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Technical Notes

Vibration of a Long, Tip Pulled Deflected Beam

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Nomenclature

A	=	base point of the beam
B	=	end point of the beam
C	=	generic point along the beam
c	=	adimensional constant ($c = -(PL^2/EI)$)
E	=	elasticity coefficient (Young's module), $\mathrm{N/m^2}$
F	=	cable pull force, N
h_x, h_y	=	coordinates of position of cable support, m
K_e, $\bar{k}_e$	=	finite element stiffness matrix in global or local coordinates, N/m
L	=	local-to-global degrees of freedom rotation matrix (dimensionless)
L, L_c	=	total length of beam, length of cable in non-deformed beam condition, m
ℓ	=	length of cable in the deformed beam condition, m
M_e, $\bar{m}_e$	=	finite element mass matrix in global or local coordinates, kg
M	=	torque, $\mathrm{N \cdot m}$
N	=	order of polynomial (dimensionless)
o_i	=	linear degree of freedom, m
P	=	pulling cable force, N
p	=	dimensionless parameter; $\sin(\alpha_B + \beta/2)$
$\boldsymbol{q}$	=	degrees of freedom of long beam finite element model, m
q_i	=	linear degree of freedom, m
$\boldsymbol{R}_e$, $\boldsymbol{r}_e$	=	global and local vectors of degrees of freedom of finite element, m
r_i	=	linear degree of freedom, m
s	=	arc length, m
T	=	local-to-global, coordinates transform matrix (dimensionless)
u	=	arc length ratio; s/L
x, x_B	=	horizontal coordinate, horizontal coordinate of the beam's end position, m
y, y_B	=	vertical coordinate, vertical coordinate of the beam's end position, m
α, α_B	=	slope angle of the beam, slope angle of beam's extremity, rad
β	=	angle of cable or applied force, rad
γ	=	cumulative angle; $\alpha + \beta$, rad
θ_{xi}, θ_{yi}, θ_{zi}	=	angular degrees of freedom, rad
μ_i	=	coefficient of polynomial expansion (dimensionless)
ϕ, ϕ_A, ϕ_B, ϕ_C	=	auxiliary angles; arc $\sin(\gamma/2)$

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I. Introduction

Long, flexible deployable structures are used in space exploration vehicles in low-gravity environment applications (Puig et al. [1], Tibert [2], and Pellegrino [3]). The use of the same types of probes in the exploration of planets with a large gravitational field would be of interest to reach regions of scientific interest of difficult access, such as cliff sides and terrain slopes. Vibration control in this case is a sensitive issue that can be addressed by the use of a reliable model of dynamics of the long exploration probe. The present work addresses the problem of building a simple dynamic model of a long beam deformed by a tip pulling cable. The simple model is sought for future use in a vibrations control strategy.

The exact static deformation shape of a long beam is obtained through the solution of the nonlinear beam governing differential equations. The exact deformed configuration for beams with a uniform cross section and transversal end load is found through the recursive solution of elliptic integrals (Frish–Fay [4] and Timoshenko and Gere [5]). The deformed configuration of beams with an inclined pulling force can be found via numerical approximations, using adequate methods such as the Runge–Kutta (Ohtsuki [6] and Shvartsman [7]), the shooting method (Holland et al. [8]), the finite element method (Howell [9]), and the quasi-linearization finite difference method (Al-Sadder and Al-Rawi [10]). A strategy to calculate the static deformation of a long beam pulled by a tip cable is summarized in the work of Yau [11].

The dynamic behavior of the long beam can be obtained through the analysis of small vibration about the static equilibrium of a finite elements model of the structure (Ferris and Afonta [12], Sallstrom et al. [13], and Santillan et al. [14–16]). Vibration analysis and experimental validation of a finely meshed finite element model of a tip pulled beam was carried out by Holland et al. [8]. The present work proposes the combination of a nonlinear static deformation model of the same structure and a courser finite element model of the deformed beam in order to assess its dynamic characteristics. Results of the different modeling strategies are compared and discussed.

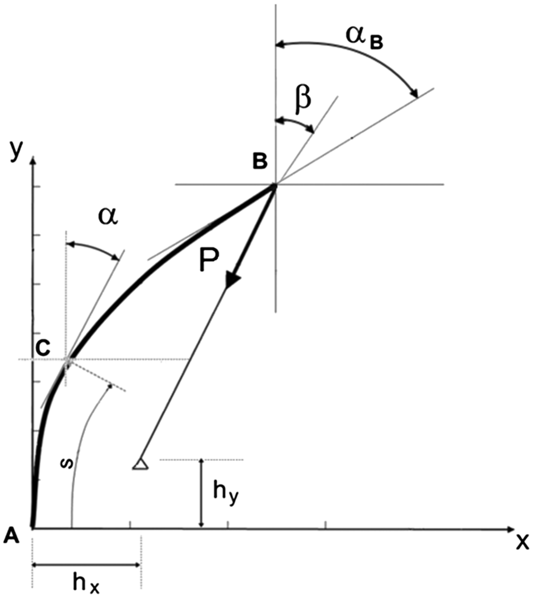

Fig. 1 Long beam pulled by an inclined cable.

II. Analytical and Numerical Models

A. Static Deformation Model

A simplified model of a long beam with a pulling cable is presented in Fig. 1. The arc length s is measured from the base (point A) to point C. The beam's slope is represented by angle α, and the inclination of the cable's pulling force P is represented by angle β. The beam's total length is L, whereas the coordinates of the cable support are (h_x, h_y). The beam's tip position is represented by coordinates (x_B, y_B).

According to the classical Bernoulli–Euler theory for the deflection of beams, the bending moment at section C of the beam is proportional to the change of curvature caused by the applied force. The equilibrium of moments of an infinitesimal portion of the homogeneous beam with constant cross section is described as

$$M = EI\frac{\mathrm{d}\alpha}{\mathrm{d}s} = P\sin\beta(y_B - y) - P\cos\beta(x_B - x) \tag{1}$$

Differentiation of Eq. (1), with respect to the arch length s, yields the governing differential equation of the model given by

$$\frac{\mathrm{d}^2\gamma}{\mathrm{d}u^2} + c\sin\gamma = 0 \tag{2}$$

where u, γ, and c are defined as

$$\left.\begin{array}{r} u = \frac{s}{L} \\ \gamma = \alpha - \beta \\ c = \frac{-PL^2}{EI} \end{array}\right\} \tag{3}$$

The boundary conditions for the configuration shown in Fig. 1 are

$$\left.\begin{array}{r} \gamma_{u=0} = -\beta \\ \left(\frac{\mathrm{d}\gamma}{\mathrm{d}u}\right)_{u=1} = 0 \end{array}\right\} \tag{4}$$

Parameters p and ϕ are introduced for manipulation of the differential equations in order to use the elliptical integrals

$$\left.\begin{array}{r} p = \sin\frac{\alpha_B - \beta}{2} \\ p^* \sin\phi = \sin\frac{\gamma}{2} \end{array}\right\} \tag{5}$$

Definition of the relationships between ds and γ; dγ and dϕ; and ds and dϕ is necessary in order to find expressions for the coordinates of the deflected beam in terms of elliptical integrals. The first of such relations is found by multiplying Eq. (2) by 2dγ and integrating it with respect to γ. The relationship between du and dγ can be expressed by

$$\mathrm{d}u = \frac{\mathrm{d}\gamma}{\sqrt{2c[\cos\gamma - \cos(\alpha_B - \beta)]}} \tag{6}$$

The relation between dγ and dϕ is obtained by differentiating Eq. (5) with respect to ϕ; that is,

$$\mathrm{d}\gamma = \frac{2p\cos\phi}{\sqrt{1 - p^2\sin^2\phi}}\mathrm{d}\phi \tag{7}$$

It can be seen from Eqs. (6) and (7) that

$$\mathrm{d}s = \frac{1}{k\sqrt{1 - p^2\sin^2\phi}}\mathrm{d}\phi \tag{8}$$

where

$$k = \frac{\sqrt{c}}{L} \tag{9}$$

Some algebraic work is done to express coordinates x and y of the beam, in terms of the variables of the elliptical integrals

$$y = \frac{\cos\beta}{k}[-F(p,\phi_c) + F(p,\phi_A) + 2E(p,\phi_c) - 2E(p,\phi_A)] + \frac{2p\sin\beta}{k}(\cos(\phi_c) - \cos(\phi_a)) \tag{10}$$

and

$$x = \frac{2p\cos\beta}{k}(\cos(\phi_a) - \cos(\phi_c)) + \frac{\sin\beta}{k}[-F(p,\phi_c) + F(p,\phi_A) + 2E(p,\phi_c) - 2E(p,\phi_A)] \tag{11}$$

where

$$\left.\begin{array}{r} \phi_A = \arcsin\left(\frac{\sin(\frac{-\beta}{2})}{p}\right) \\ \phi_B = \frac{\pi}{2} \\ \phi_C = \arcsin\left(\frac{\sin(\frac{\alpha-\beta}{2})}{p}\right) \end{array}\right\} \tag{12}$$

$$F(p,\phi) = \int_0^\phi \frac{1}{\sqrt{1 - p^2\sin^2\phi}}\mathrm{d}\phi \tag{13}$$

$$E(p,\phi) = \int_0^\phi \sqrt{1 - p^2\sin^2\phi}\,\mathrm{d}\phi \tag{14}$$

Displacements y and x of Eqs. (10) and (11) allow us to write an analytic expression that involves the forcing action angle β:

$$\frac{h_x}{L}\cos\beta - \frac{h_y}{L}\sin\beta - \frac{2p\cos(\phi_A)}{F(p,\phi_c) - F(p,\phi_A)} = 0 \tag{15}$$

Equation (15) expresses the angle of the pulling cable β with respect to an arbitrary angular tip inclination. The static deformation curve of the beam can be analytically calculated by Eqs. (10) and (11), provided the value of angle β is known. A solution for Eq. (15) can only be found in an iterative numerical manner. This problem can be simplified by using a polynomial interpolation curve, relating the pulling force angle β as a function of the length of the beam's traction cable. The iteratively calculated points of the curve of the pulling

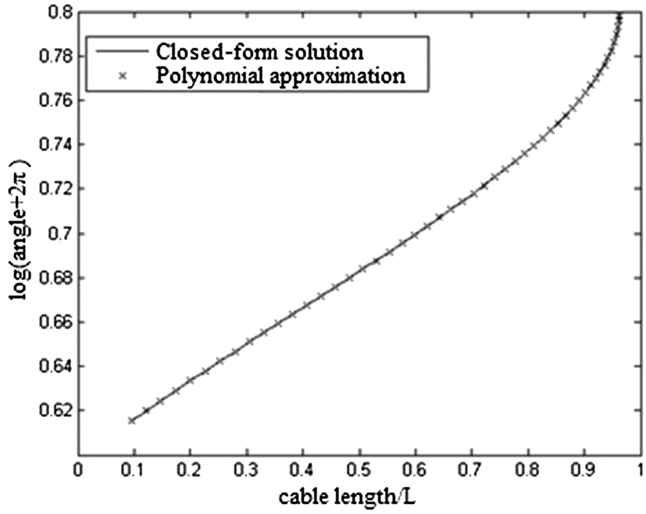

Fig. 2 Force angle β as a function of cable length.

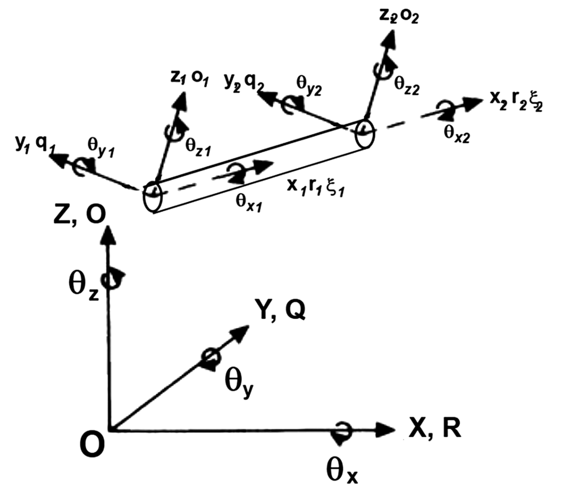

Fig. 3 Global coordinate systems and element local coordinate system.

force angleβ as a function of the length of traction cable are shown in Fig. 2. Equation (16) shows the exponential interpolation function used to fit such a curve through the interpolating coefficients μ_i

$$\beta = \sum_{i=0}^{N} \mu_i \times \left(\frac{\ell}{L_c}\right)^i \tag{16}$$

where ℓ and L_c are the lengths of the cable in the deformed and nondeformed beam conditions, respectively. Numerical tests show that a polynomial order of $N \approx 15$ is sufficient to fit the graphically described angle-length behavior curve of Fig. 2.

In a practical application, parameters μ_i, $i = 0, 1, \ldots, M$ of the force-angle $\times$ cable-length function, can be initially calculated based on physical properties of the long beam. The coordinates of the static deformation curve of the beam as a function of the current pulling cable length can be calculated in a deterministic way with the use of the polynomial parameterization and analytical expressions of Eqs. (10) and (11). Such a deterministic procedure yields a fast calculation of the static deformation curve, which can be advantageously employed into a future real-time control strategy. The static deformation curve of the beam is used as an adaptive centerline of geometry of the finite element models described next.

B. Finite Element Models for Beam and Cable

A numerical model containing stiffness and mass matrices of the long deformed beam is obtained via the finite element method (Petyt [17]). A small number of finite elements are allocated along the central line of deformation of the beam, calculated according to the approximated polynomial fit strategy described earlier. The three-dimensional finite beam element used, with its local and global coordinate systems, is shown in Fig. 3. Such a beam element is designed to respond to uncoupled axial, bending, and torsion deformations (Petyt [17], Craig and Kurdila [18], Kwon and Bang [19], and Cook et al. [20]).

The local degrees of freedom of each element are described in vector $\boldsymbol{r}_e$ as

$$\boldsymbol{r}_e^T = [\, r_1 \quad q_1 \quad o_1 \quad \theta_{x1} \quad \theta_{y1} \quad \theta_{z1} \quad r_2 \quad q_2 \quad o_2 \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2} \,] \tag{17}$$

The stiffness K_e and mass M_e contribution of each individual element is described in the global coordinate system as

$$K_e = T^T \bar{k}_e T \tag{18}$$

$$M_e = T^T \bar{m}_e T \tag{19}$$

where $\bar{k}_e$ and $\bar{m}_e$ are a single-element stiffness and mass matrix, given in the local coordinates system. The global position $\boldsymbol{R}_e$ of the local vector of degrees of freedom $\boldsymbol{r}_e$ is expressed by

$$\boldsymbol{R}_e = T^T \boldsymbol{r}_e \tag{20}$$

where

$$T = \begin{bmatrix} [L] & 0 & 0 & 0 \\ 0 & [L] & 0 & 0 \\ 0 & 0 & [L] & 0 \\ 0 & 0 & 0 & [L] \end{bmatrix} \tag{21}$$

$$L = \begin{bmatrix} \cos(x,X) & \cos(x,Y) & \cos(x,Z) \\ \cos(y,X) & \cos(y,Y) & \cos(y,Z) \\ \cos(z,X) & \cos(z,Y) & \cos(z,Z) \end{bmatrix} \tag{22}$$

and $\cos(i,j)$ represents the cosine of the angles formed between the local coordinates axis i and the global coordinates axis j.

The probe's pulling cable is modeled as an axial displacement truss element, and the same procedure previously described is done in order to calculate its stiffness and mass matrices.

III. Model Comparisons

A. Static Deformation Curve of the Long Beam

This section compares the use of the proposed technique for determining the static deformation of a pulled flexible beam using the polynomial expression given by Eq. (16). Solutions with the proposed modeling process are compared with the exact results derived from the iterative solution of Eqs. (10), (11), and (15), as well as experimental results obtained by Holland et al. [8]. The structure chosen for comparison is a polycarbonate beam with a length of 762 mm and a rectangular section of 25.4×4.8 mm. The beam is mounted with one end clamped and the other end pulled by a cable. The material's Young modulus and density are 1.656 GPa and $1120\ \mathrm{kg/m^3}$, respectively. The cable is modeled as a massless spring of stiffness 11.67 kN/m. The cable origin coordinates (see Fig. 1) are $h_x = 12.7$ mm and $h_y = 28.6$ mm. The pulling loads and deflections in the y direction are normalized by the critical buckling force and the beam's length, respectively. Comparative results for the beam's end tip deflection using the closed-form (exact) experimental and polynomial approximated solutions are shown in Fig. 4. Static baseline deformation of the beam, using the exact results and approximate polynomial, are shown in Fig. 5.

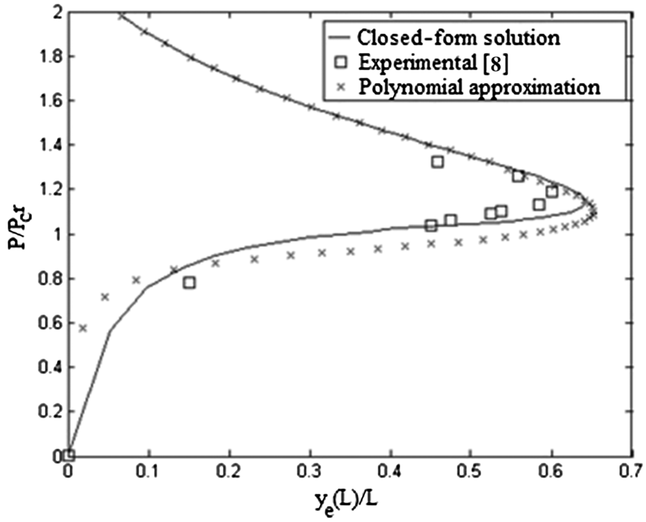

Fig. 4 Comparison of the beam's end tip deflection.

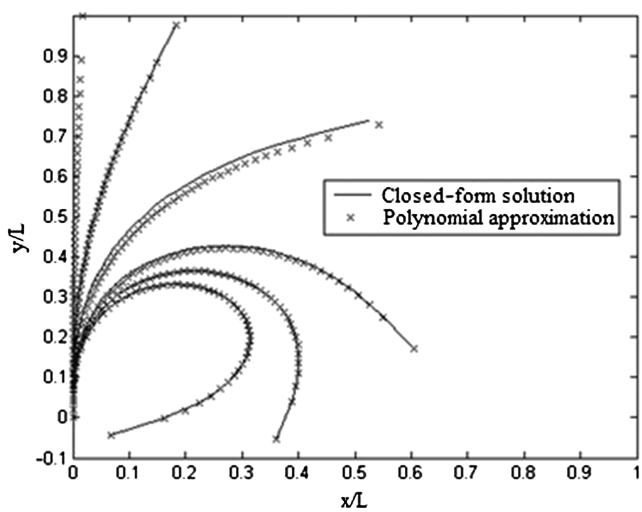

Fig. 5 Comparison of baseline deformations.

B. Dynamic Characteristics of the Statically Deformed Structure

Comparisons of the dynamic characteristics of the deformed beam models and experimental measurements are carried out in the present section. The proposed modeling strategy consists in fitting 50 planar, 6-degree-of-freedom (DOF) beam elements to the baseline geometry of the deflected beam. Holland et al.'s simulation of the same structure [8] uses a mesh of 1000 6-DOF finite beam elements. A graphic containing the values of the first four natural frequencies of the deformed structure, as a function of the normalized pulling force, is shown in Fig. 6. Experimental results of natural frequencies identified in [8] are also shown in the figure.

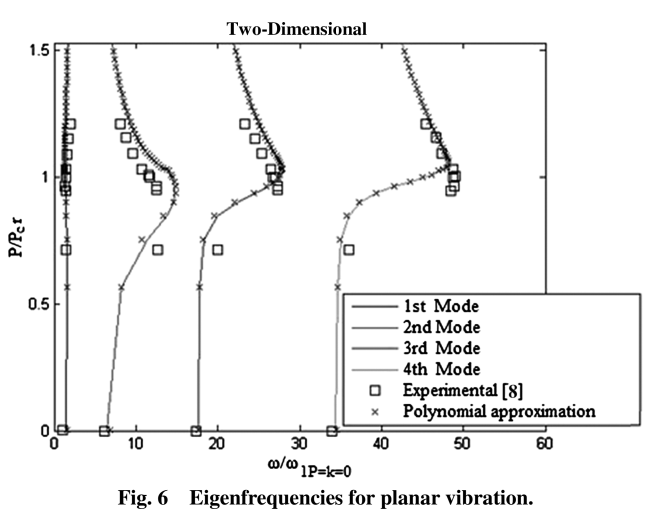

Fig. 6 Eigenfrequencies for planar vibration.

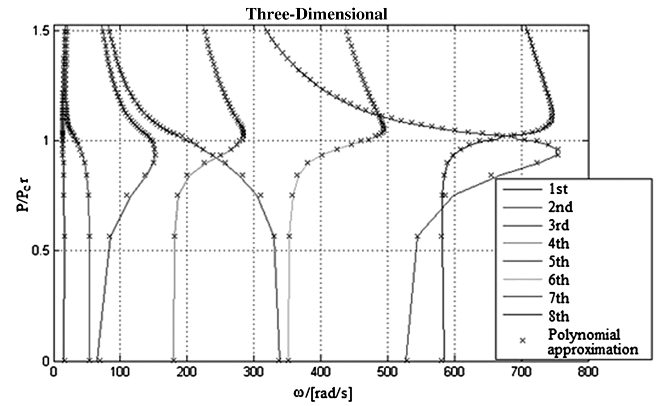

Fig. 7 Eigenfrequencies for three-dimensional vibration.

Results of the dynamic analysis of the proposed model are also made using 50 12-DOF spatial beam elements. Eight natural frequencies are calculated, as a function of the normalized pulling force using the exact deformed baseline solution of Eq. (15), and the deformation baseline is calculated from the polynomial approximations of the same analytical expression [Eq. (16)]. Figure 7 shows the results of such a comparison.

IV. Conclusions

The static baseline curve of the deflected beam is calculated from the proposed polynomial approximation of the nonlinear relation between the force application angle and cable length. Comparison between the approximated results and the exact deformation solution shows a good agreement between the two strategies. A low-order finite element mesh is superimposed to the deflected baseline of the beam to yield a useful dynamic model of the system.

When the beam is subject to small pulling forces and deformations, the polynomial approximation strategy does not yield an accurate representation of the beam's static deflection, as seen in Figs. 4 and 5. However, the eigenfrequency results displayed in Figs. 6 and 7 show that, in spite of the differences in the prediction of the beam's static deformed configuration, the dynamic behavior results are always in good agreement between the exact and approximated solutions.

The comparison of results indicate that the strategy of obtaining an approximated analytical expression for the baseline static deformation of the beam, followed by adaptation of a model of low-order finite elements, is an efficient way to obtain a fast and convenient dynamic description of the structure.

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