

Now fix the following setting: Let $X = \mathbb{P}^2$, $f \in H^0(\mathcal{O}_{\mathbb{P}^2}(d+1))$ a square-free homogeneous form for some positive integer d and $W_d := Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^0(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(d))$ as k-vector space. Let $\mathcal{I}_J$ be the Jacobian ideal given by the partial derivatives of f_0, f_1, f_2 of f, it defines a zero-dimensional scheme J with length $j := h^0(\mathcal{O}_J)$. Remark (7) gives the following lemma:

Lemma 14. *The Hilbert polynomial of $\mathcal{I}_J(d)$ is $P_{\mathcal{I}_J(d)} = \binom{d+2}{2} - j$*

Since the set of monomials of degree d form a basis for the k-vector space $H^0\mathcal{O}_{\mathbb{P}^2}(d)$ we can define (fixing basis) a linear morphism acting as a gradient map:

Definition 15. *Let $\nabla'_d : H^0\mathcal{O}_{\mathbb{P}^2}(d+1) \to W_d$ be a k-linear morphism taking each monomial to the partial derivatives $\nabla'_d = (\partial_x, \partial_y, \partial_z)$. Explicitly,*

$$x^i y^j z^k \mapsto (i.x^{i-1}y^j z^k, j.x^i y^{j-1} z^k, k.x^i y^j z^{k-1})$$

for $i + j + k = d + 1$.

We can visualize what is happening with the coordinates (a_{ij}) under each partial derivative by writing down each array.

Example 2. *Let $d + 1 = 4$, fix $B = \{m_{ij} = x^{4-i-j}y^i z^j \mid i, j \in \mathbb{Z}_{\geqslant 0}\ i + j \leqslant 4\}$ a basis for $H^0\mathcal{O}_{\mathbb{P}^2}(4)$ and $B' = \{m'_{ij} = x^{3-i-j}y^i z^j \mid i + j \leqslant 3\}$ basis for $H^0\mathcal{O}_{\mathbb{P}^2}(3)$.*

Let $f = \sum_{i,j\geqslant 0}^{i+j\leqslant 4} a_{ij}x^{4-i-j}y^i z^j \in H^0\mathcal{O}_{\mathbb{P}^2}(4)$ *and* $g = \sum_{i,j\geqslant 0}^{i+j\leqslant 3} b_{ij}x^{3-i-j}y^i z^j \in H^0\mathcal{O}_{\mathbb{P}^2}(3)$, *we can* ***formally*** *represent f and g as matrices:*

$$M_f = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \\ a_{10} & a_{11} & a_{12} & a_{13} & 0 \\ a_{20} & a_{21} & a_{22} & 0 & 0 \\ a_{30} & a_{31} & 0 & 0 & 0 \\ a_{40} & 0 & 0 & 0 & 0 \end{bmatrix}, M_g = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & 0 \\ b_{20} & b_{21} & 0 & 0 \\ b_{30} & 0 & 0 & 0 \end{bmatrix} \tag{3.1}$$

Then $\partial_x(f), \partial_y(f), \partial_z(f) \in H^0\mathcal{O}_{\mathbb{P}^2}(3)$ and:

$$M_{\partial_x(f)} = \begin{bmatrix} 4a_{00} & 3a_{01} & 2a_{02} & a_{03} \\ 3a_{10} & 2a_{11} & a_{12} & 0 \\ 2a_{20} & a_{21} & 0 & 0 \\ a_{30} & 0 & 0 & 0 \end{bmatrix}, \ M_{\partial_y(f)} = \begin{bmatrix} a_{10} & a_{11} & a_{12} & a_{13} \\ 2a_{20} & 2a_{21} & 2a_{22} & 0 \\ 3a_{30} & 3a_{31} & 0 & 0 \\ 4a_{40} & 0 & 0 & 0 \end{bmatrix},$$

$$M_{\partial_z(f)} = \begin{bmatrix} a_{01} & 2a_{02} & 3a_{03} & 4a_{04} \\ a_{11} & 2a_{12} & 2a_{13} & 0 \\ a_{21} & 2a_{22} & 0 & 0 \\ a_{31} & 0 & 0 & 0 \end{bmatrix}$$

Lemma 16. *∇'_d defines an injective morphism $\nabla_d : \mathbb{P}H^0\mathcal{O}_{\mathbb{P}^2}(d+1) \to \mathbb{P}W_d$ such that $\nabla_d([f]) = [\partial_x f, \partial_y f, \partial_z f]$. Furthermore, the image of ∇_d is a projective variety of dimension* $\dim(im\nabla_d) = \binom{d+3}{2} - 1$.

Proof. $\nabla'_d(f) = 0$ implies that every partial derivative is zero identically, which means that f does not depend on x, y, z and hence it is the zero vector. An injective linear morphism $\phi : V \to W$ defines an injective algebraic morphism of projective spaces $\mathbb{P}\phi : \mathbb{P}V \to \mathbb{P}W$ with $\dim(im(\mathbb{P}\phi)) = \dim \mathbb{P}V = \dim \mathbb{P}H^0\mathcal{O}_{\mathbb{P}^2}(d+1) = \binom{d+3}{2} - 1$. □

Definition 17. *Let $Y_d := im(\nabla_d) \subset \mathbb{P}W_d$, by our last lemma it is a projective variety of dimension $\binom{d+3}{2} - 1$*

3.1.1 Applying flattening stratification

Definition 18. *For a scheme S, a stratification of S is a finite set $\{S_1, ..., S_m\}$ of locally closed subschemes of S such that every point $s \in S$ is in exactly one subset S_i.*

We will use the following result:

Corollary 19. *(MUMFORD, 1964, Flattening Stratification p.60) Let $f : X \to S$ be a morphism which can be factored $X \to \mathbb{P}^n \times S \to S$, explicitly, $f = \pi_2 \circ i$ where i is a closed immersion and π_2 the canonical projection. Let $\mathcal{F}$ be a coherent sheaf on X, then $\mathcal{F}$ defines a stratification (Z_i) on S where Z_i are indexed by Hilbert polynomials P_i so that:*

- *The induced sheaf $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{Z_i}$ has Hilbert polynomial P_i on $\mathbb{P}^n \times Z_i$*
- *if $i \neq j$ then $P_i \neq P_j$*

The stratification is flat in the sense that $\mathcal{F} \otimes \mathcal{O}_{Z_i}$ is a flat family of sheaves on the fibers of f (Definition 10) for each i.

The next definition will fix some notations for our main theorem.

Definition 20. *Let $\mathbb{P}^2$ be the projective plane over a field k, $d \geqslant 0$ an integer and consider $W_d = (H^0\mathcal{O}_{\mathbb{P}^2}(d))^{\oplus 3}$ as a vector space over k. Let $X_d := \mathbb{P}^2 \times \mathbb{P}W_d$, $p : X_d \to \mathbb{P}^2$ and $q : X_d \to \mathbb{P}W_d$ be the projections and denote $\mathcal{O}_{\mathbb{P}^2}(d) \boxtimes \mathcal{O}_{\mathbb{P}W_d}(1) := p^*\mathcal{O}_{\mathbb{P}^2}(d) \otimes_k q^*\mathcal{O}_{\mathbb{P}W_d}(1)$*

Theorem 21. *There is a finite set of locally closed subschemes $Z_{P_i} \subset \mathbb{P}W_d$ indexed by distinct numerical polynomials P_i such that $\mathbb{P}W_d = \bigsqcup_i Z_{P_i}$ with the following property: If $h \in W_d$ has projective class $[h] \in Z_{P_i}$, then the Hilbert polynomial of $im(h) = \mathcal{I}_h(d)$ is P_i*

Proof. Since $Hom_{X\times_k Y}(A \boxtimes B, A' \boxtimes B') \cong Hom_X(A, A') \otimes_k Hom_Y(B, B')$, there is a non-zero global section $\sigma \in 3H^0\mathcal{O}_{\mathbb{P}^2}(d) \otimes_k H^0\mathcal{O}_{\mathbb{P}W_d}(1)$ that corresponds to a morphism of sheaves

$$\sigma : 3\mathcal{O}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}W_d} \to \mathcal{O}_{\mathbb{P}^2}(d) \boxtimes \mathcal{O}_{\mathbb{P}W_d}(1) \tag{3.2}$$

It defines a subscheme $T \subset \mathbb{P}^2 \times \mathbb{P}W_d$ induced by the cokernel of σ:

$$3\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}W_d} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{P}^2}(d) \boxtimes \mathcal{O}_{\mathbb{P}W_d}(1) \xrightarrow{\pi} \mathcal{O}_T(d,1) \to 0 \tag{3.3}$$

Let $\mathcal{I}_T$ be the ideal sheaf of T, then we have

$$0 \to \mathcal{I}_T \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^2} \boxtimes \mathcal{O}_{\mathbb{P}W_d} \to \mathcal{O}_T \to 0 \tag{3.4}$$

where we denote α to be the inclusion morphism. $\mathcal{I}_T$ is the kernel of a morphism of coherent sheaves thus it is also coherent. We also have that $im(\sigma) = \mathcal{I}_T(d,1)$, hence there is an exact sequence:

$$0 \to ker(\sigma) \to 3\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}W_d} \xrightarrow{\sigma} \mathcal{I}_T(d,1) \to 0 \tag{3.5}$$

Applying Corollary 19 on the coherent sheaf $\mathcal{I}_T(d,1)$ gives us a finite set of locally closed subschemes $Z_i \subset \mathbb{P}W_d$ with the property that the induced sheaf $\mathcal{I}_T(d,1)|_{\mathbb{P}^2\times\{[h]\}}$ has constant Hilbert polynomial P_i for any $[h] \in Z_i$, this property is accompanied with flatness over Z_i: The coherent sheaf $\mathcal{I}_T(d,1)|_{Z_i}$ on $\mathbb{P}^2 \times Z_i$ is a flat family over Z_i. (Definition 10)

Pick $[h] \in Z_i$ and apply the restriction to the fiber $\mathbb{P}^2 \times \{[h]\}$:

$$0 \to ker(\sigma)|_{\mathbb{P}^2\times\{[h]\}} \to 3\mathcal{O}_{\mathbb{P}^2\times\{[h]\}} \xrightarrow{\sigma} \mathcal{I}_T(d,1)|_{\mathbb{P}^2\times\{[h]\}} \to 0 \tag{3.6}$$

The sequence is exact because the first Tor appearing involves $\mathcal{I}_T(d,1)|_{Z_i}$, which is flat over Z_i and thus has no torsion on the fiber $\mathbb{P}^2 \times \{[h]\}$. □

We have defined a flattening stratification Z_i parameterizing ideal sheaves defined by three homogeneous forms with Hilbert polynomial P_i. Each Z_i is locally closed, i.e. closed inside an open set of $\mathbb{P}W_d$. In particular there are strata Z_j with Hilbert polynomial of the form $P_j = \binom{d+2}{2} - j$ corresponding to twisted ideal sheaves $\mathcal{I}_J(d)$ with J zero-dimensional scheme of length j generated by three homogeneous forms of degree d, we will now focus on Z_j with this particular Hilbert polynomial P_j.

3.1.2 Applying representability of Quot functors

In this subsection we will use the property that the Quot functors are represented by the quasi-projective Quot schemes. The objective is to define a flat family of quotient sheaves $\mathcal{I}_J(d)$ (twisted ideal sheaves) parameterized by some subscheme of the Quot scheme.

Definition 22. *Let j be a non-negative integer, Z_j be a stratum corresponding to the Hilbert polynomial $P_j = \binom{d+2}{2} - j$ and denote $\nu_{d,j} = Z_j \cap Y_d$ where $Y_d = im\nabla_d$. Then $\nu_{d,j}$ is locally closed and we also define $\nu_d := \bigsqcup_{j \geqslant 0} \nu_{d,j} \subset \mathbb{P}W_d$, note that ν_d is a finite union of locally closed subschemes.*

Remark 23. *An element in $[h] \in \nu_{d,j}$ is of the form $[h] = \nabla_d([f]) = [\nabla(f)] \in \mathbb{P}W_d$. Since $h \in W_d = Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}, \mathcal{O}_{\mathbb{P}^2}(d))$ and $[h] \in Z_j$ we have that $im(h) = \mathcal{I}_h(d)$ is a twisted ideal sheaf of a length j zero-dimensional subscheme. Furthermore Z_j induces a flat family of sheaves on $\mathbb{P}^2$ given by the coherent sheaf $\mathcal{I}_T(d,1)|_{Z_j}$ on $\mathbb{P}^2 \times Z_j$.*

Theorem 24. *Fix an integer $j \geqslant 0$ such that $P_j = \binom{d+2}{2} - j$ is the Hilbert polynomial corresponding to the stratum Z_j. There is a natural morphism of schemes $\phi_{d,j} : \nu_{d,j} \to Quot_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$ taking $[\nabla f]$ to $(\mathcal{I}_J(d), \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{I}_J(d)) \in Quot_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$ where $\mathcal{I}_J$ is the Jacobian ideal of f.*

Proof. Consider the flat family of sheaves $\mathcal{I}_T(d,1)|_{Z_j}$ parameterized by Z_j. For each $[h] \in Z_j$ we have that $im(h) = \mathcal{I}_h(d)$ is naturally a torsion-free quotient of $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$, thus our family of sheaves must be an element of $\underline{Quot}_{tf}^{P_j}\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(Z_j)$. By representability (Remark 11) of $\underline{Quot}^{P_j}\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ this family must correspond to a morphism of schemes:

$$\psi_j : Z_j \to Quot_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) \tag{3.7}$$

The restriction of ψ_j to $\nu_{d,j}$ yields the desired morphism

$$\phi_{d,j} := \psi_j|_{\nu_{d,j}} : \nu_{d,j} \to Quot_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) \tag{3.8}$$

□

Remark 25. *Consider the morphism $\phi_{d,j} : \nu_{d,j} \to Quot_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$ from our previous lemma. Following Remark 11, we have that $\phi_{d,j} \in Mor(\nu_{d,j}, Quot_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})) \cong \underline{Quot}_{tf}^{P_j}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})(\nu_{d,j})$ defines a family of ideal sheaves $\mathcal{I}_{d,j}$ flat over $\nu_{d,j}$. The kernel $\mathcal{G}_{d,j}$ in equation 3.6 also has vanishing first Tor, thus it is also flat over $\nu_{d,j}$. $\mathcal{G}_{d,j}$ represents logarithmic tangent sheaves defined by ∇f with J of length j.*

Proposition 26. *Let $f \in H^0\mathcal{O}_{\mathbb{P}^2}(d+1)$, then $\mathcal{T}_f := ker(\nabla f)$ is locally free of rank 2*

Proof. Consider the following exact sequence:

$$0 \to \mathcal{T}_f \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{I}_J(d) \to 0$$

Notice first that $rk\mathcal{I}_J(d) = 1$ and being the kernel of coherent sheaves, $\mathcal{T}_f$ is coherent of rank 2. Furthermore $\mathcal{T}_f$ is reflexive because $ker(\mathcal{F} \twoheadrightarrow \mathcal{G})$ is reflexive if $\mathcal{F}$ locally free and $\mathcal{G}$ torsion-free. Finally, it is locally free since we are in $\mathbb{P}^2$ and the singular locus of a reflexive sheaf must be at least of codimension 3 ((OKONEK; SCHNEIDER; SPINDLER, 1980)). □

Let $[q : \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{I}_J(d)]$ be an element parameterized by $\nu_{d,j} \to Quot_{tf}^{Q}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$ where Q is the polynomial $Q := P_{\mathcal{I}_{[h]}(d)} = \binom{d+2}{2} - j$. There is the following estimation for $dim_{[q]}Quot_{tf}^{Q}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$:

Proposition 27. *(HUYBRECHTS; LEHN, 2010)*

$$hom(\mathcal{T}_f, \mathcal{I}_J(d)) \geqslant dim_{[q]}Quot_{tf}^{Q}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) \geqslant hom(\mathcal{T}_f, \mathcal{I}_J(d)) - ext^1(\mathcal{T}_f, \mathcal{I}_J(d)) \tag{3.9}$$

We need one technical lemma:

Lemma 28. *Let $\mathcal{T}_f$ be locally free as before, then:*

$$H^2({\mathcal{T}_f}^* \otimes \mathcal{I}_J(d)) \cong H^2(\mathcal{T}_f(2d)) \tag{3.10}$$

Proof. Apply the exact functor $(-) \otimes {\mathcal{T}_f}^*$ to the sequence

$$0 \to \mathcal{I}_J(d) \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_J(d) \to 0$$

so that we have

$$0 \to \mathcal{I}_J(d) \otimes {\mathcal{T}_f}^* \to \mathcal{O}_{\mathbb{P}^2}(d) \otimes {\mathcal{T}_f}^* \to \mathcal{O}_J(d) \otimes {\mathcal{T}_f}^* \to 0.$$

$\mathcal{T}_f$ is a rank 2 locally free sheaf, hence ${\mathcal{T}_f}^* = \mathcal{T}_f(d)$. And $dimJ = 0$ implies that we can write $\mathcal{O}_J = \mathcal{O}_J(d)$. So

$$0 \to \mathcal{I}_J \otimes \mathcal{T}_f(2d) \to \mathcal{T}_f(2d) \to \mathcal{O}_J \otimes \mathcal{T}_f \to 0 \tag{3.11}$$

Using the functor of global sections and again the fact that J is zero-dimensional:

$$0 = H^1(\mathcal{O}_J \otimes \mathcal{T}_f) \to H^2(\mathcal{I}_J \otimes \mathcal{T}_f(2d)) \to H^2(\mathcal{T}_f(2d)) \to H^2(\mathcal{O}_J \otimes \mathcal{T}_f) = 0 \tag{3.12}$$

□

A brief computation using the Riemann-Roch equation (2.11) and the previous lemma yield the following theorem:

Theorem 29. $\chi(\mathcal{T}_f, \mathcal{I}_J(d)) = hom(\mathcal{T}_f, \mathcal{I}_J(d)) - ext^1(\mathcal{T}_f, \mathcal{I}_J(d)) = 7 - j + 3d(5+d)/2$

Proof. First notice that $\mathcal{T}_f$ is locally free. Then

$$ext^2(\mathcal{T}_f, \mathcal{I}_J(d)) = ext^2(\mathcal{O}_X, \mathcal{T}_f{}^* \otimes \mathcal{I}_J(d)) = h^2(\mathcal{T}_f{}^* \otimes \mathcal{I}_J(d)) = h^2(\mathcal{T}_f(2d))$$

where last equality comes from our previous lemma. Now use Serre Duality (2.6) so that

$$h^2(\mathcal{T}_f(2d)) = ext^0(\mathcal{T}_f(2d), \omega_{\mathbb{P}^2}) = h^0(\mathcal{T}_f(-2d+d) \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) = h^0(\mathcal{T}_f(-d-3)) = 0$$

.

Thus the Euler characteristic is just

$$\chi(\mathcal{T}_f, \mathcal{I}_J(d)) = hom(\mathcal{T}_f, \mathcal{I}_J(d)) - ext^1(\mathcal{T}_f, \mathcal{I}_J(d)) \tag{3.13}$$

Next we use the equation (2.11)

$$\chi(\mathcal{T}_f, \mathcal{I}_J(d)) = \int_{\mathbb{P}^2} ch(\mathcal{T}_f(d))ch(\mathcal{I}_J(d))td(\mathbb{P}^2) \tag{3.14}$$

and $ch(\mathcal{T}_f(d)) = (2, d, \frac{2j-d^2}{2})$, $ch(\mathcal{I}_J(d)) = (1, d, \frac{d^2-2j}{2})$, $td(\mathbb{P}^2) = (1, 3/2, 1)$.

$$ch(\mathcal{T}_f(d)).ch(\mathcal{I}_J(d)).td(\mathbb{P}^2) = (2, 3d, \frac{3d^2-2j}{2}).(1, 3/2, 1) = (2, 3+3d, \frac{3d^2-2j}{2} + 2 + \frac{9d}{2})$$

, hence

$$\int_{\mathbb{P}^2} ch(\mathcal{T}_f(d)).ch(\mathcal{I}_J(d)).td(\mathbb{P}^2) = \frac{14+15d+3d^2-2j}{2} = 7 - j + \frac{3d(5+d)}{2}$$

□

4 Properties of Log bundles

4.1 Zero-subscheme of a global section

Fix the following setting in this subsection: Let a be the least positive integer such that $H^0\mathcal{T}_f(a) \neq 0$. Since $\mathcal{T}_f(a)$ is a rank two bundle there is a zero-dimensional scheme Z satisfying the following exact sequence ((BARTH, 1977)):

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{T}_f \to \mathcal{I}_Z(a-d) \to 0 \tag{4.1}$$

We can also justify the existence of the ideal sheaf $\mathcal{I}_Z$ within the problem of Bourbaki sequences, which gives an explicit construction of Z. We cite (DIMCA; STICLARU, 2020) for the usage of Bourbaki ideals for logarithmic tangent sheaves on $\mathbb{P}^2$ and note that (DIMCA; STICLARU, 2020, Theorem 5.1 (1)) implies on equation (4.7).

4.2 Algebraic setting

4.2.1 Bourbaki sequences and Bourbaki ideals

Let R be a commutative Noetherian ring and M a finitely generated R-module. Then a Bourbaki sequence is defined to be and exact sequence of R-modules:

$$0 \to F \to M \to I \to 0 \tag{4.2}$$

such that F is free and I is an ideal of R. I is said to be a Bourbaki Ideal of M. We have the following problem concerning the existence of a Bourbaki sequence:

Given R a commutative Noetherian ring and $0 \to F \xrightarrow{\phi} M$ with F and M finitely generated R modules with F free. When does $0 \to F \to M \to coker(\phi) \to 0$ becomes a Bourbaki sequence?

Theorem 30. *(BOURBAKI, 1998, Theorem 6.8.4) Let R be a Noetherian integral domain and M a finitely generated torsion-free R-module. Then there is a free submodule F of M such that M/F is isomorphic to an ideal I of R.*

The next definition establishes the explicit algebraic connection between the first syzygy module and its sheafification $E_f = \mathcal{T}_f(-1)$:

Definition 31. *(DIMCA; STICLARU, 2020) Let $AR(f)$ denote the graded $S = \mathbb{C}[x,y,z]$-module of first syzygies defined by $AR(f)_k := \{(\sigma_0, \sigma_1, \sigma_2) \in S_k^{\oplus 3} | \sum_{i=0,1,2} \sigma_i.\partial_i(f) = 0\}$. Then the sheafification E_f of $AR(f)$ is a rank two vector bundle on $\mathbb{P}^2$ and $E_f = \mathcal{T}_f(-1)$.*

If σ is a nonzero syzygy of minimum degree a then

$$0 \to S(-a) \xrightarrow{.\sigma} AR(f) \to B(f,\sigma)(a-d) \to 0$$

where $B(f,\sigma)$ is a Bourbaki ideal.

During the next section we focus on properties of $\mathcal{T}_f$ that can be deduced using simple operations on exact sequences of sheaves.

4.3 Global sections and syzygies

Let $f \in k[x,y,z]$ be homogeneous of degree $d+1$ and $\mathcal{T}_f$ its logarithmic tangent sheaf with inclusion $\pi : \mathcal{T}_f \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$. We now have two exact sequences:

$$0 \to \mathcal{T}_f \xrightarrow{\pi} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{q_f} \mathcal{I}_J(d) \to 0 \tag{4.3}$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \xrightarrow{\sigma} \mathcal{T}_f \to \mathcal{I}_Z(a-d) \to 0 \tag{4.4}$$

where we denote q_f to be the restriction of $\nabla(f) : \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(d)$ to its image $\mathcal{I}_J(d)$ and σ' a non-zero global section of $\mathcal{T}_f(a)$ of minimum degree a.

The next proposition gives a correspondence between elements of $AR(f)_a$ (Definition 31) and morphisms $\sigma : \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{T}_f$ where we identify $\sigma \in H^0\mathcal{T}_f(a)$.

Proposition 32. *Let $\sigma' : \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}$ be in $AR(f)_a$ for some f homogeneous and a non-negative integer, that is: $\sigma' = (\sigma'_0, \sigma'_1, \sigma'_2)$ with σ'_i homogeneous of degree a satisfying $\sum_{i=0,1,2} \sigma'_i.\partial_i(f) = 0$. Then there is an unique global section $\sigma : \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{T}_f$ of degree a such that $\pi \circ \sigma = \sigma'$.*

Proof. We denote $\nabla f : \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(d)$ and consider the sequence (4.3) and apply the left exact functor $Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), -)$ so that we have an exact sequence:

$$0 \to Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{T}_f) \xrightarrow{\pi\circ -} Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}) \xrightarrow{q_f -} Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{I}_J(d)) \tag{4.5}$$

The composition $\nabla f \circ \sigma' \in Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^0(\mathcal{O}_{\mathbb{P}^2}(a+d))$ is given by the polynomial $\sum_{i=0,1,2} \sigma'_i.\partial_i(f)$ which is zero since $\sigma' \in AR(f)_a$. Then by exactness of the previous sequence we have that σ' must be an element in the image of $Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{T}_f) \xrightarrow{\pi\circ -} Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{O}_{\mathbb{P}^2}^{\oplus 3})$. In other words we have a $\sigma \in Hom_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(-a), \mathcal{T}_f)$ such that $\pi \circ \sigma = \sigma'$. The uniqueness of σ comes from the injectivity of $\pi \circ -$. $\square$

The uniqueness gives us a correspondence of global sections of $\mathcal{T}_f(a)$ and first syzygies σ' of the map ∇f corresponding to the composition $\sum_{i=0,1,2} \sigma'_i.\partial_i(f) = 0$. If a is the least integer such that $\mathcal{T}_f(a)$ has non-zero global sections and $\sigma' \in H^0\mathcal{T}_f(a)$ then we have:

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathbb{P}^2}(-a) & = & \mathcal{O}_{\mathbb{P}^2}(-a) & & & & \\
\downarrow \sigma & & \downarrow \sigma' & & \overset{\nabla f}{\frown} & & \\
\mathcal{T}_f & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \overset{q_f}{\twoheadrightarrow} & \mathcal{I}_J(d) & \hookrightarrow & \mathcal{O}_{\mathbb{P}^2}(d) \\
\downarrow & & \downarrow & & \| & & \\
\mathcal{I}_Z(a-d) & \longrightarrow & Q := coker\sigma' & \longrightarrow & \mathcal{I}_J(d) & &
\end{array}$$

where

$$0 \to \mathcal{I}_Z(a-d) \to Q \to \mathcal{I}_J(d) \to 0 \tag{4.6}$$

is exact by the snake lemma: the first morphism is injective because of the identity $\mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{O}_{\mathbb{P}^2}(-a)$ and its cokernel is $\mathcal{I}_J(d)$ because $\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to Q$ is surjective.

Example 3. *Consider the arrangement given by $f = xyz$, then $\nabla f = (yz, xz, xy)$ and denote by J the scheme defined by ∇f, $C = V(f)$ has 3 double points and 0 triple points. Thus we have the following diagram*

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathbb{P}^2}(-1) & = & \mathcal{O}_{\mathbb{P}^2}(-1) & & & & \\
\downarrow \sigma & & \sigma' := \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix} \downarrow & & \overset{\nabla f}{\frown} & & \\
\mathcal{T}_f & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \overset{q_f}{\twoheadrightarrow} & \mathcal{I}_J(2) & \hookrightarrow & \mathcal{O}_{\mathbb{P}^2}(2) \\
\downarrow & & \downarrow & & \| & & \\
\mathcal{I}_Z(-1) & \longrightarrow & Q := coker\sigma' & \longrightarrow & \mathcal{I}_J(2) & &
\end{array}$$

But $q_f \circ \sigma' = 0$ since $\nabla f \circ \sigma' = 0$, then $im(\sigma') \subset ker(q_f)$ and hence σ' defines the section σ. Let n be the length of Z, we use the Chern classes of $\mathcal{I}(2)$ and Q to conclude that n=0: $c(\mathcal{I}_Z(-1)) = 1 - t + nt^2$, $c(Q) = 1 + t + t^2$ and $c(\mathcal{I}_J(2)) = 1 + 2t + 3t^2$. But $c(\mathcal{I}_Z(-1))c(\mathcal{I}_J(2)) = c(Q)$ and hence $n = 0$. Thus $\mathcal{T}_f$ is free.

The next theorem simplifies the computations done on Example 3 into an useful formula:

Theorem 33. *Let f be a square-free homogeneous polynomial of degree $d+1$, J be the zero-dimensional subscheme defined by the Jacobian ideal $\mathcal{I}_J$, $\mathcal{T}_f = ker\nabla f$ the logarithmic tangent sheaf of f with a the least integer such that $\mathcal{T}_f(a)$ has non-zero global section with subscheme Z, zero-dimensional of length n. Then:*

$$n + j = d^2 - a.d + a^2 \tag{4.7}$$

Moreover if f is non-singular then $n = d^2$ and $a = d$.

Proof. The equation follows, from the exact sequence corresponding to the third row of the main diagram, by comparing Chern classes. We will use Chern polynomials $c(\mathcal{F})(t)$ for the computation:

$$c(\mathcal{I}_Z(a-d)).c(\mathcal{I}_J(d)) = c(Q) \tag{4.8}$$

where $c(\mathcal{I}_Z(a-d)) = 1 + (a-d).t + n.t^2$, $c(Q) = (1-a.t)^{-1} = 1 + a.t + a^2.t^2$ and $c(\mathcal{I}_J(d)) = 1 + d.t + j.t^2$, then

$$\begin{aligned} c(\mathcal{I}_Z(a-d)).c(\mathcal{I}_J(d)) &= 1 + d.t + j.t^2 + (a-d).t + d.(a-d).t^2 + n.t^2 \\ &= 1 + a.t + (n + j + a.d - d^2).t^2 \end{aligned} \tag{4.9}$$

On the other side we have $c(Q) = 1 + a.t + a^2.t^2$, by comparing the coefficients we have that $n + j + a.d - d^2 = a^2$.

If f is non-singular then $j = 0$ and $\nabla f : \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(d)$ is surjective. By dualizing

$$0 \to \mathcal{T}_f \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(d) \to 0$$

we have that

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{T}_f^* \to 0$$

. Since $\mathcal{T}_f^* = \mathcal{T}_f(-c_1(\mathcal{T}_f)) = \mathcal{T}_f(d)$ it follows that $H^0\mathcal{T}_f(d-1) = 0$ but $H^0\mathcal{T}_f(d) \neq 0$, hence $a = d$ and $n = 0$ from the equation. □

Example 4. *Consider the diagram from our previous example:*

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathbb{P}^2}(-1) & = & \mathcal{O}_{\mathbb{P}^2}(-1) & & & & \\
\big\downarrow \sigma & & \big\downarrow \sigma' := \begin{bmatrix} x \\ -y \\ 0 \end{bmatrix} & & \overset{\nabla f}{\frown} & & \\
\mathcal{T}_f & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \xrightarrow{q_f} & \mathcal{I}_J(2) & \hookrightarrow & \mathcal{O}_{\mathbb{P}^2}(2) \\
\big\downarrow & & \big\downarrow & & \Vert & & \\
\mathcal{I}_Z(-1) & \longrightarrow & Q := coker\sigma' & \longrightarrow & \mathcal{I}_J(2) & &
\end{array}$$

We have that $a = 1, d = 2, j = 3$. Using equation (4.7) we have that $n = 4 - 2 + 1 - 3 = 0$, hence $f = xyz$ is free.

It is possible to compute a free resolution of $\mathcal{T}_f$ using the free resolution of $\mathcal{I}_Z$. We can use the Horseshoe lemma to understand the relation of resolutions of $\mathcal{I}_Z$ and resolutions of $\mathcal{T}_f$. Explicitly:

Proposition 34. *Let Z be the zero-scheme associated to a section $\sigma \in H^0\mathcal{T}_f(a)$ as above and let*

$$0 \to F_1 \to F_0 \xrightarrow{\omega} \mathcal{I}_Z \to 0 \tag{4.10}$$

be a free resolution with quotient map ω. Then we have a free resolution

$$0 \to F_1(a-d) \to F_0(a-d) \oplus \mathcal{O}(-a) \overset{\tilde{\omega}\oplus\sigma}{\to} T_f \to 0 \tag{4.11}$$

where $\tilde{\omega}$ is some morphism induced by ω.

Proof. Apply the left exact functor $Hom_{\mathcal{O}_{\mathbb{P}^2}}(F_0(a-d), -)$ on the exact sequence (4.4):

$$Hom_{\mathcal{O}_{\mathbb{P}^2}}(F_0(a-d), \mathcal{T}_f) \to Hom_{\mathcal{O}_{\mathbb{P}^2}}(F_0(a-d), \mathcal{I}_Z(a-d)) \to Ext^1_{\mathcal{O}_{\mathbb{P}^2}}(F_0(a-d), \mathcal{O}_{\mathbb{P}^2}(-a)) \tag{4.12}$$

But F_0 is free and $H^1(\mathcal{O}_{\mathbb{P}^2}(k)) = 0$ for any k so the Ext term vanishes and the first morphism above is a surjection. There is then a morphism $\tilde{\omega} \in Hom_{\mathcal{O}_{\mathbb{P}^2}}(F_0(a-d), \mathcal{T}_f)$ such that the composition

$$F_0(a-d) \to \mathcal{T}_f \to \mathcal{I}_Z(a-d)$$

is ω.

We can now consider the induced morphism $\tilde{\omega} \oplus \sigma : F_0(a-d) \oplus \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{T}_f$. We use the snake lemma to show that $\tilde{\omega} \oplus \sigma$ is surjective and its kernel is $F_1(a-d)$, explaining in details: consider the two columns corresponding to the exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{T}_f \to \mathcal{I}_Z(a-d) \to 0 \tag{4.13}$$

and

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \to F_0(a-d) \oplus \mathcal{O}_{\mathbb{P}^2}(-a) \to F_0(a-d) \to 0 \tag{4.14}$$

and the morphisms corresponding to the rows $id, \tilde{\omega} \oplus \sigma$ and ω:

$$\begin{array}{ccccccccc}
 & & & & \mathcal{O}(-a) & = & \mathcal{O}(-a) & & \\
 & & & & \downarrow & & \downarrow \sigma & & \\
0 & \longrightarrow & ker\ \tilde{\omega}\oplus\sigma & \longrightarrow & F_0(a-d)\oplus\mathcal{O}(-a) & \xrightarrow{\tilde{\omega}\oplus\sigma} & \mathcal{T}_f & & \\
 & & & & \downarrow & \nearrow_{\tilde{\omega}} & \downarrow & & \\
0 & \longrightarrow & F_1(a-d) & \longrightarrow & F_0(a-d) & \xrightarrow{\omega} & \mathcal{I}_Z(a-d) & \longrightarrow & 0
\end{array}$$

Then the snake lemma states that the following sequence is exact:

$$0 \to ker(\tilde{\omega} \oplus \sigma) \to ker(\omega) \to 0 \to coker(\tilde{\omega} \oplus \sigma) \to 0 \tag{4.15}$$

where we used the vanishings of $coker(\tilde{\omega} \oplus \sigma)$ and $ker(id)$. We have then that $ker(\tilde{\omega} \oplus \sigma) \cong F_1(a-d)$ and $coker(\tilde{\omega} \oplus \sigma) = 0$, which implies that $\tilde{\omega} \oplus \sigma$ is surjective. ☐

Example 5. *For $n = 1$ we have a free resolution:*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-d-2) \to \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(a-d-1)^2 \to \mathcal{T}_f \to 0$$

Proof. One point is given by the intersection of two lines, in other words the resolution of $\mathcal{I}_Z$ must be defined by two linear forms:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{I}_Z \to 0 \quad (4.16)$$

So Proposition 34 implies that we must have

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-d-2) \to \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(a-d-1)^2 \to \mathcal{T}_f \to 0 \quad (4.17)$$

□

If $n = 2$ there is still only one resolution for $\mathcal{I}_Z$ hence we can easily describe a resolution for $\mathcal{T}_f$:

Example 6. *Let f be homogeneous of degree $d+1$ and $\mathcal{T}_f$ be its logarithmic tangent bundle. Suppose that the zero subscheme Z of a global section of minimal degree has length 2. So Proposition 34 with the resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{I}_Z \to 0$$

gives us a sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-d-3) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(a-d-1) \oplus \mathcal{O}_{\mathbb{P}^2}(a-d-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{T}_f \to 0 \quad (4.18)$$

where $M = [M_0(x,y,z), M_1(x,y,z), M_2(x,y,z)]$ with $M_0 \in H^0(\mathcal{O}_{\mathbb{P}^2}(2))$, $M_1 \in H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ and $M_2 \in H^0(\mathcal{O}_{\mathbb{P}^2}(d-2a+3))$.

Since M_1 is a linear form, we can choose $M_1 = y$ through coordinate change, hence we can eliminate any terms of M_0, M_2 with y in M using elementary row operations of matrices. In particular, we can write $M_0 = ax^2 + bxz + cz^2$ which implies $M_0 = c(x - c_1 z)(x - c_2 z)$ for some $c, c_1, c_2 \in \mathbb{C}$. Now there are two cases : $c_1 = c_2$ or $c_1 \neq c_2$.

***If** $c_1 = c_2$ **then** Z **is a double point thus we can write** $M_0 = x^2$*

Then $M = [x^2, y, \alpha z^{d-2a+3} + \beta x z^{d-2a+2}]$ for $\alpha, \beta \in \mathbb{C}$.

***If** $c_1 \neq c_2$ **then** Z **are distinct points and we can write** $f = xz$*

Then $M = [xz, y, \alpha z^{d-2a+3} + \beta x^{d-2a+3}]$ for $\alpha, \beta \in \mathbb{C}$.

In both cases using Serre duality (2.6) to justify the next isomorphism we have that $\mathcal{T}_f \in Ext^1(\mathcal{I}_Z(a-d), \mathcal{O}_{\mathbb{P}^2}(-a)) \cong H^1(\mathcal{I}_Z(2a-d-3))$, and this last vector space has dimension 2. The case $\alpha = \beta = 0$ implies that $\mathcal{T}_f$ is the trivial extension $\mathcal{T}_f = \mathcal{I}_Z(a-d) \oplus \mathcal{O}_{\mathbb{P}^2}(-a)$. But $\mathcal{T}_f$ is locally free, hence this corresponds to the free case.

4.3.1 Nearly-Free Curves

Let C be a reduced curve in $\mathbb{P}^2$ defined by a polynomial $f \in H^0(\mathcal{O}_{\mathbb{P}^2}(d))$ and $\mathcal{T}_C = \mathcal{T}_f$ as above. The curve C is said to be free if $\mathcal{T}_f$ splits as a sum of line bundles, and is nearly free if $\mathcal{T}_f$ fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-b-1) \to \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)^2 \to \mathcal{T}_f \to 0$$

where $a, b \in \mathbb{N}$, $b \geqslant a$. Nearly free vector bundles are completely defined by a point $P \in \mathbb{P}^2$ and a choice of a, b:

Theorem 35. *(MARCHESI; VALLÈS, 2019, Theorem 2.1) $\mathcal{T}_f$ is a nearly free vector bundle with exponents $(a, b) \in \mathbb{N}$ if and only if there exists a point $P \in \mathbb{P}^2$ such that $\mathcal{T}_f$ fits in the following exact sequence*

$$0 \to \mathcal{O}(-a) \to \mathcal{T}_f \to \mathcal{I}_P(-b+1) \to 0 \tag{4.19}$$

The sequence above is equivalent to

$$0 \to \mathcal{O}(-a) \to \mathcal{T}_f \to \mathcal{I}_P(a-d) \to 0$$

We then fix $deg(f) = d + 1$. By example 5, we know that a nearly free bundle has free resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(a-d-2) \to \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(a-d-1)^2 \to \mathcal{T}_f \to 0 \tag{4.20}$$

Definition 36. *Let f be reduced, then we say that $\mathcal{T}_f$ **is** n **- free** if $h^0\mathcal{O}_Z = n$.*

We pose the following question: Given d, is every n from 0 to d^2 realizable? This problem will be further discussed in the next chapter.

Theorem 37. *Let Z and J be the zero-schemes defined by $\mathcal{T}_f$ for some f homogeneous of degree $d+1$ and denote ω_J and ω_Z to be the dualizing sheaves of J and Z. Then there is an exact sequence*

$$\mathcal{O}_{\mathbb{P}^2}(d-a) \xrightarrow{\eta} \omega_J \to \mathcal{O}_S \to \omega_Z \to 0 \tag{4.21}$$

inducing a short exact sequence

$$0 \to coker(\eta) \to \mathcal{O}_S \to \omega_Z \to 0 \tag{4.22}$$

where S is the zero-dimensional subscheme given by σ.

Proof. Since Z and J are zero-dimensional their dualizing sheaves are:

$$\omega_J := \mathcal{E}xt^2(\mathcal{O}_J, \mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{E}xt^1(\mathcal{I}_J, \mathcal{O}_{\mathbb{P}^2})$$

and

$$\omega_Z := \mathcal{E}xt^2(\mathcal{O}_Z, \mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{E}xt^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}^2})$$

Consider now the exact sequence (4.6) and apply the functor $(-)^*$:

$$\mathcal{O}_{\mathbb{P}^2} \xrightarrow{\eta} \omega_J \xrightarrow{g} \mathcal{E}xt^1(Q, \mathcal{O}_{\mathbb{P}^2}) \xrightarrow{g'} \omega_Z \to 0 \tag{4.23}$$

Now dualize the sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-a) \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to Q \to 0 \tag{4.24}$$

to get

$$\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2}(a) \to \mathcal{E}xt^1(Q, \mathcal{O}_{\mathbb{P}^2}) \to 0 \tag{4.25}$$

and note that $\mathcal{E}xt^1(Q, \mathcal{O}_{\mathbb{P}^2})$ is a quotient of $\mathcal{O}_{\mathbb{P}^2}(a)$ which implies that $\mathcal{E}xt^1(Q, \mathcal{O}_{\mathbb{P}^2}) \cong \mathcal{O}_S(a)$ for a subscheme S which is the subscheme of zeroes of σ' (the subschemes are all zero dimensional, so we drop the degrees):

$$\mathcal{O}_{\mathbb{P}^2}(d-a) \to \omega_J \to \mathcal{O}_S \to \omega_Z \to 0 \tag{4.26}$$

Exactness implies $im(\eta) = ker(g)$ and thus

$$coker(\eta) = \omega_J / im(\eta) = \omega_J / ker(g) \cong im(g) = ker(g')$$

so the sequence

$$0 \to ker(g') \to \mathcal{O}_S \xrightarrow{g'} \omega_Z \to 0 \tag{4.27}$$

can be written

$$0 \to coker(\eta) \to \mathcal{O}_S \to \omega_Z \to 0 \tag{4.28}$$

□

Remark 38. *Our last theorem gives a geometrical interpretation: The support of $\mathcal{O}_S$ can be seen as the union of $Supp(coker(\eta))$ and $Supp(\omega_Z)$, but each dualizing sheaf must be supported on the corresponding zero-dimensional schemes and $coker(\eta)$ is given by a quotient $\omega_J/\eta(\mathcal{O}_{\mathbb{P}^2}(d-a))$ with support inside J.*

We finish this chapter with a result on the stability of $\mathcal{T}_f$:

4.3.2 Stability

Let r be the integer such that $\mathcal{T}_f(r)$ is normalized, that is, $c_1\mathcal{T}_f(r) = 0$ or -1. We will need to use the following lemma:

Lemma 39. *(OKONEK; SCHNEIDER; SPINDLER, 1980) Let $\mathcal{F}$ be a normalized rank 2 reflexive sheaf on $\mathbb{P}^n$. Then $\mathcal{F}$ is stable if and only if $H^0(\mathcal{F}) = 0$. If $c_1\mathcal{F}$ is even then $\mathcal{F}$ is semistable if and only if $H^0\mathcal{F}(-1) = 0$*

Since a is, by definition, the least positive integer such that $H^0\mathcal{T}_f(a) \neq 0$, it is possible to tell if $\mathcal{T}_f$ is stable looking only at d, a:

Proposition 40. *Let f be homogeneous of degree $d+1$ and a the integer such that $H^0\mathcal{T}_f(a-1) = 0$ and $H^0\mathcal{T}_f(a) \neq 0$. Define $r = d/2$ if d even and $r = (d-1)/2$ if d odd. Then $\mathcal{T}_f$ is stable if and only if $H^0\mathcal{T}_f(r) = 0$ if and only if $r < a$.*

Remark 41. *It follows then that for a cubic and quartic, the sheaf $\mathcal{T}_f$ is stable if and only if $a > 1$.*

There is another result given by (FAENZI; MARCHESI, 2021) concerning stability. Although being a general result for $\mathbb{P}^N$ we state here the particular case for $\mathbb{P}^2$.

Theorem 42. *(FAENZI; MARCHESI, 2021, Theorem C) Let f be reduced as before, then $\mathcal{T}_f$ is stable if:*

$$j < (d-r)(d) \tag{4.29}$$

where r is the same number given in Proposition 40.

For cubics and quartics this implies that $\mathcal{T}_f$ is stable if:

$$j < (d-1)d \tag{4.30}$$

The right hand side of the inequality has value $2, 6$ for cubics and quartics respectively.

Example 7. *Let $f = y^4 + 2xy^2z + x^2z^2 + yz^3$ and $g = y^2z^2 + y^4 + xz^3$ be quartics with singularities A_6 and E_6. Both curves are absolutely irreducible with $\mathcal{T}_f$ stable and $j = 6$ (see table at the end of next chapter).*

5 Numerical invariant associated to plane curves

We compute the numbers n, j, a and list the cases where $\mathcal{T}_f$ is stable as a sheaf, and whether C is GIT-stable.

5.1 Cubics

5.1.1 Reducible Cubics

Let C be a plane cubic, suppose that C is **reducible**. We can list every possible combination:

- C is the union of an irreducible conic and a line
  - $(xz + y^2)y = 0$ conic and a non tangent line (j=2, a=1, n=1)
  - $(xz + y^2)z = 0$ conic and a tangent line (j=3, a=1, n=0)
- C is the union of three lines
  - $y^3 = 0$ triple line (non reduced)
  - $y^2(y + z) = 0$ double + simple line (non reduced)
  - $yz(y + z) = 0$ three concurrent lines (j=4, a=0, n=0)
  - $xyz = 0$ generic (j=3, a=1, n=0)

5.1.2 Irreducible Cubics

If C is **irreducible**, then it is well known that we can write C in two forms:

- If C is non-singular then it is projectively equivalent to $y^2z = x(x - z)(x - \lambda z$ with $\lambda \neq 0, 1$ (Legendre form)
- C is projectively equivalent to $y^2z = x^3 + axz^2 + bz^3$ (Weierstrass form)

We also have that if C is in a Weierstrass form then it is non-singular if and only if $4a^3 + 27b^2 \neq 0$

If C is smooth we already know that it is 4-free. It is known that a irreducible singular cubic is either isomorphic to a cuspid $y^2z = x^3$ (n,j,a $=$ 1,2,1) or a nodal $y^2z = x^3 + x^2z$ (n,j,a $=$ 3,1,2).

- $j = 0 \implies$ Non-singular (n = 4, a = 2)
- $j = 1 \implies$ Nodal (n = 3, a = 2)
- $j = 2 \implies$ Cuspid and Conic + non tangent line (n = 1, a = 1)
- $j = 3 \implies$ Generic lines xyz and Conic + tangent line (n = 0, a = 1)
- $j = 4 \implies$ Concurrent lines (n = 0, a = 0)

5.2 Reduced Quartics

Here we will use the classification of singular quartic curves (where we take the normal form of each projective class) given in (NEJAD, 2010). By Proposition 40 we have that $\mathcal{T}_f$ is stable if and only if $a > 1$.

5.2.1 j=1,2

Singularities	Polynomial	(n,j,a)
$1A_1$	$(x^3ya_1 + xy^3a_2 + x^2y^2a_3 + x^3za_4 + y^3za_5 + x^4 + y^4 + xyz^2)$	$(8, 1, 3)$
$2A_1$	$(x^4a_1 + x^3ya_2 + x^3za_3 + x^2yza_4 + x^2y^2 + x^2z^2 + y^2z^2)$	$(7, 2, 3)$
$1A_2$	$(y^4a_1 + xy^3a_2 + x^2y^2a_3 + x^3za_4 + x^2yza_5 + x^4 + y^2z^2)$	$(7, 2, 3)$

5.2.2 j=3,4

Singularities	Polynomial	(n,j,a)
$(1A_1, 2A_2)$	$(xy^2za_1 + x^3za_2 + x^3ya_3 + x^4 - x^2z^2 + y^2z^2)$	$(6, 3, 3)$
$3A_1$	$(2x^2yza_1 + 2xy^2za_2 + 2xyz^2a_3 + x^2y^2 + x^2z^2 + y^2z^2)$	$(6, 3, 3)$
$1A_3$	$(xy^3a_1 + x^2y^2a_2 + y^4a_3 - x^4 + y^2z^2)$	$(4, 3, 2)$
$3A_1$	$x \cdot (x^2za_1 + xz^2a_2 + xyza_3 + xy^2 + y^2z + yz^2)$	$(4, 3, 2)$
$(2A_1, 1A_2)$	$(2x^2yza_1 + 2xy^2za_2 + x^2y^2 + x^2z^2 + 2xyz^2 + y^2z^2)$	$(3, 4, 2)$
$(1A_1, 1A_3)$	$(2y^3za_1 + 2xy^2za_2 + y^4 + x^2z^2 + y^2z^2)$	$(3, 4, 2)$
$1D_4$	$(x^2y^2a_1 + x^3ya_2 + y^4 - x^3z + xy^2z)$	$(3, 4, 2)$
$2A_2$	$(x^3ya_1 + x^3za_2 + x^2yza_3 + x^4 + y^2z^2)$	$(3, 4, 2)$
$1A_4$	$(yz^3a_1 + z^4a_2 + y^4 + 2xy^2z + y^3z + x^2z^2)$	$(3, 4, 2)$
$(1A_1, 1A_3)$	$x \cdot (xz^2a_1 + xyza_2 + y^3 + x^2z + y^2z)$	$(3, 4, 2)$
$4A_1$	$x \cdot (y^2za_1 + yz^2a_2 + xy^2 + xyz + z^3)$	$(3, 4, 2)$
$4A_1$	$(xy + xz + yz) \cdot (yza_1 + xza_2 + xy)$	$(3, 4, 2)$

5.2.3 j=5

Singularities	Polynomial	(n,j,a)
$(1A_1, 2A_2)$	$(2x^2yza_1 + x^2y^2 + 2xy^2z + x^2z^2 + 2xyz^2 + y^2z^2)$	$(2, 5, 2)$
$(1A_2, 1A_3)$	$(2xy^2za_1 + y^4 + 2y^3z + x^2z^2)$	$(2, 5, 2)$
$(1A_1, 1A_4)$	$(y^2z^2a_1 + y^4 + 2xy^2z + 2y^3z + x^2z^2)$	$(2, 5, 2)$
$1A_5$	$(z^4a_1 + y^4 - 2xy^2z + x^2z^2 + y^2z^2)$	$(2, 5, 2)$
$1D_5$	$(x^3ya_1 + x^4 + y^4 + xy^2z)$	$(2, 5, 2)$
$1A_5$	$x \cdot (xyza_1 + y^3 + x^2z + xz^2)$	$(2, 5, 2)$
$(2A_1, 1A_3)$	$x \cdot (yz^2a_1 + xy^2 + xyz + z^3)$	$(2, 5, 2)$
$(1A_1, 1D_4)$	$x \cdot (x^2za_1 + x^3 + xyz + yz^2)$	$(2, 5, 2)$
$(3A_1, 1A_2)$	$x \cdot (yz^2a_1 + xy^2 + y^2z + z^3)$	$(2, 5, 2)$
$(2A_1, 1A_3)$	$(xy + xz + yz) \cdot (yza_1 + xy + xz)$	$(2, 5, 2)$
$5A_1$	$z \cdot y \cdot (yza_1 + x^2 + xy + xz)$	$(2, 5, 2)$

5.2.4 j=6 (nearly-free curves)

Singularities	Polynomial	(n,j,a)
$3A_2$	$(x^2y^2 - 2x^2yz - 2xy^2z + x^2z^2 - 2xyz^2 + y^2z^2)$	$(1, 6, 2)$
$(1A_2, 1A_4)$	$(y^4 + 2xy^2z + 2y^3z + x^2z^2)$	$(1, 6, 2)$
$1A_6$	$(y^4 - 2xy^2z + x^2z^2 + 2yz^3)$	$(1, 6, 2)$
$1E_6$	$(x^2y^2a_1 + x^4 + y^3z)$	$(1, 6, 2)$
$(1A_1, 1A_5)$	$x \cdot (xy^2 + xyz + z^3)$	$(1, 6, 2)$
$1D_6$	$x \cdot (x^3 + xyz + z^3)$	$(1, 6, 2)$
$(1A_1, 1A_2, 1A_3)$	$x \cdot (xy^2 + yz^2 + z^3)$	$(1, 6, 2)$
$(1A_1, 1D_5)$	$x \cdot (x^2za_1 + x^3 + yz^2)$	$(1, 6, 2)$
$(1A_1, 1A_5)$	$(xy + z^2) \cdot (xy + yz + z^2)$	$(1, 6, 2)$
$2A_3$	$x \cdot (x + y) \cdot (yza_1 + x^2)$	$(1, 6, 2)$
$(3A_1, 1A_3)$	$z \cdot y \cdot (x^2 + xz + yz)$	$(1, 6, 2)$
$(2A_1, 1D_4)$	$z \cdot y \cdot (xy + xz + yz)$	$(1, 6, 2)$
$6A_1$	$z \cdot y \cdot x \cdot (x + y + z)$	$(1, 6, 2)$

5.2.5 j=7,9 (free curves)

Singularities	Polynomial	(n,j,a)
$(1A_2, 1A_5)$	$x \cdot (xy^2 + z^3)$	$(0, 7, 1)$
$1E_7$	$x \cdot (x^2y + xz^2 + z^3)$	$(0, 7, 1)$
$1A_7$	$(x^2 + yz) \cdot (x^2 + y^2 + yz)$	$(0, 7, 1)$
$(1A_1, 2A_3)$	$z \cdot y \cdot (x^2 + yz)$	$(0, 7, 1)$
$(1A_1, 1D_6)$	$z \cdot y \cdot (y^2 + xz)$	$(0, 7, 1)$
$(3A_1, 1D_4)$	$z \cdot y \cdot x \cdot (x + z)$	$(0, 7, 1)$
Four concurrent lines	$x^2z^2a_1 + xz^3a_1 + x^3z + x^2z^2$	$(0, 9, 0)$

Note that $n = 5$ does not occur, motivating our question of gaps between values of n. We also see that for the irreducible quartics with singularities A_6 and E_6 we have $j = 6, a = 2$, then $\mathcal{T}_f$ is a stable bundle, hence we show that there is an example for the projective plane with sharper conditions on j in contrast with Theorem 42.

5.3 GIT stability for cubics and quartics

Let $G = SL(3)$ be the reductive group acting on $\mathbb{P}^2$. G has a well defined action on homogeneous polynomials $f \in V_{d+1} := H^0\mathcal{O}_{\mathbb{P}^2}(d+1)$ via $g.f = f \circ g^{-1}$ which induces an action $SL(3) \times \mathbb{P}V_{d+1} \to \mathbb{P}V_{d+1}$. In this sense there is a GIT semistable locus $\mathbb{P}{V_{d+1}}^{ss}$ and a GIT quotient $\mathbb{P}{V_{d+1}}^{ss} \to (\mathbb{P}{V_{d+1}}^{ss})//G$.

The classification for cubics (HOSKINS,) and quartics (ARTEBANI, 2009) with respect to GIT stability is well known. For cubics we have the following:

- Stable locus corresponds to smooth cubics (j=0)
- Semistable orbit given by Irreducible nodal (j=1)
- Semistable orbit given by conic + non tangent line (j=2)
- Semistable orbit given by union of non concurrent lines (j=3)

And for quartics:

- Stable locus corresponds to quartics that are smooth (j=0) or have at most ordinary double points and cusps (j=1,2)
- Semistable orbit given by reduced quartics with tacnodes (j=3) and (non-reduced) double conics.

For the cases $d + 1 = 3, 4$ we can manually verify the following remark:

Remark 43. *Let $f \in V_{d+1}$ be homogeneous of degree $d + 1 = 3, 4$ defining a curve C. If C is GIT stable then $\mathcal{T}_f$ is a stable as sheaf.*

Remark 44. *For quartics we have that $\mathcal{T}_f$ is stable if $a > 1$. This includes every quartic in our table for $j = 1, ..., 6$.*

5.4 The algorithm

The main purpose for the existence of our algorithm is to be able to generate concrete examples of reduced plane curves. (See appendix for the definitions of each function)

```
class dVec():
    def __init__(self,d):
        self.d = d
        P2.<x,y,z> = ProjectiveSpace(2,QQ)
        self.P2 = P2
        self.indices = [(i,j) if i+j<=d else None for i in range(d+1) for j in range(d-i+1)]
        print("generating indices... ",self.indices)
        self.monomialdict = {}
        for i,j in self.indices:
            m = x^(d-i-j)*y^(i)*z^(j)
            self.monomialdict[(i,j)] = m
        print("with respective monomials:", self.monomialdict)
    def generate_poly(self,CONSTRAINTS = []):
        coefdict = {}
        vanishingmonomials = []
        f = 0
        for i,j in self.indices:
            if (i,j) in CONSTRAINTS:
                coef = 0
            else:
                coef = ZZ.random_element(-5,5)
            if coef==0:
                vanishingmonomials.append((i,j))
            coefdict[(i,j)] = coef*self.monomialdict[(i,j)]
            f = f + coefdict[(i,j)]
```

```
        CHECKSQUAREFREE = (f!=0 and f.is_squarefree())
        if CHECKSQUAREFREE == False:
            #print("f non square-free, rerunning...")
            f = self.generate_poly(CONSTRAINTS = CONSTRAINTS)
        return f
        # then substitute zero for i,j in CONSTRAINTS
    def runexperimentonce(self,CONSTRAINTS = []):
        f = self.generate_poly(CONSTRAINTS = CONSTRAINTS)
        delta = logtest(f)
        return {"f":f,"delta":delta}
    def endlessexperiment(self,LIMIT = 1000,CONSTRAINTS = []):
        self.experiments = []
        for k in range(LIMIT):
            output = self.runexperimentonce(CONSTRAINTS = CONSTRAINTS)
            self.experiments.append(output)
    def computefrequency(self):
        from collections import Counter
        LIMIT = len(self.experiments)
        self.frequency_sigma =
        Counter([self.experiments[k]['delta'] for k in range(len(self.experiments))]).most_common()
        PERCENTAGES = []
        for k in range(len(self.frequency_sigma)):
            PERCENTAGES.append(self.frequency_sigma[k][1]/LIMIT)
        self.percentages = PERCENTAGES
        cases = [x[0] for x in self.frequency_sigma]
        self.filterpoly = {}
        for CASE in cases:
            filterpolytemp =
            [x['f'] for x in self.experiments if x['delta']==CASE]
            self.filterpoly[CASE]=
            [factor(f) for f in filterpolytemp][0]
            print("\n Case (n,j,a):",CASE," with polynomials:",
            self.filterpoly[CASE])
```

```
        print("\n Table of occurrences: ",self.frequency_sigma,", with
        probabilities:",PERCENTAGES)

P2.<x,y,z> = ProjectiveSpace(QQ,2)
```

Then run the following code for d=3, d=4 and d=5:

```
V = dVec(d)
V.endlessexperiment(LIMIT=10000, CONSTRAINTS = [(0,0),(1,0),(0,1)])
V.computefrequency()
```

The output for each case is:

d=3:

```
generating indices...  [(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0),
(0, 3), (1, 2), (2, 1), (3, 0)]
with respective monomials: {(0, 0): z^3, (0, 1): y*z^2, (1, 0): x*z^2,
(0, 2): y^2*z, (1, 1): x*y*z,
(2, 0): x^2*z, (0, 3): y^3,
(1, 2): x*y^2, (2, 1): x^2*y, (3, 0): x^3}

 Case (n,j,a): (3, 1, 2)  f: x^3 + x^2*y + x*y^2 + y^3 + x^2*z + 2*x*y*z + 2*y^2*z with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (1, 2, 1)  f: x^3 + 2*x^2*y + 2*x*y^2 + y^3 + x^2*z
 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (0, 3, 1)  f: (2*x + 2*y + z) * (x^2 + y^2) with
 singularities: [(0 : 0 : 1)]

 Case (n,j,a): (0, 4, 0)  f: y * x * (x + y) with
 singularities: [(0 : 0 : 1)]

 Table of occurrences:
```

So we managed to recover at least one example for each of the four possible set of (n, j, a) corresponding to singular reduced cubics.

For quartics we have:

```
generating indices...  [(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), (0, 3), (1, 2), (2, 1), (3, 0), (0, 4), (1, 3), (2, 2), (3, 1), (4, 0)]
with respective monomials: {(0, 0): z^4, (0, 1): y*z^3, (1, 0): x*z^3, (0, 2): y^2*z^2, (1, 1): x*y*z^2, (2, 0): x^2*z^2, (0, 3): y^3*z, (1, 2): x*y^2*z, (2, 1): x^2*y*z, (3, 0): x^3*z, (0, 4): y^4, (1, 3): x*y^3, (2, 2): x^2*y^2, (3, 1): x^3*y, (4, 0): x^4}

 Case (n,j,a): (8, 1, 3)  f: x^4 + 2*x^3*y + x^2*y^2 + 2*x*y^3 +
 2*y^4 + x^3*z + x*y^2*z + x^2*z^2 + y^2*z^2 with singularities:
 [(0 : 0 : 1)]

 Case (n,j,a): (7, 2, 3)  f: 2*x^4 + x^2*y^2 + x*y^3 + 2*x^3*z +
 y^3*z + 2*x^2*z^2 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (3, 4, 2)  f: (2*x*y + 2*y^2 + x*z + 2*y*z) * (x^2 + y*z)
 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (4, 3, 2)  f: 2*x^4 + x^3*y + x*y^3 + x*y^2*z + y^3*z
 + 2*y^2*z^2 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (6, 3, 3)  f: 2*x^4 + x^3*y + 2*x^2*y^2 + y^4 +
 2*x^2*y*z + 2*x*y^2*z + 2*y^3*z + 2*y^2*z^2 with singularities
 : [(0 : 0 : 1)]

 Case (n,j,a): (2, 5, 2)  f: x * (2*x^2*y + x*y^2 + 2*x*y*z + x*z^2
 + 2*y*z^2) with singularities: [(-1 : 1 : 1), (0 : 0 : 1),
```

```
(0 : 1 : 0)]

Case (n,j,a): (1, 6, 2)  f: y * (x^3 + 2*x*y^2 + 2*y^3 + 2*x
   *y*z +
y^2*z) with singularities: [(0 : 0 : 1)]

Case (n,j,a): (0, 7, 1)  f: (x^2 + y^2) * (x^2 + x*y + y^2 +
    2*x*z +
2*y*z + 2*z^2) with singularities: [(0 : 0 : 1)]

Case (n,j,a): (1, 6, 1)  f: (2) * x * (x^3 + y^2*z) with
   singularities:
[(0 : 0 : 1), (0 : 1 : 0)]

Case (n,j,a): (0, 9, 0)  f: x^4 + 2*x^3*y + x^2*y^2 + 2*y^4
with singularities: [(0 : 0 : 1)]

Table of occurrences:  [((8, 1, 3), 6652), ((7, 2, 3), 1604)
   , ((3, 4, 2), 554),
((4, 3, 2), 525), ((6, 3, 3), 385), ((2, 5, 2), 212), ((1,
   6, 2), 49),
((0, 7, 1), 11), ((1, 6, 1), 4), ((0, 9, 0), 4)] , with
   probabilities:
[0.6652, 0.1604, 0.0554, 0.0525, 0.0385, 0.0212, 0.0049,
   0.0011, 0.0004, 0.0004]
```

So we recover all the 9 possible cases of reduced singular quartics of previous chapter: [(8, 1, 3), (7, 2, 3), (3, 4, 2),(4, 3, 2),(6, 3, 3), (2, 5, 2), (1, 6, 2), (0, 7, 1), (0, 9, 0)] with the addition of (1, 6, 1) corresponding to the union of a cuspidal cubic with cusp at (0 : 0 : 1) and a non-tangent line intersecting the cubic at (0 : 1 : 0) and (0 : 0 : 1), a degree 1 syzygy is $[0, y, -2z]$ with subscheme of zeroes S corresponding to a point (1 : 0 : 0).

For d=5:

```
generating indices...  [(0, 0), (0, 1), (1, 0), (0, 2), (1,
   1), (2, 0), (0, 3), (1, 2), (2, 1), (3, 0), (0, 4), (1, 3)
   , (2, 2), (3, 1), (4, 0), (0, 5), (1, 4), (2, 3), (3, 2),
   (4, 1), (5, 0)]
with respective monomials: {(0, 0): z^5, (0, 1): y*z^4, (1,
   0): x*z^4, (0, 2): y^2*z^3, (1, 1): x*y*z^3, (2, 0): x^2*z
   ^3, (0, 3): y^3*z^2, (1, 2): x*y^2*z^2, (2, 1): x^2*y*z^2,
```

```
(3, 0): x^3*z^2, (0, 4): y^4*z, (1, 3): x*y^3*z, (2, 2): x^2*y^2*z, (3, 1): x^3*y*z, (4, 0): x^4*z, (0, 5): y^5, (1, 4): x*y^4, (2, 3): x^2*y^3, (3, 2): x^3*y^2, (4, 1): x^4*y, (5, 0): x^5}

Case (n,j,a): (15, 1, 4)  f: 2*x^4*y + x^2*y^3 + x*y^4 + 2*x^4*z + x^2*y^2*z + 2*x*y^3*z + 2*y^4*z + x^3*z^2 + x^2*y*z^2 + x*y^2*z^2 + 2*x*y*z^3 + 2*y^2*z^3 with singularities: [(0 : 0 : 1)]

Case (n,j,a): (14, 2, 4)  f: x^5 + 2*x^3*y^2 + x*y^4 + 2*y^5 + x^3*y*z + 2*x*y^3*z + y^4*z + 2*x^3*z^2 + 2*x*y^2*z^2 + 2*y^3*z^2 + x^2*z^3 with singularities: [(0 : 0 : 1)]

Case (n,j,a): (13, 3, 4)  f: x^5 + x^4*y + x^2*y^3 + 2*x*y^4 + 2*y^5 + x^4*z + x^2*y^2*z + x*y^3*z + 2*y^4*z + x^3*z^2 + 2*x^2*y*z^2 + x*y^2*z^2 + x^2*z^3 with singularities: [(0 : 0 : 1)]

Case (n,j,a): (12, 4, 4)  f: x^5 + x^4*y + 2*x^3*y^2 + 2*x^2*y^3 + x*y^4 + x^2*y^2*z + 2*x*y^3*z + 2*y^4*z + 2*x^3*z^2 + x^2*y*z^2 + 2*x*y^2*z^2 with singularities: [(0 : 0 : 1)]

Case (n,j,a): (9, 4, 3)  f: x * (2*x^4 + 2*x^3*y + 2*x^2*y^2 + 2*x*y^3 + 2*y^4 + x^3*z + x^2*y*z + 2*x*y^2*z + y^3*z + 2*x^2*z^2 + x*y*z^2 + y^2*z^2 + 2*x*z^3 + y*z^3) with singularities: [(0 : 0 : 1)]

Case (n,j,a): (11, 5, 4)  f: 2*x^3*y^2 + x*y^4 + 2*x^3*y*z + x^2*y^2*z + 2*y^4*z + 2*x^3*z^2 + 2*x^2*y*z^2 + 2*x*y^2*z^2 with singularities: [(0 : 0 : 1), (1 : 0 : 0)]

Case (n,j,a): (8, 5, 3)  f: z * (x^4 + x*y^3 + x^3*z + 2*x^2*y*z + x*y^2*z + 2*y^3*z + 2*x*y*z^2) with singularities: [(-1 : 1 : 0), (0 : 0 : 1), (0 : 1 : 0)]

Case (n,j,a): (7, 6, 3)  f: (x + y) * (x^4 + x^3*y + 2*y^4 + x^3*z - x^2*y*z + x*y^2*z + x^2*z^2 + y^2*z^2) with
```

```
  singularities: [(0 : 0 : 1)]

Case (n,j,a): (6, 7, 3)  f: z * x * (2*x^3 + 2*x^2*y + x*y^2 + y^3 + x^2*z + x*y*z + y^2*z + x*z^2 + 2*y*z^2) with singularities: [(-1 : 1 : 0), (0 : 0 : 1), (0 : 1 : 0)]

Case (n,j,a): (10, 6, 4)  f: x^5 + 2*x^4*y + 2*x^2*y^3 + 2*y^5 + 2*x^4*z + x*y^3*z + y^4*z + x^3*z^2 with singularities: [(0 : 0 : 1)]

Case (n,j,a): (5, 8, 3)  f: x * (x^4 + 2*x*y^3 + 2*y^4 + x^2*y*z + 2*x*y^2*z + 2*y^2*z^2) with singularities: [(0 : 0 : 1)]

Case (n,j,a): (4, 9, 3)  f: x^4*y + x^3*y^2 + 2*x^2*y^3 + 2*y^5 + 2*x^4*z + 2*x^2*y^2*z + x*y^3*z with singularities: [(0 : 0 : 1)]

Case (n,j,a): (1, 11, 2)  f: z * x * (2*x^3 + x^2*y + y^3 + 2*x^2*z) with singularities: [(-1 : 1 : 0), (0 : 0 : 1), (0 : 1 : 0)]

Case (n,j,a): (0, 12, 2)  f: y * (x^4 + x^3*y + 2*x^2*y^2 + y^4 + 2*x^3*z + 2*x*y^2*z + x^2*z^2) with singularities: [(-1 : 0 : 1), (0 : 0 : 1)]

Table of occurrences:  [((15, 1, 4), 7009), ((14, 2, 4), 1594), ((13, 3, 4), 580), ((12, 4, 4), 365), ((9, 4, 3), 159), ((11, 5, 4), 90), ((8, 5, 3), 70), ((7, 6, 3), 58), ((6, 7, 3), 27), ((10, 6, 4), 23), ((5, 8, 3), 14), ((4, 9, 3), 7), ((1, 11, 2), 3), ((0, 12, 2), 1)] , with probabilities: [0.7009, 0.1594, 0.058, 0.0365, 0.0159, 0.009, 0.007, 0.0058, 0.0027, 0.0023, 0.0014, 0.0007, 0.0003, 0.0001]
```

So we find 14 cases, it is possible to eliminate ordinary double points (cases (15,1,4),(14,2,4),(13,3,4)) and search for triple points or higher multiplicities by adding constraints, i.e. we can demand $a_{i,j} = 0$ for some choices of i, j:

```
CONSTRAINTS = [(0,0),(1,0),(0,1),(1,1),(2,0),(0,2)]
```

which results in:

```
Case (n,j,a): (12, 4, 4)  f: x^5 + x^4*y + x^3*y^2 + 2*x^2*y^3 + 2*x*y^4 + 2*x^4*z + 2*x*y^3*z + y^4*z + x^3*z^2 + 2*x^2*y*z^2 + 2*x*y^2*z^2 + 2*y^3*z^2 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (11, 5, 4)  f: 2*x^5 + x^4*y + x*y^4 + 2*x^4*z + 2*x^3*y*z + 2*x^2*y^2*z + x*y^3*z + y^4*z + x^3*z^2 + 2*x^2*y*z^2 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (10, 6, 4)  f: x^5 + 2*x^4*y + 2*x^3*y^2 + 2*y^5 + 2*x^3*y*z + 2*x*y^3*z + 2*x^3*z^2 + 2*x^2*y*z^2 with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (7, 6, 3)  f: y * (2*x^4 + x^3*y + x^2*y^2 + x*y^3 + x^3*z + x^2*y*z + x*y^2*z + x^2*z^2 + y^2*z^2) with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (6, 7, 3)  f: 2*x^5 + 2*x^4*y + x^2*y^3 + y^5 + x*y^3*z + 2*y^4*z + x*y^2*z^2 + y^3*z^2 with singularities: [(0 : -1 : 1), (0 : 0 : 1)]

 Case (n,j,a): (5, 8, 3)  f: z * (x^4 + 2*x^3*y + x^2*y^2 + x*y^3 + 2*x^3*z + x^2*y*z + x*y^2*z + 2*y^3*z) with singularities: [(0 : 0 : 1), (0 : 1 : 0)]

 Case (n,j,a): (4, 9, 3)  f: x^5 + 2*x^4*y + x^3*y^2 + 2*x^2*y^3 + 2*x*y^4 + 2*y^5 + x^3*y*z + 2*y^4*z with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (3, 10, 3)  f: y * (2*y^4 + 2*y^3*z + x^2*z^2 + x*y*z^2 + 2*y^2*z^2) with singularities: [(0 : 0 : 1), (1 : 0 : 0)]

 Case (n,j,a): (1, 11, 2)  f: x * (2*x^4 + 2*x^3*y + 2*x^2*y^2 + 2*x*y^3 + y^4 + 2*x^2*y*z + 2*x*y^2*z + x^2*z^2) with singularities: [(0 : 0 : 1)]

 Case (n,j,a): (2, 10, 2)  f: x * (x^4 + 2*x^3*y + 2*x*y^3 +
```

```
   2*x^3*z + 2*y^3*z) with singularities: [(0 : 0 : 1), (0 :
    1 : 0)]

Case (n,j,a): (0, 13, 1)  f: (y + 2*z) * x * (x^3 + x^2*y +
   y^3) with singularities: [(0 : -2 : 1), (0 : 0 : 1)]

Case (n,j,a): (0, 12, 2)  f: (x + z) * (x + y) * (x^3 + 2*y
   ^3 + x^2*z) with singularities: [(-1 : 0 : 1), (-1 : 1 :
   1), (0 : 0 : 1), (1 : -1 : 1)]

Table of occurrences:  [((12, 4, 4), 6680), ((11, 5, 4),
   1497), ((10, 6, 4), 590), ((7, 6, 3), 475), ((6, 7, 3),
   386), ((5, 8, 3), 192), ((4, 9, 3), 122), ((3, 10, 3),
   37), ((1, 11, 2), 12), ((2, 10, 2), 6), ((0, 13, 1), 2),
   ((0, 12, 2), 1)] , with probabilities: [0.668, 0.1497,
   0.059, 0.0475, 0.0386, 0.0192, 0.0122, 0.0037, 0.0012,
   0.0006, 0.0002, 0.0001]
```

We see that new polynomials with singularity of higher multiplicity appears: [((3, 10, 3), 37), ((2, 10, 2), 6), ((0, 13, 1), 2)]

5.4.1 Irreducible nearly-free and free quintics and sextics

An interesting case occurs when we ask for irreducible plane curves of degree d with a point (say P=(0:0:1)) that is singular of multiplicity d-1. For example, if f is a quintic then we pass

$$CONSTRAINTS = (0,0),(1,0),(0,1),(1,1),(2,0),(0,2),(3,0),(0,3),(2,1),(1,2)]$$

so that $a_{i,j} = 0$ when $i + j < d - 1 = 4$:

```
Case (n,j,a): (1, 11, 2)  f: (-1) * (-2*x^5 + 4*x^4*y - 4*x
   ^3*y^2 - 4*x^2*y^3 + 4*x*y^4 - y^5 + 5*x*y^3*z - 2*y^4*z)
    with singularities: [(0 : 0 : 1)]

Case (n,j,a): (0, 12, 2)  f: (-1) * (-x^5 - 3*x^4*y + 4*x^3*
   y^2 - 2*y^5 + 4*x^4*z) with singularities: [(0 : 0 : 1)]

Case (n,j,a): (0, 16, 0)  f: (-1) * (4*x^5 - x^4*y - 4*x^3*y
   ^2 - x*y^4 + 3*y^5) with singularities: [(0 : 0 : 1)]
```

And appropriate conditions for sextics yield:

```
Case (n,j,a): (1, 18, 3)  f: (-1) * (5*x^6 - 4*x^5*y + 2*x^4*y^2 - 4*x^3*y^3 + x^2*y^4 + x*y^5 - y^6 + 2*x^5*z) with singularities: [(0 : 0 : 1)]

Case (n,j,a): (0, 25, 0)  f: (-1) * (5*x^6 + 5*x^5*y - 3*x^4*y^2 + 3*x^3*y^3 - 4*x^2*y^4 - 2*x*y^5 - 3*y^6) with singularities: [(0 : 0 : 1)]
```

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Appendix

APPENDIX A – Algorithm

```
# Basic functions
from IPython.core.interactiveshell import InteractiveShell
InteractiveShell.ast_node_interactivity = 'all'

def generate_quartics(substitute_parameter=False, field = QQ):
    # using delta_p := (mu_p + r_p - 1)/2 we have g = (1/2)(d-1)(d-2) - sum_p(delta_p) for p in Sing(X)
    R.<x,y,z,a_1,a_2,a_3,a_4,a_5> = PolynomialRing(field,8)
    DESCRIPTION = {}
    f = {}
    # IRREDUCIBLE
    # Genus 0:
    # |Sing(C)| = 3:
    ## Three nodes with a_i != +-1 and delta = 2*a_1*a_2*a_3+a_1^2+a_2^2+a_3^2-1 != 0 #TODO conditions
    f[0] = y^2*z^2+x^2*z^2+x^2*y^2+2*x*y*z*(a_1*x+a_2*y+a_3*z)
    DESCRIPTION[0] = "3.A_1"
    ## Three cusps:
    f[1] = y^2*z^2+x^2*z^2+x^2*y^2-2*x*y*z*(x+y+z)
    DESCRIPTION[1] = "3.A_2"
    ## One node two cusps with a_i != +-1
    f[2] = y^2*z^2+x^2*z^2+x^2*y^2+2*x*y*z*(a_1*x+y+z)
    DESCRIPTION[2] = "1.A_1 + 2.A_2"
    ## One cusp two nodes with a_i != +-1 and delta = (a_1+a_2)^2 != 0
    f[3] = y^2*z^2+x^2*z^2+x^2*y^2+2*x*y*z*(a_1*x+a_2*y+z)
    DESCRIPTION[3] = "2.A_1 + 1.A_2"

    # |Sing(C)| = 2:
    ## One node and Tacnode with a_2 != +-1 and delta = a_1^2 + a_2^2 - 1 != 0
    f[4] = z^2*(x^2+y^2)+y^4+2*a_1*y^3*z+2*a_2*x*y^2*z
    DESCRIPTION[4] = "1.A_1 + 1.A_3"
```

```
    ## One cusp and Tacnode with a_1 != 1
    f[5] = x^2*z^2+y^4+2*y^3*z+2*a_1*x*y^2*z
    DESCRIPTION[5] = "1.A_2 + 1.A_3"
    ## One node and Ramphoid cusp with a_1 != 0
    f[6] = x^2*z^2+y^4+2*z*y^3+2*x*y^2*z+a_1*z^2*y^2
    DESCRIPTION[6] = "1.A_1 + 1.A_4"
    ## One cusp and Ramphoid cusp
    f[7] = x^2*z^2+y^4+2*z*y^3+2*x*y^2*z
    DESCRIPTION[7] = "1.A_2 + 1.A_4"

    # |Sing(C)| = 1:
    ## One Oscnode with a_1 != 0
    f[8] = (y^2-x*z)^2+y^2*z^2+a_1*z^4
    DESCRIPTION[8] = "1.A_5"
    ## A_6 cusps
    f[9] = (y^2-x*z)^2+2*y*z^3
    DESCRIPTION[9] = "1.A_6"
    ## Ordinary triple point
    f[10] = x*(y^2-x^2)*z+y^4+a_1*x^2*y^2+a_2*y*x^3
    DESCRIPTION[10] = "1.D_4"
    ## Tacnode cusp
    f[11] = x*y^2*z+x^4+y^4+a_1*x^3*y
    DESCRIPTION[11] = "1.D_5"
    ## Multiplicity 3 cusp
    f[12] = y^3*z+x^4+a_1*x^2*y^2
    DESCRIPTION[12] = "1.E_6"

    # Genus 1
    # |Sing(C)| = 2
    ## Two nodes
    f[13] = (y^2+x^2)*z^2+x^2*y^2+a_1*x^4+a_2*x^3*y+a_3*x^3*z
       +a_4*x^2*y*z
    DESCRIPTION[13] = "2.A_1"
    ## Two cusps
    f[14] = y^2*z^2+x^4+a_1*x^3*y+a_2*x^3*z+a_3*x^2*y*z
    DESCRIPTION[14] = "2.A_2"
    ## One node and cusp
    f[15] = z^2*(y^2-x^2)+x^4+a_1*x*y^2*z+a_2*x^3*z+a_3*x^3*y
    DESCRIPTION[15] = "1.A_1 + 1.A_2"
```

```
    # |Sing(C)| = 1
    ## One tacnode
    f[16] = y^2*z^2-x^4+a_1*x*y^3+a_2*x^2*y^2+a_3*y^4
    DESCRIPTION[16] = "1.A_3"
    ## One Ramphoid cusp
    f[17] = x^2*z^2+2*x*y^2*z+y^4+y^3*z+a_1*y*z^3+a_2*z^4
    DESCRIPTION[17] = "1.A_4"

    # Genus 2
    # |Sing(C)| = 1
    ## One node
    f[18] = x*y*z^2+x^4+y^4+a_1*x^3*y+a_2*x*y^3+a_3*x^2*y^2+a_4*x^3*z+a_5*y^3*z
    DESCRIPTION[18] = "1.A_1"
    ## One cusp
    f[19] = y^2*z^2+x^4+a_1*y^4+a_2*x*y^3+a_3*x^2*y^2+a_4*x^3*z+a_5*x^2*y*z
    DESCRIPTION[19] = "1.A_2"

    # REDUCIBLE: Cubic and line
    ## Smooth cubic then:
    ## Three nodes with a_1 != 0
    f[20] = x*(x*y^2+y^2*z+y*z^2+a_1*x^2*z+a_2*x*z^2+a_3*x*y*z)
    DESCRIPTION[20] = "3.A_1"
    ## One node and tacnode with a_1 != 0
    f[21] = x*(x^2*z+a_1*x*z^2+a_2*x*y*z+y^3+y^2*z)
    DESCRIPTION[21] = "1.A_1 + 1.A_3"
    ## Oscnode with a_1 != 0
    f[22] = x*(x^2*z+x*z^2+a_1*x*y*z+y^3)
    DESCRIPTION[22] = "1.A_5"

    ## Nodal cubic then:
    ## Four nodes with a_1 != 0 and a_2-4a_1 != 0 and a_1-a_2 != -1
    f[23] = x*(z^3+x*y^2+a_1*y^2*z+a_2*y*z^2+x*y*z)
    DESCRIPTION[23] = "4.A_1"
    ## Two nodes and tacnode with a_1 != 0,1
    f[24] = x*(z^3+x*y^2+a_1*y*z^2+x*y*z)
```

```
    DESCRIPTION[24] = "2.A_1 + 1.A_3"
    ## One node and one Oscnode
    f[25] = x*(z^3+x*y^2+x*y*z)
    DESCRIPTION[25] = "1.A_1 + 1.A_5"
    ## One node and ordinary triple point
    f[26] = x*(x^3+a_1*x^2*z+y*z^2+x*y*z)
    DESCRIPTION[26] = "1.A_1 + 1.D_4"
    ## D_6
    f[27] = x*(x^3+z^3+x*y*z)
    DESCRIPTION[27] = "1.D_6"

    ## Cuspidal cubic then
    ## Three nodes and cusp with a_1 != +-2
    f[28] = x*(z^3+x*y^2+y^2*z+a_1*y*z^2)
    DESCRIPTION[28] = "3.A_1 + 1.A_2"
    ## One node one cusp one tacnode
    f[29] = x*(z^3+x*y^2+y*z^2)
    DESCRIPTION[29] = "1.A_1 + 1.A_2 + 1.A_3"
    ## One cusp one Oscnode
    f[30] = x*(z^3+x*y^2)
    DESCRIPTION[30] = "1.A_2 + 1.A_5"
    ## One node and Tacnode cusp D_5
    f[31] = x*(x^3+a_1*x^2*z+y*z^2)
    DESCRIPTION[31] = "1.A_1 + 1.D_5"
    ## E_7
    f[32] = x*(z^3+x^2*y+x*z^2)
    DESCRIPTION[32] = "1.E_7"

    # Two conics:
    ## Four nodes
    f[33] = (x*y+a_1*y*z+a_2*x*z)*(x*y+x*z+y*z)
    DESCRIPTION[33] = "4.A_1"
    ## Two nodes and tacnode
    f[34] = (x*y+x*z+a_1*y*z)*(x*y+x*z+y*z)
    DESCRIPTION[34] = "2.A_1 + 1.A_3"
    ## One node and Oscnode
    f[35] = (z^2+x*y)*(z^2+y*z+x*y)
    DESCRIPTION[35] = "1.A_1 + 1.A_5"
    ## Two tacnodes with a_1 != 0,1
```

```
    f[36] = (x^2+x*y)*(x^2+a_1*y*z)
    DESCRIPTION[36] = "2.A_3"
    ## A_7
    f[37] = (x^2+y*z)*(x^2+y*z+y^2)
    DESCRIPTION[37] = "1.A_7"

    # Conic and two lines
    ## Five nodes with a_1 != 0,1
    f[38] = y*z*(x^2+a_1*y*z+x*z+x*y)
    DESCRIPTION[38] = "5.A_1"
    ## Three nodes and tacnode
    f[39] = y*z*(x^2+y*z+x*z)
    DESCRIPTION[39] = "3.A_1 + 1.A_3"
    ## One node and two tacnodes
    f[40] = y*z*(x^2+y*z)
    DESCRIPTION[40] = "1.A_1 + 2.A_3"
    ## Two nodes and triple point D_4
    f[41] = y*z*(x*y+x*z+y*z)
    DESCRIPTION[41] = "2.A_1 + 1.D_4"
    ## One node and D_6
    f[42] = y*z*(y^2+x*z)
    DESCRIPTION[42] = "1.A_1 + 1.D_6"

    # Four lines
    ## Six nodes
    f[43] = x*y*z*(x+y+z)
    DESCRIPTION[43] = "6.A_1"
    ## Three nodes and triple point D_4
    f[44] = x*y*z*(x+z)
    DESCRIPTION[44] = "3.A_1 + 1.D_4"
    ## Four concurrent lines with a_1 != 0,1
    f[45] = x*z*(x+z)*(x+a_1*z)
    DESCRIPTION[45] = "Four lines intersecting"

    DESCRIPTION = list(DESCRIPTION.values())


    if substitute_parameter == True:
        R2.<x,y,z> = PolynomialRing(field,3)
```

```
        # generate parameters a_i and substitute
        g = list()
        for poly in f.values():
            p = tuple(ZZ.random_element(1000) for k in range(5))
            temp = poly((x,y,z)+p)
            temp = R2(temp)
            g.append(temp)
        return g, DESCRIPTION
    else:
        return list(f.values()), DESCRIPTION


def JacobianIdeal(X):
    '''
        X: Projective Subscheme defined by method ProjectiveSpace.subscheme, expected to be generated by a single f
        J: Ideal generated by all first partial derivatives \partial_i f
    '''
    J = X.Jacobian()
    J = Ideal(J.gens()[1:]) # Jacobian with only the partial derivatives as generators (doesnt matter because f is homogeneous but Sagemath carries an extra generator if we dont exclude)
    return J

def JacobianSyzygy(X):
    '''
        X: Projective subscheme
        Syz: Syzygy module of ideal J
    '''
    J = JacobianIdeal(X)
    Syz = J.syzygy_module()
    return Syz

def MinimumDegreeColumn(Syz):
    ''' Auxiliary function for MinimumDegreeSyzygy
```

```
        Syz: Syzygy module
        output: dictionary of minimum degree generators
    '''
    gens = Syz.rows() # list of generators of syzygy
    mindegree = min([max(generator[0].degree(),generator[1].
        degree(),generator[2].degree()) for generator in gens
        ])
    mindeg_generators = [generator for generator in gens if
        max(generator[0].degree(),generator[1].degree(),
        generator[2].degree())==mindegree]
    return {"mindegree_generators":mindeg_generators, "degree
        ":mindegree, "len":len(mindeg_generators)}

def MinimumDegreeSyzygy(X):
    '''
        X: Projective subscheme
        mindeg: dictionary of minimum degree generators
    '''
    Syz = JacobianSyzygy(X)
    mindeg = MinimumDegreeColumn(Syz)
    return mindeg

def logtest(polynomial):
    '''
        polynomial: Homogeneous polynomial in x,y,z
        value: [n,j,a]
    '''
    # d+1:=deg(f)
    d = polynomial.degree() - 1
    X = P2.subscheme(polynomial)
    I_J = JacobianIdeal(X)
    hilb = I_J.hilbert_polynomial()
    a = MinimumDegreeSyzygy(X)['degree']
    j = hilb(0)
    n = (d^2-a*d+a^2-j)
    value = n,j,a
    return value
```