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Study of solutions to some non-linear evolution equations of dispersive type

Estudo das soluções para algumas equações de evolução não-lineares do tipo dispersivo

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Advisor: Mahendra Prasad Panthee

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Resumo

Nesta tese, estudamos a controlabilidade e estabilização das equações de Benjamin e Intermediate Long Wave (ILW) num domínio periódico.

A primeira parte deste trabalho envolve a equação de Benjamin derivada por Benjamin em [12] que modela a propagação unidirecional de ondas longas num sistema de dois fluidos onde o fluido inferior com maior densidade é infinitamente profundo e a interface está sujeita a capilaridade. Primeiramente, estudaremos a controlabilidade e estabilidade do sistema linear não homogêneo associado à equação de Benjamin. Provamos a existência e unicidade das soluções para este sistema via teoria de semigrupos. Depois, usamos o método clássico do momento (veja [79]) para mostrar que o sistema linear é globalmente exatamente controlável e consequentemente obter um resultado de estabilização exponencial com uma taxa de decaimento arbitraria.

Em seguida, derivamos a propriedade de propagação de compacidade, propagação de regularidade e a propriedade de continuação única para a equação de Benjamin em espaços de Bourgain associado. Usamos essas propriedades para provar um resultado de estabilidade global assintótica com uma taxa de decaimento arbitraria. Finalmente, obtemos un resultado de controlabilidade global para a equação de Benjamin.

A segunda parte de este trabalho se concentra nas propriedades de controlabilidade e estabilização da equação ILW, que modela ondas dispersivas não lineares de amplitude moderada na interface entre dois fluidos de diferentes densidades positivas contidos em repouso num canal longo com uma parte superior e inferior horizontal, o fluido mais leve formando uma camada horizontal acima de uma camada da mesma profundidade do fluido mais pesado. Nós provamos que a equação ILW com condições de fronteira periódicas é exatamente controlável e exponencialmente estabilizável. Especificamente, incorporamos uma lei de feedback na forma de amortecimento localizado na equação para estabelecer um efeito regularizante. Usando este efeito regularizante e propriedades de propagação de regularidade e continuação única, conseguimos demonstrar a estabilização semi-global no espaço de Sobolev $L^2_0(\mathbb{T})$ de soluções fracas obtidas pelo método de vanishing viscocity. Também, estabelecemos a boa colocação local e a estabilidade exponencial local em $H^s_0(\mathbb{T})$ com $s > \dfrac{1}{2}$. Finalmente, a controlabilidade exata local em $H^s_0(\mathbb{T})$, com $s > \dfrac{1}{2}$, é derivada combinado a lei de feedback acima com um controle de malha aberta. Esses resultados são semelhantes aos obtidos por Linares e Rosier [59] para a equação de BO.

Palavras-chave: Equações Dispersivas; Equação de Benjamin; Equação Intermediate Long Wave; Espaços de Sobolev; Boa-colocação local e global; Controle; Estabilização; Propriedade de continuação única.

Abstract

In this thesis, we study the controllability and stabilization of Benjamin and Intermediate Long Wave (ILW) equations on a periodic domain.

In the first part of this work we consider the Benjamin equation derived by Benjamin in [12]. This model describes the unidirectional propagation of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. First we deal with the controllability and stabilization of the nonhomogenous linear system associated to the Benjamin equation. We obtain the existence and uniqueness of solutions of this system via semigroup theory. Then, we use the classical moment method (see [79]) to show that the linear system is globally exactly controllable, and consequently to get a global exponential stabilization result with an arbitrary decay rate.

Next, we derive propagation of compactness, the propagation of smoothness and the unique continuation property for the nonlinear Benjamin equation in associated Bourgain's spaces in the periodic setting. We use these properties to obtain the global exponential stability with a natural feedback law and an arbitrary decay rate. Finally, we also obtain the global controllability result for the Benjamin equation.

The second part of this work we focus on the controllability and the stabilization properties of the ILW equation which models nonlinear dispersive waves of moderate amplitude on the interface between two fluids of different positive densities contained at rest in a long channel with a horizontal top and bottom, the lighter fluid forming a horizontal layer above a layer of the same depth of the heavier fluid. We prove that the ILW equation with periodic boundary conditions is exactly controllable and exponentially stabilizable. Specifically, we incorporate a feedback law in the form of localized damping into the equation to establish a smoothing effect. Using this smoothing effect together with the propagation of regularity property and the unique continuation property we show the semi-global stabilization in $L^2(\mathbb{T})$ of weak solutions obtained by the method of vanishing viscosity. The local-well posedness and the local exponential stability in $H_0^s(\mathbb{T})$ with $s > \dfrac{1}{2}$ is also established using the contraction mapping theorem. Finally, the local exact controllability is derived in $H_0^s(\mathbb{T})$ with $s > \dfrac{1}{2}$ by combining the above feedback law with some open-loop control. These results are similar to the ones obtained by Linares and Rosier [59] for the BO.

Keywords: Dispersive equation; Benjamin equation; Intermediate Long Wave Equation; Sobolev spaces; local and Global Well-posedness; Control; Stabilization; Unique continuation property.

List of symbols

$\mathbb{N}$	the set of natural numbers.
$\mathbb{Z}$	the set of integers.
$\mathbb{Z}^*$	the set of integers different from zero.
$\mathbb{N}^*$	the set of natural numbers different from zero.
$\mathbb{R}$	the set of real numbers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{R}^n$	the n-dimensional Euclidean space.
$\mathbb{T}$	the torus, defined by $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) := \{[x] : x \in \mathbb{R}\}$, where each equivalent class $[x]$ is defined as $[x] := \{y \in \mathbb{R} : x = y \bmod(2\pi)\}$.
$\|x\|$	the euclidean norm $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.
$A \subset B$	A subset of B.
$A \subsetneq B$	A proper subset of B.
$X \hookrightarrow Y$	continuous embedding of space X into space Y.
$\|\beta\|$	$\beta_1 + \beta_2 + \cdots + \beta_n$ for each multi-index $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}^{*n}$.
$\partial_j^k f$	partial derivative of $f = f(x_1, ..., x_n)$ of order $k \in \mathbb{N}^*$ with respect to j-th variable.
$\partial^\beta f$	$\partial_1^{\beta_1} \partial_2^{\beta_2} \cdots \partial_n^{\beta_n} f$, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$.
$l^p = l^p(\mathbb{Z})$	the set of all complex sequences $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ such that $\\|\alpha\\|_{l^p}^p := \sum_{n\in\mathbb{Z}} \|\alpha_n\|^p < \infty$, for $1 \leqslant p < +\infty$. If $p = \infty$, then $\\|\alpha\\|_{l^\infty} := \sup_{n\in\mathbb{Z}} \|\alpha_n\|$.

$l_s^2 = l_s^2(\mathbb{Z})$	the set of all complex sequences $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ such that $\|\alpha\|_{l_s^2}^2 := 2\pi \sum_{n\in\mathbb{Z}} (1+\|n\|^2)^s \|\alpha_n\|^2 < \infty$, for $s \geqslant 0$.
$C(X)$	the set of continuous functions in X.
$C_p(X)$	the set of continuous-periodic functions defined on X.
$C_p^\infty(X)$	the set of continuous-periodic functions defined on X that are infinitely differentiable.
$C_c(X)$	the set of continuous functions with compact support in X.
$C_c^\infty(\mathbb{T})$	the set of continuous-periodic functions defined on $\mathbb{T}$ that are infinitely differentiable with compact support in $\mathbb{T}$.
$C_c^\infty(X)$	the set of continuous functions defined on X that are infinitely differentiable with compact support in X.
$C_0(X)$	All functions $f \in C(X)$ such that f vanishes at infinity.
$C^n(X)$	the set of functions f such that $\partial^\beta f \in C(X)$ for each multi-indexes β with $\|\beta\| \leqslant n$.
C^∞	the set $\bigcap_{n\in\mathbb{N}} C^n(X)$.
$C_c^\infty(X)$	the set of all functions in the class C^∞ with compact support in X.
$\mathcal{F}[f] = \widehat{f}$	the Fourier transform of the distribution f.
$\mathcal{F}^{-1}[f] = f^\vee$	the inverse Fourier transform of the distribution f.
$L^p(X)$	the Lebesgue space of p-integrable functions.
$\mathcal{L}(X)$	the Banach space of bounded linear operators from X into X with norm $\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{x}$.
$H_p^s(\mathbb{T})$	the Sobolev space of L^2-type, 24.
$J^s f$	$\mathcal{F}^{-1}[(1+\|\cdot\|^2)^{\frac{s}{2}}\widehat{f}]$.
$[\cdot,\cdot]$	the commutator operator.

Contents

Introduction

A control system is a dynamical system on which one can act by using suitable controls. There are a lot of problems that appear when studying controls systems. The most common ones are the controllability problem and stabilization problem. In control theory, some nonlinear evolution equations that arise in physics (electron-plasma, ion field interaction, electromagnetic waves, gravitational waves, optical fibres etc.) or in fluid dynamics and water wave theory can be formulated in the following abstract form

$$u_t = A(t)u + F(t)(h), \tag{0.0.1}$$

where h is a so-called control function in a suitable space. Therefore, roughly speaking, the controllability problem consists in finding an appropriate control input $F(t)(h)$ to guide the system (0.0.1) from a given initial state u_0 to a given terminal state u_1. On the other hand, assuming that we have an equilibrium which is unstable without an use of the control, the following question arises: can one construct a feedback control law $F(t)(u)$ which stabilizes the equilibrium? So the problem of stabilization consists in the existence and construction of such stabilizing feedback law for a given control system (see [23] and the references therein). More precisely, the following control and stabilization problems are fundamental in control theory:

- **Exact control problem.** Let $T > 0$. Given an initial state u_0 and a terminal state u_1 in a certain space, can one find an appropriate control input $f = F(t)(h)$ so that the equation (0.0.1) admits a solution u which satisfies $u(\cdot, 0) = u_0$ and $u(\cdot, T) = u_1$?
- **Stabilization problem.** Can one find a feedback law $f = F(t)u$ so that the resulting closed-loop system

  $$u_t + A(t)u = F(t)u \qquad t \in \mathbb{R}, \tag{0.0.2}$$

  is asymptotically stable as $t \longrightarrow \infty$?

Control and stabilization of dispersive equations have been widely studied in the literature, see [53, 55, 59, 52, 79, 80, 81, 30, 51, 74, 75] and references therein. In particular, for the

Korteweg-de Vries (KdV) equation we refer to [52, 80, 81, 90, 72, 24, 61, 73] and for the Benjamin-Ono (BO) equation we refer to [53, 55, 59] and the references therein.

In this thesis, we address some questions about the controllability and stabilization of the Benjamin and the Intermediate Long Wave (ILW) equations posed on a periodic domain. Both equations are of a dispersive type and appear in different physical contexts. In what follows, we summarize some results known in the literature for these equations.

The Benjamin equation can be written as

$$\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + \partial_x(u^2) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{0.0.3}$$

where $u = u(x,t)$ denotes a real-valued function, α is a positive real number, and $\mathcal{H}$ denotes the Hilbert transform defined by

$$\mathcal{H}(f)(x) = \frac{1}{\pi} p.v. \int \frac{f(x-y)}{y} dy. \tag{0.0.4}$$

The Benjamin equation (0.0.3) is an integro-differential equation that serves as a generic model for unidirectional propagation of long waves in a two-fluid system where the lower fluid with greater density is infinitely deep and the interface is subject to capillarity. It was derived by Benjamin in [12] to study gravity-capillarity surface waves of solitary type on deep water. He also showed that solutions of the equation (0.0.3) satisfy the conserved quantities,

$$I_1(u) = \frac{1}{2} \int_{\mathbb{R}} u^2(x,t)\, dx, \tag{0.0.5}$$

and

$$I_2(u) = \int_{\mathbb{R}} \left[\frac{1}{2} (\partial_x u)^2(x,t) - \frac{\alpha}{2} u(x,t) \mathcal{H} \partial_x u(x,t) - \frac{1}{3} u^3(x,t) \right] dx. \tag{0.0.6}$$

We refer to [12] and the references therein for more details about this physical model. Here, we present a brief review of some results obtained for the Benjamin equation. Several works have been devoted to study the existence, stability and asymptotic properties of solitary waves solutions of (0.0.3), see for instance [3, 6, 12, 17]. The well-posedness of the initial value problem (IVP) associated to the Benjamin equation on $H^s(\mathbb{R})$ has been extensively studied for many years, see [47, 18, 85, 57, 58]. The best known global well-posedness result in $L^2(\mathbb{R})$ is due to Linares [57]. There are further improvements of this result, viz., local well-posedness in $H^s(\mathbb{R})$ for $s \geqslant -\frac{3}{4}$ [18].

The Benjamin equation posed on a periodic spatial domain $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$ is also widely studied in the literature. Linares [57] proved the global well-posedness in $L^2(\mathbb{T})$, and Shi and Junfeng [85] proved local well-posedness in $H^s(\mathbb{T})$ for $s \geqslant -\frac{1}{2}$.

In this work, we are interested in considering the Benjamin equation posed on a periodic domain,

$$\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + \partial_x(u^2) = 0, \quad x \in \mathbb{T}, \ \ t \in \mathbb{R}, \tag{0.0.7}$$

where $\mathcal{H}$ denotes the Hilbert transform defined by

$$\mathcal{H}(f)(x) = \frac{1}{2\pi} p.v. \int_0^{2\pi} f(x-y) \cot\left(\frac{y}{2}\right) dy, \quad \forall x \in \mathbb{T}. \tag{0.0.8}$$

Note that the conserved quantities (0.0.5)-(0.0.6) hold in the periodic case as well. Our aim is to study the equation (0.0.7) in the context of control theory by adding a control term $f = f(x,t)$. Such control f will be allowed to act only on a small subset of the domain $\mathbb{T}$. This situation includes more cases of practical interest and is therefore more relevant in general. The Benjamin equation (0.0.7) contains both the third order local term $-\partial_x^3 u$, as in the KdV equation, and the second order nonlocal term $-\alpha\mathcal{H}\partial_x^2 u$, as in the BO equation. So, it is natural to analyze the Benjamin equation from the control and stabilization point of view and check whether it behaves in similar way as the KdV and BO equations. In this regard, our study is inspired by the works of Linares and Ortega [55], Russell and Zhang [81], and Laurent, Rosier and Zhang [52].

We start studying the controllability and stabilization of the linearized Benjamin equation. Using the classical moment method, and a generalization of the Ingham theorem, we show that the linearized Benjamin equation with periodic boundary conditions is exactly controllable for any time $T > 0$ with some control $h \in L^2([0,T]; H_p^s(\mathbb{T}))$, $s \geqslant 0$. We also prove the global exponential stabilization of the linearized Benjamin equation in $H_p^s(\mathbb{T})$ $(s \geqslant 0)$ with any given decay rate.

Next, we extend the linear results to the corresponding nonlinear systems following a similar approach as those implemented by Laurent et al. [51, 50, 52] (see also [29, 30]). We use Bourgain's spaces (see [14]) and some techniques motivated from microlocal analysis to get certain propagation of compactness and regularity properties to the solutions of the Benjamin equation posed on a periodic domain. These properties together with the unique continuation property of the Benjamin equation will be used to establish the global stabilization and exact controllability of the nonlinear Benjamin equation (see the nonlinear system (2.0.1)). We also show that the same feedback control law $f = -GG^*u$ (see (2.0.9)) that stabilizes the linearized Benjamin equation, stabilizes the nonlinear Benjamin equation as well.

The global controllability result is derived by a combination of the exponential stabilization result and the local control result, as is usual in control theory (see for instance [29, 30, 51, 52, 53]). Indeed, given the initial data u_0 to be controlled, by means of the damping term $f = -GG^*u$ supported in ω, i.e by solving the IVP (2.0.1) (with $f = -GG^*u$), we drive it to a state close enough to the mean value $\mu := [u_0]$ in a sufficiently large time. We do the same with the final state u_1 by solving the system backwards in time, due to the time reversibility of the Benjamin equation. This produces two states which are close enough to μ so that the local controllability result applies. We also show that it is possible to construct a time-varying feedback law, as in [26, 52], ensuring a global stabilization result with an arbitrary large decay rate for the Benjamin equation.

Finally, motivated by our results obtained for the linearized Benjamin equation in the first part of this work and the results due to Linares and Rosier [59] for the BO equation, we study the controllability and stabilization properties for the Intermediate Long Wave (ILW) equation,

$$\partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{0.0.9}$$

where $u = u(x,t)$ denotes a real-valued function, δ is a positive real number, and $\mathcal{T}$ is defined by the principle-value convolution

$$\mathcal{T}(f) = -\frac{1}{2\delta} p.v. \int_{-\infty}^{+\infty} \coth\left(\frac{\pi}{2\delta}(x-y)\right)u(y)dy. \tag{0.0.10}$$

The equation (0.0.9) arises in internal wave theory (see [42, 48]) as a mathematical model of nonlinear dispersive waves of moderate amplitude on the interface between two fluids of different positive densities contained at rest in a long channel with a horizontal top and bottom, the lighter fluid forming a horizontal layer above a layer of the same depth of the heavier fluid. The parameter $\delta > 0$ characterizes the relative depths of two homogeneous fluid layers, the deviation of the interface between which is governed approximately by the Intermediate Long-Wave equation.

In [1], Bona et. al proved that the ILW equation (0.0.9) possesses an infinite sequence of conserved quantities, the first three being

$$\tilde{I}_1(u) = \int_{\mathbb{R}} u(x,t)\, dx, \tag{0.0.11}$$

$$\tilde{I}_2(u) = \frac{1}{2}\int_{\mathbb{R}} u^2(x,t)\, dx, \tag{0.0.12}$$

and

$$\tilde{I}_3(u) = -\int_{\mathbb{R}} \left(\frac{1}{3}u^3(x,t) - u(x,t)T(\partial_x u(x,t)) + \frac{1}{\delta}u^2(x,t)\right) dx. \tag{0.0.13}$$

Some works have been devoted to study the existence, stability and asymptotic properties of solitary waves solutions of (0.0.9), see for instance [42, 7, 2] and references therein. The equation (0.0.9) can be solved in $\mathbb{R}$ via an inverse scattering transform (see [45]). Also, the well-posedness of the IVP associated to the ILW equation (0.0.9) on $H^s(\mathbb{R})$ has been studied (see [4], [1] and [2]). The best known local well-posedness result in $H^s(\mathbb{R})$

with $s > \dfrac{3}{2}$ is due to Bona et all [1].

In this work we consider the ILW equation posed on a periodic spatial domain $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$

$$\partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = 0, \quad x \in \mathbb{T}, \;\; t \in \mathbb{R}, \tag{0.0.14}$$

where the mean value of u denoted by $[u]$ is zero and $\mathcal{T}$ is a Fourier multiplier operator defined by

$$\widehat{\mathcal{T}u}(n) := i\coth(n\delta)\,\widehat{u}(n), \quad \text{for all } n \in \mathbb{Z}^*, \tag{0.0.15}$$

with $\widehat{u}(n)$ the $n^{th}-$ Fourier coefficient of u, (see [5, 1, 66]). The operator $\mathcal{T}$ defined in (0.0.15) satisfies

$$\mathcal{T}(u\mathcal{T}v + v\mathcal{T}u) = \mathcal{T}u\mathcal{T}v - uv, \tag{0.0.16}$$

and

$$\int_{-\infty}^{+\infty}(u\mathcal{T}v + v\mathcal{T}u)\,dx = 0, \tag{0.0.17}$$

where $[u] = [v] = 0$. The quantities (0.0.11)-(0.0.13) are also conserved on the torus $\mathbb{T}$. The IVP in the periodic setting has been studied by Bona et. al [1]. The best result so far is that the IVP is locally well-posed in the space

$$H_0^s(\mathbb{T}) := \left\{u \in H_p^s(\mathbb{T}) : [u] = 0, \right\}$$

for $s > \dfrac{3}{2}$. Considering initial data u_0 for the IVP associated to (0.0.14), we have the following results (see [1] and the references therein).

- If $u_0 \in H_0^{\frac{j}{2}}(\mathbb{T})$ for $j = 2$, or 3, then there exists a weak solution u of (0.0.14) with initial value u_0 such that $u \in L^{+\infty}((0,+\infty); H_0^{\frac{j}{2}}(\mathbb{T}))$.
- If $u_0 \in H_0^s(\mathbb{T})$ where $s > \dfrac{3}{2}$, then this solution is unique and, for each $T > 0$, $u \in C^k((0,T); H_0^{s-2k}(\mathbb{T}))$ for all k such that $s - 2k \geqslant -\dfrac{3}{2}$. Moreover, for each $T > 0$, the correspondence that associates u_0 to u is continuous from $H_0^s(\mathbb{T})$ to $C^k((0,T); H_0^{s-2k}(\mathbb{T}))$ for all k for which $s - 2k \geqslant -\dfrac{3}{2}$. If $s = \dfrac{n}{2}$ with $n \in \mathbb{Z}$, $n > 3$, then $u \in C_b^k([0,+\infty); H_0^{s-2k}(\mathbb{T}))$ for k with $s - 2k > -\dfrac{3}{2}$. Here $C_b^k([0,+\infty); H_0^{s-2k}(\mathbb{T}))$ stands for the space of functions $u : [0,+\infty) \longrightarrow H_0^{s-2k}(\mathbb{T})$ whose $t-$derivatives up to order k exist and are continuous and bounded with values in $H_0^{s-2k}(\mathbb{T})$.

Recall that the positive parameter δ in equation (0.0.14) characterizes the depth of the lighter fluid layer in a two-fluid system in which the light fluid rests upon a heavier fluid (see [48]). Several works have pointed out that the equation (0.0.9) reduces to the KdV equation as $\delta \longrightarrow 0$ and to the BO equation as $\delta \longrightarrow +\infty$ (see [1, 48, 83, 82, 60]).

As announced earlier, we consider the ILW equation (0.0.14) in the periodic setting. First, we prove the controllability and stabilization of the linearized system associated to the ILW equation (0.0.14). Next, using an observability inequality derived from the exact controllability result, we show that the exponential decay rate of the resulting linearized closed-loop system is as large as one desires. Finally, we deal with the control and stabilization problem for the full ILW equation. To stabilize the nonlinear ILW equation, we consider the feedback law

$$f = -G(D(Gu)),$$

where $\widehat{Du}(n) = |n|\widehat{u}(n), \ \ \forall n \in \mathbb{Z}$. A scaling argument gives (at least formally)

$$\frac{1}{2}\|u(T)\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|D^{\frac{1}{2}}(Gu)\|^2_{L^2_0(\mathbb{T})}dt = \frac{1}{2}\|u_0\|^2_{L^2_0(\mathbb{T})}. \tag{0.0.18}$$

This suggests that the energy is dissipated over time. On the other hand, (0.0.18) reveals a smoothing effect, at least in a region $\omega \subset \mathbb{T}$. Using a propagation of regularity property in the same vein as in [30, 52, 51, 50, 68], we prove that the smoothing effect holds everywhere, i.e.

$$\|u\|_{L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T}))} \leqslant C(T, \|u_0\|_{L^2_0(\mathbb{T})}). \tag{0.0.19}$$

Using this smoothing effect and the classical compactness/uniqueness argument, we first prove that the weak solutions in the sense of vanishing viscosity of the corresponding closed-loop equation is semi-globally exponentially stable in $L^2_0(\mathbb{T})$.

We also use the smoothing effect (0.0.19) to extend (at least locally) the exponential stability from $L^2_0(\mathbb{T})$ to $H^s_0(\mathbb{T})$ for $s \in (\frac{1}{2}, 2]$. Finally, we derive an exact controllability result in $H^s(\mathbb{T})$, $s \in (\frac{1}{2}, 2]$ by incorporating the same feedback law $f = -G(D(Gu))$ in the control input to obtain a smoothing effect. Thus, one can write the control input $f = Gh \in L^2(0, T; H^{s-\frac{1}{2}}(\mathbb{T}))$ with $h = -DGu + D^{\frac{1}{2}}\tilde{h}$ for some function $\tilde{h}$.

This thesis is organized as follows. Chapter 1 is dedicated to the preliminaries for the proper development of the text.

In Chapter 2, we prove the exact controllability and stabilization of the linearized Benjamin equation posed on a periodic domain (see Theorem 2.3.7 and Corollary 2.4.3).

In Chapter 3 we introduce the Bourgain's spaces associated to Benjamin equation and derive the propagation of compactness and Regularity as well as the unique continuation property for the Benjamin equation.

In Chapter 4 we present the global exact controllability and global exponential stabilization of the Benjamin equation on a periodic domain.

Finally, Chapter 5 contains the proofs of the results concerning the controllability and stabilization of the ILW equation on a periodic domain.

Chapter 1

Preliminaries

In this chapter we introduce some definitions, concepts, properties and results that are used for the development of this work. We begin with some results related to the Lebesgue spaces L^p, Fourier analysis on $\mathbb{T}$ and periodic distributions. Then, we continue with Sobolev spaces on $\mathbb{T}$, the Hilbert transform, Semigroup theory, Riesz Basis and Ingham's type inequalities. Finally, we recall some results related to non-linear interpolation theory and interpolation of L^p-spaces with change of measure.

1.1 The Lebesgue spaces L^p

In this section, we present the Lebesgue spaces. We deeply encourage the reader to revise more properties and results on these spaces in the books [15, 36].

Let $(X, \mathcal{M}, \mu)$ be a measure space with μ always being a positive measure. Here we understand that two μ-measurable functions are considered equal if they coincide except on a set of μ-measure zero. For a measurable function f on X, we define the space

$$L^p(X) = L^p(X, \mathcal{M}, \mu) = \{f\colon X \longrightarrow \mathbb{C}\colon f \text{ is measurable and } \|f\|_{L^p} < \infty\},$$

where

$$\|f\|_{L^p} := \begin{cases} \left(\displaystyle\int_X |f|^p\right)^{\frac{1}{p}} \text{ if } 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x\in X} |f(x)| \text{ if } p = \infty. \end{cases}$$

We will abbreviate $L^p(X, \mathcal{M}, \mu)$ by $L^p(X)$ or simply by L^p. For $1 \leqslant p \leqslant \infty$, $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_{L^p})$ is a normed space. It is clear that if X has finite measure and $1 \leqslant q < p \leqslant \infty$, then $L^\infty(X) \subseteq L^p(X) \subset L^q(X) \subseteq L^1(X)$.

Important results regarding duality in L^p-spaces can be found in [36, page 190] and [31]. In [64, page 40] the mixed spaces $L^p(X; L^q(Y))$, where X and Y are assumed Banach spaces, are introduced. Further properties on mixed L^p spaces can be found there.

1.2 Fourier analysis on the Torus

In this section, we record some definitions and properties related with Fourier transform and periodic distributions on $\mathbb{T}$. The content of this section is based mainly on the books [41, 37]. We also refer the references [49, 36, 37, 38] to recall the definition of Fourier transform, the Schwartz class and the space of tempered distributions, which are important tools to define Bourgain's spaces.

1.2.1 Fourier transform on $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$

We begin this subsection defining the Fourier's coefficients.

Definition 1.2.1. *For a complex-valued function f in $L^1(\mathbb{T})$ and $k \in \mathbb{Z}$, we define*

$$\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx, \quad \forall\, k \in \mathbb{Z}.$$

We call $\widehat{f}(k)$ the $k^{th}-$Fourier coefficient of f. The Fourier series of f at x is the series

$$\sum_{m\in\mathbb{Z}} \widehat{f}(k)e^{ikx}.$$

Note that $\widehat{f}$ is well defined with $\sup_{k\in\mathbb{Z}} |\widehat{f}(k)| \leqslant \frac{1}{2\pi}\|f\|_{L^1}$. We denote by $\bar{f}$ the complex conjugate of the function f, by $\tilde{f}$ the function $\tilde{f}(x) = f(-x)$, by $\tau_y f(x) = f(x-y)$ the translation for any $y \in \mathbb{R}$, and by $f * g$ the function

$$(f * g)(x) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x-y)g(y)dy, \quad \text{for all } x \in \mathbb{T}.$$

Some properties of the Fourier's transform on $L^1(\mathbb{T})$ can be found in [37, Chapter 3]. We continue with a short discussion of Fourier series of square summable coefficients. Now we consider the Hilbert space $L^2(\mathbb{T})$ (some times denoted also by $L^2_p(\mathbb{T})$) with inner product

$$(f, g)_{L^2(\mathbb{T})} = \int_{\mathbb{T}} f(x)\overline{g(x)}dx.$$

Remark 1.2.2. *It is known that $\{\psi_k\}_{k\in\mathbb{Z}} := \left\{\frac{e^{ikx}}{\sqrt{2\pi}}\right\}_{k\in\mathbb{Z}}$ forms a Fourier's orthonormal basis for $L^2(\mathbb{T})$.*

The orthonormality of the sequence $\{\psi_k\}$ is a consequence of the following simple but powerful identity:

$$\int_{\mathbb{T}} \psi_k(x)\psi_m(x)dx = \begin{cases} \frac{1}{2\pi} & \text{when } k = m, \\ 0 & \text{when } k \neq m. \end{cases}$$

Proposition 1.2.3 ([37, Prop. 3.1.16]). *The following are valid for $f, g \in L^2(\mathbb{T})$:*

(i) (Plancherel's identity)
$$\|f\|^2_{L^2(\mathbb{T})} = 2\pi \sum_{k\in\mathbb{Z}} |\widehat{f}(k)|^2,$$

(ii) (Parseval's relation)
$$(f, g)_{L^2(\mathbb{T})} = 2\pi \sum_{k\in\mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)},$$

(iii) The map $f \longmapsto \{\widehat{f}(k)\}_{k\in\mathbb{Z}}$ is an isometry from $L^2(\mathbb{T})$ onto $l^2_0(\mathbb{Z})$,

(iv) For all $k \in \mathbb{Z}$ we have
$$\widehat{fg}(k) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)\widehat{g}(k-m) = \sum_{m\in\mathbb{Z}} \widehat{f}(k-m)\widehat{g}(m).$$

1.2.2 Periodic Distributions

In this subsection, we introduce the space of periodic distributions and its basic properties. We begin by recalling the C^∞ periodic functions.

Let $C^\infty_p(\mathbb{T})$ be the collection of all functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ which are C^∞ and $2\pi-$periodic. There is no natural norm with respect to which $C^\infty_p(\mathbb{T})$ is a Banach space. Nevertheless, there exists a natural distance which turns $C^\infty_p(\mathbb{T})$ into a complete metric space. In fact, it can be shown that
$$d_1(f, g) = \sum_{j=0}^{+\infty} 2^{-j} \frac{\|f^{(j)} - g^{(j)}\|_\infty}{1 + \|f^{(j)} - g^{(j)}\|_\infty}, \quad f, g \in C^\infty_p(\mathbb{T}),$$
defines a metric in $C^\infty_p(\mathbb{T})$. Furthermore, if $\{f_n\} \subset C^\infty_p(\mathbb{T})$ and $f \in C^\infty_p(\mathbb{T})$, then $f_n \xrightarrow{d_1} f$ if and only if $\|f^{(j)}_n - f^{(j)}\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$ for all $j = 0, 1, 2, \cdots$. In this case we write $f_n \xrightarrow{C^\infty_p(\mathbb{T})} f$.

Now, we introduce the class of rapidly decreasing sequences. This space has nice properties in relation to the Fourier transform and it is fundamental for the definition of periodic distributions.

The *space of rapidly decreasing sequences*, denoted by $\mathcal{G}(\mathbb{Z})$, is the set of all complex sequences $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ such that $\displaystyle\sum_{k=-\infty}^{+\infty} |k|^j|\alpha_k| < \infty$, for all $j \in \mathbb{N}$. Note that $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \in \mathcal{G}(\mathbb{Z})$ if and only if $\|\alpha\|_{\infty,j} := \sup_{k\in\mathbb{Z}}(|\alpha_k||k|^j) < \infty$, for all $j \in \mathbb{N}$.

In what follows we will regard $\mathcal{G}(\mathbb{Z})$ as a complete metric space provided with the distance $\quad d_2(\alpha, \beta) = \displaystyle\sum_{j=0}^{+\infty} 2^{-j} \frac{\|\alpha - \beta\|_{\infty,j}}{1 + \|\alpha - \beta\|_{\infty,j}}$, $\alpha, \beta \in \mathcal{G}(\mathbb{Z})$. Thus, a sequence $\{\alpha^n\}_{n\in\mathbb{Z}} \subset \mathcal{G}(\mathbb{Z})$ converges to $\alpha \in \mathcal{G}(\mathbb{Z})$, with respect to d_2 if and only if $\|\alpha^n - \alpha\|_{\infty,j} \longrightarrow 0$ as $n \longrightarrow \infty$ for all $j = 0, 1, 2, ...$ In this case we write $\alpha^n \xrightarrow{\mathcal{G}(\mathbb{Z})} \alpha$.

The Fourier transform $\wedge : C_p^\infty(\mathbb{T}) \longrightarrow \mathcal{G}(\mathbb{Z})$ is an isomorphism and homeomorphism, that is, linear, one to one, onto $\mathcal{G}(\mathbb{Z})$, and continuous with a continuous inverse (with respect to the metrics d_1 and d_2). Further properties of the Fourier's transform in $C_p^\infty(\mathbb{T})$ can be found in [41, §3.1].

Here, we define the class of periodic distributions.

Definition 1.2.4 (Periodic Distribution, see [41, Theorem 3.168])**.** *A linear functional on $C_p^\infty(\mathbb{T})$, $T : C_p^\infty(\mathbb{T}) \to \mathbb{C}$, is called a periodic distribution if T is continuous.*

The set of all periodic distributions will be denoted by $\mathcal{D}'(\mathbb{T})$. Therefore, $\mathcal{D}'(\mathbb{T})$ is the topological dual of $C_p^\infty(\mathbb{T})$. Before proceeding it is convenient to introduce some definitions (see §3.2 in [41]).

Definition 1.2.5. *Let $f \in \mathcal{D}'(\mathbb{T})$.*

(i) ($\mathcal{D}'(\mathbb{T})$ convergence) We say that a sequence $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{T})$ converges to $T \in \mathcal{D}'(\mathbb{T})$, if $\langle T_n, g\rangle \to \langle T, g\rangle$, as $n \to \infty$, $\forall\, g \in C_p^\infty(\mathbb{T})$.

(ii) Its jth distributional derivative $f^{(j)}$ is defined by the relation

$$\langle f^{(j)}, g\rangle := (-1)^j\langle f, g^{(j)}\rangle, \;\; \forall\, g \in C_p^\infty(\mathbb{T}).$$

(iii) Let $\psi \in C_p^\infty(\mathbb{T})$. The product ψf is the periodic distribution defined by the formula $\langle \psi f, g\rangle := \langle f, \psi g\rangle$, $\forall\, g \in C_p^\infty(\mathbb{T})$.

(iv) (Fourier transform in $\mathcal{D}'(\mathbb{T})$) The Fourier transform of $f \in \mathcal{D}'(\mathbb{T})$ is the function $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ defined by the formula $\widehat{f}(k) = \dfrac{1}{2\pi}\langle f, e^{-ikx}\rangle$, $\forall\, k \in \mathbb{Z}$. Moreover, the n^{th} partial sum associated to f is

$$S_n(f)(x) = \sum_{k=-n}^{n} \widehat{f}(k)\, e^{ikx}.$$

(v) Let $\psi \in C_p^\infty(\mathbb{T})$. The convolution $f * \psi$ of f and ψ is the function

$$(f * \psi)(x) := \frac{1}{2\pi}\langle f, \tau_y\tilde{\psi}\rangle, \;\; \forall\, \psi \in C_p^\infty(\mathbb{T}).$$

The next theorem shows that the sequence $\{S_n f\}_{n\geqslant 0}$ converges to f in $\mathcal{D}'(\mathbb{T})$.

Theorem 1.2.6 ([41, Theorem 3.166])**.** *Let $f \in \mathcal{D}'(\mathbb{T})$. Then $S_n(f) \in C_p^\infty(\mathbb{T})$, $\forall n \in \mathbb{N}$, and $S_n(f) \xrightarrow{\mathcal{D}'(\mathbb{T})} f$ as $n \to \infty$. Therefore,*

$$f = \sum_{k=-\infty}^{+\infty} \widehat{f}(k)\, e^{ikx} = \lim_{k\to+\infty} S_k(f), \quad \forall\, f \in \mathcal{D}'(\mathbb{T}),$$

where the limit is taken in the distributional sense.

Next, we give a complete characterization of $\mathcal{D}'(\mathbb{T})$ in terms of the Fourier transform. To achieve this, we introduce the following definition

Definition 1.2.7. *(Slow Growth Sequences) The slow growth sequence space, denoted by $\mathcal{G}'(\mathbb{Z})$ is defined as $\{\{\alpha_m\}_{m\in\mathbb{Z}} \in \mathbb{C} : \ \exists C > 0$ and $n_0 \in \mathbb{N}$ with $|\alpha_m| \leqslant C\ |m|^{n_0},\ \forall m \in \mathbb{Z}^*\}$.*

Since $\mathcal{G}'(\mathbb{Z})$ is clearly a complex vector space, we can turn it into a topological vector space defining the topology of pointwise convergence. It can be shown that $\mathcal{G}'(\mathbb{Z})$ is the topological dual of $\mathcal{G}(\mathbb{Z})$.

Theorem 1.2.8 ([41, Theorem 3.172])**.** *The Fourier transform $\hat{}\ : \mathcal{D}'(\mathbb{T}) \to \mathcal{G}'(\mathbb{Z})$ is a linear bijection. Its inverse transform $\ ^{\vee} : \mathcal{G}'(\mathbb{Z}) \to \mathcal{D}'(\mathbb{T})$ is given by the formula*

$$(\alpha) = \{\alpha_m\}_{m\in\mathbb{Z}} \to (\alpha)^{\vee} := \sum_{k=-\infty}^{+\infty} \alpha_k e^{ikx},$$

where the series convergence is in the sense of $\mathcal{D}'(\mathbb{T})$. Moreover, the Fourier transform $\hat{}$ and its inverse are continuous maps.

Further properties of the Fourier's transform $\hat{}$ for periodic distributions and convolution are mentioned in the following proposition.

Proposition 1.2.9 ([41, Prop. 3.183])**.** *Let $f, g \in \mathcal{D}'(\mathbb{T})$, $\psi, \varphi \in C_p^{\infty}(\mathbb{T})$ and $\lambda \in \mathbb{C}$.*

(i) $(\tau_y f) * \varphi = \tau_y(f * \varphi) = f * (\tau_y \varphi)$. *In particular,* $f * \varphi$ *is* $2\pi-$*periodic. Moreover,* $f * \varphi \in C_p^{\infty}(\mathbb{T})$ *and* $(f * \varphi)^{(n)} = f^{(n)} * \varphi = f * \varphi^{(n)}$, $\forall n \in \mathbb{N}$.

(ii) *(identity for the convolution product)* $(2\pi\delta * \varphi)(x) = \varphi(x)$, $\forall x \in \mathbb{R}$.

(iii) $(f * \varphi)^{\wedge}(k) = \widehat{f}(k)\widehat{\varphi}(k)$, $\forall k \in \mathbb{Z}$.

(iv) $\widehat{f^{(n)}}(k) = (ik)^n \widehat{f}(k)$, $\forall\ k \in \mathbb{Z}$, $\forall\ n \in \mathbb{Z}^+$.

(v) $(\tau_y f)^{\wedge}(k) = e^{-ik\cdot y}\widehat{f}(k)$, $\forall k \in \mathbb{Z}$, $\forall y \in \mathbb{Z}$.

(vi) f *is real valued if and only if* $\overline{\hat{f}(k)} = \hat{f}(-k)$, $\forall k \in \mathbb{Z}$.

To close this subsection, we introduce the convolution product of two elements of $\mathcal{D}'(\mathbb{T})$. It should be noted that this is a very special property of periodic distributions.

Definition 1.2.10. *Let Let $f, g \in \mathcal{D}'(\mathbb{T})$. The convolution $f * g$ is defined by the formula*

$$\langle f * g, \varphi \rangle = \langle f, \tilde{g} * \varphi \rangle, \quad \varphi \in C_p^{\infty}(\mathbb{T}).$$

It can be shown that if $f, g \in \mathcal{D}'(\mathbb{T})$, then $f * g \in \mathcal{D}'(\mathbb{T})$. Some properties that hold for the convolution product can be found in [41, Page 200].

1.3 Sobolev Spaces of $L^2(\mathbb{T})$ type

Here we introduce the so-called Sobolev spaces of L^2-type on the Torus. As we will see in the subsequent chapters, they are fundamental in our work.

Definition 1.3.1 ([41, Section 3.6])**.** *Let $s \geqslant 0$, the Sobolev space of order s on torus is defined by*

$$H_p^s(\mathbb{T}) := \left\{ f \in \mathcal{D}'(\mathbb{T}) \mid \|f\|^2_{H_p^s(\mathbb{T})} := 2\pi \sum_{k=-\infty}^{\infty} (1+|k|^2)^s |\widehat{f}(k)|^2 < \infty \right\}.$$

For $s < 0$, we define the Sobolev space $H_p^s(\mathbb{T})$ as the topological dual of $H_p^{-s}(\mathbb{T})$. The duality is implemented by the pairing

$$(f\ ,\ g)_{L^2(\mathbb{T})} = 2\pi \sum_{k=-\infty}^{\infty} \widehat{f}(k)\ \widehat{g}(k),\quad f \in H_p^s(\mathbb{T}),\quad g \in H_p^{-s}(\mathbb{T}).$$

Note that $f \in H_p^s(\mathbb{T})$ if and only if $\{\widehat{f}(k)\}_{k\in\mathbb{Z}} \in l_s^2(\mathbb{Z})$ (see the list of symbols). Also, for all $s \in \mathbb{R}$, $H_p^s(\mathbb{T})$ is a Hilbert space with the inner product

$$(f, g)_{H_p^s(\mathbb{T})} = 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^s \widehat{f}(k)\ \overline{\widehat{g}(k)} < \infty.$$

Sometimes we write $(f, g)_s$ to denote $(f, g)_{H_p^s(\mathbb{T})}$ in order to summarize the notation. If $s = 0$ then $H_p^0(\mathbb{T})$ is isometrically isomorphic to $L_p^2(\mathbb{T})$. Also, the Sobolev spaces form a decreasing sequence of Hilbert spaces i.e. given $s, r \in \mathbb{R}$ with $s \geqslant r$ then $H_p^s(\mathbb{T}) \hookrightarrow H_p^r(\mathbb{T})$, that is, $H_p^s(\mathbb{T})$ is continuously and densely embedded in $H_p^r(\mathbb{T})$. In particular, if $s \geqslant 0$ then, $H_p^s(\mathbb{T}) \hookrightarrow L_p^2(\mathbb{T})$.

Sometimes we will write $\|f\|_s$ to denote $\|f\|_{H_p^s(\mathbb{T})}$ in order to shorten the notation.

Now we state some results on the Sobolev spaces that provide a classification of the elements of $\mathcal{D}'(\mathbb{T})$ in term of their smoothness.

Proposition 1.3.2 ([41, page 203])**.** *Let $m \in \mathbb{N}$. Then $f \in H_p^m(\mathbb{T})$ if and only if $\partial^j f \in L_p^2(\mathbb{T})$, $j \in \{0, 1, 2, ..., m\}$ where the derivatives are taken in the sense of $\mathcal{D}'(\mathbb{T})$. Moreover, $\|f\|_m$ and*

$$|||f|||_m^2 := \left(\sum_{j=0}^{m} \|\partial^j f\|_0^2 \right)^{\frac{1}{2}}$$

are equivalent, that is, there are positive constants C_m and C'_m such that

$$C_m\ \|f\|_m^2 \leqslant |||f|||_m^2 \leqslant C'_m\ \|f\|_m^2, \ \text{ for all }\ f \in H_p^m(\mathbb{T}).$$

Proposition 1.3.3 (Sobolev Lemma, see [41, page 204])**.** *If $s > \dfrac{1}{2}$ then, $H_p^s(\mathbb{T}) \hookrightarrow C_p(\mathbb{T})$ and there exists $C > 0$ such that*

$$\|f\|_\infty \leqslant \|\widehat{f}\|_{l^1} \leqslant C\|f\|_s.$$

The Sobolev lemma implies that for $s > \frac{1}{2}$, $H_p^s(\mathbb{T})$ is a Banach algebra. The Sobolev lemma for L^p-spaces can be found in [33, Lemma 1.5]. We finalize this subsection giving a characterization for the Sobolev spaces $H_p^s(\mathbb{T})$ with $s \in \mathbb{R}$ and defining the subspace $H_0^s(\mathbb{T})$.

Theorem 1.3.4 ([55, Page 205-206])**.** *Let $s \in \mathbb{R}$. $v \in H_p^s(\mathbb{T})$ if and only if,*

$$v(x) = \sum_{k\in\mathbb{Z}} v_k \psi_k(x), \quad \textit{for all } x \in \mathbb{T} \quad \textit{and} \quad \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |v_k|^2 < \infty,$$

where the first series converges in the distributional sense.

For $s \in \mathbb{R}$, we define

$$H_0^s(\mathbb{T}) := \left\{ u \in H_p^s(\mathbb{T}) : [u] = 0 \right\}. \tag{1.3.1}$$

If $s = 0$ then, we denote $H_0^0(\mathbb{T})$ by $L_0^2(\mathbb{T})$. In sequel, we record some properties of these spaces.

Proposition 1.3.5. *$H_0^s(\mathbb{T})$ is a closed subspace of $H_p^s(\mathbb{T})$ for all $s \geqslant 0$. In particular, $L_0^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$.*

Lemma 1.3.6 ([69, Page 211-212])**.** *Let $s \geqslant r \geqslant 0$. Given any $\epsilon > 0$ and $u \in H_0^r(\mathbb{T})$, there exists $v \in H_0^s(\mathbb{T})$ such that $\|u - v\|_{H_0^r(\mathbb{T})} < \epsilon$.*

Remark 1.3.7. *The Proposition 1.3.5 implies that $(H_0^s(\mathbb{T}), \|\cdot\|_{H_p^s(\mathbb{T})})$ is a Hilbert space for all $s \geqslant 0$. Furthermore, the lemma says that $H_0^s(\mathbb{T}) \hookrightarrow H_0^r(\mathbb{T})$, where the embedding is dense, whenever $s \geqslant r \geqslant 0$.*

1.4 The Hilbert transform

Here, we define the Hilbert transform and record some of its properties. The content of this section is based mainly on the books [41, 65].

Let $s \in \mathbb{R}$ and $f \in H_p^s([-\pi, \pi])$, a $2\pi-$periodic function. The Hilbert transform of f is defined by (see pag. 66 in [65]).

$$\begin{aligned} \mathcal{H}(f)(x) &:= \frac{1}{2\pi} p.v. \int_{-\pi}^{\pi} f(x-y) \cot\left(\frac{y}{2}\right) dy \\ &= \frac{1}{2\pi} \lim_{\epsilon \to 0^+} \int_{\epsilon \leqslant |y| \leqslant \pi} f(y) \cot\left(\frac{x-y}{2}\right) dy, \quad \forall x \in \mathbb{T}. \end{aligned} \tag{1.4.1}$$

Theorem 1.4.1 (See [41, pag. 210])**.** *For all $s \in \mathbb{R}$, $\mathcal{H} \in \mathcal{L}(H_p^s(\mathbb{T}))$, that is, $\mathcal{H}$ is a bounded linear operator from $H_p^s(\mathbb{T})$ into itself. Moreover, $\mathcal{H}$ is an isometry in $H_p^s(\mathbb{T})$.*

The Hilbert transform can also be defined through Fourier's transform, as follows. Given $f \in L^2_p(\mathbb{T})$ one defines $\widehat{\mathcal{H}(f)}(k) := -i\ \mathrm{sgn}(k)\widehat{f}(k), \quad \forall k \in \mathbb{Z}$. Using the inversion formula given in Theorem 1.2.8, we have

$$\mathcal{H}(f)(x) = \sum_{k=-\infty}^{+\infty} -i\ \mathrm{sng}(k)\widehat{f}(k)e^{ikx}, \quad \forall x \in \mathbb{T}. \tag{1.4.2}$$

Now, we prove important properties that hold for the Hilbert transform.

Proposition 1.4.2 (The Hilbert Transform Properties)**.** *Assume* $f,\ g \in L^2_p(\mathbb{T})$*. Then,*

$$\int_{\mathbb{T}} f(x)\ \overline{g}(x)\ dx = \int_{\mathbb{T}} \mathcal{H}(f)(x)\ \overline{\mathcal{H}(g)(x)}\ dx. \tag{1.4.3}$$

$$\int_{\mathbb{T}} f(x)\ \overline{\mathcal{H}(g)(x)}\ dx = -\int_{\mathbb{T}} \mathcal{H}(f)(x)\ \overline{g(x)}\ dx. \tag{1.4.4}$$

$$\mathcal{H}\left(f \cdot \mathcal{H}(g) + \mathcal{H}(f) \cdot g\right) = \mathcal{H}(f) \cdot \mathcal{H}(g) - f \cdot g. \tag{1.4.5}$$

Proof. Using Parseval's identity we prove (1.4.3) and (1.4.4). The proof of (1.4.5) can be found in [65, page 80]. □

1.5 Semigroup Theory

In this section we discuss the basic theory about semigroups, uniformly continuous semigroups, C_0-semigroups, and sectorial and m-dissipative operators, as well as their application to homogeneous and nonhomogeneous Cauchy problems. The content of this section is taken from the books [16, 28, 71].

We begin by giving the definition of a semigroup of bounded linear operators.

Definition 1.5.1. *Let X be a Banach space. A one-parameter family $U(t)$, with $0 \leqslant t < \infty$, in $\mathcal{L}(X)$ is a semigroup (of bounded linear operators on X) if*

(i) $U(0) = I$*, where I is the identity operator on X;*

(ii) $U(t + t') = U(t)U(t')$ *for every* $t, t' \geqslant 0$.

A semigroup of bounded linear operators, $U(t)$, is uniformly continuous if

$$\lim_{t \to 0^+} \|U(t) - I\| = 0.$$

It is easy to show that if the family $U(t)$, with $0 \leqslant t < \infty$, is an uniformly continuous semigroup then

(i) For all $T > 0$, there exists $M = M(T) > 0$ such that $\|U(t)\| \leqslant M$, for all $t \in [0, T]$;

(ii) The map $t \in [0, +\infty) \longrightarrow U(t) \in \mathcal{L}(X)$ is continuous.

Definition 1.5.2. *Let $U(t)$, $0 \leqslant t < +\infty$, be a semigroup. The linear operator A defined by*

$$D(A) = \left\{ x \in X\colon \lim_{t \to 0^+} \frac{U(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t \to 0^+} \frac{U(t)x - x}{t} \text{ for } x \in D(A)$$

is called the infinitesimal generator of the semigroup $U(t)$. $D(A)$ is the domain of A.

Proposition 1.5.3. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup, if and only if A is a bounded linear operator, i.e. $A \in \mathcal{L}(X)$.*

Some properties of uniformly continuous semigroups are summarized in Theorem 1.3 and Corollary 1.4 in [71, page 3]. Proposition 1.5.3 says that this theory does not work for unbounded operators. Therefore, we introduce the strongly continuous semigroups or C_0-semigroups.

Definition 1.5.4. *A semigroup $U(t) \in \mathcal{L}(X)$, $0 \leqslant t < \infty$, is called a strongly continuous semigroup of bounded linear operators (or simply a C_0-semigroup) if*

$$\lim_{t \to 0^+} U(t)x = x \text{ for every } x \in X.$$

For C_0-semigroups, we have an exponential type estimate and the continuity of the map $t \to U(t)x$.

Proposition 1.5.5 ([71, page 4])**.** *Let $U(t)$ be a C_0-semigroup. There exist constants $\omega \geqslant 0$ and $M \geqslant 1$ such that*

$$\|U(t)\| \leqslant M e^{\omega t} \text{ for } 0 \leqslant t < +\infty.$$

Moreover, for every $x \in X$, the map $t \in [0, +\infty) \to U(t)x$ is a continuous function.

If $\omega = 0$, then $U(t)$ is called uniformly bounded, and moreover, if $M = 1$, it is called a C_0-semigroup of contractions.

Several properties of C_0-semigroups are condensed in [71, page 4]. For C_0-semigroups we also have the uniqueness of semigroups of an infinitesimal generator (see Theorem 2.6 in [71, page 6]). Note that if A is the infinitesimal generator operator of a C_0-semigroup, then $D(A)$ is dense in X and A is a closed operator.

Definition 1.5.6. *Let X be a Banach space and $A : D(A) \subset X \to X$ be a linear operator not necessarily bounded. The resolvent set of A, denoted by $\rho(A)$, is defined by*

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible and } (\lambda I - A)^{-1} \text{ is bounded}\}.$$

The family $R(\lambda; A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$, of bounded linear operators is called the resolvent of A.

It can be showed that if the operator A is an infinitesimal generator of a C_0-semigroup of contractions $U(t)$, then $\Lambda := \{\lambda \in \mathbb{C} : Re(\lambda) > 0\} \subset \rho(A)$. Moreover, If $\lambda \in \Lambda$, then $\|R(\lambda - A)\| \leqslant \dfrac{1}{Re(\lambda)}$. An important result in the semigroups theory is the Hille-Yosida Theorem, see [71, page 8].

Next, we introduce the sectorial operators and analytic semigroups. This part of the theory will be useful to study the well-posedness of the Linearized Intermediate Long Wave equation in $H_0^s(\mathbb{T})$. It was taken from the book of Daniel Henry [28].

Definition 1.5.7. *We call a linear operator A in a Banach space X a sectorial operator if it is a closed densely defined operator such that for some θ_0 in $(0, \dfrac{\pi}{2})$ and some $M \geqslant 1$ and real a, the sector*

$$S_{a,\theta_0} := \{\lambda \in \mathbb{C} : \theta_0 \leqslant |\arg(\lambda - a)| \leqslant \pi, \quad \lambda \neq a\}$$

is in the resolvent set of A and

$$\|(\lambda - A)^{-1}\| \leqslant \frac{M}{|\lambda - a|}, \quad \textit{for all } \lambda \in S_{a,\theta_0}.$$

Definition 1.5.8. *Let X be a Banach space and $A : D(A) \subset X \to X$ be a linear operator not necessarily bounded. The continuous spectrum of A, denoted by $\sigma(A)$, is the set of all $\lambda \in \mathbb{C}$ such that the range of $\lambda I - A$ is dense in X and $\lambda I - A$ is invertible with $(\lambda I - A)^{-1}$ not bounded.*

In the following we define the fractional powers of operators.

Definition 1.5.9. *Suppose A is a sectorial operator and $Re(\sigma(A)) > 0$; then for any $\alpha > 0$*

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt.$$

Theorem 1.5.10 ([28, Theorem 1.4.2])**.** *If A is a sectorial operator in X with $Re(\sigma(A)) > 0$, then for any $\alpha > 0$, $A^{-\alpha}$ is a bounded linear operator on X which is one-one and satisfies $A^{-\alpha}A^{-\beta} = A^{-(\alpha+\beta)}$ whenever $\alpha > 0$, $\beta > 0$. Also, for $0 < \alpha < 1$,*

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda + A)^{-1} d\lambda.$$

Continuing with A a sectorial operator and $Re(\sigma(A)) > 0$, we define A^{α} as the inverse of $A^{-\alpha}$ ($\alpha > 0$), $D(A^{\alpha}) = R(A^{-\alpha})$, and A^0 as the identity on X. If $\alpha > 0$, A^{α} is closed and densely defined. Also, if $\alpha \geqslant \beta$ then $D(A^{\alpha}) \subset D(B^{\beta})$. Furthermore, $A^{\alpha}A^{\beta} = A^{\beta}A^{\alpha} = A^{\alpha+\beta}$ on $D(A^{\gamma})$ where $\gamma = \max\{\alpha, \beta, \alpha+\beta\}$ and $A^{\alpha}e^{-At} = e^{-At}A^{\alpha}$ on $D(A^{\alpha})$, $t > 0$. Further properties of fractional power operators can be found in §1.4 in [28].

The following result is a consequence of Theorems 1.3.2 and 1.4.4 in [28].

Corollary 1.5.11 ([28, Corollary 1.4.5]). *If A is a sectorial operator with $Re(\sigma(A)) > 0$ and if B is a linear operator such that $BA^{-\alpha}$ is bounded on X for some $\alpha \in [0, 1)$, then $A + B$ is sectorial.*

Definition 1.5.12. *An analytic semigroup on a Banach space X is a family $\{U(t)\}_{t\geqslant 0}$, of continuous linear operators on X, satisfying*

(i) $U(0) = I$, $\quad U(t+s) = U(t)U(s)$, *for all* $t \geqslant 0$, $s \geqslant 0$,

(ii) $\lim_{t\to 0^+} U(t)x = x$, *for each* $x \in X$,

(iii) The function $t \in (0, +\infty) \to U(t)x$ *is real analytic for each* $x \in X$.

Theorem 1.5.13 ([28, Theorem 1.3.4]). *If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-At}\}_{t\geqslant 0}$, given by*

$$e^{-At} = \frac{1}{2\pi i}\int_{\Gamma} (\lambda + A)^{-1} e^{\lambda t} d\lambda,$$

where Γ is a contour in $\rho(-A)$ with $\lim_{|\lambda|\to+\infty} \arg(\lambda) = \pm\theta$ for some $\theta \in (\frac{\pi}{2}, \pi)$.

Further, e^{-At} can be continued analytically into a sector $\{t \neq 0 : |\arg(t)| < \epsilon\}$ containing the positive real axis, and if $Re(\sigma(A)) > a$, i.e. $Re(\lambda) > a$ whenever $\lambda \in \sigma(A)$, then for $t > 0$

$$\|e^{-At}\| \leqslant Ce^{-at}, \quad \|Ae^{-At}\| \leqslant \frac{C}{t}e^{-at},$$

for some constant C. Finally,

$$\frac{d}{dt}e^{-At} = -Ae^{-At}, \quad \textit{for } t > 0.$$

The converse is also true, if $-A$ generates an analytic semigroup, then A is sectorial.

Now, we introduce the m-dissipative operators. We encourage the reader to revise the books [16, 71] to delve into this part of the theory which will be useful to proves the well-posedness of the Linearized Benjamin equation in $H_0^s(\mathbb{T})$. Here onward we suppose X is a Hilbert space and $A : D(A) \subseteq X \to X$ is a linear operator bounded or not bounded.

Definition 1.5.14. *(Dissipative and m-dissipative operators).*

(*i*) *The operator A is called dissipative in X if $\|u - \lambda Au\| \geqslant \|u\|$, $\forall u \in D(A)$ and $\lambda > 0$.*

(*ii*) *The operator A is called m-dissipative in X if A is dissipative and*

$$\forall\, \lambda > 0,\ \ \forall\, f \in X,\ \ \exists\, u \in D(A)\ \text{ such that }\ u - \lambda Au = f.$$

Observe that the operator A is m-dissipative in X if A is dissipative and $\forall\, \lambda > 0,\ \ \mathrm{Rang}(\lambda I - A) = X$.

Proposition 1.5.15 ([16, page 25]). *Let $D(A)$ be a dense subset in X. Then A and $-A$ are m-dissipative if and only if A is skew-adjoint.*

The Lumer-Phillips theorem (see Theorem 4.3 in [71, page 14]) proves that if the linear operator A is m-dissipative with domain $D(A)$ dense in X, then A is an infinitesimal generator of a C_0-semigroup of contractions $\{U(t)\}_{t\geqslant 0}$ in X. Furthermore, we have the following result.

Theorem 1.5.16 ([16, page 37]). *If the Linear operator A is skew-adjoint, with domain $D(A)$ dense in X, then $\{U(t)\}_{t\geqslant 0}$ can be extended to a one parameter group $U : \mathbb{R} \to \mathcal{L}(X)$ such that*

(*i*) $U(0) = I;\quad U(s + t) = U(s)U(t),\ \ \forall s, t \in \mathbb{R};$

(*ii*) $U(t)x \in C(\mathbb{R}, X),\ \ \forall x \in X;$

(*iii*) $\|U(t)x\| = \|x\|,\ \ \forall x \in X,\ t \in \mathbb{R}.$

In addition, for all $x \in D(A)$, $u(t) = U(t)x$ satisfies $u \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, X)$ and $u'(t) = Au(t)$, for all $t \in \mathbb{R}$.

Theorem 1.5.16 says that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group in X.

We finalize this section by introducing the abstract Cauchy problem. Given u_0 belonging to a Banach space X, the abstract Cauchy problem for A with initial data u_0 consists of finding a solution $u(t)$ to the IVP

$$\begin{cases} \dfrac{du}{dt} = Au, \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{1.5.1}$$

where by a solution we mean an X valued function $u(t)$ such that $u(t)$ is continuous for $t \geqslant 0$, continuously differentiable, $u(t) \in D(A)$ for $t > 0$ and (1.5.1) is satisfied.

It is clear that if A is the infinitesimal generator of a C_0-semigroup $U(t)$, the problem (1.5.1) has a solution, namely $u(t) = U(t)u_0$ for every $u_0 \in D(A)$. Furthermore, Theorem 1.5.16 say that, the IVP (1.5.1) has a solution if the linear operator A is skew-adjoint, with domain $D(A)$ dense in X and X is a Hilbert space.

Definition 1.5.17. *Let $U(t)$ be a C_0-semigroup on a Banach space X. The semigroup $U(t)$ is called differentiable for $t > t_0$ if for every $u_0 \in X$, $t \mapsto U(t)u_0$ is differentiable for $t > t_0$. $U(t)$ is called differentiable if it is differentiable for $t > 0$.*

The existence and uniqueness of solution to (1.5.1) is guaranteed for differentiable semigroups.

Proposition 1.5.18 ([71, page 104]). *If A is the infinitesimal generator of a differentiable semigroup, then for every $u_0 \in X$ (1.5.1) has a unique solution.*

Now if we consider the nonhomogeneous Cauchy problem associated to (1.5.1), that is, the following problem

$$\begin{cases} \dfrac{du}{dt} = Au(t) + f(t), \quad t > 0 \\ u(0) = u_0, \end{cases} \tag{1.5.2}$$

where $f : [0, T] \to X$, with X a Banach space. Here we assume that A is an infinitesimal generator of a C_0-semigroup $\{U(t)\}_{t\geqslant 0}$ so that the corresponding homogeneous equation ($f \equiv 0$) has a unique solution for every initial value $u_0 \in D(A)$. See [71] for more details. Results related with the existence of solutions to (1.5.2), we have the following.

Definition 1.5.19 (Classical Solution). *A function $u : [0, T] \to X$ is a (classical) solution of (1.5.2) on $[0, T]$ if u is continuous in $[0, T]$, continuously differentiable on $(0,T)$, $u(t) \in D(A)$ for $0 < t < T$ and (1.5.2) is satisfied on $(0, T)$.*

Corollary 1.5.20. *If $f \in L^1([0, T]; X)$ then for every $u_0 \in X$ the initial value problem (1.5.2) has at most one solution. If it has a solution, it is given by*

$$u(t) = U(t)u_0 + \int_0^t U(t-\tau)f(\tau)d\tau, \quad u_0 \in X, \;\; 0 \leqslant t \leqslant T. \tag{1.5.3}$$

Definition 1.5.21 (Mild Solution). *Let A the infinitesimal generator of a C_0-semigroup $U(t)$. Let $u_0 \in X$ and $f \in L^1([0, T]; X)$. The function $u \in C([0, T]; X)$ given by (1.5.3) is the mild solution of the initial valued problem (1.5.2) on $[0, T]$.*

We remark that not every mild solution to (1.5.2) is indeed a classical solution, even in the case $f \equiv 0$. It is clear that (1.5.2) has a unique mild solution if $f \in L^1(0, T; X)$. Thus, it is necessary to impose conditions on f so that for $x \in D(A)$, the mild solution becomes a classical solution to (1.5.2). Notice that the continuity of f, in general, is not sufficient to ensure the existence of solutions of (1.5.2), for $u_0 \in D(A)$. For example, consider A the infinitesimal generator of a C_0-semigroup $U(t)$ and let $u_0 \in X$ be such that $U(t)u_0 \notin D(A)$ for any $t \geqslant 0$. Suppose that the continuous function f is defined by $f(\tau) = U(\tau)x$ for $\tau \geqslant 0$, and consider the initial value problem

$$\frac{du}{dt}(t) = Au(t) + U(t)u_0 \text{ and } u(0) = 0. \tag{1.5.4}$$

Then (1.5.4) does not have solution, because the function

$$u(t) = \int_0^t U(t-\tau)U(\tau)u_0)\, d\tau = tU(t)u_0$$

is not differentiable for $t > 0$. So, we must give more conditions on f to guarantee the existence of solutions to the problem (1.5.2).

Theorem 2.4 in [71, page 107] gives a general criterion for the existence of classical solutions of the IVP (1.5.2). This result implies the following theorem.

Theorem 1.5.22 ([71, page 108]). *Let $f \in L^1([0,T);X)$. If u is the mild solution of (1.5.2) on $[0,T)$ then for every $T' \leqslant T$, u is the uniform limit on $[0,T']$ of (classical) solutions of (1.5.2).*

1.6 Riesz Basis and Ingham's Type Inequalities

We dedicate this section to record some definitions and results related to Riesz basis. Most of the results of this section can be found in [39]. Also, we will introduce Ingham's inequality which is the main tool to prove the controllability of the linearized Benjamin equation.

1.6.1 Riesz Basis

Assume F denotes the scalar field associated with the vector space X. In this subsection, F will be either the real line $\mathbb{R}$ or the complex plane $\mathbb{C}$.

Definition 1.6.1. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in a Banach space X. The series $\sum_{n\in\mathbb{Z}} x_n$ is unconditionally convergent if $\sum_{n\in\mathbb{Z}} x_{\sigma(n)}$ is convergent in X, for all permutation σ of $\mathbb{Z}$.*

Definition 1.6.2 ([39, page 129]). *A countable set $\{x_n\}_{n\in\mathbb{Z}}$ in a Banach space X is a basis for X if $\forall x \in X$, there exist unique scalars a_n such that*

$$x = \sum_{n\in\mathbb{Z}} a_n(x)x_n. \tag{1.6.1}$$

We say that $\{x_n\}_{n\in\mathbb{Z}}$ is an unconditional basis if the series in (1.6.1) converges unconditionally for each $x \in X$.

Definition 1.6.3 ([39, page 21]). *Let $\{x_n\}_{n\in J}$ be a sequence in a normed linear space X. The finite linear span, or simply the span of $\{x_n\}_{n\in J}$ is the set of all finite linear combinations of elements of $\{x_n\}_{n\in J}$, it means,*

$$span\{\{x_n\}_{n\in J}\} = \left\{ \sum_{n=-N}^{N} c_n x_n : \text{ for all } N > 0 \text{ and } c_1, ..., c_n \in \mathrm{F} \right\},$$

and we say that $\{x_n\}_{n\in J}$ is complete in X if $\overline{span\{\{x_n\}_{n\in J}\}} = X$

Definition 1.6.4. *Let $\{x_n\}_{n\in J}$ be a sequence in a Hilbert space X.*

(i) (Riesz Basis) $\{x_n\}_{n\in J}$ is a Riesz basis if it is equivalent (see [39, §4.4]) to some (and therefore every) orthonormal basis for X.

(ii) (Bessel Sequence) A sequence $\{x_n\}_{n\in J}$ in Hilbert space X is a Bessel sequence if

$$\forall\, x \text{ in } X, \ \sum_{n\in J} |\langle x, x_n\rangle|^2 < \infty.$$

Definition 1.6.5. *Given a Banach space X and given sequences $\{x_n\}_{n\in J} \subseteq X$ and $\{a_n\}_{n\in J} \subseteq X^*$, we say that $\{a_n\}$ is biorthogonal to $\{x_n\}$ if $\langle x_m, a_n\rangle = \delta_{nm}$ for every $n, m \in J$. We call $\{a_n\}$ a biorthogonal system or a dual system of $\{x_n\}$.*

Theorem 1.6.6 ([39, page 197]). *Let $\{x_n\}_{n\in J}$ be a sequence in a Hilbert space X. Then the following statements are equivalent.*

1. *$\{x_n\}_{n\in J}$ is a Riesz basis for X.*
2. *$\{x_n\}_{n\in J}$ is a bounded unconditional basis for X.*
3. *$\{x_n\}_{n\in J}$ is a basis for X, and*

$$\sum_{n\in J} c_n x_n \text{ converges } \Leftrightarrow \sum_{n\in J} |c_n|^2 \text{ converges}$$

4. *$\{x_n\}_{n\in J}$ is complete in X and there exists constants $A,\ B > 0$ such that*

$$\text{for all } c_1, ..., c_N \text{ scalars, } A\sum_{n=1}^{N} |c_n|^2 \leqslant \| \sum_{n=1}^{N} c_n x_n\|_X^2 \leqslant B\sum_{n=1}^{N} |c_n|^2.$$

5. *$\{x_n\}_{n\in J}$ is a complete Bessel sequence and possesses a biorthogonal system $\{y_n\}_{n\in J}$ that is also a complete Bessel sequence.*

Definition 1.6.7. *We say that a sequence $\{x_n\}_{n\in J}$ in a Banach space X is minimal if no vector x_m lies in the closed span of the other vector x_n, it means,*

$$\forall m \in J, \quad x_m \notin \overline{span\{\{x_n\}_{n\in J,\, n\neq m}\}}.$$

A sequence that is both minimal and complete is said to be exact.

1.6.2 The Ingham's inequality

Our main tool to get the controllability of the linearized Benjamin equation is the so-called Ingham's inequality, which is a generalization of Parseval's equality due to Ingham in [40].

In what follows, K represents a countable set of indices. It could be finite or infinite.

Theorem 1.6.8 ([40]). *Let $\{\lambda_k\}_{k=-\infty}^{\infty}$ be a strictly increasing sequence of real numbers, and I be a bounded interval. Consider the sums of the form*

$$f(t) = \sum_{k\in K} c_k e^{i\lambda_k t}, \quad t \in I,$$

with square-summable complex coefficients c_k. Assume that there exists $\gamma > 0$ such that the "gap condition"

$$\lambda_{n+1} - \lambda_n \geqslant \gamma, \quad \forall\, n \in \mathbb{Z},$$

holds, then there exist constants $A,\ B > 0$, such that for every bounded interval I of length $|I| > \dfrac{2\pi}{\gamma}$,

$$A \sum_{k\in K} |c_k|^2 \leqslant \int_I |f(t)|^2 dt \leqslant B \sum_{k\in K} |c_k|^2.$$

Remark 1.6.9. *On the same hypotheses of Theorem 1.6.8. If $I = [0,T]$ or $I = [-T,T]$ with $|I| > \dfrac{2\pi}{\gamma}$, then $A = T\cdot\widetilde{A}$ and $B = T\cdot\widetilde{B}$, with positive constants $\widetilde{A}$ and $\widetilde{B}$ independents of T.*

The following result is generalization of Theorem 1.6.8.

Theorem 1.6.10 (Theorem 4.6 [46, page 67]). *Let $\{\lambda_k\}_{k\in K}$ be a family of real numbers, satisfying the uniform gap condition*

$$\gamma = \inf_{k\neq n} |\lambda_k - \lambda_n| > 0,$$

and set

$$\gamma' = \sup_{A\subset K} \inf_{\substack{k,n\in K\setminus A\\ k\neq n}} |\lambda_k - \lambda_n| > 0,$$

where A rums over the finite subsets of K.

If I is a bounded interval of length $|I| > \dfrac{2\pi}{\gamma'}$, then there exist positive constants A and B such that

$$A \sum_{k\in K} |c_k|^2 \leqslant \int_I |f(t)|^2 dt \leqslant B \sum_{k\in K} |c_k|^2,$$

for all functions given by the sum $f(t) = \sum_{k\in K} c_k e^{i\lambda_k t}$ with square-summable complex coefficients c_k.

In the Theorem 1.6.10 there is no problem for the interpretation of the convergence: we can have only countably many non zero terms, and the convergence is unconditional. More generalizations of the Ingham inequality can be found in [46, Chapters 4, 6, and 8] and [10, page 558].

1.7 Nonlinear interpolation theory and interpolation of L^p-spaces with change of measure

In this section, we summarize the nonlinear interpolation theory as expounded by Bona and Scott [13]. Also, we present a result due to Tartar [88], which is the key to obtain the global well-posedness of a closed-loop system associated to the Benjamin equation. More precisely, we present a real interpolation theorem for nonlinear operators and the complex interpolation theorem of Stein-Weiss for weighted L^p spaces. For details, we refer to the general theory on interpolation spaces in Bergh and Lofstrom [11].

Let B_0 and B_1 be two Banach spaces such that $B_1 \subset B_0$ with continuous inclusion map. Let $f \in B_0$ and for $\epsilon > 0$, define $K(f,\epsilon) = \inf_{g \in B_1} \{\|f-g\|_{B_0} + \epsilon \|g\|_{B_1}\}$, where $\|\cdot\|_{B_j}$, is the norm on B_j, $j = 0, 1$. For $0 < \theta < 1$ and $1 \leqslant p \leqslant +\infty$, define

$$(B_0, B_1)_{\theta,p} = B_{\theta,p} = \left\{ f \in B_0 : \|f\|_{\theta,p} = \left(\int_0^\infty K(f,\epsilon)^p \epsilon^{-\frac{\theta}{p}} \, d\epsilon \right)^{\frac{1}{p}} < +\infty \right\},$$

with the usual modification in the case $p = +\infty$,

$$B_{\theta,\infty} = \left\{ f \in B_0 : \|f\|_{\theta,\infty} = \sup_{\epsilon>0} |K(f,\epsilon)| < +\infty \right\}.$$

Then $B_{\theta,p}$ is a Banach space with the norm $\|\cdot\|_{\theta,p}$. Given two pairs of indices as above, then $(\theta_1, p_1) < (\theta_2, p_2)$ means

$$\begin{cases} \theta_1 < \theta_2, & \text{or} \\ \\ \theta_1 = \theta_2 & \text{and} \quad p_1 > p_2. \end{cases}$$

If $(\theta_1, p_1) < (\theta_2, p_2)$, then $B_{\theta_2,p_2} \subset B_{\theta_1,p_1}$ and the inclusion map is continuous.

Theorem 1.7.1 ([13, Theorem 1])**.** *Let B_0^j and B_1^j be Banach spaces such that $B_1^j \subset B_0^j$ with continuous inclusions mappings, $j = 1, 2$. Let λ and q lie in the ranges $0 < \lambda < 1$ and $1 \leqslant q \leqslant +\infty$. Suppose S is a mapping such that*

(i) $S : B_{\lambda,q}^1 \longrightarrow B_0^2$ and for $f, g \in B_{\lambda,q}^1$,

$$\|Sf - Sg\|_{B_0^2} \leqslant C_0(\|f\|_{B_{\lambda,q}^1} + \|g\|_{B_{\lambda,q}^1})\|f-g\|_{B_0^1},$$

and

(ii) $S : B_1^1 \longrightarrow B_1^2$ *and for* $h \in B_1^1$,

$$\|Sh\|_{B_1^2} \leqslant C_1(\|h\|_{B_{\lambda,q}^1})\|h\|_{B_1^1},$$

where $C_j : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *are continuous non-decresing functions,* $j = 0, 1$.

Then if $(\theta, p) \geqslant (\lambda, q)$, S *maps* $B_{\theta,p}^1$ *into* $B_{\theta,p}^2$ *and for* $f \in B_{\theta,p}^1$

$$\|Sf\|_{B_{\theta,p}^2} \leqslant C(\|f\|_{B_{\lambda,q}^1})\|f\|_{B_{\theta,p}^1},$$

where for $r > 0$, $C(r) = 4C_0(4r)^{1-\theta}C_1(3r)^{\theta}$.

Remark 1.7.2. *Theorem 1.7.1 was used by Bona and Scott [13] to provide the original proof of the global well-posedness of the IVP for the KdV equation on the whole line in fractional order Sobolev spaces* $H^s(\mathbb{R})$.

Attention is now focused on the question of continuity of S as a mapping of intermediate spaces, assuming S is known to be continuous as a mapping of the initial spaces. For this, the following notion is useful.

Definition 1.7.3. *Let* B_0 *and* B_1 *be Banach spaces with* B_1 *continuously included in* B_0. *Let* $0 < \theta < 1$ *and* $1 \leqslant p \leqslant +\infty$. *We say that the pair* B_0, B_1 *has a* (θ, p) *approximate identity if there is a family of continuous mappings* $S_\epsilon : B_{\theta,p} \longrightarrow B_1$, *for* $0 < \epsilon \leqslant 1$, *such that*

(i) for all $f \in B_{\theta,p}$ *and* $0 < \epsilon \leqslant 1$,

$$\|S_\epsilon f\|_{B_{\theta,p}} + \epsilon^{1-\theta}\|S_\epsilon f\|_{B_1} \leqslant C\ \|f\|_{B_{\theta,p}},$$

(ii)

$$\|S_\epsilon f - f\|_{B_{\theta,p}} + \epsilon^{-\theta}\|S_\epsilon f - f\|_{B_0} \longrightarrow 0,$$

as $\epsilon \longrightarrow 0$ *for* $f \in B_{\theta,p}$ *and uniformly on compact subsets of* $B_{\theta,p}$.

Example 1.7.4. *Take* $B_0 = L_0^2(\mathbb{T}) = L_0^2$ *and, for* k *a positive integer,* $B_1 = H_0^k(\mathbb{T}) = H_0^k$, *the Sobolev space of* L_0^2 *functions whose first* k *derivatives lie in* L_0^2.

Using $(L_0^2, H_0^k)_{\theta,2} \cong H_0^s$ *with* $s = \theta k$ *and defining* $\widehat{S_\epsilon f}(n) = g_\epsilon(n)\widehat{f}(n)$, *where* $\widehat{f}$ *denotes the Fourier transform of* f, *and*

$$g_\epsilon(n) = \begin{cases} 1, & \text{if} \quad |n| \leqslant \dfrac{1}{\epsilon^{\frac{1}{k}}} \\ \\ 0 & \text{if} \quad |n| > \dfrac{1}{\epsilon^{\frac{1}{k}}}, \end{cases} \tag{1.7.1}$$

we can prove the properties (i) and (ii) of the Definition 1.7.3 by following a similar procedure as those used by Bona and Scott in [13]. Therefore, S_ϵ *is a* $(\theta, 2)$ *aproximate identity for the pair* $L_0^2(\mathbb{T})$, $H_0^k(\mathbb{T})$, *for any* $\theta \in (0, 1)$.

Theorem 1.7.5 ([13, Theorem 2]). *Let B_0^1, B_0^2, B_1^1, B_1^2, λ, q, and S be as in Theorem 1.7.1. Assume additionally that the pair B_0^1, B_1^1 has a (θ, p) approximate identity $\{S_\epsilon\}$ for some $(\theta, p) \geqslant (\lambda, q)$ and that*

(iii) S is a continuous map of B_1^1 to B_1^2.

Then S is a continuous map of $B_{\theta,p}^1$ to $B_{\theta,p}^2$.

Let B be a Banach space and $T > 0$. Denote by $C([0,T]; B)$ the Banach space of continuous functions from $[0,T]$ to B with norm given by

$$\|f\|_{C([0,T];B)} = \sup_{0 \leqslant t \leqslant T} \|f(t)\|_B.$$

The following simple fact about interpolation between spaces of the form $C([0,T]; B)$ will be used.

Proposition 1.7.6 ([13, Proposition 3]). *Let B_0 and B_1 be Banach spaces with B_1 included continuously in B_0. Let θ and p lie in the ranges $0 < \theta < 1$ and $1 \leqslant p < +\infty$. Then for any $T > 0$,*

$$(C([0,T]; B_0), C([0,T]; B_1))_{\theta,p} \subset C([0,T]; (B_0, B_1)_{\theta,p}),$$

with the inclusions mapping continuous.

We continue this section with a result due to Tartar [88]. Tartar makes more restrictive assumptions to obtain an Interpolation result. Nevertheless, we will be willing to apply this theorem.

Theorem 1.7.7 (A real interpolation theorem for nonlinear operators, see Theorem 2, page 474 [88]). *Let B_0^j and B_1^j be Banach spaces such that $B_1^j \subset B_0^j$ with continuous inclusion mappings, $j = 1, 2$. Assume that $0 < \alpha \leqslant 1$ and $\beta > 0$. Suppose S is a mapping such that*

i) $S : B_0^1 \longrightarrow B_0^2$ and for $f, g \in B_0^1$,

$$\|Sf - Sg\|_{B_0^2} \leqslant C_0(\|f\|_{B_0^1}, \|g\|_{B_0^1})\|f - g\|_{B_0^1}^\alpha,$$

and

ii) $S : B_1^1 \longrightarrow B_1^2$ and for $h \in B_1^1$,

$$\|Sh\|_{B_1^2} \leqslant C_1(\|h\|_{B_0^1})\|h\|_{B_1^1},$$

where $C_0 : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $C_1 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous functions.

Then, if $0 < \theta < 1$ and $1 \leqslant p \leqslant +\infty$, there exists a positive constant c such that S maps $B^1_{\theta,p}$ into $B^2_{\lambda,q}$ and for $f \in B^1_{\theta,p}$

$$\|Sf\|_{B^2_{\lambda,q}} \leqslant c\, C(\|f\|_{B^1_0})\|f\|_{B^1_{\theta,p}},$$

where λ, q are given by $\quad \dfrac{1-\lambda}{\lambda} = \left(\dfrac{1-\theta}{\theta}\right)\dfrac{\alpha}{\beta},$

$$q = \max\left\{1, \frac{p}{(1-\lambda)\beta + \lambda\alpha}\right\} = \max\left\{1, \left(\frac{1-\theta}{\beta} + \frac{\theta}{\alpha}\right)p\right\},$$

and C is a function given by $C(r) = C_0(r, 2r)^{\lambda} C_1(2r)^{1-\lambda}$.

We finish this section with the complex interpolation theorem of Stein-Weiss for weighted L^p-spaces. The proof of this result can be found in the Bergh and Lofstrom's book [11]. This interpolation theorem will be used to prove the multiplication property of the Bourgain's spaces associated to the Benjamin equation.

Let μ_0 and μ_1 two positive measures. We may assume that μ_0 and μ_1 are absolutely continuous with respect to a third measure μ. Thus we suppose that

$$d\mu_0(x) = w_0(x)\, d\mu(x)$$

$$d\mu_1(x) = w_1(x)\, d\mu(x).$$

Let $(U, w\, d\mu)$ a measure space and let us write $L^p(w) = L^p(U, w\, d\mu)$

Theorem 1.7.8 (The complex interpolation theorem of Stein-Weiss, see [11, page 115])**.** *Assume that $0 < p \leqslant \infty$ and that $0 < \theta < 1$. Put $w(x) = w_0^{1-\theta}(x) w_1^{\theta}(x)$, then*

$$(L^p(w_0), L^p(w_1))_{\theta,p} = L^p(w),$$

with equivalent norms. Moreover, if the operators

$$T : L^p(U, w_0\, d\mu) \longrightarrow L^p(V, \widetilde{w_0}\, d\nu)$$

$$T : L^p(U, w_1\, d\mu) \longrightarrow L^p(V, \widetilde{w_1}\, d\nu)$$

with quasi-norms M_0 and M_1 respectively, then

$$T : L^p(U, w\, d\mu) \longrightarrow L^p(V, \widetilde{w}\, d\nu)$$

with quasi-norm $M \leqslant M_0^{1-\theta} M_1^{\theta}$, with $\widetilde{w}(x) = \widetilde{w}_0^{1-\theta}(x)\, \widetilde{w}_1^{\theta}(x)$.

Chapter 2

Controllability and Stabilization of the linearized Benjamin equation on a periodic domain

In this chapter, we analyze the controllability of the linear system associated to Benjamin equation (0.0.7) on $\mathbb{T}$. We show the local and global well-posedness of this system via semigroup theory. Then, we use the classical moment method (see [79]) to show that the linearized Benjamin equation is globally exactly controllable. We also study the stabilization problem for the linear equation associated to system (0.0.7) in $H_p^s(\mathbb{T})$, with $s \geqslant 0$. First, we prove that there exists an adequate feedback law such that the trivial solution ($u = 0$) is exponentially asymptotically stable when t goes to infinity. Finally, we show that it is possible to choose an appropriate linear feedback law such that the decay rate of the resulting closed-loop system is as large as one desires.

This chapter is organized in five sections. In Section 2.1, we record some properties that the operator G defined in (2.0.9) possesses. In Section 2.2 we establish the well-possedness for the linear system associated to the Benjamin equation. In Section 2.3, we proof of an exact controllability result for this system is provided. Section 2.4 is devoted to prove the stabilization result. Then in section 2.5 we choose an appropriate linear feedback law such that the decay rate of the resulting closed-loop system is as large as one desires. The main results of this chapter are given by Theorems 2.3.7 and 2.5.5.

Before beginning with Section 2.1, we specify one of our main objectives of this work. As mentioned earlier, we are interested in the Benjamin equation (0.0.7) in the context of control theory by adding a control term $f = f(x,t)$. More precisely, we are interested in the following two problems.

1) ***Exact control problem:*** Given an initial state u_0 and a terminal state u_1 in a

certain space, with $[u_0] = [u_1]$, can one find an appropriate control input f so that the equation

$$\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + \partial_x(u^2) = f(x,t), \quad x \in \mathbb{T}, \ t \in \mathbb{R}, \tag{2.0.1}$$

admits a solution u such that $u(x,0) = u_0(x)$ and $u(x,T) = u_1(x)$, for all $x \in \mathbb{T}$ and any final time $T > 0$?

2) ***Stabilization Problem:*** Given u_0 in a certain space. Can one find a feedback control law: $f = Ku$ so that the resulting closed-loop system

$$\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + \partial_x(u^2) = Ku, \quad u(x,0) = u_0, \ \ x \in \mathbb{T}, \ \ t \in \mathbb{R}^+, \tag{2.0.2}$$

where K is a bounded linear operator in an adequate space, is asymptotically stable (see def. below) as $t \to \infty$?

Initially, we consider the linearized IVP associated to equation (0.0.7) in the periodic setting,

$$\begin{cases} \partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u = 0, & t \in \mathbb{R}, \ \ x \in \mathbb{T} \\ \partial_x^m u(2\pi,t) = \partial_x^m u(0,t), & t \in \mathbb{R}, \quad m = 0,1,2 \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.0.3}$$

with initial data $u_0(x)$ in an adequate space. The equation (2.0.3) admits the following conserved quantity

$$I_0(u) = \int_0^{2\pi} u(x,t)\, dx = \int_0^{2\pi} u_0(x)\, dx. \tag{2.0.4}$$

Also, the linearized IVP associated to equation (2.0.1) in the periodic setting, can be written as

$$\begin{cases} \partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u = f(x,t), & 0 < t < T, \ \ x \in \mathbb{T} \\ \partial_x^m u(2\pi,t) = \partial_x^m u(0,t), & 0 \leqslant t \leqslant T, \quad m = 0,1,2 \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.0.5}$$

with the initial data u_0 in an adequate space. The solution u of system (2.0.5) satisfies

$$\frac{d}{dt}\left(\int_0^{2\pi} u(x,t)\, dx \right) = \int_0^{2\pi} f(x,t)\, dx. \tag{2.0.6}$$

So, the volume (mass) $I_0(u)$ in the control system (2.0.5) is indeed conserved if we demand the function f to satisfy

$$\int_0^{2\pi} f(x,t)\, dx = 0. \tag{2.0.7}$$

In this work, the control f in (2.0.1) is allowed to act only on a small subset of the domain $\mathbb{T}$, i.e., f is considered to be supported in a nonempty open subinterval of $(0,2\pi)$. This situation includes more cases of practical interest and is therefore more relevant in general. For this reason, we consider $g(x)$ as a real non-negative smooth function defined on $\mathbb{T}$, such that,

$$2\pi[g] := \int_0^{2\pi} g(x)\, dx = 1, \tag{2.0.8}$$

where $[g]$ denotes the mean value of g over the interval $(0,2\pi)$. We assume supp $g \subset (0,2\pi)$, where $\omega = \{x \in \mathbb{T} : g(x) > 0\}$ is an open interval. We restrict our attention to controls of the form

$$f(x,t) = G(h)(x,t) := g(x)\left(h(x,t) - \int_0^{2\pi} g(y)h(y,t)\, dy\right), \ \forall x \in \mathbb{T},\ t \in [0,T], \tag{2.0.9}$$

where h is a function defined in $\mathbb{T} \times [0,T]$. Thus, $h \equiv h(x,t)$ can be considered as a new control function and for each $t \in [0,T]$, we have that (2.0.7) is satisfied. Moreover, the control input (2.0.9) keep the mass $I_0(u)$ conserved for the system (2.0.1).

2.1 Properties of the Operator G

Here, we present some properties satisfied by the operator G defined in (2.0.9).

Proposition 2.1.1. *Let $s \geqslant 0$. The operator $G : L^2\left([0,T]; H_p^s(\mathbb{T})\right) \to L^2\left([0,T]; H_p^s(\mathbb{T})\right)$ is linear and bounded.*

Proof. It's easy to see that G is a linear operator. We will show that G is bounded. Suppose that $h \in L^2([0,T]; H_p^s(\mathbb{T}))$, then

$$\begin{aligned}
\|Gh\|^2_{L^2([0,T];H_p^s(\mathbb{T}))} &= \int_0^T 2\pi \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^s \left|\widehat{Gh}(k)\right|^2 dt \\
&\leqslant 4\pi \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^s \int_0^T \left|\widehat{gh}(k)\right|^2 dt \\
&\quad + 4\pi \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^s |\widehat{g}(k)|^2 \int_0^T \left|\int_0^{2\pi} g(y)h(y,t)dy\right|^2 dt \\
&=: I_1 + I_2.
\end{aligned} \tag{2.1.1}$$

Using Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
I_2 &\leqslant 2\, \|g\|^2_{H_p^s(\mathbb{T})} \int_0^T \left(\int_0^{2\pi} |g(y)|^2\, dy\right)\left(\int_0^{2\pi} |h(y,t)|^2 dy\right) dt \\
&\leqslant 2\, \|g\|^3_{H_p^s(\mathbb{T})} \int_0^T \|h\|^2_{L_p^2(\mathbb{T})}\, dt.
\end{aligned} \tag{2.1.2}$$

On the other hand, using that g is a smooth function, we have

$$|\widehat{gh}(k)|^2 \leqslant \left|\sum_{l\in\mathbb{Z}} \widehat{g}(l)\, \widehat{h}(k-l)\right|^2 \quad \text{(see Proposition 1.2.3).}$$

Therefore, using Lemma 3.197 in [41], we have

$$I_1 := 4\pi \sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^s \int_0^T \left|\widehat{gh}(k)\right|^2 dt \leqslant C_1 \int_0^T \sum_{k\in\mathbb{Z}} \left[(1+|k|^s)\sum_{l\in\mathbb{Z}} |\widehat{g}(l)||\widehat{h}(k-l)|\right]^2 dt, \tag{2.1.3}$$

where C_1 is a positive constant depending only on s. Observe that

$$\left\{(1+|k|^s)\sum_{l\in\mathbb{Z}} |\widehat{g}(l)|\ |\widehat{h}(k-l)|\right\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}), \quad \forall\, t \in [0,T], \tag{2.1.4}$$

and

$$\begin{aligned} \left\|(1+|k|^s)\sum_{l\in\mathbb{Z}} |\widehat{g}(l)|\ |\widehat{h}(k-l)|\right\|_{l^2}^2 &\leqslant C_2\big(\|\widehat{g}\|_{l^1}^2\ \|\widehat{h}\|_{l^2}^2 + \|\ |\cdot|^s\ \widehat{g}(\cdot)\|_{l^1}^2\ \|\widehat{h}\|_{l^2}^2 \\ &\quad + \|\widehat{g}\|_{l^1}^2 \|\ |\cdot|^s\ \widehat{h}(\cdot)\|_{l^2}^2\big), \quad \forall\, t \in [0,T]. \end{aligned} \tag{2.1.5}$$

In fact, since $1+|k|^s \leqslant C_2\left(1+|k-l|^s+|l|^s\right)$, for all $k, j \in \mathbb{Z}$, then

$$\begin{aligned} (1+|k|^s)\sum_{l\in\mathbb{Z}} |\widehat{g}(l)|\ |\widehat{h}(k-l)| &= \sum_{l\in\mathbb{Z}}(1+|k|^s)|\widehat{g}(l)|\ |\widehat{h}(k-l)| \\ &\leqslant C_2 \sum_{l\in\mathbb{Z}} (1+|k-l|^s+|l|^s)\,|\widehat{g}(l)|\ \ |\widehat{h}(k-l)| \\ &\leqslant C_2\Big(\sum_{l\in\mathbb{Z}} |\widehat{g}(l)|\ |\widehat{h}(k-l)| + \sum_{l\in\mathbb{Z}} |k-l|^s|\widehat{g}(l)|\ |\widehat{h}(k-l)| \\ &\quad + \sum_{l\in\mathbb{Z}} |l|^s|\widehat{g}(l)|\ |\widehat{h}(k-l)|\Big), \\ &= C_2\Big[(|\widehat{g}| * \ |\widehat{h}|)(k) + (|\widehat{g}| *\ |\cdot|^s\ |\widehat{h}|)(k) + (|\cdot|^s\ |\widehat{g}| *\ |\widehat{h}|)(k)\Big], \end{aligned}$$

for all $t \in [0,T]$, where C_2 is a positive constant depending only on s. From Young's inequality (see [41, Page 209]), we have that (2.1.4)-(2.1.5) hold. From (2.1.3) and (2.1.5), we infer

$$I_1 \leqslant C_2 \int_0^T \left(\|\widehat{g}\|_{l^1}^2\ \|\widehat{h}\|_{l^2}^2 + \||\cdot|^s\widehat{g}(\cdot)\|_{l^1}^2\ \|\widehat{h}\|_{l^2}^2 + \|\widehat{g}(k)\|_{l^1}^2\||\cdot|^s\widehat{h}(\cdot)\|_{l^2}^2\right)\, dt, \tag{2.1.6}$$

The smoothness of function g, Proposition 1.2.9, and (2.0.8) yield

$$\|\widehat{g}\|_{l^1} = \sum_{k\in\mathbb{Z}} |\widehat{g}(k)| = |\widehat{g}(0)| + \sum_{k\in\mathbb{Z}-\{0\}} \frac{|\widehat{g''}(k)|}{|k|^2} \leqslant \frac{1}{2\pi} + \frac{\|g''\|_{L^1(0,2\pi)}}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty, \tag{2.1.7}$$

and

$$\begin{aligned} \||\cdot|^s\ \widehat{g}\|_{l^1} &= \sum_{k\in\mathbb{Z}} |k|^s|\widehat{g}(k)| = |\widehat{g}(0)| + \sum_{k\in\mathbb{Z}-\{0\}} \frac{|\widehat{g^{(n)}}(k)|}{|k|^{n-s}} \\ &\leqslant \frac{1}{2\pi} + \frac{\|g^{(n)}\|_{L^1(0,2\pi)}}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^{n-s}} < \infty, \quad \text{for some } n > s+1. \end{aligned} \tag{2.1.8}$$

From (2.1.6)-(2.1.8), we infer that there exists $C > 0$ depending only on s, and g such that

$$I_1 \leqslant C \int_0^T \|h\|^2_{H^s_p}(\mathbb{T}) dt = C\|h\|^2_{L^2_p([0,T],H^s_p(\mathbb{T}))}. \tag{2.1.9}$$

The estimates (2.1.1)-(2.1.2), and (2.1.9) imply $\|Gh\|_{L^2([0,T];H^s_p(\mathbb{T}))} \leqslant C\|h\|_{L^2([0,T];H^s_p(\mathbb{T}))}$.
□

Observe that the same conclusion as those in Proposition 2.1.1 can be drawn for any finite interval $[-T, T]$ in place of $[0, T]$.

Remark 2.1.2 ([62] Lemma 2.20). *Let G be given by (2.0.9). With similar arguments as above one can prove that*

$$\|G\phi\|_{H^s_p(\mathbb{T})} \leqslant C\|\phi\|_{H^s_p(\mathbb{T})},$$

where C is a positive constant depending only on s and g. Furthermore, if $s < 0$, then we define Gh by

$$Gh := g\, h - g\, \langle g, h\rangle_{\left(L^2([0,T];H^s_p(\mathbb{T}))\right)' \times L^2([0,T];H^s_p(\mathbb{T}))} \tag{2.1.10}$$

for any $h \in L^2([0, T]; H^s_p(\mathbb{T}))$, and the result obtained in Proposition 2.1.1 holds.

Now, we show that the operator G is self-adjoint in $L^2(\mathbb{T})$.

Proposition 2.1.3. *The operator $G : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given in (2.0.9) is linear, bounded and self-adjoint.*

Proof. It is easy to see that $G \in \mathcal{L}(L^2(\mathbb{T}))$. Moreover, there is a positive constant C_g depending only on g such that

$$\|G\varphi\|_{L^2(\mathbb{T})} \leqslant C_g\|\varphi\|_{L^2(\mathbb{T})}.$$

By the density of $\mathrm{Dom}(G)$ in $L^2(\mathbb{T})$, it is enough to prove that G is symmetric. Let $h \in L^2(\mathbb{T})$, thus

$$\begin{aligned}
(Gh\ ,\ f)_{L^2(\mathbb{T})} &= \int_0^{2\pi} g(x)h(x)\overline{f}(x)\, dx - \int_0^{2\pi} g(x)\overline{f}(x) \left[\int_0^{2\pi} g(y)h(y)\, dy\right] dx.\\
&= \int_0^{2\pi} g(y)h(y)\overline{f}(y)\, dy - \int_0^{2\pi} g(y)h(y) \left[\int_0^{2\pi} g(x)\overline{f}(x)\, dx\right] dy\\
&= \int_0^{2\pi} g(y)h(y) \left[\overline{f}(y) - \int_0^{2\pi} g(x)\overline{f}(x)\, dx\right] dy\\
&= \int_0^{2\pi} h(y)G\left(\overline{f}(y)\right)\, dy\\
&= (h\ ,\ Gf)_{L^2(\mathbb{T})}.
\end{aligned}$$

This proves the Proposition. □

Remark 2.1.4. *Note that for any $\varphi \in L^2(\mathbb{T})$, one has $\overline{G\varphi} = G\overline{\varphi}$.*

2.2 Existence of solutions for the associated linear system

In this section, we show that the linear IVP (2.0.3) associated to the Benjamin equation is globally well-posed on $H_p^s(\mathbb{T})$, with $s \in \mathbb{R}$.

Proposition 2.2.1. *Let $\alpha > 0$. The operator*

$$A = D(A) \subseteq L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad \text{defined by} \quad A\varphi = \alpha \mathcal{H} \partial_x^2 \varphi + \partial_x^3 \varphi, \tag{2.2.1}$$

generates a strongly continuous unitary group $\{U(t)\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{T})$.

Proof. Let $\varphi, \psi \in D(A) = H_p^3(\mathbb{T})$. Observe that φ is three times differentiable. Furthermore,

$$\mathcal{H}(\partial_x \varphi)(x) = \partial_x \mathcal{H}(\varphi)(x) \quad \text{and} \quad \mathcal{H}(\partial_x^2 \varphi)(x) = \partial_x^2 \mathcal{H}(\varphi)(x), \ \forall x \in \mathbb{T}.$$

Using property (1.4.4) of the Hilbert's transform, we have

$$(A\varphi \ , \ \psi)_{L^2(\mathbb{T})} = -\alpha \int_0^{2\pi} \partial_x^2 \varphi(x) \overline{\mathcal{H}\psi(x)} \, dx + \int_0^{2\pi} \partial_x^3 \varphi(x) \overline{\psi(x)} \, dx. \tag{2.2.2}$$

Integrating (2.2.2) by parts with respect to x and using the periodicity of the functions involved, we obtain

$$(A\varphi, \psi)_{L^2(\mathbb{T})} = -\int_0^{2\pi} \varphi(x) [\alpha \overline{\mathcal{H}\partial_x^2 \psi(x) + \partial_x^3 \psi(x)}] \, dx = -(\varphi \ , \ A\psi)_{L^2(\mathbb{T})},$$

which implies that A is skew-adjoint. From Proposition 1.5.15 we have that A is m-dissipative. Definition 1.5.14 implies that A is dissipative. Therefore, $(A\varphi, \varphi)_{L^2(\mathbb{T})}$ is real for all $\varphi \in D(A)$ (see Proposition 2.4.2 in [16]). Moreover, since

$$(A\varphi \ , \ \varphi)_{L^2(\mathbb{T})} = -(\varphi \ , \ A\varphi)_{L^2(\mathbb{T})},$$

we have that $(A\varphi \ , \ \varphi)_{L^2(\mathbb{T})} = 0$. Therefore, Theorem 1.5.16 implies that the operator A generates a strongly continuous unitary group of isometries (contractions) $\{U(t)\}_{t\in\mathbb{R}}$ (see [16, Definition 3.4.6]). □

The following result gives the existence of solutions in $H_p^3(\mathbb{T})$ for the linear-homogeneous system (2.0.3) associated to the Benjamin equation.

Corollary 2.2.2. *Let $u_0 \in H_p^3(\mathbb{T})$, then there exists a unique solution*

$$u \in C(\mathbb{R}, H_p^3(\mathbb{T})) \cap C^1(\mathbb{R}, L^2(\mathbb{T}))$$

for the homogeneous IVP (2.0.3).

Proof. This is a consequence of Theorem 1.5.16 and Proposition 2.2.1. □

We can generalize the last Corollary to get solutions of system (2.0.3) in $H^s_p(\mathbb{T})$ for all $s \in \mathbb{R}$. This can be stated in a formal way as following.

The homogeneous IVP (2.0.3) is equivalent to the following Cauchy problem

$$\begin{cases} u \in C(\mathbb{R}, H^s_p(\mathbb{T})) \\ \partial_t u = \alpha\mathcal{H}\partial_x^2 u + \partial_x^3 u \ \in H^{s-3}_p(\mathbb{T}), \quad t \in \mathbb{R} \\ u(0) = u_0, \end{cases} \tag{2.2.3}$$

where, the initial data $u_0 \in H^s_p(\mathbb{T})$. Taking Fourier's transform in the spatial variable, the IVP (2.2.3) is equivalent to the following ordinary differential equation (ODE)

$$\begin{cases} \partial_t \widehat{u}(k) = ik^2\left[\alpha\ sgn(k) - k\right]\widehat{u}(k), \quad t \in \mathbb{R}, \\ \widehat{u}(k,0) = \widehat{u_0}(k), \end{cases} \tag{2.2.4}$$

for all $k \in \mathbb{Z}$. The unique solution of equation (2.2.4) is given by

$$\widehat{u}(t)(k) = e^{ik^2[\alpha\,\mathrm{sgn}(k)-k]t}\widehat{u_0}(k), \quad \forall k \in \mathbb{Z}. \tag{2.2.5}$$

Note that $\{e^{ik^2[\alpha\,\mathrm{sgn}(k)-k]t}\widehat{u_0}(k)\}_{k\in\mathbb{Z}} \in \mathcal{G}'(\mathbb{Z})$ is a slow growth sequence. Taking inverse Fourier transform in (2.2.5), we get the unique solution of (2.2.3)

$$u(t) = \left(e^{ik^2[\alpha\ sgn(k)-k]t}\widehat{\varphi}(k)\right)^{\vee}, \quad \forall\, t \in \mathbb{R}. \tag{2.2.6}$$

It means that,

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{ik^2[\alpha\,\mathrm{sgn}(k)-k]t}\widehat{u_0}(k)e^{ikx}, \quad \forall\, t \in \mathbb{R}, \tag{2.2.7}$$

is the unique solution for the IVP (2.0.3), where the series convergence is in the sense of $\mathcal{D}'(\mathbb{T})$.

Now, in rigorous way, define the family of operators $U : \mathbb{R} \to \mathcal{L}(H^s_p(\mathbb{T}))$ by

$$t \longmapsto U(t)\varphi := e^{(\alpha\mathcal{H}\partial_x^2+\partial_x^3)t}\varphi = (e^{ik^2[\alpha\ sgn(k)-k]t}\widehat{\varphi}(k))^{\vee}. \tag{2.2.8}$$

Note that, with this definition the relation (2.2.6) becomes $u(t) = U(t)u_0, \ \ t \in \mathbb{R}$, and we obtain the following results.

Lemma 2.2.3. *Let $s \in \mathbb{R}$. The family of operators $\{U(t)\}_{t\in\mathbb{R}}$ given by (2.2.8) defines a strongly continuous one-parameter unitary group of contractions on $H^s_p(\mathbb{T})$. Furthermore, $U(t)$ is an isometry for all $t \in \mathbb{R}$ (see [16, Definition 3.4.6]).*

Proof. Note that $\{U(t)\}_{t\in\mathbb{R}}$ has the following properties:

i) $U(t) \in \mathcal{L}(H^s_p(\mathbb{T})), \ \forall\, t \in \mathbb{R}$.

In fact, given $f \in H^s_p(\mathbb{T})$ and $t \in \mathbb{R}$,

$$\begin{aligned} \|U(t)f\|^2_{H^s_p(\mathbb{T})} &= \left\| \sum_{k\in\mathbb{Z}} e^{ik^2[\alpha\,\mathrm{sgn}(k)-k]t}\widehat{f}(k)\ e^{ikx} \right\|^2_{H^s_p(\mathbb{T})} \\ &= 2\pi\sum_{k\in\mathbb{Z}}(1+|k|^2)^s\left|e^{ik^2[\alpha\ sgn(k)-k]t}\widehat{f}(k)\right|^2 = \|f\|^2_{H^s_p(\mathbb{T})}. \end{aligned}$$

ii) $U(t)$ is an unitary operator $\forall t \in \mathbb{R}$, it means $\|U(t)\|_{\mathcal{L}(H^s_p(\mathbb{T}))} = 1$.

iii) $U(0) = I$.

iv) $U(t+s) = U(t)U(s), \quad \forall\, t, s \in \mathbb{R}$.

Indeed,

$$\begin{aligned} U(t+s)f &= \left(e^{ik^2[\alpha\ \mathrm{sgn}(k)-k](t+s)}\widehat{f}(k)\right)^{\vee} \\ &= \left(e^{ik^2[\alpha\ \mathrm{sgn}(k)-k]t}\ e^{ik^2[\alpha\ Sgn(k)-k]s}\widehat{f}(k)\right)^{\vee} \\ &= \left(e^{ik^2[\alpha\ sgn(k)-k](t)}(U(s)f)^\wedge(k)\right)^{\vee} = U(t)U(s)f, \quad \forall\ f \in H^s_p(\mathbb{T}). \end{aligned}$$

v) $\lim\limits_{t\to t_0} \|U(t)f - U(t_0)f\|_{H^s_p(\mathbb{T})} = 0, \ \forall\ t_0 \in \mathbb{R}, \text{ and } f \in H^s_p(\mathbb{T})$.

In fact, assume $t_0 \in \mathbb{R}$ then,

$$\|U(t)f - U(t_0)f\|^2_{H^s_p(\mathbb{T})} = 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^s \left|(e^{ik^2[\alpha\ \mathrm{sgn}(k)-k]t} - e^{ik^2[\alpha\ \mathrm{sgn}(k)-k]t_0})\widehat{f}(k)\right|^2 \tag{2.2.9}$$

Note that $(1+|k|^2)^s \left|(e^{ik^2[\alpha\ \mathrm{sgn}(k)-k]t} - e^{ik^2[\alpha\ \mathrm{sgn}(k)-k]t_0})\widehat{f}(k)\right|^2 \leqslant 4(1+|k|^2)^s|\widehat{f}(k)|^2$, and

$$\sum_{k\in\mathbb{Z}} 4(1+|k|^2)^s|\widehat{f}(k)|^2 \leqslant 4\|f\|_{H^s_p(\mathbb{T})} < \infty.$$

A direct application of Weierstrass's M-test implies that the series

$$\sum_{k\in\mathbb{Z}} (1+|k|^2)^s \left|\left(e^{ik^2[\alpha\ sgn(k)-k]t} - e^{ik^2[\alpha\ sgn(k)-k]t_0}\right)\widehat{f}(k)\right|^2,$$

converges absolutely and uniformly with respect to t. Taking the limit when t goes to t_0 in (2.2.9) we get the result.

□

Remark 2.2.4. *By [16, Corollary 3.2.6, pag. 38] we have that the adjoint operator $U(t)^*$ of $U(t)$ exists. It is bounded, linear, and $U(t)^* = U(-t)$ for all $t \in \mathbb{R}$. Moreover, it is easy to prove that $\overline{U(t)\varphi} = U(t)\overline{\varphi}$ for all $t \in \mathbb{R}$.*

Next, we turn our attention to prove differentiability.

Lemma 2.2.5. *Assume $s \in \mathbb{R}$. If $u(t) = U(t)u_0$, then*

$$\lim_{h\to 0} \left\| \frac{u(t+h)-u(t)}{h} - [\alpha\mathcal{H}\partial_x^2 + \partial_x^3]u \right\|_{H^{s-3}_p(\mathbb{T})} = 0, \tag{2.2.10}$$

uniformly with respect to $t \in \mathbb{R}$.

Proof. Denote $\lambda_k = k^2(\alpha \operatorname{sgn}(k) - k)$ and define

$$I_h := \left\| \frac{u(t+h) - u(t)}{h} - [\alpha \mathcal{H}\partial_x^2 + \partial_x^3]u \right\|_{H_p^{s-3}(\mathbb{T})}^2 .$$

Observe that

$$u(t) = U(t)u_0(x) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k t}\widehat{u_0}(k)e^{ikx}, \quad \mathcal{H}(\partial_x^2 U(t)u_0)(x) = \sum_{k\in\mathbb{Z}} i \operatorname{sgn}(k)k^2 e^{i\lambda_k t}\widehat{u_0}(k)e^{ikx},$$

and

$$\partial_x^3 U(t)u_0(x) = \sum_{k\in\mathbb{Z}} -e^{i\lambda_k t}\widehat{u_0}(k)ik^3 e^{ikx}.$$

Consequently,

$$\begin{aligned} I_h &= \left\| \sum_{k\in\mathbb{Z}} \left(\frac{e^{i\lambda_k(t+h)} - e^{i\lambda_k t}}{h} - i\,\alpha \operatorname{sgn}(k)\, k^2\, e^{i\lambda_k t} + e^{i\lambda_k t}\, ik^3 \right) \widehat{u_0}(k)\, e^{ikx} \right\|_{H_p^{s-3}(\mathbb{T})}^2 \\ &= 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s-3} \left| \frac{e^{i\lambda_k(t+h)} - e^{i\lambda_k t}}{h} - i\,\alpha \operatorname{sgn}(k)\, k^2\, e^{i\lambda_k t} + e^{i\lambda_k t}\, ik^3 \right|^2 |\widehat{u_0}(k)|^2 \\ &= 2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s-3} J_h\, |\widehat{u_0}(k)|^2, \end{aligned} \tag{2.2.11}$$

where $J_h := \left| \dfrac{e^{i\lambda_k h} - 1}{h} - i\,\alpha\operatorname{sgn}(k)\, k^2 + ik^3 \right|^2$.

Note that

$$\begin{aligned} J_h &\leqslant \left(\frac{|e^{i\lambda_k h} - 1|}{|h|} + \alpha\, k^2 + |k|^3 \right)^2 \leqslant \left(\frac{|\lambda_k h|}{|h|} + \alpha\, k^2 + |k|^3 \right)^2 \\ &= \left(|k^2\alpha \operatorname{sgn}(k) - k^3| + \alpha\, k^2 + |k|^3 \right)^2 \leqslant 4\left(\alpha\, k^2 + |k|^3\right)^2 . \end{aligned}$$

Therefore,

$$(1+|k|^2)^{s-3} J_h \leqslant 4(1+|k|^2)^s \frac{(\alpha\, k^2 + |k|^3)^2}{(1+|k|^2)^3} \leqslant C(1+|k|^2)^s. \tag{2.2.12}$$

and the last term of (2.2.11) is bounded. In fact, (2.2.12) yield

$$2\pi \sum_{k\in\mathbb{Z}} (1+|k|^2)^{s-3} J_h\, |\widehat{u_0}(k)|^2 \leqslant C \sum_{k\in\mathbb{Z}} (1+|k|^2)^s |\widehat{u_0}(k)|^2 < \infty.$$

The Weierstrass' M-test implies that the series in (2.2.11) converges absolutely and uniformly with respect to h and t. Finally, taking the limit when h goes to 0 in (2.2.11), we obtain (2.2.10). □

Lemmas 2.2.3 and 2.2.5 imply that the system (2.0.3) is globally well-posed. Its unique solution, which depends continuously on the initial data, is given by (2.2.7). This result is established in the following theorem.

Theorem 2.2.6. *Let $s \in \mathbb{R}$ and $u_0 \in H_p^s(\mathbb{T})$, then there exists a unique solution $u \in C(\mathbb{R}, H_p^s(\mathbb{T}))$ for the homogeneous IVP (2.0.3).*

In the following, we are going to deal with the well-posedness of the non-homogeneous system associated to the linearized Benjamin equation with periodic boundary conditions.

$$\begin{cases} \partial_t u - \alpha \mathcal{H}\partial_x^2 u - \partial_x^3 u = Gh(x,t), & t \in (0,T), \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{2.2.13}$$

A direct application of the semigroup theory gives us the following lemma.

Lemma 2.2.7. *Let $0 \leqslant T < \infty$, $s \geqslant 0$, $u_0 \in H_p^s(\mathbb{T})$, and $h \in L^2([0,T]; H_p^s(\mathbb{T}))$ then, there exists a unique mild solution $u \in C([0,T], H_p^s(\mathbb{T}))$ for the non-homogeneous IVP (2.2.13).*

Proof. Let $h \in L^2([0,T]; H_p^s(\mathbb{T}))$. From Proposition 2.1.1 we obtain that

$$Gh \in L^2([0,T]; H_p^s(\mathbb{T})).$$

Thus, $Gh \in L^1([0,T]; H_p^s(\mathbb{T}))$. We rewrite the IVP (2.2.13) in its equivalent form,

$$\begin{cases} u \in C([0,T], H_p^s(\mathbb{T})) \\ \partial_t u = \alpha \mathcal{H}\partial_x^2 u + \partial_x^3 u + Gh(t) \ \in H_p^{s-3}(\mathbb{T}), & t \in (0,T) \\ u(0) = u_0, \end{cases} \tag{2.2.14}$$

where the initial data $u_0 \in H_p^s(\mathbb{T})$. From Corollary 1.5.20 and Definition 1.5.21, we have

$$u(t) = U(t)u_0 + \int_0^t U(t-t')Gh(t')dt',$$

is the unique solution of (2.2.14) for $s \geqslant 0$, $0 \leqslant t \leqslant T < \infty$. □

2.3 Control of the linear Benjamin equation

In this section we prove an exact controllability result for the system (2.2.13) using the classical moment method, see [79]. Without loss of generality, one can consider $u_0 = 0$. In fact, for given $u_0, u_1 \in H_p^s(\mathbb{T})$ with $[u_0] = [u_1]$, if h is the control which leads the solution v of system (2.2.13) from initial data $v_0 = 0$ to the final state $u_1 - U(T)u_0$, then v satisfies

$$\begin{cases} \partial_t v - \alpha \mathcal{H}\partial_x^2 v - \partial_x^3 v = Gh(x,t), & t \in (0,T), \quad x \in \mathbb{T} \\ v(x,0) = 0, & x \in \mathbb{T}, \end{cases}$$

and v can be written as, $v(t) = \displaystyle\int_0^t U(t-s)Gh(s)ds$.

So,

$$u_1 - U(T)u_0 = v(T) = \int_0^T U(T-s)Gh(s)ds.$$

Therefore,

$$u_1 = U(T)u_0 + \int_0^T U(T-s)Gh(s)ds = u(T),$$

where u is the solution of system (2.2.13) with initial data u_0. It means that, the control h leads the solution u of system (2.2.13) from the initial state u_0 to the final state u_1.

From this point onward in this section we assume $u_0 = 0$. Thus, $[u_1] = [u_0] = 0$. We have the following lemma.

Lemma 2.3.1. *Suppose $s \geqslant 0$ and $\psi_k(x)$ for all $k \in \mathbb{Z}, x \in \mathbb{T}$ defined as in Remark 1.2.2. Then for $u_1 \in H_p^s(\mathbb{T})$ given by*

$$u_1(x) = \sum_{m \in \mathbb{Z}} c_m \, \psi_m(x),$$

and $[u_1] = 0$, we obtain that $c_0 = 0$.

The next result is fundamental to get control for the linear system (2.2.13).

Lemma 2.3.2. *Let $s \geqslant 0$, and $T > 0$ be given. Assume $u_1 \in H_p^s(\mathbb{T})$ with $[u_1] = 0$. Then, there exists $h \in L^2([0,T], H_p^s(\mathbb{T}))$, such that the solution of the IVP (2.2.13) with initial data $u_0 = 0$ satisfies $u(T) = u_1$ if and only if*

$$\int_0^T \langle Gh(\cdot,t), \varphi(\cdot,t)\rangle_{H_p^s \times (H_p^s)'} \, dt = \langle u_1, \varphi_0 \rangle_{H_p^s \times (H_p^s)'}\,, \tag{2.3.1}$$

for any $\varphi_0 \in (H_p^s(\mathbb{T}))'$, where $(H_p^s(\mathbb{T}))'$ is the dual space of $H_p^s(\mathbb{T})$, and φ is the solution of the adjoint system

$$\begin{cases} \partial_t \varphi - \alpha \mathcal{H} \partial_x^2 \varphi - \partial_x^3 \varphi = 0, & t > 0, \quad x \in (0, 2\pi) \\ \varphi(x,T) = \varphi_0(x), & x \in \mathbb{T}. \end{cases} \tag{2.3.2}$$

Proof. ($\Rightarrow$) Let φ_0 and h be smooth functions and φ be the solution of the adjoint system (2.3.2) with final data φ_0. Multiplying the equation in (2.2.13) by $\overline{\varphi}$, integrating by parts, and using the Hilbert transform's proprieties given in Proposition 1.4.2, we have

$$\begin{aligned} \int_0^T \int_0^{2\pi} Gh \, \overline{\varphi} \, dx \, dt &= \int_0^T \int_0^{2\pi} \partial_t u \, \overline{\varphi} \, dx \, dt - \alpha \int_0^T \int_0^{2\pi} H \partial_x^2 u \, \overline{\varphi} \, dx \, dt \\ &\quad - \int_0^T \int_0^{2\pi} \partial_x^3 u \, \overline{\varphi} \, dx \, dt. \\ &= \int_0^{2\pi} u(T) \, \overline{\varphi}(T) \, dx - \int_0^T \int_0^{2\pi} u \, \left[\partial_t \overline{\varphi} - \alpha \partial_x^2 \mathcal{H} \overline{\varphi} - \partial_x^3 \overline{\varphi}\right] \, dx \, dt. \\ &= \int_0^{2\pi} u(T) \, \overline{\varphi}(T) \, dx. \end{aligned} \tag{2.3.3}$$

Therefore,

$$\int_0^T \int_0^{2\pi} Gh \, \overline{\varphi} \, dx \, dt = \int_0^{2\pi} u_1 \, \overline{\varphi_0} \, dx.$$

Now, identifying $L^2(\mathbb{T})$ with its dual (see Zeidler [32, page 254]) by means of the (conjugate linear) map $y \longmapsto (\cdot, y)_{L^2(\mathbb{T})}$ we have the following inclusion,

$$H_p^s(\mathbb{T}) \hookrightarrow L^2(\mathbb{T}) \equiv (L^2(\mathbb{T}))' \hookrightarrow (H_p^s(\mathbb{T}))',$$

where the embedding is dense and continuous. Moreover, $\langle \phi, \varphi \rangle_{H_p^s \times (H_p^s)'} = (\phi, \varphi)_{L^2(\mathbb{T})}$, for all ϕ, $\varphi \in L^2(\mathbb{T})$. Thus,

$$\begin{aligned}\int_0^T \langle Gh(\cdot,t), \varphi(\cdot,t)\rangle_{H_p^s \times (H_p^s)'}\ dt &= \int_0^T (Gh(\cdot,t), \varphi(\cdot,t))_{L^2(\mathbb{T})}\ dt \\ &= \int_0^{2\pi} u_1\ \overline{\varphi_0}\ dx \\ &= \langle u_1,\ \varphi_0 \rangle_{H_p^s \times (H_p^s)'} .\end{aligned}$$

($\Leftarrow$) Let h be a smooth function such that (2.3.1) holds for any smooth $\varphi_0 \in (H_p^s(\mathbb{T}))'$, where φ is the smooth solution of the adjoint system (2.3.2) with final data φ_0. Identifying $L^2(\mathbb{T})$ with its dual and using (2.3.3) we have

$$\begin{aligned}\int_0^{2\pi} u_1\ \overline{\varphi_0}\ dx &= \langle u_1,\ \varphi_0 \rangle_{H_p^s \times (H_p^s)'} \\ &= \int_0^T \langle Gh(\cdot,t), \varphi(\cdot,t)\rangle_{H_p^s \times (H_p^s)'}\ dt \\ &= \int_0^T \int_0^{2\pi} Gh\ \overline{\varphi}\ dx\ dt \\ &= \int_0^{2\pi} u(T)\ \overline{\varphi}(T)\ dx \\ &= \int_0^{2\pi} u(T)\ \overline{\varphi_0}\ dx.\end{aligned} \tag{2.3.4}$$

Identity (2.3.4) implies that $\displaystyle\int_0^{2\pi} (u(T) - u_1)\ \overline{\varphi_0}\ dx = 0$, for all smooth function φ_0. In consequence $u(T) = u_1$. Thus, the lemma is true for all smooth data.

In general case, we use density arguments to complete the proof. □

Lemma 2.3.3. *Let $s \geqslant 0$, $T > 0$, and $u_1 \in H_p^s(\mathbb{T})$ with $[u_1] = 0$. Then, there exists $h \in L^2([0,T], H_p^s(\mathbb{T}))$, such that the solution of the IVP (2.2.13) with initial data $u_0 = 0$ satisfies $u(T) = u_1$, if and only if, there exists $\delta > 0$ such that*

$$\int_0^T \|G^* U(\tau)^* \phi^*\|^2_{(H^s(\mathbb{T}))'}(\tau)\ d\tau \geqslant \delta^2 \|\phi^*\|^2_{(H^s(\mathbb{T}))'}, \tag{2.3.5}$$

for any $\phi^* \in (H^s(\mathbb{T}))'$.

Proof. ($\Rightarrow$) Let $T > 0$. Define a linear map $F_T : L^2([0,T]; H_p^s(\mathbb{T})) \to H_p^s(\mathbb{T})$ by

$$F_T(h) = u(\cdot, T), \tag{2.3.6}$$

where $u = u(x,t)$ is the solution (mild solution) of

$$\begin{cases} \partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u & = Gh(x,t), \quad t \in (0,T), \quad x \in \mathbb{T} \\ u(x,0) = 0, & \quad x \in \mathbb{T}. \end{cases} \tag{2.3.7}$$

Note that if $u_1 \in H_p^s(\mathbb{T})$ is given, then from hypothesis there exists h such that

$$F_T(h) = u(T) = u_1. \tag{2.3.8}$$

Therefore, F_T is onto, i.e,

$$F_T\left(L^2([0,T]; H_p^s(\mathbb{T}))\right) = H_p^s(\mathbb{T}), \tag{2.3.9}$$

trivially, $\mathrm{Ran}(F_T)$ is dense in $H_p^s(\mathbb{T})$. On the other hand, from hypothesis, for $u_1 \in H_p^s(\mathbb{T})$, we have that

$$u_1 = \int_0^T U(T-s)(Gh)(\cdot, s)\, ds. \tag{2.3.10}$$

Therefore, from (2.3.8) and (2.3.10)

$$\begin{aligned} \|F_T(h)\|_{H_p^s(\mathbb{T})} &= \left\| \int_0^T U(T-s)(Gh)(\cdot,s)\, ds \right\|_{H_p^s(\mathbb{T})} \\ &\leqslant \int_0^T \|U(T-s)(Gh)(\cdot,s)\|_{H_p^s(\mathbb{T})}\, ds \\ &\leqslant c_g \int_0^T \|h\|_{H_p^s(\mathbb{T})}\, ds \\ &\leqslant c_g\, T^{\frac{1}{2}} \|h\|_{H_p^s([0,T];L^2(\mathbb{T}))}. \end{aligned} \tag{2.3.11}$$

So, F_T is a bounded linear operator. Thus, F_T^* exists, is a bounded linear operator, and is one-to-one (see Rudin [78, Corollary b) page 99]). Also, from Theorem 4.13 in [78, page 100] (see also Coron [23, page 35]) we have that there exists $\delta > 0$ such that

$$\|F_T^*(\phi^*)\|_{\left(L^2([0,T];H_p^s(\mathbb{T}))\right)'} \geqslant \delta\, \|\phi^*\|_{\left(H_p^s(\mathbb{T})\right)'}, \qquad \text{for all } \phi^* \in \left(H_p^s(\mathbb{T})\right)'. \tag{2.3.12}$$

From Lemma 2.3.2, we have that the solution u of (2.3.7) satisfies

$$\int_0^T \langle Gh(\cdot,t), \varphi(\cdot,t)\rangle_{H_p^s \times (H_p^s)'}\, dt - \langle u_1, \varphi_0\rangle_{H_p^s \times (H_p^s)'} = 0, \tag{2.3.13}$$

for any $\varphi_0 \in (H_p^s(\mathbb{T}))'$, where φ is the solution of the adjoint system (2.3.2). Note that

$$\varphi(\cdot,t) = U(T-t)^* \varphi_0.$$

Then it follows from (2.3.13) that

$$\begin{aligned} \int_0^T \langle h(\cdot,t), G^* U(T-t)^* \varphi_0\rangle_{H_p^s(\mathbb{T}) \times (H_p^s(\mathbb{T}))'}\, dt &= \langle u(\cdot,T), \varphi_0\rangle_{H_p^s(\mathbb{T}) \times (H_p^s(\mathbb{T}))'} \\ &= \langle F_T(h), \varphi_0\rangle_{H_p^s(\mathbb{T}) \times (H_p^s(\mathbb{T}))'} \\ &= \langle h\,,\, F_T^* \varphi_0\rangle_{L^2([0,T];H_p^s(\mathbb{T})) \times \left(L^2([0,T];H_p^s(\mathbb{T}))\right)'}. \end{aligned} \tag{2.3.14}$$

Therefore, $F_T^* = G^*U(T-t)^*$ and using (2.3.12), we have

$$\|G^*U(T-t)^*(\phi^*)\|_{L^2([0,T];(H_p^s(\mathbb{T}))')} \geqslant \delta \, \|\phi^*\|_{\left(H_p^s(\mathbb{T})\right)'}, \quad \text{for all } \ \phi^* \in \left(H_p^s(\mathbb{T})\right)'.$$

It means

$$\int_0^T \|G^*U(T-t)^*(\phi^*(x))\|^2_{(H_p^s(\mathbb{T}))'} \, dt \geqslant \delta^2 \, \|\phi^*\|^2_{(H_p^s(\mathbb{T}))'}, \quad \text{for all } \ \phi^* \in (H_p^s(\mathbb{T}))'.$$

Performing a change of the temporal variable $\tau = T - t$, we obtain (2.3.5).

($\Leftarrow$) If (2.3.5) holds, then $F_T^* = G^*U(T-t)^*$ is onto. It is easy to prove that F_T^* is bounded from $(H_p^s(\mathbb{T}))'$ into $\left(L^2([0,T]; H_p^s(\mathbb{T}))\right)'$. Therefore, F_T is onto. From calculations similar to (2.3.14), we obtain that (2.3.13) holds. Then Lemma 2.3.2 imply the result. □

The following result is a characterization for the existence of control to the system (2.2.13) with initial data $u_0 = 0$.

Lemma 2.3.4. *Let $s \geqslant 0$ and $T > 0$ be given, and $\psi_k(x)$ as in Remark 1.2.2. If*

$$u_1(x) = \sum_{l \in \mathbb{Z}} c_l \, \psi_l(x) \quad \in H_p^s(\mathbb{T}),$$

is a function such that $[u_1] = 0$, then the non-homogeneous system (2.2.13) with initial data $u_0 = 0$ is exactly controllable in time T to u_1, that is, $u(x,T) = u_1(x), \ \forall x \in \mathbb{T}$, if an only if there exists $h \in L^2([0,T]; H_p^s(\mathbb{T}))$ such that

$$\int_0^T \int_{\mathbb{T}} Gh(x,t) \, e^{-i\lambda_k(T-t)} \overline{\psi_k(x)} \, dx dt = c_k, \ \forall \ k \in \mathbb{Z}, \tag{2.3.15}$$

where $\lambda_k := k^3 - \alpha \ sgn(k) k^2 = k^3 - \alpha k|k|$.

Proof. ($\Rightarrow$) In view of Lemma 2.3.2, let us to consider the adjoint system

$$\begin{cases} \partial_t \varphi - \alpha \mathcal{H} \partial_x^2 \varphi - \partial_x^3 \varphi = 0, & t > 0, \quad x \in \mathbb{T} \\ \varphi(x,T) = \varphi_0(x), & x \in \mathbb{T} \end{cases} \tag{2.3.16}$$

and let $k \in \mathbb{Z}$ be fixed. Note that $\psi_k \in (H_p^s(\mathbb{T}))'$. In fact, for $\psi_k \in \mathcal{D}'(\mathbb{T})$

$$\|\psi_k\|^2_{(H_p^s(\mathbb{T}))'} = 2\pi \sum_{l \in Z} (1 + |l|^2)^{-s} |\widehat{\psi_k}(l)|^2,$$

and

$$\widehat{\psi_k}(l) = \frac{1}{2\pi} \int_0^{2\pi} \psi_k(x) \, e^{-ilx} \, dx = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi_k(x) \, \psi_{-l}(x) \, dx = \begin{cases} \dfrac{1}{\sqrt{2\pi}}, & if k = l \\ 0, & if k \neq l. \end{cases}$$

Therefore, $\|\psi_k\|^2_{(H^s_p(\mathbb{T}))'} = 2\pi(1+|k|^2)^{-s}\dfrac{1}{2\pi} = (1+|k|^2)^{-s} < \infty$. So, we suppose $\varphi_0 = \psi_k$. Then identity (2.2.6) implies that

$$\varphi(x,t) = U(T-t)^*\varphi_0(x) = \left(e^{-il^2[\alpha\,\mathrm{sgn}(l)-l](T-t)}\widehat{\varphi_0}(l)\right)^{\vee} = \sum_{l\in\mathbb{Z}} e^{i\lambda_l(T-t)}\widehat{\varphi_0}(l)e^{ilx}, \quad (2.3.17)$$

where $\lambda_l = k^3 - \alpha k|k|$. Consequently, $\varphi(x,t) = e^{i\lambda_k(T-t)}\psi_k(x)$. Now, using identity (2.3.1) one gets

$$\int_0^T\int_{\mathbb{T}} Gh(x,t)\,\overline{\varphi}(x)\;\,dx\;dt - \int_{\mathbb{T}}\left(\sum_{l\in\mathbb{Z}} c_l\;\psi_l(x)\right)\;\overline{\varphi_0}(x)\;dx = 0.$$

Therefore,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}} Gh(x,t)\;e^{-i\lambda_k(T-t)}\psi_{-k}(x)\;\,dx\;dt &= \int_{\mathbb{T}}\left(\sum_{l\in\mathbb{Z}} c_l\;\psi_l(x)\right)\;\psi_{-k}(x)\;dx\\ &= \sum_{l\in\mathbb{Z}} c_l\int_{\mathbb{T}}\psi_l(x)\;\psi_{-k}(x)\;dx\\ &= c_k\;\forall k\in\mathbb{Z},\end{aligned}$$

as required.
($\Leftarrow$) Now, suppose that there exists $h\in L^2([0,T];H^s_p(\mathbb{T}))$ such that (2.3.15) holds. With similar calculations as above, we obtain

$$\int_0^T\int_{\mathbb{T}} Gh(x,t)\;\;\overline{e^{i\lambda_k(T-t)}\;\psi_k}(x)\;dx\;dt - \int_0^{2\pi} u_1\;\overline{\psi_k}\;dx = 0,\;\forall\;\varphi_0 = \psi_k,\;\;k\in\mathbb{Z}.$$

Multiplying both sides of the last equality by $\overline{\widehat{\varphi_0}(k)}$ and summing over $k\in\mathbb{Z}$, we get

$$\sum_{k\in\mathbb{Z}}\int_0^T\int_{\mathbb{T}} Gh(x,t)\;\;\overline{e^{i\lambda_k(T-t)}\;\psi_k}(x)\;\overline{\widehat{\varphi_0}(k)}dx\;dt = \sum_{k\in\mathbb{Z}}\int_0^{2\pi} u_1(x)\;\overline{\psi_k}(x)\;\overline{\widehat{\varphi_0}(k)}\;dx.$$

Note that

$$\varphi(x,t) = \sum_{k\in\mathbb{Z}} e^{i\lambda_k(T-t)}\widehat{\varphi_0}(l)e^{ikx},$$

is the solution of the adjoint system (2.3.16) and $\varphi_0\in C^\infty_p(\mathbb{T})$ can be expressed as

$$\varphi_0(x) = \sum_{k\in\mathbb{Z}}\widehat{\varphi_0}(k)\;\psi_k(x),$$

where the series converge uniformly. Thus

$$\int_0^T\int_{\mathbb{T}} Gh(x,t)\;\;\overline{\varphi}(x,t)\;dx\;dt - \int_0^{2\pi} u_1\;\overline{\varphi_0}\;dx = 0,\;\forall\;\varphi_0\in C^\infty_p(\mathbb{T}).$$

The result follows by using density arguments. $\square$

Lemma 2.3.5. *Let $\psi_k(x)$ be as in Remark 1.2.2, and*

$$m_{j,k} = \widehat{G(\psi_j)}(k) = \int_0^{2\pi} G(\psi_j)(x)\overline{\psi_k}(x)\, dx, \quad j,k \in \mathbb{Z}, \tag{2.3.18}$$

where G is as in (2.0.9). In addition, for any given finite sequence of nonzero integers k_j, j=1,2,3,....,n, let

$$M_n = \begin{pmatrix} m_{k_1,k_1} & \cdots & m_{k_1,k_n} \\ m_{k_2,k_1} & \cdots & m_{k_2,k_n} \\ \vdots & \vdots & \vdots \\ m_{k_n,k_1} & \cdots & m_{k_n,k_n} \end{pmatrix}$$

Then

i) there exists a constant $\beta > 0$, depending only on g, such that

$$m_{k,k} \geqslant \beta, \quad \text{for any } k \in \mathbb{Z} - \{0\}.$$

ii) $m_{j,0} = 0$, for any $j \in \mathbb{Z}$.

iii) M_n is an invertible $n \times n$ hermitian matrix.

iv) There exists $\delta > 0$, depending only on g, such that

$$\delta_k = \|G(\psi_k)\|^2_{L^2(\mathbb{T})} > \delta > 0, \quad \text{for all } k \in \mathbb{Z} - \{0\}. \tag{2.3.19}$$

Proof. The proof of items i), ii), and iii) can be found in [62, page 296]. Here, we prove the item iv) only.
If there exists $k \in \mathbb{Z} - \{0\}$ such that $\delta_k = 0$, then

$$g(x)\psi_k(x) - g(x)\int_0^{2\pi} g(y)\psi_k(y)\, dy = 0, \quad \forall x \in \mathbb{T}.$$

Therefore, there exists $C > 0$ such that $g(x)\psi_k(x) = Cg(x)$, $\forall x \in \mathbb{T}$. Thus, $\psi_k(x) = C$, $\forall x \in \omega \subset \mathbb{T}$ which is a contradiction. Hence, $\delta_k > 0$, for all $k \in \mathbb{Z} - \{0\}$, and $\delta_0 = 0$.

On the other hand, for each $k \in \mathbb{Z} - \{0\}$ we have

$$\begin{aligned}
\delta_k &= \frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2 dx - \int_0^{2\pi} |g(x)|^2 \left(\psi_k(x)\overline{\int_0^{2\pi} g(y)\psi_k(y)dy} + \overline{\psi_k(x)}\int_0^{2\pi} g(y)\psi_k(y)dy\right) dx \\
&\quad + \int_0^{2\pi} |g(x)|^2 \left|\int_0^{2\pi} g(y)\psi_k(y)dy\right|^2 dx \\
&\geqslant \frac{1}{2\pi}\int_0^{2\pi} |g(x)|^2 dx - \frac{2}{\sqrt{2\pi}}\int_0^{2\pi} |g(x)|^2 \left|\int_0^{2\pi} g(y)\psi_k(y)dy\right| dx + \left(\int_0^{2\pi} |g(x)|^2 dx\right)\left|\int_0^{2\pi} g(y)\psi_k(y)dy\right|^2 \\
&= \left(\frac{1}{2\pi} - \frac{2}{\sqrt{2\pi}}D_k + D_k^2\right) E,
\end{aligned}$$

where

$$D_k = \left| \int_0^{2\pi} g(y)\psi_k(y)\, dy \right| \quad \text{and} \quad E = \int_0^{2\pi} |g(x)|^2 dx.$$

Consider the function $f : [0, +\infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{2\pi} - \dfrac{2}{\sqrt{2\pi}} x + x^2$. Since $\delta_k > 0$, for all $k \in \mathbb{Z} - \{0\}$, then the value $D_k = \dfrac{1}{\sqrt{2\pi}}$ is never attained for any $k \in \mathbb{Z} - \{0\}$ (see Figure 1).

Using the Riemann-Lebesgue lemma, we get

$$\lim_{|k| \to +\infty} \delta_k \geqslant \lim_{|k| \to +\infty} \left(\frac{1}{2\pi} - \frac{2}{2\pi} D_k + D_k^2 \right) E = \frac{1}{2\pi} E = \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^2 dx =: \delta > 0.$$

□

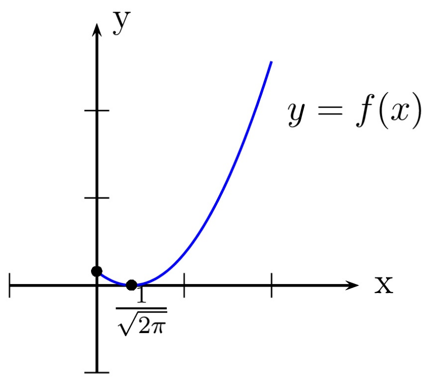

Figure 1 –

Remark 2.3.6. *The sequence of eigenvalues $\{i\lambda_k\}_{k \in \mathbb{Z}}$, with $\lambda_k = k^3 - \alpha k|k|$, satisfies the following properties:*

i) $i\lambda_{-k} = -i\lambda_k$, *and* $\lambda_{-k} = -\lambda_k$, *for all* $k \in \mathbb{Z}$.

ii) $\lim\limits_{|k| \to \infty} |i\lambda_k| = \lim\limits_{|k| \to \infty} |\lambda_k| = \infty$.

iii) $\lim\limits_{|k| \to \infty} |i\lambda_{k+1} - i\lambda_k| = \lim\limits_{|k| \to \infty} |\lambda_{k+1} - \lambda_k| = \infty$ *(asymptotic gap condition).*

iv) Observe that not all the eigenvalues of the sequence $\{i\lambda_k\}_{k \in \mathbb{Z}}$ are distinct, it depends on the value of α. For each $k_1 \in \mathbb{Z}$ set $I(k_1) = \{k \in \mathbb{Z} : \lambda_k = \lambda_{k_1}\}$ and $|I(k_1)| = m(k_1)$, where $|I(k_1)|$ denotes the numbers of elements of $I(k_1)$. Then we have the following properties for $m(k_1)$:

a) $m(k_1) \leqslant 3$, for all $k_1 \in \mathbb{Z}$. This is consequence of the fact that $m(k_1)$ is less or equal to the number of integer roots of the equation $f(x) := x^3 - \alpha x|x| = \beta$, where β is an arbitrary real number, see the format of the curve in Figure 2 below.

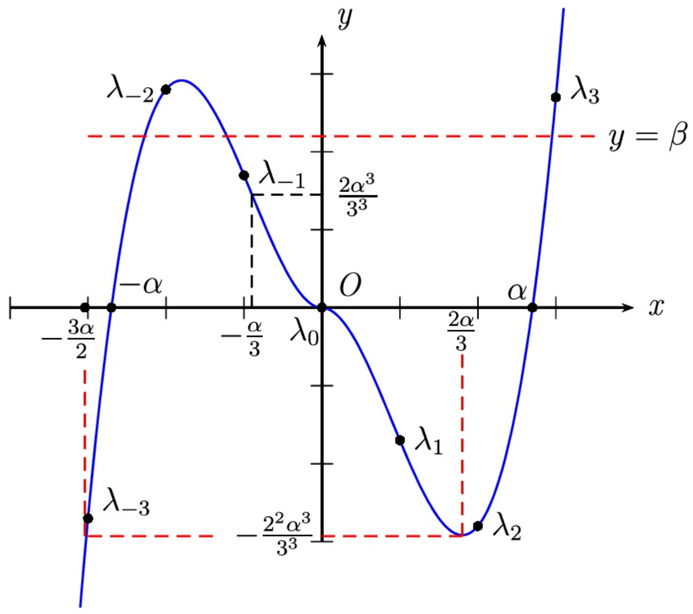

Figure 2 – Eigenvalues

b) Since the sequence of eigenvalues tend to infinity, there exists $k_1^* \in \mathbb{N}$ *such that* $m(k_1) = 1$*, for all* $|k_1| > k_1^*$*. This is a consequence of the fact that the function* $x \longmapsto x^3 - \alpha x|x|$ *is strictly increasing for* $|x|$ *large enough.*

v) If we count only the distinct eigenvalues, we obtain a sequence $\{\lambda_k\}_{k\in\mathbb{I}}$*, where* $\mathbb{I} \subseteq \mathbb{Z}$ *has the property that* $\lambda_{k_1} \neq \lambda_{k_2}$*, for any* $k_1, k_2 \in \mathbb{I}$*, with* $k_1 \neq k_2$*.*

vi) From part a) in iv) we infer that there are only finitely many integers in $\mathbb{I}$*, say,* k_j*,* $j = 1, 2, 3, ..., n$*, such that one can find another integer* $k \neq k_j$ *with* $\lambda_k = \lambda_{k_j}$*. Let*

$$\mathbb{I}_j = \{k \in \mathbb{Z} : k \neq k_j, \lambda_k = \lambda_{k_j}\}, \quad j = 1, 2, 3, ..., n.$$

Then

$$\mathbb{Z} = \mathbb{I} \cup \mathbb{I}_1 \cup \mathbb{I}_2 \cup \cdots \cup \mathbb{I}_n,$$

where the sets in the right are pairwise disjoint.

vii) From part b) in iv), we infer that

$$\gamma := \inf_{\substack{k,n\in\mathbb{I}\\ k\neq n}} |\lambda_k - \lambda_n| = \min_F |\lambda_k - \lambda_n| > 0, \tag{2.3.20}$$

where $F := \left\{n, k \in \mathbb{I} : k \neq n, \text{ and } -1 - \left[\frac{3\alpha}{2}\right] \leqslant k, n \leqslant \left[\frac{3\alpha}{2}\right] + 1\right\}$*, because* $x \longmapsto x^3 - \alpha x|x|$ *is increasing very fast for* $|x| > \left[\frac{3\alpha}{2}\right] + 1$*.*

Now we provide proof of our main theorem regarding controllability of non-homogeneous linear system (2.2.13).

Theorem 2.3.7. *Let $s \geqslant 0$, $\alpha > 0$, and $T > 0$ be given. Then for each $u_0,\ u_1 \in H_p^s(\mathbb{T})$ with $[u_0] = [u_1]$, there exists a function $h \in L^2([0,T]; H_p^s(\mathbb{T}))$ such that the solution $u \in C([0,T]; H_p^s(\mathbb{T}))$ of the non homogeneous system (2.2.13) satisfies $u(x,T) = u_1(x)$, $x \in \mathbb{T}$. Moreover, there exists a positive constant $\nu \equiv \nu(s,g,T) > 0$ such that*

$$\|h\|_{L^2([0,T];H_p^s(\mathbb{T}))} \leqslant \nu(\|u_0\|_{H_p^s(\mathbb{T})} + \|u_1\|_{H_p^s(\mathbb{T})}) \tag{2.3.21}$$

Proof. As discussed above, it is enough to consider $u_0 = 0$. We prove this theorem in five steps.

Step 1. We show that the family $\{e^{i\lambda_k t}\}_{k\in\mathbb{I}}$ is a Riesz basis (see Definition 1.6.4) for the closed span $\overline{\text{span}\{e^{i\lambda_k t} : k \in \mathbb{I}\}} =: \mathrm{H}$ in $L^2([0,T])$, where the set of indices $\mathbb{I}$ was defined in part $v)$ of Remark 2.3.6.

In fact, since $L^2([0,T])$ is a reflexive separable Hilbert space so is H. Observe that the sequence $\{e^{i\lambda_k t}\}_{k\in\mathbb{I}}$ is complete in H (see Definition 1.6.3). On the other hand, from item $iii)$ of Remark 2.3.6, the eigenvalues associated to the linearized Benjamin equation satisfy the assymptotic gap condition which implies

$$\gamma' := \sup_{S\subset\mathbb{I}} \inf_{\substack{k,n\in\mathbb{I}\setminus S\\ k\neq n}} |\lambda_k - \lambda_n| = +\infty,$$

where S runs over the finite subsets of $\mathbb{I}$. Using the generalized Ingham's Theorem 1.6.10 with γ defined by (2.3.20), we obtain that there exist positive constants A and B, such that

$$A\sum_{n\in\mathbb{I}} |b_n|^2 \leqslant \int_0^T |f(t)|^2 dt \leqslant B\sum_{n\in\mathbb{I}} |b_n|^2, \tag{2.3.22}$$

for all functions of the form $f(t) = \sum\limits_{n\in\mathbb{I}} b_n e^{i\lambda_n t}$ with square-summable complex coefficients b_n. In particular, if $b_1, ..., b_N$ are N arbitrary constants, we have

$$A\sum_{n=1}^{N} |b_n|^2 \leqslant \int_0^T \left|\sum_{n\in\mathbb{I}} b_n e^{i\lambda_n t}\right|^2 dt \leqslant B\sum_{n=1}^{N} |b_n|^2.$$

Thus

$$A\sum_{n=1}^{N} |b_n|^2 \leqslant \left\|\sum_{n\in\mathbb{I}} b_n e^{i\lambda_n t}\right\|_H^2 \leqslant B\sum_{n=1}^{N} |b_n|^2 \ \text{ for all } \ b_1, ..., b_N \ \text{ scalars.}$$

Now, applying Theorem 1.6.6 we conclude that $\{e^{i\lambda_k t}\}_{k\in\mathbb{I}}$ is a Riesz basis for the closed span H in $L^2([0,T])$.

Step 2. In this step we show the existence of a unique biorthogonal dual basis $\{q_j\}_{j\in\mathbb{I}} \subseteq H^*$.

Indeed, the Theorem 1.6.6 implies that $\{e^{i\lambda_k t}\}_{k\in\mathbb{I}}$ is a complete Bessel sequence and possesses a biorthogonal system $\{q_j\}_{j\in\mathbb{I}}$ which is also a complete Bessel sequence. Moreover, Corollary 5.22 in [39, page 171] implies that $\{q_j\}_{j\in\mathbb{I}}$ is a basis for H^* which can be identified with H. Therefore, $\{q_j\}_{j\in\mathbb{I}}$ is also a Riesz basis for H.

Thus, by Lemma 5.4 in [39, page 155] part a), we get that $\{e^{i\lambda_k t}\}_{k\in\mathbb{I}}$ is minimal. In consequence, we have the existence of a unique biorthogonal dual basis $\{q_j\}_{j\in\mathbb{I}} \subseteq H^*$ due to exactness (see definition 1.6.7) of the sequence $\{e^{i\lambda_k t}\}_{k\in\mathbb{I}}$ and Lemma 5.4 [39, page 155] part b).

Therefore,

$$(e^{i\lambda_k t}\ ,\ q_j)_H = \int_0^T e^{i\lambda_k t}\overline{q_j}(t)\ dt = \delta_{kj}, \quad \forall\ k,j \in \mathbb{I}. \tag{2.3.23}$$

Step 3. Here we will define an adequate control function h.
In fact, in Step 2, we found a sequence of functions q_j where j is running on the set of indices $\mathbb{I}$. In this step, we will need to define a sequence of functions q_j with j running on $\mathbb{Z}$. Note that $\mathbb{Z} = \mathbb{I}\cup\mathbb{I}_1\cup\mathbb{I}_2\cup\cdots\cup\mathbb{I}_n$, so it is enough to define this sequence for indices in $\mathbb{I}_j$, $j = 1,\cdots,n$. Furthermore, recall from part vi) in Remark 2.3.6 that, each $\mathbb{I}_j$ contains at most 2 integers. Without loss of generality, we may assume that

$$\mathbb{I}_j = \{k_{j,1}, k_{j,2}\}, \quad j = 1,2,3,...,n.$$

We denote k_j by $k_{j,0}$ for any $j = 1,2,3,..,n$. Therefore, for $k_{j,l}$ we define

$$q_{k_{j,l}} = q_{k_{j,0}} = q_{k_j}, \quad \text{for all } j = 1,2,3,...,n, \text{ and } l = 0,1,2.$$

Also it is important to note that

$$\lambda_{k_{j,l}} = \lambda_{k_j}, \quad \text{for all } j = 1,2,3,...,n, \text{ and } l = 0,1,2.$$

For suitable h_j's, consider a control function h defined by

$$h = \sum_{j\in\mathbb{Z}} h_j\ \overline{q_j}(t)\ \psi_j(x). \tag{2.3.24}$$

Note that, using the identity $G(\overline{q_j}(t)\ \psi_j) = \overline{q_j}(t)\ G(\psi_j)$, we obtain

$$\begin{aligned}\int_0^T\int_0^{2\pi} G(h)(x,t)e^{-i\lambda_k(T-t)}\overline{\psi_k}(x)\ dxdt &= \int_0^T\int_0^{2\pi}\left(\sum_{j\in\mathbb{Z}} h_j\overline{q_j}(t)G(\psi_j)(x)\right)e^{-i\lambda_k(T-t)}\overline{\psi_k}(x)\ dxdt\\ &= \sum_{j\in\mathbb{Z}} h_j\int_0^T \overline{q_j}(t)e^{-i\lambda_k(T-t)}\ dt\int_0^{2\pi} G(\psi_j)(x)\overline{\psi_j}(x)\ dx\\ &= \sum_{j\in\mathbb{Z}} h_j e^{-i\lambda_k T} m_{j,k}\int_0^T \overline{q_j}(t)e^{i\lambda_k t}\ dt.\end{aligned}$$

Therefore,

$$\int_0^T \int_0^{2\pi} G(h)(x,t)e^{-i\lambda_k(T-t)}\overline{\psi_k}\, dxdt = \sum_{j\in\mathbb{Z}} h_j e^{-i\lambda_k T} m_{j,k} \int_0^T \overline{q_j}(t)e^{i\lambda_k t}\, dt. \qquad (2.3.25)$$

Step 4. In this step we find h_j's such that h defined by (2.3.24) serves as a required control function. For this, we use the identity (2.3.25) and Lemma 2.3.4 applied to

$$u_1(x) = \sum_{n\in\mathbb{Z}} c_n\psi_n(x) \in H_p^s(\mathbb{T}), \quad \text{with} \quad [u_1] = 0 \quad (c_0 = 0 \quad \text{and} \quad u_0 = 0),$$

to infer that it is enough to consider h_j's satisfying

$$c_k = \sum_{j\in\mathbb{Z}} h_j e^{-i\lambda_k T} m_{j,k} \int_0^T \overline{q_j}(t)e^{i\lambda_k t}\, dt. \qquad (2.3.26)$$

Note that, part $ii)$ of Lemma 2.3.5 implies that equation (2.3.26) is satisfied for $k = 0$, independently of the values of h_j. Moreover, from (2.3.23) we obtain that

$$c_k = h_k m_{k,k} e^{-i\lambda_k T}, \quad \text{if } k \neq k_{j,l}, \quad l = 0,1,2, \quad j = 1,2,3,...,n; \qquad (2.3.27)$$

and for $k = k_{j,l}$, $l = 0,1,2$, $j = 1,2,3,...,n$.

$$\begin{cases} c_{k,0} = \sum_{l=0}^{2} h_{k_{j,l}} \; m_{k_{j,l},k_{j,0}} \; e^{-i\lambda_{k_{j,0}}T} \; ; \\ c_{k,1} = \sum_{l=0}^{2} h_{k_{j,l}} \; m_{k_{j,l},k_{j,1}} \; e^{-i\lambda_{k_{j,1}}T} \; ; \\ c_{k,2} = \sum_{l=0}^{2} h_{k_{j,l}} \; m_{k_{j,l},k_{j,2}} \; e^{-i\lambda_{k_{j,2}}T} \; . \end{cases} \qquad (2.3.28)$$

Therefore, choosing $h_0 = 0$, and using part $iii)$ of Lemma 2.3.5, we obtain

$$h_k = \frac{c_k \; e^{i\lambda_k T}}{m_{k,k}}, \quad \text{if } k \neq 0 \text{ and } k \neq k_{j,l}, \quad l = 0,1,2, \quad j = 1,2,3,...,n; \qquad (2.3.29)$$

and

$$\begin{pmatrix} h_{k_{j,0}} \\ h_{k_{j,1}} \\ h_{k_{j,2}} \end{pmatrix}^{\top} = \begin{pmatrix} c_{k_{j,0}} e^{i\lambda_{k_{j,0}}T} \\ c_{k_{j,1}} e^{i\lambda_{k_{j,1}}T} \\ c_{k_{j,2}} e^{i\lambda_{k_{j,2}}T} \end{pmatrix}^{\top} M_j^{-1}, \quad \text{for } j = 1,2,3,...,n, \qquad (2.3.30)$$

where

$$M_j = \begin{pmatrix} m_{k_{j,0},k_{j,0}} & m_{k_{j,0},k_{j,1}} & m_{k_{j,0},k_{j,2}} \\ m_{k_{j,1},k_{j,0}} & m_{k_{j,1},k_{j,1}} & m_{k_{j,1},k_{j,2}} \\ m_{k_{j,2},k_{j,0}} & m_{k_{j,2},k_{j,1}} & m_{k_{j,2},k_{j,2}} \end{pmatrix}.$$

In this way, we take h_j's given by (2.3.29)-(2.3.30).

Step 5. In this step we prove that the unique function h defined by (2.3.24) belongs to $L^2([0,T]; H_p^s(\mathbb{T}))$, where $h_0 = 0$, and h_k with $k \neq 0$ is defined by (2.3.29) and (2.3.30).

Indeed, identifying H^* with H, and using the Theorem 1.3.4, together with the fact that $\{q_j\}_{j\in\mathbb{I}}$ is a Riesz basis for H we obtain

$$\begin{aligned}
\|h\|^2_{L^2([0,T];H^s_p(\mathbb{T}))} &= \int_0^T \left\| \sum_{k\in\mathbb{Z}} h_k\ \overline{q_k}(t)\ \psi_k(x) \right\|^2_{H^s_p(\mathbb{T})} dt \\
&= \int_0^T \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |h_k\ \overline{q_k}(t)|^2\ dt \\
&= \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \int_0^T |h_k\ q_k(t)|^2\ dt \\
&\leqslant \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} B_2\ |h_k|^2,
\end{aligned} \tag{2.3.31}$$

where B_2 is the constant given by the Bessel type inequality (similar to (2.3.22)) for the Riesz basis $\{q_j\}_{j\in\mathbb{I}}$ in H. From identity (2.3.29) and lemma 2.3.5 part i), we obtain

$$\begin{aligned}
\|h\|^2_{L^2([0,T];H^s_p(\mathbb{T}))} &\leqslant CB_2 \sum_{\substack{k\in\mathbb{Z}-\{0\} \ \ k\neq k_{j,l} \\ l=0,1,2 \ \ \ j=1,2,\ldots,n}} (1+|k|)^{2s} \left| \frac{c_k\ \ e^{i\lambda_k T}}{m_{k,k}} \right|^2 + CB_2 \sum_{j=1}^{n}\sum_{l=0}^{2} (1+|k_{j,l}|)^{2s} |h_{k_{j,l}}|^2 \\
&\leqslant \frac{CB_2}{\beta^2} \sum_{\substack{k\in\mathbb{Z}-\{0\} \ \ k\neq k_{j,l} \\ l=0,1,2 \ \ \ j=1,2,\ldots,n}} (1+|k|)^{2s}\ |c_k|^2 + CB_2 \sum_{j=1}^{n}\sum_{l=0}^{2} (1+|k_{j,l}|)^{2s} |h_{k_{j,l}}|^2.
\end{aligned} \tag{2.3.32}$$

From identity (2.3.30), we obtain that for each $l = 0,1,2$ and $j = 1,2,\ldots,n$

$$|h_{j,l}|^2 \leqslant \sum_{m=0}^{2} |h_{j,m}|^2 \leqslant \left(\sum_{m=0}^{2} \left| c_{k_{j,m}} e^{i\lambda_{k_{j,m}}} \right|^2 \right) \|M_j^{-1}\|^2 \leqslant \|M_j^{-1}\|^2 \sum_{m=0}^{2} |c_{k_{j,m}}|^2,$$

where $\|M_j^{-1}\|$ is the Euclidean norm of the Matrix M_j^{-1}. This implies that for each $l = 0,1,2$ and $j = 1,2,\ldots,n$

$$\begin{aligned}
(1+|k_{j,l}|)^{2s}|h_{j,l}|^2 &\leqslant \sum_{m=0}^{2} \|M_j^{-1}\|^2 \frac{(1+|k_{j,l}|)^{2s}}{(1+|k_{j,m}|)^{2s}} (1+|k_{j,m}|)^{2s} |c_{k_{j,m}}|^2 \\
&\leqslant C(s) \sum_{m=0}^{2} (1+|k_{j,m}|)^{2s} |c_{k_{j,m}}|^2,
\end{aligned} \tag{2.3.33}$$

where $C(s) = \max\limits_{\substack{j=1,2,\ldots,n \\ m,l=0,1,2}} \left\{ \|M_j^{-1}\|^2 \dfrac{(1+|k_{j,l}|)^{2s}}{(1+|k_{j,m}|)^{2s}} \right\}$.

Therefore, using inequalities (2.3.32), and (2.3.33), we get

$$\begin{aligned}
\|h\|^2_{L^2([0,T];H^s_p(\mathbb{T}))} &\leqslant \frac{CB_2}{\beta^2} \sum_{\substack{k\in\mathbb{Z}-\{0\} \ \ k\neq k_{j,l} \\ l=0,1,2 \ \ \ j=1,2,\ldots,n}} (1+|k|)^{2s}\ |c_k|^2 + 3CB_2 C(s) \sum_{j=1}^{n}\sum_{m=0}^{2} (1+|k_{j,m}|)^{2s} |c_{k_{j,m}}|^2 \\
&\leqslant \nu^2 \|u_1\|^2_{H^s_p(\mathbb{T})},
\end{aligned}$$

where $\nu^2 \equiv \nu^2(s,g,T) = \max\left\{ \dfrac{CB_2}{\beta^2}, 3CB_2C(s) \right\}$. This completes the proof of the theorem.

$\square$

Remark 2.3.8. *The dependence of ν with respect to T is implicit in the constant B_2 of the Theorem 1.6.10.*

Remark 2.3.9. *The difficulty in the proof of Theorem 2.3.7 comes from the fact that the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$, with $\lambda_k = k^3 - \alpha k|k|$ associated to the Benjamin equation is not increasing, contrary to the case of the KdV and the BO equations (see the Figure 3 below). The increasing property of the eigenvalues is a necessary condition to apply the Ingham's Theorem 1.6.8. Due to this reason, we followed an approach implemented by Micu, Ortega, Rosier and Zhang in [62] and used a generalized form of the Ingham's inequality (see Theorem 1.6.10).*

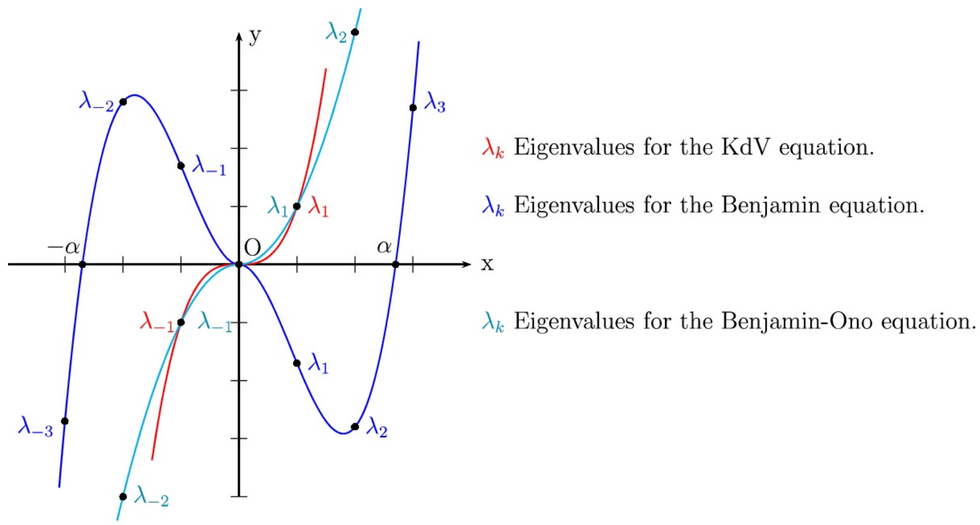

Figure 3 – Eigenvalues

Remark 2.3.10. *Theorem 2.3.7 is strong from the point of view that we do not make restrictions on the time T. It is important to point out that the so-called "asymptotic gap condition" (see condition iii) of Remark 2.3.6 below) that holds for the eigenvalues associated to the Benjamin equation was crucial to obtain the exact controllability for any positive time T.*

Equations (2.3.24), (2.3.29), (2.3.30) and (2.3.21) in Theorem 2.3.7 allow us to get the following corollary, which is fundamental to prove a small data control result for the Benjamin equation (2.0.1).

Corollary 2.3.11. *For $s \geqslant 0$, and $T > 0$ given, there exists a bounded linear operator $\Phi : H^s_p(\mathbb{T}) \times H^s_p(\mathbb{T}) \to L^2([0,T]; H^s_p(\mathbb{T}))$ defined by $\Phi(u_0, u_1) := h$, for all $(u_0,\ u_1) \in$*

$H_p^s(\mathbb{T}) \times H_p^s(\mathbb{T})$ *(see* (2.3.24) *) such that*

$$u_1 = U(T)u_0 + \int_0^T U(T-s)(G(\Phi(u_0, u_1)))(\cdot, s)\, ds, \tag{2.3.34}$$

and

$$\|\Phi(u_0, u_1)\|_{L^2([0,T];H_p^s(\mathbb{T}))} \leqslant \nu\, (\|u_0\|_{H_p^s(\mathbb{T})} + \|u_1\|_{H_p^s(\mathbb{T})}), \tag{2.3.35}$$

where ν depends only on s, T, and g (see (2.0.8)*).*

Also, Lemma 2.3.3 and Corollary 2.3.11 allow us to get the following observability inequality, which is fundamental to obtain a result on exponential asymptotic stabilization with decay rate as large as one desires for the linear system associated to (2.0.2).

Corollary 2.3.12. *Let $T > 0$ be given. There exists $\delta > 0$ such that*

$$\int_0^T \|GU(-\tau)\phi\|^2_{L^2(\mathbb{T})}(\tau)\, d\tau \geqslant \delta^2 \|\phi\|^2_{L^2(\mathbb{T})},$$

for any $\phi \in L^2(\mathbb{T})$.

Before ending this section, we record an observation in order to study the control problem for the Benjamin equation

$$\begin{cases} \partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + 2u\partial_x u = Gh(x,t), & t \in (0,T), \quad x \in \mathbb{T}, \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.3.36}$$

where $u = u(x,t)$ denotes a real-valued function, $\alpha > 0$, and $u_0 \in H_p^s(\mathbb{T})$ with $s \geqslant 0$.

Remark 2.3.13. *As in the non-homogeneous linear case, the "volume" $[u(\cdot,t)]$ for equation* (2.3.36) *continues to be conserved.*

To get some estimates in Bourgain's spaces which will be defined in Chapter 3, the assumption $[u(\cdot,t)] = [u_0] = [u_1] = 0$, where u is the solution of system (2.3.36), can not be omitted. In general, this assumption is not valid for equation (2.3.36). To solve this problem, let u be a solution of equation (2.3.36) with $[u(\cdot,t)] = [u_0] = [u_1] =: \mu$, for all $t \in [0,T]$ and let $v(x,t) = u(x,t) - \mu$. Note that v solves

$$\begin{cases} \partial_t v - \alpha \mathcal{H} \partial_x^2 v - \partial_x^3 v + 2\mu\partial_x v + 2v\partial_x v = Gh(x,t), & t \in (0,T), \quad x \in \mathbb{T} \\ v(x,0) = v_0(x) := u_0(x) - \mu, & x \in \mathbb{T}, \end{cases} \tag{2.3.37}$$

where $\mu \in \mathbb{R}$, $\alpha > 0$, and $v_0 \in H_p^s(\mathbb{T})$ with $s \geqslant 0$ are given and $[v(\cdot,t)] = [v_0] = 0$, $\forall t \in [0,T]$,

Conversely, if v is a solution of equation (2.3.37) then, $u(x,t) = v(x,t) + \mu$ is a solution of system (2.3.36). In consequence, we must resolve the controllability and stabilization problems for the system (2.3.37).

As before, we begin by considering the linear non-homogeneous system

$$\begin{cases} \partial_t v - \alpha \mathcal{H}\partial_x^2 v - \partial_x^3 v + 2\mu\partial_x v = Gh(x,t), & t \in (0,T), \quad x \in \mathbb{T} \\ v(x,0) = v_0(x), & x \in \mathbb{T} \end{cases} \tag{2.3.38}$$

where $\mu \in \mathbb{R}$ is a given number. As the operator $A_\mu : D(A_\mu) \subseteq L^2(\mathbb{T}) \to L^2(\mathbb{T})$, defined by

$$A_\mu \varphi := \alpha \mathcal{H}\partial_x^2 \varphi + \partial_x^3 \varphi - 2\mu \partial_x \varphi \tag{2.3.39}$$

is skew-adjoint, it generates a strongly continuous unitary group $\{U_\mu(t)\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{T})$. Moreover, for $s \in \mathbb{R}$, the family of operators $\{U_\mu(t)\}_{t\in\mathbb{R}}$, defined by

$$\begin{aligned} U_\mu &: \mathbb{R} \to \mathcal{L}(H_p^s(\mathbb{T})) \\ t \to U_\mu(t)\varphi &:= e^{(\alpha\mathcal{H}\partial_x^2 + \partial_x^3 - 2\mu\partial_x)t}\varphi = \left(e^{i(-k^3 - 2\mu k + \alpha k|k|)t}\widehat{\varphi}(k)\right)^{\vee}, \end{aligned} \tag{2.3.40}$$

defines a strongly continuous one-parameter unitary group of contractions on $H_p^s(\mathbb{T})$. Furthermore, $U_\mu(t)$ is an isometry (see [16, Definition 3.4.6]) for all $t \in \mathbb{R}$.

Remark 2.3.14. *For $\mu \in \mathbb{R}$, $s \in \mathbb{R}$, and $v_0 \in H_p^s(\mathbb{T})$ (respectively $v_0 \in H_p^3(\mathbb{T})$) given, there exists a unique solution $v \in C(\mathbb{R}, H_p^s(\mathbb{T}))$ (respectively, $v \in C(\mathbb{R}, H_p^3(\mathbb{T})) \cap C^1(\mathbb{R}, L^2(\mathbb{T}))$) for the homogeneous equation associated to equation (2.3.38). Furthermore, if $0 \leqslant T < \infty$, $s \geqslant 0$, $v_0 \in H_p^s(\mathbb{T})$, and $h \in L^2([0,T]; H_p^s(\mathbb{T}))$ then, there exists a unique mild solution $v \in C([0,T], H_p^s(\mathbb{T}))$ for the system (2.3.38).*

Remark 2.3.15. *For $\mu \in \mathbb{R}$ given, we get an analogous result of Lemma 2.3.4 for the system (2.3.38), just modifying $\lambda_k = k^3 - \alpha k|k|$ by $\lambda_k = k^3 + 2\mu k - \alpha k|k|$. Also, due to the "asymptotic gap condition" that holds for the eigenvalues of the operator A_μ we have an analogous result of Theorem 2.3.7 for the equation (2.3.38), it means that the system (2.3.38) is exactly controllable.*

Thus, similarly to Corollary 2.3.11, for $\mu \in \mathbb{R}$, $s \geqslant 0$ and any $T > 0$ given, there exists a bounded linear operator $\Phi_\mu : H_p^s(\mathbb{T}) \times H_p^s(\mathbb{T}) \to L^2([0,T]; H_p^s(\mathbb{T}))$ defined by $h_\mu = \Phi_\mu(v_0, v_1)$, for all $(v_0, v_1) \in H_p^s(\mathbb{T}) \times H_p^s(\mathbb{T})$ such that

$$v_1 = U_\mu(T)v_0 + \int_0^T U_\mu(T-s)(G(\Phi_\mu(v_0,v_1)))(\cdot, s)\, ds, \tag{2.3.41}$$

and

$$\|\Phi_\mu(v_0,v_1)\|_{L^2([0,T];H_p^s(\mathbb{T}))} \leqslant \nu \left(\|v_0\|_{H_p^s(\mathbb{T})} + \|v_1\|_{H_p^s(\mathbb{T})}\right), \tag{2.3.42}$$

where ν depends only on s, T, and g. Therefore, for any $T > 0$ the following observability inequality holds for some $\delta > 0$

$$\int_0^T \|G^* U_\mu(\tau)^*(\phi(x))\|^2_{L_p^2(\mathbb{T})}\, d\tau \geqslant \delta^2\, \|\phi\|^2_{L^2(\mathbb{T})}, \quad \text{for any } \phi \in L^2(\mathbb{T}), \tag{2.3.43}$$

2.4 Stabilization of the Linear Benjamin Equation

Now we move to study the stabilization of the linearized Benjamin equation. From the observation made in the final part of Section 2.3, it is enough to study the stabilization problem for the IVP (2.3.38) in $H_0^s(\mathbb{T})$, with $s \geqslant 0$ (see (1.3.1)). Thus, we consider the system

$$\begin{cases} \partial_t u - \alpha\mathcal{H}\partial_x^2 u - \partial_x^3 u + 2\mu\partial_x u = Ku, & t > 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.4.1}$$

where $u = u(x,t)$ is a real valued function, $\alpha > 0$, and K is a bounded linear operator on $H_p^s(\mathbb{T})$. In view of the discussion at the end of Section 2.3, we assume that μ is a real given number and $[u(\cdot,t)] = 0$, for all $t \geqslant 0$.

In this section we prove that there exists a feedback control law such that the system (2.4.1) is exponentially asymptotically stable when t goes to infinity. First, we prove that the system (2.4.1) is globally well-posed in $H_0^s(\mathbb{T})$, for $s \geqslant 0$.

Theorem 2.4.1. *Let $u_0 \in H_0^3(\mathbb{T})$, then the IVP (2.4.1) has a unique solution*

$$u \in C([0,\infty); H_0^3(\mathbb{T})) \cap C^1([0,\infty); L_0^2(\mathbb{T})).$$

Moreover, if $u_0 \in H_0^s(\mathbb{T})$, then we have that $u \in C([0,\infty); H_0^s(\mathbb{T}))$, for all $s \geqslant 0$.

Proof. We know that the operator $A_\mu = \alpha\mathcal{H}\partial_x^2 + \partial_x^3 - 2\mu\partial_x$, is an infinitesimal generator of a C_0-semigroup $\{U_\mu(t)\}_{t\geqslant 0}$ over $H_0^s(\mathbb{T})$. Also we know that K is a bounded linear operator on $H_0^s(\mathbb{T})$. From Theorem 1.1 in [71, page 76], we get that the operator $A_\mu + K$, which is a perturbation of A_μ by a bounded linear operator, is an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ on $H_0^s(\mathbb{T})$. It is important to observe that $A_\mu^* = -A_\mu$ is the infinitesimal generator of a C_0-semigroup $\{U_\mu(t)^*\}_{t\geqslant 0}$, with domain of A_μ^* dense in $L_0^2(\mathbb{T})$, and $U_\mu(t)^* = U_\mu(-t)$. □

In order to stabilize the equation (2.4.1) in $H_0^s(\mathbb{T})$, we employ a simple feedback control law, $Ku = -GG^*u$, where G is defined as in (2.0.9). The following theorem says that the trivial solution, u=0, of equation (2.4.1) with this feedback law is exponentially asymptotically stable when t goes to infinity.

Theorem 2.4.2. *Let $\alpha > 0$, $\mu \in \mathbb{R}$, g as in (2.0.8), and $s \geqslant 0$ be given. There exist positive constants $M = M(\alpha,\mu,g,s)$ and $\gamma = \gamma(g)$ such that for any $u_0 \in H_0^s(\mathbb{T})$, the unique solution u of (2.4.1) with $K = -GG^*$ satisfies*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant Me^{-\gamma t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \textit{for all } t \geqslant 0. \tag{2.4.2}$$

Proof. We prove this theorem in five steps.

Step 1. First we prove the case $s = 0$. In this case we use a procedure similar to [55, 80].

Let $T > 0$ be given and assume $u_0 \in H_0^3(\mathbb{T})$. Theorem 2.4.1 implies that the solution u of the IVP

$$\begin{cases} \partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u + 2\mu \partial_x u = -GG^* u, & t > 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.4.3}$$

satisfies $u \in C([0,\infty); H_0^3(\mathbb{T})) \cap C^1([0,\infty); L_0^2(\mathbb{T}))$. It means $u(\cdot,t) \in H_0^3(\mathbb{T})$, for all $t \geqslant 0$ and in particular, for $t = T$. Now we consider the IVP

$$\begin{cases} \partial_t w - \alpha \mathcal{H} \partial_x^2 w - \partial_x^3 w + 2\mu \partial_x w = Gh, & t \in (0,T), \quad x \in \mathbb{T} \\ w(x,0) = 0, & x \in \mathbb{T} \\ w(x,T) = u(x,T), & x \in \mathbb{T}. \end{cases} \tag{2.4.4}$$

Remark 2.3.15 implies that there exists a unique $h \in L^2([0,T]; H_0^3(\mathbb{T}))$ such that the unique solution $w \in C([0,\infty); H_0^3(\mathbb{T})) \cap C^1([0,\infty); L_0^2(\mathbb{T}))$ of equation (2.4.4) satisfies $w(x,T) = u(x,T)$ for all $x \in \mathbb{T}$, and there exists a positive constant $\nu = \nu(g,T)$ such that

$$\|h\|_{L^2([0,T];H_0^3(\mathbb{T}))} \leqslant \nu \|u(x,T)\|_{H_0^3(\mathbb{T})}.$$

On the other hand, note that $u_0 \in H_0^3(\mathbb{T}) \subset L_0^2(\mathbb{T})$, therefore Theorem 2.4.1 implies that $u \in C([0,\infty); L_0^2(\mathbb{T}))$ is a solution of equation (2.4.3). Furthermore, Remark 2.3.15 implies that $h \in L^2([0,T]; L_0^2(\mathbb{T}))$ and the solution $w \in C([0,\infty); L_0^2(\mathbb{T}))$ of equation (2.4.4) satisfies $w(x,T) = u(x,T)$, for all $x \in \mathbb{T}$, with

$$\|h\|_{L^2([0,T];L_0^2(\mathbb{T}))} \leqslant \nu \, \|u(x,T)\|_{L_0^2(\mathbb{T})}. \tag{2.4.5}$$

Now, multiplying the first equation in (2.4.3) by $\bar{u}$ and integrating with respect x, it follows that

$$\int_{\mathbb{T}} \partial_t u \, \bar{u} dx - \int_{\mathbb{T}} \alpha \mathcal{H} \partial_x^2 u \, \bar{u} dx - \int_{\mathbb{T}} \partial_x^3 u \, \bar{u} dx + \int_{\mathbb{T}} 2\mu \partial_x u \, \bar{u} dx = \int_{\mathbb{T}} -GG^* u \, \bar{u} dx. \tag{2.4.6}$$

Note that

$$\frac{1}{2}\frac{d}{dt}(u,u)_{L_0^2(\mathbb{T})} = (u_t, u)_{L_0^2(\mathbb{T})}. \tag{2.4.7}$$

Therefore, using the Parseval's identity, we get

$$-\int_{\mathbb{T}} \alpha \mathcal{H} \partial_x^2 u \, \bar{u} dx = -2\pi\alpha \sum_{k=-\infty}^{\infty} i \operatorname{sgn}(k) k^2 \widehat{u}(k) \widehat{u}(-k).$$

It is easy to show that the partial sums satisfy

$$\lim_{N\to\infty} S_N := \lim_{N\to\infty} \sum_{k=-N}^{N} i \operatorname{sgn}(k) k^2 \widehat{u}(k) \widehat{u}(-k) = 0.$$

Thus,

$$-\int_{\mathbb{T}} \alpha \mathcal{H} \partial_x^2 u \, \bar{u} dx = 0. \tag{2.4.8}$$

Now, using integration by parts, we obtain

$$-\int_{\mathbb{T}} \partial_x^3 u\ \bar{u}\ dx = \frac{1}{2}\left[(\partial_x u)^2 \Big|_{x=0}^{x=2\pi} \right] = 0, \tag{2.4.9}$$

and

$$\int_{\mathbb{T}} 2\mu\ \partial_x u\ \bar{u}\ dx = \mu \left[u^2 \Big|_{x=0}^{x=2\pi} \right] = 0. \tag{2.4.10}$$

Hence, from (2.4.6)-(2.4.10) and the fact that the operator G is self-adjoint on $L_0^2(\mathbb{T})$ we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{L_0^2(\mathbb{T})} \right) = -\|Gu(\cdot,t)\|^2_{L_0^2(\mathbb{T})}, \quad \text{for all } t > 0. \tag{2.4.11}$$

Integrating (2.4.11) with respect to the variable t from 0 and T, we get

$$\frac{1}{2}\|u(T)\|^2_{L_0^2(\mathbb{T})} - \frac{1}{2}\|u_0\|^2_{L_0^2(\mathbb{T})} = -\|Gu\|^2_{L^2((0,T);L_0^2(\mathbb{T}))}. \tag{2.4.12}$$

On the other hand, multiplying the first equation in (2.4.4) by $\bar{u}$ and integrating with respect to the $x-$variable, we get

$$\int_{\mathbb{T}} \partial_t w\ \bar{u}\ dx - \int_{\mathbb{T}} \left[\alpha\mathcal{H}\partial_x^2 w\ \bar{u} + \partial_x^3 w\ \bar{u}\ - 2\mu\partial_x w\ \bar{u} \right]\ dx = \int_{\mathbb{T}} Gh\ \bar{u}dx, \tag{2.4.13}$$

for all $0 < t < T$. Using integration by parts in the second term of (2.4.13) we get

$$\int_{\mathbb{T}} \partial_t w\ \bar{u}\ dx - \int_{\mathbb{T}} w\ \overline{\left[-\alpha\mathcal{H}\partial_x^2 u - \partial_x^3 u\ + 2\mu\ \partial_x u \right]}\ dx = \int_{\mathbb{T}} Gh\ \bar{u}dx, \tag{2.4.14}$$

for all $0 < t < T$. Integrating (2.4.14) with respect to t from 0 and T, and using integration by parts, we obtain

$$\int_{\mathbb{T}} w(T)\ \bar{u}(T)dx - \int_0^T \int_{\mathbb{T}} w\ \overline{\left[\partial_t u - \alpha\mathcal{H}\partial_x^2 u - \partial_x^3 u + 2\mu\partial_x u \right]} dxdt = \int_0^T \int_{\mathbb{T}} Gh\ \bar{u}dxdt.$$

Observe that u is a solution of equation (2.4.3). Thus

$$\int_{\mathbb{T}} u^2(T)dx + \int_0^T \int_{\mathbb{T}} w\ \overline{(GG^* u)} dxdt = \int_0^T \int_{\mathbb{T}} Gh\ \bar{u}dxdt.$$

Using that the solution u is real, the operator G is self-adjoint on $L_0^2(\mathbb{T})$, and the Cauchy-Schwartz inequality, we get

$$\|u(\cdot,T)\|^2_{L_0^2(\mathbb{T})} \leqslant \|h - Gw\|_{L^2((0,T);L_0^2(\mathbb{T}))}\ \|Gu\|_{L^2((0,T);L_0^2(\mathbb{T}))}. \tag{2.4.15}$$

From (2.4.5), we infer

$$\|h - Gw\|_{L^2((0,T);L_0^2(\mathbb{T}))} \leqslant \nu\|u(T)\|_{L_0^2(\mathbb{T})} + c\left(\int_0^T \|w(\cdot,t)\|^2_{L_0^2(\mathbb{T})}\ dt \right)^{\frac{1}{2}}. \tag{2.4.16}$$

Also, observe that

$$
\begin{aligned}
\|w(\cdot,t)\|^2_{L^2_0(\mathbb{T})} &\leqslant \left\| \int_0^t U_\mu(t-t')Gh(\cdot,t')\,dt' \right\|^2_{L^2_0(\mathbb{T})} \\
&\leqslant c^2 T \left[\left(\int_0^T \|h(\cdot,t')\|^2_{L^2_0(\mathbb{T})}\;\; dt' \right)^{\frac{1}{2}} \right]^2 \\
&\leqslant c^2 T \|h\|^2_{L^2((0,T);L^2_0(\mathbb{T}))} \\
&\leqslant c^2 T \nu^2 \|u(T)\|^2_{L^2_0(\mathbb{T})}.
\end{aligned}
\tag{2.4.17}
$$

It follows from (2.4.16)-(2.4.17) that

$$
\|h - Gw\|_{L^2((0,T);L^2_0(\mathbb{T}))} \leqslant c_{g,T}\; \|u(T)\|_{L^2_0(\mathbb{T})}, \tag{2.4.18}
$$

where $c_{g,T} = max\{\nu,\ c^2 T\nu\}$.

Thus, from (2.4.15) and (2.4.18), we have

$$
\|u(\cdot,T)\|^2_{L^2_0(\mathbb{T})} \leqslant c_{g,T}\; \|u(T)\|_{L^2_0(\mathbb{T})} \cdot \|Gu\|_{L^2((0,T);L^2_0(\mathbb{T}))},
$$

which implies that

$$
-\|Gu\|^2_{L^2((0,T);L^2_0(\mathbb{T}))} \leqslant -\frac{1}{c^2_{g,T}}\; \|u(\cdot,T)\|^2_{L^2_0(\mathbb{T})}. \tag{2.4.19}
$$

From identity (2.4.12) and the inequality (2.4.19), we obtain

$$
\left(1 + \frac{2}{c^2_{g,T}}\right) \|u(T)\|^2_{L^2_0(\mathbb{T})} \leqslant \|u_0\|^2_{L^2_0(\mathbb{T})}.
$$

Thus, there exists $\rho_{g,T} = \rho \in (0,1)$ such that $\|u(T)\|^2_{L^2_0(\mathbb{T})} \leqslant \rho \|u_0\|^2_{L^2_0(\mathbb{T})}$, for any $T > 0$. Moreover, we can repeat this estimate on successive intervals $[(n-1)T, nT]$ to get

$$
\|u(x,nT)\|^2_{L^2_0(\mathbb{T})} \leqslant \rho^n\; \|u_0\|^2_{L^2_0(\mathbb{T})}, \quad \text{for any} \quad T > 0,\ n \geqslant 1, \tag{2.4.20}
$$

where u is the solution of (2.4.3), and $\rho = \rho_{g,T} \in (0,1)$.

In particular, fixing $T > 0$ we obtain that for any $t \geqslant 0$, there exists $n \in \mathbb{N}$ such that $nT \leqslant t \leqslant (n+1)T$. From (2.4.11) we know that the function $t \to \|u(\cdot,t)\|^2_{L^2_0(\mathbb{T})}$, with $t \geqslant 0$ is decreasing. From (2.4.20) there exists $\rho = \rho_g \in (0,1)$ such that

$$
\begin{aligned}
\|u(\cdot,t)\|^2_{L^2_0(\mathbb{T})} &\leqslant \|u(\cdot,nT)\|^2_{L^2_0(\mathbb{T})} \\
&\leqslant \rho^n\; \|u_0\|^2_{L^2_0(\mathbb{T})}, \quad \text{for all} \;\; n \geqslant 1.
\end{aligned}
\tag{2.4.21}
$$

It is easy to show that if

$$
0 < \gamma \leqslant -\frac{ln(\rho)}{2T}, \quad \text{and} \quad M \geqslant e^{\gamma T}, \tag{2.4.22}
$$

one has

$$\rho^n \leqslant M^2\, e^{-2\gamma\, t}, \quad \text{for all } \ n \in \mathbb{N}. \tag{2.4.23}$$

In fact, inequality (2.4.23) is equivalent to $2\gamma t \leqslant \ln(M^2) - n\ln(\rho)$, for all $n \in \mathbb{N}$. As $2\,\gamma\, t \leqslant 2\,\gamma\,(n+1)\,T$, then it is enough to prove that

$$2\,\gamma\,(n+1)\,T \leqslant \ \ln(M^2) - n\ln(\rho), \quad \text{for all } \ n \in \mathbb{N}.$$

It is easy to verify that the last inequality is true if (2.4.22) holds. Therefore, (2.4.21) and (2.4.23) yield

$$\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant M\, e^{-\gamma\, t}\|u_0\|_{L^2_0(\mathbb{T})}, \quad \text{for all } t \geqslant 0, \tag{2.4.24}$$

and we get the result for smooth initial data in $H^3_0(\mathbb{T})$. We complete the proof for s=0 using density arguments.

Step 2. Here we consider $s = 3$. In this case we use a similar argument as in Proposition 2.3 of [52]. Let u be the solution of equation (2.4.3) with initial data $u_0 \in H^3_0(\mathbb{T})$, then

$$u \in C([0,\infty); H^3_0(\mathbb{T})) \cap C^1([0,\infty); L^2_0(\mathbb{T})). \tag{2.4.25}$$

Since $H^3_0(\mathbb{T}) \subset L^2_0(\mathbb{T})$, then from the $s = 0$ case we have that there exist positive constants M_1 and $\gamma = \gamma(g)$ independent of u_0, such that

$$\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant M_1 e^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0. \tag{2.4.26}$$

On the other hand, differenciating the equation in (2.4.3) with respect to t, we obtain

$$\partial_t(\partial_t u) - \alpha\mathcal{H}\partial_x^2(\partial_t u) - \partial_x^3(\partial_t u) + 2\mu\partial_x(\partial_t u) = -GG^*(\partial_t u). \tag{2.4.27}$$

Therefore, $w := \partial_t u \in C([0,+\infty); L^2_0(\mathbb{T}))$ is the unique solution of

$$\begin{cases} \partial_t w - \alpha\mathcal{H}\partial_x^2 w - \partial_x^3 w + 2\mu\partial_x w = -GG^* w, & t > 0, \quad x \in \mathbb{T}, \\ w(x,0) = w_0 = \partial_t u(x,0) = \alpha\mathcal{H}\partial_x^2 u_0 + \partial_x^3 u_0 - 2\mu\partial_x u_0 - GG^* u_0 \in L^2_0(\mathbb{T}), & x \in \mathbb{T}, \end{cases} \tag{2.4.28}$$

Again, from the case $s = 0$ applied to equation (2.4.28), there exist positive constants $M_1 = M_1(g)$, and $\gamma = \gamma(g)$, independent of w_0, such that

$$\|\partial_t u(\cdot,t)\|_{L^2_0(\mathbb{T})} = \|w(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant M_1 e^{-\gamma t}\|w_0\|_{L^2_0(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0. \tag{2.4.29}$$

Note that, for each $t \geqslant 0$

$$\|u(\cdot,t)\|_{H^3_0(\mathbb{T})} \leqslant c_0 \left(\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})}\right). \tag{2.4.30}$$

To estimate the term $\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})}$ observe that from equation (2.4.3)

$$\partial_x^3 u(\cdot,t) = w - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + GG^* u. \tag{2.4.31}$$

Thus, for each $t \geqslant 0$

$$\begin{aligned}\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant \|w(\cdot,t)\|_{L^2_0(\mathbb{T})} + \alpha\|\mathcal{H}\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})} + 2|\mu|\|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})}\\ + \|GG^* u(\cdot,t)\|_{L^2_0(\mathbb{T})}.\end{aligned} \tag{2.4.32}$$

Using Gagliardo-Niremberg inequality (see the Theorem 3.70 in [8]) and Cauchy-Schwartz inequality with ϵ, we have

$$\begin{aligned}2|\mu|\|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})} &\leqslant 2|\mu|\sqrt{2\pi}\|\partial_x u(\cdot,t)\|_{L^\infty(\mathbb{T})}\\ &\leqslant 2|\mu|\sqrt{2\pi}\ c_1\|\partial_x^3 u(\cdot,t)\|^{\frac12}_{L^2_0(\mathbb{T})}\|u(\cdot,t)\|^{\frac12}_{L^2_0(\mathbb{T})}\\ &= c_\mu\epsilon\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \frac{c_\mu}{4\epsilon}\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})},\end{aligned} \tag{2.4.33}$$

where $c_\mu = 2|\mu|\sqrt{2\pi}\ c_1$. Also, using that $\mathcal{H}$ is an isometry in $L^2_0(\mathbb{T})$, integration by parts and Cauchy-Schwartz inequality with ϵ we have

$$\begin{aligned}\|\mathcal{H}\partial_x^2 u(\cdot,t)\|^2_{L^2_0(\mathbb{T})} &= \int_{\mathbb{T}} \partial_x^2 u(x,t)\ \overline{\partial_x^2 u(x,t)}\ dx\\ &\leqslant \|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})}\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})}\\ &\leqslant \epsilon\|\partial_x u(\cdot,t)\|^2_{L^2_0(\mathbb{T})} + \frac{1}{4\epsilon}\|\partial_x^3 u(\cdot,t)\|^2_{L^2_0(\mathbb{T})}.\end{aligned}$$

Therefore,

$$\|\mathcal{H}\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant c_2\left(\epsilon^{\frac12}\|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \frac{1}{2\epsilon^{\frac12}}\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})}\right). \tag{2.4.34}$$

Using inequality (2.4.33) we obtain from (2.4.34) that

$$\|\mathcal{H}\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant c_3\ \epsilon^{\frac32}\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \frac{c_4}{\epsilon^{\frac12}}\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})}, \tag{2.4.35}$$

where $c_3 = c_2\sqrt{2\pi}\ c_1$, and $c_4 = \dfrac{c_2 c_1\sqrt{2\pi}}{4} + \dfrac{c_2}{2}$. Thus, from inequalities (2.4.29), (2.4.32), (2.4.33) and (2.4.35) we obtain

$$\begin{aligned}\left(1 - \frac{c_4\alpha}{\epsilon^{\frac12}} - \frac{c_\mu}{4\epsilon}\right)\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})} &\leqslant M_1 e^{-\gamma t}\|w_0\|_{L^2_0(\mathbb{T})} + \left(\alpha c_3\epsilon^{\frac32} + c_\mu\epsilon + c_g^2\right)\|u(\cdot,t)\|_{L^2_0(\mathbb{T})}\\ &\leqslant M_1 e^{-\gamma t}\|w_0\|_{L^2_0(\mathbb{T})} + \left(\alpha c_3\epsilon^{\frac32} + c_\mu\epsilon + c_g^2\right)M_1 e^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})}.\end{aligned}$$

Therefore, taking $\epsilon > 0$ large enough such that $1 - \dfrac{c_4\ \alpha}{\epsilon^{\frac12}} - \dfrac{c_\mu}{4\epsilon} > 0$ we infer that there exists a positive constant $c = c_{\alpha,\mu,g}$, independent of u_0, and w_0 such that

$$\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant\ c\ M_1 e^{-\gamma t}\left(\|w_0\|_{L^2_0(\mathbb{T})} + \|u_0\|_{L^2_0(\mathbb{T})}\right). \tag{2.4.36}$$

Also, note that

$$\begin{aligned}\|w_0\|_{L^2_0(\mathbb{T})} &\leqslant \alpha\ \|\partial_x^2 u_0\|_{L^2_0(\mathbb{T})} + \|\partial_x^3 u_0\|_{L^2_0(\mathbb{T})} + 2|\mu|\ \|\partial_x u_0\|_{L^2_0(\mathbb{T})} + c_g^2\|u_0\|_{L^2_0(\mathbb{T})} \\ &\leqslant c_6\ \|u_0\|_{L^2_0(\mathbb{T})},\end{aligned} \tag{2.4.37}$$

where $c_6 = c_5(\alpha + 1 + 2|\mu| + c_g^2)$. Now, from (2.4.36)-(2.4.37) we have

$$\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant M_2 e^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})}, \tag{2.4.38}$$

where $M_2 = c\ M_1(c_6+1)$. From expstabilizationinL21, (2.4.30) and (2.4.38), we get

$$\begin{aligned}\|u(\cdot,t)\|_{H^3_0(\mathbb{T})} &\leqslant c_0\ \left(M_1 e^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})} + M_2 e^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})}\right) \\ &\leqslant c_0\ (M_1+M_2)\, e^{-\gamma t}\ c_5\ \|u_0\|_{H^3_0(\mathbb{T})} \\ &\leqslant M\ e^{-\gamma t}\|u_0\|_{H^3_0(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0,\end{aligned} \tag{2.4.39}$$

where $M = M(\alpha,\mu,g) = c_0\ (M_1+M_2)\ c_5$, and $\gamma = \gamma(g)$ are positive constants independent of u_0.

Step 3. Using induction and similar arguments as above, we prove that inequality (2.4.2) holds for $s = 3n$, with $n \in \mathbb{N}$. In fact:

Inductive Base: for $n = 0$ or $n = 1$, the inequality (2.4.2) follows from steps 1, 2.

Inductive Hypothesis: Assume that inequality (2.4.2) is true for $s = 3n$, where $n = 0, 1, 2, ..., k-1, k$. Lets prove the result for $n = k+1$.

By the inductive hypothesis ($n = k$) we know that for $u_0 \in H_0^{3k}(\mathbb{T})$, there exists positive constants $M_1 = M_1(\alpha,\mu,g,k)$ and $\gamma = \gamma(g)$, such that the unique solution u of (2.4.1) with $K = -GG^*$ and $j = k$, satisfies

$$\|u(\cdot,t)\|_{H_0^{3k}(\mathbb{T})} \leqslant M_1 e^{-\gamma t}\|u_0\|_{H_0^{3k}(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0. \tag{2.4.40}$$

On the other hand, for $n = k+1$, we have that $u_0 \in H_0^{3(k+1)}(\mathbb{T}) \hookrightarrow H_0^{3k}(\mathbb{T}) \hookrightarrow L^2_0(\mathbb{T})$, with dense embedding. Then, the solution u of (2.4.1) with $K = -GG^*$ and $j = k+1$, satisfies

$$u \in C([0,\infty); H_0^{3(k+1)}(\mathbb{T})) \cap C^1([0,\infty); H_0^{3k}(\mathbb{T})),$$

and (2.4.40). Thus, defining $w := \partial_t u$, we have that $w \in C((0,\infty); H_0^{3k}(\mathbb{T}))$ is the solution of

$$\begin{cases} \partial_t w - \alpha\mathcal{H}\partial_x^2 w - \partial_x^3 w + 2\mu\partial_x w = -GG^* w, & t>0, \ \ x \in \mathbb{T} \\ \partial_x^m w(2\pi,t) = \partial_x^m w(0,t), & t>0, \quad m = 0,1,...,3k-1 \\ w(x,0) = w_0(x) = \alpha\mathcal{H}\partial_x^2 u_0 + \partial_x^3 u_0 - 2\mu\partial_x u_0 - GG^* u_0 \in H_0^{3k}(\mathbb{T}), & x \in \mathbb{T}. \end{cases}$$

From inequality (2.4.40), we infer that w satisfies

$$\|\partial_t u(\cdot,t)\|_{H_0^{3k}(\mathbb{T})} = \|w(\cdot,t)\|_{H_0^{3k}(\mathbb{T})} \leqslant M_1 e^{-\gamma t}\|w_0\|_{H_0^{3k}(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0. \tag{2.4.41}$$

It is easy to prove that

$$\|u(\cdot,t)\|_{H_0^{3(k+1)}(\mathbb{T})} \leqslant c_{k+1} \left(\|u(\cdot,t)\|_{L_0^2(\mathbb{T})} + \|\partial_x^{3k}(\partial_x^3 u(\cdot,t))\|_{L_0^2(\mathbb{T})}\right). \tag{2.4.42}$$

From step 2 there exist positive constants $M_2 = M_2(g)$, $\gamma = \gamma(g)$, such that u satisfies

$$\|u(\cdot,t)\|_{L_0^2(\mathbb{T})} \leqslant M_2 e^{-\gamma t}\|u_0\|_{L_0^2(\mathbb{T})} \leqslant M_2 e^{-\gamma t}\|u_0\|_{H_0^{3(k+1)}(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0. \tag{2.4.43}$$

Equation (2.4.31) yields

$$\partial_x^{3k}(\partial_x^3 u) = \partial_x^{3k} w - \alpha \mathcal{H}\partial_x^2(\partial_x^{3k} u) + 2\mu\partial_x(\partial_x^{3k} u) + GG^*(\partial_x^{3k} u).$$

Thus, using the estimates in (2.4.41), (2.4.33), (2.4.35) we obtain

$$\begin{aligned}
\|\partial_x^{3k}(\partial_x^3 u)\|_{L_0^2(\mathbb{T})} \leqslant{}& \|\partial_x^{3k} w\|_{L_0^2(\mathbb{T})} + \alpha\|\mathcal{H}\partial_x^2(\partial_x^{3k} u)\|_{L_0^2(\mathbb{T})} + 2|\mu|\|\partial_x(\partial_x^{3k} u)\|_{L_0^2(\mathbb{T})} \\
&+ \|GG^*(\partial_x^{3k} u)\|_{L_0^2(\mathbb{T})} \\
\leqslant{}& \|w\|_{H_0^{3k}(\mathbb{T})} + \alpha\ c_3\ \epsilon^{\frac{3}{2}}\|\partial_x^{3k} u\|_{L_0^2(\mathbb{T})} + \frac{c_4\ \alpha}{\epsilon^{\frac{1}{2}}}\|\partial_x^3\partial_x^{3k} u\|_{L_0^2(\mathbb{T})} \\
&+ c_\mu\epsilon\|\partial_x^{3k} u\|_{L_0^2(\mathbb{T})} + \frac{c_\mu}{4\epsilon}\|\partial_x^3(\partial_x^{3k} u)\|_{L_0^2(\mathbb{T})} + c_g^2\|\partial_x^{3k} u\|_{L_0^2(\mathbb{T})} \\
\leqslant{}& \|w\|_{H_0^{3k}(\mathbb{T})} + \alpha\ c_3\ \epsilon^{\frac{3}{2}}\|u\|_{H_0^{3k}(\mathbb{T})} + \frac{c_4\ \alpha}{\epsilon^{\frac{1}{2}}}\|\partial_x^3\partial_x^{3k} u\|_{L_0^2(\mathbb{T})} \\
&+ c_\mu\epsilon\|u\|_{H_0^{3k}(\mathbb{T})} + \frac{c_\mu}{4\epsilon}\|\partial_x^3(\partial_x^{3k} u)\|_{L_0^2(\mathbb{T})} + c_g^2\|u\|_{H_0^{3k}(\mathbb{T})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(1 - \frac{c_4\ \alpha}{\epsilon^{\frac{1}{2}}} - \frac{c_\mu}{4\epsilon}\right)\|\partial_x^{3k}(\partial_x^3 u)\|_{L_0^2(\mathbb{T})} \leqslant{}& \|w\|_{H_0^{3k}(\mathbb{T})} + (\alpha\ c_3\ \epsilon^{\frac{3}{2}} + c_\mu\ \epsilon + c_g^2)\|u\|_{K_0^{3k}(\mathbb{T})} \\
\leqslant{}& M_1 e^{-\gamma t}\|w_0\|_{H_0^{3k}(\mathbb{T})} + (\alpha\ c_3\ \epsilon^{\frac{3}{2}} + c_\mu\ \epsilon + c_g^2) M_1 e^{-\gamma t}\|u_0\|_{H_0^{3k}(\mathbb{T})}
\end{aligned}$$

Taking ϵ large enough such that $1 - \dfrac{c_4\ \alpha}{\epsilon^{\frac{1}{2}}} - \dfrac{c_\mu}{4\epsilon} > 0$ we obtain that there exists a positive constant $c_{\alpha,\mu,g}$, independent of u_0, and w_0 such that

$$\begin{aligned}
\|\partial_x^{3k}(\partial_x^3 u)\|_{L_0^2(\mathbb{T})} \leqslant{}& c_{\alpha,\mu,g}\ M_1 e^{-\gamma t}\left(\|w_0\|_{H_0^{3k}(\mathbb{T})} + \|u_0\|_{H_0^{3k}(\mathbb{T})}\right) \\
\leqslant{}& c_{\alpha,\mu,g}\ M_1 e^{-\gamma t}\left(c_6\|u_0\|_{H_0^{3(k+1)}(\mathbb{T})} + \|u_0\|_{H_0^{3(k+1)}(\mathbb{T})}\right) \\
\leqslant{}& M_3 e^{-\gamma t}\|u_0\|_{H_0^{3(k+1)}(\mathbb{T})}.
\end{aligned} \tag{2.4.44}$$

From (2.4.42), (2.4.43), and (2.4.44) we have that

$$\|u(\cdot,t)\|_{H_0^{3(k+1)}(\mathbb{T})} \leqslant M e^{-\gamma t}\|u_0\|_{H_0^{3(k+1)}(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0, \tag{2.4.45}$$

where $M = M(\alpha,\mu,g,k+1)$ and $\gamma = \gamma(g)$, are positive constants. This completes the proof of step 3.

Step 4. Here we consider $0 < s < 3$. In this case we use the Real Interpolation Method, specifically the K-method of Interpolation, (see Definition 2.4.3, and Theorem 3.1.2 in Bergh and Lofstrom [11]). From Theorem 6.2.4 in [11] we know that the space of interpolation between $L^2(\mathbb{R})$ and $H^3(\mathbb{R})$ is

$$(L^2(\mathbb{R}), H^3(\mathbb{R}))_{\theta,2} = H^{3\theta}(\mathbb{R}),$$

where $0 < \theta < 1$. This property is also true for the torus $\mathbb{T}$ (see Corollary 1.111 in Triebel [89]). It means,

$$(L^2_0(\mathbb{T}), H^3_0(\mathbb{T}))_{\theta,2} = H^{3\theta}_0(\mathbb{T}), \tag{2.4.46}$$

where $0 < \theta < 1$.

Let $t \geqslant 0$ fixed. From the case $s = 0$, we have that the norm of the operator $T(t) : L^2_0(\mathbb{T}) \to L^2_0(\mathbb{T})$, satisfies

$$\|T(t)\|_{L^2_0(\mathbb{T}),L^2_0(\mathbb{T})} \leqslant M_0\, e^{-\gamma\, t}, \tag{2.4.47}$$

where $M_0 = M_0(g)$, and $\gamma = \gamma(g)$ are positive constants. Also, from the case $s = 3$, we know that the norm of the operator $T(t) : H^3_0(\mathbb{T}) \to H^3_0(\mathbb{T})$, satisfies

$$\|T(t)\|_{H^3_0(\mathbb{T}),H^3_0(\mathbb{T})} \leqslant M_1\, e^{-\gamma\, t}, \tag{2.4.48}$$

where $M_1 = M_1(\alpha, \mu, g)$, and $\gamma = \gamma(g)$ are positive constants.

Interpolating (2.4.47) and (2.4.48) (see (2.4.46)), we get that the norm of operator $T(t) : H^{3\theta}_0(\mathbb{T}) \to H^{3\theta}_0(\mathbb{T})$, satisfies

$$\begin{aligned} \|T(t)u_0\|_{H^{3\theta}_0(\mathbb{T})} &\leqslant \left(M_0\, e^{-\gamma\, t}\right)^{1-\theta} \left(M_1\, e^{-\gamma\, t}\right)^{\theta} \|u_0\|_{H^{3\theta}_0(\mathbb{T})} \\ &= M_0^{1-\theta}\, M_1^{\theta}\, e^{-\gamma\, t}\|u_0\|_{H^{3\theta}_0(\mathbb{T})}, \end{aligned} \tag{2.4.49}$$

where $0 < \theta < 1$. Thus, denoting $s = 3\theta$, we have that there exists $M = M(\alpha, \mu, g, s)$, and $\gamma = \gamma(g)$ such that

$$\|u(\cdot, t)\|_{H^s_0(\mathbb{T})} \leqslant M\, e^{-\gamma\, t}\|u_0\|_{H^s_0(\mathbb{T})}, \tag{2.4.50}$$

where u is the solution of (2.4.1) with $K = -GG^*$, and $0 < s < 3$. As t was fixed but arbitrary we get the result.

Step 5. For the others values of s, we use

$$(H^{3k}_0(\mathbb{T}), H^{3(k+1)}_0(\mathbb{T}))_{\rho,2} = H^{3k+3\rho}_0(\mathbb{T}), \tag{2.4.51}$$

where $0 < \rho < 1$, to interpolate inequalities (2.4.40) and (2.4.45), similarly as in step 4. Using an induction argument and computations similar to those in the previous cases, we can prove the following claim.

Claim: For $0 < \rho < 1$, and $n \in \mathbb{N}$, there exist positive constants $M = M(\alpha, \mu, g, n, \rho)$ and $\gamma = \gamma(g)$, such that for any $u_0 \in H^{3n+3\rho}_0(\mathbb{T})$, the unique solution u of (2.4.1) with $K = -GG^*$ satisfies

$$\|u(\cdot, t)\|_{H^{3n+3\rho}_0(\mathbb{T})} \leqslant Me^{-\gamma t}\|u_0\|_{H^{3n+3\rho}_0(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0.$$

Note that, for $s \geqslant 0$ given, there exists $n \in \mathbb{N}$ and $0 \leqslant \rho \leqslant 1$, such that $s = 3n + 3\rho$. Therefore, inequality (2.4.2) for the other values of s follows from the claim and the result obtained in the third step. This completes the proof of Theorem 2.4.2. □

The following corollary is a direct consequence of Theorem 2.4.2.

Corollary 2.4.3. *Let $\mu \in \mathbb{R}$, $\alpha > 0$, g as in (2.0.8), and $s \geqslant 0$ be given. There exist positive constants $M = M(\mu, \alpha, g, s)$ and $\gamma = \gamma(g)$, such that for any $u_0 \in H_p^s(\mathbb{T})$ with $\mu = [u_0]$, the unique solution $u \in C([0,\infty); H_p^s(\mathbb{T}))$ of the closed-loop system*

$$\partial_t u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^3 u = -GG^* u, \quad u(x,0) = u_0, \quad x \in \mathbb{T}, \quad t \in \mathbb{R}^+,$$

satisfies

$$\|u(\cdot,t) - [u_0]\|_{H_p^s(\mathbb{T})} \leqslant M e^{-\gamma t} \|u_0 - [u_0]\|_{H_p^s(\mathbb{T})}, \quad \textit{for all } t \geqslant 0.$$

2.5 Stabilization of the Linear Benjamin Equation in $H_0^s(\mathbb{T})$ with Arbitrary Decay Rate

In this section, we show that it is possible to choose an appropriate linear feedback control law such that the decay rate of the resulting closed-loop system (2.4.1) is as large as one desires.

For $\lambda > 0$ and $s \geqslant 0$ given, we define the operator

$$L_\lambda \phi = \int_0^1 e^{-2\lambda\tau} U_\mu(-\tau) GG^* U_\mu(-\tau)^* \phi \, d\tau, \quad \text{for all } \phi \in H_0^s(\mathbb{T}). \tag{2.5.1}$$

We begin by establishing some properties of this operator.

Proposition 2.5.1. *L_λ is a self-adjoint positive bounded linear operator on $L_0^2(\mathbb{T})$, and so is its inverse L_λ^{-1}. L_λ is therefore an isomorphism from $L_0^2(\mathbb{T})$ onto itself.*

Proof. It is easy to show that L_λ is a linear operator. Also, for each $\phi \in L_0^2(\mathbb{T})$ we have that

$$\begin{aligned} \|L_\lambda \phi\|_{L_0^2(\mathbb{T})} &\leqslant \int_0^1 e^{-2\lambda\tau} \|U_\mu(-\tau) GG^* U_\mu(-\tau)^* \phi\|_{L_0^2(\mathbb{T})} \, d\tau \\ &\leqslant \int_0^1 e^{-2\lambda\tau} c_g^2 \|\phi\|_{L_0^2(\mathbb{T})} \, d\tau = c_{g,\lambda} \|\phi\|_{L_0^2(\mathbb{T})}, \end{aligned} \tag{2.5.2}$$

where $c_{g,\lambda} = c_g^2 \dfrac{(1 - e^{-2\lambda})}{2\lambda}$. Moreover, for each $\phi, \psi \in L_0^2(\mathbb{T})$, we have

$$\begin{aligned} (L_\lambda \phi, \psi)_{L_0^2(\mathbb{T})} &= \int_0^1 e^{-2\lambda\tau} \left(\int_{\mathbb{T}} \phi(x) U_\mu(-\tau)^{**} G^{**} G^* U_\mu(-\tau)^* \overline{\psi}(x) \, dx \right) d\tau \\ &= \int_{\mathbb{T}} \phi(x) \left(\int_0^1 \overline{e^{-2\lambda\tau} U_\mu(-\tau) GG^* U_\mu(-\tau)^* \psi}(x) \, d\tau \right) dx \\ &= (\phi, \, L_\lambda \psi)_{L_0^2(\mathbb{T})}, \end{aligned} \tag{2.5.3}$$

Thus, L_λ is self-adjoint and a bounded linear operator on $L_0^2(\mathbb{T})$. Moreover, from Theorem 16.2, in Bachman and Narici [9] we have that L_λ is a closed operator from $L_0^2(\mathbb{T})$ on itself. From the computations in (2.5.3) and the observability inequality (2.3.43) we have

$$\begin{aligned}(L_\lambda\phi,\ \phi)_{L_0^2(\mathbb{T})} &= \int_0^1 e^{-2\lambda\tau}\,\|G^*U_\mu(1-\tau)^*U_\mu(-1)^*\phi(x)\|^2_{L_0^2(\mathbb{T})}\,d\tau \\ &= \int_0^1 e^{-2\lambda(1-t)}\,\|G^*U_\mu(t)^*U_\mu(-1)^*\phi(x)\|^2_{L_0^2(\mathbb{T})}\,dt \\ &\geqslant \int_0^1 e^{-2\lambda}\,\|G^*U_\mu(t)^*U_\mu(-1)^*\phi(x)\|^2_{L_0^2(\mathbb{T})}\,dt \\ &\geqslant e^{-2\lambda}\delta^2\|\phi\|^2_{L_0^2(\mathbb{T})} \geqslant 0,\end{aligned} \tag{2.5.4}$$

for all $\phi \in L^2(\mathbb{T})$. Thus L_λ is a positive operator. Also, if $L_\lambda\phi = 0$ for some $\phi \in L_0^2(\mathbb{T})$ then from the computations in (2.5.4), we have

$$0 = (0,\ \phi)_{L_0^2(\mathbb{T})} = (L_\lambda\phi,\ \phi)_{L_0^2(\mathbb{T})} \geqslant e^{-2\lambda}\delta^2\|\phi\|^2_{L^2(\mathbb{T})} \geqslant 0.$$

Therefore $\|\phi\|^2_{L^2(\mathbb{T})} = 0$ and $\phi = 0$. Consequently, $\mathrm{Ker}(L_\lambda) = \{0\}$ and L_λ is injective and is an invertible operator. Thus, the adjoint operator $L_\lambda^* = L_\lambda$ exists, is a bounded linear injective operator. It follows from Corollary b), pag. 99, of Rudin [78] that $\mathrm{Ran}\,(L_\lambda)$ is dense in $L_0^2(\mathbb{T})$. Moreover, from (2.5.4), and Cauchy-Shwartz inequality, we get

$$e^{-2\lambda}\delta^2\|\phi\|^2_{L_0^2(\mathbb{T})} \leqslant (L_\lambda\phi,\ \phi)_{L_0^2(\mathbb{T})} \leqslant \|L_\lambda\phi\|_{L_0^2(\mathbb{T})}\|\phi\|_{L_0^2(\mathbb{T})},\quad \text{for all } \phi \in L_0^2(\mathbb{T}), \tag{2.5.5}$$

Therefore $e^{-2\lambda}\delta^2\|\phi\|_{L_0^2(\mathbb{T})} \leqslant \|L_\lambda^*\phi\|_{L_0^2(\mathbb{T})}$, for all $\phi \in L_0^2(\mathbb{T})$. From Proposition 2.16 in Coron [23], we have that L_λ is onto. Then the inverse operator $L_\lambda^{-1} : L_0^2(\mathbb{T}) \to L_0^2(\mathbb{T})$ exists, and is a self-adjoint bounded linear operator. In fact,

$$[L_\lambda^{-1}]^* = [L_\lambda^*]^{-1} = L_\lambda^{-1}.$$

Also, for each $\psi \in L_0^2(\mathbb{T})$, there exists $\phi \in L_0^2(\mathbb{T})$, such that $L_\lambda\phi = \psi$, and $L_\lambda^{-1}\psi = \phi$. Thus

$$\left(L_\lambda^{-1}\psi,\ \psi\right)_{L_0^2(\mathbb{T})} = (\phi,\ L_\lambda\phi)_{L_0^2(\mathbb{T})} = (L_\lambda^*\phi,\ \phi)_{L_0^2(\mathbb{T})} = (L_\lambda\phi,\ \phi)_{L_0^2(\mathbb{T})} \geqslant 0.$$

This completes the proof of the lemma. □

In the following Corollary we show that the operator L_λ also possesses the same properties on $H_0^s(\mathbb{T})$. Before that we record the following definition.

Definition 2.5.2. *Let $r \in \mathbb{R}$. Denote by D^r the operator $D^r : \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$ defined by*

$$\widehat{D^r v}(k) = \begin{cases} |k|^r\widehat{v}(k), & if\ k \neq 0; \\ \widehat{v}(0), & if\ k = 0. \end{cases}$$

Corollary 2.5.3. *The operator $L_\lambda : H^s_p(\mathbb{T}) \longrightarrow H^s_p(\mathbb{T})$ is linear and bounded. Moreover, L_λ is an isomorphism from $H^s_0(\mathbb{T})$ onto $H^s_0(\mathbb{T})$, for all $s > 0$.*

Proof. From computations similars to those in (2.5.2), and Remark 2.1.2 we have that for each $\phi \in L^2_0(\mathbb{T})$

$$\|L_\lambda \phi\|_{H^s_0(\mathbb{T})} \leqslant c_{g,s,\lambda} \|\phi\|_{H^s_0(\mathbb{T})}, \tag{2.5.6}$$

where $c_{g,s,\lambda} = c^2_{g,s} \dfrac{(1-e^{-2\lambda})}{2\lambda}$. Thus L_λ maps $H^s_0(\mathbb{T})$ into $H^s_0(\mathbb{T})$. Since the result is known for $s = 0$ and L_λ maps $H^s_0(\mathbb{T})$ into itself, then any $\phi \in H^s_0(\mathbb{T})$ with $L_\lambda(\phi) = 0$ satisfies

$$0 = \|L_\lambda(\phi)\|_{H^s_0(\mathbb{T})} \geqslant \|L_\lambda(\phi)\|_{L^2_0(\mathbb{T})}.$$

Thus, $L_\lambda(\phi) = 0$ in $L^2_0(\mathbb{T})$ and consequently $\phi = 0$ in $H^s_0(\mathbb{T})$. Therefore, L_λ is injective from $H^s_0(\mathbb{T})$ to $H^s_0(\mathbb{T})$.

On the other hand, for any $\varphi \in H^s_0(\mathbb{T}) \subset L^2_0(\mathbb{T})$, there exists $\phi \in L^2_0(\mathbb{T})$ such that $L_\lambda \phi = \varphi \in H^s_0(\mathbb{T})$. Thus, we must prove that $\phi \in H^s_0(\mathbb{T})$, in order to establish that L_λ is onto. Therefore, it is enough to prove that if $\phi \in L^2_0(\mathbb{T})$, and $L_\lambda \phi \in H^s_0(\mathbb{T})$, then $\phi \in H^s_0(\mathbb{T})$. See Figure 4 below. Observe that, for any $s \geqslant 0$, there exists $n \in \mathbb{N}$, and

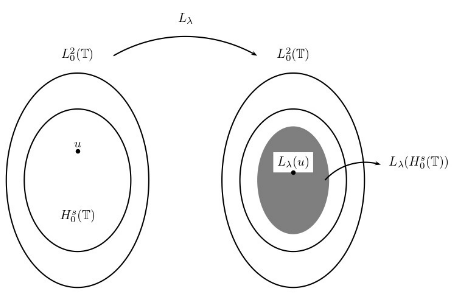

Figure 4 –

$0 \leqslant s' \leqslant 1$ such that $s = n + s'$. The following claim is necessary.

Claim: Let $0 \leqslant s' \leqslant 1$, and $n \in \mathbb{N}$. For any $\phi \in L^2_0(\mathbb{T})$ such that $L_\lambda \phi \in H^{n+s'}_0(\mathbb{T})$, we have that $\phi \in H^{n+s'}_0(\mathbb{T})$, i.e. $D^{n+s'}\phi \in L^2_0(\mathbb{T})$.

In fact, we use induction.

Inductive Base: Let $n = 0$ and $0 \leqslant s \leqslant 1$ be given. Assume $\phi \in L^2_0(\mathbb{T})$ with $L_\lambda \phi \in H^s_0(\mathbb{T})$. We must show that $\phi \in H^s_0(\mathbb{T})$. Note that

$$\|\phi\|_{H^s_0(\mathbb{T})} \leqslant c_{1,s} \left(\|\phi\|_{L^2_0(\mathbb{T})} + \|D^s \phi\|_{L^2_0(\mathbb{T})} \right).$$

Therefore, it is enough to prove that $D^s\phi \in L^2_0(\mathbb{T})$. In fact, since L_λ^{-1} is a continuous function on $L^2_0(\mathbb{T})$, we have

$$\begin{aligned}
\|D^s\phi\|_{L^2_0(\mathbb{T})} &\leqslant c_{\lambda,\delta}\, \|L_\lambda D^s\phi\|_{L^2_0(\mathbb{T})} \\
&\leqslant c_{\lambda,\delta} \left\| \int_0^1 e^{-2\lambda\tau} U_\mu(-\tau)[GG^*, D^s]U_\mu(-\tau)^*\phi\, d\tau \right\|_{L^2_0(\mathbb{T})} \\
&\quad + c_{\lambda,\delta} \left\| \int_0^1 e^{-2\lambda\tau} U_\mu(-\tau) D^s GG^* U_\mu(-\tau)^*\phi\, d\tau \right\|_{L^2_0(\mathbb{T})} \\
&\leqslant c_{\lambda,\delta} \int_0^1 e^{-2\lambda\tau}\, \|[GG^*, D^s]\psi\|_{L^2_0(\mathbb{T})}\, d\tau + c_{\lambda,\delta}\, c_s\, \|L_\lambda\phi\|_{H^s_0(\mathbb{T})},
\end{aligned} \tag{2.5.7}$$

where $c_{\lambda,\delta} = \dfrac{e^{2\lambda}}{\delta^2}$ and $\psi = U_\mu(-\tau)^*\phi$. We must estimate $\|[GG^*, D^s]\psi\|_{L^2_0(\mathbb{T})}$. Using that $G = G^*$ in $L^2_0(\mathbb{T})$, we obtain

$$\begin{aligned}
[GG^*, D^s]\psi =& g^2 D^s\psi - D^s g^2\psi - g\int_{\mathbb{T}} g^2(z) D^s\psi(z)\, dz - g^2 \int_{\mathbb{T}} g(y) D^s\psi(y)\, dy \\
&+ g\left(\int_{\mathbb{T}} g^2(\theta)\, d\theta\right)\left(\int_{\mathbb{T}} g(y) D^s\psi(y)\, dy\right) + D^s g\int_{\mathbb{T}} g^2(z)\psi(z)\, dz \\
&+ D^s g^2 \int_{\mathbb{T}} g(y)\psi(y)\, dy - D^s g\left(\int_{\mathbb{T}} g^2(\theta)\, d\theta\right)\left(\int_{\mathbb{T}} g(y)\psi(y)\, dy\right).
\end{aligned} \tag{2.5.8}$$

From (2.5.8) and Cauchy-Schwartz inequality, we infer that

$$\|[GG^*, D^s]\psi\|_{L^2_0(\mathbb{T})} \leqslant c_{1,g}\left(\left\|[g^2, D^s]\psi\right\|_{L^2_0(\mathbb{T})} + \|D^s\psi\|_{L^2_0(\mathbb{T})} + \|\psi\|_{L^2_0(\mathbb{T})}\right), \tag{2.5.9}$$

where $c_{1,g}$ is the maximum of the constants involved. From Lemma A.1, (2.5.9), and Remark 1.3.7, we obtain

$$\begin{aligned}
\|[GG^*, D^s]\psi\|_{L^2_0(\mathbb{T})} &\leqslant c_{1,g}\left(c_{2,g}\, \|\psi\|_{H^{s-1}_0(\mathbb{T})} + \|\psi\|_{H^s_0(\mathbb{T})} + \|\psi\|_{L^2_0(\mathbb{T})}\right) \\
&\leqslant c_{3,g,s}\, \|\psi\|_{H^{s-1}_0(\mathbb{T})},
\end{aligned} \tag{2.5.10}$$

where $c_{3,g}$ is the maximum of the constants involved. Therefore, inequalities (2.5.7) and (2.5.10) yield

$$\begin{aligned}
\|D^s\phi\|_{L^2_0(\mathbb{T})} &\leqslant c_{\lambda,\delta} \int_0^1 e^{-2\lambda\tau} c_{3,g,s}\, \|\psi\|_{H^{s-1}_0(\mathbb{T})}\, d\tau + c_{\lambda,\delta}\, c_s\, \|L_\lambda\phi\|_{H^s_0(\mathbb{T})} \\
&= c_{\lambda,\delta} \int_0^1 e^{-2\lambda\tau} c_{3,g,s}\, \|U_\mu(-\tau)^*\phi\|_{H^{s-1}_0(\mathbb{T})}\, d\tau + c_{\lambda,\delta}\, c_s\, \|L_\lambda\phi\|_{H^s_0(\mathbb{T})} \\
&= c_{3,g,s} \frac{(e^{2\lambda}-1)}{2\lambda\, \delta^2}\, \|\phi\|_{H^{s-1}_0(\mathbb{T})} + c_{\lambda,\delta}\, c_s\, \|L_\lambda\phi\|_{H^s_0(\mathbb{T})}.
\end{aligned}$$

Using that $0 \leqslant s \leqslant 1$, we obtain positive constants $c_{1,\lambda,\delta,g}$, and $c_{2,\lambda,\delta,s}$ such that

$$\begin{aligned}
\|D^s\phi\|_{L^2_0(\mathbb{T})} &\leqslant c_{3,g,s} \frac{(e^{2\lambda}-1)}{2\lambda\delta^2}\, \|\phi\|_{H^{s-1}_0(\mathbb{T})} + c_{2,\lambda,\delta,s}\, \|L_\lambda\phi\|_{H^s_0(\mathbb{T})} \\
&\leqslant c_{1,\lambda,\delta,g,s}\, \|\phi\|_{L^2_0(\mathbb{T})} + c_{2,\lambda,\delta,s}\, \|L_\lambda\phi\|_{H^s_0(\mathbb{T})},
\end{aligned} \tag{2.5.11}$$

Inductive Hypothesis: Assume that the result is true for $n = 1, 2, ..., k-1$. In particular, if $\phi \in L^2_0(\mathbb{T})$, $L_\lambda \phi \in H_0^{k-1+s'}(\mathbb{T})$, then $D^{k-1+s'}\phi \in L^2_0(\mathbb{T})$. Lets prove the result for $n = k$.

Let $\phi \in L^2_0(\mathbb{T})$, and $L_\lambda \phi \in H_0^{k+s'}(\mathbb{T}) \subset H_0^{k-1+s'}(\mathbb{T})$ be given. From inequality (2.5.11), we have

$$\begin{aligned}
\left\|D^{k+s'}\phi\right\|_{L^2_0(\mathbb{T})} &= \left\|D^{s'}D^k\phi\right\|_{L^2_0(\mathbb{T})} \\
&\leqslant c_{3,g,s}\frac{(e^{2\lambda}-1)}{2\lambda\delta^2}\left\|D^k\phi\right\|_{H_0^{s'-1}(\mathbb{T})} + c_{2,\lambda,\delta,s}\left\|L_\lambda D^k\phi\right\|_{H_0^{s'}(\mathbb{T})} \\
&\leqslant c_{4,g,s}\frac{(e^{2\lambda}-1)}{2\lambda\delta^2}\left\|D^{k+s'-1}\phi\right\|_{L^2_0(\mathbb{T})} + c_{2,\lambda,\delta,s}\left\|L_\lambda \phi\right\|_{H_0^{k+s'}(\mathbb{T})}.
\end{aligned}$$

This proof the Claim.

Finally, the Corollary 2.5.3 follows from (2.5.6) and the claim. $\square$

Remark 2.5.4. *Corollary 2.5.3 implies that there exists a positive constant $C = C(\delta, s, \lambda, g)$ such that*

$$\|L_\lambda^{-1}\psi\|_{H_0^s(\mathbb{T})} \leqslant C\,\|\psi\|_{H_0^s(\mathbb{T})}, \quad \textit{for all} \;\; \psi \in H_0^s(\mathbb{T}). \tag{2.5.12}$$

Choosing the feedback control law in system (2.4.1) as

$$Ku = \begin{cases} -K_\lambda u := -GG^* L_\lambda^{-1} u, & \text{if} \quad \lambda > 0 \\ \\ -K_0 u := -GG^* u, & \text{if} \quad \lambda = 0, \end{cases} \tag{2.5.13}$$

we can rewrite the resulting closed-loop system in the following form

$$\begin{cases} \partial_t u - \alpha\mathcal{H}\partial_x^2 u - \partial_x^3 u + 2\mu\partial_x u = -K_\lambda u, & t > 0, \;\; x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.5.14}$$

where $\lambda \geqslant 0$, and K_λ is a bounded linear operator on $H_p^s(\mathbb{T})$ with $s \geqslant 0$. We have the following result.

Theorem 2.5.5. *Let $\alpha > 0$, $\mu \in \mathbb{R}$, $s \geqslant 0$, and $\lambda > 0$ be given. For any $u_0 \in H_0^s(\mathbb{T})$, the system (2.5.14) admits a unique solution $u \in C([0,+\infty), H_0^s(\mathbb{T}))$. Moreover, there exists $M = M(g, \lambda, \delta, \alpha, \mu, s)$ such that*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant Me^{-\lambda t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \textit{for all}\; t \geqslant 0. \tag{2.5.15}$$

Proof. As K_λ is a bounded linear operator, the same argument used in Theorem 2.4.1 shows that for $u_0 \in H_0^s(\mathbb{T})$ the problem (2.5.14) has a unique solution $u \in C([0,\infty); H_0^s(\mathbb{T}))$, for all $s \geqslant 0$. We denote by $\{T_\lambda(t)\}_{t\geqslant 0}$ the C_0-semigroup on $H_0^s(\mathbb{T})$, with infinitesimal generator $A_\mu - K_\lambda$.

Assume $s = 0$ and let u be the solution of system (2.5.14). Observe that $v := e^{\lambda t}u$, satisfies

$$\begin{cases} \partial_t v - \lambda\, v - \alpha\mathcal{H}\partial_x^2 v - \partial_x^3 v + 2\mu\partial_x v = -GG^* L_\lambda^{-1} v, & t > 0, \quad x \in \mathbb{T} \\ v(x,0) = v_0(x) = u_0(x) \in L_0^2(\mathbb{T}), & x \in \mathbb{T}. \end{cases} \tag{2.5.16}$$

Also, as the operator $\lambda I - GG^* L_\lambda^{-1}$ is bounded on $L_0^2(\mathbb{T})$, then the problem (2.5.16) has a unique solution $v \in C([0,\infty); L_0^2(\mathbb{T}))$. We denote by $\{V(t)\}_{t\geqslant 0}$ the C_0-semigroup on $L_0^2(\mathbb{T})$, with infinitesimal generator $A_\mu + \lambda I - GG^* L_\lambda^{-1}$. Observe that $V(t)^*$ exists, is a bounded linear operator with $\|V(t)\| = \|V(t)^*\|$, for all $t \geqslant 0$, and

$$V(t) = e^{\lambda t} T_\lambda(t). \tag{2.5.17}$$

On the other hand, for $u_0 \in D(A_\mu^*) = H_0^3(\mathbb{T})$, and $w(t) = V(t)^* u_0$ (note that $w(t)$ is the solution of the adjoint system associated to (2.5.16)) we get

$$\frac{d}{dt} w(t) = \left[A_\mu + \lambda I - GG^* L_\lambda^{-1}\right]^* w(t) = \left(A_\mu^* + \lambda I - L_\lambda^{-1} G^{**} G^*\right) w(t), \tag{2.5.18}$$

and

$$\frac{d}{dt} L_\lambda w(t) = L_\lambda \frac{d}{dt} w(t) = L_\lambda \left(A_\mu^* + \lambda I - L_\lambda^{-1} G^{**} G^*\right) w(t). \tag{2.5.19}$$

From (2.5.18) and (2.5.19), we obtain

$$\begin{aligned} \frac{d}{dt} \left(w(t)\,,\, L_\lambda w(t)\right)_{L_0^2(\mathbb{T})} &= \left(\left(A_\mu^* + \lambda I - L_\lambda^{-1} G^{**} G^*\right) w(t)\,,\, L_\lambda w(t)\right)_{L_0^2(\mathbb{T})} \\ &\quad + \left(w(t)\,,\, L_\lambda \left(A_\mu^* + \lambda I - L_\lambda^{-1} G^{**} G^*\right) w(t)\right)_{L_0^2(\mathbb{T})} \\ &= 2\left(L_\lambda A_\mu^* w(t)\,,\, w(t)\right)_{L_0^2(\mathbb{T})} + 2\lambda\, \left(w(t)\,,\, L_\lambda w(t)\right)_{L_0^2(\mathbb{T})} \\ &\quad - 2\left(G^* w(t)\,,\, G^* w(t)\right)_{L_0^2(\mathbb{T})}. \end{aligned} \tag{2.5.20}$$

Note that

$$\begin{aligned} 2\left(L_\lambda A_\mu^* w(t), w(t)\right)_{L_0^2(\mathbb{T})} &= 2\int_0^1 e^{-2\lambda\tau} \left(U_\mu(-\tau) GG^* U_\mu(-\tau)^* A_\mu^* w(\cdot,t)\,,\, w(\cdot,t)\right)_{L_0^2(\mathbb{T})} d\tau \\ &= -2\int_0^1 e^{-2\lambda\tau} \left(G^* \left(\frac{d}{d\tau} U_\mu(-\tau)^* w(\cdot,t)\right)\,,\, G^* U_\mu(-\tau)^* w(\cdot,t)\right)_{L_0^2(\mathbb{T})} d\tau \\ &= -\int_0^1 e^{-2\lambda\tau} \frac{d}{d\tau}\left(\|G^* U_\mu(-\tau)^* w(\cdot,t)\|_{L_0^2(\mathbb{T})}^2\right) d\tau. \end{aligned}$$

Hence, integration by parts yields

$$\begin{aligned} 2\left(L_\lambda A_\mu^* w(t), w(t)\right)_{L_0^2(\mathbb{T})} &= -e^{-2\lambda} \|G^* U_\mu(-1)^* w(\cdot,t)\|_{L_0^2(\mathbb{T})}^2 + \|G^* U_\mu(0)^* w(\cdot,t)\|_{L_0^2(\mathbb{T})}^2 \\ &\quad - 2\lambda \int_0^1 e^{-2\lambda\tau} \left(G^* U_\mu(-\tau)^* w(\cdot,t)\,,\, G^* U_\mu(-\tau)^* w(\cdot,t)\right)_{L_0^2(\mathbb{T})} d\tau \\ &= -\left\|e^{-\lambda} G^* U_\mu(-1)^* w(\cdot,t)\right\|_{L_0^2(\mathbb{T})}^2 + \|G^* w(\cdot,t)\|_{L_0^2(\mathbb{T})}^2 \\ &\quad - 2\lambda \left(L_\lambda w(t)\,,\, w(t)\right)_{L_0^2(\mathbb{T})}. \end{aligned} \tag{2.5.21}$$

From (2.5.20) and (2.5.21), we have

$$
\begin{aligned}
\frac{d}{dt}\left(V(t)^*u_0\,,\,L_\lambda V(t)^*u_0\right)_{L^2_0(\mathbb{T})} &= \frac{d}{dt}\left(w(t)\,,\,L_\lambda w(t)\right)_{L^2_0(\mathbb{T})} \\
&= -\left\|e^{-\lambda}G^*U_\mu(-1)^*w(\cdot,t)\right\|^2_{L^2_0(\mathbb{T})} - \|G^*w(\cdot,t)\|^2_{L^2_0(\mathbb{T})} \\
&= -\left\|e^{-\lambda}G^*U_\mu(-1)^*V(t)^*u_0\right\|^2_{L^2_0(\mathbb{T})} - \|G^*V(t)^*u_0\|^2_{L^2_0(\mathbb{T})} \leqslant 0,
\end{aligned}
\tag{2.5.22}
$$

for all $u_0 \in D(A_\mu^*)$, and $t \geqslant 0$. From inequalities (2.5.22), and (2.5.2) we have

$$
\begin{aligned}
\left(V(t)^*u_0\,,\,L_\lambda V(t)^*u_0\right)_{L^2_0(\mathbb{T})} &\leqslant \left(V(0)^*u_0\,,\,L_\lambda V(0)^*u_0\right)_{L^2_0(\mathbb{T})} \\
&\leqslant \|u_0\|_{L^2_0(\mathbb{T})}\|L_\lambda u_0\|_{L^2_0(\mathbb{T})} \\
&\leqslant\ c_g^2\frac{(1-e^{-2\lambda})}{2\lambda}\|u_0\|^2_{L^2_0(\mathbb{T})}.
\end{aligned}
\tag{2.5.23}
$$

Also, inequality (2.5.5) implies that

$$
e^{-2\lambda}\delta^2\|V(t)^*u_0\|^2_{L^2_0(\mathbb{T})} \leqslant \left(V(t)^*u_0\,,\,L_\lambda V(t)^*u_0\right)_{L^2_0(\mathbb{T})}. \tag{2.5.24}
$$

From inequalities (2.5.23)-(2.5.24), we infer

$$
\|V(t)^*u_0\|_{L^2_0(\mathbb{T})} \leqslant c_g\frac{(1-e^{-2\lambda})^{\frac{1}{2}}}{(2\lambda)^{\frac{1}{2}}}e^{-\lambda}\delta\,\|u_0\|_{L^2_0(\mathbb{T})},\quad \text{for all}\ \ u_0 \in D(A_\mu^*),\ \text{and}\ \ t \geqslant 0.
$$

Finally, using density arguments we show that there exists a positive constant $M_{g,\lambda,\delta} = c_g\dfrac{(1-e^{-2\lambda})^{\frac{1}{2}}}{(2\lambda)^{\frac{1}{2}}}e^{-\lambda}\delta$, such that

$$
\|V(t)^*u_0\|_{L^2_0(\mathbb{T})} \leqslant M_{g,\lambda,\delta}\ \|u_0\|_{L^2_0(\mathbb{T})},\quad \text{for all } u_0 \in L^2_0(\mathbb{T}),\ \textit{and}\ t \geqslant 0.
$$

From identity (2.5.17) and the fact that $\|V(t)\| = \|V(t)^*\|$, we infer

$$
\|e^{\lambda\,t}T_\lambda(t)u_0\|_{L^2_0(\mathbb{T})} \leqslant M_{g,\lambda,\delta}\ \|u_0\|_{L^2_0(\mathbb{T})},\quad \text{for all}\ \ u_0 \in L^2_0(\mathbb{T}),\ \text{and}\ \ t \geqslant 0.
$$

This proves the Theorem in the case $s = 0$. The other cases of s are proved as in Theorem 2.4.2. □

Corollary 2.5.6. *Let $\mu \in \mathbb{R}$, $\lambda > 0$, $\alpha > 0$, g as in (2.0.8), and $s \geqslant 0$ be given. For any $u_0 \in H^s_p(\mathbb{T})$ with $\mu = [u_0]$, there exist positive constants $M = M(g,\lambda,\delta,\mu,\alpha,s)$ such that the unique solution $u \in C([0,\infty); H^s_p(\mathbb{T}))$ of the closed-loop system*

$$
\partial_t u - \alpha\mathcal{H}\partial_x^2 u - \partial_x^3 u = -K_\lambda u,\quad u(x,0) = u_0,\quad x \in \mathbb{T},\quad t \in \mathbb{R}^+,
$$

satisfies

$$
\|u(\cdot,t) - [u_0]\|_{H^s_p(\mathbb{T})} \leqslant Me^{-\lambda t}\|u_0 - [u_0]\|_{H^s_p(\mathbb{T})},\quad \textit{for all}\ \ t \geqslant 0.
$$

◇

Chapter 3

Bourgain's spaces associated to the Benjamin equation

In this chapter, we introduce the Bourgain's spaces associated to the Benjamin equation (2.3.37) and derive its properties. As was indicated in the final part of Section 2.3, the Strichartz estimates and multilinear estimates obtained in this chapter are fundamental to prove the existence of solutions for the Benjamin equation (2.3.37). This preliminary results will bring a small data control result for equation (2.3.37) which is proved in Chapter 4. Here we also are particularly interested in obtain the propagation of compactness and regularity for the differential operator $L := \partial_t - \alpha\, \mathcal{H}\partial_x^2 - \partial_x^3 + 2\mu\, \partial_x$ associated to the Benjamin equation. These propagation properties will play a key role when studying the global stabilizability. Furthermore, in this chapter we prove the unique continuation property for Benjamin equation (2.3.37).

This chapter is organized as follows. Section 3.1 is devoted to study the Bourgain's space associated to the Benjamin equation (2.3.37) and its properties. The main result in this section is the bilinear estimate given by Theorem 3.1.24. In section 3.2, a multiplication property is established, viz., Theorem 3.2.3. Next, in section 3.3, we present some properties of propagation of compactness and regularity. Finally, in section 3.4 we prove the unique continuation property for the Benjamin equation.

3.1 Bourgain's spaces

Here we will introduce the Bourgain's spaces associated to Benjamin type equation (2.3.37) and study its properties. To simplify the notation, we denote $U_\mu(t)$ (see (2.3.40)) by $V(t)$. We start with the following definition.

Definition 3.1.1 (see Tao [87, page 99]). *Let $v : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be a function. The spatially* *periodic, and temporally non-periodic Fourier transform is defined by*

$$\widehat{v}(k,\tau) := \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{T}} v(x,t)e^{-i(t\tau+kx)}\,dx\,dt,$$

where $k \in \mathbb{Z}$ *and* $\tau \in \mathbb{R}$.

In this case the Fourier inversion formula is given by

$$v(x,t) = \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \widehat{v}(k,\tau)e^{i(t\tau+kx)}\,d\tau. \tag{3.1.1}$$

We use $\widehat{v}(k,t)$ (respectively $\widehat{v}(x,\tau)$) to denote the Fourier transform in space variable x (respectively in time variable t).

For given $b, s \in \mathbb{R}$ and a function $v : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$, define the quantities

$$\|v\|_{X_{s,b}} := \left(\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}} \langle k\rangle^{2s}\langle \tau-\phi(k)\rangle^{2b}|\widehat{v}(k,\tau)|^2\,d\tau\right)^{\frac{1}{2}},$$

and

$$\|v\|_{Y_{s,b}} := \left(\sum_{k=-\infty}^{\infty}\left(\int_{\mathbb{R}} \langle k\rangle^{s}\langle \tau-\phi(k)\rangle^{b}|\widehat{v}(k,\tau)|\,d\tau\right)^2\right)^{\frac{1}{2}},$$

where $\phi(k) = -k^3 - 2\mu k + \alpha k|k|$ is called the phase function, $\langle\cdot\rangle := \sqrt{1+|\cdot|^2}$, and $\widehat{v}(k,\tau)$ denotes the space-time Fourier transform of v.

Definition 3.1.2 (Bourgain's space, see Tao [87, page 99])**.** *For given* $b, s \in \mathbb{R}$ *we define the Bourgain's space* $X_{s,b}$ *associated to Benjamin equation on* $\mathbb{T}$ *as the closure of the space of Schwartz functions* $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ *under the norm* $\|\cdot\|_{X_{s,b}}$.

Note that $X_{s,b}$ is a Hilbert space and for $b > \dfrac{1}{2}$

$$X_{s,b} \subset C(\mathbb{R}; H_p^s(\mathbb{T})), \tag{3.1.2}$$

the embedding being continuous. In particular,

$$\|v\|_{X_{s,0}} = \|v\|_{L^2(\mathbb{R}_t; H_p^s(\mathbb{T}))}. \tag{3.1.3}$$

Let $f : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ be a function. We define $\|f\|_{H_x^s H_t^b}$ as

$$\|f\|_{H_x^s H_t^b} := \Big\| \, \|\widehat{f}(k,t)\langle k\rangle^s\|_{H_t^b(\mathbb{R})}\Big\|_{l_k^2} = \left(\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}} \langle k\rangle^{2s}\langle\tau\rangle^{2b}|\widehat{f}(k,\tau)|^2\,d\tau\right)^{\frac{1}{2}}.$$

Proposition 3.1.3. *Let* $v : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ *be a function. Then,*

$$\|v\|_{X_{s,b}} = \|V(-t)v\|_{H_x^s H_t^b} = \Big\| \, \|\left(V(-t)v\right)^\wedge(k,t)\langle k\rangle^s\|_{H_t^b(\mathbb{R})}\Big\|_{l_k^2}.$$

Proof. The proof follows by using $(V(-t)v(x,t))^\wedge\,(k,t) = e^{-i\phi(k)t}\widehat{v}(k,t) = \widehat{v}(k,\tau+\phi(k))$ and an appropriate change of variables. We omit the details. $\square$

As noted in [14, 44, 52, 57, 87], while dealing with the bilinear estimates in the periodic case one needs to consider $b = \dfrac{1}{2}$ for which the embedding in 3.1.2 fails. To overcome this situation, we introduce the space $Z_{s,b}$.

Definition 3.1.4. *For given $b, s \in \mathbb{R}$ we define the space $Y_{s,b}$ as the closure of the space of Schwartz functions $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ under the norm $\|\cdot\|_{Y_{s,b}}$.*

For any function $v : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$, we define $\|v\|_{l^2_k L^1_\tau(\mathbb{R})} := \Big\| \|v\|_{L^1_\tau(\mathbb{R})} \Big\|_{l^2_k}$, and we note that $\|v\|_{Y_{s,b}} = \|\langle k\rangle^s \langle \tau - \phi(k)\rangle^b \hat{v}(k,\tau)\|_{l^2_k L^1_\tau(\mathbb{R})}$.

Definition 3.1.5. *For given $s, b \in \mathbb{R}$, we define the space*

$$Z_{s,b} := X_{s,b} \cap Y_{s,b-\frac{1}{2}}$$

endowed with the norm $\|\cdot\|_{Z_{s,b}} := \|\cdot\|_{X_{s,b}} + \|\cdot\|_{Y_{s,b-\frac{1}{2}}}$. For a given interval I, let $X_{s,b}(I)$ (resp. $Y_{s,b}(I)$, $Z_{s,b}(I)$) be the restriction space of $X_{s,b}$ (resp. $Y_{s,b}$, $Z_{s,b}$) to the interval I with the norm

$$\|f\|_{X_{s,b}(I)} := \inf\left\{\|\tilde{f}\|_{X_{s,b}} | \tilde{f} = f \text{ on } \mathbb{T}\times I\right\}$$

$$\left(\text{resp. } \|f\|_{Y_{s,b}(I)} := \inf\left\{\|\tilde{f}\|_{Y_{s,b}} | \tilde{f} = f \text{ on } \mathbb{T}\times I\right\}\right)$$
$$\left(\text{resp. } \|f\|_{Z_{s,b}(I)} := \inf\left\{\|\tilde{f}\|_{Z_{s,b}} | \tilde{f} = f \text{ on } \mathbb{T}\times I\right\}\right).$$

For simplicity, we will denote $X_{s,b}(I)$ (resp. $Y_{s,b}(I)$, $Z_{s,b}(I)$) by $X^T_{s,b}$ (resp. $Y^T_{s,b}$, $Z^T_{s,b}$) if $I = [0,T]$.

3.1.1 Properties of the spaces $X_{s,b}(I)$ (resp. $Z_{s,b}(I)$)

In this subsection we show important properties that the Bourgain's spaces hold. The main result of this subsection is given by Proposition 3.1.8 which establishes the continuous embedding of space $Z_{s,b}(I)$ in $C(I; H^s_p(\mathbb{T}))$. We start with the following properties whose proofs are by now classical and can be found for example in [87].

Proposition 3.1.6. *The spaces $X_{s,b}$ (resp. $X_{s,b}(I)$) have the following properties:*

i) $X_{s,b}$ and $X_{s,b}(I)$ are Hilbert's spaces.

ii) If $s_1 \leqslant s_2$ and $b_1 \leqslant b_2$, then X_{s_2,b_2} (resp. $X_{s_2,b_2}(I)$) is continuously embedded in the space X_{s_1,b_1} (resp. $X_{s_1,b_1}(I)$).

iii) For a given finite interval I, if $s_1 < s_2$ and $b_1 < b_2$, then the space $X_{s_2,b_2}(I)$ is compactly embedded in the space $X_{s_1,b_1}(I)$.

Lemma 3.1.7. *Let $s, r \in \mathbb{R}$ and D^r be the operator given by Definition 2.5.2. Then, $D^r v \in X_{s-r,b}$ (resp. $X_{s-r,b}(I)$) for any $v \in X_{s,b}$ (resp. $X_{s,b}(I)$). The same is valid for the operator ∂_x^r. Moreover, there exists a positive constant C such that*

$$\|D^r v\|_{X_{s-r,b}} \leqslant C\,\|v\|_{X_{s,b}}$$

Proof. Let $v \in X_{s,b}$. Then

$$\begin{aligned}\|D^r v\|^2_{X_{s-r,b}} &= \sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}\langle k\rangle^{2s}\langle \tau-\phi(k)\rangle^{2b}\,\langle k\rangle^{-2r}|k|^{2r}\,|\widehat{v}(k,\tau)|^2\;d\tau + \int_{\mathbb{R}}\langle \tau-\phi(0)\rangle^{2b}|\widehat{v}(0,\tau)|^2\,d\tau\\ &\leqslant C_1\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}\langle k\rangle^{2s}\langle \tau-\phi(k)\rangle^{2b}\;|\widehat{v}(k,\tau)|^2\;d\tau + \int_{\mathbb{R}}\langle \tau-\phi(0)\rangle^{2b}|\widehat{v}(0,\tau)|^2\,d\tau\\ &\leqslant C^2\|v\|^2_{X_{s,b}}.\end{aligned}$$

□

The following proposition establishes the main result of this subsection.

Proposition 3.1.8. *Let I be an interval and $s \in \mathbb{R}$. The space $Z_{s,\frac{1}{2}}$ (resp. $Z_{s,\frac{1}{2}}(I)$) is continuously embedded in the space $C(\mathbb{R}; H^s_p(\mathbb{T}))$ (resp. $C(I; H^s_p(\mathbb{T}))$).*

Proof. First assume $v \in \mathcal{S}(\mathbb{T}\times\mathbb{R})$. Then, for each $t \in \mathbb{R}$ fixed, we have

$$\|v(\cdot,t)\|^2_{H^s_p(\mathbb{T})} = 2\pi\sum_{k=-\infty}^{k=\infty}\langle k\rangle^{2s}\left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{it\tau}\widehat{v}(k,\tau)d\tau\right|^2 \leqslant C\sum_{k=-\infty}^{k=\infty}\left(\int_{\mathbb{R}}\langle k\rangle^{s}|\widehat{v}(k,\tau)|d\tau\right)^2 \leqslant C\|v\|^2_{Z_{s,\frac{1}{2}}},$$

where the positive constant C does not depend neither on t nor v. Continuity in t follows from the dominated convergence theorem. Therefore,

$$\|v\|_{C(\mathbb{R},H^s_p(\mathbb{T}))} = \sup_{t\in\mathbb{R}}\|v(\cdot,t)\|_{H^s_p(\mathbb{T})} \leqslant C\|v\|_{Z_{s,\frac{1}{2}}}.$$

Now, if v is any function in $Z_{s,\frac{1}{2}} = X_{s,\frac{1}{2}} \cap Y_{s,0}$, then there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ such that $\lim\limits_{n\to\infty} v_n = v$ in $Z_{s,\frac{1}{2}}$, due to the density of $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ in $Z_{s,\frac{1}{2}}$. Hence,

$$\begin{aligned}\|v_n - v_m\|_{C(\mathbb{R},H^s(\mathbb{T}))} &\leqslant C\|v_n - v_m\|_{Z_{s,\frac{1}{2}}}\\ &\leqslant C\left(\|v_n - v\|_{Z_{s,\frac{1}{2}}} + \|v - v_m\|_{Z_{s,\frac{1}{2}}}\right) \longrightarrow 0,\end{aligned}$$

as n, m tend to infinity. Thus, there exists $w \in C(\mathbb{R}, H^s_p(\mathbb{T}))$ such that $\lim\limits_{n\to\infty} v_n = w$ in $C(\mathbb{R}, H^s_p(\mathbb{T}))$. Note that if $s \geqslant 0$, then $\lim\limits_{n\to\infty} v_n = v$ in $L^2(\mathbb{T}\times\mathbb{R})$ and we infer $w = v$ a.e. $(x,t) \in \mathbb{T}\times\mathbb{R}$. Therefore, using the continuity of the norm we have

$$\|v\|_{C(\mathbb{R},H^s(\mathbb{T}))} = \left\|\lim_{n\to\infty} v_n\right\|_{C(\mathbb{R},H^s(\mathbb{T}))} = \lim_{n\to\infty}\|v_n\|_{C(\mathbb{R},H^s(\mathbb{T}))} \leqslant C\lim_{n\to\infty}\|v_n\|_{Z_{s,\frac{1}{2}}} = C\|v\|_{Z_{s,\frac{1}{2}}}. \tag{3.1.4}$$

Finally, if v is a function in $Z_{s,\frac{1}{2}}(I)$, we can choose, by the definition of infimum, an extension $\widetilde{v}$ to $\mathbb{T}\times\mathbb{R}$ of v such that $\|\widetilde{v}\|_{Z_{s,\frac{1}{2}}} \leqslant 2\,\|v\|_{Z_{s,\frac{1}{2}}(I)}$. From (3.1.4) we infer that there exists a positive constant C such that

$$\|v\|_{C(I,H^s(\mathbb{T}))} \leqslant \|\widetilde{v}\|_{C(\mathbb{R},H^s(\mathbb{T}))} \leqslant C\|\widetilde{v}\|_{Z_{s,\frac{1}{2}}} \leqslant 2C\|v\|_{Z_{s,\frac{1}{2}}(I)}.$$

□

Lemma 3.1.9. *Let $s,b\in\mathbb{R}$. The dual space of $X_{s,b}$ is $X_{-s,-b}$. Moreover, the space $X_{s,b}$ is reflexive.*

Proof. It follows from the fact that $\phi(k)$ is an odd function. See page 97 in Tao [87]. □

The ideas to prove the majority of the results in the following two subsections are similar to those derived in the KdV case (see [14, 44, 21, 22, 43, 33]).

3.1.2 Linear and integral estimates

To derive some estimates localized in time variable, we introduce a cut-off function $\eta\in C_c^\infty(\mathbb{R})$ such that $\eta\equiv 1$, if $t\in[-1,1]$ and $\eta\equiv 0$, if $t\notin(-2,2)$. Then, for $T>0$ given, we define

$$\eta_T\in C_c^\infty(\mathbb{R}) \quad \text{by} \quad \eta_T(t)=\eta\left(\frac{t}{T}\right).$$

We begin this subsection by proving some linear estimates.

Proposition 3.1.10. *Let $s,b\in\mathbb{R}$ and $T>0$ given. Then for all $v_0\in H_p^s(\mathbb{T})$, we have*

$$i)\ \|\eta(t)V(t)v_0\|_{X_{s,b}} \leqslant C_{\eta,b}\,\|v_0\|_{H_p^s(\mathbb{T})}, \qquad ii)\ \|\eta_T(t)V(t)v_0\|_{X_{s,b}} \leqslant C_{\eta,b}\,T^{\frac{1}{2}}\,\|v_0\|_{H_p^s(\mathbb{T})},$$

$$iii)\ \|\eta(t)V(t)v_0\|_{Y_{s,b}} \leqslant C_{\eta,b}\,\|v_0\|_{H_p^s(\mathbb{T})}, \qquad iv)\ \|\eta_T(t)V(t)v_0\|_{Y_{s,b}} \leqslant C_{\eta,b}\,T^{\frac{1}{2}}\,\|v_0\|_{H_p^s(\mathbb{T})},$$

where $C_{\eta,b}$ is a positive constant that depends only on η and b.

Proof. The estimates $i)$ and $iii)$ are immediate consequence of $ii)$ and $iv)$ by taking $T=1$. First, we prove the estimate in $ii)$.
Note that for any $v_0\in H_p^s(\mathbb{T})$, we obtain

$$\begin{aligned}\|\eta_T(t)\;V(t)v_0\|_{X_{s,b}} &= \|V(-t)V(t)\;\eta_T(t)v_0\|_{H_x^sH_t^b}\\ &= \Big\||\widehat{v_0}(k)|\;\langle k\rangle^s\;\|\eta_T(t)\|_{H_t^b(\mathbb{R})}\Big\|_{l_k^2}\\ &= \|\eta_T(t)\|_{H_t^b(\mathbb{R})}\;\;\|v_0\|_{H_p^s(\mathbb{T})}.\end{aligned} \tag{3.1.5}$$

If $b\leqslant 0$, then

$$\|\eta_T(t)\|_{H_t^b(\mathbb{R})} \leqslant c_b\,\|\eta_T(t)\|_{L^2(\mathbb{R})} = c_b\,T^{\frac{1}{2}}\,\|\eta(t)\|_{L^2(\mathbb{R})}. \tag{3.1.6}$$

On the other hand, if $b \geqslant 0$, then

$$\begin{aligned} \|\eta_T(t)\|_{H^b_t(\mathbb{R})} &\leqslant c_b \left(\|\eta_T(t)\|_{L^2(\mathbb{R})} + \|(\widehat{D^b\eta_T(\cdot)})(\tau)\|_{L^2(\mathbb{R})}\right) \\ &= c_b\, T^{\frac{1}{2}} \left(\|\eta(t)\|_{L^2(\mathbb{R})} + \|D^b\eta(t)\|_{L^2(\mathbb{R})}\right). \end{aligned} \tag{3.1.7}$$

From (3.1.5)-(3.1.7), we have that there exists $C_{b,\eta} > 0$ such that

$$\|\eta_T(t)\; V(t)v_0\|_{X_{s,b}} \leqslant C_{b,\eta}\, T^{\frac{1}{2}}\; \|v_0\|_{H^s_p(\mathbb{T})}. \tag{3.1.8}$$

Next, we prove estimate *iv*). Observe that for any $v_0 \in H^s_p(\mathbb{T})$

$$(\eta(t)\; V(t)\; v_0)^\wedge\,(k,\tau) = \left(\eta(t)\; e^{it\,\phi(k)}\; \widehat{v_0}(k)\right)^\wedge(\tau) = \widehat{v_0}(k)\; \widehat{\eta}(\tau + \phi(k)). \tag{3.1.9}$$

Thus,

$$\begin{aligned} \|\eta_T(t)\; V(t)v_0\|_{Y_{s,b}} &= \left\|\|\langle k\rangle^s\,\langle \tau - \phi(k)\rangle^b\; \widehat{v_0}(k)\; \widehat{\eta_T}(\tau+\phi(k))\|_{L^1_\tau(\mathbb{R})}\right\|_{l^2_k} \\ &= \left\|\|\langle k\rangle^s\,\langle \theta\rangle^b\; \widehat{v_0}(k)\; \widehat{\eta_T}(\theta+2\phi(k))\|_{L^1_\theta(\mathbb{R})}\right\|_{l^2_k} \\ &= \left\|\langle k\rangle^s\,|\widehat{v_0}(k)|\;\|\langle \theta\rangle^b\; e^{-2i\,\theta\,\phi(k)}\widehat{\eta_T}(\theta)\|_{L^1_\theta(\mathbb{R})}\right\|_{l^2_k} \\ &= \|\langle\theta\rangle^b\;\widehat{\eta_T}(\theta)\|_{L^1_\theta(\mathbb{R})}\;\;\|v_0\|_{H^s_p(\mathbb{T})}. \end{aligned} \tag{3.1.10}$$

Therefore, taking some $N \in \mathbb{N}$, with $N > b + \dfrac{1}{2}$, we have

$$\begin{aligned} \|\langle\theta\rangle^b\;\widehat{\eta_T}(\theta)\|_{L^1_\theta(\mathbb{R})} &\leqslant c\int_{\mathbb{R}} (1+|\theta|)^{b-N}\;(1+|\theta|)^N\;|\widehat{\eta_T}(\theta)|\;d\theta \\ &\leqslant c\;\|(1+|\theta|)^{b-N}\|_{L^2(\mathbb{R})}\;\|(1+|\theta|)^N\;\widehat{\eta_T}(\theta)\|_{L^2(\mathbb{R})} \\ &\leqslant c_b\;\|\eta_T\|_{H^N_p(\mathbb{T})}. \end{aligned} \tag{3.1.11}$$

From (3.1.7), we infer that

$$\|\langle\theta\rangle^b\;\widehat{\eta_T}(\theta)\|_{L^1_\theta(\mathbb{R})} \leqslant c_b\;T^{\frac{1}{2}}\;\left(\|\eta(t)\|_{L^2(\mathbb{R})} + \|D^N\eta(t)\|_{L^2(\mathbb{R})}\right). \tag{3.1.12}$$

Using (3.1.10) and (3.1.12), we obtain that there exists $C_{b,\eta} > 0$ such that

$$\|\eta_T(t)V(t)v_0\|_{Y_{s,b}} \leqslant C_{\eta,b}\;T^{\frac{1}{2}}\;\|v_0\|_{H^s_p(\mathbb{T})}.$$

This completes the proof of proposition. □

Corollary 3.1.11. *Let $s, b \in \mathbb{R}$ and $T > 0$ be given. Then for all $v_0 \in H^s_p(\mathbb{T})$ we have*

i) $\|V(t)v_0\|_{X^T_{s,b}} \leqslant C_{\eta,b}\;\|v_0\|_{H^s_p(\mathbb{T})}$, *if* $T \leqslant 1$,

ii) $\|V(t)v_0\|_{X_{s,b}(I)} \leqslant C_{\eta,b}\;T^{\frac{1}{2}}\;\|v_0\|_{H^s_p(\mathbb{T})}$, *if* $I = [-T, T]$, *iii)* $\|V(t)v_0\|_{Y^T_{s,b}} \leqslant C_{\eta,b}\, \|v_0\|_{H^s_p(\mathbb{T})}$, *if* $T \leqslant 1$,

iv) $\|V(t)v_0\|_{Y_{s,b}(I)} \leqslant C_{\eta,b}\, T^{\frac{1}{2}}\, \|v_0\|_{H^s_p(\mathbb{T})}$, *if* $I = [-T, T]$,

where $C_{\eta,b}$ *is a positive constant that depends only on* η *and* b.

Proof. It is enough to prove $ii)$ and $iv)$.

We first prove $ii)$. Let $v_0 \in H^s_p(\mathbb{T})$. Then $\eta_T(t)V(t)v_0$ is a particular extension to $X_{s,b}$ of $V(t)v_0$ in $X_{s,b}(I)$. Thus, appliying Proposition 3.1.10 part ii), we have

$$\|V(t)v_0\|_{X_{s,b}(I)} \leqslant \|\eta_T(t)V(t)v_0\|_{X_{s,b}} \leqslant C_{b,\eta}\, T^{\frac{1}{2}} \|v_0\|_{H^s_p(\mathbb{T})}. \tag{3.1.13}$$

This prove $ii)$. A similar argument applies to prove $iv)$. □

The folloing Corollary is a direct consequence of Proposition 3.1.10, Corollary 3.1.11 and the definition of the norm in the space $Z_{s,b}$ ($Z_{s,b}(I)$, and $Z^T_{s,b}$ respectively).

Corollary 3.1.12. *Let* $s, b \in \mathbb{R}$ *and* $T > 0$ *be given. Then for all* $v_0 \in H^s_p(\mathbb{T})$ *we have the following estimates*

i) $\|\eta(t)V(t)v_0\|_{Z_{s,b}} \leqslant C_{\eta,b}\, \|v_0\|_{H^s_p(\mathbb{T})}$,

ii) $\|\eta_T(t)V(t)v_0\|_{Z_{s,b}} \leqslant C_{\eta,b}\, T^{\frac{1}{2}}\, \|v_0\|_{H^s_p(\mathbb{T})}$,

iii) $\|V(t)v_0\|_{Z^T_{s,b}} \leqslant C_{\eta,b}\, \|v_0\|_{H^s_p(\mathbb{T})}$, *if* $T \leqslant 1$,

iv) $\|V(t)v_0\|_{Z^T_{s,b}} \leqslant C_{\eta,b}\, T^{\frac{1}{2}} \|v_0\|_{H^s_p(\mathbb{T})}$,

v) $\|V(t)v_0\|_{Z_{s,b}(I)} \leqslant C_{\eta,b}\, T^{\frac{1}{2}}\, \|v_0\|_{H^s_p(\mathbb{T})}$, *if* $I = [-T, T]$,

where $C_{\eta,b}$ *is a positive constant depending only on* η *and* b.

Now we show some integral estimates. The following lemma is necessary.

Lemma 3.1.13 (See [33, page 67-68])**.** *We have the following integral estimates:*

i) *Let* $\dfrac{1}{2} < b \leqslant 1$, *and let* f *be a function in* $H^{b-1}(\mathbb{R})$. *Then*

$$\left\|\eta_T(t)\int_0^t f(s)\, ds\right\|_{H^b_t(\mathbb{R})} \leqslant C_{\eta,b,T}\, \|f\|_{H^{b-1}_p(\mathbb{R})},$$

where $C_{\eta,b,T}$ *is a positive constant depending on* η, b, *and the final time* T.

ii) *Let* $b = \dfrac{1}{2}$. *Let* f *be a function in* $H^{-\frac{1}{2}}(\mathbb{R})$, *and* $\langle\cdot\rangle^{-1}\hat{f}(\cdot) \in L^1(\mathbb{R})$. *Then*

$$\left\|\eta_T(t)\int_0^t f(s)\, ds\right\|_{H^{\frac{1}{2}}_t(\mathbb{R})} \leqslant C_{\eta,T}\, \left(\|f\|_{H_p^{-\frac{1}{2}}(\mathbb{R})} + \|\langle z\rangle^{-1}\hat{f}(z)\|_{L^1_z(\mathbb{R})}\right),$$

where $C_{\eta,T}$ *is a positive constant depending on* η, *and the final time* T.

If $T \leqslant 1$, then the positive constant C in the estimates i) and ii) does not depend on T.

Theorem 3.1.14. *Let $b = \dfrac{1}{2}$ and $T > 0$ be given. Then, one has*

i)

$$\left\| \eta_T(t) \int_0^t V(t-t')f(t')\, dt' \right\|_{Z_{s,\frac{1}{2}}} \leqslant C_{\eta,T} \, \|f\|_{Z_{s,-\frac{1}{2}}}, \quad \forall f \in Z_{s,-\frac{1}{2}},$$

where $C_{\eta,T}$ is a positive constant depending on η, and the final time T.

ii)

$$\left\| \int_0^t V(t-t')f(t')\, dt' \right\|_{Z_{s,\frac{1}{2}}(I)} \leqslant C_{\eta,T} \, \|f\|_{Z_{s,-\frac{1}{2}}(I)}, \quad \forall f \in Z_{s,-\frac{1}{2}}(I), \text{ with } I = [-T, T],$$

where $C_{\eta,T}$ is a positive constant depending on η, and the final time T.

iii)

$$\left\| \int_0^t V(t-t')f(t')\, dt' \right\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_{\eta,T} \, \|f\|_{Z^T_{s,-\frac{1}{2}}}, \quad \forall f \in Z^T_{s,-\frac{1}{2}},$$

where $C_{\eta,T}$ is a positive constant depending on η, and the final time T.

If $T \leqslant 1$, then the positive constant C in the estimates i), ii), and iii) does not depend on T.

Proof. First, we show the estimate in *i*). Let $T > 0$ and $f \equiv f(x,t') \in Z_{s,-\frac{1}{2}}$. From Bochner theorem (see Theorem 8, page 734, [54]), we get

$$\begin{aligned} \left(\int_0^t V(-t')\, f(t')\, dt' \right)^{\wedge} (k,t') &= \frac{1}{2\pi} \left\langle \int_0^t V(-t')\, f(t')\, dt', e^{-ikx} \right\rangle \\ &= \int_0^t \left(V(-t')\, f(t') \right)^{\wedge} (k,t')\, dt' \end{aligned} \tag{3.1.14}$$

We begin estimating the norm in $X_{s,\frac{1}{2}}$. From Proposition 3.1.3, and identity (3.1.14), we obtain

$$\left\| \eta_T(t) \int_0^t V(t-t')f(t')\, dt' \right\|_{X_{s,\frac{1}{2}}} = \left\| \left\| \eta_T(t) \int_0^t \left(V(-t')f(t') \right)^{\wedge} (k,t')\, dt' \right\|_{H_t^{\frac{1}{2}}(\mathbb{R})} \langle k \rangle^s \right\|_{l^2_k},$$

where $\left(V(-t')f(t')\right)^{\wedge}(k,t')$ represents the Fourier's transform of the function $V(-t')f(t')$ only with respect to the spatial variable x. From Lemma 3.1.13 item *ii*), we get

$$\begin{aligned} \left\| \eta_T(t) \int_0^t V(t-t')f(t')\, dt' \right\|_{X_{s,\frac{1}{2}}} &\leqslant C_{\eta,T} \left\| \left\| \langle k \rangle^s \left(V(-t')f(t') \right)^{\wedge} (k,t') \right\|_{H_{t'}^{-\frac{1}{2}}(\mathbb{R})} \right\|_{l^2_k} \\ &\quad + C_{\eta,T} \left\| \left\| \langle k \rangle^s \langle \lambda \rangle^{-1} \left(V(-t')f(t') \right)^{\wedge} (k,\lambda) \right\|_{L^1_\lambda(\mathbb{R})} \right\|_{l^2_k}, \end{aligned}$$

where $(V(-t')f(t'))^{\wedge}(k,\lambda)$ is the Fourier's transform of the function $V(-t')f(t')$ with respect to the spatial variable x and the temporal variable t'. We note that

$$(V(-t')f(t'))^{\wedge}(k,\lambda) = \widehat{f}(k,\lambda+\phi(k)).$$

Therefore,

$$\begin{aligned}\left\|\eta_T(t)\int_0^t V(t-t')f(t')\,dt'\right\|_{X_{s,\frac{1}{2}}} &\leqslant C_{\eta,T}\left(\|f\|_{X_{s,-\frac{1}{2}}} + \left\|\|\langle k\rangle^s\langle\lambda\rangle^{-1}\widehat{f}(k,\lambda+\phi(k))\|_{L^1_\lambda(\mathbb{R})}\right\|_{l^2_k}\right)\\ &= C_{\eta,T}\left(\|f\|_{X_{s,-\frac{1}{2}}} + \left\|\|\langle k\rangle^s\langle\lambda-\phi(k)\rangle^{-1}\widehat{f}(k,\lambda)\|_{L^1_\lambda(\mathbb{R})}\right\|_{l^2_k}\right)\\ &= C_{\eta,T}\left(\|f\|_{X_{s,-\frac{1}{2}}} + \|f\|_{Y_{s,-1}}\right)\\ &= C_{\eta,T}\ \|f\|_{Z_{s,-\frac{1}{2}}}.\end{aligned} \tag{3.1.15}$$

Now, we will estimate the norm in $Y_{s,0}$. Set

$$D(x,t) := \eta_T(t)\int_0^t V(t-t')f(t')\,dt'.$$

We need to compute $\widehat{D}(k,\lambda)$. From the estimate (3.1.14), we obtain that

$$\widehat{D}(k,t) = \eta_T(t)\int_0^t e^{i(t-t')\phi(k)}\widehat{f}(k,t')\,dt' = e^{it\phi(k)}\eta_T(t)\int_0^t e^{-it'\phi(k)}\widehat{f}(k,t')\,dt'. \tag{3.1.16}$$

Define

$$\chi_{[0,t]}(s) = \begin{cases} 1, & \text{if } s\in[0,t]\\ 0, & \text{if } s\notin[0,t].\end{cases}$$

Using the properties of the Fourier's transform and the Fubini's Theorem (see [36, page 73]]), we get

$$\begin{aligned}\left(\eta_T(t)\int_0^t e^{-it'\phi(k)}\widehat{f}(k,t')\,dt'\right)^{\wedge}(\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t}\left(\eta_T(t)\int_0^t e^{-it'\phi(k)}\widehat{f}(k,t')\,dt'\right)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t}\left(\eta_T(t)\int_{\mathbb{R}}\chi_{[0,t]}(t')e^{-it'\phi(k)}\widehat{f}(k,t')\,dt'\right)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t}\left(\eta_T(t)\int_{\mathbb{R}}(\chi_{[0,t]}(\cdot))^{\vee}(z)\,(e^{-i(\cdot)\phi(k)}\widehat{f}(k,\cdot))^{\wedge}(z)\,dz\right)dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t}\left(\frac{\eta_T(t)}{\sqrt{2\pi}}\int_{\mathbb{R}}\frac{(e^{itz}-1)}{iz}\,\widehat{f}(k,z+\phi(k))\,dz\right)\,dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{f}(k,z+\phi(k))\left(\frac{\widehat{\eta_T}(\lambda-z)-\widehat{\eta_T}(\lambda)}{iz}\right)\,dz.\end{aligned} \tag{3.1.17}$$

From (3.1.16)-(3.1.17) and the property of tralation for the Fourier's transform, we have

$$\begin{aligned}\widehat{D}(k,\lambda) &= \left(e^{it\phi(k)}\ \eta_T(t)\int_0^t e^{-it'\phi(k)}\widehat{f}(k,t')\,dt'\right)^{\wedge}(\lambda)\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\widehat{f}(k,z+\phi(k))\left(\frac{\widehat{\eta_T}(\lambda-\phi(k)-z)-\widehat{\eta_T}(\lambda-\phi(k))}{iz}\right)\,dz.\end{aligned}$$

Using the $L^1(\mathbb{R})-$invariance under translations, we obtain

$$\begin{aligned}
\|D(x,t)\|_{Y_{s,0}} &\leqslant C \left\| \langle k\rangle^s \left\| \int_{\mathbb{R}} \widehat{f}(k,z+\phi(k)) \left(\frac{\widehat{\eta_T}(\lambda-\phi(k)-z)-\widehat{\eta_T}(\lambda-\phi(k))}{iz} \right) dz \right\|_{L^1_\lambda(\mathbb{R})} \right\|_{l^2_k} \\
&\leqslant C \left\| \int_{|z|\leqslant 1} \langle k\rangle^s |\widehat{f}(k,z+\phi(k))| \left\| \frac{|\widehat{\eta_T}(\lambda-z)-\widehat{\eta_T}(\lambda)|}{|z|} \right\|_{L^1_\lambda(\mathbb{R})} dz \right\|_{l^2_k} \\
&\quad + C \left\| \int_{|z|>1} \left(\frac{\langle k\rangle^s |\widehat{f}(k,z+\phi(k))|}{|z|} \right) \|\widehat{\eta_T}(\lambda-z)-\widehat{\eta_T}(\lambda)\|_{L^1_\lambda(\mathbb{R})} \ dz \right\|_{l^2_k} .
\end{aligned} \tag{3.1.18}$$

If $|z|\leqslant 1$, we use the Mean Value Theorem to get some t'^* between $t'-z$ and t' such that

$$\left\| \frac{|\widehat{\eta_T}(\lambda-z)-\widehat{\eta_T}(\lambda)|}{|z|} \right\|_{L^1_\lambda(\mathbb{R})} \leqslant T^2 \left\| \sup_{|\lambda *-\lambda|\leqslant|z|<1} |\widehat{\eta}'(T\ \lambda^*)| \right\|_{L^1_\lambda(\mathbb{R})} =: A_{T,\eta} < \infty. \tag{3.1.19}$$

On the other hand, if $|z|>1$, we use the $L^1(\mathbb{R})-$norm invariance under translations to get

$$\begin{aligned}
\|\widehat{\eta_T}(\lambda-z)-\widehat{\eta_T}(\lambda)\|_{L^1_\lambda(\mathbb{R})} &\leqslant \|\widehat{\eta_T}(\lambda-z)\|_{L^1_\lambda(\mathbb{R})} + \|\widehat{\eta_T}(\lambda)\|_{L^1_\lambda(\mathbb{R})} \\
&\leqslant 2\ \|\widehat{\eta_T}(\lambda)\|_{L^1_\lambda(\mathbb{R})} =: B_{T,\eta} < \infty.
\end{aligned} \tag{3.1.20}$$

From (3.1.18)-(3.1.20), we infer

$$\begin{aligned}
\|D(x,t)\|_{Y_{s,0}} &\leqslant C A_{T,\eta} \left\| \int_{|z|\leqslant 1} \langle k\rangle^s |\widehat{f}(k,z+\phi(k))|\ dz \right\|_{l^2_k} \\
&\quad + C B_{T,\eta} \left\| \int_{|z|>1} \left(\frac{\langle k\rangle^s |\widehat{f}(k,z+\phi(k))|}{|z|} \right) dz \right\|_{l^2_k} .
\end{aligned} \tag{3.1.21}$$

Note that, if $|z|\leqslant 1$, we use Lemma 3.197 in [41] to get some constant $M>0$ such that

$$\langle z\rangle \leqslant M\ (1+|z|) \leqslant 2M. \tag{3.1.22}$$

On the other hand, if $|z|>1$ we have

$$\langle z\rangle \leqslant M\ (1+|z|) \leqslant M + M\ |z| \leqslant 2M\ |z|. \tag{3.1.23}$$

From the estimates (3.1.21)-(3.1.23), we obtain that

$$\begin{aligned}
\left\| \eta_T(t) \int_0^t V(t-t')f(t')\ dt' \right\|_{Y_{s,0}} &\leqslant c A_{T,\eta} 2M \left\| \int_{|z|\leqslant 1} \frac{1}{2M} \langle k\rangle^s |\widehat{f}(k,z+\phi(k))|\ dz \right\|_{l^2_k} \\
&\quad + c B_{T,\eta} 2M \left\| \int_{|z|>1} \left(\frac{\langle k\rangle^s |\widehat{f}(k,z+\phi(k))|}{2M|z|} \right) \ dz \right\|_{l^2_k} \\
&\leqslant C_{T,\eta} \left\| \int_{|z|\leqslant 1} \langle z\rangle^{-1} \langle k\rangle^s |\widehat{f}(k,z+\phi(k))|\ dz \right\|_{l^2_k} \\
&\quad + C_{T,\eta} \left\| \int_{|z|>1} \langle z\rangle^{-1} \langle k\rangle^s |\widehat{f}(k,z+\phi(k))|\ dz \right\|_{l^2_k} \\
&\leqslant 2C_{T,\eta} \left\| \int_{\mathbb{R}} \langle z\rangle^{-1} \langle k\rangle^s |\widehat{f}(k,z+\phi(k))|\ dz \right\|_{l^2_k} .
\end{aligned}$$

Performing the change of variable $\lambda = z + \phi(k)$ in the last integral, we get that

$$\begin{aligned}\left\|\eta_T(t)\int_0^t V(t-t')f(t')\,dt'\right\|_{Y_{s,0}} &\leqslant 2C_{T,\eta}\left\|\int_{\mathbb{R}}\langle\lambda-\phi(k)\rangle^{-1}\langle k\rangle^s|\widehat{f}(k,\lambda)|\,d\lambda\right\|_{l^2_k} \\ &= 2C_{T,\eta}\,\|f\|_{Y_{s,-1}}\end{aligned} \tag{3.1.24}$$

Inequalities (3.1.15), and (3.1.24) prove the estimate in i).

Finally, let $f \in Z_{s,-\frac{1}{2}}(I)$, with $I = [-T,T]$ be given. We consider an extension $\widetilde{f}$ to $\mathbb{T}\times\mathbb{R}$ such that $\|\widetilde{f}\|_{Z_{s,-\frac{1}{2}}} \leqslant 2\|f\|_{Z_{s,-\frac{1}{2}}(I)}$. Therefore,

$$\begin{aligned}\left\|\int_0^t V(t-t')f(t')\,dt'\right\|_{Z_{s,\frac{1}{2}}(I)} &\leqslant \left\|\eta_T(t)\,\int_0^t V(t-t')\widetilde{f}(t')\,dt'\right\|_{Z_{s,\frac{1}{2}}} \\ &\leqslant C_{\eta,T}\,\|\widetilde{f}\|_{Z_{s,-\frac{1}{2}}} \\ &\leqslant 2\,C_{\eta,T}\,\|f\|_{Z_{s,-\frac{1}{2}}(I)}.\end{aligned} \tag{3.1.25}$$

This proves the estimate in ii). Similar arguments can be used to show the estimate in iii). □

The following result is consequence of the fractional Leibniz rule and the Sobolev embedding theorem. Its proof is similar to the proof of Lemma 3.11 in [33, page 66] (see also Lemma 2.11 in [87]).

Proposition 3.1.15. *Let* $0 < T < 1$, $-\frac{1}{2} < b' < b < \frac{1}{2}$, $s \in \mathbb{R}$, *and* $I = [-T,T]$ *be given. Then there exists* $C > 0$, *independent on* T, *such that*

$$\|v\|_{X_{s,b'}(I)} \leqslant C\,T^{b-b'}\|v\|_{X_{s,b}(I)}, \quad \forall v \in X_{s,b}(I).$$

In particular,

$$\|v\|_{X^T_{s,b'}} \leqslant C\,T^{b-b'}\|v\|_{X^T_{s,b}}, \quad \forall v \in X^T_{s,b}. \tag{3.1.26}$$

Proposition 3.1.16 (see [87, page 105])**.** *For all* $s,b \in \mathbb{R}$, $\epsilon > 0$, *and* $v \in X_{s,b}$, *there exists* $C_\epsilon > 0$ *such that*

$$\|v\|_{Y_{s,b-\frac{1}{2}-\epsilon}} \leqslant C_\epsilon\,\|v\|_{X_{s,b}}.$$

Furthermore, for all $T > 0$,

$$\|v\|_{Y_{s,b-\frac{1}{2}-\epsilon}(I)} \leqslant C_\epsilon\,\|v\|_{X_{s,b}(I)}, \quad \forall v \in X_{s,b}(I),\ I = [-T,T]. \tag{3.1.27}$$

Proof. Let be $s,b \in \mathbb{R}$, $\epsilon > 0$, and $v \in X_{s,b}$. Then, applying Cauchy-Schwartz inequality and using the invariance of the norm in $L^2(\mathbb{R})$ we obtain

$$\begin{aligned}\|v\|_{Y_{s,b-\frac{1}{2}-\epsilon}} &= \left\|\|\langle k\rangle^s\langle\tau-\phi(k)\rangle^{b-\frac{1}{2}-\epsilon}\,\widehat{v}(k,\tau)\|_{L^1(\mathbb{R})}\right\|_{l^2_k} \\ &\leqslant \left\|\|\langle\tau-\phi(k)\rangle^{-\frac{1}{2}-\epsilon}\|_{L^2_\tau(\mathbb{R})}\|\langle k\rangle^s\langle\tau-\phi(k)\rangle^b\,\widehat{v}(k,\tau)\|_{L^2(\mathbb{R})}\right\|_{l^2_k}.\end{aligned}$$

Performing the change of variables $\theta = \tau - \phi(k)$, one has

$$\|\langle \tau - \phi(k)\rangle^{-\frac{1}{2}-\epsilon}\|_{L^2_\tau(\mathbb{R})} = \|\langle \theta\rangle^{-\frac{1}{2}-\epsilon}\|_{L^2_\theta(\mathbb{R})} \leqslant c\left(\int_{\mathbb{R}} (1+|\theta|)^{-1-2\epsilon} d\theta\right)$$
$$\leqslant 2^{\frac{1}{2}} c\left(\int_0^\infty (1+|\theta|)^{-1-2\epsilon} d\theta\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} c \frac{1}{(2\epsilon)^{\frac{1}{2}}} < \infty,$$

where we used that $\epsilon > 0$. Hence,

$$\|v\|_{Y_{s,b-\frac{1}{2}-\epsilon}} \leqslant C_\epsilon \left\|\|\langle k\rangle^s \langle \tau - \phi(k)\rangle^b \widehat{v}(k,\tau)\|_{L^2(\mathbb{R})}\right\|_{l^2_k} = C_\epsilon \|v\|_{X_{s,b}}.$$

Finally, taking $v \in X_{s,b}(I)$ and considering $\widetilde{v}$ an extension in $X_{s,b}$ such that $\|\widetilde{v}\|_{X_{s,b}} \leqslant 2\|v\|_{X_{s,b}(I)}$, we obtain that

$$\|v\|_{Y_{s,b-\frac{1}{2}-\epsilon}(I)} \leqslant \|\widetilde{v}\|_{Y_{s,b-\frac{1}{2}-\epsilon}} \leqslant C_\epsilon \|\widetilde{v}\|_{X_{s,b}} \leqslant 2C_\epsilon \|v\|_{X_{s,b}(I)}.$$

□

Remark 3.1.17. *Similar arguments as those in the proof of Inequality (3.1.27) shows that*

$$\|v\|_{Y^T_{s,b-\frac{1}{2}-\epsilon}} \leqslant C_\epsilon \|v\|_{X^T_{s,b}},$$

for all $T > 0$, $s, b \in \mathbb{R}$, $\epsilon > 0$, and $v \in X^T_{s,b}$.

3.1.3 Nonlinear estimates

We start with the following result, which is fundamental to estimate the nonlinear term $2u\partial_x u$, in the $Z_{s,-\frac{1}{2}}-$norm.

Theorem 3.1.18. *Let $v : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be a function in $X_{0,\frac{1}{3}}$. Then, there exists $C_\alpha > 0$, depending only on α such that*

$$\|v\|_{L^4(\mathbb{T}\times\mathbb{R})} \leqslant C_\alpha \|v\|_{X_{0,\frac{1}{3}}}.$$

Proof. Let $v \in \mathcal{S}(\mathbb{T} \times \mathbb{R})$. Observe that

$$\|v\|^2_{X_{0,\frac{1}{3}}} \sim \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} (1 + |\tau - \phi(k)|)^{\frac{2}{3}} |\widehat{v}(k,\tau)|^2 \, d\tau.$$

Also note that, we can write $v(x,t) = \sum_{m=0}^{\infty} v_{2^m}(x,t)$, where

$$\widehat{v_{2^m}}(k,\tau) = \widehat{v}(k,\tau) \cdot \chi_{2^m \leqslant 1+|\tau-\phi(k)|<2^{m+1}},$$

and $\chi_{2^m \leqslant 1+|\tau-\phi(k)|<2^{m+1}}$ is the characteristic function over the set $2^m \leqslant 1+|\tau-\phi(k)| < 2^{m+1}$. In this way, we have

$$\|v\|^2_{X_{0,\frac{1}{3}}} \sim \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \left|\sum_{m=0}^{\infty} \widehat{v}(k,\tau) \cdot 2^{\frac{m}{3}} \cdot \chi_{2^m \leqslant 1+|\tau-\phi(k)|<2^{m+1}}\right|^2 \, d\tau. \tag{3.1.28}$$

Now, using Plancherel's inequality in (3.1.28), we obtain

$$\|v\|^2_{X_{0,\frac{1}{3}}} \sim \sum_{m=0}^{\infty} 2^{\frac{2m}{3}} \|v_{2^m}\|^2_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.29}$$

On the other hand,

$$\|v\|^2_{L^4(\mathbb{T}\times\mathbb{R})} = \|v^2\|_{L^2(\mathbb{T}\times\mathbb{R})} \leqslant 2\sum_{m\leqslant m'} \|v_{2^m} v_{2^{m'}}\|_{L^2(\mathbb{T}\times\mathbb{R})} = 2\sum_{m,n\geqslant 0} \|v_{2^m} v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.30}$$

Once again, using Plancherel's identity, we get

$$\|v_{2^m} v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} = \left\| \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1,\tau_1)\, \widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)\, d\tau_1 \right\|_{l^2_k L^2_\tau}. \tag{3.1.31}$$

We estimate the RHS of (3.1.31) separately in the range $|k| \leqslant 2^a([\alpha]+2)$ and $|k| > 2^a([\alpha]+2)$, where the natural number a will be determined later.

$$\begin{aligned} \|v_{2^m} v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} &\leqslant \left(\sum_{|k|\leqslant 2^a([\alpha]+2)} \left\| \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1,\tau_1)\, \widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)\, d\tau_1 \right\|^2_{L^2_\tau} \right)^{\frac{1}{2}} \\ &\quad + \left(\sum_{|k|> 2^a([\alpha]+2)} \left\| \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1,\tau_1)\, \widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)\, d\tau_1 \right\|^2_{L^2_\tau} \right)^{\frac{1}{2}} \\ &=: I + II. \end{aligned} \tag{3.1.32}$$

To estimate I, we use the triangular and Young's (see [49, Theorem 2.2]) inequalities to obtain

$$\begin{aligned} \left\| \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1,\tau_1)\, \widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)\, d\tau_1 \right\|_{L^2_\tau} &\leqslant \sum_{k_1\in\mathbb{Z}} \left\| \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1,\tau_1)\, \widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)\, d\tau_1 \right\|_{L^2_\tau} \\ &\leqslant \sum_{k_1\in\mathbb{Z}} \|\widehat{v_{2^m}}(k_1,\cdot)\|_{L^1(\mathbb{R})} \|\widehat{v_{2^{m+n}}}(k-k_1,\cdot)\|_{L^2(\mathbb{R})}. \end{aligned} \tag{3.1.33}$$

Now applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|\widehat{v_{2^m}}(k_1,\cdot)\|_{L^1(\mathbb{R})} &\leqslant \left(\mu(\{\tau_1 : 2^m \leqslant 1+|\tau_1-\phi(k_1)| < 2^{m+1}\})\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \left|\widehat{v}(k_1,\tau_1)\cdot \chi_{2^m\leqslant 1+|\tau_1-\phi(k_1)|<2^{m+1}}\right|^2 d\tau_1 \right)^{\frac{1}{2}} \\ &\leqslant C_2\, 2^{\frac{m}{2}} \|\widehat{v_{2^m}}(k_1,\cdot)\|_{L^2(\mathbb{R})}, \end{aligned} \tag{3.1.34}$$

where μ is the Lebesgue measure. Therefore, applying Cauchy-Schwarz inequality, Plancherel, and the invariance of the norm under translations, we obtain

$$\left\| \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1,\tau_1)\, \widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)\, d\tau_1 \right\|_{L^2_\tau} \leqslant C_2\, 2^{\frac{m}{2}} \|v_{2^m}\|_{L^2(\mathbb{T}\times\mathbb{R})} \cdot \|v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.35}$$

Therefore, from definition of I in (3.1.32) and inequalities (3.1.33), (3.1.34), and (3.1.35), we get

$$I \leqslant C_3(\alpha)\, 2^{\frac{a+m}{2}} \|v_{2^m}\|_{L^2(\mathbb{T}\times\mathbb{R})} \cdot \|v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.36}$$

To estimate II, define

$$\theta := \left\| \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} \widehat{v_{2^m}}(k_1, \tau_1)\, \widehat{v_{2^{m+n}}}(k - k_1, \tau - \tau_1)\, d\tau_1 \right\|_{L^2_\tau}. \tag{3.1.37}$$

First, denote $\chi_m(k,\tau) := \chi_{2^m \leqslant 1+|\tau-\phi(k)|<2^{m+1}}(\tau)$. Applying Cauchy-Schwartz inequality in k_1 and τ_1, we get

$$\theta \leqslant \sup_{|k|>2^a([\alpha]+2)} \sup_{\tau\in\mathbb{R}} (\, \chi_m * \chi_{m+n}(k,\tau)\,)^{\frac{1}{2}} \left\| \left(\sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} |\widehat{v_{2^m}}(k_1,\tau_1)|^2\, |\widehat{v_{2^{m+n}}}(k-k_1,\tau-\tau_1)|^2\, d\tau_1 \right)^{\frac{1}{2}} \right\|_{L^2_\tau}. \tag{3.1.38}$$

Therefore, using the translation invariance of the norm, and Plancherel's inequality, we obtain

$$II \leqslant \|\chi_m * \chi_{m+n}(k,\tau)\|^{\frac{1}{2}}_{l^\infty_{|k|>2^a([\alpha]+2)} L^\infty_\tau} \cdot \|v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} \cdot \|v_{2^m}\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.39}$$

To estimate the convolution term in inequality (3.1.39), we write for fixed k with $|k| > 2^a(2+[\alpha])$ and τ

$$\chi_m * \chi_{m+n}(k,\tau) = \sum_{k_1\in\mathbb{Z}} \int_{\mathbb{R}} \chi_m(k_1,\tau_1)\cdot \chi_{m+n}(k-k_1,\tau-\tau_1)\, d\tau_1. \tag{3.1.40}$$

From the support condition on χ_m and χ_{m+n} we note that for each k_1 fixed there exist $C_4 \geqslant 0$ and $C_5 > 0$ such that

$$C_4 2^m \leqslant |\tau_1 - \phi(k_1)| < C_2 2^m.$$

Thus, $\tau_1 = \phi(k_1) + O(2^m)$. In a similar way, we have that $\tau - \tau_1 = \phi(k-k_1) + O(2^{m+n})$. In consequence,

$$\tau = \phi(k_1) + \phi(k-k_1) + O(2^{m+n}), \tag{3.1.41}$$

and

$$\int_{\mathbb{R}} \chi_m(k_1,\tau_1)\cdot\chi_{m+n}(k-k_1,\tau-\tau_1)\, d\tau_1 \leqslant \mu(\{\tau_1 \in \mathbb{R} : 2^m \leqslant 1 + |\tau_1 - \phi(k_1)| < 2^{m+1}\}).$$

Therefore, for each fixed k_1, the τ_1 integral in (3.1.40) is $O(2^m)$. To calculate the numbers of $k_1's$ for which the integral is non-zero, note that (3.1.41) implies

$$\frac{\tau}{k} = -k^2 + 3kk_1 - 3k_1^2 - 2\mu + \frac{\alpha k_1|k_1|}{k} + \frac{\alpha(k-k_1)|k-k_1|}{k} + O(C_6(\alpha)2^{m+n-a}). \tag{3.1.42}$$

Thus, we should study four cases:

Case 1) $k - k_1 \geqslant 0$ and $k_1 \geqslant 0$: By identity (3.1.42), we have

$$\frac{\alpha k_1^2}{k} + \frac{\alpha(k-k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).$$

Note that $3kk_1 - 3k_1^2 + \frac{\alpha k_1^2}{k} + \frac{\alpha(k-k_1)^2}{k} = \left(\frac{2\alpha}{k} - 3\right)\left(k_1^2 - kk_1\right) + \alpha k$. Therefore,

$$\left(\frac{2\alpha}{k} - 3\right)\left(k_1 - \frac{k}{2}\right)^2 = \frac{\tau}{k} + \left(\frac{2\alpha}{k} - 3\right)\frac{k^2}{4} - \alpha k + k^2 - 2\mu + O(C_6(\alpha)2^{m+n-a}).$$

Using that $|k| > 2^a([\alpha]+2)$, we have $\left|\frac{2\alpha}{k} - 3\right| > 1$. This implies,

$$\left(k_1 - \frac{k}{2}\right)^2 = \frac{\tau}{2\alpha - 3k} + \frac{k^2}{4} - \frac{\alpha k^2}{2\alpha - 3k} + \frac{k^3}{2\alpha - 3k} - \frac{2\mu k}{2\alpha - 3k} + O(C_6(\alpha)2^{m+n-a}).$$

Case 2) $k - k_1 \geqslant 0$ and $k_1 \leqslant 0$: In this case identity (3.1.42) implies,

$$-\frac{\alpha k_1^2}{k} + \frac{\alpha(k-k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).$$

With the similar calculations as in **Case 1**, we obtain

$$\left(k_1 - \left(-\frac{2\alpha}{3k} + 1\right)\frac{k}{2}\right)^2 = -\frac{\tau}{3k} + \left(-\frac{2\alpha}{3k} + 1\right)^2\frac{k^2}{4} + \frac{\alpha k}{3} - \frac{k^2}{3} - \frac{2\mu}{3} + O(C_6(\alpha)2^{m+n-a}).$$

Case 3) $k - k_1 \leqslant 0$ and $k_1 \leqslant 0$: In this case identity (3.1.42) implies,

$$-\frac{\alpha k_1^2}{k} - \frac{\alpha(k-k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).$$

Thus,

$$\left(-\frac{2\alpha}{k} - 3\right)\left(k_1 - \frac{k}{2}\right)^2 = \frac{\tau}{k} + \left(-\frac{-2\alpha}{k} - 3\right)\frac{k^2}{4} + \alpha k + k^2 + 2\mu + O(C_6(\alpha)2^{m+n-a}).$$

Using $|k| > 2^a([\alpha]+2)$, we observe that $\left|-\frac{2\alpha}{k} - 3\right| > 1$. Therefore,

$$\left(k_1 - \frac{k}{2}\right)^2 = \frac{\tau}{-2\alpha - 3k} + \frac{k^2}{4} + \frac{\alpha k^2}{-2\alpha - 3k} + \frac{k^3}{-2\alpha - 3k} + \frac{2\mu k}{-2\alpha - 3k} + O(C_6(\alpha)2^{m+n-a}).$$

Case 4) $k - k_1 \leqslant 0$ and $k_1 \geqslant 0$: In this case identity (3.1.42) implies,

$$\frac{\alpha k_1^2}{k} - \frac{\alpha(k-k_1)^2}{k} + 3kk_1 - 3k_1^2 - k^2 - 2\mu = \frac{\tau}{k} + O(C_6(\alpha)2^{m+n-a}).$$

Thus,

$$\left(k_1 - \left(\frac{2\alpha}{3k} + 1\right)\frac{k}{2}\right)^2 = -\frac{\tau}{3k} + \left(\frac{2\alpha}{3k} + 1\right)^2\frac{k^2}{4} - \frac{\alpha k}{3} - \frac{k^2}{3} - \frac{2\mu}{3} + O(C_6(\alpha)2^{m+n-a}).$$

Therefore, in all cases k_1 takes at most $O(C_6(\alpha)2^{\frac{m+n-a}{2}})$ values. Thus,

$$\|\chi_m * \chi_{m+n}(k,\tau)\|_{l^\infty_{|k|>2^a([\alpha]+2)}L^\infty_\tau} \leqslant C_7(\alpha)2^m \cdot 2^{\frac{m+n-a}{2}} = C_7(\alpha)2^{\frac{3m+n-a}{2}}. \tag{3.1.43}$$

We can conclude from inequalities (3.1.39) and (3.1.43) that

$$II \leqslant C_7(\alpha) 2^{\frac{3m+n-a}{4}} \|v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} \|v_{2^m}\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.44}$$

Using estimates (3.1.32), (3.1.36), and (3.1.44) implies

$$\|v_{2^m} v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} \leqslant C_7(\alpha) \left(2^{\frac{m+a}{2}} + 2^{\frac{3m+n-a}{4}}\right) \|v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} \|v_{2^m}\|_{L^2(\mathbb{T}\times\mathbb{R})}.$$

Taking $a = \dfrac{m+n}{3}$, we obtain

$$\|v_{2^m} v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} \leqslant C_8(\alpha) 2^{\frac{4m+n}{6}} \|v_{2^{m+n}}\|_{L^2(\mathbb{T}\times\mathbb{R})} \|v_{2^m}\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{3.1.45}$$

Therefore, inequalities (3.1.30) and (3.1.45) imply

$$\|v\|^2_{L^4(\mathbb{T}\times\mathbb{R})} \leqslant 2C_8(\alpha) \sum_{n\geqslant 0} 2^{-\frac{n}{6}} \left(\sum_{m\geqslant 0} 2^{\frac{2(m+n)}{3}} \|v_{2^{m+n}}\|^2_{L^2(\mathbb{T}\times\mathbb{R})}\right)^{\frac{1}{2}} \left(\sum_{m\geqslant 0} 2^{\frac{2m}{3}} \|v_{2^m}\|^2_{L^2(\mathbb{T}\times\mathbb{R})}\right)^{\frac{1}{2}}. \tag{3.1.46}$$

Thus from inequality (3.1.46) and identity (3.1.29), we obtain

$$\|v\|^2_{L^4(\mathbb{T}\times\mathbb{R})} \leqslant C_9(\alpha) \|v\|^2_{X_{0,\frac{1}{3}}} \left(\sum_{n\geqslant 0} 2^{-\frac{n}{6}}\right) \leqslant C(\alpha) \|v\|^2_{X_{0,\frac{1}{3}}}.$$

□

Corollary 3.1.19. *Let $f \in L^{\frac{4}{3}}(\mathbb{T}\times\mathbb{R})$. Then, there exists $C_\alpha > 0$, such that*

$$\|f\|_{X_{0,-\frac{1}{3}}} = \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle \tau - \phi(k)\rangle^{-\frac{2}{3}} |\widehat{f}(k,\tau)|^2 \, d\tau\right)^{\frac{1}{2}} \leqslant C_\alpha \|f\|_{L^{\frac{4}{3}}(\mathbb{T}\times\mathbb{R})}.$$

Proof. It follows from Theorem 3.1.18 that $X_{0,\frac{1}{3}} \hookrightarrow L^4(\mathbb{T}\times\mathbb{R})$, so $\left(L^4(\mathbb{T}\times\mathbb{R})\right)' \hookrightarrow \left(X_{0,\frac{1}{3}}\right)'$ i.e., $L^{\frac{4}{3}}(\mathbb{T}\times\mathbb{R}) \hookrightarrow X_{0,-\frac{1}{3}}$. □

Lemma 3.1.20. *For all $k, k_1 \in \mathbb{Z}$ with $k \neq 0$, $k_1 \neq 0$, and $k \neq k_1$, we have*

i) $|3kk_1(k-k_1)| \geqslant \dfrac{3}{2}k^2$.

ii) $|k_1(k-k_1)| \geqslant \dfrac{1}{2}|k|$.

Proof. The proof of *i)* follows by simple calculations considering six possible cases:

$$\begin{aligned}
R^{+++} &:= \{k-k_1 > 0, k > 0, k_1 > 0\}\\
R^{++-} &:= \{k-k_1 > 0, k > 0, k_1 < 0\}\\
R^{+--} &:= \{k-k_1 > 0, k < 0, k_1 < 0\}\\
R^{---} &:= \{k-k_1 < 0, k < 0, k_1 < 0\}\\
R^{--+} &:= \{k-k_1 < 0, k < 0, k_1 > 0\}\\
R^{-++} &:= \{k-k_1 < 0, k > 0, k_1 > 0\}.
\end{aligned}$$

In the first case R^{+++}, observe that

$$3kk_1(k-k_1) \geqslant \frac{3}{2}k^2 \Leftrightarrow k \geqslant \frac{k_1^2}{k_1 - \frac{1}{2}}.$$

The fact $k > k_1$ implies that the right side of the last expression is always true. The others cases are similar. Finally, note that $ii)$ is consequence of $i)$ just dividing it by $|3k|$. $\square$

Lemma 3.1.21. *For any $k \in \mathbb{Z}$, $\alpha > 0$ and $\mu \in \mathbb{R}$, let $\phi(k) = -k^3 - 2\mu k + \alpha k|k|$. For all $k, k_1 \in \mathbb{Z}$ with $k \neq 0$, $k_1 \neq 0$, $k \neq k_1$, and $\max\{|k|, |k_1|, |k-k_1|\} \geqslant \max\left\{1, \frac{4\alpha}{3}\right\}$, there exists a constant $C_\alpha > 0$ depending only on α such that,*

$$|E(k,k_1)| \geqslant 3C_\alpha |kk_1(k-k_1)|,$$

where $E(k,k_1) := (\tau - \phi(k)) - (\tau_1 - \phi(k_1)) - (\tau - \tau_1 - \phi(k-k_1))$, $\forall \tau, \tau_1 \in \mathbb{R}$. Moreover,

$$C_\alpha = 1 - \frac{\alpha}{3}, \quad \textit{if} \quad 0 < \alpha < 3,$$

and

$$C_\alpha = \min\left\{-1 + \frac{2\alpha}{3[\frac{2\alpha}{3}]}, 1 - \frac{2\alpha}{3([\frac{2\alpha}{3}]+1)}\right\}, \quad \textit{if} \quad \alpha > 3.$$

Proof. Observe that

$$E(k,k_1) = -\alpha k|k| + \alpha k_1|k_1| + \alpha(k-k_1)|k-k_1| + 3kk_1(k-k_1).$$

Again, the proof follows by straightforward calculations considering the same six cases of Lemma 3.1.20. We verify three cases, others are similar.

Case 1) In the region R^{+++}, one has $k \geqslant k_1 + 1 \geqslant 2$. Thus

$$E(k,k_1) = -\alpha kk + \alpha k_1 k_1 + \alpha(k-k_1)^2 + 3kk_1(k-k_1) = 3k_1(k-k_1)\left(k - \frac{2\alpha}{3}\right).$$

Also, in this case, $\max\{|k|, |k_1|, |k-k_1|\} = k$. Therefore,

$$|E(k,k_1)| = 3k_1(k-k_1)\left|k - \frac{2\alpha}{3}\right| \geqslant 3k_1(k-k_1)C_\alpha k = 3C_\alpha|kk_1(k-k_1)|.$$

Case 2) In R^{++-}, note that $k - k_1 \in \mathbb{Z}$ with $k - k_1 \geqslant 2$.

$$E(k,k_1) = -\alpha kk + \alpha k_1(-k_1) + \alpha(k-k_1)^2 + 3kk_1(k-k_1) = 3kk_1\left((k-k_1) - \frac{2\alpha}{3}\right).$$

In this case, $\max\{|k|, |k_1|, |k-k_1|\} = k - k_1$. Thus

$$|E(k,k_1)| = 3k(-k_1)\left|(k-k_1) - \frac{2\alpha}{3}\right| \geqslant 3k(-k_1)(k-k_1)C_\alpha = 3C_\alpha|kk_1(k-k_1)|.$$

Case 5) In R^{--+}, $k \leqslant -1$, $k_1 \geqslant 1$, and $k - k_1 \leqslant -2$. Then,

$$E(k,k_1) = -\alpha k(-k) + \alpha k_1 k_1 - \alpha(k-k_1)^2 + 3kk_1(k-k_1) = 3kk_1\left((k-k_1) + \frac{2\alpha}{3}\right).$$

Note that, $\max\{|k|,|k_1|,|k-k_1|\} = -(k-k_1)$. Therefore

$$|E(k,k_1)| = 3(-k)k_1\left|-(k-k_1) - \frac{2\alpha}{3}\right| \geqslant 3(-k)k_1|k-k_1|C_\alpha = 3C_\alpha|kk_1(k-k_1)|.$$

$\square$

Remark 3.1.22. *Lemmas 3.1.20 and 3.1.21 imply the so-called non-resonance property for Benjamin equation, it means,* $|E(k,k_1)| \geqslant \frac{3}{2}C_\alpha k^2$*, provided that*

$$\max\{|k|,|k_1|,|k-k_1|\} \geqslant \max\left\{1, \frac{4\alpha}{3}\right\}.$$

Remark 3.1.23. *It follows from Lemma 3.1.21 that if* $\max\{|k|,|k_1|,|k-k_1|\} \geqslant \max\left\{1, \frac{4\alpha}{3}\right\}$, *then one of the following cases may occur*

i) $|\tau - \phi(k)| > \frac{3}{8}C_\alpha k^2$,

ii) $|\tau_1 - \phi(k_1)| > \frac{3}{8}C_\alpha k^2$,

iii) $|\tau - \tau_1 - \phi(k-k_1)| > \frac{3}{8}C_\alpha k^2$.

In fact, if not

$$|E(k,k_1)| \leqslant |(\tau - \phi(k))| + |(\tau_1 - \phi(k_1))| + |(\tau - \tau_1 - \phi(k-k_1))| \leqslant 3\frac{3C_\alpha}{8}k^2 < \frac{3C_\alpha}{2}k^2.$$

which is a contradiction.

Using similar arguments as in Bourgain [14] (see also [19, 86]), we obtain the following key bilinear estimate.

Theorem 3.1.24. *(Bilinear Estimate) Let* $s \geqslant 0$, $\alpha > 0$, *and* $u, v : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ *be functions in* $X_{s,\frac{1}{2}}$. *Assume that the mean* $[u(\cdot,t)] = [v(\cdot,t)] = 0$ *for each* $t \in \mathbb{R}$. *Then*

$$\|\partial_x(uv)\|_{Z_{s,-\frac{1}{2}}} \leqslant C_{\alpha,s}\left(\|u\|_{X_{s,\frac{1}{2}}}\|v\|_{X_{s,\frac{1}{3}}} + \|u\|_{X_{s,\frac{1}{3}}}\|v\|_{X_{s,\frac{1}{2}}}\right).$$

Proof. We prove this in two steps.

Step 1. First we estimate the $X_{s,-\frac{1}{2}}$ norm. Using duality and Plancherel, we get

$$\begin{aligned}
\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}} &= \sup_{\substack{w\in X_{-s,\frac{1}{2}}\\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left|\int_{\mathbb{T}}\int_{\mathbb{R}} \partial_x(uv)(x,t)w(x,t)dxdt\right| \\
&= \sup_{\substack{w\in X_{-s,\frac{1}{2}}\\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \widehat{\partial_x(uv)}(k,\tau)\widehat{w}(k,\tau)d\tau\right| \qquad (3.1.47)\\
&\leqslant \sup_{\substack{w\in X_{-s,\frac{1}{2}}\\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left(\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{\substack{k_1\in\mathbb{Z}\\ k_1\neq 0}}\int_{\mathbb{R}}\int_{\mathbb{R}} |k||\widehat{u}(k_1,\tau_1)||\widehat{v}(k-k_1,\tau-\tau_1)||\widehat{w}(k,\tau)|d\tau_1 d\tau\right).
\end{aligned}$$

Since $[u(\cdot,t)] = [v(\cdot,t)] = 0$, then $k = 0$, $k_1 = 0$ and $k - k_1 = 0$ do not contribute to the sum. Now we move to estimate

$$\begin{aligned}
I &:= \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\sum_{\substack{k_1\in\mathbb{Z}\\ k_1\neq 0}}\int_{\mathbb{R}}\int_{\mathbb{R}} |k||\widehat{u}(k_1,\tau_1)||\widehat{v}(k-k_1,\tau-\tau_1)||\widehat{w}(k,\tau)|d\tau_1 d\tau \\
&= \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0}}\int_{\mathbb{R}^2} \frac{|k||k_1|^s\langle\tau_1-\phi(k_1)\rangle^{\frac{1}{2}}|\widehat{u}(k_1,\tau_1)||k-k_1|^s\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}}{|k_1|^s\langle\tau_1-\phi(k_1)\rangle^{\frac{1}{2}}|k-k_1|^s\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}} \qquad (3.1.48)\\
&\qquad\times\frac{|\widehat{v}(k-k_1,\tau-\tau_1)|\langle k\rangle^{-s}\langle\tau-\phi(k)\rangle^{\frac{1}{2}}|\widehat{w}(k,\tau)|}{\langle k\rangle^{-s}\langle\tau-\phi(k)\rangle^{\frac{1}{2}}}d\tau_1 d\tau.
\end{aligned}$$

Let $u, v, w : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ with $[u(\cdot,t)] = [v(\cdot,t)] = 0, \quad \forall t \in \mathbb{R}$. We define

$$\begin{aligned}
c_u(k_1,\tau_1) &:= (1+|k_1|)^s\,\langle\tau_1-\phi(k_1)\rangle^{\frac{1}{2}}|\widehat{u}(k_1,\tau_1)|,\\
c_v(k-k_1,\tau-\tau_1) &:= (1+|k-k_1|)^s\,\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}|\widehat{v}(k-k_1,\tau-\tau_1)|, \qquad (3.1.49)\\
c_w(k,\tau) &:= \langle k\rangle^{-s}\langle\tau-\phi(k)\rangle^{\frac{1}{2}}|\widehat{w}(k,\tau)|,
\end{aligned}$$

for all $k, k_1, k-k_1 \in \mathbb{Z}^*$ and $\tau,\tau_1 \in \mathbb{R}$. Note that $c_u(0,\tau_1) = c_v(0,\tau-\tau_1) = 0$. From inequality (3.1.48) and definition (3.1.49), we obtain

$$I \leqslant \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0}}\int_{\mathbb{R}^2} \frac{|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)\langle k\rangle^s c_w(k,\tau)}{|k_1|^s|k-k_1|^s\langle\tau_1-\phi(k_1)\rangle^{\frac{1}{2}}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}\langle\tau-\phi(k)\rangle^{\frac{1}{2}}}d\tau_1 d\tau. \qquad (3.1.50)$$

From Lemma 3.1.20 part *ii)* we infer $\dfrac{|k|}{|k_1||k-k_1|} \leqslant 2$. Thus, there exists $C_s > 0$ such that

$$\frac{\langle k\rangle^s}{|k_1|^s|k-k_1|^s} \leqslant C_s \qquad (3.1.51)$$

Using the estimates (3.1.50)-(3.1.51) and separating the small frequencies from the large ones, we obtain

$$
\begin{aligned}
I &\lesssim_s \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0}} \int_{\mathbb{R}^2} \frac{|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}\langle\tau-\phi(k)\rangle^{\frac12}}d\tau_1 d\tau \\
&\lesssim_{s,\alpha} \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\leqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}\langle\tau-\phi(k)\rangle^{\frac12}}d\tau_1 d\tau \\
&\quad + \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\geqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}\langle\tau-\phi(k)\rangle^{\frac12}}d\tau_1 d\tau
\end{aligned}
\tag{3.1.52}
$$

In view of Remark 3.1.23 we must study three different cases.

Case 1. $|\tau-\phi(k)| > \dfrac{3}{8}C_\alpha k^2$: In this case, from (3.1.52), we have

$$
\begin{aligned}
I &\lesssim_{s,\alpha} \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\leqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau \\
&\quad + \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\geqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}(1+\frac{3C_\alpha}{8}k^2)^{\frac12}}d\tau_1 d\tau \\
&\lesssim_{s,\alpha} \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \left(\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}} \frac{c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1\right) c_w(k,\tau)d\tau.
\end{aligned}
\tag{3.1.53}
$$

We define functions $F, G : \mathbb{T}\times\mathbb{R}\to\mathbb{C}$ by

$$
\widehat{F}(m,\lambda) = \frac{c_u(m,\lambda)}{(1+|\lambda-\phi(m)|)^{\frac12}}, \quad \text{and} \quad \widehat{G}(m,\lambda) = \frac{c_v(m,\lambda)}{(1+|\lambda-\phi(m)|)^{\frac12}}.
$$

It means

$$
F(x,t) = \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} e^{i(mx+\lambda t)}\frac{c_u(m,\lambda)}{(1+|\lambda-\phi(m)|)^{\frac12}}\,d\lambda,
$$

and

$$
G(x,t) = \sum_{m\in\mathbb{Z}}\int_{\mathbb{R}} e^{i(mx+\lambda t)}\frac{c_v(m,\lambda)}{(1+|\lambda-\phi(m)|)^{\frac12}}\,d\lambda.
$$

From inequalities (3.1.47), (3.1.53), Cauchy-Shwartz and Plancherel, we obtain

$$\begin{aligned}
\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}} &\lesssim_{s,\alpha} \sup_{\substack{w\in X_{-s,\frac{1}{2}}\\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}(\widehat{F}*\widehat{G})(k,\tau)\, c_w(k,\tau)\, d\tau\right)\\
&\lesssim_{s,\alpha} \sup_{\substack{w\in X_{-s,\frac{1}{2}}\\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left|\widehat{F\cdot G}(k,\tau)\right|^2\, d\tau\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|c_w(k,\tau)|^2\, d\tau\right)^{\frac{1}{2}}\\
&\lesssim_{s,\alpha} \sup_{\substack{w\in X_{-s,\frac{1}{2}}\\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left(\left\|\widehat{F.G}(k,\tau)\right\|_{l^2_k L^2_\tau(\mathbb{R})}\|w\|_{X_{-s,\frac{1}{2}}}\right)\\
&= C_{s,\alpha}\, \|F\cdot G\|_{L^2(\mathbb{T}\times\mathbb{R})}\\
&\leqslant C_{s,\alpha}\, \|F\|_{L^4(\mathbb{T}\times\mathbb{R})}\|G\|_{L^4(\mathbb{T}\times\mathbb{R})}\\
&\leqslant C_{s,\alpha}\, \|F\|_{X_{0,\frac{1}{3}}}\|G\|_{X_{0,\frac{1}{3}}},
\end{aligned} \tag{3.1.54}$$

where we applied Cauchy-Schwartz, and Theorem 3.1.18 in the last two inequalities. Note that

$$\begin{aligned}
\|F\|_{X_{0,\frac{1}{3}}} &= \left(\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\langle\tau-\phi(k)\rangle^{\frac{2}{3}}\frac{|c_u(k,\tau)|^2}{1+|\tau-\phi(k)|}\, d\tau\right)^{\frac{1}{2}}\\
&\sim \left(\sum_{k=-\infty}^{\infty}\int_{\mathbb{R}}\langle k\rangle^{2s}\langle\tau-\phi(k)\rangle^{\frac{2}{3}}|\widehat{u}(k,\tau)|^2\, d\tau\right)^{\frac{1}{2}} = \|u\|_{X_{s,\frac{1}{3}}}.
\end{aligned} \tag{3.1.55}$$

In a similar way, we show that $\|G\|_{X_{0,\frac{1}{3}}} = \|v\|_{X_{s,\frac{1}{3}}}$. From the estimates (3.1.54)-(3.1.55), and immersion $X_{s,\frac{1}{2}} \hookrightarrow X_{s,\frac{1}{3}}$, we get

$$\|\partial_x(u.v)\|_{X_{s,-\frac{1}{2}}} \leqslant C_{s,\alpha}\, \|u\|_{X_{s,\frac{1}{2}}}\|v\|_{X_{s,\frac{1}{3}}}$$

where $C_{s,\alpha}$ is a positive constant depending only on s and α.

Case 2. $|\tau_1-\phi(k_1)| > \dfrac{3}{8}C_\alpha k^2$: In this case, (3.1.52) and Cauchy-Schwartz imply

$$\begin{aligned}
I &\lesssim_{s,\alpha} \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\leqslant\max\left\{1,\frac{4\alpha}{3}\right\}}}\int_{\mathbb{R}^2}\frac{c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}\langle\tau-\phi(k)\rangle^{\frac{1}{2}}}d\tau_1 d\tau\\
&+ \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\geqslant\max\left\{1,\frac{4\alpha}{3}\right\}}}\int_{\mathbb{R}^2}\frac{|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)c_w(k,\tau)}{(1+\frac{3}{8}C_\alpha k^2)^{\frac{1}{2}}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}\langle\tau-\phi(k)\rangle^{\frac{1}{2}}}d\tau_1 d\tau\\
&\lesssim_{s,\alpha} \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\frac{1}{(1+|\tau-\phi(k)|)^{\frac{1}{2}}}\left(\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}c_u(k_1,\tau_1)\,\frac{c_v(k-k_1,\tau-\tau_1)}{\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac{1}{2}}}\, d\tau_1\right)c_w(k,\tau)d_\tau\\
&\lesssim_{s,\alpha} \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}(1+|\tau-\phi(k)|)^{-1}|(\widehat{H_u}*\widehat{G})(k,\tau)|^2 d\tau\right)^{\frac{1}{2}}\left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|c_w(k,\tau)|^2\right)^{\frac{1}{2}},
\end{aligned} \tag{3.1.56}$$

where $H_f : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, is a function defined by $\widehat{H_f}(m,\lambda) = c_f(m,\lambda)$. It means,

$$H_f(x,t) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} e^{i(mx+\lambda t)} c_f(m,\lambda)\, d\lambda.$$

From relations (3.1.47)-(3.1.48), (3.1.56) and $(1+|\tau-\phi(k)|)^{-1} < (1+|\tau-\phi(k)|)^{-\frac{2}{3}}$, we have

$$\begin{aligned}
\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}} &\leqslant C_{s,\alpha} \sup_{\substack{w\in X_{-s,\frac{1}{2}} \\ \|w\|_{X_{-s,\frac{1}{2}}}=1}} \left[\left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} (1+|\tau-\phi(k)|)^{-\frac{2}{3}} \left|\widehat{H_u \cdot G}(k,\tau)\right|^2 d\tau \right)^{\frac{1}{2}} \|w\|_{X_{-s,\frac{1}{2}}} \right] \\
&\leqslant C_{s,\alpha} \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle \tau-\phi(k)\rangle^{-\frac{2}{3}} \left|\widehat{H_u \cdot G}(k,\tau)\right|^2 d\tau \right)^{\frac{1}{2}} \\
&\leqslant C_{s,\alpha} \|H_u \cdot G\|_{L^{\frac{4}{3}}(\mathbb{T}\times\mathbb{R})} \\
&\leqslant C_{s,\alpha} \|H_u\|_{L^2(\mathbb{T}\times\mathbb{R})} \|G\|_{L^4(\mathbb{T}\times\mathbb{R})},
\end{aligned} \tag{3.1.57}$$

where we have applied Corollary 3.1.19, and Hölder's inequality (see [15, page 118]) in the two last inequalities. From Theorem 3.1.18, we have that $\|G\|_{L^4(\mathbb{T}\times\mathbb{R})} \leqslant \|G\|_{X_{0,\frac{1}{3}}}$. On the other hand,

$$\begin{aligned}
\|H_u\|_{L^2(\mathbb{T}\times\mathbb{R})} &= \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} |c_u(k,\tau)|^2\, d\tau \right)^{\frac{1}{2}} \\
&\sim \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k\rangle^{2s} \langle \tau-\phi(k)\rangle^{2\cdot\frac{1}{2}} |\widehat{u}(k,\tau)|^2\, d\tau \right)^{\frac{1}{2}} = \|u\|_{X_{s,\frac{1}{2}}}.
\end{aligned} \tag{3.1.58}$$

From relations (3.1.57)-(3.1.58), we obtain

$$\|\partial_x(uv)\|_{X_{s,-\frac{1}{2}}} \leqslant C_{s,\alpha} \|u\|_{X_{s,\frac{1}{2}}} \|G\|_{X_{0,\frac{1}{3}}} \leqslant C_{s,\alpha} \|u\|_{X_{s,\frac{1}{2}}} \|v\|_{X_{s,\frac{1}{3}}}.$$

Case 3. $|\tau-\tau_1-\phi(k-k_1)| > \dfrac{3}{8} C_\alpha k^2$: Observe that, this case is similar to the second one, just substituting H_v in the place of H_u and F in the place of G. Thus, we obtain

$$\|\partial_x(u.v)\|_{X_{s,-\frac{1}{2}}} \leqslant C_{s,\alpha} \|H_v\|_{L^2(\mathbb{T}\times\mathbb{R})} \|F\|_{X_{0,\frac{1}{3}}} \leqslant C_{s,\alpha} \|v\|_{X_{s,\frac{1}{2}}} \|u\|_{X_{s,\frac{1}{3}}}.$$

Step 2. Now we will estimate the $Y_{s,-1}$ norm. Using duality we have,

$$\begin{aligned}
\|\partial_x(uv)\|_{Y_{s,-1}} &\sim \left\| \, \|(1+|k|)^s (1+|\tau-\phi(k)|)^{-1}\, \widehat{\partial_x(uv)}(k,\tau)\|_{L^1_\tau(\mathbb{R})} \right\|_{l^2_k} \\
&= \sup_{\substack{a_k\in l^2_k,\, a_k\geqslant 0 \\ \|a_k\|_{l^2_k}=1}} \sum_{k\in\mathbb{Z}} a_k \left(\int_{\mathbb{R}} (1+|k|)^s (1+|\tau-\phi(k)|)^{-1} |\widehat{\partial_x(uv)}(k,\tau)|\, d\tau \right).
\end{aligned} \tag{3.1.59}$$

We move to estimate

$$II := (1+|k|)^s (1+|\tau-\phi(k)|)^{-1} |\widehat{\partial_x(uv)}(k,\tau)|. \tag{3.1.60}$$

Note that

$$
\begin{aligned}
II &\leqslant (1+|k|)^s|k|(1+|\tau-\phi(k)|)^{-1}\left(\sum_{\substack{k_1\in\mathbb{Z}\\k_1\neq 0}}\int_{\mathbb{R}}|\widehat{u}(k_1,\tau_1)||\widehat{v}(k-k_1,\tau-\tau_1)|\,d\tau_1\right)\\
&\leqslant |k|(1+|\tau-\phi(k)|)^{-1}\left(\sum_{\substack{k_1\in\mathbb{Z}\\k_1\neq 0}}\int_{\mathbb{R}}\frac{(1+|k|)^s c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{|k_1|^s\langle\tau_1-\phi(k_1)\rangle^{\frac12}|k-k_1|^s\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1\right) \quad (3.1.61)\\
&\lesssim_s \left(\sum_{\substack{k_1\in\mathbb{Z}\\k_1\neq 0}}\int_{\mathbb{R}}\frac{|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1\right).
\end{aligned}
$$

It follows from (3.1.59), definition (3.1.60) and estimate (3.1.61) that

$$
\|\partial_x(uv)\|_{Y_{s,-1}} \lesssim_s \sup_{\substack{a_k\in l^2_k,\ a_k\geqslant 0\\ \|a_k\|_{l^2_k}=1}}\ \sum_{\substack{k,k_1\in\mathbb{Z}\\kk_1(k-k_1)\neq 0}}\int_{\mathbb{R}^2}\frac{a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau. \quad (3.1.62)
$$

We define

$$
III := \sum_{\substack{k,k_1\in\mathbb{Z}\\kk_1(k-k_1)\neq 0}}\int_{\mathbb{R}^2}\frac{a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau. \quad (3.1.63)
$$

Separating the small frequencies from the large ones, we obtain

$$
\begin{aligned}
III \lesssim_s &\sum_{\substack{k,k_1\in\mathbb{Z}\\kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\leqslant\max\left\{1,\frac{4\alpha}{3}\right\}}}\int_{\mathbb{R}^2}\frac{a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau\\
&+\sum_{\substack{k,k_1\in\mathbb{Z}\\kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\geqslant\max\left\{1,\frac{4\alpha}{3}\right\}}}\int_{\mathbb{R}^2}\frac{a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau
\end{aligned} \quad (3.1.64)
$$

Again, from Remark 3.1.23, we must study three different cases.

Case 1. $|\tau-\phi(k)|>\dfrac{3}{8}C_\alpha k^2$, provided that $\max\{|k|,|k_1|,|k-k_1|\}\geqslant\max\left\{1,\dfrac{4\alpha}{3}\right\}$: Thus, there exists $\widetilde{C}_\alpha = 1+\dfrac{8}{3C_\alpha}>0$ such that $\dfrac{\widetilde{C}_\alpha}{k^2+|\tau-\phi(k)|}>\dfrac{1}{1+|\tau-\phi(k)|}$, for large frequencies. Also, $\dfrac{k^2}{k^2+|\tau-\phi(k)|}>\dfrac{1}{1+|\tau-\phi(k)|}$, for small frequencies.

Therefore,

$$\begin{aligned}
III &\lesssim_s \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\leqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{k^2 a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(k^2+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau\\
&+ \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\geqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{\widetilde{C}_\alpha a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(k^2+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau\\
&\lesssim_{s,\alpha} \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0}} \int_{\mathbb{R}^2} \frac{a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(k^2+|\tau-\phi(k)|)\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau\\
&\lesssim_{s,\alpha} \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \frac{a_k|k|}{(k^2+|\tau-\phi(k)|)} \times \left(\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}} \frac{c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{\langle\tau_1-\phi(k_1)\rangle^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1\right)d\tau.
\end{aligned}\tag{3.1.65}$$

From the estimates (3.1.62), (3.1.65), and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\|\partial_x(uv)\|_{Y_{s,-1}} &\leqslant C_{s,\alpha} \sup_{\substack{a_k\in l_k^2,\ a_k\geqslant 0\\ \|a_k\|_{l_k^2}=1}} \left(\left\|\frac{a_k|k|}{(k^2+|\tau-\phi(k)|)}\right\|_{L^2(\mathbb{T}\times\mathbb{R})}\left\|\widehat{F\cdot G}(k,\tau)\right\|_{L^2(\mathbb{T}\times\mathbb{R})}\right)\\
&\leqslant C_{s,\alpha}\left\|\widehat{F\cdot G}(k,\tau)\right\|_{L^2(\mathbb{T}\times\mathbb{R})}\\
&\leqslant C_{s,\alpha}\|u\|_{X_{s,\frac12}}\|v\|_{X_{s,\frac13}},
\end{aligned}\tag{3.1.66}$$

where $C_{s,\alpha}$ is a positive constant depending on s and α.

Case 2. $|\tau_1-\phi(k_1)| > \dfrac38 C_\alpha k^2$, provided that $\max\{|k|,|k_1|,|k-k_1|\}\geqslant \max\left\{1,\dfrac{4\alpha}{3}\right\}$: This implies that

$$1+|\tau_1-\phi(k_1)| > 1+\frac38 C_\alpha k^2, \tag{3.1.67}$$

for large frequencies. From relations (3.1.62)-(3.1.64) and (3.1.67), we obtain

$$\begin{aligned}
III &\lesssim_{s,\alpha} \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\leqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{a_k c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau\\
&+ \sum_{\substack{k,k_1\in\mathbb{Z}\\ kk_1(k-k_1)\neq 0\\ \max\{|k|,|k_1|,|k-k_1|\}\geqslant \max\left\{1,\frac{4\alpha}{3}\right\}}} \int_{\mathbb{R}^2} \frac{a_k|k|c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{(1+|\tau-\phi(k)|)(1+\frac38 C_\alpha k^2)^{\frac12}\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1 d\tau.
\end{aligned}\tag{3.1.68}$$

Therefore, for any $\dfrac13 < \rho < \dfrac12$ we get

$$III \lesssim_{s,\alpha} \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} \frac{a_k}{(1+|\tau-\phi(k)|)^{1-\rho}}\frac{1}{(1+|\tau-\phi(k)|)^{\rho}}\left(\sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}\frac{c_u(k_1,\tau_1)c_v(k-k_1,\tau-\tau_1)}{\langle\tau-\tau_1-\phi(k-k_1)\rangle^{\frac12}}d\tau_1\right)d\tau. \tag{3.1.69}$$

Hence, there exists a positive constant $\widetilde{C}$ depending on s and α such that

$$\begin{aligned}
\|\partial_x(uv)\|_{Y_{s,-1}} &\leqslant \widetilde{C}_{s,\alpha} \sup_{\substack{a_k\in l_k^2,\ a_k\geqslant 0\\ \|a_k\|_{l_k^2}=1}} \sum_{k\in\mathbb{R}} \int_{\mathbb{R}} \frac{a_k}{(1+|\tau-\phi(k)|)^{1-\rho}} \frac{(\widehat{H_u}*\widehat{G})(k,\tau)}{(1+|\tau-\phi(k)|)^{\rho}}\, d\tau \\
&\leqslant \widetilde{C}_{s,\alpha} \sup_{\substack{a_k\in l_k^2,\ a_k\geqslant 0\\ \|a_k\|_{l_k^2}=1}} \left\| \frac{a_k}{(1+|\tau-\phi(k)|)^{1-\rho}} \right\|_{L^2(\mathbb{T}\times\mathbb{R})} \left\| \frac{\widehat{H_u\cdot G}(k,\tau)}{(1+|\tau-\phi(k)|)^{\rho}} \right\|_{L^2(\mathbb{T}\times\mathbb{R})} \\
&\leqslant C_{s,\alpha} \left\| \frac{\widehat{H_u\cdot G}(k,\tau)}{(1+|\tau-\phi(k)|)^{\rho}} \right\|_{L^2(\mathbb{T}\times\mathbb{R})},
\end{aligned} \tag{3.1.70}$$

where $\left\| \frac{a_k}{(1+|\tau-\phi(k)|)^{1-\rho}} \right\|_{L^2(\mathbb{T}\times\mathbb{R})} \leqslant C$, because $\rho < \frac{1}{2}$. Now using $\rho > \frac{1}{3}$, we obtain

$$(1+|\tau-\phi(k)|)^{-\frac{2}{3}} > (1+|\tau-\phi(k)|)^{-2\rho}. \tag{3.1.71}$$

The estimates (3.1.70)-(3.1.71) yield

$$\begin{aligned}
\|\partial_x(uv)\|_{Y_{s,-1}} &\leqslant C_{s,\alpha} \left\| \frac{\widehat{H_u\cdot G}(k,\tau)}{(1+|\tau-\phi(k)|)^{\rho}} \right\|_{L^2(\mathbb{T}\times\mathbb{R})} \\
&= C_{s,\alpha} \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} (1+|\tau-\phi(k)|)^{-2\rho} |\widehat{H_u\cdot G}|^2\, d\tau \right)^{\frac{1}{2}} \\
&\leqslant C_{s,\alpha} \left(\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} (1+|\tau-\phi(k)|)^{-\frac{2}{3}} |\widehat{H_u\cdot G}|^2\, d\tau \right)^{\frac{1}{2}} \\
&\leqslant C_{s,\alpha} \|H_u\|_{L^2(\mathbb{T}\times\mathbb{R})} \|G\|_{L^4(\mathbb{T}\times\mathbb{R})} \\
&\leqslant C_{s,\alpha} \|u\|_{X_{s,\frac{1}{2}}} \|v\|_{X_{s,\frac{1}{3}}}.
\end{aligned}$$

Case 3. $|\tau-\tau_1-\phi(k-k_1)| > \frac{3}{8}C_\alpha k^2$, provided that $\max\{|k|,|k_1|,|k-k_1|\} \geqslant \max\left\{1, \frac{4\alpha}{3}\right\}$: This case is similar to the second one, just substituting H_v in the place of H_u and F in the place of G. Therefore we obtain,

$$\|\partial_x(uv)\|_{Y_{s,-1}} \leqslant C_{s,\alpha} \|H_v\|_{L^2(\mathbb{T}\times\mathbb{R})} \|F\|_{L^4(\mathbb{T}\times\mathbb{R})} \leqslant C_{s,\alpha} \|v\|_{X_{s,\frac{1}{2}}} \|u\|_{X_{s,\frac{1}{3}}}.$$

□

Corollary 3.1.25. *Let $s \geqslant 0$, $\alpha > 0$, and $T > 0$ be given. Assume that $u, v : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ are functions in $X_{s,\frac{1}{2}}(I)$, where $I = [-T, T]$ and mean $[u(\cdot,t)] = [v(\cdot,t)] = 0$ for each $t \in \mathbb{R}$. Then*

$$\|\partial_x(uv)\|_{Z_{s,-\frac{1}{2}}(I)} \leqslant C_{\alpha,s}(\|u\|_{X_{s,\frac{1}{2}}(I)} \|v\|_{X_{s,\frac{1}{3}}(I)} + \|u\|_{X_{s,\frac{1}{3}}(I)} \|v\|_{X_{s,\frac{1}{2}}(I)}).$$

Furthermore,

$$\|\partial_x(uv)\|_{Z^T_{s,-\frac{1}{2}}} \leqslant C_{\alpha,s}(\|u\|_{X^T_{s,\frac{1}{2}}} \|v\|_{X^T_{s,\frac{1}{3}}} + \|u\|_{X^T_{s,\frac{1}{3}}} \|v\|_{X^T_{s,\frac{1}{2}}}), \quad \forall u, v \in X^T_{s,\frac{1}{2}}, \tag{3.1.72}$$

and the mean $[u(\cdot,t)] = [v(\cdot,t)] = 0$ for each $t \in \mathbb{R}$.

Proof. Let $\widetilde{u}, \widetilde{v} : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be functions in $X_{s,\frac{1}{3}}$ and $X_{s,\frac{1}{2}}$, such that $\widetilde{u}$ (resp. $\widetilde{v}$) is an extension of u (resp. v) in $X_{s,\frac{1}{3}}$ (resp. $X_{s,\frac{1}{2}}$) with

$$\begin{aligned}
\|\widetilde{u}\|_{X_{s,\frac{1}{3}}} &\leqslant 2\,\|u\|_{X_{s,\frac{1}{3}}(I)}; \quad \|\widetilde{u}\|_{X_{s,\frac{1}{2}}} \leqslant 2\,\|u\|_{X_{s,\frac{1}{2}}(I)},\\
\|\widetilde{v}\|_{X_{s,\frac{1}{3}}} &\leqslant 2\,\|v\|_{X_{s,\frac{1}{3}}(I)}; \quad \|\widetilde{v}\|_{X_{s,\frac{1}{2}}} \leqslant 2\,\|v\|_{X_{s,\frac{1}{2}}(I)}.
\end{aligned}$$

From Theorem 3.1.24, we have that $\widetilde{u}\widetilde{v} : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a function in $Z_{s,-\frac{1}{2}}$. Furthermore,

$$\begin{aligned}
\|\partial_x(uv)\|_{Z_{s,-\frac{1}{2}}(I)} &\leqslant \|\partial_x(\widetilde{u}\widetilde{v})\|_{Z_{s,-\frac{1}{2}}}\\
&\leqslant C_{\alpha,s}\left(\|\widetilde{u}\|_{X_{s,\frac{1}{2}}}\|\widetilde{v}\|_{X_{s,\frac{1}{3}}} + \|\widetilde{u}\|_{X_{s,\frac{1}{3}}}\|\widetilde{v}\|_{X_{s,\frac{1}{2}}}\right)\\
&\leqslant 4C_{\alpha,s}\left(\|u\|_{X_{s,\frac{1}{2}}(I)}\|v\|_{X_{s,\frac{1}{3}}(I)} + \|u\|_{X_{s,\frac{1}{3}}(I)}\|v\|_{X_{s,\frac{1}{2}}(I)}\right).
\end{aligned} \tag{3.1.73}$$

This proves the Corollary. □

Corollary 3.1.26. *Let $s \geqslant 0$, $\alpha > 0$, and $0 < T < 1$ be given. Assume that $v : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a function in $X^T_{s,\frac{1}{2}}$ with mean $[v(\cdot, t)] = 0$ for each $t \in \mathbb{R}$. Then for any $\epsilon \in (0, \frac{1}{6})$ we have*

$$\|\partial_x(v^2)\|_{Z^T_{s,-\frac{1}{2}}} \leqslant C_{\alpha,s}T^{\epsilon}\,\|v\|^2_{Z^T_{s,\frac{1}{2}}}.$$

In particular, the last inequality holds for $\epsilon = \frac{1}{12}$.

Proof. Let $\epsilon \in (0, \frac{1}{6})$ be given. Then applying Proposition 3.1.15 with $b' = \frac{1}{3}$, and $b = \frac{1}{3} + \epsilon$, we obtain

$$\|u\|_{X^T_{s,\frac{1}{3}}} \leqslant C\,T^{\frac{1}{3}+\epsilon-\frac{1}{3}}\|u\|_{X^T_{s,\frac{1}{3}+\epsilon}} \leqslant C\,T^{\epsilon}\|u\|_{X^T_{s,\frac{1}{2}}}. \tag{3.1.74}$$

From the estimates (3.1.72) and (3.1.74), we have

$$\|\partial_x(u^2)\|_{Z^T_{s,-\frac{1}{2}}} \leqslant C\|u\|_{X^T_{s,\frac{1}{2}}}\|u\|_{X^T_{s,\frac{1}{3}}} \leqslant C\,T^{\epsilon}\|u\|^2_{X^T_{s,\frac{1}{2}}} \leqslant C\,T^{\epsilon}\|u\|^2_{Z^T_{s,\frac{1}{2}}}.$$

□

In the following sections of this chapter, we present essential results to establish exact controllability and stabilizability of the Benjamin equation.

3.2 The multiplication property of Bourgain's Space

In this subsection we establish the multiplication property of the Bourgain space $X_{s,b}$. We begin with the following lemma. Its proof is classic and we leave it to the reader.

Lemma 3.2.1. *If $\psi = \psi(t) \in C^\infty(\mathbb{R})$, then $\psi v \in X^T_{s,b}$ for all $v \in X^T_{s,b}$. Furthermore, there exists a positive constant $C = C_{\eta,T,b,\psi}$ such that*

$$\|\psi v\|_{X^T_{s,b}} \leqslant C\|v\|_{X^T_{s,b}}.$$

If $T \leqslant 1$, then the positive constant C does not depend on the time T.

As was pointed out by Laurent et. al [52] for the KdV equation, if $\phi = \phi(x) \in C^\infty(\mathbb{T})$, then ϕv may not belong to the space $X^T_{s,b}$ for $v \in X^T_{s,b}$. For Benjamin equation too, the same is lost in the index of regularity s due to the fact that the multiplication by a (smooth) function of x does not keep the structure in the space of the harmonics. This lost is, in fact, unavoidable. For instance, consider for $j \geqslant 1$, the function $v_j(x,t) = \psi(t)e^{ijx}e^{i\phi(j)t}$, where $\psi \in C^\infty_c(\mathbb{R})$ takes the value 1 on $[-1,1]$. Note that,

$$\widehat{v_j}(k,t) = \psi(t)e^{i\phi(j)t}\widehat{e^{ijx}}(k) = \psi(t)e^{i\phi(j)t}\delta_{kj},$$

where δ_{kj} is the Kronecker delta function. Then,

$$\widehat{v_j}(k,\tau) = \delta_{kj}\left(\psi(t)e^{i\phi(j)t}\right)^\wedge(\tau) = \delta_{kj}\widehat{\psi}(\tau - \phi(j)).$$

Therefore,

$$\begin{aligned}
\|v_j\|^2_{X_{0,b}} &= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle \tau - \phi(k)\rangle^{2b} |\delta_{kj}\,\widehat{\psi}(\tau - \phi(j))|^2\, d\tau \\
&= \int_{\mathbb{R}} \langle \tau - \phi(j)\rangle^{2b} |\widehat{\psi}(\tau - \phi(j))|^2\, d\tau \\
&\leqslant c_b\|\psi\|^2_{H^b_t(\mathbb{R})}.
\end{aligned} \tag{3.2.1}$$

Thus, the sequence $\{v_j\}_{j\geqslant 1}$ is uniformly bounded in the space $X_{0,b}$, for every $b \geqslant 0$. However, multiplying v_j by $\varphi(x) = e^{ix}$, we observe that $\widehat{e^{ix}v_j}(k,t) = \psi(t)e^{i\phi(j)t}\delta_{k(1+j)}$, and $\widehat{e^{ix}v_j}(k,\tau) = \delta_{k(1+j)}\widehat{\psi}(\tau - \phi(j))$. Thus,

$$\|e^{ix}v_j\|^2_{X_{0,b}} = \int_{\mathbb{R}} \langle \tau - \phi(1+j)\rangle^{2b} |\widehat{\psi}(\tau - \phi(j))|^2\, d\tau.$$

Using that $\tau - \phi(1+j) = \tau - \phi(j) + P(j)$ with $P(j) = 3j^2 + (3-2\alpha)j + 1 + 2\mu - \alpha$, we have

$$\|e^{ix}v_j\|^2_{X_{0,b}} \sim \int_{\mathbb{R}} (1 + |\tau + P(j)|)^{2b} |\widehat{\psi}(\tau)|^2\, d\tau \approx j^{4b},$$

for j large enough.

The next theorem shows that this is the worst case. With the purpose of prove the theorem, we begin proving a necessary lemma and showing how the $X_{s,b}$ may be viewed as a weighted L^2 space.

Lemma 3.2.2. *Let $s \in \mathbb{R}$. Then $v \in X_{s,1}$ if an only if $v \in L^2(\mathbb{R}, H_p^s(\mathbb{T}))$, and*

$$\partial_t v - \partial_x^3 v - \alpha \mathcal{H} \partial_x^2 v + 2\,\mu \partial_x v \in L^2(\mathbb{R}, H_p^s(\mathbb{T})).$$

In this case we have $\|v\|^2_{X_{s,1}} = \|v\|^2_{L^2(\mathbb{R}_t, H_p^s(\mathbb{T}))} + \|\partial_t v - \partial_x^3 v - \alpha \mathcal{H}\partial_x^2 v + 2\mu \partial_x v\|^2_{L^2(\mathbb{R}_t, H_p^s(\mathbb{T}))}$.

Proof. Let $s \in \mathbb{R}$ fixed. Then, applying Plancherel's identity in time, we obtain

$$\begin{aligned}\|v\|^2_{X_{s,1}} &= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} \left(1 + |\tau + k^3 - \alpha k|k| + 2\mu k|^2\right) |\widehat{v}(k,\tau)|^2 \, d\tau \\ &= \|v\|^2_{L^2(\mathbb{R}_t, H_p^s(\mathbb{T}))} + \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}} \langle k \rangle^{2s} |\widehat{\partial_t v}(k,t) - \widehat{\partial_x^3} v(k,t) - \alpha \widehat{\mathcal{H}\partial_x^2 v}(k,t) + 2\,\mu \widehat{\partial_x v}(k,t)|^2 \, dt.\end{aligned}$$

This proves the lemma. □

Observe that, the $X_{s,b}$ spaces may be viewed as the weighted L^2-spaces. In fact, denote by $(\mathbb{R}, \mathcal{M}, \lambda)$ and $(\mathbb{Z}, \mathcal{A}, \delta)$ two measure spaces, where λ is the Lebesgue measure over $\mathbb{R}$ and δ is the discrete measure on $\mathbb{Z}$. Thus, $(\mathbb{R} \times \mathbb{Z}, \mathcal{M} \otimes \mathcal{A}, \lambda \otimes \delta)$ is the product measure space where $\mathcal{M} \otimes \mathcal{A}$ is a $\sigma-$álgebra on $\mathbb{R} \times \mathbb{Z}$, and $\lambda \otimes \delta$ is the product measure of λ and δ (see [36, page 64]). Since λ and δ are $\sigma-$finite measures, then $\lambda \otimes \delta$ is a $\sigma-$finite measure and

$$\lambda \otimes \delta(M \times A) = \lambda(M) \cdot \delta(A), \quad \text{for all rectangles } \ M \times A,$$

where a (measurable) rectangle is a set of the form $M \times A$, with $M \in \mathcal{M}$ and $A \in \mathcal{A}$. Then, using Fourier transform, $X_{s,b}$ may be viewed as the weighted L^2 space

$$L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k \rangle^{2s} \langle \tau - \phi(k) \rangle^{2b} \lambda \otimes \delta). \tag{3.2.2}$$

Theorem 3.2.3. *Let $-1 \leqslant b \leqslant 1$, $s \in \mathbb{R}$, and $\varphi \in C^\infty(\mathbb{T})$. Then for any $v \in X_{s,b}$, $\varphi v \in X_{s-2|b|,b}$. Similarly, for any $T > 0$ the multiplication by φ maps $X^T_{s,b}$, into $X^T_{s-2|b|,b}$, i.e., there exists a positive constant $C = C_{s,\alpha,\varphi,\mu}$ which does not depend on T, such that*

$$\|\varphi v\|_{X^T_{s-2|b|,b}} \leqslant C_{s,\alpha,\varphi,\mu} \, \|v\|_{X^T_{s,b}}.$$

Proof. We proceed as in [52]. First consider the cases $b = 0$ and $b = 1$. The other cases of b will be derived later by interpolation and duality.

Case 1. $b = 0$: Let $v \in \mathcal{S}(\mathbb{T} \times \mathbb{R})$. From relation (3.1.3) we know that

$$\|\varphi v\|^2_{X_{s,0}} = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} 2\pi (1 + |k|^2)^s |\left(\varphi v\right)^\wedge (k,t)|^2 dt.$$

If $s \geqslant 0$, one has $(1 + |k|)^s \leqslant c_s \, (1 + |k-j|)^s (1 + |j|)^s$. Therefore, the Cauchy-Schwarz inequality for $N > \dfrac{1}{2}$, yields

$$\|\varphi v\|^2_{X_{s,0}} \leqslant c_s \sum_{j=-\infty}^{\infty} (1 + |j|)^{2s+2N} |\widehat{\varphi}(j)|^2 \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} (1 + |k-j|)^{2s} |\widehat{v(x,t)}(k-j,t)|^2 \, dt.$$

Using the invariance of the $L^2(\mathbb{R}_t, H_p^s(\mathbb{T}))$ norm by translations, we get

$$\|\varphi v\|^2_{X_{s,0}} \leqslant c_s \|\varphi\|^2_{H_p^{s+N}(\mathbb{T})} \|v\|^2_{X_{s,0}}.$$

On the other hand, if $s < 0$, we apply $(1+|k|)^s \leqslant c_s\,(1+|k-j|)^s(1+|j|)^{-s}$, and proceed as above to obtain

$$\|\varphi v\|^2_{X_{s,0}} \leqslant c_s \|\varphi\|^2_{H_p^{-s+N}(\mathbb{T})} \|v\|^2_{X_{s,0}}.$$

In the general case, we use density and duality arguments to complete the proof.

Case 2. $b = 1$: From Lemma 3.2.2, we obtain

$$\begin{aligned}
\|\varphi v\|^2_{X_{s-2,1}} &= \|\varphi v\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} \\
&\quad + \|\partial_t(\varphi v) - \partial_x^3(\varphi v) - \alpha\mathcal{H}\partial_x^2(\varphi v) + 2\mu\partial_x(\varphi v) - \alpha\varphi\mathcal{H}\partial_x^2 v + \alpha\varphi\mathcal{H}\partial_x^2 v\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} \\
&\leqslant \|\varphi v\|^2_{X_{s-2,0}} + c\alpha\|\mathcal{H}\partial_x^2(\varphi v) + \varphi\mathcal{H}\partial_x^2 v\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} \\
&\quad + c\|\partial_t(\varphi v) - \partial_x^3(\varphi v) + 2\mu\partial_x(\varphi v) - \alpha\varphi\mathcal{H}\partial_x^2(v)\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} \\
&=: I + II + III.
\end{aligned} \tag{3.2.3}$$

From case $b = 0$, we obtain that there exists $A_{s,\varphi} > 0$ such that

$$I = \|\varphi v\|^2_{X_{s-2,0}} \leqslant A_{s,\varphi}\, \|v\|^2_{X_{s-2,0}} \leqslant A_{s,\varphi}\, \|v\|^2_{X_{s,0}}. \tag{3.2.4}$$

From the properties of operator ∂_x^r on Bourgain's spaces, and nothing that $\mathcal{H}$ is an isometry in $H_p^{s-2}(\mathbb{T})$ (see Theorem 1.4.1), we get

$$\begin{aligned}
II &\leqslant c\alpha\left(\|\partial_x^2(\varphi v)\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} + \|\varphi\mathcal{H}\partial_x^2 v\|^2_{X_{s-2,0}}\right) \\
&\leqslant c\alpha\left(\|\varphi v\|^2_{X_{s,0}} + c_{s,\varphi}\|\mathcal{H}\partial_x^2 v\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))}\right) \\
&= c\alpha\left(d_{s,\varphi}\|v\|^2_{X_{s,0}} + c_{s,\varphi}\|\partial_x^2 v\|^2_{X_{s-2,0}}\right).
\end{aligned}$$

Hence, there exists another positive constant $B_{s,\alpha,\varphi}$ such that

$$II \leqslant B_{s,\alpha,\varphi}\, \|v\|^2_{X_{s,0}}. \tag{3.2.5}$$

We estimate III. From the Leibniz's rule for derivatives (see [54, page 13]), one has

$$\begin{aligned}
\partial_t(\varphi v) - \partial_x^3(\varphi v) + 2\mu\partial_x(\varphi v) - \alpha\varphi\mathcal{H}\partial_x^2 v &= \varphi\left(\partial_t v - \partial_x^3 v + 2\mu\partial_x v - \alpha\mathcal{H}\partial_x^2 v\right) \\
&\quad - 3\partial_x\varphi\partial_x^2 v - 3\partial_x^2\varphi\partial_x v - \partial_x^3\varphi v + 2\mu\partial_x\varphi v.
\end{aligned} \tag{3.2.6}$$

Note that $-3\partial_x\varphi\partial_x^2 v - 3\partial_x^2\varphi\partial_x v - \partial_x^3\varphi v + 2\mu\partial_x\varphi v$ is an operator of second order. From identity (3.2.6), the case $b = 0$, and the fact that $\varphi \in C^\infty(\mathbb{T})$, we get

$$\begin{aligned}
III &\leqslant c\|\varphi(x)\left(\partial_t v - \partial_x^3 v + 2\mu\partial_x v - \alpha\mathcal{H}\partial_x^2 v\right)\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} \\
&\quad + c\| - 3\partial_x\varphi\partial_x^2 v - 3\partial_x^2\varphi\partial_x v - \partial_x^3\varphi v + 2\mu\partial_x\varphi v\|^2_{L^2(\mathbb{R}_t, H_p^{s-2}(\mathbb{T}))} \\
&\leqslant c_{s,\varphi}\|\partial_t v - \partial_x^3 v + 2\mu\partial_x v - \alpha\mathcal{H}\partial_x^2 v\|^2_{X_{s-2,0}} \\
&\quad + 3c\|\partial_x\varphi\partial_x^2 v\|^2_{X_{s-2,0}} + 3c\|\partial_x^2\varphi\partial_x v\|^2_{X_{s-2,0}} + c\|\partial_x^3\varphi v\|^2_{X_{s-2,0}} + 2|\mu|c\|\partial_x\varphi v\|^2_{X_{s-2,0}} \\
&\leqslant c_{s,\varphi}\|\partial_t v - \partial_x^3 v + 2|\mu|\partial_x v - \alpha\mathcal{H}\partial_x^2 v\|^2_{X_{s-2,0}} \\
&\quad + 3cd_{s,\partial_x\varphi}\|v\|^2_{X_{s,0}} + 3cd_{s,\partial_x^2\varphi}\|v\|^2_{X_{s-1,0}} + cd_{s,\partial_x^3\varphi}\|v\|^2_{X_{s-2,0}} + 2|\mu|cd_{s,\partial_x\varphi}\|v\|^2_{X_{s-2,0}}.
\end{aligned}$$

From embeddings $X_{s,0} \hookrightarrow X_{s-2,0}$, and $X_{s,0} \hookrightarrow X_{s-1,0}$, we have that there exists $D_{s,\mu,\varphi} > 0$ such that

$$III \leqslant D_{s,\mu,\varphi}\left(\|\partial_t v - \partial_x^3 v + 2\mu\partial_x v - \alpha\mathcal{H}\partial_x^2 v\|^2_{X_{s,0}} + \|v\|^2_{X_{s,0}}\right) = D_{s,\mu,\varphi}\|v\|^2_{X_{s,1}}, \tag{3.2.7}$$

where in the last step Lemma 3.2.2 is used. From (3.2.3)-(3.2.5), and (3.2.7), we get that there exists a positive constant $C_{s,\alpha,\mu,\varphi}$ such that

$$\|\varphi(x)v\|^2_{X_{s-2,1}} \leqslant C_{s,\alpha,\mu,\varphi}\,\|v\|^2_{X_{s,1}}. \tag{3.2.8}$$

This proves the case $b = 1$.

Case 3. $0 < b < 1$: In this case we use interpolation. From identification (3.2.2) we get $X_{s,0} = L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2s}\lambda\otimes\delta)$, and $X_{s',1} = L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2s'}\langle \tau - \phi(k)\rangle^2\lambda\otimes\delta)$. Therefore, applying the Complex Interpolation Theorem of Stein-Weiss (see Theorem 1.7.8) we obtain $(X_{s,0}, X_{s',1})_{\theta,2} \approx X_{s(1-\theta)+s'\theta,\theta}$, with $0 < \theta < 1$. Furthermore, from the cases $b = 0$ and $b = 1$ we infer that the operator of multiplication by $\varphi \in C^\infty(\mathbb{T})$, defined by

$$T : X_{s,\theta} \approx L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2s}\langle \tau - \phi(k)\rangle^{2\theta}\lambda\otimes\delta) \longrightarrow X_{s-2\theta,\theta} \approx L^2(\mathbb{R}_\tau \times \mathbb{Z}_k, \langle k\rangle^{2(s-2\theta)}\langle \tau - \phi(k)\rangle^{2\theta}\lambda\otimes\delta)$$
$$T(v) = \varphi(x)v,$$

satisfies

$$\|Tv\|_{X_{s-2\theta,\theta}} \leqslant C^{1-\theta}_{s,\varphi}C^{\theta}_{\alpha,s,\varphi,\mu}\|v\|_{s,\theta} \leqslant C_{\alpha,s,\varphi,\mu,\theta}\,\|v\|_{s,\theta}.$$

and have quasi-norm $M = C_{\alpha,s,\varphi,\mu,\theta}$, with $0 < \theta < 1$. Thus, we have a 2θ loss of regularity in the spatial variable, as announced.

Case 4. Let $-1 < b < 0$. In this case we use duality.

$$\|\varphi(x)v\|_{X_{s-2|b|,b}} = \sup_{\substack{u\in X_{-s-2b,-b}\\ \|u\|_{X_{-s-2b,-b}}\leqslant 1}} \left|\int_{\mathbb{T}}\int_{\mathbb{R}} u\cdot\varphi(x)v\;dtdx\right|.$$

Finally, to get the same results for the restriction spaces $X^T_{s,b}$ we write the estimates for an extension $\tilde{v}$ of v, such that $\|\tilde{v}\|_{X_{s,b}} \leqslant 2\|v\|^T_{X_{s,b}}$, which yields

$$\|\varphi(x)v\|^T_{X_{s-2|b|,b}} \leqslant \|\varphi(x)\tilde{v}\|_{X_{s-2|b|,b}} \leqslant C_{\alpha,s,\varphi,\mu,b}\,\|\tilde{v}\|_{X_{s,b}} \leqslant 2C_{\alpha,s,\varphi,\mu,b}\,\|v\|_{X_{s,b}}.$$

This completes the proof of the theorem. □

3.3 Propagation of Compactness and Regularity

In this section we show some properties of propagation of compactness and regularity for the linear differential operator

$$L := \partial_t - \alpha\mathcal{H}\partial_x^2 - \partial_x^3 + 2\mu\partial_x, \tag{3.3.1}$$

associated to the Benjamin equation. These propagation properties will play a key role when studying the global stabilizability of the Benjamin equation. We begin establishing a necessary lemma to get the result of propagation of compactness.

Lemma 3.3.1. *Let $s, r \in \mathbb{R}$. The Hilbert transform $\mathcal{H}$ commutes with the operator D^r (see (2.5.2)) in $L^2(\mathbb{T})$. Furthermore, $\mathcal{H} = -D^{-1}\partial_x$, in $L^2(\mathbb{T})$. Also, the operator ∂_x^r commutes with the operators D^r and $\mathcal{H}$ in $L^2(\mathbb{T})$.*

Proof. It can be easily shown using Fourier's transform. □

Proposition 3.3.2 (Propagation of Compactness). *Let $T > 0$ and $0 \leqslant b' \leqslant b \leqslant 1$ be given (with $b > 0$) and assume that $v_n \in X_{0,b}^T$ and $f_n \in X_{-2+2b,-b}^T$ satisfy*

$$\partial_t v_n - \alpha\, \mathcal{H}\partial_x^2 v_n - \partial_x^3 v_n + 2\mu\, \partial_x v_n = f_n, \quad \textit{for}\, n = 1, 2, 3, \cdots. \tag{3.3.2}$$

Suppouse that there exists $C > 0$ such that

$$\|v_n\|_{X_{0,b}^T} \leqslant C, \quad \textit{for all}\, n \geqslant 1, \tag{3.3.3}$$

and that

$$\|v_n\|_{X_{-2+2b,-b}^T} + \|f_n\|_{X_{-2+2b,-b}^T} + \|v_n\|_{X_{-1+2b',-b'}^T} \longrightarrow 0, \quad \textit{as}\, n \longrightarrow \infty. \tag{3.3.4}$$

Additionally, assume that for some nonempty open set $\omega \subset \mathbb{T}$

$$v_n \longrightarrow 0, \quad \textit{strongly in} \;\; L^2((0,T); L^2(\omega)). \tag{3.3.5}$$

Then,

$$v_n \longrightarrow 0, \quad \textit{strongly in} \;\; L^2_{loc}((0,T); L^2(\mathbb{T})), \quad \textit{as}\, n \longrightarrow \infty. \tag{3.3.6}$$

Proof. Let $K \subset (0, T)$ be compact and $\psi \in C_c^\infty((0, T))$, such that $0 \leqslant \psi(t) \leqslant 1$ and $\psi(t) = 1$ in K. Then,

$$\|v_n\|^2_{L^2(K,L^2(\mathbb{T}))} \leqslant \int_0^T \psi(t)\|v_n\|^2_{L^2(\mathbb{T})}\, dt = \int_0^T \psi(t)\; (v_n, v_n)_{L^2(\mathbb{T})}\; dt. \tag{3.3.7}$$

Since $\mathbb{T}$ is compact there exists a finite set of points, say $x_0^i \in \mathbb{T}$, $i = 1, 2, 3, \cdots, N$, such that we construct a partition of the unity on $\mathbb{T}$ involving functions of the form $\chi_i(x - x_0^i)$. with $\chi_i(\cdot) \in C_c^\infty(w)$. Specifically, there exists $N \in \mathbb{N}$ such that

$$\begin{cases} 0 \leqslant \chi_i(x - x_0^i) \leqslant 1, & \text{for all} \quad x \in \mathbb{T}\; \textit{and}\; i = 1, 2, .., N \\ \chi_i(\cdot) \in C_c^\infty(w) & \text{for} \quad i = 1, 2, .., N \\ \displaystyle\sum_{i=1}^N \chi_i(\cdot - x_0^i) = 1 & \text{on} \quad \mathbb{T}. \end{cases} \tag{3.3.8}$$

Therefore,

$$\|v_n\|^2_{L^2(K,L^2(\mathbb{T}))} = \int_0^T \left(\psi(t)\left(\sum_{i=1}^N \chi_i(x-x_0^i)\right)v_n, v_n\right)_{L^2(\mathbb{T})} dt = \sum_{i=1}^N \left(\psi(t)\chi_i(x-x_0^i)v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}.$$

Thus, it is sufficient to show that for any $x_0 \in \mathbb{T}$, and any $\chi(\cdot) \in C_c^\infty(\omega)$

$$(\psi(t)\chi(x-x_0)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

For this, consider $\phi(x) = \chi(x) - \chi(x-x_0)$, where $\chi \in C_c^\infty(\omega)$ and $x_0 \in \mathbb{T}$. From Lemma A.4 there exists $\varphi \in C^\infty(\mathbb{T})$ such that $\partial_x\varphi(x) = \chi(x) - \chi(x-x_0)$ for all $x \in \mathbb{T}$. Consequently,

$$(\psi(t)\chi(x-x_0)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} = (\psi(t)\chi(x)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} - (\psi(t)\partial_x\varphi(x)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))}.$$

From (3.3.5) we have that

$$\begin{aligned}\left|(\psi(t)\chi(x)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \int_0^T\int_\omega |\psi(t)|\,|\chi(x)|\,|v_n|\cdot|\overline{v_n}|\,dxdt\\ &\leqslant \|\psi\|_{C_c^\infty((0,T))}\|\chi\|_{C_c^\infty(\omega)}\|v_n\|^2_{L^2((0,T);L^2(\omega))} \longrightarrow 0,\end{aligned}$$

as $n \longrightarrow \infty$. So, we only need to show that

$$\left|(\psi(t)\partial_x\varphi(x)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))}\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.3.9}$$

where $\psi \in C_c^\infty((0,T))$ and $\varphi \in C^\infty(\mathbb{T})$.

In what follows we prove (3.3.9). Taking into consideration the definition of D^r (see Definition 2.5.2) and passing to the frequency space, it is easy to verify that

$$\begin{aligned}(\psi(t)\partial_x\varphi(x)v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} &= 2\pi\sum_{k\in\mathbb{Z}}\int_0^T \left(-\partial_x^2 D^{-2}v_n\right)^\wedge(k,t)\psi(t)\overline{\left((\partial_x\varphi)v_n\right)^\wedge(k,t)}dt\\ &+ 2\pi\int_0^T \widehat{v_n}(0,t)\psi(t)\overline{\frac{1}{2\pi}\int_0^{2\pi}\partial_x\varphi(x)v_n(x,t)dx}\,dt.\end{aligned}$$

Note that $-\partial_x^2 D^{-2}$ is the orthogonal projection on the subspace of functions with $\widehat{w}(0) = 0$. Therefore,

$$\begin{aligned}(\psi(t)\ \partial_x\varphi(x)\ v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} &= \left(\psi(t)\ \partial_x\varphi(x)(-\partial_x^2)D^{-2}\ v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &+ (\psi(t)\ \partial_x\varphi(x)\ \widehat{v_n}(0,t), v_n)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.10}$$

First, we prove

$$\lim_{n\longrightarrow\infty}\left|\left(\psi(t)\ \partial_x\varphi(x)\ (-\partial_x^2)D^{-2}v_n,\ v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.11}$$

From (3.3.1) and (3.3.2) we have $Lv_n = f_n$, for $n = 1, 2, 3, \cdots$. Set $B := \varphi(x)D^{-2}$, $A := \psi(t)B$, and for $\epsilon > 0$, let $A_\epsilon := \psi(t)B_\epsilon$, be a regularization of A, where

$$B_\epsilon := Be^{\epsilon\partial_x^2}, \quad \text{with } e^{\epsilon\partial_x^2} \quad \text{defined by } e^{\epsilon\partial_x^2}v(\cdot) = \left(e^{-\epsilon k^2}\widehat{v}(k)\right)^\vee(\cdot). \tag{3.3.12}$$

Then $A_\epsilon^* = \psi(t)e^{\epsilon\partial_x^2}D^{-2}\varphi(x)$. Define $\alpha_{n,\epsilon} := ([A_\epsilon, L]v_n, v_n)_{L^2(\mathbb{T}\times(0,T))}$. Note that, taking the *formal adjoint* in the distributional sense for the terms involved in (3.3.1), we obtain

$$\alpha_{n,\epsilon} := (f_n,\ A_\epsilon^* v_n)_{L^2(\mathbb{T}\times(0,T))} + (A_\epsilon v_n,\ f_n)_{L^2(\mathbb{T}\times(0,T))}. \tag{3.3.13}$$

We infer from Lemma 3.2.1, Theorem 3.2.3, (3.3.3), and (3.3.4) that for $0 < b \leqslant 1$, there exists a positive constant C (independent of T, if $T \leqslant 1$), such that

$$\begin{aligned}\left|(f_n,\ A_\epsilon^* v_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \|f_n\|_{X^T_{-2+2b,-b}} \|A_\epsilon^* v_n\|_{X^T_{2-2b,b}} \\ &\leqslant C\, \|f_n\|_{X^T_{-2+2b,-b}} \|v_n\|_{X^T_{0,b}} \\ &\leqslant C\, \|f_n\|_{X^T_{-2+2b,-b}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty} \sup_{0<\epsilon\leqslant 1} \left|(f_n,\ A_\epsilon^* v_n)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.14}$$

Using a similar procedure, we obtain

$$\lim_{n\longrightarrow\infty} \sup_{0<\epsilon\leqslant 1} \left|(\ A_\epsilon v_n,\ f_n)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.15}$$

Hence, (3.3.13), (3.3.14), and (3.3.15) imply that

$$\lim_{n\longrightarrow\infty} \sup_{0<\epsilon\leqslant 1} |\alpha_{n,\epsilon}| = 0. \tag{3.3.16}$$

On the other hand, using that the operator B_ϵ commutes with derivatives in time, we obtain

$$[A_\epsilon, L]v_n = -\psi'(t)B_\epsilon v_n + [A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n + [A_\epsilon, -\partial_x^3 + 2\mu\partial_x]v_n.$$

Therefore,

$$\begin{aligned}\alpha_{n,\epsilon} = &-(\psi'(t)B_\epsilon v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} + \left([A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))} \\ &+ \left([A_\epsilon, -\partial_x^3 + 2\mu\partial_x]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.17}$$

We infer from Lemma 3.2.1, Theorem 3.2.3, and (3.3.12) that for any $s \in \mathbb{R}$, and $0 < b \leqslant 1$, there exists a positive constant C (independent of T, if $T \leqslant 1$) which does not depend on ϵ, such that

$$\|\psi'(t)B_\epsilon v\|_{X^T_{s+2-2|b|,b}} \leqslant C\|v\|_{X^T_{s,b}}. \tag{3.3.18}$$

From (3.3.18), (3.3.3), (3.3.4), and the fact that $0 < b \leqslant 1$, we obtain

$$\begin{aligned}\left|(\psi'(t)B_\epsilon v_n, v_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \|\psi'(t)B_\epsilon v_n\|_{X^T_{0,-b}} \|v_n\|_{X^T_{0,b}} \\ &\leqslant C\|v_n\|_{X^T_{-2+2|-b|,-b}} \|v_n\|_{X^T_{0,b}} \\ &\leqslant C\|v_n\|_{X^T_{-2+2b,-b}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.3.19}$$

Therefore,

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|(\psi'(t)B_\epsilon v_n, v_n)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.20}$$

Also, observe that

$$[A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n = -\alpha\psi(t)\varphi D^{-2}e^{\epsilon\partial_x^2}\mathcal{H}\partial_x^2 v_n + \alpha\psi(t)\mathcal{H}\partial_x^2\left(\varphi D^{-2}e^{\epsilon\partial_x^2}v_n\right) \tag{3.3.21}$$

From the Leibniz's rule for derivatives (see [54, page 13]), we obtain

$$\partial_x^2\left(\varphi D^{-2}e^{\epsilon\partial_x^2}v_n\right) = \varphi\partial_x^2 D^{-2}e^{\epsilon\partial_x^2}v_n + 2\partial_x\varphi\,\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n + \partial_x^2\varphi\, D^{-2}e^{\epsilon\partial_x^2}v_n. \tag{3.3.22}$$

Substituting (3.3.22) into (3.3.21) and using that the Hilbert transform $\mathcal{H}$ commutes with the operator D^r (see (2.5.2)), we obtain

$$\begin{aligned}[A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n &= \alpha\psi(t)\left\{-\varphi\mathcal{H}D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n + \mathcal{H}\left(\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right)\right\}\\ &\quad + 2\alpha\psi(t)\mathcal{H}\left(\partial_x\varphi\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n\right) + \alpha\psi(t)\mathcal{H}\left(\partial_x^2\varphi\, D^{-2}e^{\epsilon\partial_x^2}v_n\right).\end{aligned} \tag{3.3.23}$$

Also, using Lemma 3.3.1, we have

$$\begin{aligned}-\varphi\mathcal{H}D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n + \mathcal{H}\left(\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right) &= \varphi D^{-1}\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n - D^{-1}\partial_x\left(\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right)\\ &= \varphi D^{-1}\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n - D^{-1}\left(\varphi\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right)\\ &\quad - D^{-1}\left(\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right)\\ &= -[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n - D^{-1}\left(\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right).\end{aligned} \tag{3.3.24}$$

Substituting (3.3.24) into (3.3.23), we get

$$\begin{aligned}[A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n &= \alpha\psi(t)\left\{-[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n - D^{-1}\left(\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right)\right\}\\ &\quad + 2\alpha\psi(t)\mathcal{H}\left(\partial_x\varphi\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n\right) + \alpha\psi(t)\mathcal{H}\left(\partial_x^2\varphi D^{-2}e^{\epsilon\partial_x^2}v_n\right).\end{aligned}$$

Therefore,

$$\begin{aligned}([A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n, v_n)_{L^2(\mathbb{T}\times(0,T))} &= -\left(\alpha\psi(t)[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad - \left(\alpha\psi(t)D^{-1}\left(\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad + 2\alpha\left(\psi(t)\mathcal{H}\left(\partial_x\varphi\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad + \alpha\left(\psi(t)\mathcal{H}\left(\partial_x^2\varphi D^{-2}e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\end{aligned} \tag{3.3.25}$$

Appliying Cauchy-Schwartz, Lemma 3.2.1, (3.3.3), Lemmas 3.1.7, 3.1.7 and using that

$0 \leqslant b' \leqslant b \leqslant 1$ be given (with $b > 0$), we obtain

$$\begin{aligned}
\left|\left(\alpha\psi(t)[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \alpha\|\psi(t)[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\|_{L^2(\mathbb{T}\times(0,T))}\|v_n\|_{L^2(\mathbb{T}\times(0,T))}\\
&\leqslant C\|[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\|_{L^2(\mathbb{T}\times(0,T))}\|v_n\|_{X^T_{0,b}}\\
&\leqslant C\left(\int_0^T \|[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\|^2_{H^0(\mathbb{T})}dt\right)^{\frac{1}{2}}\\
&\leqslant C\left(\int_0^T \|\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\|^2_{H^{-1-1}(\mathbb{T})}dt\right)^{\frac{1}{2}}\\
&\leqslant \left\|\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right\|^{\frac{1}{2}}_{X^T_{-2,-b'}}\left\|\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right\|^{\frac{1}{2}}_{X^T_{-2,b'}}\\
&\leqslant \|v_n\|^{\frac{1}{2}}_{X^T_{-1,-b'}}\|v_n\|^{\frac{1}{2}}_{X^T_{-1,b'}}.
\end{aligned} \tag{3.3.26}$$

Applying (3.3.3), (3.3.4) and using the embeddings $X^T_{0,b} \hookrightarrow X^T_{-1,b'}$, $X^T_{-1+2b',-b'} \hookrightarrow X^T_{-1,-b'}$, we obtain that there exists a positive constant C (independent of T, if $T \leqslant 1$) which does not depend on ϵ such that

$$\begin{aligned}
\left|\left(\alpha\psi(t)[D^{-1},\varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant C\,\|v_n\|^{\frac{1}{2}}_{X^T_{-1+2b',-b'}}\,\|v_n\|^{\frac{1}{2}}_{X^T_{0,b}}\\
&\leqslant C\,\|v_n\|^{\frac{1}{2}}_{X^T_{-1+2b',-b'}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{3.3.27}$$

Note that the lost of regularity in (3.3.26) and (3.3.27) is too large if one uses the estimates with the same b. Therefore, we have to use the index b' instead. Consequently,

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|\left(\alpha\psi(t)[D^{-1},\ \varphi]\partial_x D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.28}$$

Also, note that Lemma 3.2.1, Lemma 3.1.7, Theorem (3.2.3), and (3.3.3)-(3.3.4), we have that there exists a positive constant C (independent of T, if $T \leqslant 1$), which does not depend on ϵ such that

$$\begin{aligned}
\left|\left(\alpha\psi(t)D^{-1}\left(\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant C\left\|D^{-1}\left(\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right)\right\|_{X^T_{1-2b',b'}}\|v_n\|_{X^T_{-1+2b',-b'}}\\
&\leqslant C\left\|\partial_x\varphi D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right\|_{X^T_{-2b',b'}}\|v_n\|_{X^T_{-1+2b',-b'}}\\
&\leqslant C\left\|D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right\|_{X^T_{-2b'+2|b'|,b'}}\|v_n\|_{X^T_{-1+2b',-b'}}\\
&\leqslant C\,\|v_n\|_{X^T_{0,b'}}\,\|v_n\|_{X^T_{-1+2b',-b'}}\\
&\leqslant C\,\|v_n\|_{X^T_{0,b}}\,\|v_n\|_{X^T_{-1+2b',-b'}}\\
&\leqslant C\,\|v_n\|_{X^T_{-1+2b',-b'}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Hence,

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|\left(\alpha\ \psi(t)\ D^{-1}\left(\partial_x\varphi\ D^{-2}\partial_x^2 e^{\epsilon\partial_x^2}v_n\right),\ v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.29}$$

Similarly, using that the Hilbert transform $\mathcal{H}$ is an isometry in $L^2_p(\mathbb{T})$, we have

$$\begin{aligned}\left|\left(2\alpha\psi(t)\mathcal{H}\left(\partial_x\varphi\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant 2\alpha\left\|\psi(t)\mathcal{H}\left(\partial_x\varphi\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n\right)\right\|_{X^T_{0,-b'}}\|v_n\|_{X^T_{0,b'}}\\ &\leqslant C\left\|\partial_x\varphi D^{-2}\partial_x e^{\epsilon\partial_x^2}v_n\right\|_{X^T_{0,-b'}}\\ &\leqslant C\|v_n\|_{X^T_{-1+2b',-b'}} \longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned}$$

Therefore,

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|\left(2\alpha\psi(t)\mathcal{H}\left(\partial_x\varphi\partial_x D^{-2}e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.30}$$

With similar arguments, we get

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|\left(\alpha\psi(t)\mathcal{H}\left(\partial_x^2\varphi\ D^{-2}e^{\epsilon\partial_x^2}v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.31}$$

From (3.3.25) and (3.3.28)-(3.3.31), we obtain that

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|\left([A_\epsilon, -\alpha\mathcal{H}\partial_x^2]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.32}$$

Therefore, (3.3.16), (3.3.17) (3.3.20), and (3.3.32), imply that

$$\lim_{n\longrightarrow\infty}\sup_{0<\epsilon\leqslant 1}\left|\left([A_\epsilon, -\partial_x^3+2\mu\partial_x]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.33}$$

In particular,

$$\lim_{n\longrightarrow\infty}\left([A,\ -\partial_x^3+2\mu\partial_x]v_n,\ v_n\right)_{L^2(\mathbb{T}\times(0,T))} = 0. \tag{3.3.34}$$

Using the Leibniz's rule for derivatives (see page [54, page 13]), we note that

$$[A, -\partial_x^3+2\mu\partial_x]v_n = 3\psi(t)\partial_x\varphi\partial_x^2 D^{-2}v_n + 3\psi(t)\partial_x^2\varphi\partial_x D^{-2}v_n - \psi(t)\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)D^{-2}v_n.$$

Therefore,

$$\begin{aligned}\left([A, -\partial_x^3+2\mu\partial_x]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))} &= \left(3\psi(t)\partial_x\varphi\partial_x^2 D^{-2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad + \left(3\psi(t)\partial_x^2\varphi\partial_x D^{-2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad - \left(\psi(t)\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)D^{-2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.35}$$

Using Lemma 3.2.1, Lemma 3.1.7, Theorem 3.2.3, (3.3.3), and (3.3.4), we obtain that the last term in (3.3.35) satisfies

$$\begin{aligned}\left|\left(\psi(t)\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)D^{-2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \left\|\psi(t)\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)D^{-2}v_n\right\|_{X^T_{2-2b,b}}\|v_n\|_{X^T_{-2+2b,-b}}\\ &\leqslant C\left\|D^{-2}v_n\right\|_{X^T_{2,b}}\|v_n\|_{X^T_{-2+2b,-b}}\\ &\leqslant C\|v_n\|_{X^T_{0,b}}\|v_n\|_{X^T_{-2+2b,-b}}\\ &\leqslant C\|v_n\|_{X^T_{-2+2b,-b}} \longrightarrow 0 \text{ as } n\longrightarrow\infty.\end{aligned} \tag{3.3.36}$$

However, for the second term of (3.3.35), the loss of regularity is too large if we use the estimates with the same b. Using the index b' instead, we have

$$\begin{aligned}
\left|\left(3\psi(t)\partial_x^2\varphi\partial_x D^{-2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \left\|\psi(t)\partial_x^2\varphi\partial_x D^{-2}v_n\right\|_{X^T_{1-2b',b'}} \|v_n\|_{X^T_{-1+2b',-b'}} \\
&\leqslant C\left\|\partial_x D^{-2}v_n\right\|_{X^T_{1,b'}} \|v_n\|_{X^T_{-1+2b',-b'}} \\
&\leqslant C\|v_n\|_{X^T_{0,b'}} \|v_n\|_{X^T_{-1+2b',-b'}} \\
&\leqslant C\|v_n\|_{X^T_{-1+2b',-b'}} \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned} \tag{3.3.37}$$

From (3.3.34)-(3.3.37), we obtain

$$\lim_{n\longrightarrow\infty}\left|\left(\psi(t)\partial_x\varphi(-\partial_x^2)D^{-2}v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0,$$

and (3.3.11) is proved.

Second, we prove that

$$\lim_{n\longrightarrow\infty}\left|\left(\psi(t)\ \partial_x\varphi\ \widehat{v_n}(0,t), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| = 0. \tag{3.3.38}$$

Indeed, note that for $0 < b < 1$, we have

$$\|\widehat{v_n}(0,t)\|_{H^b(0,T)} = \left(\int_0^T \langle\tau\rangle^{2b}|\widehat{v_n}(0,\tau)|^2\, d\tau\right)^{\frac{1}{2}} \leqslant C\|v_n\|_{X^T_{0,b}} \leqslant C.$$

Thus, the sequence $\widehat{v_n}(0,\cdot)$ is bounded in $H^b(0,T)$, which is compactly embeded in $\in$ $L^2(0,T)$, by the Rellich's theorem (see [36, page 305]). Therefore, there exists a subsequence that converges strongly in $L^2(0,T)$. Next, it can be seen that the only weak limit of a subsequence in $L^2(0,T)$ is zero, so that the whole sequence tends strongly to 0 in $L^2(0,T)$. Thus

$$\widehat{v_n}(0,t) \longrightarrow 0\ \text{ (strongly) in } L^2(0,T),\ \text{ as } n \longrightarrow \infty,$$

and hence we hold (3.3.38).

From (3.3.10)-(3.3.11), and (3.3.38), we obtain (3.3.9). This completes the proof of the proposition. □

Next, we investigate the propagation of regularity for the operator L defined in (3.3.1).

Proposition 3.3.3 (Propagation of Regularity). *Let $T > 0$, $0 \leqslant b < 1$, $r \geqslant 0$ and $f \in X^T_{r,-b}$ be given. Let $v \in X^T_{r,b}$ be a solution of*

$$Lv := \partial_t v - \alpha\ \mathcal{H}\partial_x^2 v - \partial_x^3 v + 2\mu\ \partial_x v = f. \tag{3.3.39}$$

If there exists a nonempty open set ω of $\mathbb{T}$ such that

$$v \in L^2_{loc}((0,T); H^{r+\rho}(\omega)), \tag{3.3.40}$$

for some ρ with

$$0 < \rho \leqslant \min\left\{1-b,\ \frac{1}{2}\right\}, \tag{3.3.41}$$

then $v \in L^2_{loc}((0,T); H^{r+\rho}(\mathbb{T}))$.

Proof. Let $s = r+\rho$. Let Ω be a compact subset of the interval $(0,T)$, and $\psi(t) \in C_c^\infty(0,T)$, such that $0 \leqslant \psi(t) \leqslant 1$ and $\psi(t) = 1$ in Ω. Observe that

$$\begin{aligned}
\|v\|^2_{L^2(\Omega,H^s(\mathbb{T}))} &= \int_0^T \psi(t)\|v\|_{H^s(\mathbb{T})}dt \\
&\leqslant c_s\left(\int_0^T \psi(t)\|v\|^2_{L^2(\mathbb{T})}dt + \int_0^T \psi(t)\left(2\pi\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}|k|^{2s}|\widehat{v}(k,t)|^2\right)dt\right) \\
&\leqslant c_s\left(\|v\|^2_{L^2(\mathbb{T}\times(0,T))} - \left(\psi(t)D^{2s-2}\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right),
\end{aligned}$$

where the operator D is defined in (2.5.2). Thus, we only need to show that there exists a positive constant C such that

$$\left|\left(\psi(t)D^{2s-2}\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant C. \tag{3.3.42}$$

Note that, with a similar argument as in the proof of Proposition 3.3.2, there exists a finite set of points $x_0^i \in \mathbb{T}$, $i = 1,2,3,\cdots$, such that we can construct a partition of the unity on $\mathbb{T}$ involving functions of the form $\chi_i^2(\cdot - x_0^i)$ with $\chi_i^2(\cdot) \in C_c^\infty(\omega)$. Therefore,

$$\begin{aligned}
\left|\left(\psi(t)D^{2s-2}\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right| &= \left|\left(\psi(t)D^{2s-2}\left(\sum_{i=1}^N \chi_i^2(x-x_0^i)\right)\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\
&\leqslant \sum_{i=1}^N\left|\left(\psi(t)D^{2s-2}\chi_i^2(x-x_0^i)\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right|.
\end{aligned}$$

Then, it is sufficient to prove that for any $x_0 \in \mathbb{T}$, and any $\chi^2(\cdot) \in C_c^\infty(\omega)$ there exists a positive constant C such that

$$\left|\left(\psi(t)D^{2s-2}\chi^2(x-x_0)\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant C. \tag{3.3.43}$$

In fact, from Lemma A.4 there exists $\varphi \in C^\infty(\mathbb{T})$ such that $\partial_x\varphi(x) = \chi^2(x) - \chi^2(x-x_0)$ for all $x \in \mathbb{T}$. Consequently,

$$\begin{aligned}
\left|\left(\psi(t)D^{2s-2}\chi^2(x-x_0)\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant &\left|\left(\psi(t)D^{2s-2}\chi^2(x)\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\
&+\left|\left(\psi(t)D^{2s-2}\partial_x\varphi(x)\partial_x^2 v, v\right)_{L^2(\mathbb{T}\times(0,T))}\right|.
\end{aligned} \tag{3.3.44}$$

Now, we move to bound the right hand side (RHS) of (3.3.44). Define

$$v_n := e^{\frac{1}{n}\partial_x^2}v = E_n v = \left(e^{-\frac{1}{n}k^2}\widehat{v}(k,t)\right)^\vee, \tag{3.3.45}$$

and $f_n := E_n f = E_n Lv$, for $n = 1, 2, 3, \cdots$. Passing to the frequency space, it is easy to verify that E_n commutes with L, i.e.,

$$f_n := E_n f = E_n Lv = LE_n v = Lv_n.$$

From hypothesis and the definition of E_n we obtain that there exists $C > 0$ independent on n such that

$$\|v_n\|_{X^T_{r,b}} \leqslant C, \quad \text{and} \quad \|f_n\|_{X^T_{r,-b}} \leqslant C, \quad \text{for all} \;\; n \geqslant 1. \tag{3.3.46}$$

Set, $B = D^{2s-2}\varphi$, and $A = \psi(t)B$. We infer from Lemma 3.2.1 and Theorem 3.2.3, that for any $r \in \mathbb{R}$, and $0 \leqslant b < 1$, there exists a positive constant C (independent of T, if $T \leqslant 1$) such that

$$\|Av\|_{X^T_{r-2|b|-2s+2,b}} \leqslant C \|v\|_{X^T_{r,b}}. \tag{3.3.47}$$

With similar calculations as in the proof of the Proposition 3.3.2 (see (3.3.17)), we obtain

$$\begin{aligned}(f_n, A^* v_n)_{L^2(\mathbb{T}\times(0,T))} + (Av_n, f_n)_{L^2(\mathbb{T}\times(0,T))} = & -\left(\psi'(t)Bv_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))} \\ & + \left([\alpha\mathcal{H}\partial_x^2, A]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))} \\ & + \left([A, -\partial_x^3 + 2\mu\partial_x]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.48}$$

Using that $\rho \leqslant 1-b$, (3.3.46) and (3.3.47) we get that there exists $C > 0$ independent of n such that

$$\begin{aligned}\left|(Av_n, f_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \|Av_n\|_{X^T_{-r,b}} \|f_n\|_{X^T_{r,-b}} \\ &\leqslant C\|v_n\|_{X^T_{-r+2b+2s-2,b}} \|f_n\|_{X^T_{r,-b}} \leqslant C\|v_n\|_{X^T_{r,b}} \leqslant C,\end{aligned} \tag{3.3.49}$$

$$\begin{aligned}\left|(f_n, A^*v_n)_{L^2(\mathbb{T}\times(0,T))}\right| &= \left|(Af_n, v_n)_{L^2(\mathbb{T}\times(0,T))}\right| \\ &\leqslant \|Af_n\|_{X^T_{-r,-b}} \|v_n\|_{X^T_{r,b}} \\ &\leqslant C\|f_n\|_{X^T_{-r+2|-b|+2s-2,-b}} \|v_n\|_{X^T_{r,b}} \leqslant C\|f_n\|_{X^T_{r,-b}} \leqslant C,\end{aligned} \tag{3.3.50}$$

and

$$\begin{aligned}\left|(\psi'(t)Bv_n, v_n)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \|\psi'(t)Bv_n\|_{X^T_{-r,-b}} \|v_n\|_{X^T_{r,b}} \\ &\leqslant C\|Bv_n\|_{X^T_{-r,-b}} \|v_n\|_{X^T_{r,b}} \\ &\leqslant C\|v_n\|_{X^T_{-r+2|-b|+2s-2,-b}} \leqslant C\|v_n\|_{X^T_{r,b}} \leqslant C.\end{aligned} \tag{3.3.51}$$

Also, using the Leibniz's rule for derivatives (see [54, page 13]), and Lemma (3.3.1), we obtain

$$\begin{aligned}\mathcal{H}\partial_x^2 Av_n = & -\psi(t)D^{2s-3}\left(\partial_x\varphi\partial_x^2 v_n\right) - \psi(t)D^{2s-3}\left(\varphi\partial_x^3 v_n\right) \\ & + 2\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x\varphi\partial_x v_n\right) + \mathcal{H}\psi(t)D^{2s-2}\left(\partial_x^2\varphi v_n\right).\end{aligned} \tag{3.3.52}$$

On the other hand,

$$A\mathcal{H}\partial_x^2 v_n = \psi(t)D^{2s-2}\left(\varphi\mathcal{H}\partial_x^2 v_n\right) = -\psi(t)D^{2s-2}\left(\varphi D^{-1}\partial_x^3 v_n\right). \tag{3.3.53}$$

From (3.3.52) and (3.3.53), we have

$$\begin{aligned}\mathcal{H}\partial_x^2 A v_n - A\mathcal{H}\partial_x^2 v_n &= -\psi(t)\ D^{2s-2}\left([D^{-1},\varphi]\partial_x^3 v_n\right) - \psi(t)D^{2s-3}\left(\partial_x\varphi\ \partial_x^2 v_n\right)\\ &\quad + 2\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x\varphi\ \partial_x v_n\right) + \mathcal{H}\psi(t)D^{2s-2}\left(\partial_x^2\varphi\ v_n\right).\end{aligned}$$

Then,

$$\begin{aligned}\left([\alpha\mathcal{H}\partial_x^2, A]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))} &= \alpha\left(\mathcal{H}\partial_x^2 A v_n - A\mathcal{H}\partial_x^2 v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &= -\alpha\left(\psi(t)D^{2s-2}\left([D^{-1},\varphi]\partial_x^3 v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad - \alpha\left(\psi(t)D^{2s-3}\left(\partial_x\varphi\partial_x^2 v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad + 2\alpha\left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x\varphi\partial_x v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\ &\quad + \alpha\left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x^2\varphi v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.54}$$

Using $\rho \leqslant \dfrac{1}{2}$, Lemma 3.2.1, Theorem 3.2.3, Lemma A.1, and that $\mathcal{H}$ is an isometry in $H_p^{-r}(\mathbb{T})$, we can obtain $C > 0$ independent on n, such that

$$\begin{aligned}\alpha\left|\left(\psi(t)D^{2s-2}\left([D^{-1},\varphi]\partial_x^3 v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \alpha\left\|\psi(t)D^{2s-2}\left([D^{-1},\varphi]\partial_x^3 v_n\right)\right\|_{X^T_{-r,0}}\|v_n\|_{X^T_{r,0}}\\ &\leqslant C\left\|[D^{-1},\varphi]\partial_x^3 v_n\right\|_{X^T_{-r+2s-2,0}}\|v_n\|_{X^T_{r,b}}\\ &\leqslant C\|v_n\|_{X^T_{-r+2s-1,0}}\\ &\leqslant C\|v_n\|_{X^T_{r,b}} \leqslant C,\end{aligned} \tag{3.3.55}$$

and

$$\begin{aligned}\alpha\left|\left(\psi(t)D^{2s-3}\left(\partial_x\varphi\partial_x^2 v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| &\leqslant \alpha\left\|\psi(t)D^{2s-3}\left(\partial_x\varphi\partial_x^2 v_n\right)\right\|_{L^2((0,T);H^{-r}(\mathbb{T}))}\|v_n\|_{L^2((0,T);H^r(\mathbb{T}))}\\ &\leqslant C\left\|\partial_x\varphi\partial_x^2 v_n\right\|_{L^2((0,T);H^{-r+2s-3}(\mathbb{T}))}\|v_n\|_{X^T_{r,0}}\\ &\leqslant C\left\|\partial_x\varphi\partial_x^2 v_n\right\|_{X^T_{-r+2s-3,0}}\|v_n\|_{X^T_{r,b}}\\ &\leqslant C\|v_n\|_{X^T_{-r+2s-1,0}}\\ &\leqslant C\|v_n\|_{X^T_{r,b}} \leqslant C.\end{aligned} \tag{3.3.56}$$

In similar manner, one can get

$$2\alpha\left|\left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x\varphi\partial_x v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant C\|\partial_x\varphi\partial_x v_n\|_{X^T_{-r+2s-2,0}}\|v_n\|_{X^T_{r,0}} \leqslant C, \tag{3.3.57}$$

and

$$\alpha\left|\left(\mathcal{H}\psi(t)D^{2s-2}\left(\partial_x^2\varphi v_n\right), v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant C\|v_n\|_{X^T_{-r+2s-2,0}}\|v_n\|_{X^T_{r,b}} \leqslant C\|v_n\|_{X^T_{r,b}} \leqslant C. \tag{3.3.58}$$

From (3.3.54)-(3.3.58), we infer that

$$\left|\left([\alpha\mathcal{H}\partial_x^2, A]v_n, v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant C. \tag{3.3.59}$$

It follows from (3.3.48), (3.3.49), (3.3.50), (3.3.51), and (3.3.59), that

$$\left|\left([A,-\partial_x^3+2\mu\partial_x]v_n,v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|\leqslant C, \tag{3.3.60}$$

where $C>0$ does not depend on n. Using the Leibniz's rule, we note that

$$\begin{aligned}([A,-\partial_x^3+2\mu\partial_x]v_n,v_n)_{L^2(\mathbb{T}\times(0,T))}&=\left(3\psi(t)D^{2s-2}\partial_x\varphi\partial_x^2v_n,v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&\quad+\left(3\psi(t)D^{2s-2}\partial_x^2\varphi\partial_xv_n,v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&\quad-\left(\psi(t)D^{2s-2}\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)v_n,v_n\right)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.61}$$

Using similar arguments as above, we get

$$\begin{aligned}\left|\left(3\psi(t)D^{2s-2}\partial_x^2\varphi\partial_xv_n,v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|&\leqslant\left\|\psi(t)D^{2s-2}\partial_x^2\varphi\partial_xv_n\right\|_{X^T_{-r,0}}\|v_n\|_{X^T_{r,0}}\\&\leqslant C\left\|\partial_xv_n\right\|_{X^T_{-r+2s-2,0}}\\&\leqslant C\left\|v_n\right\|_{X^T_{-r+2s-1,0}}\leqslant C,\end{aligned} \tag{3.3.62}$$

and

$$\begin{aligned}\left|\left(\psi(t)D^{2s-2}\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)v_n,v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|&\leqslant\left\|\psi(t)D^{2s-2}\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)v_n\right\|_{X^T_{-r,0}}\|v_n\|_{X^T_{r,0}}\\&\leqslant C\left\|\left(-\partial_x^3\varphi+2\mu\partial_x\varphi\right)v_n\right\|_{X^T_{-r+2s-2,0}}\|v_n\|_{X^T_{r,b}}\\&\leqslant C\left\|v_n\right\|_{X^T_{-r+2s-2,0}}\leqslant C,\end{aligned} \tag{3.3.63}$$

From (3.3.60), (3.3.61), (3.3.62), and (3.3.63), we infer that there exists $C>0$ independent of n such that

$$\left|\left(\psi(t)\ D^{2s-2}\partial_x\varphi(x)\ \partial_x^2v_n,\ v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right|\leqslant C\ \text{ for any } n\geqslant1. \tag{3.3.64}$$

Therefore, letting $n\longrightarrow+\infty$ we get that the second term on the right side of (3.3.44) is bounded. Now, we estimate the term $\left|\left(\psi(t)D^{2s-2}\chi^2(x)\partial_x^2v,v\right)_{L^2(\mathbb{T}\times(0,T))}\right|$ in (3.3.44). As the operator D^s is self-adjoint in $L^2(\mathbb{T})$, then for any $\chi\in C_c^\infty(\mathbb{T})$, we get

$$\begin{aligned}\left(\psi(t)D^{2s-2}\chi^2\partial_x^2v_n,D^sv_n\right)_{L^2(\mathbb{T}\times(0,T))}&=\left(\psi(t)[D^{s-2},\chi]\chi\partial_x^2v_n,D^sv_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&\quad+\left(\psi(t)D^{s-2}\chi\partial_x^2v_n,\chi D^sv_n\right)_{L^2(\mathbb{T}\times(0,T))},\end{aligned} \tag{3.3.65}$$

and

$$\begin{aligned}\left(\psi(t)D^{s-2}\chi\partial_x^2v_n,[\chi,D^s]v_n\right)_{L^2(\mathbb{T}\times(0,T))}&=\left(\psi(t)D^{s-2}\chi\partial_x^2v_n,\chi D^sv_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&\quad-\left(\psi(t)D^{s-2}\chi\partial_x^2v_n,D^s\chi v_n\right)_{L^2(\mathbb{T}\times(0,T))}.\end{aligned} \tag{3.3.66}$$

Isolating the first term of the right side of the last inequality and substituting in (3.3.65), we obtain

$$\begin{aligned}\left(\psi(t)D^{2s-2}\chi^2\partial_x^2v_n,D^sv_n\right)_{L^2(\mathbb{T}\times(0,T))}&=\left(\psi(t)D^{s-2}\chi\partial_x^2v_n,D^s\chi v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&\quad+\left(\psi(t)D^{s-2}\chi\partial_x^2v_n,[\chi,D^s]v_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&\quad+\left(\psi(t)[D^{s-2},\chi]\chi\partial_x^2v_n,D^sv_n\right)_{L^2(\mathbb{T}\times(0,T))}\\&=:I+II+III.\end{aligned} \tag{3.3.67}$$

Using (3.3.40), one have

$$\|\chi v\|_{L^2_{loc}((0,T);H^s(\mathbb{T}))} = \|\chi v\|_{L^2_{loc}((0,T);H^s(\omega))} < \infty, \tag{3.3.68}$$

and

$$\left\|\chi \partial_x^2 v\right\|_{L^2_{loc}((0,T);H^{s-2}(\mathbb{T}))} = \left\|\chi \partial_x^2 v\right\|_{L^2_{loc}((0,T);H^{s-2}(\omega))} < \infty. \tag{3.3.69}$$

Observe that $\chi v_n = E_n \chi v + [\chi, E_n]v$. Using Lemma A.2 and Lemma A.3, we obtain

$$\begin{aligned} \|\chi v_n\|_{H^s(\mathbb{T})} &\leqslant \|E_n \chi v\|_{H^s(\mathbb{T})} + \|[\chi, E_n]v\|_{H^s(\mathbb{T})} \\ &\leqslant C\,\|\chi v\|_{H^s(\mathbb{T})} + C_s\,\|v\|_{H^{s-1}(\mathbb{T})}\,. \end{aligned} \tag{3.3.70}$$

From (3.3.68) and (3.3.70), there exists $C > 0$ independent on n such that

$$\begin{aligned} \|\chi v_n\|_{L^2_{loc}((0,T);H^s(\mathbb{T}))} &\leqslant C\,\|\chi v\|_{L^2_{loc}((0,T);H^s(\mathbb{T}))} + C_s\,\|v\|_{L^2_{loc}((0,T);H^{s-1}(\mathbb{T}))} \\ &\leqslant C + C_s\,\|v\|_{X^T_{s-1,0}} \\ &\leqslant C + C_s\,\|v\|_{X^T_{r,b}} \leqslant C. \end{aligned} \tag{3.3.71}$$

Hence, χv_n is uniformly bounded in $L^2_{loc}((0,T);H^s(\mathbb{T}))$. In a similar way we can estimate $\chi\partial_x^2 v_n$ and get

$$\begin{aligned} \left\|\chi \partial_x^2 v_n\right\|_{L^2_{loc}((0,T);H^{s-2}(\mathbb{T}))} &\leqslant C\left\|\chi \partial_x^2 v_n\right\|_{L^2_{loc}((0,T);H^{s-2}(\mathbb{T}))} + C_s\left\|\partial_x^2 v\right\|_{L^2_{loc}((0,T);H^{s-3}(\mathbb{T}))} \\ &\leqslant C + C_s\,\|v\|_{X^T_{s-1,0}} \leqslant C, \end{aligned} \tag{3.3.72}$$

where the last positive constant C doesn't depend on n. Next, we estimate I, II, and III. Using Lemma 3.2.1, and Lemma 3.1.7, we have

$$\begin{aligned} |I| &= \left|\left(\psi(t) D^{s-2}\chi\partial_x^2 v_n, D^s \chi v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\ &\leqslant C\left\|\psi(t) D^{s-2}\chi\partial_x^2 v_n\right\|_{L^2_{loc}(\mathbb{T}\times(0,T))} \|D^s \chi\, v_n\|_{L^2_{loc}(\mathbb{T}\times(0,T))} \\ &\leqslant C\left\|\chi\partial_x^2 v_n\right\|_{L^2_{loc}((0,T);H^{s-2}(\mathbb{T}))} \|\chi v_n\|_{L^2_{loc}((0,T);H^s(\mathbb{T}))} \leqslant C, \end{aligned} \tag{3.3.73}$$

$$\begin{aligned} |II| &= \left|\left(\psi(t) D^{s-2}\chi\partial_x^2 v_n, [\chi, D^s] v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\ &\leqslant C\left\|\psi(t) D^{r-2}\chi\partial_x^2 v_n\right\|_{L^2_{loc}(\mathbb{T}\times(0,T))} \|D^\rho[\chi, D^s] v_n\|_{L^2_{loc}(\mathbb{T}\times(0,T))} \\ &\leqslant C\left\|\chi\partial_x^2 v_n\right\|_{L^2_{loc}((0,T);H^{r-2}(\mathbb{T}))} \|[\chi, D^s] v_n\|_{L^2_{loc}((0,T);H^\rho(\mathbb{T}))} \\ &\leqslant C\,\|v_n\|_{L^2_{loc}((0,T);H^r(\mathbb{T}))} \|v_n\|_{L^2_{loc}((0,T);H^{s-1+\rho}(\mathbb{T}))} \\ &\leqslant C\,\|v\|_{X^T_{r,0}} \|v\|_{X^T_{s-1+\rho,0}} \leqslant C, \end{aligned} \tag{3.3.74}$$

and

$$\begin{aligned}
|III| &= \left|\left(\psi(t)[D^{s-2},\chi]\chi\partial_x^2 v_n, D^s v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \\
&\leqslant C\left\|D^\rho[D^{s-2},\chi]\chi\partial_x^2 v_n\right\|_{L^2_{loc}(\mathbb{T}\times(0,T))}\left\|D^r v_n\right\|_{L^2_{loc}(\mathbb{T}\times(0,T))} \\
&\leqslant C\left\|\chi\partial_x^2 v_n\right\|_{L^2_{loc}((0,T);H^{\rho+s-3}(\mathbb{T}))}\left\|v_n\right\|_{L^2_{loc}((0,T);H^r(\mathbb{T}))} \\
&\leqslant C\left\|v_n\right\|_{L^2_{loc}((0,T);H^{\rho+s-1}(\mathbb{T}))}\left\|v_n\right\|_{X^T_{r,0}} \\
&\leqslant C\left\|v\right\|_{X^T_{s+\rho-1,0}} \leqslant C.
\end{aligned} \tag{3.3.75}$$

From (3.3.67), (3.3.73), (3.3.74), and (3.3.75), we infer that there exists $C > 0$ independent of n such that

$$\left|\left(\psi(t)D^{2s-2}\chi^2\partial_x^2 v_n, D^s v_n\right)_{L^2(\mathbb{T}\times(0,T))}\right| \leqslant C. \tag{3.3.76}$$

Letting $n \longrightarrow +\infty$ we get that the first term on the right side of (3.3.44) is bounded. Thus (3.3.44), (3.3.64), and (3.3.76), imply (3.3.43) and completes the proof. □

Corollary 3.3.4. *Let $\mu \in \mathbb{R}$ and $\alpha > 0$ be given. Let $v \in X^T_{0,\frac{1}{2}}$ be a solution of*

$$\partial_t v - \partial_x^3 v - \alpha\,\mathcal{H}\partial_x^2 v + 2\mu\,\partial_x v + 2v\partial_x v = 0, \quad on\ \ (0,T), \tag{3.3.77}$$

with $[u] = 0$. Assume that $v \in C^\infty(\omega \times (0,T))$, where ω is a nonempty open set in $\mathbb{T}$. Then $v \in C^\infty(\mathbb{T} \times (0,T))$.

Proof. From Corollary 3.1.26, we have that $2v\partial_x v \in X^T_{0,-\frac{1}{2}}$. Observe that

$$v \in C^\infty(\omega \times (0,T)) \subset L^2_{loc}((0,T); H^{\frac{1}{2}}(\omega)).$$

It follows from Proposition 3.3.3 (with $f = -2v\partial_x v$) that $v \in L^2_{loc}((0,T); H^{\frac{1}{2}}(\mathbb{T}))$.

Choosing $t_0 \in (0,T)$ such that $v(t_0) \in H^{\frac{1}{2}}(\mathbb{T})$, we can solve equation (3.3.77) in the space $X^T_{\frac{1}{2},\frac{1}{2}}$ with the initial data $v(t_0)$. By the uniqueness of the solution in $X^T_{0,\frac{1}{2}}$ we conclude that $u \in X^T_{\frac{1}{2},\frac{1}{2}}$.

An iterated application of Proposition 3.3.3 yields that $v \in L^2_{loc}((0,T); H^r(\mathbb{T}))$, for all $r \in \mathbb{R}$. Thus, $v \in C^\infty(\mathbb{T} \times \Omega)$, for all compact set $\Omega \subset (0,T)$. Therefore, $v \in C^\infty(\mathbb{T} \times (0,T))$. □

3.4 Unique continuation property for Benjamin equation

In this section we prove the unique continuation property for the Benjamin equation. First, we announce a necessary lemma.

Lemma 3.4.1 (See [59, Lemma 2.9])**.** *Let $s \in \mathbb{R}$ and let $h(x) = \sum_{k\geqslant 0}\widehat{h}(k)e^{ikx}$ be such that $h \in H^s(\mathbb{T})$ and $h = 0$ in $(a,b) \subset \mathbb{T}$. Then $h \equiv 0$.*

The following is the main result of this section. The ideas of the proof are similar to those leading Proposition 2.8 in [59].

Proposition 3.4.2. *Let $\mu \in \mathbb{R}$, $\alpha > 0$, and $c(t) \in L^2(0,T)$ be given. Let $v \in L^2((0,T); L^2_0(\mathbb{T}))$ be a solution of*

$$\begin{cases} \partial_t v - \partial_x^3 v - \alpha \mathcal{H} \partial_x^2 v + 2\mu \partial_x v + 2v\partial_x v = 0, & t > 0, \text{ on } \mathbb{T} \times (0,T) \\ v(x,t) = c(t), & \text{for almost every } (x,t) \in (a,b) \times (0,T), \end{cases} \tag{3.4.1}$$

for some numbers $T > 0$ and $0 \leqslant a < b \leqslant 2\pi$. Then $v(x,t) = 0$ for almost every $(x,t) \in \mathbb{T} \times (0,T)$.

Proof. Since $v(x,t) = c(t)$, for a.e $(x,t) \in (a,b) \times (0,T)$, we have that

$$\partial_x v(x,t) = \partial_x^2 v(x,t) = (2v\partial_x v)(x,t) = 0 \text{ for a.e } (x,t) \in (a,b) \times (0,T) \tag{3.4.2}$$

From the first relation in (3.4.1), we infer that v satisfies

$$\partial_t v - \alpha \mathcal{H} \partial_x^2 v = 0 \text{ in } (a,b) \times (0,T).$$

Thus, the second relation in (3.4.1) implies that

$$\alpha \mathcal{H} \partial_x^2 v = \partial_t v = c'(t) \text{ in } (a,b) \times (0,T).$$

Therefore, for almost every $t \in (0,T)$, it holds that

$$\begin{aligned} &\partial_x^3 v(\cdot,t) \in H^{-3}(\mathbb{T}) \\ &\partial_x^3 v(\cdot,t) = 0, \quad \text{in } (a,b). \\ &\mathcal{H}\partial_x^3 v(\cdot,t) = \partial_x \mathcal{H} \partial_x^2 v(\cdot,t) = 0, \quad \text{in } (a,b). \end{aligned} \tag{3.4.3}$$

Pick a time t as above, and set $h(x) = \partial_x^3 v(x,t)$, for $x \in \mathbb{T}$. Decompose h as

$$h(x) = \sum_{k \in \mathbb{Z}} \widehat{h}(k) e^{ikx},$$

where the convergence of the series being in $H^{-3}(\mathbb{T})$. Observe that

$$(ih - \mathcal{H}h)(x) = \sum_{k \in \mathbb{Z}} (ih - \mathcal{H}h)^\wedge(k) e^{ikx} = 2i \sum_{k>0} \hat{h}(k) e^{ikx}. \tag{3.4.4}$$

From (3.4.3), we have

$$0 = ih(x) - \mathcal{H}h(x) = 2i \sum_{k>0} \widehat{h}(k) e^{ikx}, \quad \text{for all } x \in (a,b).$$

Therefore, $\sum_{k>0} \widehat{h}(k) e^{ikx} = 0$, in (a,b). Applying Lemma 3.4.1, we obtain that

$$\sum_{k>0} \widehat{h}(k) e^{ikx} = 0, \quad \text{in } \mathbb{T}.$$

Since h is real-valued, we also have that $\widehat{h}(-k) = \overline{\widehat{h}(k)}$, for all $k \in \mathbb{Z}$. Thus,

$$\sum_{k>0} \widehat{h}(-k)e^{-ikx} = 0, \text{ in } \mathbb{T}.$$

Consequently, for a.e. $t \in (0,T)$, $\partial_x^3 v(\cdot,t) = 0$ in $\mathbb{T}$. Then, for a.e. $t \in (0,T)$, $\partial_x^2 v(\cdot,t) = c_1(t)$ in $\mathbb{T}$, but from (3.4.2) we obtain that $c_1(t) = 0$, for a.e. $t \in (0,T)$. Thus, for a.e. $t \in (0,T)$, $\partial_x^2 v(\cdot,t) = 0$ in $\mathbb{T}$. Arguing in a similar way we obtain that for a.e. $t \in (0,T)$, $\partial_x v(\cdot,t) = 0$ in $\mathbb{T}$. Thus, for a.e. $t \in (0,T)$,

$$v(x,t) = c(t) \text{ in } \mathbb{T}. \tag{3.4.5}$$

Substituting (5.2.50) in the first relation of (3.4.1), we obtain that $c'(t) = 0$ for a.e. $t \in (0,T)$. Therefore, $v(x,t) = c(t) = cte = \beta$ a.e. in $\mathbb{T} \times (0,T)$.
Finally, using that $v \in L_0^2(\mathbb{T})$, especifically $[v] = 0$, we obtain that $v(x,t) = \beta = 0$ a.e. in $\mathbb{T} \times (0,T)$. $\square$

Corollary 3.4.3. *Let $T > 0$, $\mu \in \mathbb{R}$, and $\alpha > 0$ be given. Assume that ω is a nonempty open set in $\mathbb{T}$ and let $v \in X_{0,\frac{1}{2}}^T$ be a solution of*

$$\begin{cases} \partial_t v - \partial_x^3 v - \alpha \mathcal{H} \partial_x^2 v + 2\mu \partial_x v + 2v \partial_x v = 0, & t > 0, \text{ on } \mathbb{T} \times (0,T) \\ v(x,t) = c, & \text{on } \omega \times (0,T), \end{cases} \tag{3.4.6}$$

where $c \in \mathbb{R}$, denotes some constant, and $[v] = 0$. Then $v(x,t) = c = 0$ on $\mathbb{T} \times (0,T)$.

Proof. Using Corollary 3.3.4, we infer that $v \in C^\infty(\mathbb{T} \times (0,T))$. It follows that $v(x,t) = c$ on $\mathbb{T} \times (0,T)$ by the unique continuation property of Benjamin equation (see Proposition 3.4.2). From the fact that $[v] = 0$, we obtain $c = 0$. $\square$

Chapter 4

Controllability and stabilization of the Benjamin equation on a periodic domain

In this chapter, we obtain the main results regarding global controllability and exponential stabilization of Benjamin equation on a periodic domain in $H_0^s(\mathbb{T})$ with $s \geqslant 0$. The global exponential stabilizability corresponding to a natural feedback law is first established with the aid of certain properties of solution, viz., propagation of compactness and propagation of regularity in Bourgain's spaces. The global exponential stability of the system combined with a local controllability result yields the global controllability as well, as is usual in control theory (see for instance [29, 30, 51, 52, 53]).

This chapter is organized as follows. In Section 4.1, the local controllability result for the Benjamin equation is obtained. The main result on local control is given Corollary 4.1.2. Section 4.2, is devoted to study the stabilization of the Benjamin equation by a time-invariant feedback control law. Finally, in section 4.3 we prove the global controllability for the Benjamin equation.

4.1 Local control for the nonlinear Benjamin equation

From the observation made in the final part of Section 2.3, it is enough to study the control problem for the IVP (2.3.37) in $H_0^s(\mathbb{T})$, with $s \geqslant 0$ (see (1.3.1)). Thus, in this section we are concerned with the local controllability of the system

$$\begin{cases} \partial_t u - \alpha H \partial_x^2 u - \partial_x^3 u + 2\mu \partial_x u + 2u\partial_x u = Gh(x,t), & t \in (0,T), \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.1.1}$$

To get a local control result to system (4.1.1) we proceed as in [81] by rewriting the system (4.1.1) in its equivalent integral equation form:

$$u(t) = U_\mu(t)u_0 + \int_0^t U_\mu(t-\tau)(Gh)(\tau)\,d\tau - \int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau)\,d\tau, \tag{4.1.2}$$

where $U_\mu(t)$ is the semigroup generated by operator A_μ defined by (2.3.39) on the space $L^2(\mathbb{T})$. For given $u_0,\ u_1 \in H_p^s(\mathbb{T})$, let us choose $h = \Phi_\mu(u_0, u_1 + w(T,u))$, where Φ_μ is the bounded linear operator given in Remark 2.3.15, and define

$$w(T,u) := \int_0^T U_\mu(T-\tau)(2u\partial_x u)(\tau)d\tau.$$

According to Remark 2.3.15, the linear system (2.3.38) is exactly controllable in any positive time T. Therefore, for given $u_0,\ u_1 \in H_p^s(\mathbb{T})$, we have

$$u(t) = U_\mu(t)u_0 + \int_0^t U_\mu(t-\tau)(G(\Phi_\mu(u_0, u_1 + w(T,u))))(\tau)\, d\tau - \int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau)\, d\tau,$$

and $u(0) = u_0,\ \ u(T) = u_1$, by virtue of equality (2.3.41). This suggests that we should consider the map

$$\Gamma(u) := U_\mu(t)u_0 + \int_0^t U_\mu(t-\tau)(G(\Phi_\mu(u_0, u_1 + w(T,u))))(\tau)d\tau - \int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau)d\tau \tag{4.1.3}$$

and show that Γ is a contraction in an appropriate space. The fixed point u of Γ is a mild solution of IVP (4.1.1) with $h = \Phi(u_0, u_1 + w(T,u))$ and satisfies $u(x,T) = u_1(x)$. To complete this argument, we use the Bourgain's space associated to the Benjamin equation ans show that Γ is a contraction mapping. This is the content of the following result.

Theorem 4.1.1 (Small data control)**.** *Let $T > 0$, $s \geqslant 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$ be given. Then there exists a $\delta > 0$ such that for any $u_0, u_1 \in H_0^s(\mathbb{T})$ with $[u_0] = [u_1] = 0$ and*

$$\|u_0\|_{H_0^s(\mathbb{T})} \leqslant \delta, \qquad \|u_1\|_{H_0^s(\mathbb{T})} \leqslant \delta,$$

one can find a control $h \in L^2([0,T]; H_0^s(\mathbb{T}))$ such that the IVP (4.1.1) has a unique solution $u \in C([0,T]; H_0^s(\mathbb{T})$ satisfying

$$u(x,0) = u_0(x), \quad u_1(x,T) = u_1(x), \quad \textit{for all } x \in \mathbb{T}.$$

Proof. Let $T > 0$ be given. For $s \geqslant 0$ we will show that there exists $M > 0$ such that Γ defined by (4.1.3) is a contraction on the ball

$$B(0,M) := \left\{ u \in Z^T_{s,\frac{1}{2}} :\ [u] = 0,\ \|u\|_{Z^T_{s,\frac{1}{2}}} \leqslant M \right\},$$

where $\|\cdot\|_{Z^T_{s,\frac{1}{2}}}$ is given by Definition 3.1.5.

In fact, using the Corollary 3.1.12 part *iv*), the Theorem 3.1.14 part *iii*), and the Corollary 3.1.26, we obtain

$$\|\Gamma(u)\|_{Z^T_{s,\frac{1}{2}}} \leqslant c_1 \left(\|u_0\|_{H_p^s(\mathbb{T})} + \|(G\Phi_\mu(u_0, u_1 + w(T,u)))(t)\|_{Z^T_{s,-\frac{1}{2}}} + \|u\|^2_{Z^T_{s,\frac{1}{2}}} \right), \tag{4.1.4}$$

were c_1 is a positive constant that depends on s, T, and α. Using that G (see (2.0.9)) is a bounded operator, the embedding $X^T_{s,0} \hookrightarrow X_{s,-\frac{1}{2}}$ and Remark (2.3.15), we get

$$\begin{aligned}
\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{X^T_{s,-\frac{1}{2}}} &\leqslant c\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{X^T_{s,0}} \\
&= c\|\Phi_\mu(u_0, u_1 + \omega(T,u))\|_{L^2([0,T],H^s_0(\mathbb{T}))} \\
&\leqslant c_2(\|u_0\|_{H^s_0(\mathbb{T})} + \|u_1\|_{H^s_0(\mathbb{T})} + \|w(T,u)\|_{H^s_0(\mathbb{T})}),
\end{aligned} \tag{4.1.5}$$

where $c_2 > 0$ depends on s, T, and g. Using Proposition 3.1.8, Theorem 3.1.14 and Corollary 3.1.26, we obtain

$$\begin{aligned}
\|w(T,u))\|_{H^s_p(\mathbb{T})} &\leqslant \sup_{0\leqslant t\leqslant T}\left\|\int_0^t U_\mu(t-\tau)(2u\partial_x u)(\tau)d\tau\right\|_{H^s_0(\mathbb{T})} \\
&\leqslant c\left\|\int_0^t U_\mu(t-\tau)(\partial_x(u^2))(\tau)d\tau\right\|_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant c_3\|u\|^2_{Z^T_{s,\frac{1}{2}}},
\end{aligned} \tag{4.1.6}$$

where $c_3 > 0$ depends on s, α and T. From (4.1.5) and (4.1.6), we have

$$\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{X^T_{s,-\frac{1}{2}}} \leqslant c_2(\|u_0\|_{H^s_0(\mathbb{T})} + \|u_1\|_{H^s_0(\mathbb{T})} + c_3\|u\|^2_{Z^T_{s,\frac{1}{2}}}) \tag{4.1.7}$$

On the other hand, applying Remark 3.1.17 with $b = -1$ and $0 < \epsilon \leqslant \dfrac{1}{2}$, we get

$$\begin{aligned}
\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{Y^T_{s,-1}} &\leqslant c(\epsilon)\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{X^T_{s,-1+\frac{1}{2}+\epsilon}} \\
&\leqslant c(\epsilon)\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{X^T_{s,0}}.
\end{aligned} \tag{4.1.8}$$

From inequality (4.1.8) and the same calculations as above, we obtain

$$\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{Y^T_{s,-1}} \leqslant c_4(\epsilon)\left(\|u_0\|_{H^s_0(\mathbb{T})} + \|u_1\|_{H^s_0(\mathbb{T})} + c_3\,\|u\|^2_{Z^T_{s,\frac{1}{2}}}\right), \tag{4.1.9}$$

where $c_4(\epsilon) > 0$ depends on s, α, T and g. From (4.1.7) and (4.1.9), we infer that

$$\|G(\Phi_\mu(u_0, u_1 + w(T,u)))\|_{Z^T_{s,-\frac{1}{2}}} \leqslant (c_2 + c_4)(\|u_0\|_{H^s_0(\mathbb{T})} + \|u_1\|_{H^s_0(\mathbb{T})}) + (c_2c_3 + c_4c_3)\|u\|^2_{Z^T_{s,\frac{1}{2}}}. \tag{4.1.10}$$

Combining (4.1.4) and (4.1.10), we obtain that there exists $C = C_{s,\epsilon,\alpha,g,T} > 0$ such that

$$\|\Gamma(u)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C(\|u_0\|_{H^s_0(\mathbb{T})} + \|u_1\|_{H^s_0(\mathbb{T})}) + C\|u\|^2_{Z^T_{s,\frac{1}{2}}}. \tag{4.1.11}$$

Choosing $\delta > 0$ and $M > 0$ such that

$$CM < \frac{1}{4} \quad \text{and} \quad 2C\delta + CM^2 \leqslant M, \tag{4.1.12}$$

we obtain from (4.1.11) that $\|\Gamma(u)\|_{Z^T_{\frac{1}{2},s}} \leqslant M$ for each $u \in B(0,M)$, provided that $\|u_0\|_{H^s_0(\mathbb{T})} \leqslant \delta$ and $\|u_1\|_{H^s_p(\mathbb{T})} \leqslant \delta$.

Furthermore, for all $u, v \in B(0, M)$, we have

$$\Gamma(u) - \Gamma(v) = \int_0^t U_\mu(t-\tau)(G\Phi_\mu(0, w(T,u) - w(T,v)))(\tau)d\tau + \int_0^t U_\mu(t-\tau)(\partial_x((v-u)(v+u))(\tau)d\tau,$$

and

$$w(T,u) - w(T,v) = \int_0^T U_\mu(T-\tau)\left(\partial_x((u-v)(u+v))\right)(\tau)\,d\tau.$$

Thus, with similar computations as above, we can obtain

$$\|\Gamma(u) - \Gamma(v)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C\|u-v\|_{Z^T_{s,\frac{1}{2}}}\|u+v\|_{Z^T_{s,\frac{1}{2}}} \leqslant \frac{1}{2}\|u-v\|_{Z^T_{s,\frac{1}{2}}}.$$

Therefore, the map Γ is a contraction on $B(0, M)$ provided that δ and M are chosen according to (4.1.12) with $\|u_0\|_{H^s_p(\mathbb{T})} \leqslant \delta$, and $\|u_1\|_{H^s_p(\mathbb{T})} \leqslant \delta$. □

Corollary 4.1.2 (Local control). *Let $T > 0$, $s \geqslant 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$ be given. Then there exists $\delta > 0$ such that for any $u_0, u_1 \in H^s_p(\mathbb{T})$ with $[u_0] = [u_1] = \mu$ and*

$$\|u_0 - \mu\|_{H^s_p(\mathbb{T})} \leqslant \delta, \qquad \|u_1 - \mu\|_{H^s_p(\mathbb{T})} \leqslant \delta,$$

one can find a control $h \in L^2([0,T]; H^s_p(\mathbb{T}))$ such that the IVP associated to (2.0.1) with $f = Gh$ has a unique solution $u \in C([0,T]; H^s_p(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x), \quad u_1(x,T) = u_1(x), \quad \textit{for all } x \in \mathbb{T}.$$

4.2 Stabilization of the Nonlinear Benjamin equation

In this section we will study the stabilization problem for the Benjamin equation in $H^s_p(\mathbb{T})$, with $s \geqslant 0$. Consider the IVP,

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + 2u\partial_x u = -K_\lambda u, & t > 0,\ x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.2.1}$$

where $u = u(x,t)$ denotes a real valued function with $[u] = 0$. Assume $\lambda \geqslant 0$, $\mu \in \mathbb{R}$, and $\alpha > 0$ are given. The feedback control law $f \equiv -K_\lambda u$ is as defined in (2.5.13), with the operator L_λ as in (2.5.1).

We first check that the system (4.2.1) is globally well-posed in the space $H^s_0(\mathbb{T})$ for any $s \geqslant 0$. Let $U_\mu(t)$ be the group defined in (2.3.40) that describes the solution u of the linear IVP associated to (4.2.1). The following estimate is needed.

Lemma 4.2.1. *For any $0 < \epsilon < 1$ there exists a positive constant $C(\epsilon)$ such that*

$$\left\|\int_0^t U_\mu(t-\tau)(K_\lambda v)(\tau)\,d\tau\right\|_{Z^T_{s,\frac{1}{2}}} \leqslant C(\epsilon)\,T^{1-\epsilon}\|v\|_{Z^T_{s,\frac{1}{2}}}. \tag{4.2.2}$$

Proof. Let $0 < \epsilon < 1$. From Theorem 3.1.14, we infer that

$$\left\| \int_0^t U_\mu(t-\tau)(K_\lambda v)(\tau)d\tau \right\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_{\eta,s}\left(\|K_\lambda v\|_{X^T_{s,-\frac{1}{2}}} + \|K_\lambda v\|_{Y^T_{s,-1}} \right). \tag{4.2.3}$$

Using the embedding $X^T_{s,-\frac{1}{2}+\frac{\epsilon}{2}} \hookrightarrow X^T_{s,-\frac{1}{2}}$, and Remark 3.1.26 with $-\dfrac{1}{2} + \dfrac{\epsilon}{2} = b' < 0 = b$, we have

$$\begin{aligned} \|K_\lambda v\|_{X^T_{s,-\frac{1}{2}}} &\leqslant C\|GG^* L_\lambda^{-1} v\|_{X^T_{s,-\frac{1}{2}+\frac{\epsilon}{2}}} \\ &\leqslant CT^{-(-\frac{1}{2}+\frac{\epsilon}{2})}\|GG^* L_\lambda^{-1} v\|_{X^T_{s,0}} \leqslant C_{s,g,\lambda,\delta,\epsilon} T^{\frac{1}{2}-\frac{\epsilon}{2}} \|v\|_{X^T_{s,0}}. \end{aligned} \tag{4.2.4}$$

where we have used Remark 2.1.2, and Remark 2.5.4 in the last inequality. Applying again the estimate (3.1.26) with $0 = b' < \dfrac{1}{2} - \dfrac{\epsilon}{2} = b$, and using the embedding $X^T_{s,\frac{1}{2}} \hookrightarrow X^T_{s,\frac{1}{2}-\frac{\epsilon}{2}}$, we infer that

$$\|K_\lambda v\|_{X^T_{s,-\frac{1}{2}}} \leqslant C_{s,g,\lambda,\delta,\epsilon} T^{\frac{1}{2}-\frac{\epsilon}{2}}\, T^{\frac{1}{2}-\frac{\epsilon}{2}} \|v\|_{X^T_{s,\frac{1}{2}-\frac{\epsilon}{2}}} \leqslant C_{s,g,\lambda,\delta,\epsilon} T^{1-\epsilon} \|v\|_{X^T_{s,\frac{1}{2}}}. \tag{4.2.5}$$

On the other hand, from Remark 3.1.17 and similar computations as those made in (4.2.4) and (4.2.5), we obtain that

$$\|K_\lambda v\|_{Y^T_{s,-1}} = \|K_\lambda v\|_{Y^T_{s,-\frac{1}{2}+\frac{\epsilon}{2}-\frac{1}{2}-\frac{\epsilon}{2}}} \leqslant C_\epsilon \|K_\lambda v\|_{X^T_{s,-\frac{1}{2}+\frac{\epsilon}{2}}} \leqslant C_{s,g,\lambda,\delta,\epsilon} T^{1-\epsilon} \|v\|_{X^T_{s,\frac{1}{2}}}. \tag{4.2.6}$$

From (4.2.3), (4.2.5), and (4.2.6), we have

$$\left\| \int_0^t U_\mu(t-\tau)(K_\lambda v)(\tau)\, d\tau \right\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_{\eta,s,g,\lambda,\delta,\epsilon} T^{1-\epsilon} \|v\|_{X^T_{s,\frac{1}{2}}} \leqslant C(\epsilon) T^{1-\epsilon} \|v\|_{Z^T_{s,\frac{1}{2}}}. \tag{4.2.7}$$

This proof the lemma. □

Next we study the global existence of the solutions to the IVP (4.2.1).

Theorem 4.2.2 (Global well-posedness)**.** *Let $s \geqslant 0$, $\lambda \geqslant 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$, be given. For any $u_0 \in H^s_0(\mathbb{T})$ and for any $T > 0$ there exists a unique solution $u \in Z^T_{s,\frac{1}{2}} \cap C([0,T]; L^2_0(\mathbb{T}))$ of equation* (4.2.1)*. Furthermore, the following estimates hold*

$$\|u\|_{Z^T_{s,\frac{1}{2}}} \leqslant \beta_{T,s,\mu}(\|u_0\|_{L^2_0(\mathbb{T})})\, \|u_0\|_{H^s_0(\mathbb{T})}. \tag{4.2.8}$$

In particular, $u \in C([0,T]; H^s_0(\mathbb{T}))$ and

$$\|u\|_{L^\infty([0,T];H^s_0(\mathbb{T}))} \leqslant C_4 \beta_{T,s,\mu}(\|u_0\|_{L^2_0(\mathbb{T})})\, \|u_0\|_{H^s_0(\mathbb{T})}, \tag{4.2.9}$$

where C_4 is a positive constant and $\beta_{T,s,\mu}$ is a nondecresing continuous function depending only on T, s, and μ. Moreover, denoting $S(t)u_0$ the unique solution u of the IVP (4.2.1) *corresponding to the initial data u_0, the operator $S(t) : H^s_0(\mathbb{T}) \longrightarrow Z^T_{s,\frac{1}{2}}$, defined by $S(t)u_0 = u$ is continuous in the interval $[0,T]$.*

Proof. The proof of this theorem is similar to Theorem 4.1 in [52]. First, we will show the local well-posedness of system (4.2.1) in $H_0^s(\mathbb{T})$ for any $s \geqslant 0$. We rewrite the IVP (4.2.1) in its integral form and for given $u_0 \in H_0^s(\mathbb{T})$, $0 < T < 1$, we define the map

$$\Gamma(v) = U_\mu(t)u_0 - \int_0^t U_\mu(t-\tau)(2v\partial_x v)(\tau)\, d\tau - \int_0^t U_\mu(t-\tau)(K_\lambda v)(\tau)\, d\tau.$$

Observe that

$$\Gamma(v_1) - \Gamma(v_2) = \int_0^t U_\mu(t-\tau)[\partial_x((v_2+v_1)(v_2-v_1))](\tau)d\tau + \int_0^t U_\mu(t-\tau)[K_\lambda(v_2-v_1)](\tau)d\tau.$$

It follows then from Corollary 3.1.12, Theorem 3.1.14, Corollary 3.1.26, and Lemma 4.2.1 that there exists some positive constants C_1, C_2, C_3, $0 < \theta < \dfrac{1}{6}$, and $0 < \epsilon < 1$ such that

$$\|\Gamma(v)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_1\|u_0\|_{H_0^s(\mathbb{T})} + C_2T^\theta \|v\|^2_{Z^T_{s,\frac{1}{2}}} + C_3T^{1-\epsilon} \|v\|_{Z^T_{s,\frac{1}{2}}}. \tag{4.2.10}$$

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_2T^\theta \|v_2 - v_1\|_{Z^T_{s,\frac{1}{2}}} \|v_2 + v_1\|_{Z^T_{s,\frac{1}{2}}} + C_3T^{1-\epsilon} \|v_2 - v_1\|_{Z^T_{s,\frac{1}{2}}}, \tag{4.2.11}$$

for any $v, v_1, v_2 \in Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$. Pick $M = 2C_1\|u_0\|_{H_0^s(\mathbf{T})}$, and $T > 0$ such that,

$$2C_2MT^\theta + C_3T^{1-\epsilon} \leqslant \frac{1}{2}. \tag{4.2.12}$$

Note that, if we choose $0 < \epsilon < 1$ such that $0 < \theta < 1 - \epsilon$, then $T^{1-\epsilon} < T^\theta$ and the time $T > 0$ can be taken as

$$T = T(\|u_0\|_{H_0^s(\mathbb{T})}) = \left(\frac{1}{8C_1C_2\|u_0\|_{H_0^s(\mathbb{T})} + 2C_3}\right)^{\frac{1}{\theta}}. \tag{4.2.13}$$

From (4.2.10) and (4.2.11), we infer that for any $v, v_1, v_2 \in B(0,M)$, $\|\Gamma(v)\|_{Z^T_{s,\frac{1}{2}}} \leqslant M$, and $\|\Gamma(v_1) - \Gamma(v_2)\|_{Z^T_{s,\frac{1}{2}}} \leqslant \dfrac{1}{2}\|v_2 - v_1\|_{Z^T_{s,\frac{1}{2}}}$. Thus the map Γ is a contraction in the closed ball $B(0,M)$ of $Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$ for the $\|\cdot\|_{Z^T_{s,\frac{1}{2}}}$ norm. Its unique fixed point u is the desired solution of (4.2.1) in the space $Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$. It follows from the Proposition 3.1.8 that $u \in C([0,T]; H_0^s(\mathbb{T}))$ with

$$\|u\|_{L^\infty([0,T];H_0^s(\mathbb{T}))} \leqslant C_4\|u\|_{Z^T_{s,\frac{1}{2}}} \leqslant 2C_1C_4\|u_0\|_{H_0^s(\mathbb{T})},$$

for some $C_4 > 0$.

Next, we shall prove the global existence of the solution. First, we assume that $s = 0$. Multiplying the equation (4.2.1) by $\overline{u}$ and integrating in space we obtain

$$\frac{1}{2}\frac{d}{dt'}\left(\|u(\cdot,t')\|^2_{L^2_0(\mathbb{T})}\right) = -\left(GG^*L_\lambda^{-1}u(\cdot,t'), u(\cdot,t')\right)_{L^2_0(\mathbb{T})}, \quad \text{for all } t' \geqslant 0. \tag{4.2.14}$$

Now integrating in the time variable in $(0,t)$, and using the properties of operators G and L_λ^{-1}, we infer that

$$\begin{aligned}\frac{1}{2}\|u(\cdot,t)\|^2_{L^2_0(\mathbb{T})}-\frac{1}{2}\|u_0\|^2_{L^2_0(\mathbb{T})}&=-\int_0^t\left(GL_\lambda^{-1}u(\cdot,t'),Gu(\cdot,t')\right)_{L^2_0(\mathbb{T})}dt'\\&\leqslant\int_0^t\|G\|^2\|L_\lambda^{-1}\|\|u(\cdot,t')\|^2_{L^2_0(\mathbb{T})}dt',\quad\text{for all } t\geqslant 0.\end{aligned}\tag{4.2.15}$$

The Gronwall's inequality in its integral form (see [87, Theorem 1.10, page 11]) implies that

$$\|u(\cdot,t)\|_{L^2_0(\mathbb{T})}\leqslant\|u_0\|_{L^2_0(\mathbb{T})}e^{Ct},\quad\text{for all } t\geqslant 0,\ \text{where } C=\|G\|^2\|L_\lambda^{-1}\|.\tag{4.2.16}$$

From the first line of (4.2.15), we note that

$$\|u(\cdot,t)\|_{L^2_0(\mathbb{T})}\leqslant\|u_0\|_{L^2_0(\mathbb{T})},\quad\text{when } \lambda=0 \text{ and } t\geqslant 0.\tag{4.2.17}$$

An standard continuation argument shows that equation (4.2.1) is globally well-posed in $L^2_0(\mathbb{T})$ and estimate (4.2.8) holds with $s=0$.

Next, we suppose that $s=3$. In fact, we will prove that for any $T>0$ and any $u_0\in H^3_0(\mathbb{T})\subset L^2_0(\mathbb{T})$ the solution of the IVP (4.2.1) belongs to the space $u\in Z^T_{3,\frac{1}{2}}\cap C([0,T];H^3_0(\mathbb{T}))$.

For this, let $T>0$ and $u_0\in H^3_0(\mathbb{T})\subset L^2_0(\mathbb{T})$. Then, the local solution u of equation (4.2.1) belongs to the space $u\in Z^{T_1}_{0,\frac{1}{2}}\cap C([0,T_1];L^2_0(\mathbb{T}))$, where T_1 is the time of local existence given by relation (4.2.13) with $s=0$. Then u satisfies

$$\|u\|_{L^\infty([0,T_1];L^2_0(\mathbb{T}))}\leqslant C_4\|u\|_{Z^{T_1}_{0,\frac{1}{2}}}\leqslant C_4M=C_42C_1\|u_0\|_{L^2_0(\mathbb{T})}.\tag{4.2.18}$$

Define $v=\partial_t u$, so that $[v]=0$ and v satisfies

$$\begin{cases}\partial_t v-\partial_x^3v-\alpha\mathcal{H}\partial_x^2v+2\mu\partial_xv+2\partial_x(uv)=-K_\lambda v, & 0<t\leqslant T_1,\ x\in\mathbb{T}\\ v(x,0)=v_0=u_0'''+\alpha\mathcal{H}u_0''-2\mu u_0'-2u_0u_0'-K_\lambda u_0, & x\in\mathbb{T}.\end{cases}\tag{4.2.19}$$

Note that, applying the Gagliardo-Nirenberg's inequality (see [8, Theorem 3.70]) and the Cauchy's inequality, we obtain that there exists $c_1>0$ such that

$$\begin{aligned}2\|u_0u_0'\|_{L^2_0(\mathbb{T})}&\leqslant 2\|u_0\|_{L^2_0(\mathbb{T})}\|u_0'\|_{L^\infty(\mathbb{T})}\\&\leqslant 2c_1\|u_0\|_{L^2_0(\mathbb{T})}\|u_0'''\|^{\frac{1}{2}}_{L^2_0(\mathbb{T})}\|u_0\|^{\frac{1}{2}}_{L^2_0(\mathbb{T})}\\&=2c_1\|u_0\|_{L^2_0(\mathbb{T})}\left(\frac{1}{2}\|u_0\|_{L^2_0(\mathbb{T})}+\frac{1}{2}\|u_0'''\|_{L^2_0(\mathbb{T})}\right)=2c_1\|u_0\|_{L^2_0(\mathbb{T})}\|u_0\|_{H^3_0(\mathbb{T})}.\end{aligned}\tag{4.2.20}$$

Therefore, $v_0\in L^2_0(\mathbb{T})$, with

$$\begin{aligned}\|v_0\|_{L^2_0(\mathbb{T})}&\leqslant\|u_0'''\|_{L^2_0(\mathbb{T})}+\alpha\|\mathcal{H}u_0''\|_{L^2_0(\mathbb{T})}+2|\mu|\|u_0'\|_{L^2_0(\mathbb{T})}+2\|u_0u_0'\|_{L^2_0(\mathbb{T})}+\|K_\lambda u_0\|_{L^2_0(\mathbb{T})}\\&\leqslant c_2\|u_0\|_{H^3_0(\mathbb{T})}+2c_1\|u_0\|_{L^2_0(\mathbb{T})}\|u_0\|_{H^3_0(\mathbb{T})}\\&\leqslant(c_2+2c_1\|u_0\|_{L^2_0(\mathbb{T})})\|u_0\|_{H^3_0(\mathbb{T})}<+\infty,\end{aligned}\ .\tag{4.2.21}$$

where $c_2 > 0$ depends on α, μ, λ, g, and δ. On the other hand, considering the map

$$\Gamma(w) = U_\mu(t)v_0 - 2\int_0^t U_\mu(t-\tau)(\partial_x(u.w))(\tau)d\tau - \int_0^t U_\mu(t-\tau)(K_\lambda w)(\tau)d\tau,$$

using the bilinear estimate 3.1.72, and doing the same calculations as those leading to (4.2.10), yield

$$\begin{aligned}\|\Gamma(w)\|_{Z^{T_2}_{0,\frac{1}{2}}} &\leqslant C_1\|v_0\|_{L^2_0(\mathbb{T})} + 2C_2T_2^\theta\|u\|_{Z^{T_2}_{0,\frac{1}{2}}}\|w\|_{Z^{T_2}_{0,\frac{1}{2}}} + C_3T_2^{1-\epsilon}\|w\|_{Z^{T_2}_{0,\frac{1}{2}}}\\ &\leqslant C_1\|v_0\|_{L^2_0(\mathbb{T})} + \left(4C_1C_2T_2^\theta\|u_0\|_{L^2_0(\mathbb{T})} + C_3T_2^{1-\epsilon}\right)\|w\|_{Z^{T_2}_{0,\frac{1}{2}}}.\end{aligned}$$

Note that

$$\begin{aligned}\Gamma(w_1)-\Gamma(w_2) &= \Gamma(w_1-w_2)\\ &= -2\int_0^t U_\mu(t-\tau)[\partial_x(u\cdot(w_1-w_2))](\tau)d\tau - \int_0^t U_\mu(t-\tau)[K_\lambda(w_1-w_2)](\tau)d\tau.\end{aligned}$$

Thus

$$\begin{aligned}\|\Gamma(w_1-w_2)\|_{Z^{T_2}_{0,\frac{1}{2}}} &\leqslant 2C_2T_2^\theta\|u\|_{Z^{T_2}_{0,\frac{1}{2}}}\|w_1-w_2\|_{Z^{T_2}_{0,\frac{1}{2}}} + C_3T_2^{1-\epsilon}\|w_1-w_2\|_{Z^{T_2}_{0,\frac{1}{2}}}\\ &\leqslant \left(4C_1C_2T_2^\theta\|u_0\|_{L^2_0(\mathbb{T})} + C_3T_2^{1-\epsilon}\right)\|w_1-w_2\|_{Z^{T_2}_{0,\frac{1}{2}}},\end{aligned}$$

for any $w, w_1, w_2 \in Z^{T_2}_{0,\frac{1}{2}} \cap L^2([0,T_2]; L^2_0(\mathbb{T}))$. Therefore, taking $T_2 = T_1(\|u_0\|_{L^2_0(\mathbb{T})})$ (note that T_2 can be taken bigger that T_1, but we take $T_2 = T_1$ in order to guarantee the existence of solutions for systems (4.2.1) and (4.2.19) simultaneously), we obtain that the map Γ is a contraction in a closed ball

$$\tilde{B}(0,M) = \left\{w \in Z^{T_1}_{0,\frac{1}{2}} : [w] = 0,\ \|w\|_{Z^{T_1}_{0,\frac{1}{2}}} \leqslant M\right\},$$

where $M = 2C_1\|v_0\|_{L^2_0(\mathbb{T})}$. Its unique fixed point v is the desired solution of (4.2.19) in the space $Z^{T_1}_{0,\frac{1}{2}} \cap L^2([0,T_1]; L^2_0(\mathbb{T}))$. Thus, $\|v\|_{Z^{T_1}_{0,\frac{1}{2}}} \leqslant 2C_1\|v_0\|_{L^2_0(\mathbb{T})}$. From Proposition 3.1.8 we infer that $v \in C([0,T_1]; L^2_0(\mathbb{T}))$ with

$$\|v\|_{L^\infty([0,T_1];L^2_0(\mathbb{T}))} \leqslant C_4\|v\|_{Z^{T_1}_{0,\frac{1}{2}}} \leqslant 2C_4C_1\|v_0\|_{L^2_0(\mathbb{T})}. \tag{4.2.22}$$

From equation (4.2.1), we have $\partial_x^3 u = v - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + 2u\partial_x u + K_\lambda u$. Consequently,

$$\|\partial_x^3 u\|_{L^2_0(\mathbb{T})} \leqslant \|v\|_{L^2_0(\mathbb{T})} + \alpha\|\mathcal{H}\partial_x^2 u\|_{L^2_0(\mathbb{T})} + 2|\mu|\|\partial_x u\|_{L^2_0(\mathbb{T})} + 2\|u\partial_x u\|_{L^2_0(\mathbb{T})} + \|K_\lambda u\|_{L^2_0(\mathbb{T})}. \tag{4.2.23}$$

The analogous computations as those leading to (2.4.33) and (2.4.35), yield for any $\epsilon > 0$,

$$2|\mu|\|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant c_\mu\epsilon\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \frac{c_\mu}{4\epsilon}\|\partial_x^3 u(\cdot,t)\|_{L^2_0(\mathbb{T})}, \tag{4.2.24}$$

$$\alpha\|\mathcal{H}\partial_x^2 u(\cdot,t)\|_{L_0^2(\mathbb{T})} \leqslant c_\alpha \epsilon^{\frac{3}{2}}\|u(\cdot,t)\|_{L_0^2(\mathbb{T})} + \frac{c_\alpha}{\epsilon^{\frac{1}{2}}}\|\partial_x^3 u(\cdot,t)\|_{L_0^2(\mathbb{T})}. \tag{4.2.25}$$

The silmilar computations as those leading to (4.2.20), but using Cauchy's inequality with $\epsilon > 0$, yield

$$\begin{aligned} 2\|u(\cdot,t)\partial_x u(\cdot,t)\|_{L_0^2(\mathbb{T})} &\leqslant 2c_1\|u(\cdot,t)\|_{L_0^2(\mathbb{T})}\|u(\cdot,t)\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x^3 u(\cdot,t)\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}} \\ &\leqslant c_3\epsilon\|u(\cdot,t)\|_{L_0^2(\mathbb{T})}^3 + \frac{c_3}{4\epsilon}\|\partial_x^3 u(\cdot,t)\|_{L_0^2(\mathbb{T})}. \end{aligned} \tag{4.2.26}$$

We already know that

$$\|K_\lambda u(\cdot,t)\|_{L_0^2(\mathbb{T})} \leqslant c_4\|u(\cdot,t)\|_{L_0^2(\mathbb{T})}. \tag{4.2.27}$$

From (4.2.22), (4.2.21), and (4.2.24)-(4.2.27), we get that for $0 < t \leqslant T_1$

$$\begin{aligned} \left(1 - \frac{c_\alpha}{\epsilon^{\frac{1}{2}}} - \frac{(c_\mu + c_3)}{4\epsilon}\right)\|\partial_x^3 u\|_{L_0^2(\mathbb{T})} &\leqslant \|v\|_{L_0^2(\mathbb{T})} + \left(c_\alpha\epsilon^{\frac{3}{2}} + c_\mu\epsilon + c_4\right)\|u\|_{L_0^2(\mathbb{T})} + c_3\epsilon\|u\|_{L_0^2(\mathbb{T})}^3 \\ &\leqslant 2C_4C_1\|v_0\|_{L_0^2(\mathbb{T})} + \left(c_\alpha\epsilon^{\frac{3}{2}} + c_\mu\epsilon + c_4\right)\|u\|_{L_0^2(\mathbb{T})} + c_3\epsilon\|u\|_{L_0^2(\mathbb{T})}^3 \\ &\leqslant 2C_4C_1(c_2 + 2c_1\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^3(\mathbb{T})} \\ &\quad + \left(c_\alpha\epsilon^{\frac{3}{2}} + c_\mu\epsilon + c_4\right)2C_4C_1\|u_0\|_{H_0^3(\mathbb{T})} \\ &\quad + c_3\epsilon 2C_4C_1\|u_0\|_{L_0^2(\mathbb{T})}^2\|u_0\|_{H_0^3(\mathbb{T})}. \end{aligned}$$

Taking ϵ large enough, we can conclude that there exists $C > 0$ such that

$$\|\partial_x^3 u\|_{L_0^2(\mathbb{T})} \leqslant C\left(1 + \|u_0\|_{L_0^2(\mathbb{T})} + \|u_0\|_{L_0^2(\mathbb{T})}^2\right)\|u_0\|_{H_0^3(\mathbb{T})}. \tag{4.2.28}$$

Consequently,

$$\|u\|_{L^\infty([0,T_1];H_0^3(\mathbb{T}))} \leqslant \beta_{T_1,3}(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^3(\mathbb{T})}, \tag{4.2.29}$$

where $\beta_{T_1,3}$ is a nondecresing continuous function depending only on T_1.

Consequently, we can iterate the procedure leading to (4.2.29) in order to cover the compact interval $[0,T]$, thus we obtain that $u \in C([0,T];H_0^3(\mathbb{T}))$ and satisfies (4.2.8) with $s = 3$. This completes the proof of case $s = 3$.

Next, we show that equation (4.2.1) is globally well-posed in the space $H_0^s(\mathbb{T})$, for $0 < s < 3$. Let $T > 0$ and $u_0 \in H_0^s(\mathbb{T})$. Denote by $S(t)u_0$ the unique solution u of equation (4.2.1) corresponding to the initial data u_0. In order to apply the interpolation Theorem 1.7.7 due to Tartar, we choose

$$B_0^1 = L_0^2(\mathbb{T}), \qquad B_0^2 = C([0,T];L_0^2(\mathbb{T})).$$

$$B_1^1 = H_0^3(\mathbb{T}), \qquad B_1^2 = C([0,T]; H_0^3(\mathbb{T})).$$
$$\alpha = \beta = 1, \qquad \lambda = \theta = \frac{s}{3}, \qquad q = p = 2.$$

Therefore, $B^1_{\theta,p} = B^1_{\frac{s}{3},2} = H_0^s(\mathbb{T})$. From Proposition 1.7.6, we obtain

$$\begin{aligned} B^2_{\lambda,q} = B^2_{\frac{s}{3},2} &= \left(C([0,T]; L_0^2(\mathbb{T})), C([0,T]; H_0^3(\mathbb{T}))\right)_{\frac{s}{3},2} = C\left([0,T]; (L_0^2(\mathbb{T}), H_0^3(\mathbb{T}))_{\frac{s}{3},2}\right) \\ &= C([0,T]; H_0^s(\mathbb{T})). \end{aligned}$$

Then, the operator $S(t) : B_0^1 = L_0^2(\mathbb{T}) \longrightarrow B_0^2 = C([0,T]; L_0^2(\mathbb{T}))$, defined by

$$S(t)u_0 = u, \tag{4.2.30}$$

satisfies, for $u_0, \widetilde{u_0} \in L_0^2(\mathbb{T})$,

$$\begin{aligned} \|Su_0 - S\widetilde{u_0}\|_{L^\infty([0,T];L_0^2(\mathbb{T}))} &\leqslant c\,\|Su_0 - S\widetilde{u_0}\|_{Z^T_{0,\frac{1}{2}}} \leqslant C_{T,0}(\|u_0 - \widetilde{u_0}\|_{L_0^2(\mathbb{T})})\|u_0 - \widetilde{u_0}\|_{L_0^2(\mathbb{T})} \\ &\leqslant C_{T,0}(\|u_0\|_{L_0^2(\mathbb{T})} + \|\widetilde{u_0}\|_{L_0^2(\mathbb{T})})\|u_0 - \widetilde{u_0}\|_{L_0^2(\mathbb{T})}, \end{aligned}$$

by the continuous dependence of the initial data property in $L_0^2(\mathbb{T})$.

Also, the operator $S(t) : B_1^1 = H_0^3(\mathbb{T}) \longrightarrow B_1^2 = C([0,T]; H_0^3(\mathbb{T}))$, defined by (4.2.30) satisfies, for $u_0 \in H_0^3(\mathbb{T})$,

$$\|Su_0\|_{L^\infty([0,T];H_0^3(\mathbb{T}))} \leqslant c\,\|Su_0\|_{Z^T_{3,\frac{1}{2}}} \leqslant C_{T,3}(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^3(\mathbb{T})},$$

by the estimate (4.2.8) with $s = 3$, where $C_{T,j} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous non decreasing functions, for $j = 0, 3$.

Hence, a direct application of Theorem 1.7.7 yields that $S(t)$ maps $H_0^s(\mathbb{T})$ into $C([0,T]; H_0^s(\mathbb{T}))$ and for $u_0 \in H_0^s(\mathbb{T})$

$$\|S(t)u_0\|_{L^\infty([0,T];H_0^s(\mathbb{T}))} \leqslant C_{T,s,\mu}(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^s(\mathbb{T})},$$

where for $r > 0$, $C_{T,s,\mu}(r) = C_{T,3}(2r)^{1-\frac{s}{3}} C_{T,0}(r + 2r)^{\frac{s}{3}}$.

Note that a similar result as in the case $s = 3$ can be obtained for $s \in 3\mathbb{N}^*$. For other values of s, the global well-posedness follows by nonlinear interpolation.

Finally, we will prove the continuous dependence of the initial data. Let $u_0 \in H_0^s(\mathbb{T})$ and consider a sequence $u_{n,0}$ in $H_0^s(\mathbb{T})$ such that $\lim\limits_{n\longrightarrow\infty} u_{n,0} = u_0$, where the limit is taken in the $H_0^s(\mathbb{T})$ norm. Let u and u_n be the solutions of the IVP (4.2.1) in the spaces $C([0,T]; H_0^s(\mathbb{T}))$ and $C([0,T_n]; H_0^s(\mathbb{T}))$ with initial data u_0 and $u_{n,0}$ respectively. For n sufficiently large we have that $\|u_{n,0}\|_{H_0^s(\mathbb{T})} \leqslant 2\|u_0\|_{H_0^s(\mathbb{T})}$. So, by the local theory there exists T_0 with

$$T_0 = T_0(\|u_0\|_{H_0^s(\mathbb{T})}) = \left(\frac{1}{2}\right)^{\frac{1}{\theta}} T(\|u_0\|_{H_0^s(\mathbb{T})}) < T(\|u_0\|_{H_0^s(\mathbb{T})}), \tag{4.2.31}$$

fulfilling (4.2.12) such that u and $u_{n,0}$ are defined in $[0,T_0]$ for $n>N_0$. Observe that

$$\Gamma(u)-\Gamma(u_n)=U_\mu(u_0-u_{n,0})+\int_0^t U_\mu(t-\tau)[\partial_x(u_n^2-u^2)](\tau)\,d\tau$$
$$+\int_0^t U_\mu(t-\tau)[K_\lambda(u_n-u)](\tau)\,d\tau.$$

With a similar procedure leading to (4.2.11), we get

$$\begin{aligned}\|u-u_n\|_{Z^{T_0}_{s,\frac12}} &\leqslant C_1\ \|u_0-u_{n,0}\|_{H^s_0(\mathbb{T})}+\left(2C_2\,T_0^\theta\,M\ +C_3\,T_0^{1-\epsilon}\right)\ \|u_n-u\|_{Z^{T_0}_{s,\frac12}}\\ &\leqslant C_1\ \|u_0-u_{n,0}\|_{H^s_0(\mathbb{T})}+\frac12\,\|u-u_n\|_{Z^{T_0}_{s,\frac12}}.\end{aligned}$$

Thus

$$\|u-u_n\|_{Z^{T_0}_{s,\frac12}}\leqslant\ 2C_1\ \|u_0-u_{n,0}\|_{H^s_0(\mathbb{T})}. \tag{4.2.32}$$

From (4.2.32) we infer

$$\lim_{n\longrightarrow\infty}\|u-u_n\|_{L^\infty([0,T_0],H^s_0(\mathbb{T}))}\leqslant C_4\lim_{n\longrightarrow\infty}\|u-u_n\|_{Z^{T_0}_{s,\frac12}}\leqslant 2C_1C_4\lim_{n\longrightarrow\infty}\|u_0-u_{n,0}\|_{H^s_0(\mathbb{T})}=0.$$

We can Iterate this property to cover the compact set $[0,T]$. Also, the continuous dependence shows that the operator $S(t)$ is continuous. This completes the proof of the Theorem. □

Next, we prove a local exponential stability result when applying the feedback law $f=-K_\lambda u$. For this, we need to observe that the system (4.2.1) can be rewritten as

$$\partial_t u=A_\mu u-2u\partial_x u-K_\lambda u,\quad t>0,\ x\in\mathbb{T},$$

where $A_\mu=\alpha\mathcal{H}\partial_x^2+\partial_x^3-2\mu\partial_x$. Let $T_\lambda(t)=e^{(\alpha\mathcal{H}\partial_x^2+\partial_x^3-2\mu\partial_x-K_\lambda)t}$ be the C_0-semigroup on $H^s_0(\mathbb{T})$ with infinitesimal generator $A_\mu-K_\lambda$. The system (4.2.1) can be rewritten in an equivalent integral form

$$u(t)=T_\lambda(t)u_0-\int_0^t T_\lambda(t-\tau)(2u\partial_x u)(\tau)\,d\tau. \tag{4.2.33}$$

At this point we need to extend some estimates for the C_0-semigroup $\{T_\lambda(t)\}$.

Lemma 4.2.3. *Let $s\geqslant0$, $\lambda\geqslant0$, and $T>0$ be given. Assume $\mu\in\mathbb{R}$, and $\alpha>0$. Then, there exists a constant $C>0$ such that:*

$$\|T_\lambda(t)\phi\|_{Z^T_{s,\frac12}}\leqslant C\|\phi\|_{H^s_0(\mathbb{T})},\quad \textit{for any } \phi\in H^s_0(\mathbb{T}). \tag{4.2.34}$$

$$\left\|\int_0^t T_\lambda(t-\tau)\partial_x(u\cdot v)(\tau)d\tau\right\|_{Z^T_{s,\frac12}}\leqslant C\|u\|_{Z^T_{s,\frac12}}\|v\|_{Z^T_{s,\frac12}},\quad \textit{for any } u,v\in Z^T_{s,\frac12}. \tag{4.2.35}$$

where the constant C does not depend on T if $T\in[0,1]$.

Proof. We begin by proving (4.2.34). From definition of T_λ,

$$u(t) = T_\lambda(t)\phi \tag{4.2.36}$$

is a solution to

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + K_\lambda u = 0, & t > 0, \quad x \in \mathbb{T} \\ u(x,0) = \phi(x), & x \in \mathbb{T}. \end{cases} \tag{4.2.37}$$

On the other hand, using the Duhamel's formula, we can write (4.2.37) as

$$u(t) = U_\mu(t)\phi - \int_0^t U_\mu(t-\tau)[K_\lambda u](\tau)\, d\tau. \tag{4.2.38}$$

From (4.2.36) and (4.2.38), we infer that

$$T_\lambda(t)\phi = U_\mu(t)\phi - \int_0^t U_\mu(t-\tau)(K_\lambda T_\lambda(\tau)\phi)\, d\tau. \tag{4.2.39}$$

Therefore, Lemma 4.2.1 implies that

$$\begin{aligned} \|T_\lambda(t)\phi\|_{Z^T_{s,\frac{1}{2}}} &\leqslant \|U_\mu(t)\phi\|_{Z^T_{s,\frac{1}{2}}} + \left\| \int_0^t U_\mu(t-\tau)(K_\lambda T_\lambda(\tau)\phi)\, d\tau \right\|_{Z^T_{s,\frac{1}{2}}} \\ &\leqslant C_1\|\phi\|_{H_0^s(\mathbb{T})} + C(\epsilon)T^{1-\epsilon}\, \|T_\lambda(\tau)\phi\|_{Z^T_{s,\frac{1}{2}}}, \end{aligned}$$

for some $0 < \epsilon < 1$. Thus, for T_0 sufficiently small such that $1 - C(\epsilon)\, T_0^{1-\epsilon} > 0$ we get that there exists a positive constant $C = C(T_0)$ such that

$$\|T_\lambda(t)\phi\|_{Z^{T_0}_{s,\frac{1}{2}}} \leqslant C(T_0)\|\phi\|_{H_0^s(\mathbb{T})}.$$

For $T \geqslant T_0$, the result follows from an easy induction and the fact that

$$\|T_\lambda(t)\phi\|_{Z^T_{s,\frac{1}{2}}} \leqslant \|T_\lambda(t)\phi\|_{Z^{[0,T_0]}_{s,\frac{1}{2}}} + \|T_\lambda(t)\phi\|_{Z^{[T_0,2T_0]}_{s,\frac{1}{2}}} + \cdots + \|T_\lambda(t)\phi\|_{Z^{[(k-1)T_0,T]}_{s,\frac{1}{2}}}, \tag{4.2.40}$$

for some $k \in \mathbb{Z}$. This proves (4.2.34).

Now we move to prove (4.2.35). Note that, from (4.2.39), we get

$$\int_0^t T_\lambda(t-\tau)f(\tau)\, d\tau = \int_0^t U_\mu(t-\tau)f(\tau)\, d\tau - \int_0^t \int_0^{t-\tau} U_\mu(t-\tau-s)(K_\lambda T_\lambda(s)f(\tau))\, ds\, d\tau.$$

Performing the change of variable $s = -\tau + \theta$ and changing the order of integration, we obtain

$$\begin{aligned} \int_0^t T_\lambda(t-\tau)f(\tau)\, d\tau &= \int_0^t U_\mu(t-\tau)f(\tau)\, d\tau - \int_0^t \int_\tau^t U_\mu(t-\theta)(K_\lambda T_\lambda(-\tau+\theta)f(\tau))\, d\theta\, d\tau \\ &= \int_0^t U_\mu(t-\tau)f(\tau)\, d\tau - \int_0^t \int_0^\theta U_\mu(t-\theta)(K_\lambda T_\lambda(-\tau+\theta)f(\tau))\, d\tau\, d\theta. \end{aligned}$$

Therefore,

$$\int_0^t T_\lambda(t-\tau)f(\tau)\,d\tau = \int_0^t U_\mu(t-\tau)f(\tau)\,d\tau - \int_0^t U_\mu(t-\theta)\int_0^\theta [K_\lambda T_\lambda(\theta-\tau)f(\tau)]\,d\tau\,d\theta. \tag{4.2.41}$$

From the Fubini's theorem (see [36, page 73]) and the linearity of the integral, we infer

$$\begin{aligned}\int_0^\theta [K_\lambda T_\lambda(\theta-\tau)f(\tau)]\,d\tau &= GG^*\left(\int_0^\theta\int_0^1 e^{-2\lambda s}U_\mu(-s)GG^*U_\mu(-s)^*[T_\lambda(\theta-\tau)f(\tau)]dsd\tau\right)\\ &= GG^*\left(\int_0^1 e^{-2\lambda s}U_\mu(-s)GG^*U_\mu(-s)^*\left(\int_0^\theta [T_\lambda(\theta-\tau)f(\tau)]d\tau\right)ds\right)\\ &= K_\lambda\left(\int_0^\theta [T_\lambda(\theta-\tau)f(\tau)]d\tau\right).\end{aligned} \tag{4.2.42}$$

It follows, from (4.2.41) and (4.2.42) that

$$\int_0^t T_\lambda(t-\tau)f(\tau)d\tau = \int_0^t U_\mu(t-\tau)f(\tau)d\tau - \int_0^t U_\mu(t-\theta)K_\lambda\left(\int_0^\theta [T_\lambda(\theta-\tau)f(\tau)]d\tau\right)d\theta, \tag{4.2.43}$$

which gives with $f=\partial_x(u\cdot v)$

$$\begin{aligned}\left\|\int_0^t T_\lambda(t-\tau)\partial_x(u\cdot v)(\tau)d\tau\right\|_{Z^T_{s,\frac12}} &\leqslant \left\|\int_0^t U_\mu(t-\tau)\partial_x(u\cdot v)(\tau)d\tau\right\|_{Z^T_{s,\frac12}}\\ &\quad+\left\|\int_0^t U_\mu(t-\theta)K_\lambda\left(\int_0^\theta [T_\lambda(\theta-\tau)\partial_x(u\cdot v)(\tau)]d\tau\right)d\theta\right\|_{Z^T_{s,\frac12}}\\ &\leqslant C_1\left\|\partial_x(u\cdot v)\right\|_{Z^T_{s,-\frac12}} + C(\epsilon)T^{1-\epsilon}\left\|\int_0^\theta [T_\lambda(\theta-\tau)\partial_x(u\cdot v)(\tau)]d\tau\right\|_{Z^T_{s,\frac12}}\\ &\leqslant C_2\left\|u\right\|_{Z^T_{s,\frac12}}\left\|v\right\|_{Z^T_{s,\frac12}} + C(\epsilon)T^{1-\epsilon}\left\|\int_0^t [T_\lambda(t-\tau)\partial_x(u\cdot v)(\tau)]d\tau\right\|_{Z^T_{s,\frac12}},\end{aligned} \tag{4.2.44}$$

for some $0<\epsilon<1$. Thus, for T_0 sufficiently small such that $1-C(\epsilon)\,T_0^{1-\epsilon}>0$ we obtain that there exists a positive constant $C=C(T_0)$ such that

$$\left\|\int_0^t T_\lambda(t-\tau)\partial_x(u\cdot v)(\tau)\,d\tau\right\|_{Z^{T_0}_{s,\frac12}} \leqslant C(T_0)\left\|u\right\|_{Z^{T_0}_{s,\frac12}}\left\|v\right\|_{Z^{T_0}_{s,\frac12}}.$$

The result follows by induction and a similar property as in (4.2.40). □

Theorem 4.2.4. *Let $\mu\in\mathbb{R}$, $\alpha>0$, $0<\lambda'<\lambda$ and $s\geqslant 0$ be given. Then there exists $\delta>0$ such that for any $u_0\in H_0^s(\mathbb{T})$ with $\|u_0\|_{H_0^s(\mathbb{T})}\leqslant\delta$, the corresponding solution u of IVP (4.2.1) satisfies*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})}\leqslant Ce^{-\lambda' t}\|u_0\|_{H_0^s(\mathbb{T})},\quad \textit{for all}\, t\geqslant 0, \tag{4.2.45}$$

where $C(\mu)>0$ is a constant that does not depend on u_0.

Proof. We proceed as in [52, 74, 75]. For given $s \geqslant 0$, there exists by Theorem 2.5.5 some constant $C > 0$ such that

$$\|T_\lambda(t)u_0\|_{H_0^s(\mathbb{T})} \leqslant Ce^{-\lambda t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \text{for all } t \geqslant 0.$$

Pick $T > 0$ such that

$$2Ce^{-\lambda T} \leqslant e^{-\lambda' T}. \tag{4.2.46}$$

We seek a solution u of the integral equation (4.2.33) as a fixed point of the map

$$\Gamma(v) = T_\lambda(t)u_0 - \int_0^t T_\lambda(t-\tau)(2v\partial_x v)(\tau)\, d\tau, \tag{4.2.47}$$

in some closed ball $B(0, M)$ in the space $Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$ for the $\|v\|_{Z^T_{s,\frac{1}{2}}}$ norm. This will be done provided that $\|v\|_{H_0^s(\mathbb{T})} \leqslant \delta$, where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and M will be chosen in such a way that

$$\|u(T)\|_{H_0^s(\mathbb{T})} \leqslant Ce^{-\lambda' T}\|u_0\|_{H_0^s(\mathbb{T})}. \tag{4.2.48}$$

In fact. From Lemma 4.2.3 there exist some positive constants C_1, C_2 (independent of δ and M) such that

$$\|\Gamma(v)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_1\|u_0\|_{H_0^s(\mathbb{T})} + C_2\|v\|^2_{Z^T_{s,\frac{1}{2}}}, \tag{4.2.49}$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_2\|v_1 + v_2\|_{Z^T_{s,\frac{1}{2}}}\|v_1 - v_2\|_{Z^T_{s,\frac{1}{2}}}. \tag{4.2.50}$$

On the other hand, since $Z^T_{s,\frac{1}{2}} \subset C([0,T]; H_0^s(\mathbb{T}))$, we have for some constant $C' > 0$ and all $v \in B(0, M)$

$$\begin{aligned}
\|\Gamma(v)(T)\|_{H_0^s(\mathbb{T})} &\leqslant \|T_\lambda(T)u_0\|_{H_0^s(\mathbb{T})} + \left\|\int_0^t T_\lambda(t-\tau)(2v\partial_x v)(\tau)\, d\tau\right\|_{L^\infty([0,T];H_0^s(\mathbb{T}))} \\
&\leqslant \|T_\lambda(T)u_0\|_{H_0^s(\mathbb{T})} + c\left\|\int_0^t T_\lambda(t-\tau)(2v\partial_x v)(\tau)\, d\tau\right\|_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant Ce^{-\lambda T}\|u_0\|_{H_0^s(\mathbb{T})} + C'\|v\|^2_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant Ce^{-\lambda T}\delta + C'M^2.
\end{aligned} \tag{4.2.51}$$

Pick $\delta = C_4 M^2$, where C_4 and M are chosen so that

$$\frac{C'}{C_4} \leqslant Ce^{-\lambda T}, \quad (C_1C_4 + C_2)M^2 \leqslant M, \quad \text{and} \quad 2C_2M \leqslant \frac{1}{2}. \tag{4.2.52}$$

From (4.2.49), (4.2.50), and (4.2.52), we get

$$\|\Gamma(v)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_1\delta + C_2M^2 \leqslant C_1C_4M^2 + C_2M^2 \leqslant M, \quad \text{for all } v \in B(0,M),$$

and

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Z^T_{s,\frac{1}{2}}} \leqslant 2C_2M\|v_1 - v_2\|_{Z^T_{s,\frac{1}{2}}} \leqslant \frac{1}{2}\|v_1 - v_2\|_{Z^T_{s,\frac{1}{2}}}, \quad \text{for all } v_1, v_2 \in B_M(0).$$

Therefore, Γ is a contraction in $B(0,M)$. Furthermore, by (4.2.51) and (4.2.52) its unique fixed point $u \in B(0,M)$ fulfills

$$\|u(T)\|_{H^s_0(\mathbb{T})} = \|\Gamma(u)(T)\|_{H^s_0(\mathbb{T})} \leqslant (Ce^{-\lambda T}C_4 + C')M^2 \leqslant 2Ce^{-\lambda T}C_4M^2 \leqslant 2Ce^{-\lambda T}\delta \leqslant \delta e^{-\lambda' T}.$$

Assume now that $0 < \|u_0\|_{H^s_0(\mathbb{T})} < \delta$. Changing δ into $\delta' := \|u_0\|_{H^s_0(\mathbb{T})}$ and M into $M' := \left(\frac{\delta'}{\delta}\right)^{\frac{1}{2}} M \leqslant M$, we infer that $\|u(T)\|_{H^s_0(\mathbb{T})} \leqslant e^{-\lambda' T}\, \|u_0\|_{H^s_0(\mathbb{T})}$ and (4.2.48) holds. Following a similar argument as those in the demonstration of Theorem 2.4.2 ((2.4.20), and (2.4.21)), we have that an obvious induction yields

$$\|u(nT)\|_{H^s_0(\mathbb{T})} \leqslant e^{-\lambda' nT}\, \|u_0\|_{H^s_0(\mathbb{T})}, \tag{4.2.53}$$

for some $T > 0$ fixed, and for any $n \in \mathbb{N}$.

As $Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T})) \subset C([0,T]; H^s_0(\mathbb{T}))$, we infer by the semigroup property that there exists some constant $C > 0$ such that (4.2.45) holds provided that $\|u_0\|_{H^s_0(\mathbb{T})} \leqslant \delta$.

In fact, for $t \geqslant 0$, there exists $n \in \mathbb{N}$ and $s' \in \mathbb{R}$ with $0 \leqslant s' < T$ such that $t = nT + s'$. Thus,

$$\begin{aligned} u(t) &= u(nT + s') \\ &= T_\lambda(s')T_\lambda(nT)u_0 - \int_0^{nT} T_\lambda(s')T_\lambda(nT-\tau)(2u\partial_x u)(\tau)d\tau - \int_{nT}^{nT+s'} T_\lambda(nT+s'-\tau)(2u\partial_x u)(\tau)d\tau \\ &= T_\lambda(s')\left[T_\lambda(nT)u_0 - \int_0^{nT} T_\lambda(nT-\tau)(2u\partial_x u)(\tau)d\tau\right] \\ &\quad - \int_0^{s'} T_\lambda(nT+s'-\theta-nT)(2u\partial_x u)(\theta+nT)d\theta, \end{aligned} \tag{4.2.54}$$

where in the last line we performed the change of variables $\theta = \tau - nT$. Define

$$I_2 := -T_\lambda(nT)\int_0^{s'} T_\lambda(s' - (\theta + nT))(2u\partial_x u)(\theta + nT)d\theta. \tag{4.2.55}$$

From (4.2.46) and (4.2.52), we obtain

$$
\begin{aligned}
\|I_2\|_{H_0^s(\mathbb{T})} &= \left\| \int_0^{s'} T_\lambda(s' - (\theta + nT))(2u\partial_x u)(\theta + nT)d\theta \right\|_{H_0^s(\mathbb{T})} \\
&= \left\| \int_0^{t} T_\lambda(t - (\theta + nT))(2u\partial_x u)(\theta + nT)d\theta \right\|_{L^\infty([0,T];H_0^s(\mathbb{T}))} \\
&\leqslant c \left\| \int_0^{t} T_\lambda(t - (\theta + nT))(2u\partial_x u)(\theta + nT)d\theta \right\|_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant C' \left\| u(\theta + nT) \right\|^2_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant C' \left\| u \right\|^2_{Z^{[nT,(n+1)T]}_{s,\frac{1}{2}}} \\
&\leqslant C e^{-\lambda T(n+1)} C_4 \, (M')^2 \\
&\leqslant C e^{-\lambda T(n+1)} \delta' \\
&\leqslant \frac{e^{-\lambda' T(n+1)}}{2} \delta'.
\end{aligned} \tag{4.2.56}
$$

Therefore, (4.2.53), (4.2.54) and (4.2.56) implies that

$$
\begin{aligned}
\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} &\leqslant \|T_\lambda(s')[u(\cdot,nT)]\|_{H_0^s(\mathbb{T})} + \|I_2\|_{H_0^s(\mathbb{T})} \\
&\leqslant e^{-\lambda' nT} \|u_0\|_{H_0^s(\mathbb{T})} + \frac{e^{-\lambda' T(n+1)}}{2} \|u_0\|_{H_0^s(\mathbb{T})} \\
&= \left(e^{-\lambda' Tn} + \frac{e^{-\lambda' T(n+1)}}{2} \right) \|u_0\|_{H_0^s(\mathbb{T})}.
\end{aligned}
$$

Finally, it is easy to prove that there exists a positive constant C such that

$$
e^{-\lambda' T\, n} + \frac{e^{-\lambda' T(n+1)}}{2} \leqslant C\, e^{-\lambda' t}, \quad \text{for all } \; t \geqslant 0 \; \text{ and for all } \; n \in \mathbb{N}.
$$

This proves the theorem. □

Corollary 4.2.5. *Let $\alpha > 0$, $0 < \lambda' < \lambda$, $s \geqslant 0$, and $\mu \in \mathbb{R}$ be given. Then there exist $\delta = \delta(\mu) > 0$ and a linear bounded operator $K_\lambda : H_p^s(\mathbb{T}) \to H_p^s(\mathbb{T})$ such that for any $u_0 \in H_p^s(\mathbb{T})$ with $[u_0] = \mu$, and $\|u_0 - [u_0]\|_{H_p^s(\mathbb{T})} \leqslant \delta$, the corresponding solution $u \in C([0,+\infty), H_p^s(\mathbb{T}))$ of the closed-loop system (2.0.2) with $Ku = K_\lambda u$, satisfies*

$$
\|u(\cdot,t) - [u_0]\|_{H_p^s(\mathbb{T})} \leqslant C\, e^{-\lambda' t} \|u_0 - [u_0]\|_{H_p^s(\mathbb{T})}, \quad \textit{for all } \; t \geqslant 0,
$$

where $C = C(\mu) > 0$ is a constant independent of u_0.

The stability result presented in Theorem 4.2.4 is local. We will extend it to a global stability result. In order to do that, the following observability inequality is needed.

Proposition 4.2.6. *Let $s \geqslant 0$, $\lambda = 0$, $\mu \in \mathbb{R}$, $\alpha > 0$, $T > 0$, and R_0 be given. Then there exists a constant $\beta > 1$ such that for any $u_0 \in L^2(\mathbb{T})$ satisfying*

$$
\|u_0\|_{L_0^2(\mathbb{T})} \leqslant R_0, \tag{4.2.57}
$$

the corresponding solution u of equation (4.2.1) satisfies

$$\|u_0\|^2_{L^2_0(\mathbb{T})} \leqslant \beta \int_0^T \|Gu\|^2_{L^2_0(\mathbb{T})}(t)\, dt. \tag{4.2.58}$$

Proof. We argue by contradiction assuming that (4.2.58) is not true, then for any $n \geqslant 1$, equation (4.2.1) admits a solution u_n satisfying

$$u_n \in Z^T_{0,\frac{1}{2}} \cap C([0,T]; L^2_0(\mathbb{T})), \tag{4.2.59}$$

$$\|u_n(0)\|_{L^2_0(\mathbb{T})} \leqslant R_0, \tag{4.2.60}$$

and

$$\int_0^T \|Gu_n\|^2_{L^2_0(\mathbb{T})}(t)\, dt < \frac{1}{n}\|u_{0,n}\|^2_{L^2_0(\mathbb{T})}, \tag{4.2.61}$$

where $u_{0,n} = u_n(0)$. As $\alpha_n := \|u_{0,n}\|_{L^2_0(\mathbb{T})} \leqslant R_0$, one can choose a subsequence of $\{\alpha_n\}$, still denoted by $\{\alpha_n\}$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. There are two possible cases, viz., $\alpha > 0$, and $\alpha = 0$.

Case 1. $\alpha > 0$:

From (4.2.59) and (4.2.60) we obtain that the sequence $\{\alpha_n\}$ is bounded in both spaces $L^\infty([0,T]; L^2(\mathbb{T}))$ and $X^T_{0,\frac{1}{2}}$ with

$$\|u_n(t)\|_{L^2_0(\mathbb{T})} \leqslant \|u_n\|_{L^\infty([0,T];L^2_0(\mathbb{T}))} \leqslant C\|u_n\|_{Z^T_{0,\frac{1}{2}}} \leqslant \beta_{T,0}(\|u_{0,n}\|_{L^2_0(\mathbb{T})})\|u_{0,n}\|_{L^2_0(\mathbb{T})} \leqslant \beta_{T,0}(R_0)R_0,$$

for all $t \in [0,T]$. From 3.1.72, the sequence $\{\partial_x(u_n^2)\}$ is bounded in the space $X^T_{0,-\frac{1}{2}}$ and

$$\|\partial_x(u_n^2)\|_{X^T_{0,-\frac{1}{2}}} \leqslant C\|u_n\|^2_{X^T_{0,\frac{1}{2}}}.$$

On the other hand, from Proposition 3.1.6 we infer that the embedding $X^T_{0,\frac{1}{2}} \hookrightarrow X^T_{-1,0}$ is compact (see [54, pages 271 and 639]). Therefore, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \ \text{ in } X^T_{0,\frac{1}{2}}, \tag{4.2.62}$$

$$u_n \longrightarrow u \ \text{ in } X^T_{-1,0}, \tag{4.2.63}$$

and

$$-\partial_x(u_n^2) \rightharpoonup f \ \text{ in } X^T_{0,-\frac{1}{2}}, \tag{4.2.64}$$

where $u \in X^T_{0,\frac{1}{2}}$ and $f \in X^T_{0,-\frac{1}{2}}$. Also, from Theorem 3.1.18, $X^T_{0,\frac{1}{2}}$ is continuously embedded in $L^4(\mathbb{T}\times[0,T])$ and

$$\|u_n^2\|_{L^2(\mathbb{T}\times[0,T])} = \|u_n\|^2_{L^4(\mathbb{T}\times[0,T])} \leqslant C\|u_n\|^2_{X^T_{0,\frac{1}{3}}} \leqslant C\|u_n\|^2_{X^T_{0,\frac{1}{2}}}.$$

Thus, u_n^2 is bounded in $L^2(\mathbb{T}\times[0,T])$ and it follows that

$$\|\partial_x(u_n^2)\|_{L^2([0,T];H^{-1}(\mathbb{T}))} = \|\partial_x(u_n^2)\|_{X^T_{-1,0}} \leqslant \|u_n^2\|_{L^2(\mathbb{T}\times[0,T])}.$$

Therefore, $\partial_x(u_n^2)$ is bounded in $L^2([0,T];H^{-1}(\mathbb{T})) = X^T_{-1,0}$. Conducting interpolation between $X^T_{0,-\frac{1}{2}}$, and $X^T_{-1,0}$ (see proof of Theorem 3.2.3) we conclude that $\partial_x(u_n^2)$ is bounded in $X^T_{-\theta,-\frac{(1-\theta)}{2}} = X^T_{-\theta,-\frac{1}{2}+\frac{\theta}{2}}$, for $0<\theta<1$. As $X^T_{-\theta,-\frac{1}{2}+\frac{\theta}{2}}$ is compactly embedded in $X^T_{-1,-\frac{1}{2}}$, for $0<\theta<1$, we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$-\partial_x(u_n^2) \longrightarrow f \text{ in } X^T_{-1,-\frac{1}{2}}.$$

It follows from (4.2.61) that

$$\int_0^T \|Gu_n\|^2_{L^2_0(\mathbb{T})}(t)dt \longrightarrow \int_0^T \|Gu\|^2_{L^2_0(\mathbb{T})}(t)dt = 0, \tag{4.2.65}$$

which implies that $Gu(x,t)=0$, in $\mathbb{T}\times[0,T]$. Consequently,

$$u(x,t) = c(t) = \int_0^T g(y)u(y,t)\,dy,$$

on $\operatorname{supp}(g)\times(0,T) = \omega\times(0,T)$ (see (2.0.9)). Thus, passing to the limit in (4.2.1) in the distributional sense, we obtain

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u = f, & \text{on } \mathbb{T}\times(0,T) \\ u(x,t) = c(t), & \text{on } \omega\times(0,T). \end{cases} \tag{4.2.66}$$

Let

$$w_n = u_n - u \quad\text{and}\quad f_n = -\partial_x(u_n^2) - f - K_0 u_n. \tag{4.2.67}$$

Note first that (4.2.65) implies

$$\begin{aligned}\int_0^T \|Gw_n\|^2_{L^2_0(\mathbb{T})}(t)dt &= \int_0^T \|Gu_n\|^2_{L^2_0(\mathbb{T})}(t)dt + \int_0^T \|Gu\|^2_{L^2_0(\mathbb{T})}(t)dt - 2\int_0^T (Gu_n, Gu)_{L^2_0(\mathbb{T})}(t)dt \\ &\longrightarrow 0, \quad \text{as } n\longrightarrow\infty,\end{aligned} \tag{4.2.68}$$

Also, from (4.2.62) we obtain that

$$w_n \rightharpoonup 0 \text{ in } X^T_{0,\frac{1}{2}}. \tag{4.2.69}$$

Furthermore, from equation (4.2.1), equation (4.2.66), and (4.2.67) we have that w_n satisfies

$$\partial_t w_n - \partial_x^3 w_n - \alpha\mathcal{H}\partial_x^2 w_n + 2\mu\partial_x w_n = f_n, \text{ on } \mathbb{T}\times(0,T). \tag{4.2.70}$$

Note that

$$\int_0^T\int_{\mathbb{T}}|Gw_n|^2dxdt = \int_0^T\int_{\mathbb{T}} g^2(x)w_n^2(x,t)dxdt - 2\int_0^T\left(\int_{\mathbb{T}} g(y)w_n(y,t)dy\right)\left(\int_{\mathbb{T}} g^2(x)w_n(x,t)dx\right)dt + \int_0^T\left(\int_{\mathbb{T}} g(y)w_n(y,t)dy\right)^2\left(\int_{\mathbb{T}} g^2(x)dx\right)dt. \tag{4.2.71}$$

At this point we need the following lemma.

Lemma 4.2.7. *Let $\{w_n\}_{n\geqslant 1}$ be a sequence of solutions of equation (4.2.70), g defined in (2.0.8), and $c_n := \int_{\mathbb{T}} g(y)w_n(y,t)dy$, $t\in(0,T)$, $n\in\mathbb{N}$. If $w_n \rightharpoonup 0$ in $X^T_{0,\frac{1}{2}}$, then the sequence $\{c_n\}_{n\geqslant 1}$ satisfies $c_n \longrightarrow 0$ in $L^2(0,T)$ as $n\longrightarrow\infty$.*

Proof. From hypothesis we infer that $w_n \rightharpoonup 0$ in $X^T_{0,0}$. So, $\{w_n\}_{n\geqslant 1}$ is bounded in $X^T_{0,0}$. From (4.2.70), (4.2.67) and integration by parts, we have

$$\begin{aligned}\frac{d}{dt}c_n(t) &= \int_{\mathbb{T}} g(y)\frac{d}{dt}[w_n](y,t)dy\\ &= \int_{\mathbb{T}} g(y)\left(\partial_y^3 w_n + \alpha\mathcal{H}\partial_y^2 w_n - 2\mu\partial_y w_n + f_n\right)(y)dy\\ &= \int_{\mathbb{T}} w_n(y)\left(-\partial_y^3 g - \alpha\mathcal{H}\partial_y^2 g + 2\mu\partial_y g\right)(y) + \partial_y g(y)\left(u_n^2 + \partial_y^{-1}f\right)(y) - GGg(y)u_n(y)dy,\end{aligned}$$

Using Cauchy-Schwarz inequality on space variable, we obtain $\left\|\frac{d}{dt}c_n(t)\right\|_{L^2(0,T)} < +\infty$. On the other hand,

$$\|c_n(t)\|_{L^2(0,T)} \leqslant \left(\int_0^T\left|\int_{\mathbb{T}} g(y)w_n(y,t)dy\right|^2 dt\right)^{\frac{1}{2}} \leqslant \|g\|_{L^2(\mathbb{T})}\|w_n\|_{X^T_{0,0}} < +\infty.$$

Therefore, the sequence $\{c_n(t)\}_{n\geqslant 1}$ is bounded in $H^1(0,T)$. The Rellich's Theorem (see [36, page 305]), and the fact that $w_n \rightharpoonup 0$ in $X^T_{0,0}$, imply the desired conclusion. ☐

We infer from (4.2.68), (4.2.71), and Lemma 4.2.7 that

$$\int_0^T\int_{\mathbb{T}} g^2(x)\; w_n^2(x,t)\; dx\; dt \longrightarrow 0. \tag{4.2.72}$$

Hence, if $\widetilde{\omega} := \left\{g(x) > \frac{\|g\|_{L^\infty(\mathbb{T})}}{2}\right\}$, then

$$\begin{aligned}\|w_n\|_{L^2((0,T);L^2(\widetilde{\omega}))} &= \int_0^T\int_{\widetilde{\omega}}\frac{|g(x)|^2}{|g(x)|^2}w_n^2(x,t)dxdt\\ &\leqslant \frac{4}{\|g\|^2_{L^\infty(\mathbb{T})}}\int_0^T\int_{\widetilde{\omega}} g^2(x)w_n^2(x,t)dxdt \longrightarrow 0, \quad \text{as } n\longrightarrow+\infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|f_n\|_{X^T_{-1,-\frac{1}{2}}} &\leqslant \|-\partial_x(u_n^2) - f\|_{X^T_{-1,-\frac{1}{2}}} + C\|GG^*u_n\|_{X^T_{0,0}}\\ &\leqslant \|-\partial_x(u_n^2) - f\|_{X^T_{-1,-\frac{1}{2}}} + C\left(\int_0^T\|Gu_n\|^2dt\right)^{\frac{1}{2}} \longrightarrow 0, \quad \text{as } n\longrightarrow+\infty.\end{aligned}$$

Applying the propagation of compactness property for the Benjamin equation (see Proposition 3.3.2 with $b = \frac{1}{2}$, and $b' = 0$) we obtain that

$$\|w_n\|_{L^2_{loc}((0,T);L^2(\mathbb{T}))} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow +\infty. \tag{4.2.73}$$

Therefore, for each compact set $K \subset (0,T)$

$$\begin{aligned}\int_K \int_{\mathbb{T}} |u_n^2(x,t) - u^2(x,t)| \, dx \, dt &= \int_K \int_{\mathbb{T}} |u_n(x,t) + u(x,t)| \cdot |u_n(x,t) - u(x,t)| \, dx \, dt \\ &\leqslant \|u_n + u\|_{L^2(K;L^2(\mathbb{T}))} \|w_n\|_{L^2(K;L^2(\mathbb{T}))} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow +\infty,\end{aligned}$$

where we used the Cauchy-Schwarz inequality, first in space and then in time. Thus, $u_n^2 \longrightarrow u^2$ in $L^1_{loc}((0,T);L^1(\mathbb{T}))$. This implies that

$$\begin{aligned}\left| \int_0^T \int_{\mathbb{T}} \partial_x(u_n^2 - u^2)(x,t) \, \varphi(x,t) \, dx \, dt \right| &\leqslant \int_0^T \int_{\mathbb{T}} \left|u_n^2 - u^2\right| \, |\partial_x \varphi| \, dx \, dt \\ &\leqslant \|\partial_x \varphi\|_{L^\infty(\mathbb{T}\times(0,T))} \int_K \int_{\mathbb{T}} \left|u_n^2 - u^2\right| \, dx \, dt \\ &\longrightarrow 0 \ \text{ as } n \longrightarrow \infty,\end{aligned} \tag{4.2.74}$$

for all $\varphi \in C_c^\infty(\mathbb{T} \times (0,T))$ with $K = \operatorname{supp}(\partial_x \varphi)$.

Then $\partial_x(u_n^2) \longrightarrow \partial_x(u^2)$ in the distributional sense. Therefore, $f = -\partial_x(u^2)$ and $u \in X^T_{0,\frac{1}{2}}$ satisfy

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha \mathcal{H} \partial_x^2 u + 2\mu \partial_x u + \partial_x(u^2) = 0, & \text{on } \mathbb{T} \times (0,T) \\ u(x,t) = c(t), & \text{on } \omega \times (0,T), \end{cases} \tag{4.2.75}$$

in a distributional sense. From the unique continuation property (see Proposition 3.4.2) we get that $u = 0$. Now (4.2.73) implies that $u_n \longrightarrow 0$ in $L^2_{loc}((0,T);L^2(\mathbb{T}))$. We can pick some time $t_0 \in [0,T]$ such that $u_n(t_0) \longrightarrow 0$ in $L^2(\mathbb{T})$. From (4.2.14) with $\lambda = 0$ we have that

$$\|u_n(0)\|^2_{L^2_0(\mathbb{T})} = \|u_n(t_0)\|^2_{L^2_0(\mathbb{T})} + \int_0^{t_0} \|Gu_n\|_{L^2_0(\mathbb{T})} \, dt' \longrightarrow 0 \ \text{ as } \ n \longrightarrow +\infty,$$

which is a contradiction with the assumption $\alpha > 0$.

Case 2. $\alpha = 0$.

From (4.2.61) we infer that $\alpha_n > 0$, for all $n \in \mathbb{N}^*$. Set $v_n = \dfrac{u_n}{\alpha_n}$, for all $n \geqslant 1$, then

$$\int_0^T \|Gv_n\|^2_{L^2_0(\mathbb{T})} \, dt < \frac{1}{n}, \tag{4.2.76}$$

holds by (4.2.61) and we get the identity $\dfrac{\|Gu_n\|^2_{L^2_0(\mathbb{T})}}{\|u_{0,n}\|^2_{L^2_0(\mathbb{T})}} = \left\| \dfrac{Gu_n}{\|u_{0,n}\|_{L^2_0(\mathbb{T})}} \right\|^2_{L^2_0(\mathbb{T})} = \|Gv_n\|^2_{L^2_0(\mathbb{T})}$.

Therefore, v_n satisfies

$$\partial_t v_n - \partial_x^3 v_n - \alpha \mathcal{H} \partial_x^2 v_n + 2\mu \partial_x v_n + K_0 v_n + \alpha_n \partial_x(v_n^2) = 0.$$

Because of

$$\|v_n(0)\|_{L^2_0(\mathbb{T})} = 1, \tag{4.2.77}$$

the sequence $\{v_n\}$ is bounded in both spaces $L^\infty((0,T);L^2(\mathbb{T}))$ and $X^T_{0,\frac{1}{2}}$. Indeed, from (4.2.14) with $\lambda = 0$ we have that the map $t \longmapsto \|v_n(t)\|_{L^2_0(\mathbb{T})}$ in a nonincreasing function. Therefore, $\|v_n\|_{L^\infty((0,T);L^2_0(\mathbb{T}))} \leqslant 1$. The boundedness of $\|v_n\|_{X^T_{0,\frac{1}{2}}}$ follows from an estimate similar to (4.2.8) (since α_n is bounded).

We can extract a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, such that

$$v_n \rightharpoonup v \text{ in } X^T_{0,\frac{1}{2}},$$

$$v_n \longrightarrow v \text{ in } X^T_{-1,-\frac{1}{2}},$$

and

$$v_n \longrightarrow v \text{ in } X^T_{-1,0}.$$

Moreover, the sequence $\{\partial_x(v_n^2)\}$ is bounded in the space $X^T_{0,-\frac{1}{2}}$, and therefore $\alpha_n\partial_x(v_n^2)$ tends to 0 in the space $X^T_{0,-\frac{1}{2}}$. Finally,

$$\int_0^T \|Gv\|_{L^2_0(\mathbb{T})}\,dt = 0.$$

Hence, v solves,

$$\begin{cases} \partial_t v - \partial_x^3 v - \alpha\mathcal{H}\partial_x^2 v + 2\mu\partial_x v = 0, & \text{on } \mathbb{T}\times(0,T) \\ v(x,t) = c(t), & \text{on } \omega\times(0,T). \end{cases}$$

Therefore, using the unique continuation property for the linearized Benjamin equation, which can be proved in a similar way to Proposition 3.4.2, we obtain that $v(x,t) = c(t) = c = 0$ because $[v] = 0$.

According to (4.2.76)

$$\int_0^T \|Gv_n\|^2_{L^2_0(\mathbb{T})}\,dt \longrightarrow 0 \tag{4.2.78}$$

and so $K_0 v_n$ converges strongly to 0 in $X^T_{-1,-\frac{1}{2}}$, by the embedding of $X^T_{0,0}$ into $X^T_{-1,-\frac{1}{2}}$. Then, an application of Proposition 3.3.2 as in **Case 1**, shows that v_n converges to 0 in $L^2_{loc}((0,T);L^2(\mathbb{T}))$. Thus we can pick a time $t_0 \in (0,T)$ such that $v_n(t_0)$ converges to 0 strongly in $L^2(\mathbb{T})$. Since

$$\|v_n(0)\|^2_{L^2_0(\mathbb{T})} = \|v_n(t_0)\|^2_{L^2_0(\mathbb{T})} + \int_0^{t_0} \|Gv_n\|^2_{L^2_0(\mathbb{T})}\,dt',$$

we infer from (4.2.76) that $\|v_n(0)\|_{L^2_0(\mathbb{T})} \longrightarrow 0$ which is a contradiction with (4.2.77). The proof is complete. □

Theorem 4.2.8. *Let $\lambda = 0$ in (4.2.1). Assume $\mu \in \mathbb{R}$, and $\alpha > 0$ are given. Then there exists $k > 0$ such that for any $R_0 > 0$, there exists a constant $C = C(\mu) > 0$, independent of u_0, such that for any $u_0 \in L^2_0(\mathbb{T})$ with $\|u_0\|_{L^2_0(\mathbb{T})} \leqslant R_0$, the corresponding solution u of equation (4.2.1) (with $\lambda = 0$) satisfies*

$$\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant C\, e^{-kt}\|u_0\|_{L^2_0(\mathbb{T})}, \quad \textit{for all}\, t \geqslant 0. \tag{4.2.79}$$

Proof. This theorem is a direct consequence of the observability inequality (4.2.58). Indeed, from (4.2.14), we infer the following energy estimate

$$\|u(\cdot,t)\|^2_{L^2_0(\mathbb{T})} = \|u_0\|^2_{L^2_0(\mathbb{T})} - \int_0^t \|Gu\|^2_{L^2_0(\mathbb{T})}(t')\, dt', \quad \text{for all } t \geqslant 0.$$

Therefore, (4.2.58) implies that

$$\|u(\cdot,T)\|^2_{L^2_0(\mathbb{T})} = (1-\beta^{-1})\|u_0\|^2_{L^2_0(\mathbb{T})}.$$

Hence, following a procedure similar to the estimates in the proof of Theorem 2.4.2 we have

$$\|u(\cdot,mT)\|^2_{L^2_0(\mathbb{T})} = (1-\beta^{-1})^m\, \|u_0\|^2_{L^2_0(\mathbb{T})}, \tag{4.2.80}$$

which gives (4.2.79) by the semigroup property. We obtain a constant k independent of R_0 by noticing that for $t > c(\|u_0\|_{L^2_0(\mathbb{T})})$, the L^2-norm of $u(\cdot,t)$ is smaller than 1, so that we can take the k corresponding to $R_0 = 1$. □

Now, we prove that the solution u of (4.2.1) (with $\lambda = 0$) decays exponentially in any space $H^s_0(\mathbb{T})$. For this, we need an exponential stability result for the linearized system

$$\begin{cases} \partial_t w - \partial_x^3 w - \alpha\mathcal{H}\partial_x^2 w + 2\mu\partial_x w + 2\partial_x(aw) = -K_0 w, & x \in \mathbb{T},\ t > 0, \\ w(x,0) = w_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.2.81}$$

where $a \in Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$ is a given function. It is established in the following two Lemmas.

Lemma 4.2.9. *Let $s \geqslant 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$, be given. Assume $a \in Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$ for all $T > 0$, and that there exists $T' > 0$ such that*

$$\sup_{n\geqslant 1} \|a\|_{Z^{[nT',(n+1)T']}_{s,\frac{1}{2}}} \leqslant \beta. \tag{4.2.82}$$

Then for any $w_0 \in H^s_0(\mathbb{T})$ and for any $T > 0$ there exists a unique solution $w \in Z^T_{s,\frac{1}{2}} \cap C([0,T]; H^s_0(\mathbb{T}))$ of the IVP (4.2.81). Furthermore, the following estimate holds

$$\|w\|_{Z^T_{s,\frac{1}{2}}} \leqslant \upsilon(\|a\|_{Z^T_{s,\frac{1}{2}}})\, \|w_0\|_{H^s_0(\mathbb{T})}, \tag{4.2.83}$$

where $\upsilon : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function. Moreover, denote by $S(t)u_0$ the unique solution u of the IVP (4.2.81) corresponding to the initial data w_0. Then the operator $S(t) : H_0^s(\mathbb{T}) \longrightarrow C([0,T]; H_0^s(\mathbb{T}))$, defined by $S(t)w_0 = w$ is continuous in the interval $[0,T]$.

Proof. We first establish the existence and uniqueness of a solution

$$w \in Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$$

of (4.2.81) for $0 < T \leqslant 1$ small enough and then show that T can be taken arbitrarily large. Let us rewrite system (4.2.81) in its integral form and for given initial datas $w_0, w_1 \in H_0^s(\mathbb{T})$ we define the map

$$\Gamma(v_j) = U_\mu(t)w_j - \int_0^t U_\mu(t-\tau)(2\partial_x(av_j))(\tau)d\tau - \int_0^t U_\mu(t-\tau)(K_0 v_j)(\tau)d\tau,$$

where $j = 0, 1$ and $U_\mu(t) = e^{(\partial_x^3 + \alpha\mathcal{H}\partial_x^2 - \mu\partial_x)t}$. Assume $0 < T \leqslant T'$. Then, calculations similar to those of Theorem 4.2.2 yield

$$\|\Gamma(v_1) - \Gamma(v_2)\|_{Z^T_{s,\frac{1}{2}}} \leqslant C_1\|w_0 - w_1\|_{H_0^s} + 2C_2T^\theta \|a\|_{Z^T_{s,\frac{1}{2}}} \|v_1 - v_2\|_{Z^T_{s,\frac{1}{2}}} + C_3T^{1-\epsilon} \|v_2 - v_1\|_{Z^T_{s,\frac{1}{2}}}, \tag{4.2.84}$$

for any $a, v_1, v_2 \in Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$. Choosing $w_1 = 0$, $M = 2C_1\|w_0\|_{H_0^s(\mathbb{T})}$, and $T > 0$ such that,

$$2C_2T^\theta \|a\|_{Z^T_{s,\frac{1}{2}}} + C_3T^{1-\epsilon} \leqslant 2C_2T^\theta \|a\|_{Z^{T'}_{s,\frac{1}{2}}} + C_3T^{1-\epsilon} \leqslant 2C_2T^\theta\beta + C_3T^{1-\epsilon} \leqslant \frac{1}{2},$$

we obtain that the map Γ is a contraction in a closed ball

$$B(0,M) = \left\{ v \in Z^T_{s,\frac{1}{2}} : \|v\|_{Z^T_{s,\frac{1}{2}}} \leqslant M \right\},$$

with $M = 2C_1\|w_0\|_{H_0^s(\mathbb{T})}$ of $Z^T_{s,\frac{1}{2}}$. Its unique fixed point w is the desired solution of (4.2.81). Note that, the time of existence, can be taken as

$$T = \min\left\{ \frac{1}{2},\ T',\ \left(\frac{1}{2C_2\ \beta + C_3} \right)^{\frac{1}{\theta}} \right\}. \tag{4.2.85}$$

Furthermore, (4.2.84) shows that the solution depends continuously on the initial data and satisfies (4.2.83). Following a similar argument as in the proof of Theorem 4.2.2 we prove the global existence of the solution. This proof the Lemma. □

Lemma 4.2.10. *Let $s \geqslant 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$, be given. Assume $a \in Z^T_{s,\frac{1}{2}} \cap L^2([0,T]; L^2_0(\mathbb{T}))$ for all $T > 0$. Then for any $k' \in (0,k)$ there exists $T > 0$, and $\beta > 0$ such that if*

$$\sup_{n \geqslant 1} \|a\|_{Z^{[nT,(n+1)T]}_{s,\frac{1}{2}}} \leqslant \beta, \tag{4.2.86}$$

then the solution of the IVP (4.2.81) satisfies

$$\|w(\cdot,t)\|_{H^s_p(\mathbb{T})} \leqslant Ce^{-k't}\|w_0\|_{H^s_p(\mathbb{T})}, \quad \textit{for all}\, t \geqslant 0, \tag{4.2.87}$$

where $C > 0$ is a constant independent of w_0.

Proof. From Lemma 4.2.9 we have that for any $T > 0$ the IVP (4.2.81) admits a unique solution $w \in Z^T_{s,\frac{1}{2}} \cap C([0,T]; H^s_0(\mathbb{T}))$ and

$$\|w\|_{Z^T_{s,\frac{1}{2}}} \leqslant \upsilon(\|a\|_{Z^T_{s,\frac{1}{2}}})\ \|w_0\|_{H^s_0(\mathbb{T})}, \tag{4.2.88}$$

where $\upsilon : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function. Rewrite (4.2.81) in its integral form

$$w(t) = T_0(t)w_0 - \int_0^t T_0(t-\tau)(2\partial_x(a\cdot w))(\tau)d\tau,$$

where $T_0(t) = e^{(\alpha\mathcal{H}\partial_x^2+\partial_x^3-2\mu\partial_x-K_0)t}$ is the C_0-semigroup on $H^s_0(\mathbb{T})$ with infinitesimal generator $A_\mu - K_0$. Thus, for any $T > 0$, we infer from Theorem 2.4.2, Lemma 4.2.3, and (4.2.88) that

$$\begin{aligned}
\|w(\cdot,T)\|_{H^s_0(\mathbb{T})} &\leqslant \|T_0(T)w_0\|_{H^s_0(\mathbb{T})} + \left\|\int_0^T T_0(T-\tau)(2\partial_x(aw))(\tau)d\tau\right\|_{H^s_0(\mathbb{T})}\\
&\leqslant C_1e^{-kT}\|w_0\|_{H^s_0(\mathbb{T})} + 2\left\|\int_0^t T_0(t-\tau)(\partial_x(aw))(\tau)d\tau\right\|_{C([0,T];H^s_0(\mathbb{T}))}\\
&\leqslant C_1e^{-kT}\|w_0\|_{H^s_0(\mathbb{T})} + c\left\|\int_0^t T_\lambda(t-\tau)(2\partial_x(aw))(\tau)d\tau\right\|_{Z^T_{s,\frac{1}{2}}}\\
&\leqslant C_1e^{-kT}\|w_0\|_{H^s_0(\mathbb{T})} + C_2\,\|a\|_{Z^T_{s,\frac{1}{2}}}\ \upsilon(\|a\|_{Z^T_{s,\frac{1}{2}}})\|w_0\|_{H^s_0(\mathbb{T})},
\end{aligned}$$

where $C_1 > 0$ is independent of T and $C_2 > 0$ may depend on T. Let

$$y_n = w(\cdot, nT), \quad \text{for } \ n = 1,2,3,\cdots$$

Using the semigroup property of system (4.2.81), (see (4.2.54) and (4.2.55)), we have

$$\begin{aligned}
y_{n+1} &= w(\cdot, nT+T)\\
&= T_0(T)\left(T_0(nT)w_0 - \int_0^{nT} T_0(nT-\tau)(2\partial_x(aw))(\tau)d\tau\right)\\
&\quad - T_0(nT)\int_0^T T_0(T-(\theta+nT))(2\partial_x(aw))(\theta+nT)d\theta.
\end{aligned} \tag{4.2.89}$$

Define

$$I_2 := -T_0(nT)\int_0^T T_0(T-(\theta+nT))(2\partial_x(aw))(\theta+nT)d\theta. \tag{4.2.90}$$

Note that

$$
\begin{aligned}
\|I_2\|_{H_0^s(\mathbb{T})} &\leqslant c \left\| \int_0^t T_0(t-(\theta+nT))(2\partial_x(a\cdot w))(\theta+nT)d\theta \right\|_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant C_2 \left\|a(\theta+nT)\right\|_{Z^T_{s,\frac{1}{2}}} \left\|w(\theta+nT)\right\|_{Z^T_{s,\frac{1}{2}}} \\
&\leqslant C_2 \left\|a\right\|_{Z^{[nT,(n+1)T]}_{s,\frac{1}{2}}} \left\|w\right\|_{Z^{[nT,(n+1)T]}_{s,\frac{1}{2}}} \\
&\leqslant C_2 \left\|a\right\|_{Z^{[nT,(n+1)T]}_{s,\frac{1}{2}}} \upsilon\left(\left\|a\right\|_{Z^{[nT,(n+1)T]}_{s,\frac{1}{2}}} \right) \|w(\cdot,nT)\|_{H_0^s(\mathbb{T})} \\
&\leqslant C_2\beta\upsilon(\beta)\|y_n\|_{H_0^s(\mathbb{T})}.
\end{aligned}
\tag{4.2.91}
$$

From (4.2.89)-(4.2.91), we get

$$
\begin{aligned}
\|y_{n+1}\|_{H_0^s(\mathbb{T})} &\leqslant \|T_0(T)y_n\|_{H_0^s(\mathbb{T})} + C_2\beta\upsilon(\beta)\|y_n\|_{H_0^s(\mathbb{T})} \\
&\leqslant \left(C_1e^{-kT} + C_2\beta\upsilon(\beta)\right)\|y_n\|_{H_0^s(\mathbb{T})}, \quad \text{for } n \geqslant 1.
\end{aligned}
$$

Since $k' \in (0,k)$, we note that for $T>0$ large enough $C_1e^{-kT} < e^{-k't}$ if an only if $\ln(C_1) < (k-k')T$. Therefore, choosing $T>0$ large enough and β small enough so that

$$
C_1e^{-kT} + C_2\beta\upsilon(\beta) = e^{-k't}, \tag{4.2.92}
$$

we get

$$
\|y_{n+1}\|_{H_0^s(\mathbb{T})} \leqslant e^{-k'T}\|y_n\|_{H_0^s(\mathbb{T})}, \quad \text{for } n \geqslant 1,
$$

as long as (4.2.86) holds. Thus,

$$
\|y_n\|_{H_0^s(\mathbb{T})} \leqslant e^{-nk'T}\|y_0\|_{H_0^s(\mathbb{T})}, \quad \text{for any } n \geqslant 1.
$$

This implies that (4.2.87) holds. The proof is complete. □

Theorem 4.2.11. *Let $\mu \in \mathbb{R}$, $s \geqslant 0$, and $\alpha > 0$ be given. Assume $\lambda = 0$ in* (4.2.1)*. Let $k_0 > 0$ be the infimum of the numbers γ, k given respectively in Theorem 2.4.2 and Theorem 4.2.8. Let $k' \in (0, k_0]$ be given. Then there exists a nondecreasing continuous function $\alpha_{s,\mu} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for any $u_0 \in H_0^s(\mathbb{T})$, the corresponding solution u of the IVP* (4.2.1) *(with $\lambda = 0$) satisfies*

$$
\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant \alpha_{s,\mu}(\|u_0\|_{H_0^s(\mathbb{T})})\, e^{-k't}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \textit{for all}\, t \geqslant 0. \tag{4.2.93}
$$

Proof. The result for $s=0$ has already been established in Theorem 4.2.8 with $k'=k$. Let us consider the case $s=3$. Pick any number $R_0>0$ and any $u_0 \in H_0^3(\mathbb{T})$ with $\|u_0\|_{L_0^2(\mathbb{T})} \leqslant R_0$. Let u denote the solution of (4.2.1) (with $\lambda = 0$) emanating from u_0 at $t=0$, and let $v = \partial_t u$. Then v solves

$$
\begin{cases}
\partial_t v - \partial_x^3 v - \alpha\mathcal{H}\partial_x^2 v + 2\mu\partial_x v + 2\partial_x(uv) = -K_0 v, & x \in \mathbb{T},\ t>0, \\
v(x,0) = v_0 = u_0''' + \alpha\mathcal{H}u_0'' - 2\mu u_0' - 2u_0u_0' - K_0u_0 \in L_0^2(\mathbb{T}), & x \in \mathbb{T}.
\end{cases}
\tag{4.2.94}
$$

From Theorem 4.2.2 and Theorem 4.2.8, we infer that for any $T > 0$ there exists a number $C = C_{R_0,T} > 0$ that depends only on R_0 and T such that

$$\begin{aligned}\|u\|_{Z_{0,\frac{1}{2}}^{[t,t+T]}} &\leqslant \beta_{T,0}\left(Ce^{-kt}\|u_0\|_{L_0^2(\mathbb{T})}\right)Ce^{-kt}\|u_0\|_{L_0^2(\mathbb{T})}\\ &\leqslant \beta_{T,0}\left(CR_0\right)Ce^{-kt}\|u_0\|_{L_0^2(\mathbb{T})}\\ &\leqslant C_{R_0,T}e^{-kt}\|u_0\|_{L_0^2(\mathbb{T})},\quad \text{for all } t \geqslant 0.\end{aligned}$$

Thus, for any $\epsilon > 0$, there exists $t^* > 0$ such that if $t \geqslant t^*$, one has

$$\|u\|_{Z_{0,\frac{1}{2}}^{[t,t+T]}} \leqslant \epsilon. \tag{4.2.95}$$

At this point we use the exponential stability result for the linearized system

$$\begin{cases} \partial_t w - \partial_x^3 w - \alpha\mathcal{H}\partial_x^2 w + 2\mu\partial_x w + 2\partial_x(uw) = -K_0 w, & x \in \mathbb{T},\ t > t^*,\\ w(x,0) = w_0(x) = v(t^*), & x \in \mathbb{T}, \end{cases} \tag{4.2.96}$$

where $u \in Z_{s,\frac{1}{2}}^{T} \cap L^2([0,T]; L_0^2(\mathbb{T}))$ is a given function and $w = v(t - t^*)$.

Choosing $\epsilon < \beta$ in (4.2.95), where β is given by (4.2.92), and applying Lemma 4.2.10 to system (4.2.96), we obtain

$$\|v(\cdot, t - t^*)\|_{L_0^2(\mathbb{T})} \leqslant Ce^{-k'(t-t^*)}\|v(\cdot, t^*)\|_{L_0^2(\mathbb{T})},\quad \text{for all } t \geqslant t^*,$$

or

$$\|v(\cdot, t)\|_{L_0^2(\mathbb{T})} \leqslant Ce^{-k't}\|v_0\|_{L_0^2(\mathbb{T})},\quad \text{for any } t \geqslant 0.$$

where $C > 0$ depends only on R_0. It follows from Theorem 4.2.8 and the equation

$$\partial_x^3 u = v - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + 2u\partial_x u + K_0 u$$

that

$$\|u(\cdot, t)\|_{H_0^3(\mathbb{T})} \leqslant Ce^{-k't}\|u_0\|_{H_0^3(\mathbb{T})},\quad \text{for any } t \geqslant 0.$$

where $C > 0$ depends only on R_0.

Thus the Theorem has been proved for $s = 0$ and $s = 3$. Using the same argument for $u_1 - u_2$ and $a = u_1 + u_2$ for two different solutions u_1 and u_2, we obtain the Lipchitz stability estimate needed for interpolation:

$$\|(u_1 - u_2)(\cdot, t)\|_{L_0^2(\mathbb{T})} \leqslant Ce^{-k't}\|(u_1 - u_2)(\cdot, 0)\|_{L_0^2(\mathbb{T})},\quad \text{for any } t \geqslant 0.$$

The case $0 < s < 3$ follows by an interpolation argument similar to those applied in Theorem 4.2.2. The other cases of s can be proved similarly. □

Corollary 4.2.12. *Let $s \geqslant 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$ be given. There exists a constant $\gamma > 0$ such that for any $u_0 \in H_p^s(\mathbb{T})$ with $[u_0] = \mu$, the corresponding solution $u \in C([0, +\infty), H_p^s(\mathbb{T}))$ of the closed-loop system (2.0.2) with $Ku = -GG^*u$, satisfies*

$$\|u(\cdot, t) - [u_0]\|_{H_p^s(\mathbb{T})} \leqslant \alpha_{s,\mu}(\|u_0 - [u_0]\|_{H_p^s(\mathbb{T})})\, e^{-\gamma t}\|u_0 - [u_0]\|_{H_p^s(\mathbb{T})},\quad \textit{for all } t \geqslant 0,$$

where $\alpha_{s,\mu} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function depending on s and μ.

4.3 Controllability of the Benjamin Equation

In this section we will study the controllability problem for the full Benjamin equation for large data in $H^s_p(\mathbb{T})$, with $s \geqslant 0$. This control result will be a combination of the stabilization result presented in Corollary 4.2.12 and the local control presented in Corollary 4.1.2, as is usual in control Theory (see for instance [29, 30, 51, 52, 53]).

Theorem 4.3.1. *(Large data control) Let $s \geqslant 0$, $\alpha > 0$, $\mu \in \mathbb{R}$, and $R > 0$ be given. Then there exists a time $T > 0$, such that for any $u_0, u_1 \in H^s_p(\mathbb{T})$ with $[u_0] = [u_1] = \mu$ and*

$$\|u_0\|_{H^s_p(\mathbb{T})} \leqslant R, \qquad \|u_1\|_{H^s_p(\mathbb{T})} \leqslant R,$$

one can find a control input $h \in L^2([0,T]; H^s_p(\mathbb{T}))$ such that the IVP (2.0.1) with $f = Gh$ admits a unique solution $u \in C([0,T]; H^s_p(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x), \quad u_1(x,T) = u_1(x), \quad \textit{for all } x \in \mathbb{T}.$$

Then, the system (2.0.1) is globally exactly controllable.

Proof. This result is a direct consequence of the local controllability Corollary 4.1.2 and the stabilization Corollary 4.2.12. Indeed, given the initial data u_0 to be controlled, by means of the damping term $Ku = -GG^*u$ supported in ω, i.e by solving the IVP (2.0.2), we drive it to a state close enough to the mean value μ in a sufficiently large time. We do the same with the final state u_1 by solving the system backwards in time, due to the time reversibility of the Benjamin equation. This produces two states which are close enough to μ so that the local controllability result applies. □

4.4 Stabilization of the Benjamin equation with an arbitrary decay rate

In this section we construct a continuous time-varying feedback law ensuring a semiglobal stabilization with an arbitrary large decay rate.

Let $\lambda > 0$, $\mu \in \mathbb{R}$, $\alpha > 0$, and $s \geqslant 0$ be given. According to Theorem 4.2.11, there exists $\kappa > 0$ and a nondecreasing continuous function $\alpha_s : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for any $u_0 \in H^s_0(\mathbb{T})$, the corresponding solution u of IVP

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha \mathcal{H} \partial_x^2 u + 2\mu \partial_x u + 2u\partial_x u = -GG^*u, & t > t_0,\ x \in \mathbb{T} \\ u(x,t_0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.4.1}$$

satisfies

$$\|u(\cdot,t)\|_{H^s_0(\mathbb{T})} \leqslant \alpha_{s,\mu}(\|u_0\|_{L^2_0(\mathbb{T})})\, e^{-\kappa(t-t_0)} \|u_0\|_{H^s_0(\mathbb{T})}, \quad \text{for all } t \geqslant t_0. \tag{4.4.2}$$

On the other hand, Theorem 4.2.4 asserts that for any fixed $\lambda' \in (0,\lambda)$ and any $u_0 \in H_0^s(\mathbb{T})$, the solution of IVP

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + 2u\partial_x u = -K_\lambda u, & t > t_0,\ x \in \mathbb{T} \\ u(x,t_0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{4.4.3}$$

fulfills

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant C_{s,\mu}\, e^{-\lambda'(t-t_0)}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \text{for all } t \geqslant t_0, \tag{4.4.4}$$

for some $C_{s,\mu} > 0$, provided that $\|u_0\|_s \leqslant r_0$ for some number $r_0 \in (0,1)$. Pick a function $\rho \in C^\infty(\mathbb{R}^+;[0,1])$ such that

$$\rho(r) = 1,\ \text{for } r \leqslant r_0, \qquad \rho(r) = 0,\ \text{for } r \geqslant 1. \tag{4.4.5}$$

Pick any function $\theta \in C^\infty(\mathbb{R};[0,1])$ holding the following properties:

$$\begin{cases} \theta(t+2) = \theta(t) & \text{for all} \quad t \in \mathbb{R}, \\ \theta(t) = 1 & \text{for} \quad \delta \leqslant 1 \leqslant 1-\delta, \\ \theta(t) = 0 & \text{for} \quad 1 \leqslant t \leqslant 2, \end{cases} \tag{4.4.6}$$

where $\delta \in (0, \frac{1}{10})$ is a number whose value will be specified later.

Let $T > 0$ be given. We define the following time-varying feedback law

$$\begin{aligned} K(u,t) &:= \rho(\|u\|^2_{H_0^s(\mathbb{T})})\left[\theta(\frac{t}{T})K_\lambda u + \theta(\frac{t}{T} - T)GG^* u\right] + (1-\rho(\|u\|^2_{H_0^s(\mathbb{T})}))GG^* u \\ &= GG^*\left\{\rho(\|u\|^2_{H_0^s(\mathbb{T})})\left[\theta(\frac{t}{T})L_\lambda^{-1} u + \theta(\frac{t}{T} - T)u\right] + (1-\rho(\|u\|^2_{H_0^s(\mathbb{T})}))u\right\}. \end{aligned} \tag{4.4.7}$$

Observe that K has the following behavior of the trajectories. In a first time, when $\|u\|_{H_0^s(\mathbb{T})}$ s is large, we choose $K = GG^*$ to guarantee the decay of the solution. Then, after a transient period, we have $\|u\|^2_{H_0^s(\mathbb{T})} \leqslant r_0$ and we get in an oscillatory regime. During each period of length $2T$, we have three steps:

- A period of time for which the damping K_λ is active, leading to a decay like $e^{-\lambda'(t-t_0)}$;
- A short transition time of order δ where a deviation from the origin may occur;
- A period of time for which the damping GG^* is active, leading to a decay like $e^{-\kappa(t-t_0)}$.

The expected decay is a "mean value" of the two decays above. We consider the system

$$\begin{cases} \partial_t u - \partial_x^3 u - \alpha\mathcal{H}\partial_x^2 u + 2\mu\partial_x u + 2u\partial_x u = -K(u,t), & t > t_0, \quad x \in \mathbb{T} \\ u(x,t_0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{4.4.8}$$

The following estimates are needed.

Lemma 4.4.1. *Let $s \geqslant 0$, $\alpha > 0$, $0 < T \leqslant 1$, and $\mu \in \mathbb{R}$ be given. For any $0 < \epsilon < 1$ there exists a positive constant C_ϵ such that*

$$\left\| \int_{t_0}^{t} U_\mu(t-\tau) K(u(\tau),\tau) d\tau \right\|_{Z^{[t_0,t_0+T]}_{s,\frac{1}{2}}} \leqslant C_\epsilon \|u\|_{Z^{[t_0,t_0+T]}_{s,\frac{1}{2}}}.$$

Proof. This is a direct consequence of Proposition 3.1.15, Remark 2.5.4, and estimate (3.1.27), by using similar arguments as those in the proof of Lemma 4.2.1 (see also Lemma 4.2 in [52]). □

Lemma 4.4.2. *Let $s \geqslant 0$ and $\lambda > 0$ be given. Then*

$$\|K(v_1,t) - K(v_2,t)\|_{H^s_0(\mathbb{T})} \leqslant C\|v_1 - v_2\|_{H^s_0(\mathbb{T})}, \quad \text{for any } v_1, v_2 \in H^s_0(\mathbb{T}),\ t \in \mathbb{R},$$

where C denotes a positive constant independent of v_1, v_2 and t.

Proof. This is consequence of the following identity

$$\begin{aligned} K(v_1,t) - K(v_2,t) &= \rho(\|v_1\|_s^2)\left[\theta(\frac{t}{T})K_\lambda(v_1 - v_2) + \theta(\frac{t}{T} - T)GG^*(v_1 - v_2)\right] \\ &\quad + \left(\rho(\|v_1\|_s^2) - \rho(\|v_2\|_s^2)\right)\left[\theta(\frac{t}{T})K_\lambda(v_2) + \theta(\frac{t}{T} - T)GG^*(v_2) - GG^*(v_1)\right] \\ &\quad + GG^*(v_1 - v_2) + \rho(\|v_2\|_s^2)GG^*(v_2 - v_1). \end{aligned}$$

□

Note that the involved constants in both Lemmas above, only depends on θ for its L^∞ norm and not on δ. The global well-posedness of IVP (4.4.8) in $H^s_0(\mathbb{T})$ with $s \geqslant 0$ is stated in the following theorem.

Theorem 4.4.3. *Let $s \geqslant 0$, $\lambda > 0$, $\alpha > 0$, and $\mu \in \mathbb{R}$, be given. For any pair $(u_0, t_0) \in H^s_0(\mathbb{T}) \times \mathbb{R}$ there exists a unique solution $u : \mathbb{T} \times [t_0, +\infty) \to \mathbb{R}$ of IVP (4.4.8) fulfilling*

$$u \in Z^{[t_0,t_0+T]}_{s,\frac{1}{2}} \cap L^2([t_0, t_0+T]; L^2_0(\mathbb{T})) \ \ \text{for all } T > 0.$$

Furthermore, u depends continuously on the initial data and if we denote $S(t)u_0$ the unique solution u of equation (4.4.8) corresponding to the initial data u_0, then the operator $S(t) : H^s_0(\mathbb{T}) \longrightarrow Z^{[t_0,t_0+T]}_{s,\frac{1}{2}}$ defined by $S(t)u_0 = u$ is continuous on $[t_0, t_0+T]$.

The following a priori *estimates hold true:*

$$\text{If } \|u_0\|_{H^s_0(\mathbb{T})} \leqslant 1, \quad \text{then } \|u(\cdot,t)\|_{H^s_0(\mathbb{T})} \leqslant \alpha_{s,\mu}(1) \ \ \text{for all } t \geqslant t_0; \tag{4.4.9}$$

$$\text{If } \|u_0\|_{H^s_0(\mathbb{T})} > 1, \quad \text{then } \|u(\cdot,t)\|_{H^s_0(\mathbb{T})} \leqslant \alpha_{s,\mu}(\|u_0\|_{L^2_0(\mathbb{T})})\|u_0\|_{H^s_0(\mathbb{T})} \ \ \text{for all } t \geqslant t_0; \tag{4.4.10}$$

$$\textit{If } \|u_0\|_{H_0^s(\mathbb{T})} \leqslant R, \textit{ then } \|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant K_s e^{d_s(t-t_0)} \|u_0\|_{H_0^s(\mathbb{T})} \textit{ for all } t \geqslant t_0, \tag{4.4.11}$$

where K_s and $d_s$denote some positive constants depending only on s and R.

Proof. First, we prove the local well-posedness in $H_0^s(\mathbb{T})$ with $s \geqslant 0$. Pick any pair $(u_0, t_0) \in H_0^s(\mathbb{T}) \times \mathbb{R}$ and define the map

$$\Gamma(v) = U_\mu(t-t_0)u_0 - \int_{t_0}^t U_\mu(t-\tau)(2v\partial_x v)(\tau)d\tau - \int_{t_0}^t U_\mu(t-\tau)K(v(\tau),\tau)d\tau.$$

Using Lemmas 4.4.1-4.4.2 and similar procedure as in the proof of Theorem 4.2.2 we infer that (4.2.10) and (4.2.11) hold for any $v, v_1, v_2 \in Z_{s,\frac{1}{2}}^{[t_0,t_0+T_1]} \cap L^2([t_0,t_0+T_1]; L_0^2(\mathbb{T}))$. Pick $M = 2C_1\|u_0\|_{H_0^s(\mathbb{T})}$, and $T_1 > 0$ such that

$$2C_2 M T_1^\theta + C_3 T_1^{1-\epsilon} \leqslant \frac{1}{2}.$$

Then the map Γ is a contraction in the closed ball $B(0,M)$ of $Z_{s,\frac{1}{2}}^{[t_0,t_0+T_1]} \cap L^2([t_0, t_0 + T_1]; L_0^2(\mathbb{T}))$ for the $\|\cdot\|_{Z^T_{s,\frac{1}{2}}}$ norm. Its unique fixed point u is the desired solution of (4.4.8). It follows from the Proposition 3.1.8 that $u \in C([t_0, t_0+T_1]; H_0^s(\mathbb{T}))$ and u satisfies

$$\|u\|_{L^\infty([t_0,t_0+T_1];H_0^s(\mathbb{T}))} \leqslant C_4 \|u\|_{Z_{s,\frac{1}{2}}^{[t_0,t_0+T_1]}} \leqslant 2C_1C_4\|u_0\|_{H_0^s(\mathbb{T})}. \tag{4.4.12}$$

Procedding as for Theorem 4.2.2, we check that the IVP (4.4.8) is globally well-posed in H_0^s with $s \geqslant 0$.

On the other hand, (4.4.4) yields (4.4.9) and (4.4.10). It remains to prove (4.4.11). Assume $\|u_0\|_{H_0^s(\mathbb{T})} \leqslant R$ and let $M' = 2C_1 \max\left\{\alpha_s(1), \alpha_s(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^s(\mathbb{T})}\right\}$. Note that M' depends only on R and s. Replacing T_1 by T' satisfying

$$2C_2 M' T'^\theta + C_3 T'^{1-\epsilon} \leqslant \frac{1}{2},$$

in the application of the contraction mapping principle, we infer that the (unique) solution u of (4.4.8) fulfills

$$\|u\|_{L^\infty([t_0+kT',t_0+(k+1)T'];H_0^s(\mathbb{T}))} \leqslant C_4\|u\|_{Z_{s,\frac{1}{2}}^{[t_0+kT',t_0+(k+1)T']}} \leqslant 2C_1C_4\|u(\cdot,t_0+kT')\|_{H_0^s(\mathbb{T})},$$

for any $k \in \mathbb{N}$. Therefore, $\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant K_s e^{d_s(t-t_0)}\|u_0\|_{H_0^s(\mathbb{T})}$ for all $t \geqslant t_0$, for some constants $K_s > 0$ and $d_s > 0$ depending only on s and R. $\square$

Finally, we prove that a semiglobal stabilization with an arbitrary decay rate can be obtained.

Theorem 4.4.4. *Let $s \geqslant 0$, $\lambda > 0$, $\alpha > 0$ and $\mu \in \mathbb{R}$ be given. Pick any $\lambda' \in (0, \lambda)$ and any $\lambda'' \in \left(\frac{\lambda'}{2}, \frac{\lambda' + \kappa}{2}\right)$ where κ is given in (4.4.2). Then there exists a time $T_0 > 0$ such that for $T > T_0$, $t_0 \in \mathbb{R}$ and $u_0 \in H_0^s(\mathbb{T})$, the unique solution of the closed-loop system (4.4.8) satisfies*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant \gamma_{s,\mu}(\|u_0\|_{H_0^s(\mathbb{T})})\, e^{-\lambda''(t-t_0)}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \textit{for all } t \geqslant t_0, \tag{4.4.13}$$

where $\gamma_{s,\mu}$ is a nondecreasing continuous function.

Proof. For λ'' under the above assumptions, we choose some $\delta > 0$ small enough such that

$$\delta d_s - (1-2\delta)\kappa < 0, \tag{4.4.14}$$

and

$$\lambda'' < -2\delta d_s + (1-2\delta)\frac{(\kappa + \lambda')}{2}. \tag{4.4.15}$$

We define $0 < \xi := \min\left\{\frac{r_0}{K_s^2 e^{d_s(1+2\delta)T}\alpha_s(\alpha_s(1))}, \frac{r_0}{\alpha_s(\alpha_s(1))K_s e^{d_s(1+2\delta)T}}, \frac{r_0}{K_0 e^{d_s(1+2\delta)T}}\right\}$, and choose $r_1 \in (0, r_0)$ such that

$$r_1 < \xi. \tag{4.4.16}$$

From (4.4.14) and (4.4.15) we infer that there exists $T_0 > 0$ large enough such that

$$\alpha_s(1)\alpha_s(\alpha_s(1))K_s e^{[\delta d_s - (1-2\delta)\kappa]T} \leqslant r_1, \tag{4.4.17}$$

$$\alpha_s(1)C_s K_s^4 e^{[4\delta d_s - (1-2\delta)(\kappa+\lambda')]T} \leqslant e^{-2\lambda'' T}, \tag{4.4.18}$$

for all $T \geqslant T_0$. Pick any pair $(u_0, t_0) \in H_0^s(\mathbb{T}) \times \mathbb{R}$. The proof is completed by showing the following claims.

Claim 1. There exists a time $t_1 \in [t_0, t_0 + \kappa^{-1}\ln(\alpha_s(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^s(\mathbb{T})})]$ such that

$$\|u(t_1)\|_{H_0^s(\mathbb{T})} \leqslant 1. \tag{4.4.19}$$

Proof. If $\|u(t_0)\|_{H_0^s(\mathbb{T})} \leqslant 1$, then (4.4.19) follows immediately with $t_1 = t_0$. On the other hand, if $\|u(t_0)\|_{H_0^s(\mathbb{T})} > 1$, then the dynamics of u is governed by (4.4.1) as long as $\|u(t)\|_{H_0^s(\mathbb{T})} \geqslant 1$. From (4.4.2), we infer that (4.4.19) holds for $t_1 = t_0 + \kappa^{-1}\ln(\alpha_s(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^s(\mathbb{T})})$. □

Claim 2. There exists a time $t_2 \in 2\mathbb{Z}T \cap [t_1, t_1 + 3T]$ such that

$$\|u(t_2)\|_{H_0^s(\mathbb{T})} \leqslant r_1. \tag{4.4.20}$$

Proof. From (4.4.19) and (4.4.9), we obtain that $\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant \alpha_s(1)$ for all $t \geqslant t_1$. Pick $R = \alpha_s(1)$ and let K_s and d_s be as given in Theorem 4.4.3 for that choice of R. Let $t_1' \geqslant t_1$ denote the first time of the form $t_1' = (2k+1)T + \delta T$ with $k \in \mathbb{Z}$, and let $t_2 = (2k+2)T$. Using (4.4.11), (4.4.2) and (4.4.17), we obtain that

$$\begin{aligned}\|u(t_2)\|_{H_0^s(\mathbb{T})} &\leqslant K_s e^{d_s \delta T} \|u((2k+2)T - \delta T)\|_{H_0^s(\mathbb{T})} \\ &\leqslant K_s e^{d_s \delta T} \alpha_s(\|u(t_1')\|_{L_0^2(\mathbb{T})}) e^{-\kappa(1-2\delta)T} \|u(t_1')\|_{H_0^s(\mathbb{T})} \\ &\leqslant \alpha_s(1)\alpha_s(\alpha_s(1)) K_s e^{[\delta d_s - (1-2\delta)\kappa]T} \\ &\leqslant r_1.\end{aligned}$$

□

Claim 3. $\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant r_0$ for all $t \geqslant t_2$ and $\|u(t_2 + 2kT)\|_{H_0^s(\mathbb{T})} \leqslant e^{-2k\lambda'' T} \|u(t_2)\|_{H_0^s(\mathbb{T})}$, for all $k \in \mathbb{N}$.

Proof. Note that, the dynamics of the solution u is governed by (4.4.3) (resp. (4.4.1)) when $t \in (t_2 + \delta T, t_2 + (1-\delta)T)$ (resp. $t \in (t_2 + (1-\delta)T, t_2 + (2-\delta)T)$) as long as $\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant r_0$.

Let $t \in [t_2, t_2 + 2T]$. We analyze three cases.

i) If $t \in (t_2 + 2T - \delta T, t_2 + 2T]$, then (4.4.11) and (4.4.2) yield

$$\begin{aligned}\|u(t)\|_{H_0^s(\mathbb{T})} &\leqslant K_s e^{d_s \delta T} \|u(t_2 + 2T - \delta T)\|_{H_0^s(\mathbb{T})} \\ &\leqslant K_s e^{d_s \delta T} \alpha_s(\alpha_s(1)) \|u(t_2 + T + \delta T)\|_{H_0^s(\mathbb{T})} \\ &\leqslant \alpha_s(\alpha_s(1)) K_s^2 e^{d_s(1+2\delta)T} \|u(t_2)\|_{H_0^s(\mathbb{T})} \\ &\leqslant \alpha_s(\alpha_s(1)) K_s^2 e^{d_s(1+2\delta)T} r_1 \\ &\leqslant r_0,\end{aligned}$$

where we used (4.4.16) in the last inequality.

ii) If $t \in (t_2 + T + \delta T, t_2 + 2T - \delta T]$, then (4.4.11) and (4.4.2) yield

$$\begin{aligned}\|u(t)\|_{H_0^s(\mathbb{T})} &\leqslant \alpha_s(\alpha_s(1)) \|u(t_2 + T + \delta T)\|_{H_0^s(\mathbb{T})} \\ &\leqslant \alpha_s(\alpha_s(1)) K_s e^{d_s(1+\delta)T} \|u(t_2)\|_{H_0^s(\mathbb{T})} \\ &\leqslant \alpha_s(\alpha_s(1)) K_s e^{d_s(1+2\delta)T} r_1 \\ &\leqslant r_0,\end{aligned}$$

where we used (4.4.16) in the last inequality.

iii) If $t \in (t_2, t_2 + T + \delta T]$, then (4.4.11) yields

$$\begin{aligned}\|u(t)\|_{H_0^s(\mathbb{T})} &\leqslant K_s e^{d_s(t-t_2)} \|u(t_2)\|_{H_0^s(\mathbb{T})} \\ &\leqslant K_s e^{d_s(1+2\delta)T} r_1 \\ &\leqslant r_0,\end{aligned}$$

where we used (4.4.16) in the last inequality.

Hence, $\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant r_0$ for all $t \in [t_2, t_2 + 2T]$. On the other hand, from (4.4.18) we infer

$$\begin{aligned}
\|u(t_2+2T)\|_{H_0^s(\mathbb{T})} &\leqslant K_s e^{d_s\delta T}\|u(t_2+2T-\delta T)\|_{H_0^s(\mathbb{T})}\\
&\leqslant K_s e^{d_s\delta T}\alpha_s(1)e^{-\kappa(1-2\delta)T}\|u(t_2+T+\delta T)\|_{H_0^s(\mathbb{T})}\\
&\leqslant \alpha_s(1)K_s^2 e^{-\kappa(1-2\delta)T}e^{d_s 2\delta T}\|u(t_2+T)\|_{H_0^s(\mathbb{T})}\\
&\leqslant \left(\alpha_s(1)K_s^2 e^{-\kappa(1-2\delta)T}e^{d_s 2\delta T}\right) K_s e^{d_s\delta T}\|u(t_2+T-\delta T)\|_{H_0^s(\mathbb{T})}\\
&\leqslant \left(\alpha_s(1)K_s^2 e^{-\kappa(1-2\delta)T}e^{d_s 2\delta T}\right) K_s e^{d_s\delta T}C_s e^{-\lambda'(1-2\delta)T}\|u(t_2+\delta T)\|_{H_0^s(\mathbb{T})}\\
&\leqslant \left(\alpha_s(1)K_s^2 e^{-\kappa(1-2\delta)T}e^{d_s 2\delta T}\right)\left(K_s^2 e^{d_s 2\delta T}C_s e^{-\lambda'(1-2\delta)T}\right)\|u(t_2)\|_{H_0^s(\mathbb{T})}\\
&\leqslant e^{-2\lambda'' T}\|u(t_2)\|_{H_0^s(\mathbb{T})}\\
&\leqslant r_1.
\end{aligned}$$

The Claim 3 follows by induction. □

It follows from Claim 3 that

$$\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant Ce^{-\lambda''(t-t_2)}\|u(t_2)\|_{H_0^s(\mathbb{T})} \text{ for all } t \geqslant t_2,$$

for some constant $C > 0$ independent of t and u_0. Therefore, using that

$$t_2 - t_0 < 3T + \kappa^{-1}\ln(\alpha_s(\|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^s(\mathbb{T})}),$$

we have

$$\begin{aligned}
\|u(t)\|_{H_0^s(\mathbb{T})} &\leqslant Ce^{-\lambda''(t-t_0)+\lambda''(t_2-t_0)}K_s e^{d_s(t_2-t_0)}\|u(t_0)\|_{H_0^s(\mathbb{T})}\\
&\leqslant Ce^{(\lambda''+d_s)(t_2-t_0)}K_s e^{-\lambda''(t-t_0)}\|u(t_0)\|_{H_0^s(\mathbb{T})}\\
&\leqslant Ce^{(\lambda''+d_s)3T}\left[\alpha_s(\|u_0\|_{H_0^s(\mathbb{T})})\|u_0\|_{H_0^s(\mathbb{T})}\right]K_s e^{-\lambda''(t-t_0)}\|u(t_0)\|_{H_0^s(\mathbb{T})}.
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 4.4.5. *Let $s \geqslant 0$, $\alpha > 0$, $\lambda > 0$, and $\mu \in \mathbb{R}$ be given. Then there exists a continuous map $Q_\lambda : H_p^s(\mathbb{T}) \times \mathbb{R} \to H_p^s(\mathbb{T})$ which is periodic in the second variable, and such that for any $u_0 \in H_p^s(\mathbb{T})$ with $[u_0] = \mu$, the unique solution $u \in C([0, +\infty), H_p^s(\mathbb{T}))$ of the closed-loop system (2.0.2) with $Ku = -GQ_\lambda(u, t)$ satisfies*

$$\|u(\cdot,t) - [u_0]\|_{H_p^s(\mathbb{T})} \leqslant \gamma_{s,\lambda,\mu}(\|u_0 - [u_0]\|_{H_p^s(\mathbb{T})})\, e^{-\lambda t}\|u_0 - [u_0]\|_{H_p^s(\mathbb{T})}, \quad \textit{for all } t \geqslant 0,$$

where $\gamma_{s,\lambda,\mu} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function depending on s, λ and μ.

Chapter 5

Controllability and stabilization of the Intermediate Long Wave equation on a periodic domain

In this chapter we study the controllability and stabilization problems for the Intermediate Long Wave (ILW) Equation in the Sobolev space $H_0^s(\mathbb{T})$ with $s \geqslant 0$. Specifically, we will be mainly interested in the following two problems for equation (0.0.14).

1) ***Exact control problem:*** Given an initial state u_0 and a terminal state u_1 in a certain space with $[u_0] = [u_1] = 0$, can one find an appropriate control input f given by (2.0.9) so that the equation

$$\partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = f(x,t), \quad x \in \mathbb{T}, \ \ t \in \mathbb{R}, \tag{5.0.1}$$

admits a solution u such that $u(x,0) = u_0(x)$ and $u(x,T) = u_1(x)$ for all $x \in \mathbb{T}$ and any final time $T > 0$?

2) ***Stabilization Problem:*** Given u_0 in a certain space. Can one find a feedback control law $f = Fu$ so that the resulting closed-loop system

$$\partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = Fu, \ \ u(x,0) = u_0, \ \ x \in \mathbb{T}, \ \ t \in \mathbb{R}^+ \tag{5.0.2}$$

is asymptotically stable as $t \to \infty$?

Recall that the operator $\mathcal{T}$ is defined in (0.0.15).

In sequel, we summarize the main results obtained in this chapter. As usual, the first main results deals with the controllability and stabilization of linearized ILW equation.

Theorem 5.0.1. *Let $s \geqslant 0$, $\delta > 0$, and $T > 0$ be given. Then for each $u_0,\ u_1 \in H_0^s(\mathbb{T})$ with $[u_0] = [u_1] = 0$, there exists a function $h \in L^2([0,T]; H_0^s(\mathbb{T}))$ such that the unique solution $u \in C([0,T]; H_0^s(\mathbb{T}))$ of the linearized non homogeneous system associated to (5.0.1) with $f(x,t) = G(h)(x,t)$ (see (2.0.9)) satisfies $u(x,T) = u_1(x)$, $x \in \mathbb{T}$. Moreover, there exists a positive constant $\nu \equiv \nu(s,g,T) > 0$ such that*

$$\|h\|_{L^2([0,T];H_0^s(0,2\pi))} \leqslant \nu \left(\|u_0\|_{H_0^s(0,2\pi)} + \|u_1\|_{H_0^s(0,2\pi)}\right).$$

Regarding stabilization of of linearized system, we prove the following results.

Theorem 5.0.2. *Let $\delta > 0$, g as in (2.0.8), and $s \geqslant 0$ be given. There exist positive constants $M = M(\delta, g, s)$ and $\gamma = \gamma(g)$ such that for any $u_0 \in H_0^s(\mathbb{T})$ the unique solution $u \in C([0,\infty); H_0^s(\mathbb{T}))$ of the linearized closed-loop system associated to (5.0.2) with $Fu = -GG^*u$ (see (2.0.9)) satisfies*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant Me^{-\gamma t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \textit{for all}\ t \geqslant 0.$$

Furthermore, using an observability inequality derived from the exact controllability result we can prove that the exponential decay rate of the resulting linearized closed-loop system is as large as one desires. This is stated in the following theorem.

Theorem 5.0.3. *Let $s \geqslant 0$, $\delta > 0$, $\lambda > 0$, and $u_0 \in H_0^s(\mathbb{T})$ be given. There exists a bounded linear operator F_λ from $H_0^s(\mathbb{T})$ to $H_0^s(\mathbb{T})$ such that the unique solution $u \in C([0,+\infty), H_0^s(\mathbb{T}))$ of the linearized closed-loop system associated to (5.0.2) with $Fu = F_\lambda u$ satisfies*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant M\ e^{-\lambda\, t}\|u_0\|_{H_0^s(\mathbb{T})},$$

for all $t \geqslant 0$, and some positive constant $M = M(g,\lambda,\delta,s)$.

This theorem implies that for any given number $\lambda > 0$ we can design a linear feedback control law such that the exponential decay rate of the resulting linearized closed-loop system is λ.

Next, we deal with the control and stabilization problem for the full ILW equation. To stabilize the ILW equation, we consider the feedback law

$$f = -G(D(Gu)),$$

where for given $r \in \mathbb{R}$, we define an operator $D^r : \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$ by

$$\widehat{D^r v}(n) = |n|^r \widehat{v}(n), \quad \forall n \in \mathbb{Z}. \tag{5.0.3}$$

Scaling in (5.0.2) by u gives (at least formally)

$$\frac{1}{2}\|u(T)\|^2_{L_0^2(\mathbb{T})} + \int_0^T \|D^{\frac{1}{2}}(Gu)\|^2_{L_0^2(\mathbb{T})}dt = \frac{1}{2}\|u_0\|^2_{L_0^2(\mathbb{T})}. \tag{5.0.4}$$

This suggest that the energy is dissipated over time. On the other hand, (5.0.4) reveals a smoothing effect, at least in the region $\omega \subset \mathbb{T}$. Using a propagation of regularity property in the same vein as in [30, 52, 51, 50, 68], we shall prove that the smoothing effect holds everywhere, i.e.

$$\|u\|_{L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T}))} \leqslant C(T, \|u_0\|_{L_0^2(\mathbb{T})}). \tag{5.0.5}$$

Using this smoothing effect and the classical compactness/uniqueness argument, we shall first prove that the corresponding closed-loop equation is semi-globally exponentially stable.

Theorem 5.0.4. *Let $\delta > 0$, and $R > 0$ be given. Then there exist some constants $C = C(R)$ and $\lambda = \lambda(R)$ such that for any $u_0 \in L_0^2(\mathbb{T})$ with $\|u_0\|_{L_0^2(\mathbb{T})} \leqslant R$, the weak solutions in the sense of vanishing viscosity of system (5.0.2) with $Fu = -GDGu$ (see (2.0.9)) satisfy*

$$\|u(\cdot,t)\|_{L_0^2(\mathbb{T})} \leqslant C\, e^{-\lambda\, t}\|u_0\|_{L_0^2(\mathbb{T})},$$

for all $t \geqslant 0$.

A weak solution of (5.0.2) with $Fu = -GDGu$ in the sense of vanishing viscosity is a distributional solution of (5.0.2) $u \in C_w((0,+\infty), L_0^2(\mathbb{T})) \cap L^2_{loc}((0,+\infty), H_0^{\frac{1}{2}}(\mathbb{T}))$ that may be obtained as a weak limit in a certain space of solutions of the ILW equation with viscosity

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) - \varepsilon\partial_x^2 u + \partial_x(u^2) = -GDGu, & t \geqslant 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.0.6}$$

as $\varepsilon \longrightarrow 0^+$ (see Definition 5.2.11 below for a precise definition).

Also, we use the smoothing effect (5.0.5) to extend (at least locally) the exponential stability from $L_0^2(\mathbb{T})$ to $H_0^s(\mathbb{T})$ for $s > \dfrac{1}{2}$.

Theorem 5.0.5. *Let $s \in \left(\dfrac{1}{2}, 2\right]$ and $\delta > 0$ be given. Then there exists $\rho > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $\|u_0\|_{H_0^s(\mathbb{T})} < \rho$, there exists for all $T > 0$ a unique solution $u(t)$ of (5.0.2) with $Fu = -GDGu$ in the class $u \in C([0,T]; H_0^s(\mathbb{T})) \cap L^2(0,T; H_0^{s+\frac{1}{2}}(\mathbb{T}))$. Furthermore, there exist some constants $C > 0$ and $\lambda > 0$ such that*

$$\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant Ce^{-\lambda t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \forall t \geqslant 0.$$

Finally, one can derive an exact controllability result for the ILW equation by incorporating the same feedback law $f = -G(D(Gu))$ (see (2.0.9)) in the control input to obtain a smoothing effect.

Theorem 5.0.6. *Let $s \in \left(\frac{1}{2}, 2\right]$, $\delta > 0$ and $T > 0$ be given. Then there exists $\rho > 0$ such that for any $u_0, u_1 \in H_0^s(\mathbb{T})$ satisfying*

$$\|u_0\|_{H_0^s(\mathbb{T})} < \rho, \quad \|u_1\|_{H_0^s(\mathbb{T})} < \rho$$

one can find a control input $f = Gh \in L^2(0,T; H^{s-\frac{1}{2}}(\mathbb{T}))$ with $h = -DGu + D^{\frac{1}{2}}\tilde{h}$ such that the system (5.0.1) admits a solution $u \in C([0,T]; H_0^s(\mathbb{T})) \cap L^2(0,T; H_0^{s+\frac{1}{2}}(\mathbb{T}))$ satisfying

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x).$$

These results will be proved throughout this chapter which is organized as follows: In Section 5.1, the well-posedness, the exact controllability and stabilizability results are presented for the associated linear systems of (5.0.1) and (5.0.2). Finally, the main results regarding stabilization and controllability are respectively proved in Sections 5.2 and 5.3.

5.1 Linear Systems

Initially, we consider the IVP to the associated open-loop control system in the periodic setting,

$$\begin{cases} \partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) = G(h), & t \in \mathbb{R}, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.1.1}$$

where the operator G is defined in (2.0.9) and h is the applied control function. Following the same approach appearing in [1] §7, we observe that

$$\begin{aligned} \left(\frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u)\right)^{\wedge}(n) &= -\left(\frac{|n|}{n}n - |n| + n\coth(n\delta) - \frac{1}{\delta}\right)\widehat{\partial_x u}(n) \\ &= -\widehat{\mathcal{H}(\partial_x^2 u)}(n) + \widehat{K(\partial_x u)}(n), \quad \forall\, n \in \mathbb{Z}^*, \end{aligned} \tag{5.1.2}$$

where $\mathcal{H}$ is the Hilbert transform (see Section 1.4) and K is a Fourier multiplier operator with symbol a given by

$$a(n) = |n| - n\coth(n\delta) + \frac{1}{\delta}, \tag{5.1.3}$$

and for all n, $0 \leqslant a(n) \leqslant \frac{1}{\delta}$. Furthermore, the operator K is self-adjoint of order 0 and so linear and bounded on all the L_0^2-based Sobolev spaces $H_0^s(\mathbb{T})$ (see (1.3.1)). Specifically,

$$\|K\varphi\|_{H_0^s(\mathbb{T})} \leqslant \frac{1}{\delta}\|\varphi\|_{H_0^s(\mathbb{T})}, \quad \forall \varphi \in H_0^s(\mathbb{T}). \tag{5.1.4}$$

Hence, the IVP (5.1.1) can be written in the form

$$\begin{cases} \partial_t u - \mathcal{H}(\partial_x^2 u) + K(\partial_x u) = G(h), & t \in \mathbb{R}, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{5.1.5}$$

Let $A : D(A) \subseteq L^2_0(\mathbb{T}) \longrightarrow L^2_0(\mathbb{T})$ be the operator

$$Aw := \mathcal{H}(\partial_x^2 w) - K(\partial_x w), \tag{5.1.6}$$

with its domain $D(A) = H^2_0(\mathbb{T})$. Following the line of argument appearing in Section 2.2, we prove that A is sknew-adjoint. Therefore, A generates a strongly continuous unitary group of isometries (contractions) $\{U(t)\}_{t\in\mathbb{R}}$ on $L^2_0(\mathbb{T})$ (see Theorem 1.5.16 and [16, Definition 3.4.6]); the eigenfucntions are the orthonormal Fourier basis functions $\{\psi_n\}_{n\in\mathbb{Z}^*}$ in $L^2_0(\mathbb{T})$, given by Remark 1.2.2. The corresponding eigenvalue of ψ_n is

$$\lambda_n = in\left(n\coth(n\delta) - \frac{1}{\delta}\right) = in\left(\operatorname{sgn}(n)n - \frac{1}{\delta}\right) - in\left(|n| - n\coth(n\delta)\right),\ \forall n \in \mathbb{Z}^*. \tag{5.1.7}$$

Note that

$$|n(|n| - n\coth(n\delta))| \leqslant c_\delta |n|^2 e^{-|n|\delta},\ \forall n \in \mathbb{Z}^*, \tag{5.1.8}$$

for some $c_\delta > 0$. Furthermore, if we define the family of operators $U : \mathbb{R} \to \mathcal{L}(H^s_0(\mathbb{T}))$ by

$$t \to U(t)\varphi := e^{(\mathcal{H}(\partial_x^2 w) - K(\partial_x w))t}\varphi = (e^{\lambda_n t}\widehat{\varphi}(n))^\vee, \tag{5.1.9}$$

we have that $\{U(t)\}_{t\in\mathbb{R}}$ defines a strongly continuous one-parameter unitary group of contractions on $H^s_0(\mathbb{T})$, for all $s \in \mathbb{R}$. In addition, $A^* = -A$, $U^*(-t) = U(t)$, and $U(t)$ is an isometry for all $t \in \mathbb{R}$ (see [16, page 38]). From this we infer that the system (5.1.1) is well-posed on $H^s_0(\mathbb{T})$ for $s \geqslant 0$. This is stated in the following lemma.

Lemma 5.1.1. *Let $0 \leqslant T < \infty$, $s \geqslant 0$, $u_0 \in H^s_0(\mathbb{T})$, and $h \in L^2([0,T]; H^s_0(\mathbb{T}))$. Then, there exists a unique mild solution $u \in C([0,T], H^s_0(\mathbb{T}))$ for the IVP (5.1.1) and we can write u in the form*

$$u(t) = U(t)u_0 + \int_0^t U(t-t')Gh(t')dt',$$

for $s \geqslant 0,\ 0 \leqslant t \leqslant T < \infty$.

Remark 5.1.2. *The sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}^*}$, given by (5.1.7) satisfies the following properties:*

i) $\lambda_{-n} = -\lambda_n$, *for all* $n \in \mathbb{Z}^*$.

ii) $\lim\limits_{|n|\to\infty} |\lambda_n| = \infty$.

iii) $\lim\limits_{|n|\to\infty} |\lambda_{n+1} - \lambda_n| = \infty$ *(asymptotic gap condition). This is consequence of* (5.1.7), (5.1.8), *and* (34) *in [55].*

iv) Observe that all the eigenvalues of the sequence $\{\lambda_n\}_{n\in\mathbb{Z}^*}$ are distinct, independently on the value of $\delta > 0$. In fact, For each $n_1 \in \mathbb{Z}^*$ set $I(n_1) := \{n \in \mathbb{Z}^* : \lambda_n = \lambda_{n_1}\}$ and $|I(n_1)| =: m(n_1)$, where $|I(n_1)|$ denotes the numbers of elements of $I(n_1)$. Then we have $m(n_1) = 1$, for all $n_1 \in \mathbb{Z}^*$. This is a consequence of the fact

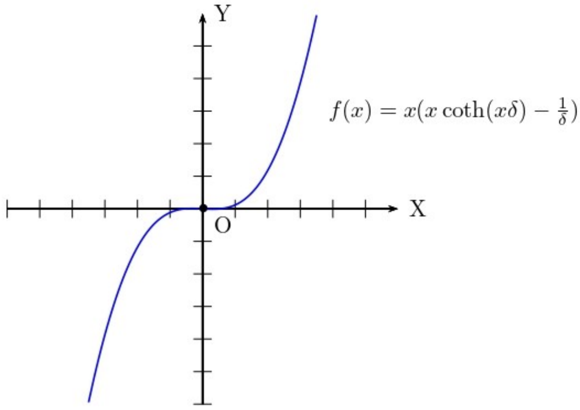

Figure 5 – Curve

that the function $x \to x\left(x\coth(x\delta) - \dfrac{1}{\delta}\right)$ *is strictly increasing for* $x \in \mathbb{R}$*. see the format of the curve in Figure 5 below.*

From Remark 5.1.2 and Ingham lemma (see Theorem 1.6.10), we deduce that the system (5.1.1) is exactly controllable in $H_0^s(\mathbb{T})$ for any $s \geqslant 0$ in small time (see the proof of Theorem 2.3.7). This proves Theorem 5.0.1.

Remark 5.1.3. *This result is similar with the ones obtained for the linearized KdV, BO, and Benjamin equations. Theorem 5.0.1 is strong from the point of view that we do not make restrictions on the time T. The so-called "asymptotic gap condition" (see condition iii) of Remark 5.1.2) that holds for the eigenvalues associated to ILW equation was crucial to obtain the exact controllability for any positive time T.*

Theorem 5.0.1 allows us to get the following corollary.

Corollary 5.1.4. *For $s \geqslant 0$, and $T > 0$ given, there exists a unique bounded linear operator $\Phi : H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T}) \to L^2([0,T]; H_0^s(\mathbb{T}))$ such that for any $u_0,\ u_1 \in H_0^s(\mathbb{T})$,*

$$u_1 = U(T)u_0 + \int_0^T U(T-s)(G(\Phi(u_0,u_1)))(\cdot,s)\,ds, \tag{5.1.10}$$

and

$$\|\Phi(u_0,u_1)\|_{L^2([0,T];H_0^s(\mathbb{T}))} \leqslant \nu\left(\|u_0\|_{H_0^s(\mathbb{T})} + \|u_1\|_{H_0^s(\mathbb{T})}\right), \tag{5.1.11}$$

where ν depends only on s, T, and g (see (2.0.8)).

The last Corollary 5.1.4 allows us to get the following observability inequality, which is fundamental to obtain a result on exponential asymptotic stabilization with decay rate as large as one desires for the linearized closed-loop system associated to (5.0.2).

Corollary 5.1.5. *Let $T > 0$ be given. There exists $\delta > 0$ such that*

$$\int_0^T \|GU(-\tau)\phi\|^2_{L^2_0(\mathbb{T})}(\tau)\, d\tau \geqslant \delta^2 \|\phi\|^2_{L^2_0(\mathbb{T})}, \quad \textit{for any } \phi \in L^2_0(\mathbb{T}).$$

Proof. Similar to the proof of Lemma 2.3.3. □

On the other hand, if one chooses the simple feedback law

$$h(u) = -G^* u,$$

the resulting closed-loop system (5.1.1) is exponentially asymptotically stable when t goes to infinity. First, we prove that the system (5.1.1), with $h(u) = -G^*u$ is globally well-posed in $H^s_0(\mathbb{T})$, $s \geqslant 0$.

Theorem 5.1.6. *Let $u_0 \in H^2_0(\mathbb{T})$, then the IVP (5.1.1), with $h(u) = -G^*u$ has a unique solution*

$$u \in C([0, \infty); H^2_0(\mathbb{T})) \cap C^1([0, \infty); L^2_0(\mathbb{T})).$$

Moreover, if $u_0 \in H^s_0(\mathbb{T})$, then we have that $u \in C([0, \infty); H^s_0(\mathbb{T}))$, for all $s \geqslant 0$.

Proof. We know that the operator $A = \dfrac{1}{\delta}\partial_x + \partial_x^2 \mathcal{T} = \mathcal{H}\partial_x^2 - K\partial_x$ is an infinitesimal generator of a C_0-semigroup $\{U(t)\}_{t\geqslant 0}$ over $H^s_0(\mathbb{T})$. Also we know that $-GG^*$ is a bounded linear operator on $H^s_0(\mathbb{T})$. From the semigroup theory (see Pazy [71, Theorem 1.1, pag. 76]), we get that the operator $A - GG^*$, which is a perturbation of A by a bounded linear operator, is an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ on $H^s_0(\mathbb{T})$. □

The following proposition proves Theorem 5.0.2. Its proof is similar to the proof of Theorem 2.4.2.

Proposition 5.1.7. *Let $\delta > 0$, g as in (2.0.8), and $s \geqslant 0$ be given. Then, there exist positive constants $M = M(\delta, g, s)$ and $\gamma = \gamma(g)$, such that for any $u_0 \in H^s_0(\mathbb{T})$ the unique solution $u \in C([0, \infty); H^s_0(\mathbb{T}))$ of the closed-loop system (5.1.1) with $h(u) = -G^*u$ satisfies*

$$\|u(\cdot, t)\|_{H^s_0(\mathbb{T})} \leqslant M e^{-\gamma t} \|u_0\|_{H^s_0(\mathbb{T})}, \quad \textit{for all } t \geqslant 0.$$

Proof. First we prove the case $s = 0$. In this case we use a procedure similar to [67, 55, 80]. Let $T > 0$ be given and assume $u_0 \in H_0^2(\mathbb{T})$. Theorem 5.1.6 implies that the solution u of the IVP

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) = -GG^*u, & t > 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.1.12}$$

satisfies $u \in C([0,\infty); H_0^2(\mathbb{T})) \cap C^1([0,\infty); L_0^2(\mathbb{T}))$. It means $u(\cdot,t) \in H_0^2(\mathbb{T})$, for all $t \geqslant 0$ and in particular, for $t = T$. Now we consider the IVP

$$\begin{cases} \partial_t w + \dfrac{1}{\delta}\partial_x w + \partial_x^2(\mathcal{T}w) = Gh, & t \in (0,T), \quad x \in \mathbb{T} \\ w(x,0) = 0, & x \in \mathbb{T}. \end{cases} \tag{5.1.13}$$

Theorem 5.0.1 implies that there exists a unique $h \in L^2([0,T]; H_0^2(\mathbb{T}))$ such that the unique solution $w \in C([0,\infty); H_0^2(\mathbb{T})) \cap C^1([0,\infty); L_0^2(\mathbb{T}))$ of equation (5.1.13) satisfies $w(x,T) = u(x,T)$ for all $x \in \mathbb{T}$, and there exists a positive constant $\nu = \nu(g)$ such that

$$\|h\|_{L^2([0,T];H_0^2(\mathbb{T}))} \leqslant \nu \|u(x,T)\|_{H_0^2(\mathbb{T})}.$$

On the other hand, note that $u_0 \in H_0^2(\mathbb{T}) \subset L_0^2(\mathbb{T})$, therefore Theorem 5.1.6 implies that $u \in C([0,\infty); L_0^2(\mathbb{T}))$ is a solution of equation (5.1.12). Furthermore, Theorem 5.0.1 implies that $h \in L^2([0,T]; L_0^2(\mathbb{T}))$ and the solution $w \in C([0,\infty); L_0^2(\mathbb{T}))$ of equation (5.1.13) satisfies $w(x,T) = u(x,T)$, for all $x \in \mathbb{T}$, with

$$\|h\|_{L^2([0,T];L_0^2(\mathbb{T}))} \leqslant \nu \|u(x,T)\|_{L_0^2(\mathbb{T})}. \tag{5.1.14}$$

Now, multiplying the first equation in (5.1.12) by $\bar{u}$ and integrating with respect to x, it follows that

$$\int_{\mathbb{T}} \partial_t u \bar{u} dx + \int_{\mathbb{T}} \left(\frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) \right) \bar{u} dx = \int_{\mathbb{T}} -GG^* u \bar{u} dx. \tag{5.1.15}$$

Using the Parseval's identity and the fact that the operator G is self-adjoint on $L_0^2(\mathbb{T})$, it is easy to obtain from (5.1.15) that

$$\frac{1}{2}\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{L_0^2(\mathbb{T})} \right) = -\|Gu(\cdot,t)\|^2_{L_0^2(\mathbb{T})}, \quad \text{for all } t > 0. \tag{5.1.16}$$

Integrating (5.1.16) with respect to the variable t from 0 and T, we get

$$\frac{1}{2}\|u(T)\|^2_{L_0^2(\mathbb{T})} - \frac{1}{2}\|u_0\|^2_{L_0^2(\mathbb{T})} = -\|Gu\|^2_{L^2((0,T);L_0^2(\mathbb{T}))}. \tag{5.1.17}$$

On the other hand, multiplying (5.1.13) by $\bar{u}$, integrating with respect to the $x-$variable, using integration by parts, the property (1.4.4) of the Hilbert transform, and the fact that K is self-adjoint, we get

$$\int_{\mathbb{T}} \partial_t w \bar{u} dx - \int_{\mathbb{T}} w \overline{(-\mathcal{H}(\partial_x^2 u) + K(\partial_x u))} dx = \int_{\mathbb{T}} Gh \bar{u} dx, \quad \text{for all } 0 < t < T. \tag{5.1.18}$$

Integrating (5.1.18) with respect to t from 0 and T, and using integration by parts, we obtain

$$\int_{\mathbb{T}} w(x,T)\bar{u}(x,T)dx - \int_0^T \int_{\mathbb{T}} w\overline{(\partial_t u - \mathcal{H}(\partial_x^2 u) + K(\partial_x u)}dxdt = \int_0^T \int_{\mathbb{T}} Gh\bar{u}dxdt.$$

Observe that u is a solution of equation (5.1.12). Thus

$$\int_{\mathbb{T}} u^2(T)\ dx + \int_0^T \int_{\mathbb{T}} w\ \overline{(GG^*u)}\ dx\ dt = \int_0^T \int_{\mathbb{T}} Gh\ \bar{u}\ dx\ dt.$$

Using that the solution u is real, the operator G is self-adjoint on $L_0^2(\mathbb{T})$, and the Cauchy-Schwarz inequality, we get

$$\|u(\cdot,T)\|_{L_0^2(\mathbb{T})}^2 \leqslant \|h - Gw\|_{L^2((0,T);L_0^2(\mathbb{T}))}\ \|Gu\|_{L^2((0,T);L_0^2(\mathbb{T}))}. \tag{5.1.19}$$

From (5.1.14), we have

$$\|h - Gw\|_{L^2((0,T);L_0^2(\mathbb{T}))} \leqslant \nu\|u(T)\|_{L_0^2(\mathbb{T})} + c\left(\int_0^T \|w(\cdot,t)\|_{L_0^2(\mathbb{T})}^2\ dt\right)^{\frac{1}{2}}. \tag{5.1.20}$$

Also, observe that

$$\|w(\cdot,t)\|_{L_0^2(\mathbb{T})}^2 \leqslant \left\|\int_0^t U_\mu(t-t')Gh(\cdot,t')dt'\right\|_{L_0^2(\mathbb{T})}^2 \leqslant c^2T\|h\|_{L^2((0,T);L_0^2(\mathbb{T}))}^2 \leqslant c^2T\nu^2\|u(T)\|_{L_0^2(\mathbb{T})}^2.$$

From this and (5.1.20), we get

$$\|h - Gw\|_{L^2((0,T);L_0^2(\mathbb{T}))} \leqslant c_{g,T}\|u(T)\|_{L_0^2(\mathbb{T})}, \tag{5.1.21}$$

where $c_{g,T} = max\{\nu, c^2T\nu\}$. From (5.1.19) and (5.1.21), we have

$$\|u(\cdot,T)\|_{L_0^2(\mathbb{T})}^2 \leqslant c_{g,T}\|u(T)\|_{L_0^2(\mathbb{T})}\|Gu\|_{L^2((0,T);L_0^2(\mathbb{T}))},$$

which implies that

$$-\|Gu\|_{L^2((0,T);L_0^2(\mathbb{T}))}^2 \leqslant -\frac{1}{c_{g,T}^2}\ \|u(\cdot,T)\|_{L_0^2(\mathbb{T})}^2. \tag{5.1.22}$$

From the relation (5.1.17) and the estimate (5.1.22), we obtain

$$\left(1 + \frac{2}{c_{g,T}^2}\right)\|u(T)\|_{L_0^2(\mathbb{T})}^2 \leqslant \|u_0\|_{L_0^2(\mathbb{T})}^2. \tag{5.1.23}$$

Thus, there exists $\rho_{g,T} = \rho \in (0,1)$ such that $\|u(T)\|_{L_0^2(\mathbb{T})}^2 \leqslant \rho\|u_0\|_{L_0^2(\mathbb{T})}^2$, for any $T > 0$. Moreover, we can repeat this estimate on successive intervals $[(n-1)T, nT]$, to get

$$\|u(x,nT)\|_{L_0^2(\mathbb{T})}^2 \leqslant \rho^n\|u_0\|_{L_0^2(\mathbb{T})}^2, \quad \text{for any } \ T > 0,\ n \geqslant 1, \tag{5.1.24}$$

where u is the solution of (5.1.12), and $\rho = \rho_{g,T} \in (0,1)$. From (5.1.16) and (5.1.24), we infer that there exists $M > 0$ and $\gamma > 0$ such that

$$\|u(x,t)\|_{L^2_0(\mathbb{T})} \leqslant Me^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})}, \quad \text{for all } t \geqslant 0,$$

and we get the result for smooth initial data in $H^2_0(\mathbb{T})$. We complete the proof for $s = 0$ by using density arguments.

Next, we prove the case $s = 2$. Let u be the solution of equation (5.1.12) with initial data $u_0 \in H^2_0(\mathbb{T})$, then $u \in C([0,\infty); H^2_0(\mathbb{T})) \cap C^1([0,\infty); L^2_0(\mathbb{T}))$. As $H^2_0(\mathbb{T}) \subset L^2_0(\mathbb{T})$, then from case $s = 0$, we have that there exist positive constants M_1 and $\gamma = \gamma(g)$ independent of u_0, such that

$$\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant M_1 e^{-\gamma t}\|u_0\|_{L^2_0(\mathbb{T})}, \quad \text{for all } \ t \geqslant 0. \tag{5.1.25}$$

On the other hand, differentiating the equation (5.1.12) with respect to t, we obtain

$$\partial_t(\partial_t u) - \mathcal{H}\partial_x^2(\partial_t u) + K\partial_x(\partial_t u) = -GG^*(\partial_t u).$$

Therefore, $w := \partial_t u \in C([0,+\infty); L^2_0(\mathbb{T}))$ is the unique solution of

$$\begin{cases} \partial_t w - \mathcal{H}\partial_x^2 w + K\partial_x w = -GG^* w, & t > 0,\ x \in \mathbb{T} \\ w(x,0) = w_0 = \partial_t u(x,0) = \mathcal{H}\partial_x^2 u_0 - K\partial_x u_0 - GG^* u_0 \in L^2_0(\mathbb{T}), & x \in \mathbb{T}, \end{cases} \tag{5.1.26}$$

Again, from the case $s = 0$ applied to equation (5.1.26), there exist positive constants $M_1 = M_1(g)$ and $\gamma = \gamma(g)$, independent of w_0, such that

$$\|\partial_t u(\cdot,t)\|_{L^2_0(\mathbb{T})} = \|w(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant M_1 e^{-\gamma t}\|w_0\|_{L^2_0(\mathbb{T})}, \quad \forall t \geqslant 0. \tag{5.1.27}$$

Note that, for each $t \geqslant 0$

$$\|u(\cdot,t)\|_{H^2_0(\mathbb{T})} \leqslant c_0 \left(\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \|\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})}\right). \tag{5.1.28}$$

To estimate the term $\|\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})}$ we use that the Hilbert transform $\mathcal{H}$ is an isometry in $H^s_0(\mathbb{T})$, and from equation (5.1.12)

$$\mathcal{H}\partial_x^2 u(\cdot,t) = w + K\partial_x u + GG^* u.$$

Hence, for each $t \geqslant 0$

$$\|\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})} = \|\mathcal{H}\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant \|w(\cdot,t)\|_{L^2_0(\mathbb{T})} + c_\delta\|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \|GG^* u(\cdot,t)\|_{L^2_0(\mathbb{T})}. \tag{5.1.29}$$

Using Gagliardo-Nirenberg inequality (see the Theorem 3.70 in [8]) and Cauchy-Schwarz inequality with ϵ, we have

$$c_\delta\|\partial_x u(\cdot,t)\|_{L^2_0(\mathbb{T})} \leqslant C_\delta\|\partial_x^2 u(\cdot,t)\|^{\frac{1}{2}}_{L^2_0(\mathbb{T})}\|u(\cdot,t)\|^{\frac{1}{2}}_{L^2_0(\mathbb{T})} = C_\delta\epsilon\|u(\cdot,t)\|_{L^2_0(\mathbb{T})} + \frac{C_\delta}{4\epsilon}\|\partial_x^2 u(\cdot,t)\|_{L^2_0(\mathbb{T})},$$

where C_δ is a positive constant depending on δ. Using this estimate and estimates (5.1.27), and (5.1.29), we obtain

$$\begin{aligned}\left(1-\frac{C_\delta}{4\epsilon}\right)\|\partial_x^2 u(\cdot,t)\|_{L_0^2(\mathbb{T})} &\leqslant M_1 e^{-\gamma t}\|w_0\|_{L_0^2(\mathbb{T})} + (C_\delta\,\epsilon + c_g)\,\|u(\cdot,t)\|_{L_0^2(\mathbb{T})}\\ &\leqslant M_1 e^{-\gamma t}\|w_0\|_{L_0^2(\mathbb{T})} + (C_\delta\,\epsilon + c_g)\,M_1 e^{-\gamma t}\|u_0\|_{L_0^2(\mathbb{T})}.\end{aligned}$$

Therefore, taking $\epsilon > 0$ large enough such that $1 - \dfrac{C_\delta}{4\epsilon} > 0$, we infer that there exists a positive constant $c = c_{\delta,g}$, independent of u_0, and w_0 such that

$$\|\partial_x^2 u(\cdot,t)\|_{L_0^2(\mathbb{T})} \leqslant\; c\,M_1 e^{-\gamma t}\left(\|w_0\|_{L_0^2(\mathbb{T})} + \|u_0\|_{L_0^2(\mathbb{T})}\right). \tag{5.1.30}$$

Also, note that

$$\|w_0\|_{L_0^2(\mathbb{T})} \leqslant \|\partial_x^2 u_0\|_{L_0^2(\mathbb{T})} + c_\delta\|\partial_x u_0\|_{L_0^2(\mathbb{T})} + c_g\|u_0\|_{L_0^2(\mathbb{T})} \leqslant c_1\|u_0\|_{L_0^2(\mathbb{T})}, \tag{5.1.31}$$

for some positive constant c_1. Thus from (5.1.30) and (5.1.31), we have

$$\|\partial_x^2 u(\cdot,t)\|_{L_0^2(\mathbb{T})} \leqslant M_2 e^{-\gamma t}\|u_0\|_{L_0^2(\mathbb{T})}, \tag{5.1.32}$$

where $M_2 = c\,M_1(c_1+1)$. From (5.1.25), (5.1.28) and (5.1.32), we get

$$\|u(\cdot,t)\|_{H_0^2(\mathbb{T})} \leqslant c_0\left(M_1 e^{-\gamma t}\|u_0\|_{L_0^2(\mathbb{T})} + M_2 e^{-\gamma t}\|u_0\|_{L_0^2(\mathbb{T})}\right) \leqslant M e^{-\gamma t}\|u_0\|_{H_0^2(\mathbb{T})}, \quad \forall t \geqslant 0,$$

where $M = M(\delta, g)$, and $\gamma = \gamma(g)$ are positive constants independent of u_0. This proves the case $s = 2$.

The case $0 < s < 2$ follows by interpolation. The other cases of s can be proved similarly. The proof is complete. □

Finally, in this section, we show that it is possible to choose an appropriate linear feedback control law such that the decay rate of the resulting closed-loop system is as large as one desires. Choosing the feedback control law as

$$h(u) := \begin{cases} -G^* L_\lambda^{-1} u, & \text{if} \quad \lambda > 0 \\ \\ -G^* u, & \text{if} \quad \lambda = 0, \end{cases}$$

where L_λ is defined in (2.5.1). We can rewrite the resulting closed-loop system in the following form

$$\begin{cases} \partial_t u - \mathcal{H}(\partial_x^2 u) + K(\partial_x u) = -F_\lambda u, & t > 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.1.33}$$

where $\lambda \geqslant 0$ and $F_\lambda := GG^* L_\lambda^{-1}$. If $\lambda = 0$, we define $F_0 = GG^*$. Thus, F_λ is a bounded linear operator on $H_0^s(\mathbb{T})$ with $s \geqslant 0$. We have the following result.

Theorem 5.1.8. *Let $\delta > 0$, $s \geqslant 0$ and $\lambda > 0$ be given. For any $u_0 \in H_0^s(\mathbb{T})$, the system (5.1.33) admits a unique solution $u \in C([0,+\infty), H_0^s(\mathbb{T}))$. Moreover, there exists $M = M(g,\lambda,\delta,s) > 0$ such that*

$$\|u(\cdot,t)\|_{H_0^s(\mathbb{T})} \leqslant Me^{-\lambda t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \textit{for all}\, t \geqslant 0.$$

Proof. As F_λ is a bounded linear operator the same argument used in Theorem 5.1.6 shows that for $u_0 \in H_0^s(\mathbb{T})$ the problem (5.1.33) has a unique solution $u \in C([0,\infty); H_0^s(\mathbb{T}))$ for all $s \geqslant 0$. We denote by $\{T_\lambda(t)\}_{t\geqslant 0}$ the C_0-semigroup on $H_0^s(\mathbb{T})$ with infinitesimal generator $A - F_\lambda$.

The case $s = 0$ follows from Theorem 2.1 in [84] (the arguments are similar to the proof of Theorem 2.5.5). The other cases of s are proved as in Proposition 5.1.7. □

Observe that Theorem 5.0.3 is a direct consequence of Theorem 5.1.8.

5.2 Stabilization of ILW equation with a localized damping

In this section we are interested in the stability properties of the ILW equation with localized damping

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = -GDGu, & t \geqslant 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.1}$$

where $\delta > 0$, D and G are defined in (5.0.3) and (2.0.9).

5.2.1 Semi-global exponential stabilization in $L_0^2(\mathbb{T})$

Assuming that $u_0 \in L_0^2(\mathbb{T})$, and using Parseval's identity we note that

$$\int_{\mathbb{T}} \partial_x^2(\mathcal{T}u)\, \bar{u} dx = 0.$$

From this, we infer (formally) by scaling in (5.2.1) by u that

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L_0^2(\mathbb{T})}^2\right) + \|D^{\frac{1}{2}}Gu(t)\|_{L_0^2(\mathbb{T})}^2 = 0, \tag{5.2.2}$$

and (5.0.4) is satisfied. This suggests that the energy is dissipated over time. A rigorous derivation of (5.0.4) requires enough regularity for u, e.g.

$$u \in L^2(0,T; H_0^1(\mathbb{T})) \cap C([0,T], L_0^2(\mathbb{T})). \tag{5.2.3}$$

Since there is a gap between (5.0.5) and (5.2.3), we put some artificial viscosity in (5.2.1) (parabolic regularization method) in the same vein as in [59] to derive in a rigorous way the energy identity for the ε−ILW equation

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = \varepsilon\partial_x^2 u - GDGu, & t \geqslant 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{5.2.4}$$

We will show the global well-posedness (GWP) of (5.2.4) in $L_0^2(\mathbb{T})$, together with the semi-global exponential stability in $L_0^2(\mathbb{T})$ with a decay rate uniform in $\varepsilon > 0$. Letting $\varepsilon \longrightarrow 0$, we obtain the semi-global exponential stability in $L_0^2(\mathbb{T})$ of the weak solutions $u \in C_w([0,+\infty), L_0^2(\mathbb{T}))$ of (5.2.1) obtained as limits of the (strong) solutions of (5.2.4). The following result provides the GWP of the system (5.2.4)

Theorem 5.2.1. *Let $\varepsilon > 0$, $\delta > 0$, and $u_0 \in L_0^2(\mathbb{T})$). Then for any $T > 0$ there exists a unique solution $u \in C([0,T], L_0^2(\mathbb{T})) \cap L^2(0,T; H_0^1(\mathbb{T}))$ of (5.2.4). Moreover,*

$$u \in C((0,T], H_0^2(\mathbb{T})) \cap C^1((0,T], L_0^2(\mathbb{T})), \tag{5.2.5}$$

and for any $t \geqslant 0$,

$$\frac{1}{2}\|u(t)\|^2_{L_0^2(\mathbb{T})} + \varepsilon\int_0^t \|\partial_x u(\tau)\|_{L_0^2(\mathbb{T})} d\tau + \int_0^t \|D^{\frac{1}{2}}(Gu)(\tau)\|^2_{L_0^2(\mathbb{T})} d\tau = \frac{1}{2}\|u_0\|^2_{L_0^2(\mathbb{T})}. \tag{5.2.6}$$

Proof. We prove this theorem in five steps.
Step 1. (Linear theory). We consider the linear system. Using (5.1.2) we can rewrite it as

$$\begin{cases} \partial_t u - \mathcal{H}(\partial_x^2 u) - \varepsilon\partial_x^2 u + K(\partial_x u) + G(D(Gu)) = 0, & t > 0, \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.7}$$

Let $Au := (-\mathcal{H} - \varepsilon)\partial_x^2 u$ with domain $D(A) = H_0^2(\mathbb{T}) \subset L_0^2(\mathbb{T})$, and $Bu := G(D(Gu)) + K(\partial_x u)$. We know that $G \in \mathcal{L}(H_p^r(\mathbb{T}), H_0^r(\mathbb{T}))$ for all $r \in \mathbb{R}$. Using (5.1.4) we infer that $B \in \mathcal{L}(H_0^1(\mathbb{T}), L_0^2(\mathbb{T}))$. Let $\theta_0 \in \left(\arctan(\varepsilon^{-1}), \dfrac{\pi}{2}\right)$. Thus, for $\theta_0 < |\arg(\lambda)| \leqslant \pi$, we have

$$\|(A-\lambda)^{-1}\| \leqslant \sup_{n\neq 0}\{|(\varepsilon - i\mathrm{sgn}(n))n^2 - \lambda|^{-1}\} \leqslant \frac{C}{|\lambda|}.$$

Then A is a sectorial operator in $L_0^2(\mathbb{T})$ (see Definition 1.5.7). Note that $\sigma(A) = \{(\varepsilon - i\,\mathrm{sgn(n)})n^2; n \in \mathbb{Z}^*\}$. In consequence, $Re(\sigma(A)) \geqslant \varepsilon$ and $A^{-\alpha}$ (see Definition 1.5.9) is meaningful for all $\alpha > 0$. Since for all $s > 0$

$$\|A^{-\frac{s}{2}}u\|^2_{H_0^s(\mathbb{T})} \leqslant C\sum_{n\in\mathbb{Z}^*} |(\varepsilon - i\mathrm{sgn}(n))|^{-s}|\widehat{u}(n)|^2 \leqslant C\|u\|^2_{L_0^2(\mathbb{T})}.$$

Therefore, $BA^{-\frac{1}{2}} \in \mathcal{L}(L_0^2(\mathbb{T}))$. It follows from Corollary 1.5.11 that the operator $\mathcal{A} := A+B$ is also sectorial, so that $-\mathcal{A}$ generates an analytic semigroup (see Definition 1.5.12) $\{\mathcal{S}(t)\}_{t\geqslant 0} = \{e^{-t\mathcal{A}}\}_{t\geqslant 0}$ on $L_0^2(\mathbb{T})$ according to Theorem 1.5.13. Furthermore, from [28, Theorem 1.4.8] we have $D((A+B+\lambda)^{\alpha}) = D(A^{\alpha}) = H_0^{2\alpha}(\mathbb{T})$ for all $\alpha \geqslant 0$ and $\lambda > 0$ large enough; hence

$$\mathcal{S}(t)H_0^s(\mathbb{T}) \subset H_0^s(\mathbb{T}), \quad \forall t > 0,\ \forall s \geqslant 0.$$

Now, we derive estimates for the solutions of the IVP

$$\begin{cases} \partial_t u + \mathcal{A}u = f, & t > 0, \ \ x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{5.2.8}$$

For any $T > 0$ and any $s \in \mathbb{N}$, let

$$Y_{s,T} = C([0,T]; H_0^s(\mathbb{T})) \cap L^2(0,T; H_0^{s+1}(\mathbb{T})) \tag{5.2.9}$$

be endowed with the norm

$$\|u\|_{Y_{s,T}} = \|u\|_{L^{\infty}(0,T;H_0^s(\mathbb{T}))} + \|u\|_{L^2(0,T;H_0^{s+1}(\mathbb{T}))}. \tag{5.2.10}$$

Lemma 5.2.2. *We have for some constant $C_0 = C_0(\varepsilon, s, T)$*

$$\|u\|_{Y_{s,T}} \leqslant C_0(\|u_0\|_{H_0^s(\mathbb{T})} + \|f\|_{L^1(0,T;H_0^s(\mathbb{T}))}),$$

with u denoting the mild solution of (5.2.8) associated with

$$(u_0, f) \in H_0^s(\mathbb{T}) \times L^1(0,T; H_0^s(\mathbb{T})).$$

Proof. From classical semigroup theory we have

$$\|u\|_{L^{\infty}(0,T;H_0^s(\mathbb{T}))} \leqslant C(\|u_0\|_{H_0^s(\mathbb{T})} + \|f\|_{L^1(0,T;H_0^s(\mathbb{T}))}).$$

To estimate $\|u\|_{L^2(0,T;H_0^{s+1}(\mathbb{T}))}$ we use Parseval's identity to have that for any $u \in H_0^s(\mathbb{T})$

$$\begin{aligned}\left(-H\partial_x^2 u + K(\partial_x u), u\right)_{H_0^s(\mathbb{T})} &= \left(\frac{1}{\delta}\partial_x u + \mathcal{T}\partial_x^2 u, u\right)_{H_0^s(\mathbb{T})} \\ &= 2\pi \sum_{n\in\mathbb{Z}^*} \langle n \rangle^{2s} \left(-n\coth(n\delta) + \frac{1}{\delta} in\ \widehat{u}(n)\overline{\widehat{u}(n)}\right) = 0.\end{aligned} \tag{5.2.11}$$

Using a density argument we prove, taking the scalar product of each term of (5.2.8) by $u \in H_0^s(\mathbb{T})$, that

$$\frac{1}{2}\|u(t)\|_{H_0^s(\mathbb{T})}^2 + \varepsilon\int_0^t \|\partial_x u(\tau)\|_{H_0^s(\mathbb{T})} d\tau + \int_0^t (G(D(Gu)),u)_{H_0^s(\mathbb{T})}\, d\tau = \frac{1}{2}\|u_0\|_{H_0^s(\mathbb{T})}^2 + \int_0^t (f,u)_{H_0^s(\mathbb{T})}$$

is true for any $u_0 \in H_0^s(\mathbb{T})$ and any $f \in L^1(0,T; H_0^s(\mathbb{T}))$. Combining this relation with Lemma A.5 in the appendix, we conclude the proof (see Lemma 2.2 in [59]). □

Remark 5.2.3. *We observe that when $u_0 \equiv 0$ in (5.2.8), then*

$$\left\|\int_0^t \mathcal{S}(t-\tau)f(\tau)d\tau\right\|_{Y_{s,T}} \leqslant C(\varepsilon,s,T)\|f\|_{L^1(0,T;H_0^s((T)))}, \tag{5.2.12}$$

and when $f \equiv 0$ in (5.2.8),

$$\|\mathcal{S}(t)u_0\|_{Y_{s,T}} \leqslant C(\varepsilon,s,T)\|u_0\|_{H_0^s((T))}, \tag{5.2.13}$$

Step 2. (Local well-posedness in $H_0^s(\mathbb{T})$, $s \geqslant 0$). We prove the following proposition.

Proposition 5.2.4. *Let $s \geqslant 0$ and $\delta > 0$ be given. For any $u_0 \in H_0^s(\mathbb{T})$, there exists some $T > 0$ such that the IVP (5.2.4) admits a unique solution $u \in Y_{s,T}$.*

Proof. We write (5.2.4) in its integral form $u(t) = \mathcal{S}(t)u_0 - \int_0^t \mathcal{S}(t-\tau)(2u\partial_x u)(\tau)d\tau$. For given $u_0 \in H_0^s(\mathbb{T})$, let $r > 0$ and $T > 0$ be constants to be determined. Define the map Γ on the closed ball $B(0,r) = \{v \in Y_{s,T}; \|v\|_{Y_{s,T}} \leqslant r\}$ of $Y_{s,T}$ by

$$\Gamma(v)(t) = \mathcal{S}u_0 - \int_0^t \mathcal{S}(t-\tau)(2v\partial_x v)(\tau)d\tau.$$

Using Gagliardo-Nirenberg inequality (see [8, Theorem 3.70]) and Theorem 1.7 in [33], we note that

$$\begin{aligned}\int_0^T \|v(\tau)\|^2_{L^\infty(\mathbb{T})} &\leqslant C\int_0^T \|v(\tau)\|_{H_0^1(\mathbb{T})}\|v(\tau)\|_{L_0^2(\mathbb{T})}d\tau \\ &\leqslant C\sqrt{T}\|v\|_{L^\infty(0,T;L_0^2(\mathbb{T}))}\|v\|_{L^2(0,T;H_0^1(\mathbb{T}))}.\end{aligned} \tag{5.2.14}$$

From Lemma 5.2.2, Remark 5.2.3, Lemma 1.10 in [33], and (5.2.14) we infer that

$$\begin{aligned}\|\Gamma(v^1) - \Gamma(v^2)\|_{Y_{s,T}} &\leqslant C\|v^1\partial_x(v^1) - v^2\partial_x(v^2)\|_{L^1(0,T;H_0^s(\mathbb{T}))} \\ &\leqslant C\int_0^T \left(\|v^1 - v^2\|_{L^\infty(\mathbb{T})}\|v^1 + v^2\|_{H_0^{s+1}(\mathbb{T})} + \|v^1 + v^2\|_{L^\infty(\mathbb{T})}\|v^1 - v^2\|_{H_0^{s+1}(\mathbb{T})}\right)d\tau \\ &\leqslant CT^{\frac14}\|v^1 - v^2\|_{Y_{s,T}}\left(\|v^1\|_{Y_{s,T}} + \|v^2\|_{Y_{s,T}}\right), \quad \forall\, v^1, v^2 \in B.\end{aligned} \tag{5.2.15}$$

Therefore, Lemma 5.2.2, Remark 5.2.3, and (5.2.15) imply that

$$\|\Gamma(v)\|_{Y_{s,T}} \leqslant C_0\|u_0\|_{H_0^s(\mathbb{T})} + C_1T^{\frac14}\|v\|^2_{Y_{s,T}}, \quad \forall\, v \in B, \tag{5.2.16}$$

and

$$\|\Gamma(v^1) - \Gamma(v^2)\|_{Y_{s,T}} \leqslant C_1T^{\frac14}\|v^1 - v^2\|_{Y_{s,T}}\left(\|v^1\|_{Y_{s,T}} + \|v^2\|_{Y_{s,T}}\right), \quad \forall\, v^1, v^2 \in B. \tag{5.2.17}$$

Choosing $r > 0$ and $T > 0$ so that

$$\begin{cases} r = 2C_0\|u_0\|_{H_0^s(\mathbb{T})}, \\ 2rC_1T^{\frac14} \leqslant \dfrac12, \end{cases} \tag{5.2.18}$$

we obtain that $\|\Gamma(v^1)\|_{Y_{s,T}} \leqslant r$, $\|\Gamma(v^1) - \Gamma(v^2)\|_{Y_{s,T}} \leqslant \frac12\|v^1 - v^2\|_{Y_{s,T}}$, for all $v^1, v^2 \in B$.

Therefore, with this choice of r and T, Γ is a contraction in $B(0,r)$. Its fixed-point is the unique solution of (5.2.4) in $B(0,r)$. □

Step 3. (Global well-posedness in $L_0^2(\mathbb{T})$). Assume that $u_0 \in L_0^2(\mathbb{T})$. We first establish (5.2.6) for $0 \leqslant t \leqslant T$. Since $u \in Y_{0,T}$, we infer by using the immersion $L_0^4(\mathbb{T}) \subset L_0^2(\mathbb{T})$ and Gagliardo-Nirenberg inequality that

$$\begin{aligned}\int_0^t \|2u\partial_x u\|^2_{H_0^{-1}(\mathbb{T})} d\tau \leqslant C \int_0^t \|u^2\|^2_{L_0^2(\mathbb{T})} d\tau \leqslant C \int_0^t \|u\|^4_{L_0^4(\mathbb{T})} d\tau \\ \leqslant C \int_0^t \|u\|^3_{L_0^2(\mathbb{T})} \|\partial_x u\|_{L_0^2(\mathbb{T})} d\tau \leqslant C\sqrt{t}\|u\|^4_{Y_{0,t}}.\end{aligned}$$

Therefore, each term in (5.2.4) belongs to $L^2(0,t;H_0^{-1}(\mathbb{T}))$. Scaling in (5.2.4) by u, we obtain

$$\int_0^t \left\langle \partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + 2u\partial_x u - \varepsilon\partial_x^2 u + G(D(Gu)), u \right\rangle_{H_0^{-1}(\mathbb{T}), H_0^1(\mathbb{T})} d\tau = 0,$$

Using Parseval's identity, we get that for a.e. $\tau \in (0,t)$

$$\begin{aligned}\left\langle \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u), u \right\rangle_{H_0^{-1}(\mathbb{T}), H_0^1(\mathbb{T})} &= -\left(\frac{1}{\delta}u + \partial_x(\mathcal{T}u), \partial_x u\right)_{L_0^2(\mathbb{T})} \\ &= -2\pi i \sum_{n\in\mathbb{Z}^*} \left(\frac{1}{\delta} - n\coth(n\delta)\right) \widehat{u}(n)\, n\, \overline{\widehat{u}(n)} = 0,\end{aligned}$$

Thus, we have for a.e. $\tau \in (0,t)$

$$\left\langle \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) - \varepsilon\partial_x^2 u, u \right\rangle_{H_0^{-1}(\mathbb{T}), H_0^1(\mathbb{T})} = -\left(\frac{1}{\delta}u + \partial_x(\mathcal{T}u) - \varepsilon\partial_x u, \partial_x u\right)_{L_0^2(\mathbb{T})} = \varepsilon\|\partial_x u\|^2_{L_0^2(\mathbb{T})},$$

$$\langle 2u\partial_x u, u\rangle_{H_0^{-1}(\mathbb{T}), H_0^1(\mathbb{T})} = 2\,(u\partial_x u, u)_{L_0^2(\mathbb{T})} = 0,$$

and

$$\langle G(D(Gu)), u\rangle_{H_0^{-1}(\mathbb{T}), H_0^1(\mathbb{T})} = (G(D(Gu)), u)_{L_0^2(\mathbb{T})} = \|D^{\frac{1}{2}}Gu\|^2_{L_0^2(\mathbb{T})}.$$

Hence, (5.2.6) follows at once, and we infer that $\|u(t)\|_{L_0^2(\mathbb{T})} \leqslant \|u_0\|_{L_0^2(\mathbb{T})}$. Using the standard extension argument, one sees that u is defined on $(0,+\infty)$ with $u \in Y_{0,T}$ for all $T > 0$. Furthermore, with the constants C_0 and C_1 given in Step 2 for $s = 0$ and $T = (8C_0C_1\|u_0\|_{L_0^2(\mathbb{T})})^{-4}$, we get

$$\|u(nT + \cdot)\|_{Y_{0,T}} \leqslant 2C_0\|u(nT)\|_{L_0^2(\mathbb{T})} \leqslant 2C_0\|u_0\|_{L_0^2(\mathbb{T})}$$

Step 4. (Global well-posedness in $H_0^2(\mathbb{T})$). Pick any $u_0 \in H_0^2(\mathbb{T})$. From Proposition 5.2.4 and Step 3, we have that equation (5.2.4) admits a unique solution $u \in Y_{0,T}$ for each $T > 0$, which belongs to T_{2,T_0} for some $T_0 > 0$. We will show that T_0 can be taken as large as desired. Let $v = \partial_t u$. If $u \in Y_{2,T}$, then $v \in Y_{0,T}$ and it satisfies

$$\begin{cases} \partial_t v + \dfrac{1}{\delta}\partial_x v + \partial_x^2(\mathcal{T}v) + 2\partial_x(uv)\varepsilon\partial_x^2 v = -GDGv, & t \geqslant 0, \quad x \in \mathbb{T} \\ v(x,0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.19}$$

where

$$v_0 = \left(-\frac{1}{\delta}\partial_x u(x,0) - \partial_x^2(\mathcal{T}u)(x,0) - 2u\partial_x(u)(x,0) + \varepsilon\partial_x^2 u(x,0) - G(D(Gu))(x,0)\right) \in L_0^2(\mathbb{T}).$$

We rewrite (5.2.19) in its integral form $v(t) = \mathcal{S}(t)v_0 - \int_0^t \mathcal{S}(t-\tau)(2\partial_x(uv))(\tau)d\tau$, and define the map

$$\Gamma(w)(t) := \mathcal{S}(t)v_0 - \int_0^t \mathcal{S}(t-\tau)(2\partial_x(uw))(\tau)d\tau, \quad \text{for } w \in Y_{0,T}.$$

Similar computations as in Step 2 yields

$$\|\Gamma(w)\|_{Y_{0,T}} \leqslant C_0\|v_0\|_{L_0^2(\mathbb{T})} + C_1 T^{\frac{1}{4}}\|u\|_{Y_{0,T}}\|w\|_{Y_{0,T}},$$

$$\|\Gamma(w^1) - \Gamma(w^2)\|_{Y_{0,T}} \leqslant C_1 T^{\frac{1}{4}}\|w^1 - w^2\|_{Y_{0,T}}\|u\|_{Y_{0,T}},$$

where the positive constants C_0 and C_1 depend only on ε for $T < 1$. Therefore, Γ contracts in $B(0,r) = \{w \in Y_{0,\theta}; \|w\|_{Y_{0,\theta}} \leqslant r := 2C_0\|v_0\|_{L_0^2(\mathbb{T})}\}$, as long as $C_1\theta^{\frac{1}{4}}\|u\|_{Y_{0,\theta}} \leqslant \frac{1}{2}$. Its fixed point gives the unique solution of the integral equation in the ball B. Pick θ fulfilling

$$\theta < \min\left\{1, (8C_0C_1\|u_0\|_{L_0^2(\mathbb{T})})^{-4}\right\}.$$

Hence, from Step 2, we have that $\|u(n\theta + \cdot)\|_{Y_{0,\theta}} \leqslant 2C_0\|u_0\|_{L_0^2(\mathbb{T})}$, for all $n \in \mathbb{N}$ and that w can be extended to $[n\theta, (n+1)\theta]$ inductively by using the contraction mapping theorem (replacing v_0 by $w(\theta)$, $w(2\theta)$, etc.). Thus, w is defined on $(0, +\infty)$ and it holds that

$$\|w(nT + \cdot)\|_{Y_{0,\theta}} \leqslant 2C_0\|w(nT)\|_{L_0^2(\mathbb{T})} \leqslant (2C_0)^{n+1}\|v_0\|_{L_0^2(\mathbb{T})}. \tag{5.2.20}$$

By uniqueness of the solution of the integral equation, we have that $v(t) = w(t)$ as long as $0 < t < T$ and $v \in Y_{0,T}$. From (5.2.20) we obtain that $\|v(t)\|_{L_0^2(\mathbb{T})} = \|w(t)\|_{L_0^2(\mathbb{T})}$ is uniformly bounded on compacts sets of $(0, +\infty)$, namely $\|v\|_{Y_{0,T}} \leqslant C(T, \|u_0\|_{L_0^2(\mathbb{T})})\|v_0\|_{L_0^2(\mathbb{T})}$.

This is also true for $\|u(t)\|_{H_0^2(\mathbb{T})}$, by (5.2.4). In fact, using Gagliardo-Nirenberg inequality two times an Young's inequality with μ (see [54, Page 706]), we note that

$$\begin{aligned}\|2u\partial_x u\|_{L_0^2(\mathbb{T})} &\leqslant 2\|u\|_{L^\infty(\mathbb{T})}\|\partial_x u\|_{L_0^2(\mathbb{T})} \leqslant 2\|u\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x u\|_{L_0^2(\mathbb{T})}^{\frac{3}{2}} \\ &\leqslant 2\|u\|_{L_0^2(\mathbb{T})}^{\frac{5}{4}}\|\partial_x^2 u\|_{L_0^2(\mathbb{T})}^{\frac{3}{4}} \leqslant 2C(\mu)\|u\|_{L_0^2(\mathbb{T})}^5 + 2\mu\|\partial_x^2 u\|_{L_0^2(\mathbb{T})}.\end{aligned} \tag{5.2.21}$$

Also, using Cauchy-Schwarz inequality with μ, one have

$$\begin{aligned}\|G(D(Gu))\|_{L_0^2(\mathbb{T})} &\leqslant C_g\|\partial_x u\|_{L_0^2(\mathbb{T})} \\ &\leqslant C\|u\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_x^2 u\|_{L_0^2(\mathbb{T})}^{\frac{1}{2}} \leqslant C\|u\|_{L_0^2(\mathbb{T})} + C\mu\|\partial_x^2 u\|_{L_0^2(\mathbb{T})}.\end{aligned} \tag{5.2.22}$$

From (5.1.2) and (5.2.4), we infer that

$$(-\mathcal{H}-\varepsilon)\partial_x^2 u = -v - K(\partial_x u) + 2u\partial_x u - G(D(Gu)). \tag{5.2.23}$$

Therefore, using (5.2.21)-(5.2.23) and (5.1.4) we obtain that

$$\begin{aligned}\sqrt{1+\varepsilon^2}\|\partial_x^2 u(t)\|_{L_0^2(\mathbb{T})} = \|(-\mathcal{H}-\varepsilon)\partial_x^2 u(t)\|_{L_0^2(\mathbb{T})} \leqslant C(T, \|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^2(\mathbb{T})} + C_{\delta,\mu}\|u\|_{L_0^2(\mathbb{T})} \\ + C_\mu\|u\|_{L_0^2(\mathbb{T})}^5 + (2+C_{g,\delta})\mu\|\partial_x^2 u\|_{L_0^2(\mathbb{T})}.\end{aligned}$$

Taking μ small enough, we obtain

$$\|u(t)\|_{L_0^2(\mathbb{T})} \leqslant C(T, \|u_0\|_{L_0^2(\mathbb{T})})\|u_0\|_{H_0^2(\mathbb{T})}.$$

Using the standard extension argument, we show that $u(t) \in H_0^2(\mathbb{T})$ for all $t \geqslant 0$ with $u \in Y_{2,T}$ for all $T > 0$.

Step 5. (Smoothing effect from $L_0^2(\mathbb{T})$ to $H_0^2(\mathbb{T})$). A similar procedure as in Step 5 of Theorem 2.1 in [59] prove that

$$u \in C((0,+\infty), H_0^2(\mathbb{T})) \cap C^1((0,+\infty), L_0^2(\mathbb{T})).$$

The proof of Theorem 5.2.1 is complete. □

Next, we prove the propagation of regularity property. The following lemma is needed.

Lemma 5.2.5. *Let $\mathcal{N} \subset \mathbb{Z}$ be a set such that for some constant $C > 0$,*

$$\langle n\rangle + \langle k\rangle \leqslant C\langle n-k\rangle, \quad \forall n \notin \mathcal{N}, \ \forall k \in \mathcal{N}. \tag{5.2.24}$$

Let P be the projector on the closure of Span$\{e^{ikx}; k \in \mathcal{N}\}$ in $L^2(\mathbb{T})$, namely

$$P\left(\sum_{k\in\mathbb{Z}} \widehat{u}(k)e^{ikx}\right) = \sum_{k\in\mathcal{N}} \widehat{u}(k)e^{ikx}.$$

Let $\varphi \in C^\infty(\mathbb{T})$ and let $p \in \mathbb{N}$, $q \in \mathbb{N}$. Then, there exists some constant $C = C(\varphi, p, q) > 0$ such that for all $v \in L^2(\mathbb{T})$,

$$\|\partial_x^p[\varphi, P]\partial_x^q v\|_{L_p^2(\mathbb{T})} \leqslant C\|v\|_{L_p^2(\mathbb{T})}. \tag{5.2.25}$$

Proof. See Lemma 2.5 in [59]. □

Remark 5.2.6. *Condition (5.2.24) is fulfilled for $\mathcal{N} = \mathbb{N}^*$, and (5.2.25) is true for*

$$P = \mathcal{H} = (-i)(P_{\mathbb{N}*} - P_{-\mathbb{N}*}).$$

Using (5.1.2), we have the following result.

Proposition 5.2.7. *Let $g \in C^\infty(\mathbb{T},(0,+\infty))$, $\varepsilon > 0$, $\alpha \in \mathbb{R}$, $T > 0$, and $R > 0$ be given. Pick any $v_0 \in L^2_0(\mathbb{T})$ with $\|v_0\|_{L^2_0(\mathbb{T})} \leqslant R$ and let*

$$v \in C([0,T]; L^2_0(\mathbb{T})) \cap L^2(0,T; H^1_0(\mathbb{T})) \cap C((0,T]; H^2_0(\mathbb{T}))$$

be such that

$$\begin{cases} \partial_t v - \mathcal{H}(\partial_x^2 v) - \varepsilon\partial_x^2 v + K(\partial_x v) + 2\alpha v\partial_x v = -G(D(Gv)), & t > 0, \quad x \in \mathbb{T} \\ v(x,0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.26}$$

then there exists some constant $C = C(T) > 0$ (independent of ε, α, and R) such that

$$\int_0^T \|D^{\frac{1}{2}} v\|^2_{L^2_0(\mathbb{T})} dt \leqslant C(R^2 + \alpha^4 R^6). \tag{5.2.27}$$

Proof. Pick any $t_0 \in (0,T)$, and let

$$(f,q)_{L^2_{t,x}} := \int_{t_0}^T \int_{\mathbb{T}} f(x,t) q(x,t)\, dx dt$$

denotes the scalar product in $L^2(t_0,T; L^2_0(\mathbb{T}))$. Let C be a constant which may vary from line to line and may depend on δ, T, but independent of t_0, ε, α and R. Note that,

$$\int_{t_0}^T \|D^{\frac{1}{2}} v\|^2_{L^2_0(\mathbb{T})} dt = \int_{t_0}^T (Dv, v)_{L^2_0(\mathbb{T})}\, dt.$$

Since $\mathbb{T}$ is compact there exists a finite set of points, say $x_0^i \in \mathbb{T}$, $i = 1, \cdots, N$, such that we can construct a partition of the unity on $\mathbb{T}$ involving functions of the form $\chi_i^2(\cdot - x_0^i)$ with $\chi_i^2(\cdot) \in C_c^\infty(\omega)$. Specifically, there exists $N \in \mathbb{N}$ such that

$$\begin{cases} 0 \leqslant \chi_i^2(x - x_0^i) \leqslant 1, \quad \text{for all} \quad x \in \mathbb{T} \quad \text{and} \quad i = 1, 2, .., N \\ \displaystyle\sum_{i=1}^N \chi_i^2(\cdot - x_0^i) = 1 \qquad \text{on} \qquad \mathbb{T}. \end{cases}$$

Therefore,

$$\int_{t_0}^T \|D^{\frac{1}{2}} v\|^2_{L^2_0(\mathbb{T})} dt = \int_{t_0}^T \left(\left(\sum_{i=1}^N \chi_i^2(x - x_0^i) \right) Dv, v \right)_{L^2_0(\mathbb{T})} dt = \sum_{i=1}^N \int_{t_0}^T \left(\chi_i^2(x - x_0^i) Dv, v\right)_{L^2_0(\mathbb{T})} dt. \tag{5.2.28}$$

We shall show that for any $\chi^2(\cdot) \in C_c^\infty(\omega)$ and any $x_0 \in \mathbb{T}$ there exists a positive constant C such that

$$\left| \int_{t_0}^T \left(\chi^2(x - x_0) Dv, v\right)_{L^2_0(\mathbb{T})} \right| \leqslant C(R^2 + \alpha^4 R^6) + \frac{1}{2N} \int_{t_0}^T \|D^{\frac{1}{2}} v\|^2_{L^2_0(\mathbb{T})}. \tag{5.2.29}$$

For this, consider $\phi(x) = \chi^2(x) - \chi^2(x - x_0)$, where $\chi^2 \in C_c^\infty(\omega)$ and $x_0 \in \mathbb{T}$. From Lemma A.4 there exists $\varphi \in C^\infty(\mathbb{T})$ such that $\partial_x \varphi(x) = \chi^2(x) - \chi^2(x - x_0)$, for all $x \in \mathbb{T}$. Consequently,

$$\left(\chi^2(x - x_0) Dv, v\right)_{L^2_0(\mathbb{T})} = \left(\chi^2(x) Dv, v\right)_{L^2_0(\mathbb{T})} - \left(\partial_x \varphi(x) Dv, v\right)_{L^2_0(\mathbb{T})}.$$

Noticing that $D = \mathcal{H}\partial_x$ and using a similar argument as in (2.39) in [59] (Substituting χ by b and a by g) we prove that

$$\left|\int_{t_0}^T \left(\chi^2(x)Dv, v\right)_{L_0^2(\mathbb{T})}\right| = \left|\int_{t_0}^T \left(\chi^2(x)\mathcal{H}\partial_x v, v\right)_{L_0^2(\mathbb{T})}\right| \leqslant CR^2. \tag{5.2.30}$$

Therefore, to achieve (5.2.29) it is enough to prove that

$$\left|\int_{t_0}^T (\partial_x\varphi(x)Dv, v)_{L_0^2(\mathbb{T})}\right| = \left|\int_{t_0}^T (\partial_x\varphi(x)\mathcal{H}\partial_x v, v)_{L_0^2(\mathbb{T})}\right| \leqslant C(R^2+\alpha^4R^6) + \frac{1}{2N}\int_{t_0}^T \|D^{\frac{1}{2}}v\|^2_{L_0^2(\mathbb{T})}. \tag{5.2.31}$$

For this, set $Lv := \partial_t v - \mathcal{H}\partial_x^2 v$, $f := \varepsilon\partial_x^2 v - G(D(Gv))$ $q := -2\alpha v\partial_x v - K\partial_x v$, and $A\varphi = \varphi(x)v$, we have that $Lv = f + q$. Noticing that L is formally anti-skew-adjoint, we have that

$$\begin{aligned}([L,A]v,v)_{L^2_{t,x}} &= (L(\varphi v), v)_{L^2_{t,x}} - (\varphi Lv, v)_{L^2_{t,x}}\\ &= (\varphi v, v)_{L^2_{t,x}}\Big|_{t_0}^T + (\varphi v, L^* v)_{L^2_{t,x}} - (Lv, \varphi v)_{L^2_{t,x}},\end{aligned}$$

Thus

$$\left|([L,A]v,v)_{L^2_{t,x}}\right| \leqslant 2\left|(f+q,\varphi v)_{L^2_{t,x}}\right| + 2\|\varphi\|_{L^\infty(\mathbb{T})}R^2. \tag{5.2.32}$$

Using Lemmas A.1-A.2, (5.2.6) and the fact that the L^2-norm is nonincreasing, we infer that

$$\begin{aligned}\left|(f,\varphi v)_{L^2_{t,x}}\right| &\leqslant \varepsilon\left|(\partial_x v, \partial_x(\varphi v))_{L^2_{t,x}}\right| + \left|(D(Gv), G(\varphi v))_{L^2_{t,x}}\right|\\ &\leqslant C\varepsilon\int_0^T\int_{\mathbb{T}}(|v|^2+|\partial_x v|^2)dxdt + \int_0^T\left|\left(D^{\frac{1}{2}}(Gv), \varphi D^{\frac{1}{2}}(Gv)\right)_{L_0^2(\mathbb{T})}\right|dt\\ &\quad + \int_0^T\left\{\left|(D(Gv),[G,\varphi]v)_{L_0^2(\mathbb{T})}\right| + \left|\left(D^{\frac{1}{2}}(Gv),[D^{\frac{1}{2}},\varphi]G(v)\right)_{L_0^2(\mathbb{T})}\right|\right\}dt\\ &\leqslant C\varepsilon\int_0^T\|\partial_x v\|^2_{L_0^2(\mathbb{T})}dt + C\int_0^T\|D^{\frac{1}{2}}(Gv)\|^2_{L_0^2(\mathbb{T})}dt + C_{\|\varphi\|_{H_p^1(\mathbb{T})}}\int_0^T\|v\|^2_{L_0^2(\mathbb{T})}dt\\ &\leqslant C\left(\|v_0\|^2_{L_0^2(\mathbb{T})} - \|v(T)\|^2_{L_0^2(\mathbb{T})}\right) + CR^2\\ &\leqslant CR^2.\end{aligned} \tag{5.2.33}$$

Also, note that Theorem 5.2.1 is still true when $\alpha = 1$ is replaced by any value $\alpha \in \mathbb{R}$. Using Parseval's identity, we note that

$$\begin{aligned}\left|(q,\varphi v)_{L^2_{t,x}}\right| &\leqslant \left|(-2\alpha v\partial_x v, \varphi v)_{L^2_{t,x}}\right| + \left|(K\partial_x v, \varphi v)_{L^2_{t,x}}\right|\\ &\leqslant \frac{|\alpha|}{3}\left|\left(v^3,\partial_x\varphi\right)_{L^2_{t,x}}\right| + \left|\int_{t_0}^T\left(2\pi\sum_{n\in\mathbb{Z}^*}\widehat{K\partial_x v}(n)\overline{\widehat{\varphi v}(n)}\right)dt\right|.\end{aligned} \tag{5.2.34}$$

From Interpolation inequality for L^p-norms (see [54, page 707]) and Sobolev embedding theorem (see [33, Lemma 1.5]), we have

$$\|v\|_{L^3(\mathbb{T})} \leqslant \|v\|^{\frac{1}{2}}_{L_0^2(\mathbb{T})}\|v\|^{\frac{1}{2}}_{L_0^6(\mathbb{T})} \leqslant CR^{\frac{1}{2}}\|v\|^{\frac{1}{2}}_{H_0^{\frac{1}{2}}(\mathbb{T})}.$$

Hence, using Young's inequality with $\epsilon = \dfrac{1}{2N}$ (see [54, page 622])

$$\begin{aligned}\frac{|\alpha|}{3}\left|\left(v^3, \partial_x\varphi\right)_{L^2_{t,x}}\right| &\leqslant C|\alpha|\int_{t_0}^{T}\|v\|^3_{L^3(\mathbb{T})}dt \leqslant C|\alpha|R^{\frac{3}{2}}T^{\frac{1}{4}}\left(\int_{t_0}^{T}\|v\|^2_{H_0^{\frac{1}{2}}(\mathbb{T})}dt\right)^{\frac{3}{4}} \\ &\leqslant C_N|\alpha|^4R^6T + \frac{1}{2N}\int_{t_0}^{T}\|D^{\frac{1}{2}}v\|^2_{L^2_0(\mathbb{T})}dt.\end{aligned} \tag{5.2.35}$$

On the other hand, from (5.1.3), (5.1.8), Young's inequality (see [49, Theorem 2.2]), and Cauchy-Schwarz, we get

$$\begin{aligned} I &:= \left|\int_{t_0}^{T}\left(2\pi\sum_{n\in\mathbb{Z}*}\widehat{K\partial_x v}(n)\overline{\widehat{\varphi v}(n)}\right)dt\right| \\ &\leqslant \left|\int_{t_0}^{T}\left(2\pi\sum_{n\in\mathbb{Z}*}(|n| - n\coth(n\delta))(in)\widehat{v}(n)\overline{\widehat{\varphi}*\widehat{v}(n)}\right)dt\right| + \left|\int_{t_0}^{T}\left(\frac{2\pi}{\delta}\sum_{n\in\mathbb{Z}*}\widehat{\partial_x v}(n)\overline{\widehat{\varphi v}(n)}\right)dt\right| \\ &\leqslant C_\delta\int_{t_0}^{T}\left(\sum_{n\in\mathbb{Z}*}|n|^2e^{-|n|\delta}|\widehat{v}(n)|\|\varphi\|_{L^2(\mathbb{T})}\|v\|_{L^2_0(\mathbb{T})}\right)dt + \left|\int_{t_0}^{T}\int_{\mathbb{T}}\partial_x v\ \varphi v\ dt\right| \\ &\leqslant CR\int_{t_0}^{T}\left(\sum_{n\in\mathbb{Z}*}|n|^4e^{-4|n|\delta}\right)^{\frac{1}{2}}\|v\|_{L^2_0(\mathbb{T})}dt + C\left|\int_{t_0}^{T}\int_{\mathbb{T}}v^2\ \partial_x\varphi\ dt\right| \\ &\leqslant CR^2. \end{aligned} \tag{5.2.36}$$

Thus, (5.2.34)-(5.2.36), we have

$$\left|(q, \varphi v)_{L^2_{t,x}}\right| \leqslant C(R^2 + |\alpha|^4R^6) + \frac{1}{2N}\int_{t_0}^{T}\|D^{\frac{1}{2}}v\|^2_{L^2_0(\mathbb{T})} \tag{5.2.37}$$

Finally, note that

$$\begin{aligned}[L, A]v &= -\mathcal{H}\left((\partial_x^2\varphi)v + 2(\partial_x\varphi)(\partial_x v) + \varphi\partial_x^2 v\right) + \varphi\mathcal{H}\partial_x^2 v \\ &= -[H, \varphi]\partial_x^2 v - H((\partial_x^2\varphi)v) - 2[\mathcal{H}, \partial_x\varphi]\partial_x v - 2\partial_x\varphi\mathcal{H}\partial_x v.\end{aligned} \tag{5.2.38}$$

From Lemma 5.2.5, and Remark 5.2.6 we have

$$\left|-\left([\mathcal{H},\varphi]\partial_x^2 v, v\right)_{L^2_{t,x}}\right| + \left|-\left(\mathcal{H}((\partial_x^2\varphi)v), v\right)_{L^2_{t,x}}\right| + \left|-2\left([\mathcal{H},\partial_x\varphi]\partial_x v, v\right)_{L^2_{t,x}}\right| \leqslant CR^2. \tag{5.2.39}$$

From (A.3), (A.4), and (5.2.37)-(5.2.39) we infer that (5.2.31) holds and

$$\left|(\partial_x\varphi\mathcal{H}\partial_x v, v)_{L^2_{t,x}}\right| \leqslant C(R^2 + |\alpha|^4R^6) + \frac{1}{2N}\int_{t_0}^{T}\|D^{\frac{1}{2}}v\|^2_{L^2_0(\mathbb{T})}.$$

Therefore, (5.2.29) is proved. From (5.2.28) and (5.2.29) we infer that

$$\int_{t_0}^{T}\|D^{\frac{1}{2}}v\|^2_{L^2_0(\mathbb{T})}dt \leqslant C(R^2 + \alpha^4R^6) + \frac{1}{2}\int_{t_0}^{T}\|D^{\frac{1}{2}}v\|^2_{L^2_0(\mathbb{T})},$$

where $C = C(T, \delta)$. Letting $t_0 \longrightarrow 0$ we obtain (5.2.27). □

Now we turn our attention to prove the unique continuation property. The following lemma is needed.

Lemma 5.2.8. *Let* $f(x) = \sum_{n\in\mathbb{Z}^*} c_n e^{inx}, x \in \mathbb{T}$ *such that*

i) $\sum_{n\in\mathbb{Z}^*} |c_n|^2 < +\infty$,

ii) $\exists\delta > 0, \exists k > 0$ *such that* $|c_n| \leqslant ke^{-\delta|n|}, \ \forall n < 0$,

iii) $\exists x_0 \in \mathbb{R}, \exists\epsilon > 0$ *such that* $f(x) = 0$ *for a.e.* $x \in (x_0 - \epsilon, x_0 + \epsilon)$.

Then $c_n = 0, \ \ \forall n \in \mathbb{Z}^*$.

Proof. Writing

$$f(x) = \sum_{n\in\mathbb{Z}^*} (e^{inx_0} c_n) e^{in(x-x_0)}$$

we may, without lost of generality, assume that $x_0 = 0$. Replacing f by its primitive

$$F(x) = \int_0^x f(s)ds = \sum_{n\in\mathbb{Z}^*} c_n \frac{(e^{inx} - 1)}{in} = \sum_{n\in\mathbb{Z}} \widetilde{c}_n e^{inx},$$

with $\widetilde{c}_n = \dfrac{c_n}{in}$, for $n \neq 0$, and $\widetilde{c}_0 = -\sum_{n\in\mathbb{Z}^*} \dfrac{c_n}{in}$. Note that $\widetilde{c}_0$ is well defined because

$$\sum_{n\in\mathbb{Z}^*} \left|\frac{c_n}{in}\right| \leqslant \left(\sum_{n\in\mathbb{Z}^*} |c_n|^2\right)^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}^*} \frac{1}{n^2}\right)^{\frac{1}{2}} < \infty.$$

Also, $\sum_{n\in\mathbb{Z}^*} |\widetilde{c}_n| < +\infty$, by the same estimate. Thus

$$\sum_{n\in\mathbb{Z}} |\widetilde{c}_n| < +\infty. \tag{5.2.40}$$

Let $h(z) := \sum_{n\in\mathbb{Z}} c_n e^{inz}$, then h is well defined and continuous on the set

$$D := \{z = x + iy : \ x \in \mathbb{R}, \ 0 \leqslant y < \delta\}.$$

Note that

$$|\widetilde{c}_n e^{inz}| = |\widetilde{c}_n e^{in(x+iy)}| = |\widetilde{c}_n| e^{-ny}. \tag{5.2.41}$$

From (5.2.41) and ii), we have

$$|\widetilde{c}_n e^{inz}| \leqslant ke^{-\delta|n|} e^{\delta'|n|}, \ \text{ if } 0 \leqslant y \leqslant \delta' < \delta, \text{ and } n < 0,$$

and

$$|\widetilde{c}_n| e^{-ny} \leqslant |\widetilde{c}_n|, \ \text{ if } 0 \leqslant y \leqslant \delta' < \delta, \text{ and } n \geqslant 0.$$

Also, we have that h is holomorphic on $\mathring{D} := \{z = x + iy : \; x \in \mathbb{R}, \; 0 < y < \delta\}$. Finally, h takes real values (since it vanishes) for $z = x + 0i, \;\; \forall x \in (-\epsilon, \epsilon)$. From Schwarz principle of reflection (see [27, Chapter IX]), h can be extended as an holomorphic function in an open neighborhood of 0. Therefore, h has to vanish on D, by the principle of isolated zeros (see [27, Corollay 3.10]). Hence, f vanish on D and $c_n = 0, \;\; \forall n \in \mathbb{Z}^*$. $\square$

Proposition 5.2.9. *Let $\alpha \in \mathbb{R}$, $\varepsilon \geqslant 0$, $\delta > 0$ $c \in L^2(0,T)$, and $u \in L^2((0,T); L^2_0(\mathbb{T}))$ be such that*

$$\begin{cases} \partial_t u - \mathcal{H}(\partial_x^2 u) + K\partial_x u + \alpha\partial_x(u^2) - \varepsilon\partial_x^2 u = 0, & \text{in } \mathbb{T} \times (0,T), \\ u(x,t) = c(t), & \text{for a.e. } (x,t) \in (a,b) \times (0,T). \end{cases} \tag{5.2.42}$$

for some numbers $T > 0$ and $0 \leqslant a < b \leqslant 2\pi$. Then $u(x,t) = 0$ for a.e. $(x,t) \in \mathbb{T} \times (0,T)$.

Proof. Without loss of generality, we can assume that $(a,b) = (-\mu, \mu)$ for some $\mu > 0$. In fact, setting $\widetilde{u}(x,t) = u\left(x + \dfrac{a+b}{2}, t\right), \;\; \forall (x,t) \in \mathbb{T} \times (0,T)$, we still have that $\widetilde{u}$ satisfies

$$\begin{cases} \partial_t \widetilde{u} - \mathcal{H}(\partial_x^2 \widetilde{u}) + K\partial_x \widetilde{u} + \alpha\partial_x(\widetilde{u}^2) - \varepsilon\partial_x^2 \widetilde{u} = 0, & \text{in } \mathbb{T} \times (0,T), \\ \widetilde{u}(x,t) = c(t), & \text{for a.e. } (x,t) \in (-\mu,\mu) \times (0,T), \end{cases} \tag{5.2.43}$$

where $\mu = \dfrac{b-a}{2}$. Therefore, if $\widetilde{u} = 0$ for a.e. $(x,t) \in \mathbb{T} \times (0,T)$, then $u(x,t) = 0$ for a.e. $(x,t) \in \mathbb{T} \times (0,T)$.

With this assumption and (5.2.42), we obtain that

$$\partial_x u(x,t) = \partial_x^2 u(x,t) = (u\partial_x u)(x,t) = 0, \;\; \text{for a.e. } (x,t) \in (-\mu,\mu) \times (0,T),$$

and

$$-\mathcal{H}(\partial_x^2 u) + K(\partial_x u) = -\partial_t u = -c'(t), \;\; \text{for a.e. } (x,t) \in (-\mu,\mu) \times (0,T).$$

Therefore, for almost every $t \in (0,T)$, it holds that

$$\partial_x^3 u(\cdot,t) \in H^{-3}(\mathbb{T}), \tag{5.2.44}$$

$$\partial_x^3 u(\cdot,t) = 0, \;\text{in } (-\mu,\mu), \tag{5.2.45}$$

and

$$-\mathcal{H}(\partial_x^3 u)(\cdot,t) + K(\partial_x^2 u)(\cdot,t) = 0 \;\; \text{in } (-\mu,\mu). \tag{5.2.46}$$

Pick a time t as above, and set $v(x) = \partial_x^3 u(x,t)$. Decompose v as

$$v(x) = \sum_{n \in \mathbb{Z}^*} \widehat{v}(n) e^{inx},$$

where the convergence of the Fourier series being in $H_0^{-3}(\mathbb{T})$. Observe that,

$$\partial_x^{-1} v(x) = \partial_x^{-1}\partial_x^3 u(x) = \int_0^x \partial_x^3 u(s)ds = 0, \;\; \forall x \in (-\mu,\mu),$$

We define, $f := iv - \mathcal{H}(v) + K(\partial_x^{-1}v) - \dfrac{1}{\delta}\partial_x^{-1}v \in H_0^{-3}(\mathbb{T})$. From (5.2.46), we have

$$f(x) = \left(iv - \mathcal{H}(v) + K(\partial_x^{-1}v) - \frac{1}{\delta}\partial_x^{-1}v\right)(x) = 0, \text{ in } (-\mu,\mu) \subset \mathbb{T}.$$

Note that

$$f(x) = \sum_{n\in\mathbb{Z}^*} \widehat{f}(n)e^{inx}, \quad x \in \mathbb{T}. \tag{5.2.47}$$

where $\widehat{f}(n) = iC_n\widehat{v}(n)$, with $C_n = 1 + \operatorname{sgn}(n) - \left(\dfrac{|n|}{n} - \coth(n\delta)\right)$. Set

$$F(x) := \partial_x^{-3}f(x) = \sum_{n\in\mathbb{Z}^*} \frac{\widehat{f}(n)}{(in)^3}e^{inx}, \quad x \in \mathbb{T}.$$

Note that,

$$\partial_x^{-1}f(x) = \int_0^x f(s)ds = 0, \quad \forall x \in (-\mu,\mu),$$

and

$$\partial_x^{-2}f(x) = \partial_x^{-1}(\partial_x^{-1}f(x)) = 0, \quad \forall x \in (-\mu,\mu).$$

Hence, $F(x) = \partial_x^{-3}f(x) = 0, \ \ \forall\, x \in (-\mu,\mu)$. Since $f \in H_0^{-3}(\mathbb{T})$, then $F \in L_0^2(\mathbb{T})$. Observe that, for all $n < 0$,

$$|\widehat{f}(n)| \leqslant C_\delta e^{-\delta|n|}|\widehat{v}(n)|, \tag{5.2.48}$$

and using that $\partial_x^{-3}v \in L_0^2(\mathbb{T})$, we have

$$\left|\frac{\widehat{f}(n)}{(in)^3}\right| \leqslant C_\delta \sup_{n\in\mathbb{Z}^*}\left\{\left|\widehat{\partial_x^{-3}v}(n)\right|\right\} e^{-\delta|n|} \leqslant C\, e^{-\delta|n|}, \quad \forall\, n < 0. \tag{5.2.49}$$

A direct application of Lemma 5.2.8 shows that $F \equiv 0$ in $\mathbb{T}$. Therefore, $\widehat{f}(n) = 0, \ \ \forall n \in \mathbb{Z}^*$. Thus,

$$\left(1 + \operatorname{sgn}(n) - \left(\frac{|n|}{n} - \coth(n\delta)\right)\right)\widehat{v}(n) = 0, \quad \forall\, n \in \mathbb{Z}^*.$$

This implies that $\widehat{v}(n) = 0, \ \ \forall n \in \mathbb{Z}^*$. Hence $v \equiv 0$ in $\mathbb{T}$.

Consequently, for a.e. $t \in (0,T)$, $\partial_x^3 u(\cdot,t) = 0$ in $\mathbb{T}$. Hence, for a.e. $t \in (0,T)$,

$$u(x,t) = c(t) \ \text{ in } \mathbb{T}. \tag{5.2.50}$$

Finally, substituting (5.2.50) in the first relation of (5.2.42), we obtain that $c'(t) = 0$ for a.e. $t \in (0,T)$. Therefore, $u(x,t) = c(t) = cte =: \beta$ a.e. in $\mathbb{T}\times(0,T)$. Using $u \in L^2((0,T); L_0^2(\mathbb{T}))$ we have $[u] = 0$. Consequently, we obtain $u(x,t) = \beta = 0$ a.e. in $\mathbb{T}\times(0,T)$. $\square$

Here, we state a stabilization result for the $\varepsilon-$ILW equation with a decay rate independent on ε.

Theorem 5.2.10. *Let $R > 0$ and $\delta > 0$ be given. There exists some numbers $\lambda > 0$ and $C > 0$ such that for any $\varepsilon \in (0,1)$ and any $u_0 \in L^2_0(\mathbb{T})$ with $\|u_0\|_{L^2_0(\mathbb{T})} \leqslant R$, the solution of (5.2.4) satisfies*

$$\|u(t)\|_{L^2_0(\mathbb{T})} \leqslant Ce^{-\lambda t}\|u_0\|_{L^2_0(\mathbb{T})}, \quad \forall\, t \geqslant 0.$$

Proof. From (5.2.6), $\|u(t)\|_{L^2_0(\mathbb{T})}$ is nonincresing. Thus, the exponential decay is ensured if

$$\|u((n+1)T)\|_{L^2_0(\mathbb{T})} \leqslant \kappa\|u(nT)\|_{L^2_0(\mathbb{T})}, \text{ for some } \kappa < 1.$$

To prove the theorem, it is enough (with (5.2.6)) to establish the following observability inequality: for any $T > 0$ and any $R > 0$ there exists some constant $C(T,R) > 1$ such that for any $\varepsilon \in (0,1)$ and any $u_0 \in L^2_0(\mathbb{T})$ with $\|u_0\|_{L^2_0(\mathbb{T})} \leqslant R$, it holds that

$$\|u_0\|^2_{L^2_0(\mathbb{T})} \leqslant C\left(\varepsilon\int_0^T \|\partial_x u(t)\|^2_{L^2_0(\mathbb{T})}dt + \int_0^T \|D^{\frac{1}{2}}Gu\|^2_{L^2_0(\mathbb{T})}dt\right), \tag{5.2.51}$$

where u denotes the solution of (5.2.4). Fix any $T > 0$ and any $R > 0$, and assume that (5.2.51) fails. Then there exists a sequence $\{u_0^n\}_{n>1}$ in $L^2_0(\mathbb{T})$ and a sequence $\{\varepsilon^n\}_{n>1}$ in $(0,1)$ such that for each n we have $\|u_0^n\|_{L^2_0(\mathbb{T})} \leqslant R$, and

$$\|u_0^n\|^2_{L^2_0(\mathbb{T})} > n\left(\varepsilon^n\int_0^T \|\partial_x u^n(t)\|^2_{L^2_0(\mathbb{T})}dt + \int_0^T \|D^{\frac{1}{2}}(Gu^n)\|^2_{L^2_0(\mathbb{T})}dt\right).$$

Let $\alpha^n = \|u_0^n\|_{L^2_0(\mathbb{T})} \in (0,R]$. Extracting a sequence if needed, we may assume that $\alpha^n \longrightarrow \alpha \in [0,R]$ and $\varepsilon^n \longrightarrow \varepsilon \in [0,1]$. Let $v^n = \dfrac{u^n}{\alpha^n}$. Using (5.1.2), v^n solves

$$\begin{cases} \partial_t v^n - \mathcal{H}(\partial_x^2 v^n) + K(\partial_x v^n) - \varepsilon\partial_x^2 v^n + 2\alpha^n v^n\partial_x v^n = -G(D(Gv^n)) = 0, & t > 0, \quad x \in \mathbb{T} \\ v^n(x,0) = v_0^n(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.52}$$

with $v_0^n \in L^2_0(\mathbb{T})$ and $\|v_0^n\|_{L^2_0(\mathbb{T})} = 1$. From Theorem (5.2.1) $v^n \in C([0,T], L^2_0(\mathbb{T})) \cap L^2(0,T; H^1_0(\mathbb{T}))$. Moreover, $v^n \in C((0,T], H^2_0(\mathbb{T})) \cap C^1((0,T], L^2_0(\mathbb{T}))$.

Thus, we have

$$\frac{1}{2}\|v^n(t)\|^2_{L^2_0(\mathbb{T})} + \varepsilon^n\int_0^t \|\partial_x v^n(\tau)\|_{L^2_0(\mathbb{T})}d\tau + \int_0^t \|D^{\frac{1}{2}}(Gv^n)(\tau)\|^2_{L^2_0(\mathbb{T})}d\tau = \frac{1}{2}\|v_0^n\|^2_{L^2_0(\mathbb{T})}, \quad \forall t > 0, \tag{5.2.53}$$

and

$$1 = \|v_0^n\|^2_{L^2_0(\mathbb{T})} > n\left(\varepsilon^n\int_0^T \|\partial_x v^n(t)\|^2_{L^2_0(\mathbb{T})}dt + \int_0^T \|D^{\frac{1}{2}}(Gv^n)\|^2_{L^2_0(\mathbb{T})}dt\right). \tag{5.2.54}$$

From Proposition 5.2.7, we have that

$$\int_0^T \|D^{\frac{1}{2}}(Gv^n)\|^2_{L^2_0(\mathbb{T})}dt \leqslant C. \tag{5.2.55}$$

Hence, $v_0^n \in H_0^{\frac{1}{2}(\mathbb{T})}$. This yields

$$\|G(D(Gv^n))\|_{L^2(0,T;H_0^{-\frac{1}{2}}(\mathbb{T}))} + \|(-\mathcal{H}-\varepsilon)\partial_x^2 v^n\|_{L^2(0,T;H_0^{-\frac{3}{2}}(\mathbb{T}))} \leqslant C.$$

On the other hand, for any $\mu > 0$

$$\|2v^n\partial_x v^n\|_{H_0^{\frac{-3}{2}-\mu}(\mathbb{T})} = \|\partial_x((v^n)^2)\|_{H_0^{\frac{-3}{2}-\mu}(\mathbb{T})} \leqslant C\|(v^n)^2\|_{H_0^{\frac{-1}{2}-\mu}(\mathbb{T})} \leqslant C\|(v^n)^2\|_{L^1(\mathbb{T})} \leqslant C\|v^n\|^2_{L^2_0(\mathbb{T})},$$

and from (5.1.4)

$$\|K(\partial_x v^n)\|_{H_0^{\frac{-3}{2}-\mu}(\mathbb{T})} \leqslant C_\delta\|\partial_x v^n\|_{H_0^{\frac{-3}{2}-\mu}(\mathbb{T})} \leqslant C\|v^n\|_{H_0^{\frac{-1}{2}-\mu}(\mathbb{T})} \leqslant C\|v^n\|_{L^1(\mathbb{T})} \leqslant C\|v^n\|_{L^2_0(\mathbb{T})} \leqslant C;$$

Therefore,

$$\|2\alpha^n v^n\partial_x v^n\|_{L^2(0,T;H_0^{\frac{-3}{2}-\mu}(\mathbb{T}))} \leqslant C,$$

and

$$\|K(\partial_x v^n)\|_{L^2(0,T;H_0^{\frac{-3}{2}-\mu}(\mathbb{T}))} \leqslant C.$$

Hence, $\{\partial_t v^n\}_{n>1}$ is bounded in $L^2(0,T;H_0^{\frac{-3}{2}-\mu}(\mathbb{T}))$. Combined with (5.2.55) and Aubin-Lions' lemma (see §7.3 in [77] and the references therein) , we obtain that for a subsequence still denoted by $\{v^n\}_{n>1}$, we have

$$v^n \longrightarrow v \text{ in } L^2(0,T;H_0^{\alpha}(\mathbb{T})), \ \ \forall\alpha < \frac{1}{2},$$

$$v^n \longrightarrow v \text{ in } L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T})), \ \ \text{weak},$$

$$v^n \longrightarrow v \text{ in } L^\infty(0,T;L^2_0(\mathbb{T})), \ \ \text{weak}*,$$

for some function $v \in L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T})) \cap L^\infty(0,T;L^2_0(\mathbb{T}))$. In particular,

$$\{(v^n)^2\} \longrightarrow v^2 \text{ in } L^1(\mathbb{T}\times(0,T)).$$

Letting $n \longrightarrow +\infty$ in (5.2.54), we obtain that

$$\int_0^T \|D^{\frac{1}{2}}(Gv)\|^2_{L^2_0(\mathbb{T})}dt = 0.$$

Therefore, $Gv = 0$ a.e. on $\mathbb{T}\times(0,T)$. Thus,

$$v(x,t) = \int_{\mathbb{T}} g(y)v(y,t)dy =: c(t) \ \text{ for a.e. } (x,t) \in \omega\times(0,T) \ \ (\text{see } (2.0.8)).$$

Note that $c \in L^\infty(0,T)$. Taking the limit in (5.2.52), we get

$$\begin{cases} \partial_t v - \mathcal{H}(\partial_x^2 v) + K\partial_x v + \alpha\partial_x(v^2) - \varepsilon\partial_x^2 v = 0, & \text{in } \mathbb{T}\times(0,T), \\ v(x,t) = c(t), & \text{for a.e. } (x,t) \in \omega\times(0,T). \end{cases} \tag{5.2.56}$$

From Proposition 5.2.9 we infer that $v \equiv 0$. Thus, extracting a subsequence still denoted by $\{v^n\}_{n>1}$, we have that $v^n(\cdot,t) \longrightarrow 0$ in $L^2_0(\mathbb{T})$ for a.e. $t \in (0,T)$. Finally, using (5.2.53)-(5.2.54), we infer that $v_0^n \longrightarrow 0$ in $L^2_0(\mathbb{T})$ which is a contradiction with the fact that $\|v_0^n\|_{L^2_0(\mathbb{T})} = 1$, for all $n > 1$. $\square$

Now we define the weak solutions of (5.2.1) obtained by the method of vanishing viscosity.

Definition 5.2.11. *For $u_0 \in L_0^2(\mathbb{T})$, we call a weak solution of (5.2.1) in the sense of vanishing viscosity to any function $u \in C_w((0,+\infty), L_0^2(\mathbb{T}))$ with $u \in L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T}))$ for all $T > 0$ which solves (5.2.1) in the distributional sense and such that for some sequence $\varepsilon^n \searrow 0$ we have for all $T > 0$*

$$u^n \longrightarrow u \;\; \text{in } L^2(0,T;H_0^{\frac{1}{2}}(\mathbb{T})), \;\; \text{weak},$$

$$u^n \longrightarrow u \;\; \text{in } L^\infty(0,T;L_0^2(\mathbb{T})), \;\; \text{weak}*,$$

where u^n solves (5.2.4) for $\varepsilon = \varepsilon^n$.

The following is the main result of this section.

Theorem 5.2.12. *Let $\delta > 0$. For any $u_0 \in L_0^2(\mathbb{T})$ there exists (at least) one weak solution of (5.2.1) in the sense of vanishing viscosity. On the other hand, for all $R > 0$ there exist some positive constants $\lambda = \lambda(R)$ and $C = C(R)$ such that for any weak solution $u(t)$ of (5.2.1) in the sense of vanishing viscosity, it holds that*

$$\|u(t)\|_{L_0^2(\mathbb{T})} \leqslant Ce^{-\lambda t}\|u_0\|_{L_0^2(\mathbb{T})}, \;\; \text{for all } t \geqslant 0, \tag{5.2.57}$$

whenever $\|u_0\|_{L_0^2(\mathbb{T})} \leqslant R$.

Proof. This theorem is consequence of (5.2.6), (5.2.27) and Theorem 5.2.10 (see Theorem 2.12 in [59]). □

Observe that Theorem 5.0.4 is consequence of Theorem 5.2.12.

5.2.2 Local stabilization in $H_0^s(\mathbb{T})$

In this subsection we are interested in studying the stability properties of the ILW equation with localized damping (5.2.1) in the space $H_0^s(\mathbb{T})$ with $s \geqslant 0$. We begin by stating the well-posedness and the smoothing effect results for the linearized ILW equation with localized damping.

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + GDGu = 0, & t \geqslant 0, \;\; x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.58}$$

where $\delta > 0$, and D and G are defined in (5.0.3) and (2.0.9) (respectively). Let $s \in \mathbb{R}$ and define

$$Au = -\left(\frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + GDGu\right), \tag{5.2.59}$$

with domain $D(A) = H_0^{s+2}(\mathbb{T}) \subset H_0^s(\mathbb{T})$.

Lemma 5.2.13. *The operator A defined in (5.2.59) generates a continuous semigroup in $H_0^s(\mathbb{T})$, denoted by $\{S(t)\}_{t\geqslant 0}$.*

Proof. Let $C = C(s) > 0$ be the constant in Lemma A.5. Note that $A - C$ is a densely defined closed operator in $H_0^s(\mathbb{T})$. Using Parseval's identity and lemma A.5, we have

$$\begin{aligned}(Au - Cu, u)_{H_0^s(\mathbb{T})} &= -\left(\frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u), u\right)_{H_0^s(\mathbb{T})} - (GDGu, u)_{H_0^s(\mathbb{T})} - (Cu, u)_{H_0^s(\mathbb{T})}\\ &= -\|D^{\frac{1}{2}}(Gu)\|^2_{H_0^s(\mathbb{T})} - C\|u\|^2_{H_0^s(\mathbb{T})}\\ &\leqslant -\|D^{\frac{1}{2}}(Gu)\|^2_{H_0^s(\mathbb{T})}, \quad \forall u \in H_0^{s+2}(\mathbb{T}),\end{aligned}$$

Hence $A - C$ is dissipative. Note that $D(A^*) = D(A) = H_0^{s+2}(\mathbb{T})$. Thus,

$$(A^*u - Cu, u)_{H_0^s(\mathbb{T})} = (u, Au - Cu)_{H_0^s(\mathbb{T})} \leqslant 0, \quad \forall u \in H_0^{s+2}(\mathbb{T}).$$

Therefore, $A^* - C$ is dissipative and $A - C$ generates a semigroup of contractions in $H_0^s(\mathbb{T})$ by Corollary 4.4 [71, p. 15]. $\square$

In the following result we prove the smoothing effect property. For $s \geqslant 0$ and $T > 0$, let

$$Z_{s,T} = C([0,T]; H_0^s(\mathbb{T})) \cap L^2(0,T; H_0^{s+\frac{1}{2}}(\mathbb{T})) \tag{5.2.60}$$

be endowed with the norm

$$\|u\|_{Z_{s,T}} = \|u\|_{L^\infty(0,T;H_0^s(\mathbb{T}))} + \|u\|_{L^2(0,T;H_0^{s+\frac{1}{2}}(\mathbb{T}))}.$$

Proposition 5.2.14. *Let $s \geqslant 0$, $\delta > 0$, $v_0 \in H_0^s(\mathbb{T})$ and $q \in L^2(0,T; H_0^{s-\frac{1}{2}}(\mathbb{T}))$. Then the solution v of*

$$\begin{cases} \partial_t v + \dfrac{1}{\delta}\partial_x v + \partial_x^2(\mathcal{T}v) + GDGv = q, & t \in (0,T), \quad x \in \mathbb{T}\\ v(x,0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.61}$$

satisfies $v \in Z_{s,T}$ with

$$\|v\|_{Z_{s,T}} \leqslant C(s,T)\left(\|v_0\|_{H_0^s(\mathbb{T})} + \|q\|_{L^2(0,T;H_0^{s-\frac{1}{2}}(\mathbb{T}))}\right), \tag{5.2.62}$$

with $C(s,T)$ nondecreasing in T.

Proof. First we prove the case $s = 0$. Let $T > 0$ and assume that $v_0 \in H_0^2(\mathbb{T})$ and that $q \in C([0,T]; H_0^2(\mathbb{T}))$. Therefore, the solution v of (5.2.61) satisfies $v \in C([0,T]; H_0^2(\mathbb{T})) \cap C^1([0,T]; L_0^2(\mathbb{T}))$. We use a procedure similar to the proof of Proposition 5.2.7. Using (5.1.2) we set $Lv := \partial_t v - \mathcal{H}\partial_x^2 v$, $f := -G(D(Gv)) - K\partial_x v$, so that $Lv = f + q$. Pick any $\varphi \in C^\infty(\mathbb{T})$, and let $Av = \varphi(x)v$. Then

$$\left|\int_0^T ([L,A]v, v)_{L_0^2(\mathbb{T})}\,dt\right| \leqslant 2\left|\int_0^T (f + q, \varphi v)_{L_0^2(\mathbb{T})}\,dt\right| + \|\varphi\|_{L^\infty(\mathbb{T})}\left(\|v_0\|^2_{L_0^2(\mathbb{T})} + \|v(T)\|^2_{L_0^2(\mathbb{T})}\right).$$

Scaling in (5.2.61) by v yields

$$\frac{1}{2}\|v(t)\|^2_{L^2_0(\mathbb{T})} + \int_0^t \|D^{\frac{1}{2}}(Gv)(\tau)\|^2_{L^2_0(\mathbb{T})} d\tau = \frac{1}{2}\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^t (q, v)_{L^2_0(\mathbb{T})}\, d\tau$$
$$\leqslant \frac{1}{2}\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt.$$

Hence,

$$\|v(t)\|^2_{L^\infty(0,T;L^2_0(\mathbb{T}))} + \int_0^T \|D^{\frac{1}{2}}(Gv)\|^2_{L^2_0(\mathbb{T})} dt \leqslant \frac{3}{2}\|v_0\|^2_{L^2_0(\mathbb{T})} + 3\int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt. \quad (5.2.63)$$

From and (5.2.63) and similar computations as those appearing in Proposition 5.2.7 (see (A.4) and (5.2.36)), we get that

$$\left|\int_0^T (f+q, \varphi v)_{L^2_0(\mathbb{T})}\, dt\right| \leqslant C\|\varphi\|_{H^2_0(\mathbb{T})} \int_0^T \left(\|D^{\frac{1}{2}}(Gv)\|^2_{L^2_0(\mathbb{T})} + \|v\|^2_{L^2_0(\mathbb{T})}\right) dt + \|\varphi\|_{L^\infty(\mathbb{T})} \int_o^T \int_{\mathbb{T}} |q||v| dt$$
$$\leqslant C(T, \|\varphi\|_{H^2_0(\mathbb{T})}) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right).$$

Thus

$$\left|\int_0^T ([L, A]v, v)_{L^2_0(\mathbb{T})}\, dt\right| \leqslant C(T, \|\varphi\|_{H^2_0(\mathbb{T})}) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right).$$

The last inequality combined with (5.2.38)-(5.2.39) gives

$$\left|\int_0^T (\partial_x\varphi Dv, v)_{L^2_0(\mathbb{T})}\, dt\right| \leqslant C(T, \|\varphi\|_{H^2_0(\mathbb{T})}) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right). \quad (5.2.64)$$

On the other hand, we pick $x_0 \in \mathbb{T}$ and $\chi \in C_0^\infty(\omega)$, where ω is given in (2.0.8). Writing again $\partial_x\varphi(x) = \chi^2(x) - \chi^2(x - x_0)$, for all $x \in \mathbb{T}$, and using similar computations as those leading to (5.2.30), we obtain successively, with (5.2.63) that

$$\int_0^T \left\|D^{\frac{1}{2}}(\chi v)\right\|^2_{L^2_0(\mathbb{T})} dt \leqslant C(T) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right),$$

and

$$\left|\int_0^T \left(\chi^2 Dv, v\right)_{L^2_0(\mathbb{T})} dt\right| \leqslant C \int_0^T \left(\|v\|^2_{L^2_0(\mathbb{T})} + \|D^{\frac{1}{2}}(\chi v)\|^2_{L^2_0(\mathbb{T})}\right) dt,$$
$$\leqslant C(T) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right).$$

From this and (5.2.64) we infer that

$$\left|\int_0^T \left(\chi^2(x - x_0) Dv, v\right)_{L^2_0(\mathbb{T})} dt\right| \leqslant C(T) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|v\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right).$$

Therefore, using a partition of unity and Cauchy's inequality we get

$$\int_0^T \|v\|^2_{H_0^{\frac{1}{2}}(\mathbb{T})} dt \leqslant C(T) \left(\|v_0\|^2_{L^2_0(\mathbb{T})} + \int_0^T \|q\|^2_{H_0^{-\frac{1}{2}}(\mathbb{T})} dt\right) + \frac{1}{2}\int_0^T \|v\|^2_{H_0^{\frac{1}{2}}(\mathbb{T})} dt.$$

The last inequality combined with (5.2.63) gives (5.2.62) for $s = 0$ when $v_0 \in H_0^2(\mathbb{T})$ and $q \in C([0,T]; H_0^2(\mathbb{T}))$. Using density we obtain the result for all $v_0 \in L_0^2(\mathbb{T})$ and $q \in L^1(0,T; H_0^{-\frac{1}{2}}(\mathbb{T}))$.

Next we prove the case $s > 0$. Pick any $v_0 \in H_0^{s+2}(\mathbb{T})$ and $q \in C([0,T]; H_0^{s+2}(\mathbb{T}))$, and let $v \in C([0,T]; H_0^{s+2}(\mathbb{T})) \cap C^1([0,T]; H_0^{s+2}(\mathbb{T}))$ denote the solution of (5.2.61). Set $w = D^s v$ and $h = D^s q$. Note that

$$D^s(GDGv) = GDGw + Ew$$

with $E = [D^s, G]DGD^{-s} + GD[D^s, G]D^{-s}$. Note that $\|Ew\|_{L_0^2(\mathbb{T})} \leqslant C\|w\|_{L_0^2(\mathbb{T})}$. Using (5.1.2) we have that w solves

$$\begin{cases} \partial_t w + \dfrac{1}{\delta}\partial_x w + \partial_x^2(\mathcal{T}w) + GDGw + Ew = h, & \text{for } t \in (0,T), \text{ and } x \text{ in } \mathbb{T}, \\ w(x,0) = w_0(x) = D^s v_0(x) \in H_0^2(\mathbb{T}), & \text{for } x \in \mathbb{T}. \end{cases} \tag{5.2.65}$$

Scaling in (5.2.65) by w, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|w(t)\|_{L_0^2(\mathbb{T})}^2\right) + \|D^{\frac{1}{2}}Gw(t)\|_{L_0^2(\mathbb{T})}^2 + (Ew, w)_{L_0^2(\mathbb{T})} = (h, w)_{L_0^2(\mathbb{T})}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\left(\|w(t)\|_{L_0^2(\mathbb{T})}^2\right) \leqslant \left|(Ew, w)_{L_0^2(\mathbb{T})}\right| + \left|(h, w)_{L_0^2(\mathbb{T})}\right| \leqslant C\|w(t)\|_{L_0^2(\mathbb{T})}^2 + \left|(h, w)_{L_0^2(\mathbb{T})}\right|.$$

Using the Gronwall's lemma in its differential form (see [87, Theorem 1.12, page 12]) in the last inequality, we have that

$$\int_0^T \|w(t)\|_{L_0^2(\mathbb{T})}^2 dt \leqslant C(T)\left(\|w_0\|_{L_0^2(\mathbb{T})}^2 + \int_0^T \|h\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|w\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right).$$

Form this, we infer that

$$\begin{aligned} \left|\int_0^T (\varphi w, Ew)_{L_0^2(\mathbb{T})}\, dt\right| &\leqslant C\|\varphi\|_{H_0^1(\mathbb{T})}\|w\|_{L^2(0,T;L_0^2(\mathbb{T}))}^2) \\ &\leqslant C(T, \|\varphi\|_{H_0^1(\mathbb{T})})\left(\|w_0\|_{L_0^2(\mathbb{T})}^2 + \int_0^T \|h\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}\|w\|_{H_0^{\frac{1}{2}}(\mathbb{T})} dt\right). \end{aligned}$$

Therefore, using the same estimates as in the case $s = 0$, we obtain

$$\|w\|_{L^\infty(0,T;L_0^2(\mathbb{T}))}^2\, dt + \int_0^T \|w\|_{H_0^{\frac{1}{2}}(\mathbb{T})}^2 \leqslant C(T)\left(\|w_0\|_{L_0^2(\mathbb{T})}^2 + \int_0^T \|h\|_{H_0^{-\frac{1}{2}}(\mathbb{T})}^2 dt\right),$$

or equivalently,

$$\|v\|_{L^\infty(0,T;H_0^s(\mathbb{T}))}^2 + \|v\|_{L^2(0,T;H_0^{s+\frac{1}{2}}(\mathbb{T}))}^2 \leqslant C(T)\left(\|v_0\|_{H_0^s(\mathbb{T})}^2 + \|q\|_{L^2(0,T;H_0^{s-\frac{1}{2}}(\mathbb{T}))}^2\right). \tag{5.2.66}$$

Inequality (5.2.66) and the fact that $v \in C([0,T]; H_0^s(\mathbb{T}))$ are also true for $v_0 \in H_0^s(\mathbb{T})$ and $q \in L^2(0,T; H_0^{s-\frac{1}{2}}(\mathbb{T}))$ by density.

□

Corollary 5.2.15. *Let $s \geqslant 0$, $\delta > 0$, and $B \in \mathcal{L}(H_0^s(\mathbb{T}))$. Then for any $v_0 \in H_0^s(\mathbb{T})$, the solution v of*

$$\begin{cases} \partial_t v + \frac{1}{\delta}\partial_x v + \partial_x^2(\mathcal{T}v) + GDGv = Bv, & t \geqslant 0, \quad x \in \mathbb{T} \\ v(x,0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.2.67}$$

satisfies $v \in Z_{s,T}$ with

$$\|v\|_{Z_{s,T}} \leqslant C(s,T)\left(\|v_0\|_{H_0^s(\mathbb{T})}\right).$$

Proof. This Corollary is consequence of Proposition 5.2.14 (see Corollary 2.17 in [59]). □

Now we prove the local well-posedness of (5.2.1) in $H_0^s(\mathbb{T})$ with $s \geqslant \frac{1}{2}$.

Theorem 5.2.16. *Let $s \in \left(\frac{1}{2}, 2\right]$ and $\delta > 0$ be given. Then there exists $\rho > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $\|u_0\|_{H_0^s(\mathbb{T})} < \rho$, there exists some time $T > 0$ such that (5.2.1) admits a unique solution in the space $Z_{s,T}$.*

Proof. An application of the fixed point theorem, together with Proposition 5.2.14 and the Sobolev embedding $H_0^s(\mathbb{T}) \subset L^\infty(\mathbb{T})$ for $s > \frac{1}{2}$ imply this result (see Theorem 2.13 in [59]). □

Finally, we derive a local exponential stability result in H_0^s for $s > \frac{1}{2}$.

Theorem 5.2.17. *Let $s \in \left(\frac{1}{2}, 2\right]$ and $\delta > 0$ be given. Then there exist some numbers $\rho > 0$, $\lambda > 0$ and $C > 0$ such that for any $u_0 \in H_0^s(\mathbb{T})$ with $\|u_0\|_{H_0^s(\mathbb{T})} < \rho$, there is a unique solution $u : (0,+\infty) \longrightarrow H_0^s(\mathbb{T})$ of (5.2.1) with $u \in Z_{s,T}$ for all $T > 0$ and such that*

$$\|u(t)\|_{H_0^s(\mathbb{T})} \leqslant Ce^{-\lambda t}\|u_0\|_{H_0^s(\mathbb{T})}, \quad \forall t \geqslant 0.$$

Proof. Similar to the proof of Theorem 2.14 in [59]. □

Observe that Theorem 5.0.5 is consequence of theorem 5.2.17.

5.3 Control of Intermediate Long Wave equation

In this section we study the control properties of the ILW equation. Specifically, we prove the exact controllability of the system

$$\begin{cases} \partial_t u + \frac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + \partial_x(u^2) = Gh, & t \in (0,T), \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.3.1}$$

where $\delta > 0$, D and G are defined in (5.0.3) and (2.0.9) respectively, and h is the control input. As the contraction mapping can not be applied directly to ILW equation, we incorporate the feedback $-DGu$ into the control input h to obtain a strong enough smoothing effect to apply the contraction principle.

Setting

$$h(t) = -DGu(t) + D^{\frac{1}{2}}\tilde{h}(t),$$

we are thus led to investigate the controllability of the system

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + GDGu + \partial_x(u^2) = GD^{\frac{1}{2}}\tilde{h}, & t \in (0,T), \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{5.3.2}$$

Therefore, we will prove the following local exact controllability result.

Theorem 5.3.1. *Let $\delta > 0$, $s \in \left(\frac{1}{2}, 2\right]$, and $T > 0$. Then there exists $\mu > 0$ such that for any $u_0,\ u_1 \in H_0^s(\mathbb{T})$ with*

$$\|u_0\|_{H_0^s(\mathbb{T})} < \mu, \quad \|u_1\|_{H_0^s(\mathbb{T})} < \mu$$

one may find a control $\tilde{h} \in L^2(0,T;H^s(\mathbb{T}))$ such that the system (5.3.2) admits a unique solution u in the class $Z_{s,T}$ for which $u(x,T) = u_1(x)$.

Proof. Following a similar procedure as in the proof of Theorem 3.1 in [59], we prove Theorem 5.3.1 in two steps.

Step 1. First, we prove the exact controllability of the linearized system

$$\begin{cases} \partial_t u + \dfrac{1}{\delta}\partial_x u + \partial_x^2(\mathcal{T}u) + GDGu = GD^{\frac{1}{2}}\tilde{h}, & t \in (0,T), \quad x \in \mathbb{T} \\ u(x,0) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{5.3.3}$$

in $H_0^s(\mathbb{T})$ for $s \geqslant 0$. For this, we use the Hilbert Uniqueness Method (HUM) following the same approach as in [76, 59]. Observe that this theorem is a consequence of Proposition 5.2.14, Proposition 5.2.9 and the fact that the operator $(1-\partial_x^2)^{-\frac{s}{2}}$ commutes with $\mathcal{T}$ (see steps 1 and 2 in the proof of Theorem 3.1 in [59]).

Step 2. Here we use the fixed-point argument as in [72, 76, 59] to prove the exact controllability of (5.3.2) in $H_0^s(\mathbb{T})$. Pick any $\delta > 0$, $s \in \left(\frac{1}{2}, 2\right]$, and $T > 0$. We consider $\{S(t)\}_{t\geqslant 0}$ the semigroup introduced in the Lemma 5.2.13 and $Z_{s,T}$ the space defined in (5.2.60). For $v \in Z_{s,T}$, we set

$$w(v) = \int_0^T S(T-t)(2v\partial_x v)(t)\, dt.$$

From step 1, we have that the linearized system (5.3.3), with initial data $u_0 \in H_0^s(\mathbb{T})$ and the control function $\tilde{h} \in L^2(0,T;H_0^s(\mathbb{T}))$, is well-posed and exactly controllable in

$H_0^s(\mathbb{T})$. Using classical functional analysis argument (see e.g. [25, Lemma 2.48 p. 58]), we can construct a continuous operator $\Phi : H_0^s(\mathbb{T}) \longrightarrow L^2(0,T;H_0^s(\mathbb{T}))$ such that for any $u_1 \in H_0^s(\mathbb{T})$ the solution u of (5.3.3) associated with $u_0 = 0$ and $\tilde{h} = \Phi(u_1)$ satisfies $u(T) = u_1$. Let us denote by $u = W(\tilde{h})$ the corresponding trajectory, i.e.

$$W(\tilde{h})(t) = u(t) = \int_0^t S(t-\tau)GD^{\frac{1}{2}}\tilde{h}(\tau)d\tau.$$

From Proposition 5.2.14 we have that W is continuous from $L^2(0,T;H_0^s(\mathbb{T}))$ into $Z_{s,T}$. Let $v \in Z_{s,T}$ and choose

$$\tilde{h} = \Phi(u_1 - S(T)u_0 + w(v)),$$

then

$$S(t)u_0 - \int_0^t S(t-\tau)(2v\partial_x v)(\tau)d\tau + W(\tilde{h})(t) = \begin{cases} u_0, & if\ t=0, \\ u_1, & if\ t=T. \end{cases}$$

This suggests that we consider the nonlinear map $v \longrightarrow \Gamma(v)$, where

$$\Gamma(v)(t) = S(t)u_0 - \int_0^t S(t-\tau)(2v\partial_x v)(\tau)d\tau + W(\Phi(u_1 - S(T)u_0 + w(v)))(t).$$

We will prove that Γ has a fixed point in the space $Z_{s,T}$. Using similar estimates as in the proof of Theorem 5.2.16, we obtain that

$$\|w(v)\|_{H_0^s(\mathbb{T})} \leqslant C\left\|\int_0^t S(t-\tau)(2v\partial_x v)(\tau)d\tau\right\|_{Z_{s,T}} \leqslant C\|v\|_{Z_{s,T}}^2$$

and that there are some constants $C_0 > 0$ and $C_1 > 0$ such that

$$\|\Gamma(v)\|_{Z_{s,T}} \leqslant C_0\left(\|u_0\|_{H_0^s(\mathbb{T})} + \|u_1\|_{H_0^s(\mathbb{T})}\right) + C_1\|v\|_{Z_{s,T}}^2, \quad \forall v \in Z_{s,T},$$

$$\|\Gamma(v^1) - \Gamma(v^2)\|_{Z_{s,T}} \leqslant C_1\left(\|v^1\|_{Z_{s,T}} + \|v^2\|_{Z_{s,T}}\right)\|v^1 - v^2\|_{Z_{s,T}}, \quad \forall v^1, v^2 \in Z_{s,T}.$$

Let $B := \{v \in Z_{s,T};\ \|v\|_{Z_{s,T}} \leqslant R\}$. We choose the radious R in such a way that the ball B is left invariant by Γ and Γ contracts in B, i.e.

$$C_0\left(\|u_0\|_{H_0^s(\mathbb{T})} + \|u_1\|_{H_0^s(\mathbb{T})}\right) + C_1R^2 \leqslant R,$$

and

$$2C_1R < 1.$$

It is sufficient to take $R = (4C_1)^{-1}$ and $\mu := \dfrac{R}{4C_0}$. Therefore, the unique fixed-point u of Γ satisfies $u(x,T) = u_1$ and the proof of Theorem 5.3.1 is complete. □

Finally, note that Theorem 5.0.6 follows at once from Theorem 5.3.1 by letting

$$h = -DGu + D^{\frac{1}{2}}\tilde{h} \in L^2(0,T;H_0^{s-\frac{1}{2}}(\mathbb{T})).$$

Concluding Remarks

In this thesis, we considered the Benjamin and the Intermediate Long Wave (ILW) equations posed on a periodic domain $\mathbb{T}$ and proved that the both are exactly controllable and exponentially stabilizable. These results are in accordance with the controllability and stabilization results for the KdV and BO equations obtained respectively in [52] and [59]. From this work, three articles have emerged:

- *On the controllability and stabilization of the linearized Benjamin equation on a Periodic Domain*, Nonlinear Analysis: Real World Applications, **51** (2020) 102978 (see [67]).
- *On the controllability and stabilization of the Benjamin equation*, submitted for publication and is available in ArXiv:1904.03492 (see [68]).
- *On the controllability and stabilization of the Intermediate Long Wave equation on a periodic domain*, in preparation.

Extending these results about the controllability and stabilization to the bidimensional dispersive models is a challenging task. In this direction, consider the bidimensional Benjamin equation

$$\partial_t u + \partial_x^3 u - \alpha \mathcal{H} \partial_x^2 u - \partial_x^{-1} \partial_y^2 u + \frac{1}{2}\partial_x(u^2) = 0, \quad (x, y) \in \mathbb{T}^2, \ t \geqslant 0, \tag{C5.1}$$

posed on a periodic domain $\mathbb{T}^2$.

In order to prove the controllability of the linearized bidimensional Benjamin equation associated to (C5.1), we tried to follow the approach in [20]. In this approach the uniform gap condition for the eigenvalues associated to the linear operator $\partial_x^3 - \alpha\mathcal{H}\partial_x^2 - \lambda^2\partial_x^{-1}$ for any $\lambda \in \mathbb{Z}$ is required. This gap condition goes to zero when λ goes to infinity. To overcome this situation, we tried to use another approach known in the literature involving for instance the Theorem of Kahane (see page 153 of [46]). However, it was not possible to prove the exact controllability of the linearized bidimensional Benjamin equation associated to (C5.1) using that tool. As far as we know, the controllability of the Linearized bidimensional Benjamin equation associated to (C5.1) is still an open problem.

Another bidimensional model of interest is the Benjamin-Ono equation

$$\partial_t u - \mathcal{H}^{(x)}(\partial^2_{xy} u) + u\partial_y u = 0, \quad (x, y) \in \mathbb{T}^2, \ t \geqslant 0, \tag{C5.2}$$

posed on $\mathbb{T}^2$, where $\mathcal{H}^{(x)}$ is the Hilbert transform in the x-variable. For the details about this model we refer to [63] and the references therein.

It is important to point out that, during the development of our doctoral project we studied the controllability and stabilization properties for the bidimensional Benjamin-Ono equation (C5.2). In fact, this was the first problem we dealt with. Due to the lack of the spectral gap condition in higher dimension we could not succeed and shifted our attention to the Benjamin equation. However, with the knowledge acquired during the development of this thesis, we think that it might be possible to prove the exact controllability of the (C5.2) by using the approach in [20]. This is a future work.

Recently, Flores et al. [34, 35], studied the controllability and stabilization properties for the dispersion-generalized Benjamin-Ono equation in $\mathbb{T}$. As a future work, we believe it is possible to apply the techniques used in [34, 35] to obtain the controllability and stabilization for a generalized Benjamin type equation on a periodic domain.

$$\partial_t u - \alpha\mathcal{H}^{2r}\partial_x u - \beta D^{2m}\partial_x u + u^p \partial_x u = 0, \quad x \in \mathbb{T}, \ \ t > 0, \tag{C5.3}$$

where $\widehat{D^{2m}u}(k) = |k|^{2m}\hat{u}(k)$, $m \geqslant 1$, $\widehat{\mathcal{H}^{2r}u}(k) = -|k|^{2r}\hat{u}(k)$, $0 < r < m$, $p \geqslant 1$ is an integer and α, β are non negative constants (see [58] and the references therein).

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Appendix

Lemma A.1. *Let $s, r \in \mathbb{R}$. Let f denotes the operator of multiplication by $f \in C^\infty(\mathbb{T})$. Then, $[D^r, f] := D^r\ f - f\ D^r$ maps any $H^s(\mathbb{T})$ into $H^{s-r+1}(\mathbb{T})$. That is, there exists a constant $c = c_f$ depending only on f such that*

$$\|[D^r, f]\phi\|_{H^{s-r+1}(\mathbb{T})} \leqslant c_f \|\phi\|_{H^s(\mathbb{T})}. \tag{A.1}$$

Proof. This result is proved in [51] (see Lemma A.1 in the appendix). □

Lemma A.2. *Let $f \in C^\infty(\mathbb{T})$. Then, there exists some constant C such that for every $s \in \mathbb{R}$, there exists C_s such that the following estimate holds*

$$\|fv\|_{H^s(\mathbb{T})} \leqslant C \|v\|_{H^s(\mathbb{T})} + C_s \|v\|_{H^{s-1}(\mathbb{T})}, \tag{A.2}$$

for all $v \in H^s(\mathbb{T})$.

Proof. This result is proved in [51] (see Corollary A.2 in the appendix), which follows just writing

$$D^s(fv) = fD^s v + [D^s, f]v,$$

where $[D^s, f] := D^s\ f - f\ D^s$. □

Lemma A.3. *Let $f \in C^\infty(\mathbb{T})$ and $\rho_\epsilon = e^{\epsilon^2 \partial_x^2}$ with $0 \leqslant \epsilon \leqslant 1$. Then $[\rho_\epsilon, f]$ is uniformly bounded as an operator from H^s into H^{s+1} and*

$$\|[\rho_\epsilon, f]v\|_{H^{s+1}(\mathbb{T})} \leqslant C_s \|v\|_{H^s(\mathbb{T})},$$

for all $v \in H^s(\mathbb{T})$.

Proof. This result is proved in [51] (see Lemma A.3 in the appendix). The proof is exactly the same as for Lemma A.1 using

$$\left| e^{-\epsilon^2 n^2} - e^{-\epsilon^2 k^2} \right| \leqslant C\ |n-k|\ \left(\langle n \rangle^{-1} + \langle k \rangle^{-1} \right), \tag{A.3}$$

because

$$|\partial_\xi \left(e^{-\epsilon^2\xi^2}\right)| \leqslant C \langle \xi \rangle^{-1}. \tag{A.4}$$

□

Lemma A.4. *A function $\phi \in C^\infty(\mathbb{T})$ can be written in the form $\partial_x \varphi$ for some function $\varphi \in C^\infty(\mathbb{T})$, if and only if,*

$$\int_{\mathbb{T}} \phi(x)\, dx = 0. \tag{A.5}$$

Proof. ($\Rightarrow$) Let $\phi \in C^\infty(\mathbb{T})$ and assume that $\phi(x) = \partial_x\varphi(x)$, for all $x \in \mathbb{T}$, where $\varphi \in C^\infty(\mathbb{T})$. Then φ is a $2\pi-$periodic function. Thus,

$$\int_{\mathbb{T}} \phi(x)\, dx = \int_{\mathbb{T}} \partial_x\varphi(x)\, dx = \varphi(2\pi) - \varphi(0) = 0. \tag{A.6}$$

($\Leftarrow$) Let $\phi \in C^\infty(\mathbb{T})$, with

$$\int_{\mathbb{T}} \phi(x)\, dx = \int_0^{2\pi} \phi(x)\, dx = 0.$$

Define

$$\varphi(x) = \int_0^x \phi(s)\, ds,$$

for all $x \in \mathbb{T}$. Note that $\varphi(0) = \varphi(2\pi) = 0$, thus φ is a $2\pi-$periodic function in $\mathbb{R}$. Then by Fundamental Theorem of Calculus we obtain that $\partial_x\varphi(x) = \phi(x)$, for all $x \in \mathbb{T}$, and $\varphi \in C^\infty(\mathbb{T})$. □

Lemma A.5. *For any $s \in \mathbb{R}$, there exists a constant $C = C(s) > 0$ such that*

$$-\left(G(D(Gu)), u\right)_{H_0^s(\mathbb{T})} \leqslant C\|u\|^2_{H_0^s(\mathbb{T})} - \|D^{\frac{1}{2}}(Gu)\|^2_{H_0^s(\mathbb{T})}, \quad \forall u \in H_0^{s+1}(\mathbb{T}).$$

Proof. See Claim 1 of Lemma 2.2 in [59]. □