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INVARIANT GEOMETRY OF REAL FLAG MANIFOLDS: GEODESICS AND EINSTEIN METRICS

GEOMETRIA INVARIANTE DE VARIEDADES BANDEIRA REAIS: GEODÉSICAS E MÉTRICAS DE EINSTEIN

CAMPINAS
2020

Brian David Grajales Triana

Invariant geometry of real flag manifolds: geodesics and Einstein metrics

Geometria invariante de variedades bandeira reais: geodésicas e métricas de Einstein

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Resumo

Nós descrevemos as métricas Riemannianas invariantes numa variedade bandeira real de uma álgebra de Lie clássica que é uma forma real normal de uma álgebra de Lie simples complexa. Isso nos permite classificar aquelas variedades com a propriedade de que toda geodésica é a órbita de um subgrupo a um parâmetro. Tal variedade é chamada de g.o. Também estudamos o problema de encontrar métricas de Einstein invariantes para variedades bandeira reais cuja representação de isotropia se decompõe em dois ou três sub-representações irredutíveis. Em contraste ao caso complexo, a representação de isotropia de uma variedade bandeira real pode ter submódulos equivalentes, consequentemente, métricas invariantes não diagonais e uma grande família de variedades g.o. aparecem.

Palavras-chave: Variedades bandeira, geodésicas homogêneas, métricas de Einstein, álgebras de Lie.

Abstract

We describe the invariant Riemannian metrics on a real flag manifold of a classical Lie algebra which is a split real form of a complex simple Lie algebra. This allows us to classify those manifolds with the property that every geodesic is the orbit of a one-parameter subgroup. Such a manifold is called g.o. We also study the problem of finding Einstein invariant metrics for real flag manifolds whose isotropy representation decomposes into two or three irreducible sub-representations. In contrast to the complex case, the isotropy representation of a real flag manifold may have equivalent submodules, consequently, non-diagonal invariant metrics and a large family of g.o. manifolds appear.

Keywords: Flag manifolds, homogeneous geodesics, Einstein metrics, Lie algebras.

Contents

Introduction **10**

1 Preliminaries: The isotropy representation of real flag manifolds **13**
1.1 Flags of A_l . . . 14
1.2 Flags of B_l . . . 17
1.3 Flags of C_l . . . 19
1.4 Flags of D_l . . . 21

2 Invariant Riemannian metrics **23**
2.1 Flags of A_l . . . 24
2.2 Flags of B_l . . . 26
2.3 Flags of C_l . . . 29
2.4 Flags of D_l . . . 36

3 Geodesic orbit spaces **40**
3.1 Flags of A_l . . . 41
3.2 Flags of B_l . . . 45
3.3 Flags of C_l . . . 53
3.4 Flags of D_l . . . 58

4 Invariant Einstein metrics on real flag manifolds with two or three isotropy summands **65**
4.1 Flags with two isotropy summands . . . 67
4.1.1 $SO(4)/S(O(2) \times O(2))$ . . . 67
4.1.2 $(SO(l) \times SO(l+1))/(SO(l-1) \times SO(l)),\ l \geq 3$ . . . 68
4.1.3 $(SO(l) \times SO(l+1))/SO(l),\ l \geq 3,\ l \neq 4$ . . . 68
4.1.4 $U(l)/O(l),\ l \geq 3$ . . . 69
4.1.5 $U(l)/(O(1) \times U(l-1)),\ l \geq 3$ . . . 70
4.1.6 $(SO(4) \times SO(4))/SO(4)$ . . . 71
4.2 Flags with three isotropy summands . . . 72
4.2.1 $SO(4)/S(O(2) \times O(1) \times O(1))$ . . . 72
4.2.2 $SO(l+1)/S(O(l_1) \times O(l_2) \times O(l_3)),\ l \geq 2,\ l \neq 3,\ l_1 + l_2 + l_3 = l+1$ . . . 77
4.2.3 $(SO(4) \times SO(5))/SO(4)$ . . . 83
4.2.4 $(SO(l) \times SO(l+1))/(SO(d) \times SO(l-d) \times SO(l-d+1)),\ l \geq 3,\ 2 \leq d \leq l-1$ 86
4.2.5 $U(l)/(O(d) \times U(l-d)),\ l \geq 3,\ 2 \leq d \leq l-1$ . . . 89
4.2.6 $(SO(l) \times SO(l))/S(O(l-1) \times O(1))$ . . . 91
4.3 Equivalent metrics . . . 97

5 Conclusions and future work **99**

Bibliography **101**

Introduction

In this work, we deal with Riemannian geometry of *real flag manifolds*. By a real flag manifold we will refer the quotient space given by the split real form of a complex Lie group by some parabolic subgroup. We are interested in the study of geodesics and Einstein Riemannian metrics in these manifolds.

A real form $\mathfrak{g}$ of a complex simple Lie algebra is called *split* if for every Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, there exists a Cartan subalgebra $\mathfrak{a}$ contained in $\mathfrak{s}$. In this case, we consider the associated set of roots Π and the corresponding root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha.$$

Fixing a set Π^+ of positive roots and taking Σ its corresponding set of simple roots, we can parametrize parabolic subalgebras $\mathfrak{p}_\Theta \subseteq \mathfrak{g}$ with subsets $\Theta \subset \Sigma$ (see Chapter 1 for details). A real flag manifold of $\mathfrak{g}$ is the homogeneous space $\mathbb{F}_\Theta = G/P_\Theta$, where G is a connected Lie group whose Lie algebra is $\mathfrak{g}$ and P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G. If K is the connected subgroup of G generated by $\mathfrak{k}$, then K acts transitively on $\mathbb{F}_\Theta$ with isotropy $K_\Theta = K \cap P_\Theta$, so $\mathbb{F}_\Theta = K/K_\Theta$ as well. For a fixed element $o \in \mathbb{F}_\Theta$, there exists a one-to-one correspondence between $K-$invariant tensor fields on $\mathbb{F}_\Theta$ and tensors on $T_o\mathbb{F}_\Theta$ that are invariant with respect to the isotropy representation of K_Θ. Thus, the understanding of the isotropy representation is fundamental in the study of invariant geometry of real flag manifolds. It is important to highlight a fundamental difference between real and complex flag manifolds. Recall that a complex flag manifold is the homogeneous space K/K_Θ, where K is a *compact real form* of a simple, complex Lie group $G^{\mathbb{C}}$ and K_Θ is the centralizer of a 1-parameter subgroup of K. In the complex case, the isotropy representation decomposes into irreducible, pairwise inequivalent submodules. In contrast, real flag manifolds can admit equivalent sub-representations. This fact suggests that the invariant geometry in the real case is richer than in the complex case. Invariant geometry of complex flag manifolds has been extensively studied (see, for instance [1], [2], [4], [5], [6], [7], [12], [15], [16], [18], [24], [26], [27]). For real flag manifolds, topological and geometric aspects have been considered by several authors ([8], [13], [23], [25], [31]). In particular, the isotropy representation of real flag manifolds was recently described by Patrão and San Martin in [22]. This description was essential to get the $K-$invariant Riemannian metrics on a flag $\mathbb{F}_\Theta$ of a classical Lie algebra (see Chapter 2).

In the context of Riemannian homogeneous spaces $(G/H, g)$, an interesting class of geodesics are *homogeneous geodesics*. A geodesic γ is called homogeneous if it is the orbit of a 1-parameter subgroup of G, that is,

$$\gamma(t) = \exp(tX) \cdot o,$$

where X is in the Lie algebra of G and o is a fixed element in G/H. The vector X is called *geodesic vector*. An algebraic criterion to decide when a vector X is a geodesic vector was proved by Kowalski and Vanhecke in [19]. The homogeneous Riemannian space $(G/H, g)$ is called a *g.o. space* if every geodesic is homogeneous. Examples of such spaces are naturally reductive spaces, symmetric and weakly symmetric spaces. The first example of a g.o. space which is not naturally reductive was given by Kaplan in [17]. In [19], Kowalski and Vanhecke classified the g.o. spaces in low dimensions. In [14], Gordon described all g.o. spaces which are nilmanifolds (nilpotent Lie groups provided with left-invariant Riemannian metrics). For complex flag manifolds, the classification was given by Alekseevsky and Arvanitoyeorgos in [2]. One of our goals is to provide necessary and sufficient conditions for a real flag manifold of a classical Lie group to be a g.o. space. This will be done using a subtle variation of the criterion in [19] given by Souris in [28]. In particular, we obtain a large number of examples of g.o. spaces where the metric is not normal.

Another classical problem in differential geometry is the description of Einstein metrics on a differentiable manifold M. A Riemannian metric g is an *Einstein metric* if it satisfies

$$\mathrm{Ric} = cg, \tag{0.0.1}$$

where c is a real number and Ric is the Ricci tensor associated to the metric g. The equation (0.0.1) is usually called *Einstein equation* and the number c is called *Einstein constant*. Those metrics are important to study several problems of Geometry and Physics (see [9]). In particular, they appear as solutions of the Einstein field equations for the interaction between gravity and space-time in the vacuum. From a variational point of view, we can see Einstein metrics on a differentiable manifold as critical points of the total *scalar curvature functional* restricted to the set of Riemannian metrics of volume 1 (see [9], [11]).

In general, equation (0.0.1) becomes a system of partial differential equations. In the context of a homogeneous space provided with an invariant metric, this equation is equivalent to a system of algebraic equations. A complete classification of invariant Einstein metrics on homogeneous spaces is still an open problem. There are examples of homogeneous spaces admitting infinitely many invariant Einstein metrics up to homotheties (see for instance [3], [29]), in this case, the isotropy representation has equivalent irreducible submodules. In [11]; Böhm, Wang and Ziller conjectured that for a compact homogeneous space whose isotropy representation decomposes into parwise inequivalent irreducible summands, the number of invariant Einstein metrics is finite. In this work, we study the Ricci equation for real flag manifolds of classical Lie algebras whose isotropy representation splits into two or three irreducible submodules. The complex case was studied in [5], [7] and [18].

This thesis is organized as follows: in Chapter 1, we review the results about the isotropy representation done by Patrão and San Martin in [22]. Chapter 2 is dedicated to the description of the invariant metrics on real flag manifolds associated to classical Lie groups. In Chapter 3, we provide a complete classification of real flag manifolds of classical type which are *g.o. spaces*. In Chapter 4, we study the problem of finding invariant Einstein metrics for real flag manifolds of classical Lie groups where the isotropy representation has two or three isotropy summands. In this situation, we only have two cases admitting equivalent submodules. We also treat the isometry problem between Einstein metrics (Section 4.5). In Chapter 5, we give the main conclusions of this thesis and discuss the important directions of future work.

Chapter 1

Preliminaries: The isotropy representation of real flag manifolds

Let $\mathfrak{g}$ be a non-compact, simple real Lie algebra. We consider the case where $\mathfrak{g}$ is a split real form of a complex Lie algebra. A generalized flag manifold of $\mathfrak{g}$ is the homogeneous space $\mathbb{F}_\Theta = G/P_\Theta$ where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and $P_\Theta \subset G$ is a parabolic subgroup. The Lie algebra $\mathfrak{p}_\Theta$ of P_Θ is a parabolic subalgebra, which is the direct sum of the eigenspaces associated with the non-negative eigenvalues of $\mathrm{ad}(H_\Theta)$, where $H_\Theta \in \mathfrak{g}$ is an element chosen in an appropriate way. If $K \subset G$ is a maximal compact subgroup and $K_\Theta = K \cap P_\Theta$ then we have and identification $\mathbb{F}_\Theta = K/K_\Theta$. By compactness of K, all flag manifolds are reductive homogeneous spaces and, therefore, we have a reductive decomposition of the Lie algebra of K given by

$$\mathfrak{k} = \mathfrak{k}_\Theta \oplus \mathfrak{m}_\Theta,$$

where $\mathfrak{m}_\Theta$ is an $\mathrm{Ad}(K_\Theta)-$invariant complement of $\mathfrak{k}_\Theta$. We fix an $\mathrm{Ad}(K)-$invariant inner product $(\cdot,\cdot)$ on $\mathfrak{k}$ such that the reductive decomposition $\mathfrak{k} = \mathfrak{k}_\Theta \oplus \mathfrak{m}_\Theta$ is $(\cdot,\cdot)-$orthogonal, since K_Θ is compact, the adjoint representation of K_Θ in $\mathfrak{m}_\Theta$ induces a $(\cdot,\cdot)-$orthogonal splitting

$$\mathfrak{m}_\Theta = \bigoplus_{i=1}^{s} \mathfrak{m}_i \tag{1.0.1}$$

of $\mathfrak{m}_\Theta$ into $K_\Theta-$invariant, irreducible submodules $\mathfrak{m}_i$, $i = 1, ..., s$. We say that the submodules $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are *equivalent* if there exists an $\mathrm{Ad}(K_\Theta)-$equivariant isomorphism $T : \mathfrak{m}_i \longrightarrow \mathfrak{m}_j$, that is, $\mathrm{Ad}(k)|_{\mathfrak{m}_j} \circ T = T \circ \mathrm{Ad}(k)|_{\mathfrak{m}_i}$, for all $k \in K_\Theta$. Evidently, this equivalence relation induces a partition

$$\{\mathfrak{m}_1, ..., \mathfrak{m}_s\} = C_1 \cup ... \cup C_S \text{ with } S \leq s,$$

so, we have a new $(\cdot,\cdot)-$orthogonal splitting

$$\mathfrak{m}_\Theta = \bigoplus_{i=1}^{S} M_i \tag{1.0.2}$$

where

$$M_i = \bigoplus_{\mathfrak{m}_j \in C_i} \mathfrak{m}_j, \quad i = 1, ..., S.$$

We call each $\mathfrak{m}_j$ an *isotropy summand* and each M_i an *isotypical summand* of the decomposition (1.0.1).

For an alternative description of the parabolic subalgebra we consider a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$. Denote by Π the associated set of roots and by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha,$$

the corresponding root space decomposition. Fixing a set Π^+ of positive roots let Σ be the corresponding set of simple roots. Any $\Theta \subseteq \Sigma$ defines a parabolic subalgebra

$$\mathfrak{p}_\Theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Pi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \langle \Theta \rangle^-} \mathfrak{g}_\alpha,$$

where $\langle \Theta \rangle^-$ is the set of negative roots generated by Θ. We say that $H_\Theta \in \mathfrak{a}$ is *characteristic* for Θ if $\alpha(H_\Theta) \geq 0$ for every $\alpha \in \Sigma$ and $\Theta = \{\alpha \in \Sigma : \alpha(H_\Theta) = 0\}$. The subalgebra

$$\mathfrak{n}_\Theta^- = \bigoplus_{\alpha \in \Pi^- \setminus \langle \Theta \rangle^-} \mathfrak{g}_\alpha,$$

is identified with the tangent space of $\mathbb{F}_\Theta$ at the origin eP_Θ. If $\mathfrak{z}_\Theta = Cent_{\mathfrak{g}}(H_\Theta)$, then the adjoint representation of $\mathfrak{z}_\Theta$ on $\mathfrak{n}_\Theta^-$ is completely reducible and we can decompose

$$\mathfrak{n}_\Theta^- = \bigoplus_\sigma V_\Theta^\sigma,$$

into $\mathfrak{z}_\Theta-$invariant, irreducible and non-equivalent subspaces.

With this notation we have that $K_\Theta = Cent_K(H_\Theta)$ and $\mathfrak{k}_\Theta = Cent_{\mathfrak{k}}(H_\Theta)$. The tangent space at $o = eK_\Theta \in \mathbb{F}_\Theta$ is identified with $\mathfrak{m}_\Theta$ and there exists a one-to-one correspondence between $G-$invariant tensors on $\mathbb{F}_\Theta$ and tensors on $T_o\mathbb{F}_\Theta \approx \mathfrak{m}_\Theta$ which are invariant with respect to the isotropy representation of K_Θ. If H_α, $\alpha \in \Sigma$ and $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Pi$ is a Weyl basis for $\mathfrak{g}_{\mathbb{C}}$, we identify $\mathfrak{n}_\Theta^-$ with $\mathfrak{m}_\Theta$ via

$$X_\alpha \mapsto X_\alpha - X_{-\alpha}, \alpha \in \Pi^- \setminus \langle \Theta \rangle^-.$$

The $\mathfrak{z}_\Theta-$invariant subspaces V_Θ^σ are $K_\Theta-$invariant but not necessarily $K_\Theta-$irreducible. In the following sections we present the description given by Patrão and San Martin in [22] of the $K_\Theta-$invariant irreducible subspaces of each V_Θ^σ and their equivalences by the adjoint representation of K_Θ.

1.1 Flags of A_l

The special linear Lie algebra $A_l = \mathfrak{sl}(l+1, \mathbb{R})$ is composed by the real $(l+1) \times (l+1)$ matrices with trace zero. In this case, $\mathfrak{a}$ is the subalgebra of traceless diagonal matrices. The roots are given by $\alpha_{ij} = \lambda_i - \lambda_j,\ 1 \leq i \neq j \leq l+1$, where

$$\lambda_i : \mathfrak{a} \longrightarrow \mathbb{R},\ \lambda_i(\mathrm{diag}(a_1, ..., a_{l+1})) = a_i,\ i = 1, ..., l+1.$$

The simple roots are $\alpha_i = \alpha_{i,i+1}$, $i = 1, ..., l$. The Lie algebra $\mathfrak{k}$ is the set $\mathfrak{so}(l+1)$ of skew-symmetric real matrices of order $l+1$. We fix $(\cdot,\cdot) = -\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the Killing form of $\mathfrak{so}(l+1)$. Let $\mathfrak{m}_{ij}$ be the subspace spanned by $w_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the real $(l+1)\times(l+1)$ matrix with value equal to 1 in the $(i,j)-$entry and zero elsewhere. The set $\{w_{ij} : 1 \leq j < i \leq l+1\}$ is an $(\cdot,\cdot)-$orthogonal basis for $\mathfrak{so}(l+1)$. For every $\Theta \subseteq \Sigma$, there exist positive integers $l_1, ..., l_r$ such that $l+1 = l_1 + ... + l_r$, and if we set $\tilde{l_0} = 0$, $\tilde{l_i} = \tilde{l}_{i-1} + l_i$, $i = 1, ..., r$, then Θ is written as the union of its connected components as

$$\Theta = \bigcup_{l_i > 1} \left\{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i - 1}\right\}. \tag{1.1.1}$$

By writing Θ in this form, we have that $K_\Theta = S(O(l_1) \times ... \times O(l_r))$.

Proposition 1.1.1. *([22]) For any flag manifold $\mathbb{F}_\Theta$ of A_l, $l \neq 3$, the $K_\Theta-$invariant irreducible subspaces of $\mathfrak{m}_\Theta$ are*

$$M_{mn} = \bigoplus_{\substack{1\leq i\leq l_m \\ 1\leq j\leq l_n}} \mathfrak{m}_{\tilde{l}_{m-1}+i, \tilde{l}_{n-1}+j}, \ 1 \leq n < m \leq r. \tag{1.1.2}$$

Two such subspaces are not equivalent. □

When $l = 3$, we have the following possibilities for Θ :

• $\Theta = \emptyset$, $\mathbb{F}_\emptyset = SO(4)/S(O(1) \times O(1) \times O(1) \times O(1))$.

In this case, $\mathfrak{k}_\emptyset = \{0\}$ and the tangent space $\mathfrak{m}_\emptyset$ has a decomposition

$$\mathfrak{m}_\emptyset = \mathfrak{m}_{21} \oplus \mathfrak{m}_{43} \oplus \mathfrak{m}_{31} \oplus \mathfrak{m}_{42} \oplus \mathfrak{m}_{32} \oplus \mathfrak{m}_{41}$$

where all the $\mathfrak{m}_{ij}$, $1 \leq j < i \leq 4$ are $K_\emptyset-$invariant, irreducible and

$$\begin{aligned} M_1 &= \mathfrak{m}_{21} \oplus \mathfrak{m}_{43}, \\ M_2 &= \mathfrak{m}_{31} \oplus \mathfrak{m}_{42}, \\ M_3 &= \mathfrak{m}_{32} \oplus \mathfrak{m}_{41} \end{aligned}$$

are the corresponding isotypical summands.

• $\Theta = \{\alpha_1\}$, $\mathbb{F}_{\{\alpha_1\}} = SO(4)/S(O(2) \times O(1) \times O(1))$.

In this case, $\mathfrak{k}_{\{\alpha_1\}} = \mathfrak{m}_{21}$ and

$$\mathfrak{m}_{\{\alpha_1\}} = \mathfrak{m}_{43} \oplus (\mathfrak{m}_{31} \oplus \mathfrak{m}_{32}) \oplus (\mathfrak{m}_{42} \oplus \mathfrak{m}_{41})$$

where $\mathfrak{m}_{43}$, $\mathfrak{m}_{31} \oplus \mathfrak{m}_{32}$ and $\mathfrak{m}_{42} \oplus \mathfrak{m}_{41}$ are $K_{\{\alpha_1\}}-$invariant, irreducible and

$$\begin{aligned} M_1 &= \mathfrak{m}_{43}, \\ M_2 &= (\mathfrak{m}_{31} \oplus \mathfrak{m}_{32}) \oplus (\mathfrak{m}_{42} \oplus \mathfrak{m}_{41}) \end{aligned}$$

are the corresponding isotypical summands.

• $\Theta = \{\alpha_2\}$, $\mathbb{F}_{\{\alpha_2\}} = SO(4)/S(O(1) \times O(2) \times O(1))$.

We have $\mathfrak{k}_{\{\alpha_2\}} = \mathfrak{m}_{32}$ and

$$\mathfrak{m}_{\{\alpha_2\}} = \mathfrak{m}_{41} \oplus (\mathfrak{m}_{21} \oplus \mathfrak{m}_{31}) \oplus (\mathfrak{m}_{43} \oplus \mathfrak{m}_{42})$$

where $\mathfrak{m}_{21}$, $\mathfrak{m}_{31} \oplus \mathfrak{m}_{41}$ and $\mathfrak{m}_{42} \oplus \mathfrak{m}_{32}$ are $K_{\{\alpha_2\}}-$invariant, irreducible and

$$\begin{array}{rcl} M_1 & = & \mathfrak{m}_{41}, \\ M_2 & = & (\mathfrak{m}_{21} \oplus \mathfrak{m}_{31}) \oplus (\mathfrak{m}_{43} \oplus \mathfrak{m}_{42}) \end{array}$$

are the corresponding isotypical summands.

• $\Theta = \{\alpha_3\}$, $\mathbb{F}_{\{\alpha_3\}} = SO(4)/S(O(1) \times O(1) \times O(2))$.

In this case, $\mathfrak{k}_{\{\alpha_3\}} = \mathfrak{m}_{43}$ and

$$\mathfrak{m}_{\{\alpha_3\}} = \mathfrak{m}_{21} \oplus (\mathfrak{m}_{31} \oplus \mathfrak{m}_{41}) \oplus (\mathfrak{m}_{42} \oplus \mathfrak{m}_{32})$$

where $\mathfrak{m}_{41}$, $\mathfrak{m}_{21} \oplus \mathfrak{m}_{31}$ and $\mathfrak{m}_{43} \oplus \mathfrak{m}_{42}$ are $K_{\{\alpha_3\}}-$invariant, irreducible and

$$\begin{array}{rcl} M_1 & = & \mathfrak{m}_{41}, \\ M_2 & = & (\mathfrak{m}_{21} \oplus \mathfrak{m}_{31}) \oplus (\mathfrak{m}_{43} \oplus \mathfrak{m}_{42}) \end{array}$$

are the corresponding isotypical summands.

• $\Theta = \{\alpha_1, \alpha_2\}$ or $\{\alpha_2, \alpha_3\}$, $\mathbb{F}_\Theta = SO(4)/S(O(3) \times O(1))$ or $SO(4)/S(O(1) \times O(3))$, respectively.

For these sets, the adjoint representation of K_Θ on $\mathfrak{m}_\Theta$ is irreducible.

• $\Theta = \{\alpha_1, \alpha_3\}$, $\mathbb{F}_{\{\alpha_1,\alpha_3\}} = SO(4)/S(O(2) \times O(2))$.

We have $\mathfrak{k}_{\{\alpha_1,\alpha_3\}} = \mathfrak{m}_{21} \oplus \mathfrak{m}_{43}$, and $\mathfrak{m}_{\{\alpha_1,\alpha_3\}} = M_1 \oplus M_2$, where

$$M_1 = \left\{ \begin{pmatrix} 0 & -B \\ B & 0 \end{pmatrix} : \ B \text{ has the form } B = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \ a, b \in \mathbb{R} \right\}$$

and

$$M_2 = \left\{ \begin{pmatrix} 0 & -B^T \\ B & 0 \end{pmatrix} : \ B \text{ has the form } B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \ a, b \in \mathbb{R} \right\}$$

are not equivalent $K_{\{\alpha_1,\alpha_3\}}-$invariant, irreducible subspaces.

1.2 Flags of B_l

The Lie algebra $B_l = \mathfrak{so}(l+1,l)$ is the set

$$\mathfrak{so}(l+1,l) = \left\{ \begin{pmatrix} 0 & -a & -b \\ b^T & A & B \\ a^T & C & -A^T \end{pmatrix} \in \mathfrak{gl}(2l+1,\mathbb{R}) \ : \ B + B^T = C + C^T = \mathbf{0} \right\}.$$

We consider the abelian subalgebra

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\Lambda \end{pmatrix} \in \mathfrak{so}(l+1,l) \ : \ \Lambda = \mathrm{diag}(a_1,...,a_l) \right\}.$$

The set of roots is described as follows:

- The long ones $\pm(\lambda_i - \lambda_j)$, $\pm(\lambda_i + \lambda_j)$, $1 \le i < j \le l$ and
- the short ones $\pm\lambda_i$, $1 \le i \le l$,

where

$$\lambda_i : \left\{ H = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\Lambda \end{pmatrix} : \Lambda = \mathrm{diag}(a_1,...,a_l) \right\} \longrightarrow \mathbb{R}, \ \ \lambda_i(H) = a_i, \ i = 1,...,l.$$

The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$, $1 \le i \le l-1$ and $\alpha_l = \lambda_l$. The subalgebra $\mathfrak{k}$ is the set of $(2l+1)\times(2l+1)$ skew-symmetric matrices

$$\begin{pmatrix} 0 & -a & -a \\ a^T & A & B \\ a^T & B & A \end{pmatrix}, \ A + A^T = B + B^T = \mathbf{0}.$$

It is isomorphic to $\mathfrak{so}(l+1) \oplus \mathfrak{so}(l)$. The isomorphism is provided by the decomposition

$$\begin{pmatrix} 0 & -a & -a \\ a^T & A & B \\ a^T & B & A \end{pmatrix} = \begin{pmatrix} 0 & -a & -a \\ a^T & \frac{(A+B)}{2} & \frac{(A+B)}{2} \\ a^T & \frac{(A+B)}{2} & \frac{(A+B)}{2} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{(A-B)}{2} & -\frac{(A-B)}{2} \\ \mathbf{0} & -\frac{(A-B)}{2} & \frac{(A-B)}{2} \end{pmatrix}.$$

We fix the $\mathrm{Ad}(K)-$invariant inner product $(\cdot,\cdot)$ on $\mathfrak{k}$ defined by

$$\left(\begin{pmatrix} 0 & -a & -a \\ a^T & A & B \\ a^T & B & A \end{pmatrix}, \begin{pmatrix} 0 & -c & -c \\ c^T & C & D \\ c^T & D & C \end{pmatrix} \right) = ac^T - \frac{1}{2}(\mathrm{Tr}(BD) + \mathrm{Tr}(AC)). \tag{1.2.1}$$

The matrices

$$\begin{aligned} v_k &= E_{1+k,1} - E_{1,1+k} + E_{1+l+k,1} - E_{1,1+l+k}, & 1 \le k \le l, \\ w_{ij} &= E_{1+i,1+j} - E_{1+j,1+i} + E_{1+l+i,1+l+j} - E_{1+l+j,1+l+i}, & \\ u_{ij} &= E_{1+l+i,1+j} - E_{1+l+j,1+i} + E_{1+i,1+l+j} - E_{1+j,1+l+i}, & 1 \le j < i \le l, \end{aligned} \tag{1.2.2}$$

where E_{ij} is the $(2l+1)\times(2l+1)$-matrix with value equal to 1 in the $(i,j)-$entry and zero elsewhere, form a $(\cdot,\cdot)-$orthonormal basis for $\mathfrak{k}$.

We take r positive integers $l_1, ..., l_r$ such that $l = l_1 + ... + l_r$ and if $\tilde{l}_0 = 0,\ \tilde{l}_i = \tilde{l}_{i-1} + l_i$ then

$$\Theta = \bigcup_{l_i>1}\{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i-1}\} \text{ or } \bigcup_{l_i>1}\{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i-1}\}\cup\{\alpha_l\}. \tag{1.2.3}$$

Proposition 1.2.1. *([22]) Let $\mathbb{F}_\Theta$ be a flag manifold of B_l, with $l \geq 5$. Then the following subspaces are $K_\Theta-$invariant and irreducible:*

a)

$$V_i = \text{span}\{v_{\tilde{l}_{i-1}+s} : 1 \leq s \leq l_i\},$$

with $1 \leq i \leq r$ if $\alpha_l \notin \Theta$. All these subspaces are not equivalent.

b)

$$W_{mn} = \bigoplus_{\substack{1\leq i\leq l_m\\ 1\leq j\leq l_n}} \text{span}\{w_{\tilde{l}_{m-1}+i,\tilde{l}_{n-1}+j}\} \textit{ and } U_{mn} = \bigoplus_{\substack{1\leq i\leq l_m\\ 1\leq j\leq l_n}} \text{span}\{u_{\tilde{l}_{m-1}+i,\tilde{l}_{n-1}+j}\},$$

with $1 \leq n < m \leq r$ if $\alpha_l \notin \Theta$ and $1 \leq n < m \leq r-1$ if $\alpha_l \in \Theta$. For each (m,n), W_{mn} and U_{mn} are equivalent. We denote by $M_{mn} = W_{mn} \oplus U_{mn}$.

c)

$$U_i = \text{span}\{u_{\tilde{l}_{i-1}+s,\tilde{l}_{i-1}+t} : 1 \leq t < s \leq l_i\}$$

for i such that $l_i > 1$ and $1 \leq i \leq r$ if $\alpha_l \notin \Theta$ and $1 \leq i \leq r-1$ if $\alpha_l \in \Theta$. All these subspaces are not equivalent.

d)

$$\begin{aligned}(V_i)_1 &= \text{span}\{w_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} - u_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} : 1 \leq s \leq l_r, 1 \leq t \leq l_i\}\\ (V_i)_2 &= \text{span}\{v_{\tilde{l}_{i-1}+t},\ w_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} + u_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} : 1 \leq s \leq l_r, 1 \leq t \leq l_i\}\end{aligned}$$

with $1 \leq i \leq r-1$ when $\alpha_l \in \Theta$. All these subspaces are not equivalent. □

For B_2, B_3, B_4 and some subsets $\Theta \subseteq \Sigma$, one can have different equivalences among the $K_\Theta-$invariant irreducible subspaces above. Even more, there are $K_\Theta-$invariant subspaces different from those in Proposition 1.2.1.

Example 1.2.2. *Consider the flag of B_4 given by $\Theta = \{\alpha_1, \alpha_2, \alpha_3\}$. Then $\mathfrak{m}_\Theta$ decomposes into $K_\Theta-$invariant, irreducible and non-equivalent subspaces as $\mathfrak{m}_\Theta = V_1 \oplus T_1 \oplus T_2$, where V_1 is defined as in Proposition 1.2.1 and*

$$T_1 = \text{span}\{u_{21} + u_{43}, u_{31} - u_{42}, u_{41} + u_{32}\},\ T_2 = \text{span}\{u_{21} - u_{43}, u_{31} + u_{42}, u_{41} - u_{32}\}.$$

1.3 Flags of C_l

The symplectic real Lie algebra $C_l = \mathfrak{sp}(l, \mathbb{R})$ is the set

$$\left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{gl}(2l, \mathbb{R}) \ : \ B - B^T = C - C^T = \mathbf{0} \right\}$$

and the subalgebra $\mathfrak{a}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & -\Lambda \end{pmatrix} \in \mathfrak{sp}(l, \mathbb{R}) \ : \ \Lambda = \mathrm{diag}(a_1, ..., a_l) \right\}.$$

The set of roots is described as follows:

- The long ones $\pm 2\lambda_i$, $1 \leq i \leq l$, and
- the short ones $\pm(\lambda_i - \lambda_j)$ and $\pm(\lambda_i + \lambda_j)$, $1 \leq i < j \leq l$

where

$$\lambda_i : \ \left\{ H = \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & -\Lambda \end{pmatrix} : \Lambda = \mathrm{diag}(a_1, ..., a_l) \right\} \ \longrightarrow \ \mathbb{R}, \ \ \lambda_i(H) = a_i, \ i = 1, ..., l.$$

The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$, $1 \leq i \leq l-1$ and $\alpha_l = 2\lambda_l$. The subalgebra $\mathfrak{k}$ is the set of $2l \times 2l$ matrices

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \ A + A^T = B - B^T = \mathbf{0}$$

which is isomorphic to $\mathfrak{u}(l)$, where the isomorphism associates the above matrix to $A + \sqrt{-1}B$. We fix the $\mathrm{Ad}(K)-$invariant inner product $(\cdot, \cdot)$ on $\mathfrak{k}$ defined by

$$\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \begin{pmatrix} C & -D \\ D & C \end{pmatrix} \right) = \frac{1}{2}(\mathrm{Tr}(BD) - \mathrm{Tr}(AC)). \tag{1.3.1}$$

The $2l \times 2l$ matrices

$$\begin{array}{ll} u_{kk} = E_{l+k,k} - E_{k,l+k}, & 1 \leq k \leq l, \\ w_{ij} = E_{ij} - E_{ji} + E_{l+i,l+j} - E_{l+j,l+i}, & \\ u_{ij} = E_{l+i,j} + E_{l+j,i} - E_{i,l+j} - E_{j,l+i}, & 1 \leq j < i \leq l \end{array} \tag{1.3.2}$$

form a $(\cdot, \cdot)-$orthogonal basis for $\mathfrak{k}$. In what follows, we describe the $K_\Theta-$invariant, irreducible subspaces. As in the previous section, for each Θ, take positive integers $l_1, ..., l_r$ such that $l = l_1 + ... + l_r$ and Θ is written as disjoint union of its connected components as

$$\Theta = \bigcup_{l_i > 1} \{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i - 1}\} \text{ or } \bigcup_{l_i > 1} \{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i - 1}\} \cup \{\alpha_l\} \tag{1.3.3}$$

where $\tilde{l}_0 = 0$, $\tilde{l}_{i-1} + l_i$, $i = 1, ..., r$. If $\alpha_l \notin \Theta$, then $K_\Theta \overset{\text{diff.}}{\approx} O(l_1) \times ... \times O(l_r)$, otherwise, we have $K_\Theta \overset{\text{diff.}}{\approx} O(l_1) \times ... \times O(l_{r-1}) \times U(l_r)$.

Proposition 1.3.1. *([22]) Let $\mathbb{F}_\Theta$ be a flag manifold of C_l, with $l \neq 4$. The following subspaces are $K_\Theta-$invariant irreducible:*

a)

$$V_i = \text{span}\{u_{\tilde{l}_{i-1}+1,\tilde{l}_{i-1}+1} + ... + u_{\tilde{l}_i,\tilde{l}_i}\},$$

with $1 \leq i \leq r$ if $\alpha_l \notin \Theta$ and $1 \leq i \leq r-1$ if $\alpha_l \in \Theta$. All these subspaces are equivalent. Set $M_0 = V_1 \oplus ... \oplus V_{\tilde{r}}$, where $\tilde{r} = r$ if $\alpha_l \notin \Theta$ and $\tilde{r} = r-1$ if $\alpha_l \in \Theta$.

b)

$$W_{mn} = \bigoplus_{\substack{1\leq i\leq l_m \\ 1\leq j\leq l_n}} \text{span}\{w_{\tilde{l}_{m-1}+i,\tilde{l}_{n-1}+j}\} \text{ and } U_{mn} = \bigoplus_{\substack{1\leq i\leq l_m \\ 1\leq j\leq l_n}} \text{span}\{u_{\tilde{l}_{m-1}+i,\tilde{l}_{n-1}+j}\},$$

with $1 \leq n < m \leq r$ if $\alpha_l \notin \Theta$ and $1 \leq n < m \leq r-1$ if $\alpha_l \in \Theta$. For each (m,n), W_{mn} and U_{mn} are equivalent. We denote $M_{mn} = W_{mn} \oplus U_{mn}$.

c)

$$M_{rn} = \bigoplus_{\substack{1\leq i\leq l_r \\ 1\leq j\leq l_n}} \text{span}\{w_{\tilde{l}_{r-1}+i,\tilde{l}_{n-1}+j}\} \oplus \bigoplus_{\substack{1\leq i\leq l_r \\ 1\leq j\leq l_n}} \text{span}\{u_{\tilde{l}_{r-1}+i,\tilde{l}_{n-1}+j}\},$$

with $1 \leq n \leq r-1$, if $\alpha_l \in \Theta$. All these subspaces are not equivalent.

d)

$$U_i = \text{span}\{u_{\tilde{l}_{i-1}+s,\tilde{l}_{i-1}+s} - u_{\tilde{l}_{i-1}+s+1,\tilde{l}_{i-1}+s+1} : 1 \leq s \leq l_i - 1\} \cup \{u_{\tilde{l}_{i-1}+s,\tilde{l}_{i-1}+t} : 1 \leq t < s \leq l_i\},$$

for i such that $l_i > 1$ and $1 \leq i \leq r$ if $\alpha_l \notin \Theta$, $1 \leq i \leq r-1$ if $\alpha_l \in \Theta$. All these subspaces are not equivalent.

Any other pair of subspaces are not equivalent. □

When $l = 4$, in addition to the subspaces described in Proposition 1.3.1, we have more equivalent subspaces for some subsets Θ. The table below shows the equivalence classes for the flags $\mathbb{F}_\Theta$ of C_4 where there exist equivalences and irreducible submodules different to those presented in Proposition 1.3.1.

Θ	Equivalence classes
$\emptyset$	$\{V_1, V_2, V_3, V_4\}$, $\{W_{21}, W_{43}, U_{21}, U_{43}\}$, $\{W_{31}, W_{42}, U_{31}, U_{42}\}$, $\{W_{32}, W_{41}, U_{32}, U_{41}\}$
$\{\alpha_1\}$	$\{V_1, V_2, V_3\}, \{W_{21}, W_{31}, U_{21}, U_{31}\}, \{W_{32}, U_{32}\}, \{U_1\}$
$\{\alpha_2\}$	$\{V_1, V_2, V_3\}, \{W_{21}, W_{32}, U_{21}, U_{32}\}, \{W_{31}, U_{31}\}, \{U_2\}$
$\{\alpha_3\}$	$\{V_1, V_2, V_3\}, \{W_{31}, W_{32}, U_{31}, U_{32}\}, \{W_{21}, U_{21}\}, \{U_3\}$

Table 1.

1.4 Flags of D_l

The Lie algebra $D_l = \mathfrak{so}(l,l)$ is the set

$$\left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{gl}(2l,\mathbb{R}) \; : \; B + B^T = C + C^T = \mathbf{0} \right\}.$$

In this case

$$\mathfrak{a} = \left\{ \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & -\Lambda \end{pmatrix} \in \mathfrak{so}(l,l) \; : \; \Lambda = \mathrm{diag}(a_1,...,a_l) \right\}.$$

The roots are $\pm(\lambda_i - \lambda_j)$, $\pm(\lambda_i + \lambda_j)$, $1 \le i < j \le l$, where

$$\lambda_i : \left\{ H = \begin{pmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & -\Lambda \end{pmatrix} : \Lambda = \mathrm{diag}(a_1,.,..,a_l) \right\} \longrightarrow \mathbb{R},\; \lambda_i(H) = a_i,\; i = 1,...,l.$$

The simple roots are $\alpha_i = \lambda_i - \lambda_{i+1}$, $i = 1,...,l-1$ and $\alpha_l = \lambda_{l-1} + \lambda_l$. The subalgebra $\mathfrak{k}$ is the set of skew-symmetric $2l \times 2l$ matrices of the form

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},\; A + A^T = B + B^T = 0$$

which is isomorphic to $\mathfrak{so}(l) \oplus \mathfrak{so}(l)$ via the decomposition

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \begin{pmatrix} \frac{A+B}{2} & \frac{A+B}{2} \\ \frac{A+B}{2} & \frac{A+B}{2} \end{pmatrix} + \begin{pmatrix} \frac{A-B}{2} & -\frac{A-B}{2} \\ -\frac{A-B}{2} & \frac{A-B}{2} \end{pmatrix}.$$

We fix the $\mathrm{Ad}(K)-$invariant inner product $(\cdot,\cdot)$ on $\mathfrak{k}$ given by

$$\left(\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \begin{pmatrix} C & D \\ D & C \end{pmatrix} \right) = -\frac{\mathrm{Tr}(AC) + \mathrm{Tr}(BD)}{2}. \tag{1.4.1}$$

The matrices

$$\begin{array}{l} w_{ij} = E_{ij} - E_{ji} + E_{l+i,l+j} - E_{l+j,l+i}, \\ u_{ij} = E_{l+i,j} - E_{l+j,i} + E_{i,l+j} - E_{j,l+i},\; 1 \le j < i \le l \end{array} \tag{1.4.2}$$

form a $(\cdot,\cdot)-$orthogonal basis for $\mathfrak{k}$. Given $\Theta \subseteq \Sigma$, we take $l_1,...,l_r$ as in (1.2.3).

Proposition 1.4.1. *([22]) Let $\mathbb{F}_\Theta$ be a flag manifold of D_l, $l \ge 5$. Then the following subspaces are $K_\Theta-$invariant and irreducible:*

a)

$$\begin{array}{l} W_{mn} = \mathrm{span}\{w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} : 1 \le s \le l_m,\; 1 \le t \le l_n\}, \\ U_{mn} = \mathrm{span}\{u_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} : 1 \le s \le l_m,\; 1 \le t \le l_n\}, \end{array}$$

with $1 \le n < m \le r$ if $\alpha_l \notin \Theta$, $1 \le n < m \le r-1$ if $\alpha_l, \alpha_{l-1} \in \Theta$ and $1 \le n < m \le r-2$ if $\alpha_l \in \Theta$ and $\alpha_{l-1} \notin \Theta$. For each (m,n), W_{mn} is equivalent to U_{mn}. We set $M_{mn} = W_{mn} \oplus U_{mn}$.

b)

$$U_i = \mathrm{span}\{u_{\tilde{l}_{i-1}+s,\tilde{l}_{i-1}+t} : 1 \le t < s \le l_i\},$$

with $l_i > 1$, $1 \leq i \leq r$ *if* $\alpha_l \notin \Theta$, $1 \leq i \leq r-1$ *if* $\alpha_l, \alpha_{l-1} \in \Theta$ *and* $1 \leq i \leq r-2$ *if* $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \notin \Theta$. *All these subspaces are not equivalent.*

c)

$$M_{rn} = \bigoplus_{\substack{1\leq s\leq l_r \\ 1\leq t\leq l_n}} \text{span}\{w_{\tilde{l}_{r-1}+s,\tilde{l}_{n-1}+t}\} \oplus \bigoplus_{\substack{1\leq s\leq l_r \\ 1\leq t\leq l_n}} \text{span}\{u_{\tilde{l}_{r-1}+s,\tilde{l}_{n-1}+t}\},$$

with $1 \leq n \leq r-1$ *when* $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \in \Theta$. *All these subspaces are not equivalent.*

d)

$$\begin{aligned} M_n &= \text{span}\{w_{\tilde{l}_{r-2}+s,\tilde{l}_{n-1}+t} : 1 \leq s \leq l_{r-1},\ 1 \leq t \leq l_n\} \cup \{u_{l,\tilde{l}_{n-1}+t} : 1 \leq t \leq l_n\}, \\ N_n &= \text{span}\{u_{\tilde{l}_{r-2}+s,\tilde{l}_{n-1}+t} : 1 \leq s \leq l_{r-1},\ 1 \leq t \leq l_n\} \cup \{w_{l,\tilde{l}_{n-1}+t} : 1 \leq t \leq l_n\}, \end{aligned}$$

with $1 \leq n \leq r-2$ *when* $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \notin \Theta$. *For each* $n \in \{1, ..., r-2\}$, M_n *is equivalent to* N_n. *We set* $S_n = M_n \oplus N_n$.

e)

$$V_{r-1} = \text{span}\{u_{\tilde{l}_{r-2}+s,\tilde{l}_{r-2}+t} : 1 \leq t < s \leq l_{r-1}\} \cup \{w_{l,\tilde{l}_{r-2}+t} : 1 \leq t \leq l_{r-1}\},$$

when $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \notin \Theta$. □

The case $l = 4$ is different from the general case since $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(4)$ decomposes into four copies of $\mathfrak{so}(3)$ which yield new invariant subspaces.

Example 1.4.2. *Consider the flag* $\mathbb{F}_{\{\alpha_1,\alpha_2,\alpha_3\}}$ *of* D_4. *The isotropy representation decomposes into two non-equivalent* $K_{\{\alpha_1,\alpha_2,\alpha_3\}}-$*invariant irreducible subspaces given by*

$$T_1 = \text{span}\{u_{21} + u_{43}, u_{31} - u_{42}, u_{41} + u_{32}\} \text{ and } S_1 = \text{span}\{u_{43} - u_{21}, u_{31} + u_{42}, u_{41} - u_{32}\}.$$

Chapter 2

Invariant Riemannian metrics

It is known that there exists a one-to-one correspondence between $K-$invariant metrics on $\mathbb{F}_\Theta = K/K_\Theta$ and $\mathrm{Ad}(K_\Theta)-$invariant inner products g on $\mathfrak{m}_\Theta$, that is,

$$g(\mathrm{Ad}(k)X, \mathrm{Ad}(k)Y) = g(X, Y) \text{ for all } k \in K_\Theta,\ X, Y \in \mathfrak{m}_\Theta. \tag{2.0.1}$$

For every such g, there exists a unique $(\cdot,\cdot)-$self-adjoint, positive operator $A : \mathfrak{m}_\Theta \longrightarrow \mathfrak{m}_\Theta$ commuting with $\mathrm{Ad}(k)$ for all $k \in K_\Theta$ such that

$$g(X, Y) = (AX, Y) \text{ for all } X, Y \in \mathfrak{m}_\Theta. \tag{2.0.2}$$

Any $\mathrm{Ad}(K_\Theta)-$invariant inner product is determined by such an operator. We call A the *metric operator* corresponding to g. We will make no distinction between g and its metric operator A. If we consider a $(\cdot,\cdot)-$orthogonal ordered basis $\mathcal{B} = \mathcal{B}_1 \cup ... \cup \mathcal{B}_S$ adapted to the decomposition (1.0.2) such that for every i, all vectors in $\mathcal{B}_i$ have the same norm with respect to $(\cdot,\cdot)$. Then, any metric operator A can be written in the basis $\mathcal{B}$ as a block-diagonal matrix of the form

$$[A]_{\mathcal{B}} = \begin{pmatrix} [A|_{M_1}]_{\mathcal{B}_1} & 0 & \dots & 0 \\ 0 & [A|_{M_2}]_{\mathcal{B}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & [A|_{M_S}]_{\mathcal{B}_S} \end{pmatrix}. \tag{2.0.3}$$

If $M_i = \mathfrak{m}_{i_1} \oplus ... \oplus \mathfrak{m}_{i_n}$, then $[A|_{M_i}]_{\mathcal{B}_i}$ has the form

$$[A|_{M_i}]_{\mathcal{B}_i} = \begin{pmatrix} \mu_1 \mathrm{I}_{\mathfrak{m}_{i_1}} & B_{21}^T & \dots & B_{n1}^T \\ B_{21} & \mu_2 \mathrm{I}_{\mathfrak{m}_{i_2}} & \dots & B_{n2}^T \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & \mu_n \mathrm{I}_{\mathfrak{m}_{i_n}} \end{pmatrix} \tag{2.0.4}$$

where B_{ab}^T represents the transpose of B_{ab} and $\mu_1, ..., \mu_n > 0$. We shall use the facts above and the results presented in Chapter 1 to obtain the invariant metrics on classical real flag manifolds.

2.1 Flags of A_l

Recall that, for flags of A_l, we are considering $(\cdot,\cdot)$ as the negative of the Killing form of $\mathfrak{so}(l+1)$. Let $\Theta \subseteq \Sigma$ written as in (1.1.1), by Proposition 1.1.1, we have that

$$\mathfrak{m}_\Theta = \bigoplus_{1\leq n<m\leq r} M_{mn}, \tag{2.1.1}$$

where the subspaces M_{mn} are described in (1.1.2).

Proposition 2.1.1. *Let $\mathbb{F}_\Theta$ be a flag of A_l, $l \neq 3$. Then, every metric operator A is determined by $\frac{r(r-1)}{2}$ positive numbers μ_{mn}, $1 \leq n < m \leq r$ such that*

$$A|_{M_{mn}} = \mu_{mn} I_{M_{mn}},\ 1 \leq n < m \leq r. \tag{2.1.2}$$

Proof. By Proposition 1.1.1, we know that the subspaces M_{mn} are $K_\Theta-$invariant, irreducible, $(\cdot,\cdot)-$orthogonal and pairwise non-equivalent. Let $\mathcal{B}$ be any $(\cdot,\cdot)-$orthogonal basis adapted to (2.1.1), then every isotypical summand M_{mn} is irreducible, by (2.0.4) we have that $A|_{M_{mn}}$ is a positive multiple of the identity map for each (m,n), so we have the result. □

Now, consider the case when $l = 3$. For each $\Theta \subseteq \Sigma$, we fix an ordered $(\cdot,\cdot)-$orthogonal basis for $\mathfrak{m}_\Theta$:

$$\begin{array}{lcl}
\mathcal{B}_\emptyset & = & \{w_{21}, w_{43}, w_{31}, w_{42}, w_{32}, w_{41}\},\\
\mathcal{B}_{\{\alpha_1\}} & = & \{w_{43}, w_{31}, w_{32}, w_{42}, w_{41}\},\\
\mathcal{B}_{\{\alpha_2\}} & = & \{w_{41}, w_{21}, w_{31}, w_{43}, w_{42}\},\\
\mathcal{B}_{\{\alpha_3\}} & = & \{w_{21}, w_{31}, w_{41}, w_{42}, w_{32}\},\\
\mathcal{B}_{\{\alpha_1,\alpha_2\}} & = & \{w_{41}, w_{42}, w_{43}\},\\
\mathcal{B}_{\{\alpha_2,\alpha_3\}} & = & \{w_{21}, w_{31}, w_{41}\},\\
\mathcal{B}_{\{\alpha_1,\alpha_3\}} & = & \{w_{31} - w_{42}, w_{41} + w_{32}, w_{31} + w_{42}, w_{41} - w_{32}\}.
\end{array} \tag{2.1.3}$$

Proposition 2.1.2. *Let A be an invariant metric on a flag $\mathbb{F}_\Theta$ of A_3. Then, A is written in the basis* (2.1.3) *in the following form:*

$$[A]_{\mathcal{B}_\emptyset} = \begin{pmatrix}
\mu_1^{(1)} & b_1 & 0 & 0 & 0 & 0\\
b_1 & \mu_2^{(1)} & 0 & 0 & 0 & 0\\
0 & 0 & \mu_1^{(2)} & b_2 & 0 & 0\\
0 & 0 & b_2 & \mu_2^{(2)} & 0 & 0\\
0 & 0 & 0 & 0 & \mu_1^{(3)} & b_3\\
0 & 0 & 0 & 0 & b_3 & \mu_2^{(3)}
\end{pmatrix},\ \textit{if } \Theta = \emptyset,$$

$$[A]_{\mathcal{B}_\Theta} = \begin{pmatrix}
\mu_1^{(1)} & 0 & 0 & 0 & 0\\
0 & \mu_1^{(2)} & 0 & b & 0\\
0 & 0 & \mu_1^{(2)} & 0 & -b\\
0 & b & 0 & \mu_2^{(2)} & 0\\
0 & 0 & -b & 0 & \mu_2^{(2)}
\end{pmatrix},\ \textit{if } \Theta = \{\alpha_1\}, \{\alpha_2\} \textit{ or } \{\alpha_3\},$$

$$[A]_{\mathcal{B}_\Theta} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \ \textit{if } \Theta = \{\alpha_1, \alpha_2\} \textit{ or } \{\alpha_2, \alpha_3\},$$

$$[A]_{\mathcal{B}_{\{\alpha_1,\alpha_3\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}, \ \textit{if } \Theta = \{\alpha_1, \alpha_3\},$$

where the numbers $\mu_j^{(i)}$, μ *and* μ_i *are positive.*

Proof. For $\Theta = \emptyset$, $\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_1, \alpha_2\}$ and $\{\alpha_2, \alpha_3\}$, it is obvious from (2.0.3), (2.0.4) and the description of the isotypical summands given in chapter 1. When $\Theta = \{\alpha_1\}$, we have that A is written with respect to $\mathcal{B}_{\{\alpha_1\}}$ in the form

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1^{(1)} & 0 & 0 & 0 & 0 \\ 0 & \mu_1^{(2)} & 0 & b & d \\ 0 & 0 & \mu_1^{(2)} & c & e \\ 0 & b & c & \mu_2^{(2)} & 0 \\ 0 & d & e & 0 & \mu_2^{(2)} \end{pmatrix}.$$

Given $k \in K_{\{\alpha_1\}} = S(O(2) \times O(1) \times O(1))$, we have that k has the form

$$k = \begin{pmatrix} r & s & 0 & 0 \\ t & u & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & z \end{pmatrix},$$

where its columns are orthonormal and $\det(k) = 1$. It is easy to verify that

$$[\mathrm{Ad}(k)]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} vz & 0 & 0 & 0 & 0 \\ 0 & vr & vs & 0 & 0 \\ 0 & vt & vu & 0 & 0 \\ 0 & 0 & 0 & zu & zt \\ 0 & 0 & 0 & zs & zr \end{pmatrix}.$$

Since A commutes with $\mathrm{Ad}(k)$ for all $k \in K_{\{\alpha_1\}}$, then for $r = t = u = -s = \frac{1}{\sqrt{2}}$ and $v = z = 1$ we have

$$[A]_{\mathcal{B}_{\{\alpha_1\}}}[\mathrm{Ad}(k)]_{\mathcal{B}_{\{\alpha_1\}}} - [\mathrm{Ad}(k)]_{\mathcal{B}_{\{\alpha_1\}}}[A]_{\mathcal{B}_{\{\alpha_1\}}} = \mathbf{0}$$

$$\iff \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c-d}{\sqrt{2}} & -\frac{b+e}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{b+e}{\sqrt{2}} & -\frac{c-d}{\sqrt{2}} \\ 0 & \frac{c-d}{\sqrt{2}} & \frac{b+e}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{b+e}{\sqrt{2}} & \frac{c-d}{\sqrt{2}} & 0 & 0 \end{pmatrix} = \mathbf{0}$$

thus, $b = -e$ and $c = d$. If $r = v = -u = -z = 1$ and $t = s = 0$, then

$$[A]_{\mathcal{B}_{\{\alpha_1\}}}[\mathrm{Ad}(k)]_{\mathcal{B}_{\{\alpha_1\}}} - [\mathrm{Ad}(k)]_{\mathcal{B}_{\{\alpha_1\}}}[A]_{\mathcal{B}_{\{\alpha_1\}}} = \mathbf{0}$$

$$\iff \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2d \\ 0 & 0 & 0 & 2c & 0 \\ 0 & 0 & -2c & 0 & 0 \\ 0 & 2d & 0 & 0 & 0 \end{pmatrix} = \mathbf{0},$$

hence $c = d = 0$, as we wanted to prove. Analogously for $\Theta = \{\alpha_2\}$ and $\Theta = \{\alpha_3\}$. □

2.2 Flags of B_l

In this section, we consider $l \geq 5$ and fix $(\cdot,\cdot)$ as in (1.2.1). Given $\Theta \subseteq \Sigma$, we take $l_1, ..., l_r$ satisfying (1.2.3).

Proposition 2.2.1. *Let $\Theta \subseteq \Sigma$, and $l \geq 5$.*

a) If $\alpha_l \notin \Theta$ then

$$\mathcal{B}_\Theta = \left(\bigcup_{i=1}^{r} \mathcal{B}_{0,i}\right) \cup \left(\bigcup_{1\leq n<m\leq r} \mathcal{B}_{mn}\right) \cup \left(\bigcup_{l_i>1} \mathcal{B}_i\right) \tag{2.2.1}$$

is a $(\cdot,\cdot)-$orthonormal basis for $\mathfrak{m}_\Theta$ adapted to the subspaces of Proposition 1.2.1. Where

$$\mathcal{B}_{0,i} = \left\{v_{\tilde{l}_{i-1}+s} : 1 \leq s \leq l_i\right\},$$

$$\mathcal{B}_{mn} = \{w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} : s = 1, ..., l_m,\ t = 1, ..., l_n\} \cup \{u_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} : s = 1, ..., l_m,\ t = 1, ..., l_n\},$$

$$\mathcal{B}_i = \left\{u_{\tilde{l}_{i-1}+s,\tilde{l}_{i-1}+t} : 1 \leq t < s \leq l_i\right\}.$$

b) If $\alpha_l \in \Theta$ then

$$\mathcal{B}_\Theta = \left(\bigcup_{i=1}^{r-1} (\mathcal{B}_i)_1 \cup (\mathcal{B}_i)_2\right) \cup \left(\bigcup_{1\leq n<m\leq r-1} \mathcal{B}_{mn}\right) \cup \left(\bigcup_{\substack{1\leq i\leq r-1 \\ l_i>1}} \mathcal{B}_i\right) \tag{2.2.2}$$

is a $(\cdot,\cdot)-$orthogonal basis for $\mathfrak{m}_\Theta$ adapted to the subspaces of Proposition 1.2.1. Here $\mathcal{B}_{mn}$ and $\mathcal{B}_i$ are as before and

$$(\mathcal{B}_i)_1 = \left\{w_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} - u_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} : 1 \leq s \leq l_r,\ 1 \leq t \leq l_i\right\},$$

$$(\mathcal{B}_i)_2 = \left\{v_{\tilde{l}_{i-1}+t},\ w_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} + u_{\tilde{l}_{r-1}+s,\tilde{l}_{i-1}+t} : 1 \leq s \leq l_r,\ 1 \leq t \leq l_i\right\}.$$

□

Proposition 2.2.2. *Every invariant metric A on a flag of B_l, $l \geq 5$ is written in the basis of Proposition 2.2.1 in the following form:*

a) If $\alpha_l \notin \Theta$ then

$$A|_{V_i} = \mu^{(i)} I_{V_i},\ 1 \leq i \leq r,$$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \lambda_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ & \\ b_{mn} I & \lambda_2^{(mn)} I_{U_{mn}} \end{pmatrix},\ 1 \leq n < m \leq r,$$

$$A|_{U_i} = \gamma^{(i)} I_{U_i},\ 1 \leq i \leq r \text{ and } l_i > 1.$$

b) If $\alpha_l \in \Theta$ then

$$A|_{(V_i)_1} = \rho^{(i)} I_{(V_i)_1},\ 1 \leq i \leq r-1,$$

$$A|_{(V_i)_2} = \mu^{(i)} I_{(V_i)_2},\ 1 \leq i \leq r-1,$$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \lambda_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ & \\ b_{mn} I & \lambda_2^{(mn)} I_{U_{mn}} \end{pmatrix},\ 1 \leq n < m \leq r-1,$$

$$A|_{U_i} = \gamma^{(i)} I_{U_i},\ 1 \leq i \leq r-1 \text{ and } l_i > 1.$$

Proof. Due to (2.0.3), (2.0.4) and Proposition 1.2.1, it is enough to prove the result for $A|_{M_{mn}}$. By (2.0.4) we have

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \lambda_1^{(mn)} \mathrm{I}_{W_{mn}} & B^T \\ & \\ B & \lambda_2^{(mn)} \mathrm{I}_{U_{mn}} \end{pmatrix}.$$

Let us take

$$k = \begin{pmatrix} 1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & P \end{pmatrix} \in K_\Theta$$

where $\det(P) = 1$ and P is a block diagonal matrix

$$P = \begin{pmatrix} P_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & P_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & P_r \end{pmatrix},\ P_i \in O(l_i) \text{ for } i = 1, ..., r.$$

If $P_i = (p_{st}^i)_{l_i \times l_i}$, then

$$\mathrm{Ad}(k)w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} = \sum_{e=1}^{l_m}\sum_{f=1}^{l_n} p^m_{es}p^n_{ft}\ w_{\tilde{l}_{m-1}+e,\tilde{l}_{n-1}+f}$$

and

$$\mathrm{Ad}(k)u_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} = \sum_{e=1}^{l_m}\sum_{f=1}^{l_n} p^m_{es}p^n_{ft}\ u_{\tilde{l}_{m-1}+e,\tilde{l}_{n-1}+f},$$

Also, we have that for every (s,t) there exist a set of real numbers $\{b^{st}_{ef} : 1 \leq e \leq l_m,\ 1 \leq f \leq l_n\}$ (each b^{st}_{ef} is an entry of the matrix B) such that

$$Aw_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} = \lambda_1^{(mn)} w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} + \sum_{e=1}^{l_m}\sum_{f=1}^{l_n} b^{st}_{ef}\ u_{\tilde{l}_{m-1}+e,\tilde{l}_{n-1}+f},$$

so

$$\begin{aligned}\mathrm{Ad}(k)\circ A\ w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} &= \lambda_1^{(mn)} \sum_{e=1}^{l_m}\sum_{f=1}^{l_n} p^m_{es}p^n_{ft}\ w_{\tilde{l}_{m-1}+e,\tilde{l}_{n-1}+f} \\ &\quad + \sum_{\tilde{e},e=1}^{l_m}\sum_{\tilde{f},f=1}^{l_n} b^{st}_{ef}p^m_{\tilde{e}e}p^n_{\tilde{f}f}\ u_{\tilde{l}_{m-1}+\tilde{e},\tilde{l}_{n-1}+\tilde{f}}\end{aligned}$$

and

$$\begin{aligned}A\circ \mathrm{Ad}(k)w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t} &= \lambda_1^{(mn)} \sum_{e=1}^{l_m}\sum_{f=1}^{l_n} p^m_{es}p^n_{ft}\ w_{\tilde{l}_{m-1}+e,\tilde{l}_{n-1}+f} \\ &\quad + \sum_{\tilde{e},e=1}^{l_m}\sum_{\tilde{f},f=1}^{l_n} b^{ef}_{\tilde{e}\tilde{f}}p^m_{es}p^n_{ft}\ u_{\tilde{l}_{m-1}+\tilde{e},\tilde{l}_{n-1}+\tilde{f}}.\end{aligned}$$

Since A commutes with $\mathrm{Ad}(k)$ then

$$\sum_{e=1}^{l_m}\sum_{f=1}^{l_n} b^{st}_{ef}p^m_{\tilde{e}e}p^n_{\tilde{f}f} = \sum_{e=1}^{l_m}\sum_{f=1}^{l_n} b^{ef}_{\tilde{e}\tilde{f}}p^m_{es}p^n_{ft}, \tag{2.2.3}$$

for $1 \leq s, \tilde{e} \leq l_m$, and $1 \leq t, \tilde{f} \leq l_n$. Fixing $s, t, \tilde{e}, \tilde{f}$, we shall show that $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $(s,t) \neq (\tilde{e}, \tilde{f})$. First, we suppose $r > 2$. By taking $P_m = \mathrm{diag}(1, .., 1, -1, 1, ..., 1)$, with -1 in the $(\tilde{e}, \tilde{e})-$entry, $P_i = \mathrm{diag}(-1, 1, ..., 1)$ for some $i \notin \{m, n\}$ (which exists because $r > 2$) and $P_j = \mathrm{I}$, for $j \notin \{m, i\}$ we obtain

$$-b^{st}_{\tilde{e}\tilde{f}} = b^{st}_{\tilde{e}\tilde{f}}p^m_{ss}.$$

Since $p^m_{ss} = 1$ for $s \neq \tilde{e}$, then $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $s \neq \tilde{e}$. By taking $P_n = \mathrm{diag}(1, ..., 1, -1, 1, ..., 1)$, with -1 in the $(\tilde{f}, \tilde{f})-$entry, $P_i = \mathrm{diag}(-1, 1, ..., 1)$ for some $i \notin \{m, n\}$ and $P_j = \mathrm{I}$ for $j \notin \{n, i\}$ we have

$$-b^{st}_{\tilde{e}\tilde{f}} = b^{st}_{\tilde{e}\tilde{f}}p^n_{tt}.$$

Again, $p^n_{tt} = 1$ for $t \neq \tilde{f}$, then $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $t \neq \tilde{f}$. We conclude that $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $(s,t) \neq (\tilde{e}, \tilde{f})$. If $r = 2$ then $m = 2$, $n = 1$ and we have two possibilities:

• $l_1 > 2$: In this case, we take $P_2 = \mathrm{diag}(1, ..., 1, -1, 1, ..., 1)$ and $P_1 = \mathrm{diag}(1, ..., 1, -1, 1, ..., 1)$, where P_2 has -1 in the $(\tilde{e}, \tilde{e})-$entry and P_1 has -1 in the $(j, j)-$entry for some $j \notin \{\tilde{f}, t\}$ and we obtain from (2.2.3) that

$$-b^{st}_{\tilde{e}\tilde{f}} = b^{st}_{\tilde{e}\tilde{f}} p^2_{ss}.$$

Then $s \neq \tilde{e} \implies b^{st}_{\tilde{e}\tilde{f}} = 0$. For $s = \tilde{e}$, we take $P_2 = \mathrm{diag}(1, ..., 1, -1, 1, ..., 1)$ with -1 in the $(\tilde{e}, \tilde{e})-$entry, $P_1 = \mathrm{diag}(1, ..., 1, -1, 1, ..., 1)$ with -1 in the $(\tilde{f}, \tilde{f})-$entry and therefore

$$b^{\tilde{e}t}_{\tilde{e}\tilde{f}} = -b^{\tilde{e}t}_{\tilde{e}\tilde{f}} p^1_{tt},$$

so $t \neq \tilde{f} \implies b^{\tilde{e}t}_{\tilde{e}\tilde{f}} = 0$.

• $l_1 \leq 2$: Since $l \geq 5$ and $l_1 + l_2 = l$, we have that $l_2 > 2$ and we can proceed analogously as before.

We conclude that $b^{st}_{\tilde{e}\tilde{f}} = 0$ of $(s, t) \neq (\tilde{e}, \tilde{f})$, thus equation (2.2.3) becomes

$$b^{st}_{st} p^m_{\tilde{e}s} p^n_{\tilde{f}t} = b^{\tilde{e}\tilde{f}}_{\tilde{e}\tilde{f}} p^m_{\tilde{e}s} p^n_{\tilde{f}t}. \tag{2.2.4}$$

By taking $P_m \in SO(l_m)$ with non-zero $(\tilde{e}, s)-$entry and $P_n \in SO(l_n)$ with non-zero $(\tilde{f}, t)-$entry, we have that $b^{st}_{st} = b^{\tilde{e}\tilde{f}}_{\tilde{e}\tilde{f}} =: b_{mn}$ for all $s, t, \tilde{e}, \tilde{f}$. Hence

$$Aw_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} \; = \; \lambda^{(mn)}_1 w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} + b_{mn} \, u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}$$

and $B = b_{mn}\mathrm{I}$. □

2.3 Flags of C_l

We consider the invariant inner product $(\cdot, \cdot)$ given in (1.3.1) and $l_1, ..., l_r$ as in (1.3.3).

Proposition 2.3.1. *Let $\Theta \subseteq \Sigma$, $l \neq 4$, $\alpha_l \notin \Theta$ and $l_1, ..., l_r$ as in* (1.3.3)*. Then*

$$\mathcal{B} = \mathcal{B}_0 \cup \left(\bigcup_{1 \leq n < m \leq r} \mathcal{B}_{mn} \right) \cup \left(\bigcup_{\substack{i=1 \\ l_i > 1}}^{r} \mathcal{B}_i \right) \tag{2.3.1}$$

is a $(\cdot, \cdot)-$orthogonal basis for $\mathfrak{m}_\Theta$ adapted to the subspaces of Proposition 1.3.1, where

$$\mathcal{B}_0 = \left\{ \frac{1}{\sqrt{l_j}} \left(u_{\tilde{l}_{j-1}+1, \tilde{l}_{j-1}+1} + ... + u_{\tilde{l}_j, \tilde{l}_j} \right) : j = 1, ..., r \right\},$$

$$\mathcal{B}_{mn} = \{ w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} : s = 1, ..., l_m, \; t = 1, ..., l_n \} \cup \{ u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} : s = 1, ..., l_m, \; t = 1, ..., l_n \}$$

$$\mathcal{B}_i = \left\{ \frac{1}{s} \sum_{t=1}^{s} u_{\tilde{l}_{i-1}+t, \tilde{l}_{i-1}+t} - u_{\tilde{l}_{i-1}+s+1, \tilde{l}_{i-1}+s+1} : s = 1, ..., l_i - 1 \right\} \cup \left\{ u_{\tilde{l}_{i-1}+s, \tilde{l}_{i-1}+t} : 1 \leq t < s \leq l_i \right\}.$$

If $\alpha_l \in \Theta$, then i extends only over $\{1, ..., r-1\}$. □

Proposition 2.3.2. *Every invariant metric A on a flag of C_l, $l \neq 4$ is written in the basis* (2.3.1) *in the following form:*

a) If $\alpha_l \notin \Theta$ then

$$[A|_{M_0}]_{\mathcal{B}_0} = \begin{pmatrix} \mu_1^{(0)} & a_{21} & a_{31} & \dots & a_{r1} \\ a_{21} & \mu_2^{(0)} & a_{32} & \dots & a_{r2} \\ a_{31} & a_{32} & \mu_3^{(0)} & \dots & a_{r3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & a_{r3} & \dots & \mu_r^{(0)} \end{pmatrix},$$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \mu_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ & \\ b_{mn} I & \mu_2^{(mn)} I_{U_{mn}} \end{pmatrix}, \ 1 \leq n < m \leq r,$$

$$[A|_{U_i}]_{\mathcal{B}_i} = \mu^{(i)} I_{U_i}, \ 1 \leq i \leq r \ \textit{and} \ l_i > 1.$$

b) If $\alpha_l \in \Theta$ then

$$[A|_{M_0}]_{\mathcal{B}_0} = \begin{pmatrix} \mu_1^{(0)} & a_{21} & a_{31} & \dots & a_{r-1,1} \\ a_{21} & \mu_2^{(0)} & a_{32} & \dots & a_{r-1,2} \\ a_{31} & a_{32} & \mu_3^{(0)} & \dots & a_{r-1,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r-1,1} & a_{r-1,2} & a_{r-1,3} & \dots & \mu_{r-1}^{(0)} \end{pmatrix},$$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \mu_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ & \\ b_{mn} I & \mu_2^{(mn)} I_{U_{mn}} \end{pmatrix}, \ 1 \leq n < m \leq r-1,$$

$$[A|_{M_{rn}}]_{\mathcal{B}_{rn}} = \mu^{(rn)} I_{M_{rn}}, \ 1 \leq n \leq r-1,$$

$$[A|_{U_i}]_{\mathcal{B}_i} = \mu^{(i)} I_{U_i}, \ 1 \leq i \leq r-1 \ \textit{and} \ l_i > 1.$$

Proof. Case 1. $\alpha_l \notin \Theta$.

By (2.0.3), (2.0.4) and Proposition 1.3.1, we only have to prove the result for $A|_{M_{mn}}$. In fact, we know $[A|_{M_{mn}}]_{\mathcal{B}_{mn}}$ has the form

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \mu_1^{(mn)} \mathrm{I}_{W_{mn}} & B^T \\ & \\ B & \mu_2^{(mn)} \mathrm{I}_{U_{mn}} \end{pmatrix}.$$

Given $k \in K_\Theta \overset{\text{dif.}}{\approx} O(l_1) \times ... \times O(l_r)$, we have that k has the form

$$k = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix},$$

where P is a block diagonal matrix

$$P = \begin{pmatrix} P_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & P_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & P_r \end{pmatrix}, \ P_i \in O(l_i) \text{ for } i = 1, ..., r.$$

Writing $P_i = (p^i_{st})_{l_i \times l_i}$, we can easily verify that for any pair (s,t) with $1 \leq s \leq l_m$, $1 \leq t \leq l_n$

$$\mathrm{Ad}(k) w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} = \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p^m_{es} p^n_{ft} \ w_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f} \tag{2.3.2}$$

and

$$\mathrm{Ad}(k) u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} = \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p^m_{es} p^n_{ft} \ u_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f}. \tag{2.3.3}$$

Because of the form of $A|_{M_{mn}}$, we also have that for every (s,t) there exists a set of real numbers $\{b^{st}_{ef} : 1 \leq e \leq l_m, \ 1 \leq f \leq l_n\}$ such that

$$A w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} = \mu_1^{(mn)} w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} + \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} b^{st}_{ef} \ u_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f}. \tag{2.3.4}$$

From (2.3.2), (2.3.3) and (2.3.4) we get

$$\begin{aligned} \mathrm{Ad}(k) \circ A \ w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} &= \mu_1^{(mn)} \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p^m_{es} p^n_{ft} \ w_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f} \\ &+ \sum_{\tilde{e},e=1}^{l_m} \sum_{\tilde{f},f=1}^{l_n} b^{st}_{ef} p^m_{\tilde{e}e} p^n_{\tilde{f}f} \ u_{\tilde{l}_{m-1}+\tilde{e}, \tilde{l}_{n-1}+\tilde{f}} \end{aligned}$$

and

$$\begin{aligned} A \circ \mathrm{Ad}(k) w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} &= \mu_1^{(mn)} \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p^m_{es} p^n_{ft} \ w_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f} \\ &+ \sum_{\tilde{e},e=1}^{l_m} \sum_{\tilde{f},f=1}^{l_n} b^{ef}_{\tilde{e}\tilde{f}} p^m_{es} p^n_{ft} \ u_{\tilde{l}_{m-1}+\tilde{e}, \tilde{l}_{n-1}+\tilde{f}}. \end{aligned}$$

Since $A \circ \mathrm{Ad}(k) = \mathrm{Ad}(k) \circ A$, then

$$\sum_{e=1}^{l_m} \sum_{f=1}^{l_n} b^{st}_{ef} p^m_{\tilde{e}e} p^n_{\tilde{f}f} = \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} b^{ef}_{\tilde{e}\tilde{f}} p^m_{es} p^n_{ft}, \tag{2.3.5}$$

for $1 \leq s, \tilde{e} \leq l_m$, and $1 \leq t, \tilde{f} \leq l_n$. Fixing $s, t, \tilde{e}, \tilde{f}$, we shall show that $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $(s,t) \neq (\tilde{e}, \tilde{f})$. Equation (2.3.5) is true for every $k \in K_\Theta$ in particular, if $P_m = \mathrm{diag}(1, .., 1, -1, 1, ..., 1)$, with -1 in the $(\tilde{e}, \tilde{e})-$entry and $P_i = \mathrm{I}$, $i \neq m$ we have

$$-b^{st}_{\tilde{e}\tilde{f}} = b^{st}_{\tilde{e}\tilde{f}} p^m_{ss}.$$

Since $p^m_{ss} = 1$ for $s \neq \tilde{e}$, then $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $s \neq \tilde{e}$. By taking $P_n = \mathrm{diag}(1, ..., 1, -1, 1, ..., 1)$, with -1 in the $(\tilde{f}, \tilde{f})-$entry and $P_i = \mathrm{I}$, $i \neq n$ we have

$$-b^{st}_{\tilde{e}\tilde{f}} = b^{st}_{\tilde{e}\tilde{f}} p^n_{tt}.$$

But $p^n_{tt} = 1$ for $t \neq \tilde{f}$, so $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $t \neq \tilde{f}$. We conclude that $b^{st}_{\tilde{e}\tilde{f}} = 0$ of $(s, t) \neq (\tilde{e}, \tilde{f})$, whereupon equation (2.3.5) becomes

$$b^{st}_{st} p^m_{\tilde{e}s} p^n_{\tilde{f}t} = b^{\tilde{e}\tilde{f}}_{\tilde{e}\tilde{f}} p^m_{\tilde{e}s} p^n_{\tilde{f}t}. \tag{2.3.6}$$

By taking $P_m \in O(l_m)$ with non-zero $(\tilde{e}, s)-$entry and $P_n \in O(l_n)$ with non-zero $(\tilde{f}, t)-$entry, we have $b^{st}_{st} = b^{\tilde{e}\tilde{f}}_{\tilde{e}\tilde{f}} =: b_{mn}$ for all $s, t, \tilde{e}, \tilde{f}$. Hence, equation (2.3.4) implies

$$Aw_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} \quad = \quad \mu_1^{(mn)} w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} + b_{mn}\ u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}. \tag{2.3.7}$$

as we wanted. The same argument works for the case $\alpha_l \in \Theta$ considering that

$$O(l_1) \times ... \times O(l_r) \subseteq O(l_1) \times ... \times O(l_{r-1}) \times U(l_r) = K_\Theta$$

□

Consider $l = 4$, Θ and its corresponding $K_\Theta-$invariant spaces in Table 1, we fix the following orthogonal bases:

- $\mathcal{B}_\emptyset = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where

$$\begin{aligned} \mathcal{B}_0 &= \{u_{11}, u_{22}, u_{33}, u_{44}\}, \\ \mathcal{B}_1 &= \{w_{21}, w_{43}, u_{21}, u_{43}\}, \\ \mathcal{B}_2 &= \{w_{31}, w_{42}, u_{31}, u_{42}\}, \\ \mathcal{B}_3 &= \{w_{32}, w_{41}, u_{32}, u_{41}\}, \end{aligned}$$

which is adapted to the isotypical summands

$$\begin{aligned} M_0 &= V_1 \oplus V_2 \oplus V_3 \oplus V_4, \\ N_1 &= W_{21} \oplus W_{43} \oplus U_{21} \oplus U_{43}, \\ N_2 &= W_{31} \oplus W_{42} \oplus U_{31} \oplus U_{42}, \\ N_3 &= W_{32} \oplus W_{41} \oplus U_{32} \oplus U_{41}. \end{aligned}$$

- $\mathcal{B}_{\{\alpha_1\}} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where

$$\begin{aligned} \mathcal{B}_0 &= \left\{\tfrac{1}{\sqrt{2}}(u_{11} + u_{22}), u_{33}, u_{44}\right\}, \\ \mathcal{B}_1 &= \{w_{31}, w_{32}, w_{41}, w_{42}, u_{31}, u_{32}, u_{41}, u_{42}\}, \\ \mathcal{B}_2 &= \{w_{43}, u_{43}\}, \\ \mathcal{B}_3 &= \{u_{21}, u_{11} - u_{22}\}, \end{aligned}$$

which is adapted to the isotypical summands

$$\begin{aligned} M_0 &= V_1 \oplus V_2 \oplus V_3, \\ N &= W_{21} \oplus W_{31} \oplus U_{21} \oplus U_{31}, \\ M &= W_{32} \oplus U_{32}, \\ U_1&. \end{aligned}$$

- $\mathcal{B}_{\{\alpha_2\}} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where

$$\begin{aligned} \mathcal{B}_0 &= \left\{u_{11}, \tfrac{1}{\sqrt{2}}(u_{22}+u_{33}), u_{44}\right\}, \\ \mathcal{B}_1 &= \{w_{21}, w_{31}, w_{42}, w_{43}, u_{21}, u_{31}, u_{42}, u_{43}\}, \\ \mathcal{B}_2 &= \{w_{41}, u_{41}\}, \\ \mathcal{B}_3 &= \{u_{32}, u_{22}-u_{33}\}, \end{aligned}$$

which is adapted to the isotypical summands

$$\begin{aligned} M_0 &= V_1 \oplus V_2 \oplus V_3, \\ N &= W_{21} \oplus W_{32} \oplus U_{21} \oplus U_{32}, \\ M &= W_{31} \oplus U_{31}, \\ U_2&. \end{aligned}$$

- $\mathcal{B}_{\{\alpha_3\}} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where

$$\begin{aligned} \mathcal{B}_0 &= \left\{u_{11}, u_{22}, \tfrac{1}{\sqrt{2}}(u_{33}+u_{44})\right\}, \\ \mathcal{B}_1 &= \{w_{31}, w_{41}, w_{32}, w_{42}, u_{31}, u_{41}, u_{32}, u_{42}\}, \\ \mathcal{B}_2 &= \{w_{21}, u_{21}\}, \\ \mathcal{B}_3 &= \{u_{43}, u_{33}-u_{44}\}, \end{aligned}$$

which is adapted to the isotypical summands

$$\begin{aligned} M_0 &= V_1 \oplus V_2 \oplus V_3, \\ N &= W_{31} \oplus W_{32} \oplus U_{31} \oplus U_{32}, \\ M &= W_{21} \oplus U_{21}, \\ U_3&. \end{aligned}$$

Proposition 2.3.3. *Let A be an invariant metric on a flag $\mathbb{F}_\Theta$ of C_4 where Θ is in the Table 1.*

a) If $\Theta = \emptyset$, then A is written in the basis $\mathcal{B}_\emptyset$ as

$$[A|_{M_0}]_{\mathcal{B}_0} = \begin{pmatrix} \mu_1^{(0)} & a_{21} & a_{31} & a_{41} \\ a_{21} & \mu_2^{(0)} & a_{32} & a_{42} \\ a_{31} & a_{32} & \mu_3^{(0)} & a_{43} \\ a_{41} & a_{42} & a_{43} & \mu_4^{(0)} \end{pmatrix},$$

$$[A|_{N_i}]_{\mathcal{B}_i} = \begin{pmatrix} \mu_1^{(i)} & 0 & b_1^{(i)} & 0 \\ 0 & \mu_2^{(i)} & 0 & b_2^{(i)} \\ b_1^{(i)} & 0 & \mu_3^{(i)} & 0 \\ 0 & b_2^{(i)} & 0 & \mu_4^{(i)} \end{pmatrix}, \quad i = 1, 2, 3.$$

b) If $\Theta = \{\alpha_1\}, \{\alpha_2\}$, *or* $\{\alpha_3\}$, *then* A *is written in the basis* $\mathcal{B}_\Theta$ *as*

$$[A|_{M_0}]_{\mathcal{B}_0} = \begin{pmatrix} \mu_1^{(0)} & a_{21} & a_{31} \\ a_{21} & \mu_2^{(0)} & a_{32} \\ a_{31} & a_{32} & \mu_3^{(0)} \end{pmatrix},$$

$$[A|_N]_{\mathcal{B}_1} = \begin{pmatrix} \mu_1^N & 0 & 0 & 0 & b_1^N & 0 & 0 & 0 \\ 0 & \mu_1^N & 0 & 0 & 0 & b_1^N & 0 & 0 \\ 0 & 0 & \mu_2^N & 0 & 0 & 0 & b_2^N & 0 \\ 0 & 0 & 0 & \mu_2^N & 0 & 0 & 0 & b_2^N \\ b_1^N & 0 & 0 & 0 & \mu_3^N & 0 & 0 & 0 \\ 0 & b_1^N & 0 & 0 & 0 & \mu_3^N & 0 & 0 \\ 0 & 0 & b_2^N & 0 & 0 & 0 & \mu_4^N & 0 \\ 0 & 0 & 0 & b_2^N & 0 & 0 & 0 & \mu_4^N \end{pmatrix},$$

$$[A|_M]_{\mathcal{B}_2} = \begin{pmatrix} \mu_1^M & b_M \\ b_M & \mu_2^M \end{pmatrix},$$

$$A|_{U_i} = \mu^{(i)} I_{U_i}, \ i = 1, 2, 3 \qquad (if \Theta = \{\alpha_i\}).$$

Proof. a) For each $i \in \{1, 2, 3\}$ we write $A|_{N_i}$ in the basis $\mathcal{B}_i$ in the form

$$[A|_{N_i}]_{\mathcal{B}_i} = \begin{pmatrix} \mu_1^{(i)} & b_{21}^{(i)} & b_{31}^{(i)} & b_{41}^{(i)} \\ b_{21}^{(i)} & \mu_2^{(i)} & b_{32}^{(i)} & b_{42}^{(i)} \\ b_{31}^{(i)} & b_{32}^{(i)} & \mu_3^{(i)} & b_{43}^{(i)} \\ b_{41}^{(i)} & b_{42}^{(i)} & b_{43}^{(i)} & \mu_4^{(i)} \end{pmatrix}.$$

Given $k \in K_\emptyset \overset{\text{dif.}}{\approx} O(1) \times O(1) \times O(1) \times O(1)$, we have that k has the form

$$k = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix},$$

where $P = \text{diag}(p_1, p_2, p_3, p_4)$, $p_i = \pm 1$, $i = 1, 2, 3, 4$. It is easy to see that

$$[\text{Ad}(k)|_{N_i}]_{\mathcal{B}_i} = \begin{pmatrix} p_{i_1}p_{i_2} & 0 & 0 & 0 \\ 0 & p_{i_3}p_{i_4} & 0 & 0 \\ 0 & 0 & p_{i_1}p_{i_2} & 0 \\ 0 & 0 & 0 & p_{i_3}p_{i_4} \end{pmatrix},$$

where $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. Since A commutes with $\text{Ad}(k)$ for all $k \in K_\emptyset$, then

$$[A|_{N_i}]_{\mathcal{B}_i}[\text{Ad}(k)|_{N_i}]_{\mathcal{B}_i} - [\text{Ad}(k)|_{N_i}]_{\mathcal{B}_i}[A|_{N_i}]_{\mathcal{B}_i} = \mathbf{0}$$

$$\Longleftrightarrow (p_{i_3}p_{i_4} - p_{i_1}p_{i_2}) \begin{pmatrix} 0 & -b_{21}^{(i)} & 0 & -b_{41}^{(i)} \\ b_{21}^{(i)} & 0 & b_{32}^{(i)} & 0 \\ 0 & -b_{32}^{(i)} & 0 & -b_{43}^{(i)} \\ b_{41}^{(i)} & 0 & b_{43}^{(i)} & 0 \end{pmatrix} = \mathbf{0}.$$

Taking $-p_{i_1} = 1 = p_{i_2} = p_{i_3} = p_{i_4}$, we can conclude that $b_{ab}^{(i)} = 0$ if $(a,b) \notin \{(3,1),(4,2)\}$. Defining $b_1^{(i)} := b_{31}^{(i)}$ and $b_2^{(i)} := b_{42}^{(i)}$, we have the result.

b) Let us consider $\Theta = \{\alpha_1\}$. Again, it is enough to show the result for $A|_N$. We know that $A|_N$ is written in the basis $\mathcal{B}_1$ in the form

$$[A|_N]_{\mathcal{B}_1} = \begin{pmatrix} \mu_1^N & 0 & b_{31}^N & b_{41}^N & b_{51}^N & b_{61}^N & b_{71}^N & b_{81}^N \\ 0 & \mu_1^N & b_{32}^N & b_{42}^N & b_{52}^N & b_{62}^N & b_{72}^N & b_{82}^N \\ b_{31}^N & b_{32}^N & \mu_2^N & 0 & b_{53}^N & b_{63}^N & b_{73}^N & b_{83}^N \\ b_{41}^N & b_{42}^N & 0 & \mu_2^N & b_{54}^N & b_{64}^N & b_{74}^N & b_{84}^N \\ b_{51}^N & b_{52}^N & b_{53}^N & b_{54}^N & \mu_3^N & 0 & b_{75}^N & b_{85}^N \\ b_{61}^N & b_{62}^N & b_{63}^N & b_{64}^N & 0 & \mu_3^N & b_{76}^N & b_{86}^N \\ b_{71}^N & b_{72}^N & b_{73}^N & b_{74}^N & b_{75}^N & b_{76}^N & \mu_4^N & 0 \\ b_{81}^N & b_{82}^N & b_{83}^N & b_{84}^N & b_{85}^N & b_{86}^N & 0 & \mu_4^N \end{pmatrix}.$$

Given $k \in K_{\{\alpha_1\}} \overset{\text{dif.}}{\approx} O(2) \times O(1) \times O(1)$, k has the form

$$k = \begin{pmatrix} P & \mathbf{0} \\ \mathbf{0} & P \end{pmatrix}, \text{ with } P = \begin{pmatrix} r & s & 0 & 0 \\ t & u & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & z \end{pmatrix},$$

where the columns of P are orthonormal. Then

$$[\mathrm{Ad}(k)|_N]_{\mathcal{B}_1} = \begin{pmatrix} vr & vs & 0 & 0 & 0 & 0 & 0 & 0 \\ vt & vu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & zr & zs & 0 & 0 & 0 & 0 \\ 0 & 0 & zt & zu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & vr & vs & 0 & 0 \\ 0 & 0 & 0 & 0 & vt & vu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & zr & zs \\ 0 & 0 & 0 & 0 & 0 & 0 & zt & zu \end{pmatrix}.$$

For $-r = 1 = u = v = z$ and $s = t = 0$

$$[A|_N]_{\mathcal{B}_1}[\mathrm{Ad}(k)|_N]_{\mathcal{B}_1} - [\mathrm{Ad}(k)|_N]_{\mathcal{B}_1}[A|_N]_{\mathcal{B}_1} = \mathbf{0}$$

$$\Longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 2b_{41}^N & 0 & 2b_{61}^N & 0 & 2b_{81}^N \\ 0 & 0 & -2b_{32}^N & 0 & -2b_{52}^N & 0 & -2b_{72}^N & 0 \\ 0 & 2b_{32}^N & 0 & 0 & 0 & 2b_{63}^N & 0 & 2b_{83}^N \\ -2b_{41}^N & 0 & 0 & 0 & -2b_{54}^N & 0 & -2b_{74}^N & 0 \\ 0 & 2b_{52}^N & 0 & 2b_{54}^N & 0 & 0 & 0 & 2b_{85}^N \\ -2b_{61}^N & 0 & -2b_{63}^N & 0 & 0 & 0 & -2b_{76}^N & 0 \\ 0 & 2b_{72}^N & 0 & 2b_{74}^N & 0 & 2b_{76}^N & 0 & 0 \\ -2b_{81}^N & 0 & -2b_{83}^N & 0 & -2b_{85}^N & 0 & 0 & 0 \end{pmatrix} = \mathbf{0},$$

therefore $b_{32}^N = b_{41}^N = b_{52}^N = b_{54}^N = b_{61}^N = b_{63}^N = b_{72}^N = b_{74}^N = b_{76}^N = b_{81}^N = b_{83}^N = b_{85}^N = 0$. By taking $r = u = z = 1 = -v$ and $s = t = 0$, we have

$$[A|_N]_{\mathcal{B}_1}[\mathrm{Ad}(k)|_N]_{\mathcal{B}_1} - [\mathrm{Ad}(k)|_N]_{\mathcal{B}_1}[A|_N]_{\mathcal{B}_1} = \mathbf{0}$$

$$\Longleftrightarrow \begin{pmatrix} 0 & 0 & 2b_{31}^N & 0 & 0 & 0 & 2b_{71}^N & 0 \\ 0 & 0 & 0 & 2b_{42}^N & 0 & 0 & 0 & 2b_{82}^N \\ -2b_{31}^N & 0 & 0 & 0 & -2b_{53}^N & 0 & 0 & 0 \\ 0 & -2b_{42}^N & 0 & 0 & 0 & -2b_{64}^N & 0 & 0 \\ 0 & 0 & 2b_{53}^N & 0 & 0 & 0 & 2b_{75}^N & 0 \\ 0 & 0 & 0 & 2b_{64}^N & 0 & 0 & 0 & 2b_{86}^N \\ -2b_{71}^N & 0 & 0 & 0 & -2b_{75}^N & 0 & 0 & 0 \\ 0 & -2b_{82}^N & 0 & 0 & 0 & -2b_{86}^N & 0 & 0 \end{pmatrix} = \mathbf{0},$$

which implies $b_{31}^N = b_{42}^N = b_{53}^N = b_{64}^N = b_{71}^N = b_{75}^N = b_{82}^N = b_{86}^N = 0$. If $r = s = t = -u = \frac{1}{\sqrt{2}}$ and $v = z = 1$, then

$$[A|_N]_{\mathcal{B}_1}[\mathrm{Ad}(k)|_N]_{\mathcal{B}_1} - [\mathrm{Ad}(k)|_N]_{\mathcal{B}_1}[A|_N]_{\mathcal{B}_1} = \mathbf{0}$$

$$\Longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{b_{51}^N - b_{62}^N}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{b_{62}^N - b_{51}^N}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b_{73}^N - b_{84}^N}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{b_{84}^N - b_{73}^N}{\sqrt{2}} & 0 \\ 0 & \frac{b_{51}^N - b_{62}^N}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{b_{62}^N - b_{51}^N}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b_{73}^N - b_{84}^N}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b_{84}^N - b_{73}^N}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0},$$

thus, $b_{51}^N = b_{62}^N =: b_1^N$ and $b_{73}^N = b_{84}^N =: b_2^N$. The cases $\Theta = \{\alpha_2\}$ and $\Theta = \{\alpha_3\}$ are analogous. □

2.4 Flags of D_l

For this section we consider $(\cdot,\cdot)$ as in (1.4.1) and for each $\Theta \subseteq \Sigma$, take $l_1, ..., l_r$ as in (1.2.3).

Proposition 2.4.1. *Let $\Theta \subseteq \Sigma$ and $l \geq 5$.*

a) If $\alpha_l \notin \Theta$ then

$$\mathcal{B}_\Theta = \left(\bigcup_{1 \leq n < m \leq r} \mathcal{B}_{mn} \right) \cup \left(\bigcup_{l_i > 1} \mathcal{B}_i \right) \tag{2.4.1}$$

is a $(\cdot,\cdot)-$orthonormal basis for $\mathfrak{m}_\Theta$ adapted to the subspaces of Proposition 1.4.1, where

$$\mathcal{B}_{mn} = \{w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} : 1 \leq s \leq l_m,\ 1 \leq t \leq l_n\} \cup \{u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} : 1 \leq s \leq l_m,\ 1 \leq t \leq l_n\}$$

and

$$\mathcal{B}_i = \{u_{\tilde{l}_{i-1}+s,\tilde{l}_{i-1}+t} : 1 \le t < s \le l_i\}, \quad 1 \le i \le r.$$

If we have $\{\alpha_{l-1}, \alpha_l\} \subseteq \Theta$ *instead of* $\alpha_l \notin \Theta$ *then* i *extends only over* $\{1, ..., r-1\}$.

b) If $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \notin \Theta$ *then*

$$\mathcal{B}_\Theta = \left(\bigcup_{1 \le n < m \le r-2} \mathcal{B}_{mn} \right) \cup \left(\bigcup_{\substack{l_i > 1 \\ 1 \le i \le r-2}} \mathcal{B}_i \right) \cup \left(\bigcup_{n=1}^{r-2} \mathcal{B}_n^M \cup \mathcal{B}_n^N \right) \cup \mathcal{B}^V, \tag{2.4.2}$$

is a $(\cdot,\cdot)-$*orthonormal basis for* $\mathfrak{m}_\Theta$ *adapted to the subspaces of Proposition 1.4.1, where* $\mathcal{B}_{mn}$ *and* $\mathcal{B}_i$ *are as before and*

$$\mathcal{B}_n^M = \{w_{\tilde{l}_{r-2}+s,\tilde{l}_{n-1}+t} : 1 \le s \le l_{r-1},\ 1 \le t \le l_n\} \cup \{u_{l,\tilde{l}_{n-1}+t} : 1 \le t \le l_n\},$$

$$\mathcal{B}_n^N = \{u_{\tilde{l}_{r-2}+s,\tilde{l}_{n-1}+t} : 1 \le s \le l_{r-1},\ 1 \le t \le l_n\} \cup \{w_{l,\tilde{l}_{n-1}+t} : 1 \le t \le l_n\},$$

$$\mathcal{B}^V = \{u_{\tilde{l}_{r-2}+s,\tilde{l}_{r-2}+t} : 1 \le t < s \le l_{r-1}\} \cup \{w_{l,\tilde{l}_{r-2}+t} : 1 \le t \le l_{r-1}\}.$$

□

Proposition 2.4.2. *Every invariant metric* A *on a flag of* D_l, $l \ge 5$ *is written in the basis of Proposition 1.4.1 in the following form:*

a) If $\alpha_l \notin \Theta$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \lambda_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ b_{mn} I & \lambda_2^{(mn)} I_{U_{mn}} \end{pmatrix}, \ 1 \le n < m \le r,$$

$$A|_{U_i} = \gamma^{(i)} I_{U_i},\ 1 \le i \le r \ \textit{and}\ l_i > 1.$$

b) If $\{\alpha_{l-1}, \alpha_l\} \subseteq \Theta$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \lambda_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ b_{mn} I & \lambda_2^{(mn)} I_{U_{mn}} \end{pmatrix}, \ 1 \le n < m \le r-1,$$

$$A|_{M_{rn}} = \lambda^{(rn)} I_{M_{rn}},\ 1 \le n \le r-1,$$

$$A|_{U_i} = \gamma^{(i)} I_{U_i},\ 1 \le i \le r-1 \ \textit{and}\ l_i > 1.$$

c) If $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \notin \Theta$

$$[A|_{M_{mn}}]_{\mathcal{B}_{mn}} = \begin{pmatrix} \lambda_1^{(mn)} I_{W_{mn}} & b_{mn} I \\ b_{mn} I & \lambda_2^{(mn)} I_{U_{mn}} \end{pmatrix}, \ 1 \le n < m \le r-2,$$

$$A|_{U_i} = \gamma^{(i)} I_{U_i},\ 1 \le i \le r-2 \ \textit{and}\ l_i > 1,$$

$$[A|_{S_n}]_{\mathcal{B}_n^M \cup \mathcal{B}_n^N} = \left(\begin{array}{cc|cc} \multicolumn{2}{c}{\lambda_1^{(r-1,n)} I_{M_n}} & b_{r-1,n}^{(1)} I & \boldsymbol{0} \\ & & \boldsymbol{0} & b_{r-1,n}^{(2)} \boldsymbol{I} \\ b_{r-1,n}^{(1)} I & \boldsymbol{0} & \multicolumn{2}{c}{\lambda_2^{(r-1,n)} I_{N_n}} \\ \boldsymbol{0} & b_{r-1,n}^{(2)} \boldsymbol{I} & & \end{array} \right),\ 1 \le n \le r-2,$$

$$A|_{V_{r-1}} = \gamma^{(r-1)} I_{V_{r-1}}.$$

Proof. We have the result for U_i, for M_{rn} when $\{\alpha_{l-1}, \alpha_l\} \subseteq \Theta$ and for V_{r-1} when $\alpha_l \in \Theta$ and $\alpha_{l-1} \notin \Theta$, because these subspaces are $K_\Theta-$invariant, irreducible and not equivalent to any other. For M_{mn}, let us take

$$k = \begin{pmatrix} P_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & P_r \end{pmatrix} \in K_\Theta,$$

where $P_i = (p_{st}^i)_{l_i \times l_i} \in O(l_i)$ and $\det(P) = 1$, then

$$\mathrm{Ad}(k) w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} = \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p_{es}^m p_{ft}^n w_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f}$$

and

$$\mathrm{Ad}(k) u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} = \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p_{es}^m p_{ft}^n u_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f}.$$

There exist real numbers $\lambda_1^{mn} > 0$ and b_{ef}^{st} such that

$$A w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} = \lambda_1^{(mn)} w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} + \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} b_{ef}^{st} u_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f},$$

so

$$\begin{aligned} \mathrm{Ad}(k) \circ A w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} &= \lambda_1^{(mn)} \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p_{es}^m p_{ft}^n \ w_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f} \\ &+ \sum_{\tilde{e}, e=1}^{l_m} \sum_{\tilde{f}, f=1}^{l_n} b_{ef}^{st} p_{\tilde{e}e}^m p_{\tilde{f}f}^n \ u_{\tilde{l}_{m-1}+\tilde{e}, \tilde{l}_{n-1}+\tilde{f}} \end{aligned}$$

and

$$\begin{aligned} A \circ \mathrm{Ad}(k) w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} &= \lambda_1^{(mn)} \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} p_{es}^m p_{ft}^n \ w_{\tilde{l}_{m-1}+e, \tilde{l}_{n-1}+f} \\ &+ \sum_{\tilde{e}, e=1}^{l_m} \sum_{\tilde{f}, f=1}^{l_n} b_{\tilde{e}\tilde{f}}^{ef} p_{es}^m p_{ft}^n \ u_{\tilde{l}_{m-1}+\tilde{e}, \tilde{l}_{n-1}+\tilde{f}}. \end{aligned}$$

Since A commutes with $\mathrm{Ad}(k)$, we obtain

$$\sum_{e=1}^{l_m} \sum_{f=1}^{l_n} b_{ef}^{st} p_{\tilde{e}e}^m p_{\tilde{f}f}^n = \sum_{e=1}^{l_m} \sum_{f=1}^{l_n} b_{\tilde{e}\tilde{f}}^{ef} p_{es}^m p_{ft}^n \ \text{ for all } s, t, \tilde{e}, \tilde{f}.$$

We can proceed as in the proof of Proposition 2.2.2 to show that $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $(s,t) \neq (\tilde{e}, \tilde{f})$ and that $b^{st}_{st} = b^{\tilde{e}\tilde{f}}_{\tilde{e}\tilde{f}} =: b_{mn}$ for all $s, t, \tilde{e}, \tilde{f}$. Now, we shall prove the result for S_n. When $\alpha_l \in \Theta$ and $\alpha_{l-1} \notin \Theta$ we have that $l_r = 1$. Also,

$$Aw_{\tilde{l}_{r-2}+s, \tilde{l}_{n-1}+t} = \lambda_1^{(r-1,n)} w_{\tilde{l}_{r-2}+s, \tilde{l}_{n-1}+t} + \sum_{e=1}^{l_{r-1}} \sum_{f=1}^{l_n} b^{st}_{ef} u_{\tilde{l}_{r-2}+e, \tilde{l}_{n-1}+f} + \sum_{f=1}^{l_n} b^{st}_{f} w_{l, \tilde{l}_{n-1}+f},$$

and

$$Au_{l, \tilde{l}_{n-1}+t} = \lambda_1^{(r-1,n)} u_{l, \tilde{l}_{n-1}+t} + \sum_{e=1}^{l_{r-1}} \sum_{f=1}^{l_n} b^{t}_{ef} u_{\tilde{l}_{r-2}+e, \tilde{l}_{n-1}+f} + \sum_{f=1}^{l_n} b^{t}_{f} w_{l, \tilde{l}_{n-1}+f}.$$

As before, since $\mathrm{Ad}(k) \circ Aw_{\tilde{l}_{r-2}+s, \tilde{l}_{n-1}+t} = A \circ \mathrm{Ad}(k) w_{\tilde{l}_{r-2}+s, \tilde{l}_{n-1}+t}$, then

$$\sum_{e=1}^{l_{r-1}} \sum_{f=1}^{l_n} b^{st}_{ef} p^{r-1}_{\tilde{e}e} p^{n}_{\tilde{f}f} = \sum_{e=1}^{l_{r-1}} \sum_{f=1}^{l_n} b^{ef}_{\tilde{e}\tilde{f}} p^{r-1}_{es} p^{n}_{ft} \tag{2.4.3}$$

and

$$\sum_{f=1}^{l_n} b^{st}_{f} p^{r}_{11} p^{n}_{\tilde{f}f} = \sum_{e=1}^{l_{r-1}} \sum_{f=1}^{l_n} b^{ef}_{\tilde{f}} p^{r-1}_{es} p^{n}_{ft}. \tag{2.4.4}$$

By the same arguments in the proof of Proposition 2.2.2, equations (2.4.3) and (2.4.4) implies $b^{st}_{\tilde{e}\tilde{f}} = 0$ if $(s,t) \neq (\tilde{e}, \tilde{f})$, $b^{st}_{st} = b^{\tilde{e}\tilde{f}}_{\tilde{e}\tilde{f}} =: b^{(1)}_{r-1,n}$ and $b^{st}_{\tilde{f}} = 0$ for all $s, t, \tilde{e}, \tilde{f}$. On the other hand, $\mathrm{Ad}(k) \circ Au_{l, \tilde{l}_{n-1}+t} = A \circ \mathrm{Ad}(k) u_{l, \tilde{l}_{n-1}+t}$ implies

$$\sum_{f=1}^{l_n} b^{t}_{f} p^{n}_{\tilde{f}f} = \sum_{f=1}^{l_n} b^{f}_{\tilde{f}} p^{n}_{ft} \tag{2.4.5}$$

and

$$\sum_{e=1}^{l_{r-1}} \sum_{f=1}^{l_n} b^{t}_{ef} p^{r-1}_{\tilde{e}e} p^{n}_{\tilde{f}f} = \sum_{f=1}^{l_n} b^{f}_{\tilde{e}\tilde{f}} p^{r}_{11} p^{n}_{ft}. \tag{2.4.6}$$

Analogously to the proof of Proposition 2.2.2, we obtain $b^{t}_{\tilde{e}\tilde{f}} = 0$, $b^{t}_{\tilde{f}} = 0$ if $t \neq \tilde{f}$ and $b^{t}_{t} = b^{\tilde{f}}_{\tilde{f}} =: b^{(2)}_{r-1,n}$ for all $t, \tilde{e}, \tilde{f}$. □

Chapter 3

Geodesic orbit spaces

In this chapter, we shall find all the invariant metrics on real flag manifolds of classical type with the property that every homogeneous geodesic is the orbit of a one-parameter subgroup. Such manifolds are called g.o. spaces (geodesic orbit spaces). Examples of g.o. spaces are compact Lie groups equipped with the bi-invariant metric, naturally reductive homogeneous space, the normal metric on homogeneous spaces of compact Lie groups and so on. The classification of *complex* flag manifolds which are *g.o. spaces* is given in [2].

Definition 3.0.1. *Let G/H be a homogeneous space with a $G-$invariant metric g. A geodesic γ starting at eH is called homogeneous if it is the orbit of a $1-$parameter subgroup of G, that is*

$$\gamma(t) = \exp(tX)H \tag{3.0.1}$$

where X is in the Lie algebra of G. In this case, X is called a geodesic vector.

Definition 3.0.2. *We say that a Riemannian homogeneous space $(G/H, g)$ (g an invariant metric) is a g.o. space if every geodesic starting at eH is homogeneous. In this case g is called a g.o. metric.*

Let us consider a homogeneous compact manifold G/H, $(\cdot, \cdot)$ an $\mathrm{Ad}(G)-$invariant inner product in the Lie algebra $\mathfrak{g}$ of G and an $(\cdot, \cdot)-$orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In order to determine the real flag manifolds $(\mathbb{F}_\Theta, g)$ which are g.o. spaces, we shall use the following propositions:

Proposition 3.0.3. *([28]) $(G/H, g)$ is a g.o. space if and only if for every $X \in \mathfrak{m}$, there exist a vector $Z = Z_X \in \mathfrak{h}$ such that*

$$[Z + X, AX] = 0 \tag{3.0.2}$$

where A is the metric operator corresponding to g. □

Since every invariant metric A is a positive $(\cdot, \cdot)-$self-adjoint operator, we have that $\mathfrak{m}$ admits an $(\cdot, \cdot)-$orthogonal decomposition into eigenspaces of A. Given ξ an eigenvalue of A, denote by $\mathfrak{m}_\xi$ its corresponding eigenspace.

Proposition 3.0.4. *([28]) Let $(G/H, A)$ be a g.o. space.*

a) If ξ_1, ξ_2 are eigenvalues of A such that there exist $\mathrm{Ad}(H)-$invariant, pairwise $(\cdot, \cdot)-$orthogonal subspaces $\mathfrak{m}_1$, $\mathfrak{m}_2$ of $\mathfrak{m}$ with

$$\mathfrak{m}_i \subseteq \mathfrak{m}_{\xi_i},\ \ i = 1, 2 \ \textit{and}\ [\mathfrak{m}_1, \mathfrak{m}_2]_{(\mathfrak{m}_1 \oplus \mathfrak{m}_2)^\perp} \neq \{0\},$$

where the orthogonal complement is taking with respect to $(\cdot, \cdot)$, *then* $\xi_1 = \xi_2$.

b) If $\xi_1,\ \xi_2, \xi_3$ *are eigenvalues of* A *such that there exist* $\mathrm{Ad}(H)-$*invariant, pairwise* $(\cdot, \cdot)-$*orthogonal subspaces* $\mathfrak{m}_1,\ \mathfrak{m}_2\ \mathfrak{m}_3$ *of* $\mathfrak{m}$ *with*

$$\mathfrak{m}_i \subseteq \mathfrak{m}_{\xi_i},\ \ i = 1, 2, 3 \ \textit{and}\ [\mathfrak{m}_1, \mathfrak{m}_2]_{\mathfrak{m}_3} \neq \{0\},$$

then $\xi_1 = \xi_2 = \xi_3$. □

Remark 3.0.5. *Every invariant metric A which is a scalar multiple of the identity endomorphism is a trivial solution for equation* (3.0.2) *(by taking* $Z = 0$ *for every* X*), such a metric is called* ***normal.***

3.1 Flags of A_l

As in Section 2.1, we fix $(\cdot, \cdot) = -\langle \cdot, \cdot \rangle$, where $\langle \cdot, \cdot \rangle$ is the Killing form of $\mathfrak{so}(l + 1)$ and the $(\cdot, \cdot)-$orthogonal basis $\{w_{ij} : 1 \leq j < i \leq l + 1\}$.

Proposition 3.1.1. *Let* $\mathbb{F}_\Theta$ *be a flag of* A_l*, with* $l \neq 3$ *or* $l = 3$ *and* $\Theta \in \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}\}$. *Then,* $(\mathbb{F}_\Theta, A)$ *is a g.o. space if and only if A is normal, i.e.,* $A = \mu I_{\mathfrak{m}_\Theta}$, $\mu > 0$.

Proof. Let A be a g.o. metric in a flag $\mathbb{F}_\Theta$ of A_l. If $l = 3$ and $\Theta \in \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}\}$, then, by Proposition 2.1.2, A has the form $A = \mu \mathbf{I}_{\mathfrak{m}_\Theta}$. If $l \neq 3$, we have by Proposition 2.1.1 that A is determined by $\frac{r(r-1)}{2}$ positive numbers μ_{mn}, $1 \leq n < m \leq r$, such that $A|_{M_{nm}} = \mu_{mn} \mathrm{I}_{M_{mn}}$, with M_{mn} as in (1.1.2). We shall prove that $\mu_{mm} = \mu_{m'n'}$ for all (m, n), (m', n'). First, we prove $\mu_{mn} = \mu_{m'n}$ for $1 \leq n < m, m' \leq r$. In fact, without loss of generality, let us suppose $m < m'$, then

$$w_{\tilde{l}_{m'-1}+1, \tilde{l}_{m-1}+1} = \left[w_{\tilde{l}_{m-1}+1, \tilde{l}_{n-1}+1}, w_{\tilde{l}_{m'-1}+1, \tilde{l}_{n-1}+1}\right] \in [M_{mn}, M_{m'n}]\,.$$

Since $\tilde{l}_{m-1} + 1 \neq \tilde{l}_{n-1} + t$ for all $t \in \{1, ..., l_n\}$, then

$$(w_{\tilde{l}_{m'-1}+1, \tilde{l}_{m-1}+1}, w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}) = 0,\ 1 \leq s \leq l_m,\ 1 \leq t \leq l_n$$

and

$$(w_{\tilde{l}_{m'-1}+1, \tilde{l}_{m-1}+1}, w_{\tilde{l}_{m'-1}+s, \tilde{l}_{n-1}+t}) = 0,\ 1 \leq s \leq l_{m'},\ 1 \leq t \leq l_n,$$

thus, $w_{\tilde{l}_{m'-1}+1, \tilde{l}_{m-1}+1} \in (M_{mn} \oplus M_{m'n})^\perp$. Evidently, M_{mn} and $M_{m'n}$ are contained in the eigenspaces of A corresponding to the eigenvalues μ_{mn} and $\mu_{m'n}$ respectively and, by Proposition 1.1.1, they are $K_\Theta-$invariant. Also M_{mn} and $M_{m'n}$ are $(\cdot, \cdot)-$orthogonal. By Proposition 3.0.4, we conclude that $\mu_{mn} = \mu_{m'n}$. Now, we will show that $\mu_{mn} = \mu_{mn'}$, $1 \leq n, n' < m \leq r$. Let us suppose $n < n'$, then

$$w_{\tilde{l}_{n'-1}+1, \tilde{l}_{n-1}+1} = \left[w_{\tilde{l}_{m-1}+1, \tilde{l}_{n-1}+1}, w_{\tilde{l}_{m-1}+1, \tilde{l}_{n'-1}+1}\right] \in [M_{mn}, M_{mn'}]\,.$$

Since $\tilde{l}_{n-1} + 1 \neq \tilde{l}_{m-1} + t$ for all $t \in \{1, ..., l_m\}$, then

$$(w_{\tilde{l}_{n'-1}+1,\tilde{l}_{n-1}+1}, w_{\tilde{l}_{m-1}+s,\tilde{l}_{n-1}+t}) = 0,\ 1 \le s \le l_m,\ 1 \le t \le l_n$$

and

$$(w_{\tilde{l}_{n'-1}+1,\tilde{l}_{n-1}+1}, w_{\tilde{l}_{m-1}+s,\tilde{l}_{n'-1}+t}) = 0,\ 1 \le s \le l_m,\ 1 \le t \le l_{n'},$$

thus, $w_{\tilde{l}_{n'-1}+1,\tilde{l}_{n-1}+1} \in (M_{mn} \oplus M_{mn'})^{\perp}$. By Proposition 3.0.4, $\mu_{mn} = \mu_{mn'}$. Hence, $\mu_{mn} = \mu_{m'n} = \mu_{m'n'}$. □

Proposition 3.1.2. *Let $\mathbb{F}_\Theta$ be a flag of A_3 with $\Theta \notin \{\{\alpha_1, \alpha_2\}, \{\alpha_2, \alpha_3\}\}$. Then, $(\mathbb{F}_\Theta, A)$ is a g.o. space if and only if A is written in the basis (2.1.3) as*

$$[A]_{\mathcal{B}_\emptyset} = \begin{pmatrix} \mu & b & 0 & 0 & 0 & 0 \\ b & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & -b & 0 & 0 \\ 0 & 0 & -b & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & b \\ 0 & 0 & 0 & 0 & b & \mu \end{pmatrix},\ [A]_{\mathcal{B}_{\{\alpha_1,\alpha_3\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix} \text{ and}$$

$$[A]_{\mathcal{B}_\Theta} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & \pm\sqrt{\mu_2(\mu_2-\mu_1)} & 0 \\ 0 & 0 & \mu_2 & 0 & \mp\sqrt{\mu_2(\mu_2-\mu_1)} \\ 0 & \pm\sqrt{\mu_2(\mu_2-\mu_1)} & 0 & \mu_2 & 0 \\ 0 & 0 & \mp\sqrt{\mu_2(\mu_2-\mu_1)} & 0 & \mu_2 \end{pmatrix},\ \mu_2 \ge \mu_1$$

if $\Theta = \{\alpha_1\}, \{\alpha_2\}$ or $\{\alpha_3\}$.

Proof. Let us analyse case by case:

- $\Theta = \emptyset$.

Since $\mathfrak{k}_\emptyset = \{0\}$, by Proposition 3.0.3, A is a g.o. metric if and only if $[X, AX] = 0$ for every $X \in \mathfrak{m}_\emptyset$. We write A as in Proposition 2.1.2. For $X = w_{21} + w_{31} + w_{41}$, we have that $AX = \mu_1^{(1)} w_{21} + b_1 w_{43} + \mu_1^{(2)} w_{31} + b_2 w_{42} + b_3 w_{32} + \mu_2^{(3)} w_{41}$ and $[X, AX] = 0$ if and only if

$$(b_2+b_3)w_{21} + (\mu_2^{(3)} - \mu_1^{(2)})w_{43} + (b_1 - b_3)w_{31} + (\mu_2^{(3)} - \mu_1^{(1)})w_{42} + (\mu_1^{(2)} - \mu_1^{(1)})w_{32} - (b_1 + b_2)w_{41} = 0,$$

thus, $\mu_1^{(1)} = \mu_1^{(2)} = \mu_2^{(3)}$ and $b_1 = -b_2 = b_3 =: b$. For $X = w_{43} + w_{42} + w_{41}$, we have that $AX = b_1 w_{21} + \mu_2^{(1)} w_{43} + +b_2 w_{31} + \mu_2^{(2)} w_{42} + b_3 w_{32} + \mu_2^{(3)} w_{41}$ and $[X, AX] = 0$ if and only if

$$(\mu_2^{(2)} - \mu_2^{(3)})w_{21} - (b_2 + b_3)w_{43} + (\mu_2^{(1)} - \mu_2^{(3)})w_{31} + (b_3 - b_1)w_{42} + (\mu_2^{(1)} - \mu_2^{(2)})w_{32} + (b_1 + b_2)w_{41} = 0,$$

concluding that $\mu_1^{(1)} = \mu_2^{(1)} = \mu_1^{(2)} = \mu_2^{(2)} = \mu_1^{(3)} = \mu_2^{(3)} =: \mu$. It is easy to verify that if $\mu_1^{(1)} = \mu_2^{(1)} = \mu_1^{(2)} = \mu_2^{(2)} = \mu_1^{(3)} = \mu_2^{(3)} =: \mu$ and $b_1 = -b_2 = b_3 =: b$, then $[X, AX] = 0$ for all $X \in \mathfrak{m}_\emptyset$.

• $\Theta = \{\alpha_1, \alpha_3\}$

Let A be a g.o. metric. By Proposition 2.1.2, A has the form

$$[A]_{\mathcal{B}_{\{\alpha_1,\alpha_3\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}.$$

Given $X = x_1(w_{31} - w_{42}) + x_2(w_{41} + w_{32}) + x_3(w_{31} + w_{42}) + x_4(w_{41} - w_{32})$ we have that

$$\begin{aligned} [X, AX] &= \mu_1 x_1 x_2 [w_{31} - w_{42}, w_{41} + w_{32}] + \mu_2 x_1 x_4 [w_{31} - w_{42}, w_{41} - w_{32}] \\ &\quad + \mu_1 x_1 x_2 [w_{41} + w_{32}, w_{31} - w_{42}] + \mu_2 x_2 x_3 [w_{41} + w_{32}, w_{31} + w_{42}] \\ &\quad + \mu_1 x_2 x_3 [w_{31} + w_{42}, w_{41} + w_{32}] + \mu_2 x_3 x_4 [w_{31} + w_{42}, w_{41} - w_{32}] \\ &\quad + \mu_1 x_1 x_4 [w_{41} - w_{32}, w_{31} - w_{42}] + \mu_2 x_3 x_4 [w_{41} - w_{32}, w_{31} + w_{42}] \\ &= 2\mu_1 x_1 x_2 (w_{21} + w_{43}) - 2\mu_1 x_1 x_2 (w_{21} + w_{43}) \\ &\quad + 2\mu_2 x_3 x_4 (w_{43} - w_{21}) - 2\mu_2 x_3 x_4 (w_{43} - w_{21}) \\ &= 0. \end{aligned}$$

Therefore $[0 + X, AX] = 0$ for all $X \in \mathfrak{m}_{\{\alpha_1,\alpha_3\}}$ and A is g.o.

• $\Theta = \{\alpha_1\}$.

Since $\mathfrak{k}_{\{\alpha_1\}} = \text{span}\{w_{21}\}$, then A is a g.o. metric if and only if for all $X \in \mathfrak{m}_{\{\alpha_1\}}$, there exists $\lambda \in \mathbb{R}$ such that

$$[\lambda w_{21} + X, AX] = 0.$$

Assume that A is a g.o. metric, then, by Proposition 2.1.2, we have that

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1^{(1)} & 0 & 0 & 0 & 0 \\ 0 & \mu_1^{(2)} & 0 & b & 0 \\ 0 & 0 & \mu_1^{(2)} & 0 & -b \\ 0 & b & 0 & \mu_2^{(2)} & 0 \\ 0 & 0 & -b & 0 & \mu_2^{(2)} \end{pmatrix}.$$

Set $\mu_1 := \mu_1^{(1)}$. For $X = w_{31} + w_{32} + w_{42} + w_{41}$ take $\lambda \in \mathbb{R}$ such that $[\lambda w_{21} + X, AX] = 0$, then

$$\begin{aligned} 0 &= [\lambda w_{21} + X, AX] \\ &= 2(\mu_2^{(2)} - \mu_1^{(2)}) w_{43} - \lambda(\mu_1^{(2)} - b) w_{31} + \lambda(\mu_2^{(2)} - b) w_{42} + \lambda(\mu_1^{(2)} + b) w_{32} - \lambda(\mu_2^{(2)} + b) w_{41}, \end{aligned}$$

by linear independence we have $\mu_2^{(2)} - \mu_1^{(2)} = 0$, i.e., $\mu_2^{(2)} = \mu_1^{(2)} =: \mu_2$. Now, let us consider the vector $X = w_{43} + w_{32} + 2w_{41}$ and $\lambda \in \mathbb{R}$ such that $[\lambda w_{21} + X, AX] = 0$, then

$$\begin{aligned}0 &= [\lambda w_{21} + X, AX] \\ &= (2(\mu_1 - \mu_2) + b - \lambda(\mu_2 - 2b))w_{31} + (\mu_2 - \mu_1 - 2b + \lambda(2\mu_2 - b))w_{42},\end{aligned}$$

by linear independence

$$2(\mu_1 - \mu_2) + b - \lambda(\mu_2 - 2b) = 0 \qquad (3.1.1)$$

and

$$\mu_2 - \mu_1 - 2b + \lambda(2\mu_2 - b) = 0. \qquad (3.1.2)$$

Multiplying equation (3.1.1) by $(2\mu_2 - b)$ and equation (3.1.2) by $(\mu_2 - 2b)$, we obtain respectively

$$-4\mu_2^2 + 4\mu_1\mu_2 + 4b\mu_2 - 2b\mu_1 - b^2 - \lambda(2\mu_2 - b)(\mu_2 - 2b) = 0 \qquad (3.1.3)$$

and

$$\mu_2^2 - 4b\mu_2 - \mu_1\mu_2 + 2b\mu_1 + 4b^2 + \lambda(2\mu_2 - b)(\mu_2 - 2b) = 0. \qquad (3.1.4)$$

Adding equations (3.1.3) and (3.1.4) we have

$$3(b^2 - \mu_2(\mu_2 - \mu_1)) = 0, \qquad (3.1.5)$$

therefore $b^2 - \mu_2(\mu_2 - \mu_1) = 0$, i.e., $b = \pm\sqrt{\mu_2(\mu_2 - \mu_1)}$.

Conversely, let us suppose that $\mu_2 := \mu_1^{(2)} = \mu_2^{(2)}$, $\mu_1 := \mu_1^{(1)}$ and $b = \sqrt{\mu_2(\mu_2 - \mu_1)}$, then, for $X = x_{43}w_{43} + x_{31}w_{31} + x_{42}w_{42} + x_{32}w_{32} + x_{41}w_{41}$ in $\mathfrak{m}_{\{\alpha_1\}}$ we have

$$\begin{aligned}[X, AX] = \; & x_{43}\sqrt{\mu_2 - \mu_1}(x_{32}\sqrt{\mu_2} - x_{41}\sqrt{\mu_2 - \mu_1})w_{31} \\ & - x_{43}\sqrt{\mu_2 - \mu_1}(x_{41}\sqrt{\mu_2} - x_{32}\sqrt{\mu_2 - \mu_1})w_{42} \\ & - x_{43}\sqrt{\mu_2 - \mu_1}(x_{31}\sqrt{\mu_2} + x_{42}\sqrt{\mu_2 - \mu_1})w_{32} \\ & + x_{43}\sqrt{\mu_2 - \mu_1}(x_{42}\sqrt{\mu_2} + x_{31}\sqrt{\mu_2 - \mu_1})w_{41}\end{aligned}$$

and for every $\lambda \in \mathbb{R}$

$$\begin{aligned}[\lambda w_{21}, AX] &= -\lambda\sqrt{\mu_2}(x_{32}\sqrt{\mu_2} - x_{41}\sqrt{\mu_2 - \mu_1})w_{31} + \lambda\sqrt{\mu_2}(x_{41}\sqrt{\mu_2} - x_{32}\sqrt{\mu_2 - \mu_1})w_{42} \\ &\quad + \lambda\sqrt{\mu_2}(x_{31}\sqrt{\mu_2} + x_{42}\sqrt{\mu_2 - \mu_1})w_{32} - \lambda\sqrt{\mu_2}(x_{42}\sqrt{\mu_2} + x_{31}\sqrt{\mu_2 - \mu_1})w_{41},\end{aligned}$$

thus

$$\begin{aligned}[\lambda w_{21} + X, AX] &= (x_{43}\sqrt{\mu_2 - \mu_1} - \lambda\sqrt{\mu_2})(x_{32}\sqrt{\mu_2} - x_{41}\sqrt{\mu_2 - \mu_1})w_{31} \\ &\quad - (x_{43}\sqrt{\mu_2 - \mu_1} - \lambda\sqrt{\mu_2})(x_{41}\sqrt{\mu_2} - x_{32}\sqrt{\mu_2 - \mu_1})w_{42} \\ &\quad - (x_{43}\sqrt{\mu_2 - \mu_1} - \lambda\sqrt{\mu_2})(x_{31}\sqrt{\mu_2} + x_{42}\sqrt{\mu_2 - \mu_1})w_{32} \\ &\quad + (x_{43}\sqrt{\mu_2 - \mu_1} - \lambda\sqrt{\mu_2})(x_{42}\sqrt{\mu_2} + x_{31}\sqrt{\mu_2 - \mu_1})w_{41}.\end{aligned}$$

By taking $\lambda = \sqrt{\frac{\mu_2-\mu_1}{\mu_2}}x_{43}$, we obtain $[\lambda w_{21} + X, AX] = 0$. If $b = -\sqrt{\mu_2(\mu_2-\mu_1)}$, the argument is analogous.

• $\Theta = \{\alpha_2\}$ or $\{\alpha_3\}$.

Let us consider the diffeomorphisms

$$\begin{array}{rccc} \varphi_i: & \mathbb{F}_{\{\alpha_1\}} & \longrightarrow & \mathbb{F}_{\{\alpha_i\}} \\ & kK_{\{\alpha_1\}} & \longmapsto & e_i^T k e_i K_{\{\alpha_i\}} \end{array}, \; i = 2, 3$$

where $e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. It is easy to verify that for every invariant metric

$$[A]_{\mathcal{B}_{\{\alpha_1\}}} = \begin{pmatrix} \mu_1^{(1)} & 0 & 0 & 0 & 0 \\ 0 & \mu_1^{(2)} & 0 & b & 0 \\ 0 & 0 & \mu_1^{(2)} & 0 & -b \\ 0 & b & 0 & \mu_2^{(2)} & 0 \\ 0 & 0 & -b & 0 & \mu_2^{(2)} \end{pmatrix}$$

we have

$$[\varphi_2^* A]_{\mathcal{B}_{\{\alpha_2\}}} = \begin{pmatrix} \mu_1^{(1)} & 0 & 0 & 0 & 0 \\ 0 & \mu_1^{(2)} & 0 & -b & 0 \\ 0 & 0 & \mu_1^{(2)} & 0 & b \\ 0 & -b & 0 & \mu_2^{(2)} & 0 \\ 0 & 0 & b & 0 & \mu_2^{(2)} \end{pmatrix} \text{ and}$$

$$[\varphi_3^* A]_{\mathcal{B}_{\{\alpha_3\}}} = \begin{pmatrix} \mu_1^{(1)} & 0 & 0 & 0 & 0 \\ 0 & \mu_1^{(2)} & 0 & b & 0 \\ 0 & 0 & \mu_1^{(2)} & 0 & -b \\ 0 & b & 0 & \mu_2^{(2)} & 0 \\ 0 & 0 & -b & 0 & \mu_2^{(2)} \end{pmatrix}.$$

By Proposition 2.1.2, every invariant metric in $\mathbb{F}_{\{\alpha_i\}}$ has the form $\varphi_i^* A$. Since A is g.o. if and only if $\varphi_i^* A$ is g.o. then we obtain the result. □

3.2 Flags of B_l

We consider $(\cdot,\cdot)$ as in (1.2.1) and the $(\cdot,\cdot)-$orthonormal basis in (1.2.2).

Proposition 3.2.1. *Let $\mathbb{F}_\Theta$ be a flag of B_l, $l \geq 5$ and A an invariant metric as in Proposition 2.2.2.*

a) If $\alpha_l \notin \Theta$, then $(\mathbb{F}_\Theta, A)$ is a g.o. space if and only if

$$\begin{cases} \mu^{(1)} = ... = \mu^{(r)} =: \mu, \\ \lambda_1^{(mn)} = \lambda_2^{(mn)} =: \lambda, \ \textit{for all } (m,n), \\ b_{mn} =: b, \ \textit{for all } (m,n), \\ \gamma^{(i)} =: \gamma \textit{ for all } i \in \{1,...,r\} \textit{ with } l_i > 1, \\ \mu - \lambda = b \\ \gamma = \frac{\lambda^2 - b^2}{\lambda} \end{cases} \tag{3.2.1}$$

b) If $\alpha_l \in \Theta$, then $(\mathbb{F}_\Theta, A)$ is a g.o. space if and only if

$$\begin{cases} \mu^{(1)} = ... = \mu^{(r-1)} =: \mu, \\ \rho^{(1)} = ... = \rho^{(r-1)} =: \rho \\ \lambda_1^{(mn)} = \lambda_2^{(mn)} =: \lambda, \ \textit{for all } (m,n), \\ b_{mn} =: b, \ \textit{for all } (m,n), \\ \gamma^{(i)} =: \gamma \textit{ for all } i \in \{1,...,r-1\} \textit{ with } l_i > 1, \\ \mu - \lambda = b = \lambda - \rho \\ \gamma = \frac{2\mu\rho}{\mu+\rho} = \frac{\lambda^2 - b^2}{\lambda} \end{cases} \tag{3.2.2}$$

Proof. *a*) Let us suppose that $(\mathbb{F}_\Theta, A)$ is a g.o. space. We have that for $i < j$, V_i and V_j are $(\cdot,\cdot)-$orthogonal, $K_\Theta-$invariant, contained in the eigenspaces corresponding to the eigenvalues $\mu^{(i)}$ and $\mu^{(j)}$ respectively and

$$w_{\tilde{l}_{j-1}+1,\tilde{l}_{i-1}+1} + u_{\tilde{l}_{j-1}+1,\tilde{l}_{i-1}+1} = \left[v_{l_{i-1}+1}, v_{l_{j-1}+1}\right] \in (V_i \oplus V_j)^\perp \cap [V_i, V_j]$$

then, by Proposition 3.0.4, $\mu^{(i)} = \mu^{(j)} = \mu$. For $X = w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + w_{\tilde{l}_{m-1}+1,\tilde{l}_n+1}$, there exists a $Z \in \mathfrak{k}_\Theta$ such that $[Z + X, AX] = 0$, but

$$\begin{aligned}
[X,AX] &= \left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}+w_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1},\lambda_1^{(mn)}w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}+b_{mn}u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}\right]\\
&\quad+\left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}+w_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1},\lambda_1^{(m,n+1)}w_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1}+b_{m,n+1}u_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1}\right]\\
&= \lambda_1^{(m,n+1)}\left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1},w_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1}\right]+b_{m,n+1}\left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1},u_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1}\right]\\
&\quad+\lambda_1^{(mn)}\left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1},w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}\right]+b_{mn}\left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1},u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}\right]\\
&= (\lambda_1^{(m,n+1)}-\lambda_1^{(mn)})w_{\tilde{l}_{n}+1,\tilde{l}_{n-1}+1}+(b_{m,n+1}-b_{mn})u_{\tilde{l}_{n}+1,\tilde{l}_{n-1}+1}\in M_{n+1,n}
\end{aligned}$$

and since $M_{mn}\oplus M_{m,n+1}$ is $\mathfrak{k}_\Theta-$invariant, then $[Z,AX]\in M_{mn}\oplus M_{m,n+1}$. By linear independence, $[Z+X,AX]=0\Rightarrow[Z,AX]=0=[X,AX]$, thus $b_{m,n+1}=b_{mn}$ and $\lambda_1^{(m,n+1)}=\lambda_1^{(mn)}$. Taking $X=u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}+u_{\tilde{l}_{m-1}+1,\tilde{l}_{n}+1}$, $w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}+w_{\tilde{l}_{m}+1,\tilde{l}_{n-1}+1}$ or $u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}+u_{\tilde{l}_{m}+1,\tilde{l}_{n-1}+1}$ we obtain that

$$\begin{aligned}
[X,AX] &= (\lambda_2^{(m,n+1)}-\lambda_2^{(mn)})w_{\tilde{l}_{n}+1,\tilde{l}_{n-1}+1}+(b_{m,n+1}-b_{mn})u_{\tilde{l}_{n}+1,\tilde{l}_{n-1}+1},\\
&\quad(\lambda_1^{(m+1,n)}-\lambda_1^{(mn)})w_{\tilde{l}_{m}+1,\tilde{l}_{m-1}+1}+(b_{m+1,n}-b_{mn})u_{\tilde{l}_{m}+1,\tilde{l}_{m-1}+1}\\
\mathit{or}&\quad(\lambda_2^{(m+1,n)}-\lambda_2^{(mn)})w_{\tilde{l}_{m}+1,\tilde{l}_{m-1}+1}+(b_{m+1,n}-b_{mn})u_{\tilde{l}_{m}+1,\tilde{l}_{m-1}+1}
\end{aligned}$$

respectively. As before, this implies $\lambda_2^{(m,n+1)}=\lambda_2^{(mn)}$, $\lambda_1^{(m+1,n)}=\lambda_1^{(mn)}$, $\lambda_2^{(m+1,n)}=\lambda_2^{(mn)}$ and $b_{m+1,n}=b_{mn}$. Since this argument works for every pair (m,n), then

$$\begin{aligned}
&\lambda_1^{(mn)}=\lambda_1^{(m'n')}=:\lambda_1,\\
&\lambda_2^{(mn)}=\lambda_2^{(m'n')}=:\lambda_2\text{ and}\\
&b_{mn}=b_{m'n'}=:b
\end{aligned}$$

for all m,n,m',n'. For $X=v_{\tilde{l}_1+1}+w_{\tilde{l}_1+1,1}$, take $Z\in\mathfrak{k}_\Theta$ such hat $[Z+X,AX]=0$, then

$$\begin{aligned}
[X,AX] &= \left[v_{\tilde{l}_1+1}+w_{\tilde{l}_1+1,1},\mu\, v_{\tilde{l}_1+1}+\lambda_1 w_{\tilde{l}_1+1,1}+b\, u_{\tilde{l}_1+1,1}\right]\\
&= \lambda_1\left[v_{\tilde{l}_1+1},w_{\tilde{l}_1+1,1}\right]+b\left[v_{\tilde{l}_1+1},u_{\tilde{l}_1+1,1}\right]+\mu\left[w_{\tilde{l}_1+1,1},v_{\tilde{l}_1+1}\right]\\
&= (\lambda_1+b-\mu)v_1\in V_1
\end{aligned}$$

and $[Z,AX]\in V_2\oplus M_{21}$. Thus, by linear independence we conclude that $\lambda_1+b-\mu=0$. Using the same argument for $X=v_{\tilde{l}_1+1}+u_{\tilde{l}_1+1,1}$, we can show that $\lambda_2+b-\mu=0$. Therefore, $\lambda_1=\lambda_2=:\lambda$ and $b=\mu-\lambda$.

Next, we will prove that $\gamma^{(i)}=\gamma^{(j)}$ for all i,j with $l_i,l_j>1$. In fact, if $l_i>1$ and $i\leq r-1$, take

$$X=v_{\tilde{l}_{i-1}+1}+u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}+w_{\tilde{l}_{i}+1,\tilde{l}_{i-1}+1}$$

and $Z\in\mathfrak{k}_\Theta$ such that $[Z+X,AX]=0$. Then, $AX=\mu\, v_{\tilde{l}_{i-1}+1}+\gamma^{(i)}u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}+\lambda\, w_{\tilde{l}_{i}+1,\tilde{l}_{i-1}+1}+b\, u_{\tilde{l}_{i}+1,\tilde{l}_{i-1}+1}$ and

$$\begin{aligned}
[X, AX] &= \gamma^{(i)} \left[v_{\tilde{l}_{i-1}+1}, u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}\right] + \lambda \left[v_{\tilde{l}_{i-1}+1}, w_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}\right] + b \left[v_{\tilde{l}_{i-1}+1}, u_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}\right] \\
&\quad +\mu \left[u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}, v_{\tilde{l}_{i-1}+1}\right] + \lambda \left[u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}, w_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}\right] \\
&\quad +b \left[u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}, u_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}\right] + \mu \left[w_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}, v_{\tilde{l}_{i-1}+1}\right] \\
&\quad +\gamma^{(i)} \left[w_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}, u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}\right] \\
&= (\mu - \gamma^{(i)})v_{\tilde{l}_{i-1}+2} + (\mu - \lambda - b)v_{\tilde{l}_i+1} + (\lambda - \gamma^{(i)})u_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} + b\, w_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} \\
&= (\mu - \gamma^{(i)})v_{\tilde{l}_{i-1}+2} + (\lambda - \gamma^{(i)})u_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} + b\, w_{\tilde{l}_i+1,\tilde{l}_{i-1}+2}.
\end{aligned}$$

We can write $Z = \sum_{l_j>1} \left(\sum_{1\le t<s\le l_j} z_{st}^{(j)} w_{\tilde{l}_{j-1}+s,\tilde{l}_{j-1}+t} \right)$, $z_{st}^{(j)} \in \mathbb{R}$, so

$$[Z, AX] = \mu z_{21}^{(i)} v_{\tilde{l}_{i-1}+2} + b z_{21}^{(i)} u_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} + \lambda z_{21}^{(i)} w_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} + Z',$$

where Z' is linearly independent of $\{v_{\tilde{l}_{i-1}+2}, u_{\tilde{l}_i+1,\tilde{l}_{i-1}+2}, w_{\tilde{l}_i+1,\tilde{l}_{i-1}+2}\}$. Therefore,

$$[Z+X, AX] = (\mu - \gamma^{(i)} + \mu z_{21}^{(i)})v_{\tilde{l}_{i-1}+2} + (z_{21}^{(i)} b + \lambda - \gamma^{(i)})u_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} + (b + \lambda z_{21}^{(i)})w_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} + Z',$$

so $Z' = 0$ and

$$\begin{cases} \mu - \gamma^{(i)} + \mu z_{21}^{(i)} = 0 \\ z_{21}^{(i)} b + \lambda - \gamma^{(i)} = 0 \\ b + \lambda z_{21}^{(i)} = 0 \end{cases}.$$

This implies $\gamma^{(i)} = \frac{\lambda^2 - b^2}{\lambda}$ (and $z_{21}^{(i)} = -\frac{b}{\lambda}$). If $i = r$ and $l_r > 1$, we use the a similar argument with $X = v_{\tilde{l}_{r-1}+1} + u_{\tilde{l}_{r-1}+2,\tilde{l}_{r-1}+1} + w_{\tilde{l}_{r-1}+1,\tilde{l}_{r-2}+1}$ to conclude that $\gamma^{(r)} = \frac{\lambda^2-b^2}{\lambda}$. Therefore $\gamma^{(i)} = \gamma^{(j)} =: \gamma$ for all i, j with $l_i, l_j > 1$ and $\gamma = \frac{\lambda^2-b^2}{\lambda}$.

Conversely, let us suppose that A satisfies the relations (3.2.1). We can write every $X \in \mathfrak{m}_\Theta$ as

$$X = \sum_{i=1}^{r} v^{(i)} + \sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i + \sum_{1\le n<m\le r} X_{mn} + \sum_{1\le n<m\le r} Y_{mn},$$

where $v^{(i)} \in V_i$, $Y_i \in U_i$, $X_{mn} \in W_{mn}$, and $Y_{mn} \in U_{mn}$. If

$$X_{mn} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & C_{mn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{mn} \end{pmatrix}, \; Y_{mn} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{mn} \\ \mathbf{0} & D_{mn} & \mathbf{0} \end{pmatrix} \text{ and } Y_i = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_i \\ \mathbf{0} & D_i & \mathbf{0} \end{pmatrix}$$

we denote

$$\tilde{X}_{mn} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_{mn} \\ \mathbf{0} & C_{mn} & \mathbf{0} \end{pmatrix}, \tilde{Y}_{mn} = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{mn} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{mn} \end{pmatrix} \text{ and } \tilde{Y}_i = \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_i \end{pmatrix},$$

so we have $AX_{mn} = \lambda X_{mn} + b\tilde{X}_{mn}$, $AY_{mn} = b\tilde{Y}_{mn} + \lambda Y_{mn}$ and

$$AX = \mu \sum_{i=1}^{r} v^{(i)} + \gamma \sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i + \lambda \sum_{1\le n<m\le r} X_{mn} + b \sum_{1\le n<m\le r} \tilde{X}_{mn} + b \sum_{1\le n<m\le r} \tilde{Y}_{mn} + \lambda \sum_{1\le n<m\le r} Y_{mn}.$$

Then

$$\begin{aligned}
[X, AX] &= \gamma \left[\sum_{i=1}^{r} v^{(i)}, \sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i \right] + \lambda \left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\le n<m\le r} X_{mn} \right] + b \left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\le n<m\le r} \tilde{X}_{mn} \right] \\
&\quad + b \left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\le n<m\le r} \tilde{Y}_{mn} \right] + \lambda \left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\le n<m\le r} Y_{mn} \right] + \mu \left[\sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i, \sum_{i=1}^{r} v^{(i)} \right] \\
&\quad + \lambda \left[\sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} X_{mn} \right] + b \left[\sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{X}_{mn} \right] + b \left[\sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{Y}_{mn} \right] \\
&\quad + \lambda \left[\sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} Y_{mn} \right] + \mu \left[\sum_{1\le n<m\le r} X_{mn}, \sum_{i=1}^{r} v^{(i)} \right] + \gamma \left[\sum_{1\le n<m\le r} X_{mn}, \sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i \right] \\
&\quad + b \left[\sum_{1\le n<m\le r} X_{mn}, \sum_{1\le n<m\le r} \tilde{X}_{mn} \right] + b \left[\sum_{1\le n<m\le r} X_{mn}, \sum_{1\le n<m\le r} \tilde{Y}_{mn} \right] \\
&\quad + \lambda \left[\sum_{1\le n<m\le r} X_{mn}, \sum_{1\le n<m\le r} Y_{mn} \right] + \mu \left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{i=1}^{r} v^{(i)} \right] \\
&\quad + \gamma \left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{\substack{i=1 \\ l_i>1}}^{r} Y_i \right] + \lambda \left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{1\le n<m\le r} X_{mn} \right] \\
&\quad + b \left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{1\le n<m\le r} \tilde{X}_{mn} \right] + b \left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{1\le n<m\le r} \tilde{Y}_{mn} \right],
\end{aligned}$$

since

$$\left[\sum_{1\leq n<m\leq r} X_{mn}, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right] = 0 = \left[\sum_{1\leq n<m\leq r} Y_{mn}, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right],$$

$$\left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\leq n<m\leq r} X_{mn}\right] = \left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right],$$

$$\left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\leq n<m\leq r} Y_{mn}\right] = \left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right] \text{ and}$$

$$\left[\sum_{1\leq n<m\leq r} X_{mn}, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right] = -\left[\sum_{1\leq n<m\leq r} Y_{mn}, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right],$$

then

$$\begin{aligned}
[X, AX] &= (\gamma-\mu)\left[\sum_{i=1}^{r} v^{(i)}, \sum_{\substack{i=1\\ l_i>1}}^{r} Y_i\right] + (\lambda+b-\mu)\left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\leq n<m\leq r} X_{mn}\right]\\
&\quad +(\lambda+b-\mu)\left[\sum_{i=1}^{r} v^{(i)}, \sum_{1\leq n<m\leq r} Y_{mn}\right] + (\lambda-\gamma)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} X_{mn}\right]\\
&\quad +(\lambda-\gamma)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} Y_{mn}\right] + b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right]\\
&\quad +b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right]\\
&= (\gamma-\mu)\left[\sum_{i=1}^{r} v^{(i)}, \sum_{\substack{i=1\\ l_i>1}}^{r} Y_i\right] + (\lambda-\gamma)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} X_{mn}\right]\\
&\quad +(\lambda-\gamma)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} Y_{mn}\right] + b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right]\\
&\quad +b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right].
\end{aligned}$$

Last equality holds because $\lambda + b - \mu = 0$. By taking $Z = -\frac{b}{\lambda}\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i \in \mathfrak{k}_\Theta$, we obtain

$$
\begin{aligned}
[Z, AX] &= -\frac{b\mu}{\lambda}\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{i=1}^{r} v^{(i)}\right] - b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} X_{mn}\right] - b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} Y_{mn}\right] \\
&\quad -\frac{b^2}{\lambda}\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right] - \frac{b^2}{\lambda}\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right],
\end{aligned}
$$

but

$$
\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{i=1}^{r} v^{(i)}\right] = \left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{i=1}^{r} v^{(i)}\right], \quad \left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} X_{mn}\right] = \left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right],
$$

$$
\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} Y_{mn}\right] = \left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right],
$$

$$
\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right] = \left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} X_{mn}\right],
$$

$$
\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right] = \left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} Y_{mn}\right],
$$

then

$$
\begin{aligned}
[Z+X, AX] &= \left(\gamma - \mu + \frac{b\mu}{\lambda}\right)\left[\sum_{i=1}^{r} v^{(i)}, \sum_{\substack{i=1\\ l_i>1}}^{r} Y_i\right] + \left(\lambda - \gamma - \frac{b^2}{\lambda}\right)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} X_{mn}\right] \\
&\quad + \left(\lambda - \gamma - \frac{b^2}{\lambda}\right)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} Y_{mn}\right] \\
&= 0 \qquad \text{(by (3.2.1))}.
\end{aligned}
$$

By Proposition 3.0.3, A is a g.o. metric.

b) Let us suppose that A is a g.o. metric. Interchanging V_i by $(V_i)_2$, the arguments of item *a*) work to show that

$$\mu^{(1)} = ... = \mu^{(r-1)} =: \mu,$$

$$\lambda_1^{(mn)} = \lambda_2^{(mn)} =: \lambda, \text{ for all } (m,n),$$

$$b_{mn} = \mu - \lambda =: b, \text{ for all } (m,n),$$

$$\gamma^{(i)} = \tfrac{\lambda^2 - b^2}{\lambda} =: \gamma \text{ for all } i \in \{1, ..., r-1\} \text{ with } l_i > 1.$$

For $i < j$, we have that $(V_i)_1$ and $(V_j)_1$ are $K_\Theta-$invariant, $(\cdot,\cdot)-$orthogonal and are contained in the $A-$eigenspaces corresponding to $\rho^{(i)}$ and $\rho^{(j)}$ respectively. Also

$$2(w_{\tilde{l}_{j-1}+1,\tilde{l}_{i-1}+1} - u_{\tilde{l}_{j-1}+1,\tilde{l}_{i-1}+1}) = \left[w_{\tilde{l}_{r-1}+1,\tilde{l}_{i-1}+1} - u_{\tilde{l}_{r-1}+1,\tilde{l}_{i-1}+1}, w_{\tilde{l}_{r-1}+1,\tilde{l}_{j-1}+1} - u_{\tilde{l}_{r-1}+1,\tilde{l}_{j-1}+1}\right]$$

is an non-zero element in $[(V_i)_1, (V_j)_1] \cap ((V_i)_1 \oplus (V_j)_1)^\perp$, then, by Proposition 3.0.4, $\rho^{(i)} = \rho^{(j)}$. Therefore, $\rho^{(1)} = ... = \rho^{(r-1)} =: \rho$. For $X = w_{\tilde{l}_{r-1}+1,1} - u_{\tilde{l}_{r-1}+1,1} + w_{\tilde{l}_{r-2}+1,1}$, there exists a $Z \in \mathfrak{k}_\Theta$ such that $[Z + X, AX] = 0$, but

$$AX = \rho(w_{\tilde{l}_{r-1}+1,1} - u_{\tilde{l}_{r-1}+1,1}) + \lambda w_{\tilde{l}_{r-2}+1,1} + b u_{\tilde{l}_{r-2}+1,1}$$

and

$$[X, AX] \;=\; (-\lambda + \rho + b)(w_{\tilde{l}_{r-1}+1,\tilde{l}_{r-2}+1} - u_{\tilde{l}_{r-1}+1,\tilde{l}_{r-2}+1}) \in M_{r,r-1}.$$

Since $AX \in (V_1)_1 \oplus M_{r-1,1}$ which is $\mathfrak{k}_\Theta-$invariant, we have $[Z, AX] \in (V_1)_1 \oplus M_{r-1,1}$, so, by linear independence $-\lambda + \rho + b = 0$, i.e., $b = \lambda - \rho$. Thus, A satisfies (3.2.2).

Conversely, if A satisfies (3.2.2) and we write

$$X = \sum_{i=1}^{r-1} v^{(i)} + \sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i + \sum_{1\le n<m\le r-1} X_{mn} + \sum_{1\le n<m\le r-1} Y_{mn} + \sum_{n=1}^{r-1} (X_{rn} + \tilde{X}_{rn}) + \sum_{n=1}^{r-1} (X'_{rn} - \tilde{X}'_{rn}),$$

where $v^{(i)} \in \text{span}\{v_{\tilde{l}_{i-1}+s} : 1 \le s \le l_i\}$, $Y_i \in U_i$, $X_{mn} \in W_{mn}$, $Y_{mn} \in U_{mn}$, for $1 \le n < m \le r-1$, $X_{rn}, X'_{rn} \in \text{span}\{w_{\tilde{l}_{r-1}+s,\tilde{l}_{n-1}+t} : 1 \le t \le l_n,\ 1 \le s \le l_r\}$ and $\tilde{X}_{rn}, \tilde{X}'_{rn}$ are as in the proof of item a), then

$$\begin{aligned} AX \;=\; & \mu \sum_{i=1}^{r-1} v^{(i)} + \gamma \sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i + \lambda \sum_{1\le n<m\le r-1} X_{mn} + b \sum_{1\le n<m\le r-1} \tilde{X}_{mn} + b \sum_{1\le n<m\le r-1} \tilde{Y}_{mn} \\ & + \lambda \sum_{1\le n<m\le r-1} Y_{mn} + \mu \sum_{n=1}^{r-1} (X_{rn} + \tilde{X}_{rn}) + \rho \sum_{n=1}^{r-1} (X'_{rn} - \tilde{X}'_{rn}). \end{aligned}$$

Similar to item a), we have that

$$[X, AX] = (\gamma - \mu) \left[\sum_{i=1}^{r-1} v^{(i)}, \sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i\right] + (\lambda - \gamma) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\le n<m\le r-1} X_{mn}\right]$$

$$+ b \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} \tilde{X}_{mn} \right] + b \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} \tilde{Y}_{mn} \right]$$

$$+ (\lambda-\gamma) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} Y_{mn} \right] + (\mu-\gamma) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{n=1}^{r-1} (X_{rn} + \tilde{X}_{rn}) \right]$$

$$+ (\rho-\gamma) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{n=1}^{r-1} (X'_{rn} - \tilde{X}'_{rn}) \right].$$

By taking $Z = -\frac{b}{\lambda} \sum_{\substack{i=1 \\ l_i>1}}^{r-1} \tilde{Y}_i \in \mathfrak{k}_\Theta$, we have

$$[Z+X, AX] = \left(\gamma - \mu + \frac{b\mu}{\lambda}\right) \left[\sum_{i=1}^{r-1} v^{(i)}, \sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i \right] + \left(\lambda - \gamma - \frac{b^2}{\lambda}\right) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} X_{mn} \right]$$

$$+ \left(\lambda - \gamma - \frac{b^2}{\lambda}\right) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} Y_{mn} \right]$$

$$+ \left(\mu - \gamma - \frac{b\mu}{\lambda}\right) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{n=1}^{r-1} (X_{rn} + \tilde{X}_{rn}) \right]$$

$$+ \left(\rho - \gamma + \frac{b\rho}{\lambda}\right) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{n=1}^{r-1} (X'_{rn} - \tilde{X}'_{rn}) \right]$$

$$= 0 \qquad \text{(by (3.2.2))}.$$

Thus, A is a g.o. metric. □

Remark 3.2.2. *The normal metric is obtained from conditions* (3.2.1) *and* (3.2.2) *when* $b = 0$.

3.3 Flags of C_l

We fix $(\cdot,\cdot)$ as in equation (1.3.1) and the $(\cdot,\cdot)-$orthogonal basis (1.3.2).

Proposition 3.3.1. *Let $\mathbb{F}_\Theta$ be a flag of C_l with $l \neq 4$. Then, $(\mathbb{F}_\Theta, A)$ is a g.o. space if and only if A is written in the basis* (2.3.1) *as:*

$$[A|_{M_0}]_{\mathcal{B}_0} = \begin{pmatrix} \mu_1^{(0)} & a_{21} & a_{31} & \dots & a_{\tilde{r}1} \\ a_{21} & \mu_2^{(0)} & a_{32} & \dots & a_{\tilde{r}2} \\ a_{31} & a_{32} & \mu_3^{(0)} & \dots & a_{\tilde{r}3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\tilde{r}1} & a_{\tilde{r}2} & a_{\tilde{r}3} & \dots & \mu_{\tilde{r}}^{(0)} \end{pmatrix}, \quad A|_{M_{mn}} = \mu I_{M_{mn}}, \quad A|_{U_i} = \mu I_{U_i} \ (l_i > 1)$$

where $\tilde{r} = r$ if $\alpha_l \notin \Theta$ and $\tilde{r} = r-1$ if $\alpha_l \in \Theta$, $1 \leq n < m \leq r$, $1 \leq i \leq \tilde{r}$, and

$$\begin{aligned} \mu_{n'}^{(0)} &= \tfrac{l_{n'}}{l_{m'}}\mu_{m'}^{(0)} + \left(1 - \tfrac{l_{n'}}{l_{m'}}\right)\mu \\ a_{m'n'} &= \sqrt{\tfrac{l_{m'}}{l_{n'}}}(\mu_{n'}^{(0)} - \mu) = \sqrt{\tfrac{l_{n'}}{l_{m'}}}(\mu_{m'}^{(0)} - \mu), \end{aligned} \tag{3.3.1}$$

for all $1 \leq n' < m' \leq \tilde{r}$.

Proof. Let us suppose that A is a g.o. metric. We take Θ as in equation (1.3.3) and we write A as in Proposition 2.3.2. First, we will show that $b_{mn} = 0$. In fact, given $s \in \{1, ..., l_m\}$ and $t \in \{1, ..., l_n\}$, there exists $Z \in \mathfrak{k}_\Theta$ such that

$$\left[Z + w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}, Aw_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}\right] = 0, \tag{3.3.2}$$

but,

$$\begin{aligned} \left[w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}, Aw_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}\right] &= \left[w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}, \mu_1^{(mn)} w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t} + b_{mn} u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}\right] \\ &= b_{mn}\left[w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}, u_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}\right] \\ &= 2b_{mn}(u_{\tilde{l}_{m-1}+s, \tilde{l}_{m-1}+s} - u_{\tilde{l}_{n-1}+t, \tilde{l}_{n-1}+t}) \end{aligned}$$

and $[Z, Aw_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}] \in M_{mn}$ (this is because M_{mn} is $K_\Theta-$invariant, so it is $\mathfrak{k}_\Theta-$invariant). Since M_{mn} is $(\cdot,\cdot)-$orthogonal to $2b_{mn}(u_{\tilde{l}_{m-1}+s, \tilde{l}_{m-1}+s} - u_{\tilde{l}_{n-1}+t, \tilde{l}_{n-1}+t})$, then equation (3.3.2) implies $[w_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}, Aw_{\tilde{l}_{m-1}+s, \tilde{l}_{n-1}+t}] = 0$, concluding that $b_{mn} = 0$. Next, we prove that $\mu_1^{(mn)} = \mu_2^{(mn)}$. Since $b_{mn} = 0$, we have that W_{mn} and U_{mn} are contained in the eigenspaces of A corresponding to the eigenvalues $\mu_1^{(mn)}$ and $\mu_2^{(mn)}$ respectively. Also, they are $K_\Theta-$invariant and

$$u_{\tilde{l}_{m-1}+1, \tilde{l}_{m-1}+1} - u_{\tilde{l}_{n-1}+1, \tilde{l}_{n-1}+1} = \left[w_{\tilde{l}_{m-1}+1, \tilde{l}_{n-1}+1}, u_{\tilde{l}_{m-1}+1, \tilde{l}_{n-1}+1}\right] \in [W_{mn}, U_{mn}] \cap (W_{mn} \oplus U_{mn})^\perp .$$

By Proposition 3.0.4 we have $\mu_1^{(mn)} = \mu_2^{(mn)} =: \mu^{(mn)}$. Analogously to the proof of Proposition 3.1.1, we can show that $\mu^{(mn)} = \mu^{(m'n')} =: \mu$ for all (m,n), (m',n'). To show the result for $A|_{M_0}$, we consider the vector $X = \frac{1}{\sqrt{l_n}}(u_{\tilde{l}_{n-1}+1, \tilde{l}_{n-1}+1} + ... + u_{\tilde{l}_n, \tilde{l}_n}) + w_{\tilde{l}_{m-1}+1, \tilde{l}_{n-1}+1}$, where $1 \leq n < m \leq \tilde{r}$, then

$$\begin{aligned}
[X, AX] &= \Bigg[\tfrac{1}{\sqrt{l_n}} \sum_{i=1}^{l_n} u_{\tilde{l}_{n-1}+i,\tilde{l}_{n-1}+i} + w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}, \mu_n^{(0)} \tfrac{1}{\sqrt{l_n}} \sum_{i=1}^{l_n} u_{\tilde{l}_{n-1}+i,\tilde{l}_{n-1}+i} \\
&\quad + \sum_{\substack{j=1\\ j\neq n}}^{\tilde{r}} a_{jn} \tfrac{1}{\sqrt{l_j}} \sum_{i=1}^{l_j} u_{\tilde{l}_{j-1}+i,\tilde{l}_{j-1}+i} + \mu\, w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}\Bigg] \text{ (here } a_{jn} = a_{nj} \text{ for } j < n) \\
&= \tfrac{(\mu_n^{(0)}-\mu)}{\sqrt{l_n}} \left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}, \sum_{i=1}^{l_n} u_{\tilde{l}_{n-1}+i,\tilde{l}_{n-1}+i}\right] + \tfrac{a_{mn}}{\sqrt{l_m}} \left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}, \sum_{i=1}^{l_m} u_{\tilde{l}_{m-1}+i,\tilde{l}_{m-1}+i}\right] \\
&= \tfrac{(\mu_n^{(0)}-\mu)}{\sqrt{l_n}} u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} - \tfrac{a_{mn}}{\sqrt{l_m}} u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} \\
&= \left(\tfrac{\mu_n^{(0)}-\mu}{\sqrt{l_n}} - \tfrac{a_{mn}}{\sqrt{l_m}}\right) u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} \in U_{mn}.
\end{aligned}$$

Since M_0 and W_{mn} are $K_\Theta-$invariant, they are $\mathfrak{k}_\Theta-$invariant too, so is $M_0 \oplus W_{mn}$. Also, we have $AX = \mu_n^{(0)} \frac{1}{\sqrt{l_n}} \sum_{i=1}^{l_n} u_{\tilde{l}_{n-1}+i,\tilde{l}_{n-1}+i} + \sum_{\substack{j=1\\ j\neq n}}^{r} a_{jn} \frac{1}{\sqrt{l_j}} \sum_{i=1}^{l_j} u_{\tilde{l}_{j-1}+i,\tilde{l}_{j-1}+i} + \mu\, w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} \in M_0 \oplus W_{mn}$, therefore

$$[Z, AX] \in M_0 \oplus W_{mn}.$$

By linear independence $[Z, AX] + [X, AX] = 0$ implies $[X, AX] = 0$, thus, $\frac{\mu_n^{(0)}-\mu}{\sqrt{l_n}} - \frac{a_{mn}}{\sqrt{l_m}} = 0$. By taking $X = \frac{1}{\sqrt{l_m}} \sum_{i=1}^{l_m} u_{\tilde{l}_{m-1}+i,\tilde{l}_{m-1}+i} + w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}$, where $1 \leq n < m \leq \tilde{r}$, we have

$$[X, AX] \;=\; \left(\tfrac{\mu-\mu_m^{(0)}}{\sqrt{l_m}} + \tfrac{a_{mn}}{\sqrt{l_n}}\right) u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} \in U_{mn}.$$

As before, we conclude $\frac{\mu-\mu_m^{(0)}}{\sqrt{l_m}} + \frac{a_{mn}}{\sqrt{l_n}} = 0$. Summarizing,

$$\begin{cases} \frac{\mu_n^{(0)}-\mu}{\sqrt{l_n}} - \frac{a_{mn}}{\sqrt{l_m}} = 0 \\[2ex] \frac{\mu_m^{(0)}-\mu}{\sqrt{l_m}} - \frac{a_{mn}}{\sqrt{l_n}} = 0 \end{cases}, \; 1 \leq n < m \leq \tilde{r},$$

hence, $\mu_n^{(0)} = \frac{l_n}{l_m}\mu_m^{(0)} + \left(1 - \frac{l_n}{l_m}\right)\mu$ and $a_{mn} = \sqrt{\frac{l_m}{l_n}}(\mu_n^{(0)} - \mu) = \sqrt{\frac{l_n}{l_m}}(\mu_m^{(0)} - \mu)$ for all $1 \leq n < m \leq \tilde{r}$. Now, we take i such that $l_i > 1$. If $1 \leq i \leq r-1$, we have that U_i and $W_{i+1,i}$ are contained in the eigenspaces corresponding to $\mu^{(i)}$ and μ, respectively. Since

$$-u_{\tilde{l}_i+1,\tilde{l}_{i-1}+2} = \left[u_{\tilde{l}_{i-1}+2,\tilde{l}_{i-1}+1}, w_{\tilde{l}_i+1,\tilde{l}_{i-1}+1}\right] \in [U_i, W_{i+1,i}] \cap (U_i \oplus W_{i+1,i})^\perp,$$

then, by Proposition 3.0.4 we obtain $\mu^{(i)} = \mu$. If $i = r$, then

$$u_{\tilde{l}_{r-1}+2,\tilde{l}_{r-2}+1} = \left[u_{\tilde{l}_{r-1}+2,\tilde{l}_{r-1}+1}, w_{\tilde{l}_{r-1}+1,\tilde{l}_{r-2}+1}\right] \in [U_r, W_{r,r-1}] \cap (U_r \oplus W_{r,r-1})^\perp,$$

so $\mu^{(r)} = \mu$. Conversely, let us suppose A has the form of the statement. Given $X \in \mathfrak{m}_\Theta$, we can write

$$X=\sum_{j=1}^{\tilde r}\frac{x_j}{\sqrt{l_j}}\sum_{s=1}^{l_j}u_{\tilde l_{j-1}+s,\tilde l_{j-1}+s}+\sum_{\substack{i=1\\ l_i>1}}^{\tilde r}X_i+\sum_{1\le n<m\le r}X_{mn}+\sum_{1\le n<m\le r}Y_{mn},$$

where $x_j\in\mathbb{R}$, $X_i\in U_i$, $X_{mn}\in W_{mn}$ and $Y_{mn}\in U_{mn}$. So

$$\begin{aligned}AX &= \sum_{j=1}^{\tilde r}\left(\frac{x_j}{\sqrt{l_j}}\mu_j^{(0)}\sum_{s=1}^{l_j}u_{\tilde l_{j-1}+s,\tilde l_{j-1}+s}+\sum_{\substack{t=1\\ t\neq j}}^{\tilde r}\frac{a_{tj}}{\sqrt{l_t}}\sum_{s=1}^{l_t}u_{\tilde l_{t-1}+s,\tilde l_{t-1}+s}\right)+\mu\sum_{\substack{i=1\\ l_i>1}}^{\tilde r}X_i\\ &\quad+\mu\sum_{1\le n<m\le r}X_{mn}+\mu\sum_{1\le n<m\le r}Y_{mn}.\end{aligned}$$

We have two cases:

Case 1. $\alpha_l\notin\Theta$: In this case

$$\begin{aligned}[X,AX]=&\ \mu\sum_{1\le n<m\le r}\frac{x_n}{\sqrt{l_n}}\left[\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s},X_{mn}\right]+\mu\sum_{1\le n<m\le r}\frac{x_m}{\sqrt{l_m}}\left[\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s},X_{mn}\right]\\
&+\mu\sum_{1\le n<m\le r}\frac{x_n}{\sqrt{l_n}}\left[\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s},Y_{mn}\right]+\mu\sum_{1\le n<m\le r}\frac{x_m}{\sqrt{l_m}}\left[\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s},Y_{mn}\right]\\
&+\sum_{1\le n<m\le r}\left(\frac{x_n}{\sqrt{l_n}}\mu_n^{(0)}\left[X_{mn},\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s}\right]+\frac{x_m}{\sqrt{l_m}}\mu_m^{(0)}\left[X_{mn},\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s}\right]\right)\\
&+\sum_{1\le n<m\le r}\left(\frac{x_n}{\sqrt{l_m}}a_{mn}\left[X_{mn},\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s}\right]+\frac{x_m}{\sqrt{l_n}}a_{mn}\left[X_{mn},\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s}\right]\right)\\
&+\sum_{1\le n<m\le r}\sum_{\substack{j=1\\ j\neq m,n}}^{r}\left(\frac{x_j}{\sqrt{l_m}}a_{mj}\left[X_{mn},\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s}\right]+\frac{x_j}{\sqrt{l_n}}a_{nj}\left[X_{mn},\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s}\right]\right)\\
&+\sum_{1\le n<m\le r}\left(\frac{x_n}{\sqrt{l_n}}\mu_n^{(0)}\left[Y_{mn},\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s}\right]+\frac{x_m}{\sqrt{l_m}}\mu_m^{(0)}\left[Y_{mn},\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s}\right]\right)\\
&+\sum_{1\le n<m\le r}\left(\frac{x_n}{\sqrt{l_m}}a_{mn}\left[Y_{mn},\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s}\right]+\frac{x_m}{\sqrt{l_n}}a_{mn}\left[Y_{mn},\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s}\right]\right)\\
&+\sum_{1\le n<m\le r}\sum_{\substack{j=1\\ j\neq m,n}}^{r}\left(\frac{x_j}{\sqrt{l_m}}a_{mj}\left[Y_{mn},\sum_{s=1}^{l_m}u_{\tilde l_{m-1}+s,\tilde l_{m-1}+s}\right]+\frac{x_j}{\sqrt{l_n}}a_{nj}\left[Y_{mn},\sum_{s=1}^{l_n}u_{\tilde l_{n-1}+s,\tilde l_{n-1}+s}\right]\right).\end{aligned}$$

Since $\left[\sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}, \zeta\right] = -\left[\sum_{s=1}^{l_m} u_{\tilde{l}_{m-1}+s,\tilde{l}_{m-1}+s}, \zeta\right]$, $\zeta \in \{X_{mn}, Y_{mn}\}$, $1 \leq n < m \leq r$, then

$$\begin{aligned}
[X, AX] = & \sum_{1\leq n<m\leq r} x_n \left(\frac{\mu_n^{(0)} - \mu}{\sqrt{l_n}} - \frac{a_{mn}}{\sqrt{l_m}}\right) \left[\sum_{s=1}^{l_m} u_{\tilde{l}_{m-1}+s,\tilde{l}_{m-1}+s}, X_{mn}\right] \\
& + \sum_{1\leq n<m\leq r} x_m \left(\frac{\mu_m^{(0)} - \mu}{\sqrt{l_m}} - \frac{a_{mn}}{\sqrt{l_n}}\right) \left[\sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}, X_{mn}\right] \\
& + \sum_{1\leq n<m\leq r} x_n \left(\frac{\mu_n^{(0)} - \mu}{\sqrt{l_n}} - \frac{a_{mn}}{\sqrt{l_m}}\right) \left[\sum_{s=1}^{l_m} u_{\tilde{l}_{m-1}+s,\tilde{l}_{m-1}+s}, Y_{mn}\right] \\
& + \sum_{1\leq n<m\leq r} x_m \left(\frac{\mu_m^{(0)} - \mu}{\sqrt{l_m}} - \frac{a_{mn}}{\sqrt{l_n}}\right) \left[\sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}, Y_{mn}\right] \\
& + \sum_{1\leq n<m\leq r} \sum_{\substack{j=1\\ j\neq m,n}}^{r} x_j \left(\frac{a_{nj}}{\sqrt{l_n}} - \frac{a_{mj}}{\sqrt{l_m}}\right) \left[X_{mn}, \sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}\right] \\
& + \sum_{1\leq n<m\leq r} \sum_{\substack{j=1\\ j\neq m,n}}^{r} x_j \left(\frac{a_{nj}}{\sqrt{l_n}} - \frac{a_{mj}}{\sqrt{l_m}}\right) \left[Y_{mn}, \sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}\right].
\end{aligned}$$

By (3.3.1), $\frac{a_{nj}}{\sqrt{l_n}} - \frac{a_{mj}}{\sqrt{l_m}} = \frac{\mu_m^{(0)}-\mu}{\sqrt{l_m}} - \frac{a_{mn}}{\sqrt{l_n}} = \frac{\mu_n^{(0)}-\mu}{\sqrt{l_n}} - \frac{a_{mn}}{\sqrt{l_m}} = 0$, thus $[X, AX] = 0$.

Case 2. $\alpha_l \in \Theta$: Similarly as before we obtain

$$\begin{aligned}
[X, AX] \;\; = \;\; & \sum_{n=1}^{r-1} \frac{1}{\sqrt{l_n}} \left(x_n(\mu_n^{(0)} - \mu) + \sum_{\substack{j=1\\ j\neq n}}^{r-1} a_{nj} x_j\right) \left[X_{rn}, \sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}\right] \\
& \sum_{n=1}^{r-1} \frac{1}{\sqrt{l_n}} \left(x_n(\mu_n^{(0)} - \mu) + \sum_{\substack{j=1\\ j\neq n}}^{r-1} a_{nj} x_j\right) \left[Y_{rn}, \sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}\right].
\end{aligned}$$

We consider $Z = \frac{x}{\sqrt{l_r}} \sum_{s=1}^{l_r} u_{\tilde{l}_{r-1}+s,\tilde{l}_{r-1}+s} \in \mathfrak{k}_\Theta$, for some $x \in \mathbb{R}$, then

$$[Z, AX] \;\; = \;\; \sum_{n=1}^{r-1} \frac{x}{\sqrt{l_r}} \mu \left[\sum_{s=1}^{l_r} u_{\tilde{l}_{r-1}+s,\tilde{l}_{r-1}+s}, X_{rn}\right] + \sum_{n=1}^{r-1} \frac{x}{\sqrt{l_r}} \mu \left[\sum_{s=1}^{l_r} u_{\tilde{l}_{r-1}+s,\tilde{l}_{r-1}+s}, Y_{rn}\right].$$

Since $\left[\sum\limits_{s=1}^{l_r} u_{\tilde{l}_{r-1}+s,\tilde{l}_{r-1}+s}, W\right] = -\left[\sum\limits_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}, W\right]$ for $W \in \{X_{rn}, Y_{rn}\}$ and $1 \leq n \leq r-1$, we have

$$\begin{aligned}[Z+X, AX] &= \sum_{n=1}^{r-1}\left(\frac{1}{\sqrt{l_n}}\left(x_n(\mu_n^{(0)}-\mu)+\sum_{\substack{j=1\\j\neq n}}^{r-1} a_{nj}x_j\right)+\frac{x\mu}{\sqrt{l_r}}\right)\left[X_{rn}, \sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}\right]\\ &+\sum_{n=1}^{r-1}\left(\frac{1}{\sqrt{l_n}}\left(x_n(\mu_n^{(0)}-\mu)+\sum_{\substack{j=1\\j\neq n}}^{r-1} a_{nj}x_j\right)+\frac{x\mu}{\sqrt{l_r}}\right)\left[Y_{rn}, \sum_{s=1}^{l_n} u_{\tilde{l}_{n-1}+s,\tilde{l}_{n-1}+s}\right].\end{aligned}$$

We observe that

$$\frac{1}{\sqrt{l_n}}\left(x_n(\mu_n^{(0)}-\mu)+\sum_{\substack{j=1\\j\neq n}}^{r-1} a_{nj}x_j\right)+\frac{x\mu}{\sqrt{l_r}}=0,\ n=1,...,r-1 \Longrightarrow [Z+X, AX]=0,$$

but

$$\frac{1}{\sqrt{l_n}}\left(x_n(\mu_n^{(0)}-\mu)+\sum_{\substack{j=1\\j\neq n}}^{r-1} a_{nj}x_j\right)+\frac{x\mu}{\sqrt{l_r}}=0 \Longleftrightarrow x=\frac{\sqrt{l_r}}{\mu}\left(\sum_{j=1}^{r-1}\sqrt{l_j}x_j\right)\left(\frac{\mu_n^{(0)}-\mu}{l_n}\right).$$

Thus, it is enough to show that the number $\frac{\sqrt{l_r}}{\mu}\left(\sum\limits_{j=1}^{r-1}\sqrt{l_j}x_j\right)\left(\frac{\mu_n^{(0)}-\mu}{l_n}\right)$ does not depend on n. In fact,

$$\frac{\mu_n^{(0)}-\mu}{l_n}=\frac{\mu_{n'}^{(0)}-\mu}{l_{n'}} \Longleftrightarrow \frac{\sqrt{l_{n'}}}{\sqrt{l_n}}(\mu_n^{(0)}-\mu)=\frac{\sqrt{l_n}}{\sqrt{l_{n'}}}(\mu_{n'}^{(0)}-\mu)$$

which is true by (3.3.1). □

Remark 3.3.2. *Under the coditions of the proposition above, if* $a_{mn}=0$ *for all* (m,n)*, then* A *is the normal metric.*

3.4 Flags of D_l

We consider the invariant inner product $(\cdot,\cdot)$ in (1.4.1) and the $(\cdot,\cdot)-$orthonormal basis (1.4.2).

Proposition 3.4.1. *Let* $\mathbb{F}_\Theta$ *be a flag of* D_l, $l \geq 5$ *and* A *an invariant metric as in Proposition 2.4.2.*

a) If $\alpha_l \notin \Theta$*, then* $(\mathbb{F}_\Theta, A)$ *is a g.o.space if and only if*

$$\begin{cases} \lambda_1^{(mn)} = \lambda_2^{(mn)} =: \lambda, \ \textit{for all } (m,n), \\ b_{mn} = b_{m'n'} =: b, \ \textit{for all } (m,n), (m',n'), \\ \gamma^{(i)} =: \gamma \textit{ for all } i \in \{1, ..., r\} \textit{ with } l_i > 1, \\ \gamma = \frac{\lambda^2 - b^2}{\lambda}. \end{cases} \tag{3.4.1}$$

b) If $\{\alpha_{l-1}, \alpha_l\} \in \Theta$*, then* $(\mathbb{F}_\Theta, A)$ *is a g.o. space if and only if* A *is normal.*

c) If $\alpha_l \in \Theta$ *and* $\alpha_{l-1} \notin \Theta$*, then* $(\mathbb{F}_\Theta, A)$ *is a g.o. space if and only if*

$$\begin{cases} \lambda_1^{(mn)} = \lambda_2^{(mn)} = \lambda_1^{(r-1,n)} = \lambda_2^{(r-1,n)} =: \lambda, \ \textit{for all } (m,n), \\ b_{mn} = b^{(1)}_{r-1,n} = b^{(2)}_{r-1,n} =: b, \ \textit{for all } (m,n), \\ \gamma^{(i)} =: \gamma \textit{ for all } i \in \{1, ..., r-1\} \textit{ with } l_i > 1, \\ \gamma = \frac{\lambda^2 - b^2}{\lambda} \end{cases} \tag{3.4.2}$$

Proof. *a*) Let us suppose that $(\mathbb{F}_\Theta, A)$ is a g.o. space. We take $Z \in \mathfrak{k}_\Theta$ such that $[Z + X, AX]$, where $X = w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + w_{\tilde{l}_m+1,\tilde{l}_{n-1}+1}$. Then we have

$$AX = \lambda_1^{(mn)} w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + b_{mn} u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + \lambda_1^{(m+1,n)} w_{\tilde{l}_m+1,\tilde{l}_{n-1}+1} + b_{m+1,n} u_{\tilde{l}_m+1,\tilde{l}_{n-1}+1},$$

and

$$\begin{aligned} [X, AX] &= \lambda_1^{(m+1,n)} \left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}, w_{\tilde{l}_m+1,\tilde{l}_{n-1}+1}\right] + b_{m+1,n} \left[w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}, u_{\tilde{l}_m+1,\tilde{l}_{n-1}+1}\right] \\ &\quad + \lambda_1^{(mn)} \left[w_{\tilde{l}_m+1,\tilde{l}_{n-1}+1}, w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}\right] + b_{mn} \left[w_{\tilde{l}_m+1,\tilde{l}_{n-1}+1}, u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1}\right] \\ &= (\lambda_1^{(m+1,n)} - \lambda_1^{(mn)}) w_{\tilde{l}_m+1,\tilde{l}_{m-1}+1} + (b_{m+1,n} - b_{mn}) u_{\tilde{l}_m+1,\tilde{l}_{m-1}+1} \in M_{m+1,m}. \end{aligned}$$

Since $AX \in M_{mn} \oplus M_{m+1,n}$, we have that $[Z, AX] \in M_{mn} \oplus M_{m+1,n}$, thus,

$$[Z + X, AX] = 0 \Longrightarrow [X, AX] = 0 = [Z, AX],$$

in particular, $\lambda_1^{(m+1,n)} = \lambda_1^{(mn)}$ and $b_{m+1,n} = b_{mn}$. We can use the same argument for $X = u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + u_{\tilde{l}_m+1,\tilde{l}_{n-1}+1}$, $w_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + w_{\tilde{l}_{m-1}+1,\tilde{l}_n+1}$ and $u_{\tilde{l}_{m-1}+1,\tilde{l}_{n-1}+1} + u_{\tilde{l}_{m-1}+1,\tilde{l}_n+1}$ to conclude that

$$\lambda_2^{(m+1,n)} = \lambda_2^{(mn)},\ \lambda_1^{(m,n+1)} = \lambda_1^{(mn)},\ \lambda_2^{(m,n+1)} = \lambda_1^{(mn)} \text{ and } b_{m,n+1} = b_{mn}.$$

Then,

$$\begin{gathered} \lambda_1^{(mn)} = \lambda_1^{(m'n')} =: \lambda_1 \\ \lambda_2^{(mn)} = \lambda_2^{(m'n')} =: \lambda_2 \\ b_{mn} = b_{m'n'} =: b \end{gathered}$$

for all m, n, m', n'. Next, we will show that $\gamma^{(i)} = \frac{\lambda_1^2 - b^2}{\lambda_1} = \frac{\lambda_2^2 - b^2}{\lambda_2}$ for all $i \in \{1, ..., r\}$ with $l_i > 1$. In fact, if $i \leq r-1$ we take $X = u_{\tilde{l}_{i-1}+2, \tilde{l}_{i-1}+1} + w_{\tilde{l}_i+1, \tilde{l}_{i-1}+1}$ and $Z \in \mathfrak{k}_\Theta$ such that $[Z + X, AX] = 0$. In this case, we can write

$$Z = \sum_{l_j > 1} \left(\sum_{1 \leq t < s \leq l_j} z_{st}^{(j)} w_{\tilde{l}_{j-1}+s, \tilde{l}_{j-1}+t} \right), \; z_{st}^{(j)} \in \mathbb{R},$$

therefore

$$\begin{aligned} [Z, AX] &= \left[Z, \gamma^{(i)} u_{\tilde{l}_{i-1}+2, \tilde{l}_{i-1}+1} + \lambda_1 w_{\tilde{l}_i+1, \tilde{l}_{i-1}+1} + b u_{\tilde{l}_i+1, \tilde{l}_{i-1}+1}\right] \\ &= z_{21}^{(i)} b \, u_{\tilde{l}_i+1, \tilde{l}_{i-1}+2} + z_{21}^{(i)} \lambda_1 w_{\tilde{l}_i+1, \tilde{l}_{i-1}+2} + Z', \end{aligned}$$

where $\{u_{\tilde{l}_i+1, \tilde{l}_{i-1}+2}, w_{\tilde{l}_i+1, \tilde{l}_{i-1}+2}, Z'\}$ is linear independent. On the other hand,

$$\begin{aligned} [X, AX] &= \left[X, \gamma^{(i)} u_{\tilde{l}_{i-1}+2, \tilde{l}_{i-1}+1} + \lambda_1 w_{\tilde{l}_i+1, \tilde{l}_{i-1}+1} + b u_{\tilde{l}_i+1, \tilde{l}_{i-1}+1}\right] \\ &= (\lambda_1 - \gamma^{(i)}) u_{\tilde{l}_i+1, \tilde{l}_{i-1}+2} + b \, w_{\tilde{l}_i+1, \tilde{l}_{i-1}+2}. \end{aligned}$$

Thus, $[Z + X, AX] = 0 \Longrightarrow Z' = 0, \; \lambda_1 - \gamma^{(i)} + z_{21}^{(i)} b = b + z_{21}^{(i)} \lambda_1 = 0$, so $\gamma^{(i)} = \frac{\lambda_1^2 - b^2}{\lambda_1}$. If $i = r$ we take $X = u_{\tilde{l}_{r-1}+2, \tilde{l}_{r-1}+1} + w_{\tilde{l}_{r-1}+1, \tilde{l}_{r-2}+1}$ and proceeding as before we obtain $\gamma^{(r)} = \frac{\lambda_1^2 - b^2}{\lambda_1}$. To show that $\gamma^{(i)} = \frac{\lambda_2^2 - b^2}{\lambda_2}$ we can use the same argument but taking $X = u_{\tilde{l}_{i-1}+2, \tilde{l}_{i-1}+1} + u_{\tilde{l}_i+1, \tilde{l}_{i-1}+1}$ instead of $u_{\tilde{l}_{i-1}+2, \tilde{l}_{i-1}+1} + w_{\tilde{l}_i+1, \tilde{l}_{i-1}+1}$ (when $i \leq r-1$) and $X = u_{\tilde{l}_{r-1}+2, \tilde{l}_{r-1}+1} + u_{\tilde{l}_{r-1}+1, \tilde{l}_{r-2}+1}$ instead of $u_{\tilde{l}_{r-1}+2, \tilde{l}_{r-1}+1} + w_{\tilde{l}_{r-1}+1, \tilde{l}_{r-2}+1}$ (when $i = r$). Summarizing,

$$\gamma^{(i)} = \frac{\lambda_1^2 - b^2}{\lambda_1} = \frac{\lambda_2^2 - b^2}{\lambda_2}, \text{ for all } i \in \{1, ..., r\} \text{ with } l_i > 1.$$

We observe that $\frac{\lambda_1^2 - b^2}{\lambda_1} = \frac{\lambda_2^2 - b^2}{\lambda_2} \Longleftrightarrow \lambda_1 \lambda_2 (\lambda_1 - \lambda_2) = -b^2 (\lambda_1 - \lambda_2)$, therefore

$$\lambda_1 \neq \lambda_2 \Longrightarrow 0 < \lambda_1 \lambda_2 = -b^2 \leq 0,$$

which is absurd. Thus, $\lambda_1 = \lambda_2$. We point that when $\Theta = \emptyset$, the previous argument does not work to show $\lambda_1 = \lambda_2 =: \lambda$ (because there is not i with $l_i > 1$). In that case, we have $[X, AX] = 0$ for all $X \in \mathfrak{m}_\Theta$ (because $\mathfrak{k}_\Theta = \{0\}$), in particular, when $X = w_{21} + u_{31}$, $[X, AX] = (\lambda_2 - \lambda_1) u_{32}$, so $\lambda_1 = \lambda_2$. Now, we suppose A satisfies (3.4.1). Let

$$X = \sum_{\substack{i=1 \\ l_i > 1}}^{r} Y_i + \sum_{1 \leq n < m \leq r} X_{mn} + \sum_{1 \leq n < m \leq r} Y_{mn}$$

be a vector in $\mathfrak{m}_\Theta$, where $Y_i \in U_i$, $X_{mn} \in W_{mn}$ and $Y_{mn} \in U_{mn}$. If

$$X_{mn} = \begin{pmatrix} C_{mn} & \mathbf{0} \\ \mathbf{0} & C_{mn} \end{pmatrix}, \; Y_{mn} = \begin{pmatrix} \mathbf{0} & D_{mn} \\ D_{mn} & \mathbf{0} \end{pmatrix} \text{ and } \; Y_i = \begin{pmatrix} \mathbf{0} & D_i \\ D_i & \mathbf{0} \end{pmatrix}$$

we set

$$\tilde{X}_{mn} = \begin{pmatrix} \mathbf{0} & C_{mn} \\ C_{mn} & \mathbf{0} \end{pmatrix}, \; \tilde{Y}_{mn} = \begin{pmatrix} D_{mn} & \mathbf{0} \\ \mathbf{0} & D_{mn} \end{pmatrix} \text{ and } \; \tilde{Y}_i = \begin{pmatrix} D_i & \mathbf{0} \\ \mathbf{0} & D_i \end{pmatrix}.$$

With this notation we have $AX_{mn} = \lambda X_{mn} + b\tilde{X}_{mn}$, $AX_{mn} = b\tilde{Y}_{mn} + \lambda Y_{mn}$,

$$\left[\sum_{1\le n<m\le r} X_{mn}, \sum_{1\le n<m\le r} \tilde{X}_{mn}\right] = 0 = \left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{1\le n<m\le r} \tilde{Y}_{mn}\right] \text{ and}$$

$$\left[\sum_{1\le n<m\le r} X_{mn}, \sum_{1\le n<m\le r} \tilde{Y}_{mn}\right] = -\left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{1\le n<m\le r} \tilde{X}_{mn}\right].$$

Therefore

$$AX = \gamma\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i + \lambda\sum_{1\le n<m\le r} X_{mn} + b\sum_{1\le n<m\le r} \tilde{X}_{mn} + b\sum_{1\le n<m\le r} \tilde{Y}_{mn} + \lambda\sum_{1\le n<m\le r} Y_{mn}$$

and

$$\begin{aligned}
[X, AX] &= \lambda\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} X_{mn}\right] + b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{X}_{mn}\right] + b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{Y}_{mn}\right]\\
&\quad +\lambda\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} Y_{mn}\right] + \gamma\left[\sum_{1\le n<m\le r} X_{mn}, \sum_{\substack{i=1\\ l_i>1}}^{r} Y_i\right] + \gamma\left[\sum_{1\le n<m\le r} Y_{mn}, \sum_{\substack{i=1\\ l_i>1}}^{r} Y_i\right]\\
&= (\lambda-\gamma)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} X_{mn}\right] + (\lambda-\gamma)\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} Y_{mn}\right]\\
&\quad +b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{X}_{mn}\right] + b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{Y}_{mn}\right].
\end{aligned}$$

Let $Z = -\frac{b}{\lambda}\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i \in \mathfrak{k}_\Theta$, then

$$\begin{aligned}
[Z, AX] &= -b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\le n<m\le r} X_{mn}\right] - b\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\le n<m\le r} Y_{mn}\right]\\
&\quad -\frac{b^2}{\lambda}\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\le n<m\le r} \tilde{X}_{mn}\right] - \frac{b^2}{\lambda}\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\le n<m\le r} \tilde{Y}_{mn}\right].
\end{aligned}$$

Since

$$\left[\sum_{\substack{i=1\\ l_i>1}}^{r} \tilde{Y}_i, \sum_{1\le n<m\le r} X_{mn}\right] = \left[\sum_{\substack{i=1\\ l_i>1}}^{r} Y_i, \sum_{1\le n<m\le r} \tilde{X}_{mn}\right],$$

$$\left[\sum_{\substack{i=1\\l_i>1}}^{r}\tilde{Y}_i, \sum_{1\leq n<m\leq r} Y_{mn}\right] = \left[\sum_{\substack{i=1\\l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r}\tilde{Y}_{mn}\right],$$

$$\left[\sum_{\substack{i=1\\l_i>1}}^{r}\tilde{Y}_i, \sum_{1\leq n<m\leq r} \tilde{X}_{mn}\right] = \left[\sum_{\substack{i=1\\l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} X_{mn}\right],$$

$$\left[\sum_{\substack{i=1\\l_i>1}}^{r}\tilde{Y}_i, \sum_{1\leq n<m\leq r} \tilde{Y}_{mn}\right] = \left[\sum_{\substack{i=1\\l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} Y_{mn}\right],$$

then

$$\begin{aligned}[Z+X, AX] &= \left(\lambda-\gamma-\frac{b^2}{\lambda}\right)\left[\sum_{\substack{i=1\\l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} X_{mn}\right] + \left(\lambda-\gamma-\frac{b^2}{\lambda}\right)\left[\sum_{\substack{i=1\\l_i>1}}^{r} Y_i, \sum_{1\leq n<m\leq r} Y_{mn}\right]\\ &= 0.\end{aligned}$$

Thus, A is a g.o. metric.

b) Let A be a g.o. metric on $\mathbb{F}_\Theta$. Analogously to item *a*) we have that

$$\lambda_1^{(mn)} = \lambda_1^{(m'n')} =: \lambda_1$$
$$\lambda_2^{(mn)} = \lambda_2^{(m'n')} =: \lambda_2$$
$$b_{mn} = b_{m'n'} =: b$$

for $1 \leq n < m \leq r-1$ and $1 \leq n' < m' \leq r-1$. Given $n < n' \leq r-1$, the subspaces M_{rn} and $M_{rn'}$ are $K_\Theta-$invariant, $(\cdot,\cdot)-$orthogonal and are contained in the eigenspaces of A corresponding to the eigenvalues $\lambda^{(rn)}$ and $\lambda^{(rn')}$ respectively. Also,

$$w_{\tilde{l}_{n'-1}+1,\tilde{l}_{n-1}+1} = \left[w_{\tilde{l}_{r-1}+1,\tilde{l}_{n-1}+1}, w_{\tilde{l}_{r-1}+1,\tilde{l}_{n'-1}+1}\right] \in [M_{rn}, M_{rn'}] \cap M_{n'n},$$

and $M_{n'n} \subseteq (M_{rn}\oplus M_{rn'})^\perp$, therefore, by Proposition 3.0.4, $\lambda^{(rn)} = \lambda^{(rn')} =: \lambda^{(r)}$. Next, we shall show that $b = 0$ and $\lambda_1 = \lambda_2 = \lambda^{(r)}$. In fact, if $X = w_{\tilde{l}_{r-1}+1,1} + w_{\tilde{l}_{r-2}+1,1}$, then

$$\begin{aligned}[X, AX] &= \left[w_{\tilde{l}_{r-1}+1,1} + w_{\tilde{l}_{r-2}+1,1}, \lambda^{(r)} w_{\tilde{l}_{r-1}+1,1} + \lambda_1 w_{\tilde{l}_{r-2}+1,1} + b\, u_{\tilde{l}_{r-2}+1,1}\right]\\ &= (\lambda^{(r)}-\lambda_1) w_{\tilde{l}_{r-1}+1,\tilde{l}_{r-2}+1} - b\, u_{\tilde{l}_{r-1}+1,\tilde{l}_{r-2}+1} \in M_{r,r-1}.\end{aligned}$$

Let $Z \in \mathfrak{k}_\Theta$ such that $[Z+X, AX] = 0$, since $AX \in M_{r1}\oplus M_{r-1,1}$ and $[X, AX] \in M_{r,r-1}$, then $[Z, AX] = 0 = [X, AX]$, so $\lambda^{(r)} = \lambda_1$ and $b = 0$. By taking $X = w_{\tilde{l}_{r-1}+1,1} + u_{\tilde{l}_{r-2}+1,1}$ instead of $w_{\tilde{l}_{r-1}+1,1} + w_{\tilde{l}_{r-2}+1,1}$ and proceeding analogously, we obtain $\lambda^{(r)} = \lambda_2$. We define $\lambda := \lambda_1 = \lambda_2 = \lambda^{(r)}$. The same arguments of item *a*) allow us to show that for each $i \in \{1, ..., r-1\}$ with $l_i > 1$, $\gamma^{(i)} = \frac{\lambda^2-b^2}{\lambda} = \frac{\lambda^2}{\lambda} = \lambda$. Therefore $A = \lambda\mathrm{I}$, i.e., A is normal.

c) Analogously to item *a*), we have that

$$\begin{array}{l} \lambda_1^{(mn)} =: \lambda_1 \\ \lambda_2^{(mn)} =: \lambda_2 \ , \quad 1 \leq n < m \leq r-2 \\ b_{mn} =: b \end{array}$$

and $\gamma^{(i)} = \frac{\lambda_1^2 - b^2}{\lambda_1} = \frac{\lambda_2^2 - b^2}{\lambda_2}$ for every $i \in \{1, ..., r-2\}$ with $l_i > 1$, in particular, $\lambda_1 = \lambda_2 =: \lambda$. By taking $X = w_{\tilde{l}_{r-2}+1, \tilde{l}_{n-1}+1} + w_{\tilde{l}_{r-2}+1, \tilde{l}_n+1}$, we have

$$AX = \lambda_1^{(r-1,n)} w_{\tilde{l}_{r-2}+1, \tilde{l}_{n-1}+1} + b^{(1)}_{r-1,n} u_{\tilde{l}_{r-2}+1, \tilde{l}_{n-1}+1} + \lambda_1^{(r-1,n+1)} w_{\tilde{l}_{r-2}+1, \tilde{l}_n+1} + b^{(1)}_{r-1,n} u_{\tilde{l}_{r-2}+1, \tilde{l}_n+1},$$

thus

$$[X, AX] = (\lambda_1^{(r-1,n+1)} - \lambda_1^{(r-1,n)}) w_{\tilde{l}_n+1, \tilde{l}_{n-1}+1} + (b^{(1)}_{r-1,n+1} - b^{(1)}_{r-1,n}) u_{\tilde{l}_n+1, \tilde{l}_{n-1}+1} \in M_{n+1,n}.$$

There exists $Z \in \mathfrak{k}_\Theta$ such that $[Z + X, AX] = 0$, but $AX \in S_n \oplus S_{n+1} \Longrightarrow [Z, AX] \in S_n \oplus S_{n+1}$, then, $[X, AX] = 0 \Longrightarrow \lambda_1^{(r-1,n+1)} = \lambda_1^{(r-1,n)}$ and $b^{(1)}_{r-1,n+1} = b^{(1)}_{r-1,n}$. By using the same argument for $X = u_{\tilde{l}_{r-2}+1, \tilde{l}_{n-1}+1} + u_{\tilde{l}_{r-2}+1, \tilde{l}_n+1}$ and $w_{l, \tilde{l}_{n-1}+1} + w_{l, \tilde{l}_n+1}$ we obtain that $\lambda_2^{(r-1,n+1)} = \lambda_2^{(r-1,n)}$ and $b^{(2)}_{r-1,n+1} = b^{(2)}_{r-1,n}$, respectively. Thus

$$\left\{ \begin{array}{l} \lambda_i^{(r-1,1)} = \cdots = \lambda_i^{(r-1,r-2)} =: \lambda_i^{(r-1)} \\ \\ b^{(i)}_{r-1,1} = \cdots = b^{(i)}_{r-1,r-2} =: b^{(i)} \end{array} \right. , \quad i = 1, 2.$$

For $X = w_{\tilde{l}_{r-3}+1,1} + w_{\tilde{l}_{r-2}+1,1}$, $w_{\tilde{l}_{r-3}+1,1} + w_{l1}$ and $u_{\tilde{l}_{r-3}+1,1} + w_{l1}$, we have $\lambda_1^{(r-1)} = \lambda_1$, $b^{(1)} = b^{(2)} = b$ and $\lambda_2^{(r-1)} = \lambda_2$. Now we will show that $\gamma^{(r-1)} = \frac{\lambda_1^2 - b^2}{\lambda_1} = \frac{\lambda_2^2 - b^2}{\lambda_2}$, in fact, if $X = w_{l1} + w_{l, \tilde{l}_{r-2}+1}$, then

$$AX = b\ u_{l1} + \lambda_2 w_{l1} + \gamma^{(r-1)} w_{l, \tilde{l}_{r-2}+1}$$

and

$$[X, AX] = (\gamma^{(r-1)} - \lambda_2) w_{\tilde{l}_{r-2}+1,1} - b\ u_{\tilde{l}_{r-2}+1,1}.$$

Let $Z \in \mathfrak{k}_\Theta$ such that $[Z + X, AX] = 0$. In this case we write Z as

$$Z = \sum_{\substack{j \leq r-1 \\ l_j > 1}} \left(\sum_{1 \leq t < s \leq l_j} z^{(j)}_{st} w_{\tilde{l}_{j-1}+s, \tilde{l}_{j-1}+t} \right) + \sum_{t=1}^{l_{r-1}} z_t u_{l, \tilde{l}_{r-2}+t}, \quad z^{(j)}_{st}, z_t \in \mathbb{R},$$

therefore

$$[Z, AX] = -z_1 b\ w_{\tilde{l}_{r-2}+1,1} - z_1 \lambda_2 u_{\tilde{l}_{r-2}+1,1} + Z',$$

where $\{w_{\tilde{l}_{r-2}+1,1}, u_{\tilde{l}_{r-2}+1,1}, Z'\}$ is linear independent. Since $[Z + X, AX] = 0$, then

$$\left\{ \begin{array}{l} \lambda_2 - \gamma^{(r-1)} + z_1 b = 0 \\ \\ b + z_1 \lambda_2 = 0, \end{array} \right.$$

where we have $\gamma^{(r-1)} = \frac{\lambda_2^2 - b^2}{\lambda_2}$. When $X = u_{l1} + w_{l, \tilde{l}_{r-2}+1}$, we have $\gamma^{(r-1)} = \frac{\lambda_1^2 - b^2}{\lambda_1}$. Conversely, if A satisfies (3.4.2), every $X \in \mathfrak{m}_\Theta$ can be written as

$$X = \sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i + \sum_{1\leq n<m\leq r-1} X_{mn} + \sum_{1\leq n<m\leq r-1} Y_{mn},$$

where $Y_i \in U_i$ if $1 \leq i \leq r-2$, $Y_{r-1} \in V_{r-1}$, $X_{mn} \in W_{mn}$, $Y_{mn} \in U_{mn}$ if $m \leq r-2$, $X_{r-1,n} \in M_n$ and $Y_{r-1,n} \in N_n$. For $i, m \leq r-2$ we consider $\tilde{X}_{mn}$, $\tilde{Y}_{mn}$ and $\tilde{Y}_i$ as in item b), if

$$X_{r-1,n} = \begin{pmatrix} A_{r-1,n} & B_{r-1,n} \\ B_{r-1,n} & A_{r-1,n} \end{pmatrix}, \; Y_{r-1,n} = \begin{pmatrix} C_{r-1,n} & D_{r-1,n} \\ D_{r-1,n} & C_{r-1,n} \end{pmatrix} \text{ and } \; Y_{r-1} = \begin{pmatrix} C_{r-1} & D_{r-1} \\ D_{r-1} & C_{r-1} \end{pmatrix}$$

we set

$$\tilde{X}_{r-1,n} = \begin{pmatrix} B_{r-1,n} & A_{r-1,n} \\ A_{r-1,n} & B_{r-1,n} \end{pmatrix}, \; \tilde{Y}_{r-1,n} = \begin{pmatrix} D_{r-1,n} & C_{r-1,n} \\ C_{r-1,n} & D_{r-1,n} \end{pmatrix} \text{ and } \; \tilde{Y}_{r-1} = \begin{pmatrix} D_{r-1} & C_{r-1} \\ C_{r-1} & D_{r-1} \end{pmatrix}.$$

Thus,

$$AX = \gamma \sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i + \lambda \sum_{1\leq n<m\leq r-1} X_{mn} + b \sum_{1\leq n<m\leq r-1} \tilde{X}_{mn} + b \sum_{1\leq n<m\leq r-1} \tilde{Y}_{mn} + \lambda \sum_{1\leq n<m\leq r-1} Y_{mn},$$

we can proceed exactly as in the proof of item a) to conclude that $Z = -\frac{b}{\lambda} \sum_{\substack{i=1 \\ l_i>1}}^{r-1} \tilde{Y}_i$ implies

$$\begin{aligned}[Z+X, AX] &= \left(\lambda - \gamma - \frac{b^2}{\lambda}\right) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} X_{mn}\right] + \left(\lambda - \gamma - \frac{b^2}{\lambda}\right) \left[\sum_{\substack{i=1 \\ l_i>1}}^{r-1} Y_i, \sum_{1\leq n<m\leq r-1} Y_{mn}\right] \\ &= 0.\end{aligned}$$

Hence A is a g.o. metric. □

Remark 3.4.2. *If $b = 0$, conditions* (3.4.1) *and* (3.4.2) *give us the normal metric.*

Chapter 4

Invariant Einstein metrics on real flag manifolds with two or three isotropy summands

A Riemannian metric g on a differentiable manifold M is called an *Einstein metric* if its associated Ricci tensor satisfies the condition $Ric = cg$ for some constant c. When M is a compact homogeneous space, the problem of finding invariant Einstein metrics reduces to solve a system of polynomial equations. This problem has been widely studied for the case where M is a complex generalized flag manifold (e.g. [6], [5] and [26]), in this case, the isotropy representation is multiplicity free and M admits only invariant metrics of diagonal type. In this chapter, we study the Einstein equation for generalized real flag manifolds associated to a split real form $\mathfrak{g}$ of a complex simple Lie algebra of classical type whose isotropy representation decomposes into two or three irreducible summands. As we have seen, we have non-diagonal invariant metrics and, moreover, we can find non-diagonal Einstein metrics (see sections 4.2.1 and 4.2.6). The following table contains the information obtained in this chapter about the number of invariant Einstein metrics.

$\mathbb{F}_\Theta$	# Isotropy summands	Equivalent summands?	# Einstein metrics	Normal
$\frac{SO(4)}{S(O(2)\times O(2))}$	2	—	1	✓
$\frac{SO(l)\times SO(l+1)}{SO(l-1)\times SO(l)}$, $l\geq 3$	2	—	1	—
$\frac{SO(l)\times SO(l+1)}{SO(l)}$, $l\geq 3$, $l\neq 4$	2	—	2	—
$\frac{U(l)}{O(l)}$, $l\geq 3$	2	—	0	—
$\frac{U(l)}{O(1)\times U(l-1)}$, $l\geq 3$	2	—	1	—
$\frac{SO(4)\times SO(4)}{SO(4)}$	2	—	1	✓
$\frac{SO(4)}{S(O(2)\times O(1)\times O(1))}$	3	✓	5	—
$\frac{SO(l+1)}{S(O(l_1)\times O(l_2)\times O(l_3))}$, $l\geq 2$, $l\neq 3$, $l_1+l_2+l_3=l+1$	3	—	≤ 4	
$\frac{SO(4)\times SO(5)}{SO(4)}$	3	—	2	✓
$\frac{SO(l)\times SO(l+1)}{SO(d)\times SO(l-d)\times SO(l-d+1)}$, $l\geq 3$, $2\leq d\leq l-1$	3	—	≤ 4	
$\frac{U(l)}{O(d)\times U(l-d)}$, $l\geq 3$, $2\leq d\leq l-1$	3	—	≤ 2	
$\frac{SO(l)\times SO(l)}{S(O(l-1)\times O(1))}$, $l\geq 4$	3	✓	6 (5 when l=4)	only for l=4

Remark 4.0.1. *In [10], Böhm and Kerr proved that every compact simply connected homogeneous space up to dimension 11 admits at least one invariant Einstein metric. For* $l = 3$ *and*

$l = 4$, the manifold $U(l)/O(l)$ has dimension 6 and 10 respectively, this is not a contradiction since these manifolds are not simply connected (see [31]).

Proposition 4.0.2. *Suppose that $\mathbb{F}_\Theta$ is a real flag manifold of classical type whose isotropy representation decomposes into two or three irreducible submodules. Then $\mathbb{F}_\Theta$ is one of the manifolds in the table above.*

Proof. Assume that $\mathbb{F}_\Theta$ is a flag of A_l, $l \neq 3$ and let $l_1, ..., l_r$ be positive integers such that $l_1 + ... + l_r = l + 1$ and

$$\Theta = \bigcup_{l_i > 1} \left\{ \alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i - 1} \right\},$$

where $\tilde{l}_0 = 0$, $\tilde{l}_i = \tilde{l}_{i-1} + l_i$, $i = 1, ..., r$. By Proposition 1.1.1, the isotropy representation of $\mathbb{F}_\Theta$ decomposes into the $r(r-1)/2$ irreducible submodules M_{mn}, $1 \leq n \leq r$ defined in (1.1.2). Therefore, it has two or three isotropy summands if and only if $r(r-1)/2 = 2$ or $r(r-1)/2 = 3$ respectively. Evidently, there exists no positive integer r satisfying $r(r-1)/2 = 2$ and for $r = 3$ we have that $r(r-1)/2 = 3(3-1)/2 = 3$, in which case, $K_\Theta = S(O(l_1) \times O(l_2) \times O(l_3))$ and

$$\mathbb{F}_\Theta = K/K_\Theta = SO(l+1)/S(O(l_1) \times O(l_2) \times O(l_3)).$$

Now suppose that $\mathbb{F}_\Theta$ is a flag of B_l, $l \geq 5$. Take $l_1, ..., l_r$ such that

$$\Theta = \bigcup_{l_i > 1} \{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i - 1}\} \text{ or } \bigcup_{l_i > 1} \{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i - 1}\} \cup \{\alpha_l\}.$$

where $l = l_1 + ... + l_r$ and $\tilde{l}_0 = 0$, $\tilde{l}_i = \tilde{l}_{i-1} + l_i$, $i = 1, ..., r$. If $\alpha_l \notin \Theta$, the irreducible $K_\Theta-$invariant subspaces of $\mathfrak{m}_\Theta$ are given by

$$V_i, \ i = 1, ..., r, \ W_{mn}, \ U_{mn}, \ 1 \leq n < m \leq r, \ U_j, \ l_j > 1, \ 1 \leq j \leq r$$

where V_i W_{mn}, U_{mn} and U_j are defined in Proposition 1.2.1. Hence, we have $r + r(r-1) + h$ isotropy summands, where h is the number of indices j such that $l_j > 1$. If $r \geq 3$ then $r + r(r-1) + h \geq 3 + 6 + h \geq 9$. If $r = 2$, then $r + r(r-1) + h \geq 2 + 2 + h \geq 4$. If $r = 1$, then $h = 1$ and $r + r(r-1) + h = 2$, in this case $\Theta = \{\alpha_1, ..., \alpha_{l-1}\}$ and $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} (SO(l) \times SO(l+1))/SO(l)$.

When $\alpha_l \in \Theta$, the isotropy representation splits into the irreducible submodules

$$(V_i)_1, \ (V_i)_2, \ i = 1, ..., r-1, \ W_{mn}, \ U_{mn}, \ 1 \leq n < m \leq r-1, \ U_j, \ l_j > 1, \ 1 \leq j \leq r-1,$$

so we have $2(r-1) + (r-1)(r-2) + h = (r-1)r + h$ summands. Assume that $r \geq 3$. Then $(r-1)r + h \geq 6 + h \geq 6$. When $r = 1$ we obtain the degenerated case $\Theta = \Sigma$ and when $r = 2$ we have $(r-1)r + h = 2 + h$. If $h = 0$ we have that $l_1 = 1$, $\Theta = \{\alpha_2, ..., \alpha_l\}$ and $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} (SO(l) \times SO(l+1))/(SO(l-1) \times SO(l))$. If $h = 1$ then $l_1 > 1$, $\Theta = \Sigma - \{\alpha_{l_1}\}$ and $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} (SO(l) \times SO(l+1))/(SO(l_1) \times SO(l_2) \times SO(l_2+1))$, or, equivalently, $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} (SO(l) \times SO(l+1))/(SO(d) \times SO(l-d) \times SO(l-d+1))$, where $d = l_1$.

We can proceed analogously to obtain $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} U(l)/O(l)$, $U(l)/(O(1) \times U(l-1))$ or $U(l)/(O(d) \times U(l-d))$ for a flag of C_l, $l \geq 5$ and $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} (SO(l) \times SO(l))/S(O(l-1) \times O(1))$ for a flag of D_l, $l \geq 5$. The cases A_3, B_2, B_3, B_4, C_4, and D_4 are treated case by case depending on the new invariant subspaces they have and the equivalences between them. $\square$

For a $G-$invariant metric g on a homogeneous space G/H with reductive decompsition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, define $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by the formula

$$2g(U(X,Y),W) = g([W,X]_{\mathfrak{m}},Y) + g([W,Y]_{\mathfrak{m}},X) \tag{4.0.1}$$

for all $W \in \mathfrak{m}$. We may apply Corollary 7.38 of [9] and obtain an explicit formula for the Ricci tensor:

$$\begin{aligned} \mathrm{Ric}(X,Y) &= -\frac{1}{2}\sum_i g([X,X_i]_{\mathfrak{m}},[Y,X_i]_{\mathfrak{m}}) - \frac{1}{2}\langle X,Y\rangle \\ &\quad +\frac{1}{4}\sum_{i,j} g([X_i,X_j]_{\mathfrak{m}},X)g([X_i,X_j]_{\mathfrak{m}},Y) - g(U(X,Y),Z) \end{aligned} \tag{4.0.2}$$

where $\{X_i\}$ is an $g-$orthonormal basis of $\mathfrak{m}$, $\langle\cdot,\cdot\rangle$ is the Killing form of $\mathfrak{g}$, $Z = \sum_i U(X_i,X_i)$ and $X,\ Y \in \mathfrak{m}$.

4.1 Flags with two isotropy summands

In this section we find all invariant Einstein metrics for real flag manifolds with two isotropy summands. The complex case was studied by Arvanitoyeorgos and Chrysikos in [7].

4.1.1 $SO(4)/S(O(2) \times O(2))$

Let $\mathfrak{g} = A_3$ and $\Theta = \{\alpha_1, \alpha_3\}$, then, the associated flag manifold $\mathbb{F}_{\{\alpha_1,\alpha_3\}}$ is diffeomorphic to $SO(4)/S(O(2)\times O(2))$. As before, we fix the invariant inner product $(\cdot,\cdot)$ considered in Section 2.1 and the $(\cdot,\cdot)-$orthogonal basis $\{w_{ij} = E_{ij} - E_{ji} : 1 \leq j < i \leq 4\}$ of $\mathfrak{k} = \mathfrak{so}(4)$. The Lie algebra of $K_{\{\alpha_1,\alpha_3\}} = S(O(2)\times O(1)\times O(1))$ is $\mathfrak{k}_{\{\alpha_1,\alpha_3\}} = \mathrm{span}\{w_{21}, w_{43}\}$ and its $(\cdot,\cdot)-$orthogonal complement $\mathfrak{m}_{\{\alpha_1,\alpha_3\}} = \mathrm{span}\{w_{31}, w_{32}, w_{41}, w_{42}\}$ decomposes into the two $K_{\{\alpha_1,\alpha_3\}}-$invariant, irreducible and non-equivalent subspaces

$$M_1 = \mathrm{span}\{w_{31} - w_{42}, w_{41} + w_{32}\} \text{ and } M_2 = \mathrm{span}\{w_{31} + w_{42}, w_{41} - w_{32}\}.$$

According to Proposition 2.1.2, every invariant metric on A can be written in the basis $\mathcal{B}_{\{\alpha_1,\alpha_3\}}$ of (2.1.3) as

$$[A]_{\mathcal{B}_{\{\alpha_1,\alpha_3\}}} = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}, \quad \mu_1, \mu_2 > 0. \tag{4.1.1}$$

Proposition 4.1.1. *Let A be an invariant metric on $SO(4)/S(O(2) \times O(2))$ written as above. Then A is an Einstein metric if and only if A is normal, i.e., $\mu_1 = \mu_2$.*

Proof. The vectors

$$X_1 = \frac{w_{31} - w_{42}}{\sqrt{8\mu_1}},\ X_2 = \frac{w_{41} + w_{32}}{\sqrt{8\mu_1}},\ Y_1 = \frac{w_{31} + w_{42}}{\sqrt{8\mu_2}},\ Y_2 = \frac{w_{41} - w_{32}}{\sqrt{8\mu_2}} \tag{4.1.2}$$

form an $A-$orthonormal basis of $\mathfrak{m}_\Theta$. It is easy to verify that $[X,Y]_{\mathfrak{m}_{\{\alpha_1,\alpha_3\}}}=0$ for X and Y in the basis (4.1.2), in particular, $Z=0$ and, therefore, by (4.0.2) we have that

$$\begin{array}{rcccccc} r_1 & := & \mathrm{Ric}(X_1,X_1) & = & \mathrm{Ric}(X_2,X_2) & = & \dfrac{1}{2\mu_1} \\ r_2 & := & \mathrm{Ric}(Y_1,Y_1) & = & \mathrm{Ric}(Y_2,Y_2) & = & \dfrac{1}{2\mu_2} \end{array}.$$

So A is Einstein if and only if $r_1=r_2$, i.e., $\mu_1=\mu_2$. □

4.1.2 $(SO(l)\times SO(l+1))/(SO(l-1)\times SO(l)),\ l\geq 3$

Let $\mathfrak{g}=B_l$, $l\geq 3$ and $\Theta=\{\alpha_2,...,\alpha_l\}$ so that $\mathbb{F}_{\{\alpha_2,...,\alpha_l\}}\overset{\text{diff.}}{\approx}(SO(l)\times SO(l+1))/(SO(l-1)\times SO(l))$. Fix the $\mathrm{Ad}(K)-$invariant inner product $(\cdot,\cdot)$ in (1.2.1) and the $(\cdot,\cdot)-$orthonormal basis (1.2.2). By Proposition 1.2.1 we have that $\mathfrak{m}_\Theta=(V_1)_1\oplus(V_1)_2$, where $(V_1)_1=\mathrm{span}\{w_{s1}-u_{s1}: 2\leq s\leq l\}$ and $(V_1)_2=\mathrm{span}\{v_1,w_{s1}+u_{s1}:2\leq s\leq l\}$. This two subspaces are $K_\Theta-$invariant, irreducible and non-equivalent. According to Proposition 2.2.2, every invariant metric A (with respect to the inner product $(\cdot,\cdot)$) is determined by two positive numbers ρ, μ such that

$$A\Big|_{(V_1)_1}=\rho I_{(V_1)_1}\ \text{ and } A\Big|_{(V_1)_2}=\mu I_{(V_1)_2}\ . \tag{4.1.3}$$

Proposition 4.1.2. *The invariant metric A above is an Einstein metric if and only if $\mu=\left(\frac{l-1}{l-2}\right)\rho$.*

Proof. Consider the $A-$orthonomal basis

$$F_s=\frac{w_{s1}-u_{s1}}{\sqrt{2\rho}},\ G_s=\frac{w_{s1}+u_{s1}}{\sqrt{2\mu}},\ Y=\frac{v_1}{\sqrt{\mu}},\ s=2,...,l.$$

Then

$$[Y,F_s]_{\mathfrak{m}_\Theta}=[Y,G_s]_{\mathfrak{m}_\Theta}=[F_s,G_t]_{\mathfrak{m}_\Theta}=[F_s,F_t]_{\mathfrak{m}_\Theta}=[G_s,G_t]_{\mathfrak{m}_\Theta}=0,\ s,t=2,...,l.$$

By (4.0.2) we have that

$$\begin{array}{rcccccl} r_1 & := & \mathrm{Ric}(F_s,F_s) & = & \dfrac{2(l-2)}{\rho}, & & s=2,...,l \\ r_2 & := & \mathrm{Ric}(Y,Y) & = & \mathrm{Ric}(G_s,G_s) & = & \dfrac{2(l-1)}{\mu},\ s=2,...,l, \end{array}$$

so A is an Einstein metric if and only if $\frac{2(l-2)}{\rho}=\frac{2(l-1)}{\mu}$ or, equivalently, $\mu=\left(\frac{l-1}{l-2}\right)\rho$. □

4.1.3 $(SO(l)\times SO(l+1))/SO(l),\ l\geq 3,\ l\neq 4$

The flag $\mathbb{F}_\Theta=SO(l)\times SO(l+1)/SO(l)$ is obtained when we consider $\mathfrak{g}=B_l$ and $\Theta=\{\alpha_1,...,\alpha_{l-1}\}$. For $l\geq 3$ and $l\neq 4$, the isotropy representation of $\mathbb{F}_\Theta$ splits into two non-equivalent, irreducible submodules given by $V_1=\mathrm{span}\{v_1,...,v_l\}$ and $U_1=\mathrm{span}\{u_{st}:1\leq t<s\leq l\}$, where v_j,u_{st} are defined in (1.2.2). By Proposition 2.2.2, every invariant metric A has the form

$$A\,|_{V_1}=\mu I_{V_1}\ ,\ A\,|_{U_1}=\gamma I_{U_1}\ ,\mu,\gamma>0. \tag{4.1.4}$$

Proposition 4.1.3. *Let A be the invariant metric on $(SO(l) \times SO(l+1))/SO(l)$ given by the parameters $\mu, \gamma > 0$. Then A is an Einstein metric if and only $\mu = \frac{\gamma}{2}$ or $\mu = \left(\frac{l}{2l-4}\right)\gamma$.*

Proof. An $A-$orthonormal basis for $\mathfrak{m}_\Theta$ is given by

$$Y_{st} = \frac{u_{st}}{\sqrt{\gamma}},\ Z_j = \frac{v_j}{\sqrt{\mu}},\ 1 \leq t < s \leq,\ j = 1, ..., l,$$

which satisfies following relations

$$[Z_s, Z_t]_{\mathfrak{m}_\Theta} = -\frac{\sqrt{\gamma}}{\mu} Y_{st},\ [Z_s, Y_{st}]_{\mathfrak{m}_\Theta} = \frac{Z_t}{\sqrt{\gamma}},\ [Z_t, Y_{st}]_{\mathfrak{m}_\Theta} = -\frac{Z_s}{\sqrt{\gamma}},\ 1 \leq t < s \leq l,$$

$$[Y_{st}, Y_{ij}]_{\mathfrak{m}_\Theta} = 0.$$

Using formula (4.0.2) we obtain

$$r_1 := \mathrm{Ric}(Z_j, Z_j) = \frac{2(l-1)}{\mu} - \frac{(l-1)\gamma}{2\mu^2},\ j = 1, ..., l$$

$$r_2 := \mathrm{Ric}(Y_{st}, Y_{st}) = \frac{2(l-2)}{\gamma} + \frac{\gamma}{2\mu^2},\ 1 \leq t < s \leq l.$$

Therefore, A is an Einstein metric if and only if

$$\frac{2(l-1)}{\mu} - \frac{(l-1)\gamma}{2\mu^2} = \frac{2(l-2)}{\gamma} + \frac{\gamma}{2\mu^2} \Longleftrightarrow -\frac{2(l-1)}{\mu} + \frac{2(l-2)}{\gamma} + \frac{l\gamma}{2\mu^2} = 0$$

$$\Longleftrightarrow -4(l-1)\gamma\mu + 4(l-2)\mu^2 + l\gamma^2 = 0$$

$$\Longleftrightarrow l(2\mu - \gamma)^2 - 4\mu(2\mu - \gamma) = 0$$

$$\Longleftrightarrow (2\mu - \gamma)(l(2\mu - \gamma) - 4\mu) = 0$$

$$\Longleftrightarrow \mu = \frac{\gamma}{2} \text{ or } \mu = \left(\frac{l}{2l-4}\right)\gamma.$$

□

4.1.4 $U(l)/O(l),\ l \geq 3$

Let $\mathfrak{g} = C_l,\ l \geq 3$ and $\Theta = \{\alpha_1, ..., \alpha_{l-1}\}$, then $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} U(l)/O(l)$. We consider the product $(\cdot, \cdot)$ defined in (1.3.1) and the $(\cdot, \cdot)-$orthogonal basis (1.3.2). Proposition 1.3.1 implies that the isotropy representation of $U(l)/O(l)$ decomposes into the two non-equivalent, irreducible submodules $V_1 = \mathrm{span}\{u_{11} + ... + u_{ll}\}$ and $U_1 = \mathrm{span}\{u_{11} - u_{22}, ..., u_{l-1,l-1} - u_{ll}\} \cup \{u_{st} : 1 \leq t < s \leq l\}$. By Proposition 2.3.2, every invariant metric A has the form

$$A\Big|_{V_1} = \mu^{(0)} I_{V_1}\ ,\ A\Big|_{U_1} = \mu^{(1)} I_{U_1}\ , \mu^{(0)}, \mu^{(1)} > 0. \tag{4.1.5}$$

Proposition 4.1.4. *There is no $U(l)-$invariant Einstein metric on $U(l)/O(l)$.*

Proof. Let A be an invariant metric as in (4.1.5) and consider any $A-$orthonormal basis $\{X_i\}$ of $\mathfrak{m}_\Theta$. Each matrix X_i has the form

$$X_i = \begin{pmatrix} \mathbf{0} & -B_i \\ B_i & \mathbf{0} \end{pmatrix}, \; B_i - B_i^T = 0,$$

therefore

$$[X_i, X_j]_{\mathfrak{m}_\Theta} = \left[\begin{pmatrix} \mathbf{0} & -B_i \\ B_i & \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} & -B_j \\ B_j & \mathbf{0} \end{pmatrix}\right]_{\mathfrak{m}_\Theta} = \begin{pmatrix} B_jB_i - B_iB_j & \mathbf{0} \\ \mathbf{0} & B_jB_i - B_iB_j \end{pmatrix}_{\mathfrak{m}_\Theta} = \mathbf{0}$$

for every i, j. Take $Z_1 = u_{11} + ... + u_{ll} \in V_1$ and $Y = u_{21} \in U_1$, by formula (4.0.2) we have

$$\begin{aligned} \mathrm{Ric}(Z_1, Z_1) &:= -\frac{1}{2}\langle Z_1, Z_1\rangle = 0 \\ \mathrm{Ric}(Y, Y) &:= -\frac{1}{2}\langle Y, Y\rangle = 2l. \end{aligned}$$

Since $2l \neq 0$, then A cannot be an Einstein metric. □

4.1.5 $U(l)/(O(1) \times U(l-1)),\; l \geq 3$

Let $\mathfrak{g} = C_l,\; l \geq 3$ and $\Theta = \{\alpha_2, ..., \alpha_l\}$ so that $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} U(l)/(O(1) \times U(l-1))$. In this case, we consider $(\cdot,\cdot)$ as in (1.3.1) and the basis (1.3.2) of $\mathfrak{k} \cong \mathfrak{u}(l)$. By Proposition 1.3.1, we have that $\mathfrak{m}_\Theta = V_1 \oplus M_{21}$, where $V_1 = \mathrm{span}\{u_{11}\}$ and $M_{21} = \mathrm{span}\{w_{s1}, u_{s1} : s = 2, ..., l\}$ are not equivalent. Every invariant metric A has the form

$$A\Big|_{V_1} = \mu^{(0)} I_{V_1}\;,\; A\Big|_{M_{21}} = \mu^{(21)} I_{M_{21}}\;, \mu^{(0)}, \mu^{(21)} > 0. \tag{4.1.6}$$

Proposition 4.1.5. *The metric* (4.1.6) *is an Einstein metric if and only if* $\mu^{(0)} = 2\mu^{(21)}$.

Proof. An $A-$orthonormal basis of $\mathfrak{m}_\Theta$ is given by

$$X_{s1} = \frac{w_{s1}}{\sqrt{\mu^{(21)}}},\; Y_{s1} = \frac{u_{s1}}{\sqrt{\mu^{(21)}}},\; Z_1 = \sqrt{\frac{2}{\mu^{(0)}}} u_{11},\; s = 2, ..., l$$

and satisfy

$$\begin{aligned} &[Z_1, X_{s1}]_{\mathfrak{m}_\Theta} = -\sqrt{\frac{2}{\mu^{(0)}}} Y_{s1},\; [Z_1, Y_{s1}]_{\mathfrak{m}_\Theta} = \sqrt{\frac{2}{\mu^{(0)}}} X_{s1},\; [X_{s1}, Y_{s1}]_{\mathfrak{m}_\Theta} = -\frac{\sqrt{2\mu^{(0)}}}{\mu^{(21)}} Z_1,\; s = 2, ..., l, \\ &[X_{s1}, X_{t1}]_{\mathfrak{m}_\Theta} = [X_{s1}, Y_{t1}]_{\mathfrak{m}_\Theta} = [Y_{s1}, Y_{t1}]_{\mathfrak{m}_\Theta} = 0,\; s \neq t. \end{aligned}$$

As before, we can apply (4.0.2) to obtain

$$\begin{aligned} \mathrm{Ric}(Z_1, Z_1) &= \frac{(l-1)\mu^{(0)}}{(\mu^{(21)})^2}, \\ \mathrm{Ric}(X_{s1}, X_{s1}) &= \mathrm{Ric}(Y_{s1}, Y_{s1}) = \frac{2l}{\mu^{(21)}} - \frac{\mu^{(0)}}{(\mu^{(21)})^2},\; s = 2, ..., l, \end{aligned}$$

so A is an Einstein metric if and only if

$$\frac{(l-1)\mu^{(0)}}{(\mu^{(21)})^2} = \frac{2l}{\mu^{(21)}} - \frac{\mu^{(0)}}{(\mu^{(21)})^2} \Longleftrightarrow (l-1)\mu^{(0)} = 2l\mu^{(21)} - \mu^{(0)}$$

$$\Longleftrightarrow l\mu^{(0)} = 2l\mu^{(21)}$$

$$\Longleftrightarrow \mu^{(0)} = 2\mu^{(21)}.$$

□

4.1.6 $(SO(4) \times SO(4))/SO(4)$

The manifold $\mathbb{F}_\Theta = (SO(4) \times SO(4))/SO(4)$ is a flag of D_4 obtained when $\Theta = \{\alpha_1, \alpha_2, \alpha_3\}$ or $\{\alpha_1, \alpha_2, \alpha_4\}$. Let us assume that $\Theta = \{\alpha_1, \alpha_2, \alpha_3\}$, then the isotropy representation decomposes into two non-equivalent $SO(4)-$invariant irreducible subspaces given by

$$T_1 = \text{span}\{u_{21} + u_{43}, u_{31} - u_{42}, u_{41} + u_{32}\} \text{ and } S_1 = \text{span}\{u_{43} - u_{21}, u_{31} + u_{42}, u_{41} - u_{32}\},$$

where the matrices u_{ij} are defined in (1.4.2). By (2.0.3) and (2.0.4) we have that every invariant metric A on $SO(4) \times SO(4)/SO(4)$ is determined by positive real numbers μ_1, μ_2 such that

$$A\,|_{T_1} = \mu_1 I_{T_1}\ ,\ A\,|_{S_1} = \mu_2 I_{S_1}\ . \tag{4.1.7}$$

Proposition 4.1.6. *Let A be an invariant metric on $(SO(4) \times SO(4))/SO(4)$. Then A is an Einstein metric if and only if A is normal.*

Proof. In this case, an $A-$orthonormal basis of $\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}$ is given by

$$X_1 = \frac{u_{21} + u_{43}}{\sqrt{2\mu_1}},\ X_2 = \frac{u_{31} - u_{42}}{\sqrt{2\mu_1}},\ X_3 = \frac{u_{41} + u_{32}}{\sqrt{2\mu_1}},$$

$$Y_1 = \frac{u_{43} - u_{21}}{\sqrt{2\mu_1}},\ Y_2 = \frac{u_{31} + u_{42}}{\sqrt{2\mu_1}},\ Y_3 = \frac{u_{41} - u_{32}}{\sqrt{2\mu_1}}$$

and we have that

$$[X_i, X_j]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = [Y_i, X_j]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = [Y_i, Y_j]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = 0,\ i, j = 1, 2, 3.$$

Therefore

$$\begin{cases} \text{Ric}(X_i, X_i) = -\frac{1}{2}\langle X_i, X_i\rangle = \frac{2}{\mu_1} \\ \text{Ric}(Y_i, Y_i) = -\frac{1}{2}\langle Y_i, Y_i\rangle = \frac{2}{\mu_2} \end{cases},\ i = 1, 2, 3.$$

So A is an Einstein metric if and only if $\frac{2}{\mu_1} = \frac{2}{\mu_2}$, i.e., $\mu_1 = \mu_2$ (A is normal). □

For $\Theta = \{\alpha_1, \alpha_2, \alpha_4\}$ we consider the automorphism η of $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$ given by

$$\eta(w_{ij}) = u_{ij},\ \eta(u_{ij}) = w_{ij},\ 1 \leq j < i \leq 3$$

$$\eta(w_{4j}) = u_{4j},\ \eta(u_{4j}) = w_{4j},\ j = 1, 2, 3.$$

Observe that $\eta(\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_4\}}) = \mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}$ and η maps isotropy summands into isotropy summands, therefore, every invariant metric on $\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_4\}}$ has the form $\left(\eta|_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_4\}}}\right)^* A$, where A is an invariant metric on $\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}$. Using the formula (4.0.2), it is easy to see that η preserves the Ricci components, thus, an invariant metric on $\mathbb{F}_{\{\alpha_1,\alpha_2,\alpha_4\}}$ is Einstein if and only if it is normal.

4.2 Flags with three isotropy summands

Homogeneous Einstein metrics on complex generalized flag manifolds with three isotropy summands were completely classified in [5] and [18]. In this section we study the Einstein equation for real flag manifolds of classical type whose isotropy representation decomposes into three isotropy summands.

4.2.1 $SO(4)/S(O(2) \times O(1) \times O(1))$

The flag $SO(4)/S(O(2) \times O(1) \times O(1))$ is obtained when $\mathfrak{g} = A_3$ and $\Theta = \{\alpha_1\},\ \{\alpha_2\}$ or $\{\alpha_3\}$. Let us normalize the $\mathrm{Ad}(K)-$invariant inner product $(\cdot,\cdot)$ of section 2.1 by setting

$$g_0 = \tfrac{1}{4}(\cdot,\cdot) = -\tfrac{1}{4}\langle\cdot,\cdot\rangle.$$

Suppose that $\Theta = \{\alpha_1\}$, then the Lie algebra of $K_{\{\alpha_1\}}$ is the subalgebra $\mathfrak{k}_{\{\alpha_1\}} = \mathrm{span}\{w_{21}\}$ and the isotropy representation of $K_{\{\alpha_1\}}$ on $\mathfrak{m}_{\{\alpha_1\}}$ decomposes into the irreducible sub-representations $W_0 = \mathrm{span}\{w_{43}\}$, $W_1 = \mathrm{span}\{w_{31}, w_{32}\}$ and $W_2 = \mathrm{span}\{w_{42}, w_{41}\}$, where W_1 and W_2 are equivalent. By Proposition 2.1.2, every invariant metric A with respect to g_0 can be written in the basis $\mathcal{B} = \{w_{43}, w_{31}, w_{32}, w_{42}, w_{41}\}$ as

$$[A]_{\mathcal{B}} = \begin{pmatrix} \mu_0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & b & 0 \\ 0 & 0 & \mu_1 & 0 & -b \\ 0 & b & 0 & \mu_2 & 0 \\ 0 & 0 & -b & 0 & \mu_2 \end{pmatrix},\ \mu_0, \mu_1, \mu_2 > 0. \tag{4.2.1}$$

Proposition 4.2.1. *Let A be an invariant metric on the flag $\mathbb{F}_{\{\alpha_1\}}$ of A_3 written as in* (4.2.1). *Then A is an Einstein metric if and only if its entries satisfy one of the following conditions:*

(E1) $b = 0,\ \mu_1 = \mu_2 = \frac{3}{4}\mu_0$

(E2) $b > 0,\ b = \mu_1 = \frac{\mu_2}{3} = \frac{\mu_0}{2}$

(E3) $b > 0,\ b = \mu_2 = \frac{\mu_1}{3} = \frac{\mu_0}{2}$

(E4) $b < 0,\ -b = \mu_1 = \frac{\mu_2}{3} = \frac{\mu_0}{2}$

(E5) $b < 0,\ -b = \mu_2 = \frac{\mu_1}{3} = \frac{\mu_0}{2}$

Proof. We separate two cases:

• A diagonal ($b = 0$):

The vectors
$$X_0 = \frac{w_{43}}{\sqrt{\mu_0}},\ X_1 = \frac{w_{31}}{\sqrt{\mu_1}},\ X_2 = \frac{w_{32}}{\sqrt{\mu_1}},\ X_3 = \frac{w_{42}}{\sqrt{\mu_2}},\ X_4 = \frac{w_{41}}{\sqrt{\mu_2}}$$
form an $A-$orthonormal basis of $\mathfrak{m}_{\{\alpha_1\}}$. The non-zero bracket relations between these vectors are given by

$$\begin{aligned}
[X_0, X_1]_{\mathfrak{m}_{\{\alpha_1\}}} &= \sqrt{\tfrac{\mu_2}{\mu_0\mu_1}}X_4, & [X_0, X_2]_{\mathfrak{m}_{\{\alpha_1\}}} &= \sqrt{\tfrac{\mu_2}{\mu_0\mu_1}}X_3,\\
[X_0, X_3]_{\mathfrak{m}_{\{\alpha_1\}}} &= -\sqrt{\tfrac{\mu_1}{\mu_0\mu_2}}X_2, & [X_0, X_4]_{\mathfrak{m}_{\{\alpha_1\}}} &= -\sqrt{\tfrac{\mu_1}{\mu_0\mu_2}}X_1,\\
[X_1, X_4]_{\mathfrak{m}_{\{\alpha_1\}}} &= \sqrt{\tfrac{\mu_0}{\mu_1\mu_2}}X_0, & [X_2, X_3]_{\mathfrak{m}_{\{\alpha_1\}}} &= \sqrt{\tfrac{\mu_0}{\mu_1\mu_2}}X_0.
\end{aligned}$$

Since $[X_i, X_j]$ is $A-$orthogonal to X_i and X_j for all $i, j \in \{0, 1, 2, 3, 4\}$, then, equation (4.0.1) implies
$$Z = \sum_i U(X_i, X_i) = 0.$$
A direct application of the formula (4.0.2) gives us the components of the Ricci tensor:
$$\begin{aligned}
r_0 &:= \mathrm{Ric}(X_0, X_0) = \frac{2}{\mu_0} + \frac{\mu_0}{\mu_1\mu_2} - \frac{\mu_2}{\mu_0\mu_1} - \frac{\mu_1}{\mu_0\mu_2}\\
r_1 &:= \mathrm{Ric}(X_1, X_1) = \mathrm{Ric}(X_2, X_2) = \frac{2}{\mu_1} + \frac{\mu_1}{2\mu_0\mu_2} - \frac{\mu_2}{2\mu_0\mu_1} - \frac{\mu_0}{2\mu_1\mu_2}\\
r_2 &:= \mathrm{Ric}(X_3, X_3) = \mathrm{Ric}(X_4, X_4) = \frac{2}{\mu_2} + \frac{\mu_2}{2\mu_0\mu_1} - \frac{\mu_1}{2\mu_0\mu_2} - \frac{\mu_0}{2\mu_1\mu_2}.
\end{aligned}$$
Thus, Einstein condition for the diagonal metric A reduces to $r_0 = r_1 = r_2$, which gives us
$$\mu_1 = \mu_2 = \frac{3}{4}\mu_0.$$

• A non-diagonal ($b \neq 0$):

The eigenvalues of A are
$$\xi_0 = \mu_0, \quad \xi_1 = \frac{\mu_1 + \mu_2 - \sqrt{4b^2 + (\mu_1 - \mu_2)^2}}{2}, \quad \xi_2 = \frac{\mu_1 + \mu_2 + \sqrt{4b^2 + (\mu_1 - \mu_2)^2}}{2}$$
and satisfy the relations
$$\xi_1 + \xi_2 = \mu_1 + \mu_2, \quad \xi_1\xi_2 = \mu_1\mu_2 - b^2,$$
$$\begin{aligned}
b^2 &= (\xi_2 - \mu_2)(\xi_2 - \mu_1) = (\mu_1 - \xi_1)(\xi_2 - \mu_1)\\
&= (\mu_2 - \xi_1)(\mu_1 - \xi_1) = (\mu_2 - \xi_1)(\xi_2 - \mu_2).
\end{aligned}$$
Since A is positive definite then $\xi_1, \xi_2 > 0$. Let us set

$$c_1 = \xi_1((\xi_1 - \mu_2)^2 + b^2) = \xi_1(\xi_1 - \mu_2)(\xi_1 - \xi_2) \text{ and}$$

$$c_2 = \xi_2((\xi_2 - \mu_2)^2 + b^2) = \xi_2(\xi_2 - \mu_2)(\xi_2 - \xi_1).$$

Then the vectors

$$X_0 = \frac{w_{43}}{\sqrt{\xi_0}}, \quad X_1 = \frac{(\xi_1 - \mu_2)w_{31} + bw_{42}}{\sqrt{c_1}}, \ X_2 = \frac{(\xi_2 - \mu_1)w_{32} + bw_{41}}{\sqrt{c_1}},$$

$$X_3 = \frac{(\xi_2 - \mu_2)w_{31} + bw_{42}}{\sqrt{c_2}}, \ X_4 = \frac{(\xi_1 - \mu_1)w_{32} + bw_{41}}{\sqrt{c_2}}$$

form an $A-$orthonormal basis for $\mathfrak{m}_{\{\alpha_1\}}$. The non-zero bracket relations are given by

$$[X_0, X_1]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{b}{\xi_1 - \xi_2}\left(\frac{2}{\sqrt{\xi_0}}X_2 + \sqrt{\frac{\xi_2}{\xi_0\xi_1}}\left(\frac{\mu_2 - \mu_1}{|b|}\right)X_4\right),$$

$$[X_0, X_2]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{b}{\xi_2 - \xi_1}\left(\frac{2}{\sqrt{\xi_0}}X_1 + \sqrt{\frac{\xi_2}{\xi_0\xi_1}}\left(\frac{\mu_2 - \mu_1}{|b|}\right)X_3\right),$$

$$[X_0, X_3]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{b}{\xi_2 - \xi_1}\left(\sqrt{\frac{\xi_1}{\xi_0\xi_2}}\left(\frac{\mu_1 - \mu_2}{|b|}\right)X_2 + \frac{2}{\sqrt{\xi_0}}X_4\right),$$

$$[X_0, X_4]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{b}{\xi_1 - \xi_2}\left(\sqrt{\frac{\xi_1}{\xi_0\xi_2}}\left(\frac{\mu_1 - \mu_2}{|b|}\right)X_1 + \frac{2}{\sqrt{\xi_0}}X_3\right),$$

$$[X_1, X_2]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{2b\sqrt{\xi_0}}{\xi_1(\xi_1 - \xi_2)}X_0,$$

$$[X_1, X_4]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{b(\mu_2 - \mu_1)}{|b|(\xi_1 - \xi_2)}\sqrt{\frac{\xi_0}{\xi_1\xi_2}}X_0,$$

$$[X_2, X_3]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{b(\mu_2 - \mu_1)}{|b|(\xi_2 - \xi_1)}\sqrt{\frac{\xi_0}{\xi_1\xi_2}}X_0,$$

$$[X_3, X_4]_{\mathfrak{m}_{\{\alpha_1\}}} = \frac{2b\sqrt{\xi_0}}{\xi_2(\xi_2 - \xi_1)}X_0.$$

Observe that $[X_i, X_j]$ is $A-$orthogonal to X_i and X_j for all $i, j \in \{0, 1, 2, 3, 4\}$, thus, $Z = 0$. If A is an Einstein metric, then $\mathrm{Ric}(X_1, X_3) = 0$ (because $\mathrm{Ric} = cg$). By (4.0.2) we have

$$\mathrm{Ric}(X_1, X_3) = -\frac{1}{2}g\left([X_1, X_2]_{\mathfrak{m}_{\{\alpha_1\}}}, [X_3, X_2]_{\mathfrak{m}_{\{\alpha_1\}}}\right) - \frac{1}{2}g\left([X_1, X_4]_{\mathfrak{m}_{\{\alpha_1\}}}, [X_3, X_4]_{\mathfrak{m}_{\{\alpha_1\}}}\right)$$

$$
\begin{aligned}
&-\frac{1}{2}g\left([X_1,X_0]_{\mathfrak{m}_{\{\alpha_1\}}},[X_3,X_0]_{\mathfrak{m}_{\{\alpha_1\}}}\right)-\frac{1}{2}\langle X_1,X_3\rangle\\
&+\frac{1}{2}g\left([X_0,X_2]_{\mathfrak{m}_{\{\alpha_1\}}},X_1\right)g\left([X_0,X_2]_{\mathfrak{m}_{\{\alpha_1\}}},X_3\right)\\
&+\frac{1}{2}g\left([X_0,X_4]_{\mathfrak{m}_{\{\alpha_1\}}},X_1\right)g\left([X_0,X_4]_{\mathfrak{m}_{\{\alpha_1\}}},X_3\right)\\
=&-\left(\frac{b\sqrt{\xi_0}}{\xi_1(\xi_1-\xi_2)}\right)\left(\frac{b(\mu_1-\mu_2)}{|b|(\xi_2-\xi_1)}\sqrt{\frac{\xi_0}{\xi_1\xi_2}}\right)-\left(\frac{b\sqrt{\xi_0}}{\xi_2(\xi_2-\xi_1)}\right)\left(\frac{b(\mu_2-\mu_1)}{|b|(\xi_1-\xi_2)}\sqrt{\frac{\xi_0}{\xi_1\xi_2}}\right)\\
&-\left(\frac{b}{\sqrt{\xi_0}(\xi_2-\xi_1)}\right)\left(\frac{b(\mu_2-\mu_1)}{|b|(\xi_2-\xi_1)}\sqrt{\frac{\xi_1}{\xi_0\xi_2}}\right)-\left(\frac{b}{\sqrt{\xi_0}(\xi_2-\xi_1)}\right)\left(\frac{b(\mu_2-\mu_1)}{|b|(\xi_1-\xi_2)}\sqrt{\frac{\xi_2}{\xi_0\xi_1}}\right)\\
&+\left(\frac{b}{\sqrt{\xi_0}(\xi_2-\xi_1)}\right)\left(\frac{b(\mu_2-\mu_1)}{|b|(\xi_2-\xi_1)}\sqrt{\frac{\xi_2}{\xi_0\xi_1}}\right)+\left(\frac{b}{\sqrt{\xi_0}(\xi_1-\xi_2)}\right)\left(\frac{b(\mu_1-\mu_2)}{|b|(\xi_1-\xi_2)}\sqrt{\frac{\xi_1}{\xi_0\xi_2}}\right)\\
=&\frac{b^2(\mu_2-\mu_1)}{|b|(\xi_2-\xi_1)^2}\left(-\frac{\xi_0}{\xi_1\sqrt{\xi_1\xi_2}}+\frac{\xi_0}{\xi_2\sqrt{\xi_1\xi_2}}-\frac{2\sqrt{\xi_1}}{\xi_0\sqrt{\xi_2}}+\frac{2\sqrt{\xi_2}}{\xi_0\sqrt{\xi_1}}\right)\\
=&\frac{b^2(\mu_2-\mu_1)}{|b|(\xi_2-\xi_1)^2}\left(\frac{-\xi_0^2(\xi_2-\xi_1)+2\xi_1\xi_2(\xi_2-\xi_1)}{\xi_0\xi_1\xi_2\sqrt{\xi_1\xi_2}}\right)\\
=&\frac{|b|(\mu_2-\mu_1)(2\xi_1\xi_2-\xi_0^2)}{\xi_0(\xi_1\xi_2)^{\frac{3}{2}}(\xi_2-\xi_1)},
\end{aligned}
$$

so, a necessary condition for A to be an Einstein metric is that $\mu_1=\mu_2$ or $2\xi_1\xi_2-\xi_0^2=0$.

Case 1. $\mu:=\mu_1=\mu_2$.

We can use formula (4.0.2) to obtain

$$r_0=\operatorname{Ric}(X_0,X_0)=\frac{\xi_0(\xi_1\xi_2+2b^2)}{(\xi_1\xi_2)^2},$$

$$r_1=\operatorname{Ric}(X_1,X_1)=\operatorname{Ric}(X_2,X_2)=\frac{4\xi_1-\xi_0}{2\xi_1^2},$$

$$r_2=\operatorname{Ric}(X_3,X_3)=\operatorname{Ric}(X_4,X_4)=\frac{4\xi_2-\xi_0}{2\xi_2^2}.$$

If $r_0=r_1=r_2$, then $\xi_1=\xi_2$, which is not possible since $b\neq 0$. Therefore, we have no solutions in this case.

Case 2. $2\xi_1\xi_2 - \xi_0^2 = 0$.

The components of the Ricci tensor are $r_0 = \mathrm{Ric}(X_0, X_0)$, $r_1 = \mathrm{Ric}(X_1, X_1) = \mathrm{Ric}(X_2, X_2)$ and $r_2 = \mathrm{Ric}(X_3, X_3) = \mathrm{Ric}(X_4, X_4)$. By (4.0.2) we have

$$
\begin{aligned}
r_1 - r_2 &= \frac{\xi_1\xi_2(\mu_2-\mu_1)^2}{\xi_0(\xi_2-\xi_1)^2}\left(\frac{1}{\xi_2^2}-\frac{1}{\xi_1^2}\right) + \frac{2b^2\xi_0}{(\xi_2-\xi_1)^2}\left(\frac{1}{\xi_2^2}-\frac{1}{\xi_1^2}\right) - 2\left(\frac{1}{\xi_2}-\frac{1}{\xi_1}\right)\\
&= \left(\frac{1}{\xi_2^2}-\frac{1}{\xi_1^2}\right)\left(\frac{\xi_1\xi_2(\mu_2-\mu_1)^2(\xi_1+\xi_2)+2b^2\xi_0^2(\xi_1+\xi_2)-2\xi_1\xi_2\xi_0(\xi_2-\xi_1)^2}{\xi_0(\xi_2-\xi_1)^2(\xi_1+\xi_2)}\right)\\
&= \left(\frac{1}{\xi_2^2}-\frac{1}{\xi_1^2}\right)\left(\frac{\xi_1\xi_2(\xi_1+\xi_2)((\mu_2-\mu_1)^2+4b^2)-2\xi_1\xi_2\xi_0(\xi_2-\xi_1)^2}{\xi_0(\xi_2-\xi_1)^2(\xi_1+\xi_2)}\right)\\
&= \left(\frac{1}{\xi_2^2}-\frac{1}{\xi_1^2}\right)\left(\frac{\xi_1\xi_2(\xi_1+\xi_2)(\xi_2-\xi_1)^2-2\xi_1\xi_2\xi_0(\xi_2-\xi_1)^2}{\xi_0(\xi_2-\xi_1)^2(\xi_1+\xi_2)}\right)\\
&= \left(\frac{1}{\xi_2^2}-\frac{1}{\xi_1^2}\right)\left(\frac{\xi_1\xi_2(\xi_1+\xi_2-2\xi_0)}{\xi_0(\xi_1+\xi_2)}\right)\\
&= \frac{(\xi_1-\xi_2)(\xi_1+\xi_2-2\sqrt{2\xi_1\xi_2})}{\sqrt{2}(\xi_1\xi_2)^{\frac{3}{2}}}
\end{aligned}
$$

and

$$
\begin{aligned}
r_0 - \frac{r_1+r_2}{2} &= \frac{b^2}{(\xi_2-\xi_1)^2}\left(\frac{3\xi_0}{\xi_1^2}+\frac{3\xi_0}{\xi_2^2}-\frac{8}{\xi_0}\right) + \frac{(\mu_2-\mu_1)^2}{2(\xi_2-\xi_1)^2}\left(\frac{3\xi_0}{\xi_1\xi_2}-\frac{2\xi_2}{\xi_0\xi_1}-\frac{2\xi_1}{\xi_0\xi_2}\right)\\
&\quad -\left(\frac{2}{\xi_0}-\frac{1}{\xi_1}-\frac{1}{\xi_2}\right)\\
&= \frac{10\sqrt{2}b^2+3\sqrt{2}\xi_1\xi_2-\sqrt{2}(\xi_1-\xi_2)^2-2(\xi_1+\xi_2)\sqrt{\xi_1\xi_2}}{(\xi_1\xi_2)^{\frac{3}{2}}}.
\end{aligned}
$$

Since the systems of equations

$$
\begin{cases} r_0 - r_1 = 0 \\ r_1 - r_2 = 0 \end{cases} \qquad\qquad \begin{cases} r_1 - r_2 = 0 \\ r_0 - \frac{r_1+r_2}{2} = 0 \end{cases}
$$

are equivalent, we have that A is an Einstein metric if and only if ξ_1, ξ_2 satisfy the equations

$$\begin{cases} 10\sqrt{2}b^2 + 3\sqrt{2}\xi_1\xi_2 - \sqrt{2}(\xi_1 - \xi_2)^2 - 2(\xi_1 + \xi_2)\sqrt{\xi_1\xi_2} = 0 \\ \\ \xi_1 + \xi_2 - 2\sqrt{2\xi_1\xi_2} = 0. \end{cases}$$

By solving this system for $\xi_1 < \xi_2$, we obtain the solutions:

$$\begin{cases} \xi_1 = (-2+\sqrt{2})b \\ \xi_2 = (-2-\sqrt{2})b \end{cases}, b < 0, \qquad \begin{cases} \xi_1 = (2-\sqrt{2})b \\ \xi_2 = (2+\sqrt{2})b \end{cases}, b > 0,$$

or, equivalently,

$$\begin{cases} \mu_1 = -3b \\ \mu_2 = -b \end{cases}, \ b < 0, \quad \begin{cases} \mu_1 = -b \\ \mu_2 = -3b \end{cases}, \ b < 0,$$

$$\begin{cases} \mu_1 = b \\ \mu_2 = 3b \end{cases}, \ b > 0, \quad \begin{cases} \mu_1 = 3b \\ \mu_2 = b \end{cases}, \ b > 0.$$

In all these cases, $\mu_0 = \xi_0 = \sqrt{2\xi_1\xi_2} = \sqrt{4b^2} = 2|b|$. □

When $\Theta = \{\alpha_2\}$ or $\{\alpha_3\}$, consider the maps

$$\varphi_i : \mathbb{F}_{\{\lambda_i - \lambda_{i+1}\}} \longrightarrow \mathbb{F}_{\{\lambda_1 - \lambda_2\}}; \quad \varphi_i\left(kK_{\{\lambda_i - \lambda_{i+1}\}}\right) = e_i k e_i^T K_{\{\lambda_1 - \lambda_2\}}, \ i = 1, 2,$$

where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

These maps are diffeomorphisms. It is easy to show that every invariant metric on $\mathbb{F}_{\{\alpha_i\}}$ has the form $\varphi_i^* g$, where g is an invariant metric on $\mathbb{F}_{\{\alpha_1\}}$. By [20, Lemma 7.2] we have that the Ricci tensor associated to $\varphi_i^* g$ is equal to the pull-back by φ_i of the Ricci tensor associated to g, thus, $\varphi_i^* g$ is an Einstein metric if and only if is so g. Therefore, Einstein metrics on $\mathbb{F}_{\{\alpha_i\}}$ are obtained by taking the pullback by φ_i of the Einstein metrics on $\mathbb{F}_{\{\alpha_1\}}$.

4.2.2 $SO(l+1)/S(O(l_1) \times O(l_2) \times O(l_3))$, $l \geq 2$, $l \neq 3$, $l_1 + l_2 + l_3 = l + 1$

In this section, we shall present a more general result about invariant Einstein metrics on homogeneous spaces with three isotropy summands; it will be useful to study Einstein metrics on our particular case. Let G be a compact connected Lie group, H a closed subgroup of G and let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H respectively. Assume that the isotropy representation of G/H is decomposed into non-equivalent three irreducible components and consider an $(\cdot,\cdot)-$orthogonal reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $(\cdot,\cdot)$ is an $\mathrm{Ad}(G)-$invariant inner product on $\mathfrak{g}$. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ be the irreducible decomposition of $\mathfrak{m}$. Each invariant metric A with respect to $(\cdot,\cdot)$ on G/H can be represented by positive numbers x_1, x_2, x_3 such that $A|_{\mathfrak{m}_i} = x_i I_{\mathfrak{m}_i}$, $i = 1, 2, 3$. Let $\mathcal{M}$ be the set of all $G-$invariant metrics on G/H with volume 1, i.e.,

$$\mathcal{M} = \{A = (x_1, x_2, x_3) \in (\mathbb{R}^+)^3 : x_1^{dim\ \mathfrak{m}_1} x_2^{dim\ \mathfrak{m}_2} x_3^{dim\ \mathfrak{m}_3} = \frac{1}{v_0^2}\}$$

where $v_0 = Vol(G/H, (\cdot,\cdot)|_{\mathfrak{m}\times\mathfrak{m}})$. Denote by $S(A)$ the scalar curvature of $(G/H, A)$. By Corollary 7.39 of [9] we have the following formula for the scalar curvature:

$$S(A) = -\frac{1}{4}\sum_{i,j}|[X_i, X_j]_{\mathfrak{m}}|^2 - \frac{1}{2}\sum_i \langle X_i, X_i\rangle - |Z|^2 \tag{4.2.2}$$

where $|\cdot|$ is the norm with respect to A, $\{X_i\}$ is an $A-$orthonormal basis of $\mathfrak{m}$ and $Z = \sum_i U(X_i, X_i)$. The following result gives us a tool to study existence of invariant Einstein metrics.

Proposition 4.2.2. *([30]) $A \in \mathcal{M}$ is an Einstein metric if and only if*

$$\frac{\partial S}{\partial u}(A) = \frac{\partial S}{\partial v}(A) = 0,$$

where $u = \frac{x_2}{x_1}$ and $v = \frac{x_3}{x_1}$.

For the flag $\mathbb{F}_{\Sigma-\{\alpha_{d_1},\alpha_{d_2}\}}$, $1 \leq d_1 < d_2 \leq l$ of A_l, $l \neq 3$ the isotropy representation decomposes into the non-equivalent irreducible submodules $M_{21} = \text{span}\{w_{st} : s = d_1{+}1, ..., d_2,\ t = 1, ..., d_1\}$, $M_{31} = \text{span}\{w_{st} : s = d_2{+}1, ..., l{+}1,\ t = 1, ..., d_1\}$ and $M_{32} = \text{span}\{w_{st} : s = d_2{+}1, ..., l{+}1,\ t = d_1 + 1, ..., d_2\}$. Let $l_1 := d_1$, $l_2 := d_2 - d_1$ and $l_3 := l + 1 - d_2$, then

$$\Theta = \bigcup_{l_i>1}\left\{\alpha_{\tilde{l}_{i-1}+1}, ..., \alpha_{\tilde{l}_i-1}\right\} \text{ and } \mathbb{F}_\Theta = SO(l+1)/S(O(l_1)\times O(l_2)\times O(l_3)).$$

Fix the $\text{Ad}(K)-$invariant inner product g_0 on $\mathfrak{k} = \mathfrak{so}(l+1)$ given by

$$g_0 = \frac{1}{2(l-1)}(\cdot,\cdot) = -\frac{1}{2(l-1)}\langle\cdot,\cdot\rangle.$$

Any invariant metric A (with respect to g_0) is defined by positive numbers $\mu_{21}, \mu_{31}, \mu_{32}$ such that

$$A|_{M_{mn}} = \mu_{mn} I_{M_{mn}},\ 1 \leq n < m \leq 3. \tag{4.2.3}$$

Proposition 4.2.3. *The flag manifold $\mathbb{F}_\Theta = SO(l+1)/S(O(l_1)\times O(l_2)\times O(l_3))$ has at most four $SO(l{+}1)-$invariant Einstein metrics up to homotheties. Moreover, when $l_2 = l_3 =: m \geq 3$ we have*

a) If $1 \leq l_1 < 2\sqrt{m-1}$ or $l_1 > \frac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$ then $\mathbb{F}_\Theta$ has exactly two invariant Einstein metrics up to homotheties.

b) If $l_1 = 2\sqrt{m-1}$ or $l_1 = \frac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$ then $\mathbb{F}_\Theta$ has exactly three invariant Einstein metrics up to homotheties.

c) If $2\sqrt{m-1} < l_1 < \frac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$ then $\mathbb{F}_\Theta$ has exactly four invariant Einstein metrics up to homotheties.

Proof. Let A be an invariant metric on $SO(l+1)/S(O(l_1)\times O(l_2)\times O(l_3))$ as in (4.2.3). We consider the $A-$orthonormal basis

$$X_{st}=\frac{w_{st}}{\sqrt{\mu_{21}}},\ s=l_1+1,...,l_1+l_2,\ t=1,...,l_1$$

$$X_{st}=\frac{w_{st}}{\sqrt{\mu_{31}}},\ s=l_1+l_2+1,...,l+1,\ t=1,...,l_1 \tag{4.2.4}$$

$$X_{st}=\frac{w_{st}}{\sqrt{\mu_{32}}},\ s=l_1+l_2+1,...,l+1,\ t=l_1+1,...l_1+l_2.$$

Then, we have two cases:

Case 1. $l_2\neq l_3$.

Applying formula (4.2.2) to basis (4.2.4) we have that

$$S(A)=-\frac{l_1l_2l_3}{2}\left(\frac{\mu_{21}}{\mu_{31}\mu_{32}}+\frac{\mu_{31}}{\mu_{21}\mu_{32}}+\frac{\mu_{32}}{\mu_{21}\mu_{31}}\right)+(l-1)\left(\frac{l_1l_2}{\mu_{21}}+\frac{l_1l_3}{\mu_{31}}+\frac{l_2l_3}{\mu_{32}}\right).$$

Now, assume that A has volume 1 and let $u:=\dfrac{\mu_{31}}{\mu_{21}}$, $v:=\dfrac{\mu_{32}}{\mu_{21}}$, then

$$\frac{S(A)}{v_0^{\frac{2}{N}}}=\frac{S(u,v)}{v_0^{\frac{2}{N}}}=-\frac{l_1l_2l_3}{2}\left(u^{\frac{l_1l_3}{N}-1}v^{\frac{l_2l_3}{N}-1}+u^{\frac{l_1l_3}{N}+1}v^{\frac{l_2l_3}{N}-1}+u^{\frac{l_1l_3}{N}-1}v^{\frac{l_2l_3}{N}+1}\right)$$
$$+(l-1)\left(l_1l_2u^{\frac{l_1l_3}{N}}v^{\frac{l_2l_3}{N}}+l_1l_3u^{\frac{l_1l_3}{N}-1}v^{\frac{l_2l_3}{N}}+l_2l_3u^{\frac{l_1l_3}{N}}v^{\frac{l_2l_3}{N}-1}\right),$$

where $N:=l_1l_2+l_1l_3+l_2l_3$ and $v_0=Vol(\mathbb{F}_\Theta,g_0)$. Computing the partial derivatives of S we obtain

$$v_0^{-\frac{2}{N}}\frac{\partial S}{\partial u}=-\frac{l_1l_2l_3}{2}\left(\left(\frac{l_1l_3}{N}-1\right)u^{\frac{l_1l_3}{N}-2}v^{\frac{l_2l_3}{N}-1}+\left(\frac{l_1l_3}{N}+1\right)u^{\frac{l_1l_3}{N}}v^{\frac{l_2l_3}{N}-1}\right)$$
$$-\frac{l_1l_2l_3}{2}\left(\frac{l_1l_3}{N}-1\right)u^{\frac{l_1l_3}{N}-2}v^{\frac{l_2l_3}{N}+1}$$
$$+(l-1)\left(\frac{l_1^2l_2l_3}{N}u^{\frac{l_1l_3}{N}-1}v^{\frac{l_2l_3}{N}}+l_1l_3\left(\frac{l_1l_3}{N}-1\right)u^{\frac{l_1l_3}{N}-2}v^{\frac{l_2l_3}{N}}+\frac{l_1l_2l_3^2}{N}u^{\frac{l_1l_3}{N}-1}v^{\frac{l_2l_3}{N}-1}\right),$$

$$v_0^{-\frac{2}{N}}\frac{\partial S}{\partial v}=-\frac{l_1l_2l_3}{2}\left(\left(\frac{l_2l_3}{N}-1\right)u^{\frac{l_1l_3}{N}-1}v^{\frac{l_2l_3}{N}-2}+\left(\frac{l_2l_3}{N}-1\right)u^{\frac{l_1l_3}{N}+1}v^{\frac{l_2l_3}{N}-2}\right)$$

$$- \frac{l_1 l_2 l_3}{2}\left(\frac{l_2 l_3}{N}+1\right) u^{\frac{l_1 l_3}{N}-1} v^{\frac{l_2 l_3}{N}}$$

$$+ (l-1)\left(\frac{l_1 l_2^2 l_3}{N} u^{\frac{l_1 l_3}{N}} v^{\frac{l_2 l_3}{N}-1} + \frac{l_1 l_2 l_3^2}{N} u^{\frac{l_1 l_3}{N}-1} v^{\frac{l_2 l_3}{N}-1} + l_2 l_3 \left(\frac{l_2 l_3}{N}-1\right) u^{\frac{l_1 l_3}{N}} v^{\frac{l_2 l_3}{N}-2}\right),$$

thus

$$d_1 := v_0^{-\frac{2}{N}} \frac{N}{l_1 l_3} u^{-\frac{l_1 l_3}{N}+2} v^{-\frac{l_2 l_3}{N}+1} \frac{\partial S}{\partial u} = - \frac{l_1 l_2 l_3}{2}\left(-\left(\frac{N}{l_1 l_3}-1\right)+\left(\frac{N}{l_1 l_3}+1\right)u^2-\left(\frac{N}{l_1 l_3}-1\right)v^2\right)$$

$$+ (l-1)\left(l_1 l_2 uv - l_1 l_3\left(\frac{N}{l_1 l_3}-1\right)v + l_2 l_3 u\right),$$

$$d_2 := v_0^{-\frac{2}{N}} \frac{N}{l_2 l_3} u^{-\frac{l_1 l_3}{N}+1} v^{-\frac{l_2 l_3}{N}+2} \frac{\partial S}{\partial v} = - \frac{l_1 l_2 l_3}{2}\left(-\left(\frac{N}{l_2 l_3}-1\right)-\left(\frac{N}{l_2 l_3}-1\right)u^2+\left(\frac{N}{l_2 l_3}+1\right)v^2\right)$$

$$+ (l-1)\left(l_1 l_2 uv + l_1 l_3 v - l_2 l_3\left(\frac{N}{l_2 l_3}-1\right)u\right).$$

If A is an Einstein metric then, by Proposition 4.2.2, $\dfrac{\partial S}{\partial u} = \dfrac{\partial S}{\partial v} = 0$, so $d_1 = d_2 = 0$ and

$$\left(\frac{N}{l_2 l_3}+1\right) d_1 + \left(\frac{N}{l_1 l_3}-1\right) d_2 = 0 \iff C_1 u^2 + C_2 uv + C_3 v + C_4 u + C_5 = 0, \tag{4.2.5}$$

where

$$C_1 = -N(l_1+l_2),\ C_2 = \frac{(l-1)N(l_1+l_2)}{l_3},\ C_3 = -\frac{(l-1)N(l_1+l_3)}{l_3},$$

$$C_4 = \frac{(l-1)N(l_3-l_2)}{l_3},\ C_5 = \frac{N l_2 (l_1+l_3)}{l_3}.$$

Now, we will show that $u \neq -\frac{C_3}{C_2}$. In fact if $u = -\frac{C_3}{C_2}$ then

$$C_1 u^2 + C_2 uv + C_3 v + C_4 u + C_5 = 0 \Longrightarrow \frac{C_1 C_3^2}{C_2^2} - \frac{C_4 C_3}{C_2} + C_5 = 0$$

$$\Longrightarrow \frac{C_1 C_3^2 - C_4 C_3 C_2 + C_5 C_2^2}{C_2^2} = 0$$

$$\Longrightarrow \frac{2N^3 (l-1)^2 (l_1+l_3)(l_1+l_2)(l_2-l_3)}{l_3^3 C_2^2} = 0$$

$$\Longrightarrow l_2 = l_3 = 0,$$

which is absurd. This fact allows us to isolate v in (4.2.5), obtaining

$$v = \frac{-C_1u^2 - C_4u - C_5}{C_2u + C_3}, \tag{4.2.6}$$

thus

$$\begin{aligned}
0 = (C_2u + C_3)^2 d_1 = &- \frac{l_2^2(l_1 + l_3)}{2}(C_2u + C_3)^2 - \frac{l_2(N + l_1l_3)}{2}u^2(C_2u + C_3)^2 \\
&+ \frac{l_2^2(l_1 + l_3)}{2}(C_1u^2 + C_4u + C_5)^2 \\
&- (l-1)l_1l_2u(C_2u + C_3)(C_1u^2 + C_4u + C_5) \\
&+ (l-1)l_2(l_1 + l_3)(C_2u + C_3)(C_1u^2 + C_4u + C_5) \\
&+ (l-1)l_2l_3u(C_2u + C_3)^2 \\
=:& f(u),
\end{aligned}$$

where f is a fourth-degree polynomial with real coefficients. Since f has at most four positive roots and v is determined by u (because of (4.2.6)), then we have at most four possibilities for (u, v). i.e., we have at most four invariant Einstein metrics on $SO(l+1)/S(O(l_1)\times O(l_2)\times O(l_3))$ up to homotheties.

Case 2. $l_2 = l_3 =: m$.

In this case, we use the basis (4.2.4) and formula (4.0.2) to obtain the Ricci components

$$r_1 = \frac{m}{2}\left(\frac{\mu_{21}}{\mu_{31}\mu_{32}} - \frac{\mu_{31}}{\mu_{21}\mu_{32}} - \frac{\mu_{32}}{\mu_{21}\mu_{31}}\right) + \frac{l-1}{\mu_{21}},$$

$$r_2 = \frac{m}{2}\left(\frac{\mu_{31}}{\mu_{21}\mu_{32}} - \frac{\mu_{32}}{\mu_{21}\mu_{31}} - \frac{\mu_{21}}{\mu_{31}\mu_{32}}\right) + \frac{l-1}{\mu_{31}},$$

$$r_3 = \frac{l_1}{2}\left(\frac{\mu_{32}}{\mu_{21}\mu_{31}} - \frac{\mu_{21}}{\mu_{31}\mu_{32}} - \frac{\mu_{31}}{\mu_{21}\mu_{32}}\right) + \frac{l-1}{\mu_{32}}.$$

Since every invariant metric which is homothetic to an Einstein metric is also an Einstein metric, we can assume that $\mu_{32} = 1$. The solutions in $\mathbb{C}\times\mathbb{C}$ of the system

$$\begin{cases} r_1 - r_2 = 0 \\ r_2 - r_3 = 0 \end{cases}$$

are given by

$$\begin{cases} \mu_{21} = \dfrac{a_1 + \sqrt{\Delta_1}}{4(m-1)} \\ \\ \mu_{31} = \dfrac{a_1 + \sqrt{\Delta_1}}{4(m-1)} \end{cases} \quad (1)$$

$$\begin{cases} \mu_{21} = \dfrac{a_1 - \sqrt{\Delta_1}}{4(m-1)} \\ \\ \mu_{31} = \dfrac{a_1 - \sqrt{\Delta_1}}{4(m-1)} \end{cases} \quad (2)$$

$$\begin{cases} \mu_{21} = \dfrac{a_2 + m\sqrt{\Delta_2}}{2m^2(m+l_1-1)} \\ \\ \mu_{31} = \dfrac{a_2 - m\sqrt{\Delta_2}}{2m^2(m+l_1-1)} \end{cases} \quad (3)$$

$$\begin{cases} \mu_{21} = \dfrac{a_2 - m\sqrt{\Delta_2}}{2m^2(m+l_1-1)} \\ \\ \mu_{31} = \dfrac{a_2 + m\sqrt{\Delta_2}}{2m^2(m+l_1-1)} \end{cases} \quad (4),$$

where

$$a_1 = 2m + l_1 - 2,\ a_2 = m(m+l_1-1)(2m+l_1-2),\ \Delta_1 = l_1^2 - 4(m-1),$$
$$\Delta_2 = (m+l_1-1)(-l_1^2 + l_1(m-2)^2 + m^3 - 4m^2 + 8m - 4).$$

Suppose that $m \geq 3$. Since l_1 is a positive integer, we have that $\Delta_1 \geq 0$ if and only if $l_1 \geq 2\sqrt{m-1}$ and $\Delta_2 \geq 0$ if and only if

$$-l_1^2 + l_1(m-2)^2 + m^3 - 4m^2 + 8m - 4 \geq 0 \Longleftrightarrow 1 \leq l_1 \leq \frac{(m-2)^2 + m\sqrt{m^2-4m+8}}{2}.$$

Also, we note that $a_1 > \sqrt{\Delta_1}$ and $a_2 > m\sqrt{\Delta_2}$ when $\Delta_1, \Delta_2 \geq 0$, in fact

$$a_1 > \sqrt{\Delta_1} \Longleftrightarrow 2m + l_1 - 2 > \sqrt{l_1^2 + 4(1-m)}$$

$$\Longleftrightarrow (2m + l_1 - 2)^2 > l_1^2 + 4(1-m)$$

$$\Longleftrightarrow l_1^2 + 4(m-1)l_1 + 4(m-1)^2 > l_1^2 + 4(1-m)$$

$$\iff 4(m-1)(l_1+1)+4(m-1)^2>0$$

which holds for $m \geq 3$, and

$$a_2 > m\sqrt{\Delta_2} \iff \sqrt{m+l_1-1}(2m+l_1-2) > \sqrt{-l_1^2+l_1(m-2)^2+m^3-4m^2+8m-4}$$

$$\iff (m+l_1-1)(2m+l_1-2)^2 > -l_1^2+l_1(m-2)^2+m^3-4m^2+8m-4$$

$$\iff l_1^3+(5m-4)l_1^2+(7m^2-12m+4)l_1+(3m^3-8m^2+4m)>0$$

which is true since $5m-4,\ 7m^2-12m+4,\ 3m^3-8m^2+4m>0$ when $m \geq 3$. Observe that $2\sqrt{m-1} < \dfrac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$, so we have:

• If $1 \leq l_1 < 2\sqrt{m-1}$, then solutions (1), (2) are complex and solutions (3), (4) are positive, thus, only (3), (4) are invariant Einstein metrics.

• If $l_1 = 2\sqrt{m-1}$, then (1), (2), (3), (4) are all positive solutions and (1) = (2), so we have three invariant Einstein metrics.

• If $2\sqrt{m-1} < l_1 < \dfrac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$, then (1), (2), (3), (4) are positive distinct solutions and we have four invariant Einstein metrics.

• If $l_1 = \dfrac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$, then (1), (2), (3), (4) are all positive solutions and (3) = (4), so we have three invariant Einstein metrics.

• If $l_1 > \dfrac{(m-2)^2+m\sqrt{m^2-4m+8}}{2}$, then (1), (2) are positive and (3), (4) are positive, therefore, only (1), (2) are invariant Einstein metrics. □

4.2.3 $(SO(4)\times SO(5))/SO(4)$

Let $\mathfrak{g} = B_4$ and consider the flag given by $\Theta = \{\alpha_1,\alpha_2,\alpha_3\}$, then $\mathbb{F}_\Theta \overset{\text{diff.}}{\approx} (SO(4)\times SO(5))/SO(4)$. We fix the $\mathrm{Ad}(SO(4)\times SO(5))-$invariant inner product $(\cdot,\cdot)$ defined in (2.1.2) and the matrices w_{ij}, u_{ij}, v_j defined in (1.2.2). The Lie algebra of $K_{\{\alpha_1,\alpha_2,\alpha_3\}} = SO(4)$ is given by $\mathfrak{k}_{\{\alpha_1,\alpha_2,\alpha_3\}} = \mathrm{span}\{w_{st} : 1 \leq t < s \leq 4\}$ and $\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}} = V_1 \oplus T_1 \oplus T_2$, where $V_1 = \mathrm{span}\{v_1,v_2,v_3,v_4\}$, $T_1 = \mathrm{span}\{u_{21}+u_{43}, u_{31}-u_{42}, u_{41}+u_{32}\}$, $T_2 = \mathrm{span}\{u_{21}-u_{43}, u_{31}+u_{42}, u_{41}-u_{32}\}$ are not equivalent. Every invariant metric A with respect to $(\cdot,\cdot)$ has the form

$$A|_{V_1} = \mu I_{V_1},\ A|_{T_1} = \gamma_1 I_{T_1},\ A|_{T_2} = \gamma_2 I_{T_2} \tag{4.2.7}$$

Proposition 4.2.4. *Let A be an invariant metric on $(SO(4)\times SO(5))/SO(4)$ as in (4.2.7). Then A is an Einstein metric if and only if $\gamma_1 = \gamma_2 =: \gamma$ and $\mu = \frac{\gamma}{2}$ or $\mu = \gamma$.*

Proof. We consider the $A-$orthonormal basis

$$X_1 = \frac{u_{21}+u_{43}}{\sqrt{2\gamma_1}},\ X_2 = \frac{u_{31}-u_{42}}{\sqrt{2\gamma_1}},\ X_3 = \frac{u_{41}+u_{32}}{\sqrt{2\gamma_1}}$$

$$Y_1 = \frac{u_{43}-u_{21}}{\sqrt{2\gamma_2}},\ Y_2 = \frac{u_{31}+u_{42}}{\sqrt{2\gamma_2}},\ Y_3 = \frac{u_{41}-u_{32}}{\sqrt{2\gamma_1}},\ Z_j = \frac{v_j}{\sqrt{\mu}},\ j=1,2,3,4.$$

Then

$$[Z_1,Z_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \sqrt{\frac{\gamma_1}{2\mu^2}}X_1 - \sqrt{\frac{\gamma_2}{2\mu^2}}Y_1,\ [Z_1,Z_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \sqrt{\frac{\gamma_1}{2\mu^2}}X_2 + \sqrt{\frac{\gamma_2}{2\mu^2}}Y_2,$$

$$[Z_1,Z_4]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \sqrt{\frac{\gamma_1}{2\mu^2}}X_3 + \sqrt{\frac{\gamma_2}{2\mu^2}}Y_3,\ [Z_2,Z_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \sqrt{\frac{\gamma_1}{2\mu^2}}X_3 - \sqrt{\frac{\gamma_2}{2\mu^2}}Y_3,$$

$$[Z_2,Z_4]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\sqrt{\frac{\gamma_1}{2\mu^2}}X_2 + \sqrt{\frac{\gamma_2}{2\mu^2}}Y_2,\ [Z_3,Z_4]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \sqrt{\frac{\gamma_1}{2\mu^2}}X_1 + \sqrt{\frac{\gamma_2}{2\mu^2}}Y_1,$$

$$[Z_1,X_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_2}{\sqrt{2\gamma_1}},\ [Z_1,X_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_3}{\sqrt{2\gamma_1}},\ [Z_1,X_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_4}{\sqrt{2\gamma_1}},$$

$$[Z_1,Y_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_2}{\sqrt{2\gamma_2}},\ [Z_1,Y_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_3}{\sqrt{2\gamma_2}},\ [Z_1,Y_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_4}{\sqrt{2\gamma_2}},$$

$$[Z_2,X_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_1}{\sqrt{2\gamma_1}},\ [Z_2,X_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_4}{\sqrt{2\gamma_1}},\ [Z_2,X_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_3}{\sqrt{2\gamma_1}},$$

$$[Z_2,Y_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_1}{\sqrt{2\gamma_2}},\ [Z_2,Y_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_4}{\sqrt{2\gamma_2}},\ [Z_2,Y_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_3}{\sqrt{2\gamma_2}},$$

$$[Z_3,X_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_4}{\sqrt{2\gamma_1}},\ [Z_3,X_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_1}{\sqrt{2\gamma_1}},\ [Z_3,X_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_2}{\sqrt{2\gamma_1}},$$

$$[Z_3,Y_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_4}{\sqrt{2\gamma_2}},\ [Z_3,Y_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_1}{\sqrt{2\gamma_2}},\ [Z_3,Y_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_2}{\sqrt{2\gamma_2}},$$

$$[Z_4,X_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_3}{\sqrt{2\gamma_1}},\ [Z_4,X_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = -\frac{Z_2}{\sqrt{2\gamma_1}},\ [Z_4,X_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_1}{\sqrt{2\gamma_1}},$$

$$[Z_4,Y_1]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_3}{\sqrt{2\gamma_2}},\ [Z_4,Y_2]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_2}{\sqrt{2\gamma_2}},\ [Z_4,Y_3]_{\mathfrak{m}_{\{\alpha_1,\alpha_2,\alpha_3\}}} = \frac{Z_1}{\sqrt{2\gamma_2}}.$$

The Ricci components are given by

$$\begin{aligned}
\mathrm{Ric}(Z_j, Z_j) &= -\frac{3\gamma_1}{4\mu^2} - \frac{3\gamma_2}{4\mu^2} + \frac{6}{\mu},\ j = 1,2,3,4,\\
\mathrm{Ric}(X_j, X_j) &= \frac{4}{\gamma_1} + \frac{\gamma_1}{2\mu^2},\ j = 1,2,3,\\
\mathrm{Ric}(Y_j, Y_j) &= \frac{4}{\gamma_2} + \frac{\gamma_2}{2\mu^2},\ j = 1,2,3,
\end{aligned}$$

therefore, if A is an Einstein metric then

$$\begin{aligned}
\frac{4}{\gamma_1} + \frac{\gamma_1}{2\mu^2} = \frac{4}{\gamma_2} + \frac{\gamma_2}{2\mu^2} &\Longleftrightarrow 8\mu^2\gamma_2 + \gamma_1^2\gamma_2 = 8\mu^2\gamma_1 + \gamma_1\gamma_2^2\\
&\Longleftrightarrow 8\mu^2(\gamma_2 - \gamma_1) - \gamma_1\gamma_2(\gamma_2 - \gamma_1) = 0\\
&\Longleftrightarrow (\gamma_2 - \gamma_1)(8\mu^2 - \gamma_1\gamma_2) = 0\\
&\Longleftrightarrow \gamma_1 = \gamma_2 =: \gamma \text{ or } \mu = \sqrt{\frac{\gamma_1\gamma_2}{8}}.
\end{aligned}$$

We study the two cases:

Case 1. $\mu = \sqrt{\frac{\gamma_1\gamma_2}{8}}$.

In this case $\mathrm{Ric}(Z_j, Z_j) = -\frac{3\gamma_1}{4\mu^2} - \frac{3\gamma_2}{4\mu^2} + \frac{6}{\mu} = \frac{12\sqrt{2}}{\sqrt{\gamma_1\gamma_2}} - \frac{6}{\gamma_1} - \frac{6}{\gamma_2}$, $\mathrm{Ric}(X_j, X_j) = \mathrm{Ric}(Y_j, Y_j) = \frac{4}{\gamma_1} + \frac{4}{\gamma_2}$ so

$$\begin{aligned}
\frac{12\sqrt{2}}{\sqrt{\gamma_1\gamma_2}} - \frac{6}{\gamma_1} - \frac{6}{\gamma_2} = \frac{4}{\gamma_1} + \frac{4}{\gamma_2} &\Longleftrightarrow \frac{12\sqrt{2\gamma_1\gamma_2} - 10\gamma_1 - 10\gamma_2}{\gamma_1\gamma_2} = 0\\
&\Longleftrightarrow 5\gamma_1 - 6\sqrt{2\gamma_1\gamma_2} + 5\gamma_2 = 0\\
&\Longleftrightarrow 5(\sqrt{\gamma_1} - \sqrt{\gamma_2})^2 + (10 - 6\sqrt{2})\sqrt{\gamma_1\gamma_2} = 0\\
&\Longrightarrow \gamma_1 = \gamma_2 = 0,
\end{aligned}$$

which contradicts that $\gamma_1, \gamma_2 > 0$. Hence, there is no Einstein metrics satisfying $\mu = \sqrt{\frac{\gamma_1\gamma_2}{8}}$.

Case 2. $\gamma_1 = \gamma_2 =: \gamma$.

In this case $\mathrm{Ric}(Z_j, Z_j) = \dfrac{6}{\mu} - \dfrac{3\gamma}{2\mu^2}$ and $\mathrm{Ric}(X_j, X_j) = \mathrm{Ric}(Y_j, Y_j) = \dfrac{4}{\gamma} + \dfrac{\gamma}{2\mu^2}$. Thus

$$\begin{aligned}
\frac{6}{\mu} - \frac{3\gamma}{2\mu^2} = \frac{4}{\gamma} + \frac{\gamma}{2\mu^2} &\iff 12\mu\gamma - 3\gamma^2 = 8\mu^2 + \gamma^2 \\
&\iff \gamma^2 - 3\mu\gamma + 2\mu^2 = 0 \\
&\iff (\gamma - 2\mu)(\gamma - \mu) = 0 \\
&\iff \mu = \frac{\gamma}{2} \text{ or } \gamma = \mu,
\end{aligned}$$

as we wanted to prove. □

4.2.4 $(SO(l) \times SO(l+1))/(SO(d) \times SO(l-d) \times SO(l-d+1)),\ l \geq 3,\ 2 \leq d \leq l-1$

Let us consider $\mathfrak{g} = B_l$, $l \geq 3$. Given $d \in \{2, ..., l-1\}$, the flag manifold $\mathbb{F}_{\Sigma-\{\alpha_d\}}$ associated to $\Theta = \Sigma - \{\alpha_d\}$ is diffeomorphic to the homogeneous space

$$(SO(l) \times SO(l+1))/(SO(d) \times SO(l-d) \times SO(l-d+1))$$

In this case, the isotropy representation decomposes into the submodules $U_1 = \mathrm{span}\{u_{st} : 1 \leq t < s \leq d\}$, $(V_1)_1 = \mathrm{span}\{w_{st} - u_{st} : d+1 \leq s \leq l,\ 1 \leq t \leq d\}$ and $(V_1)_2 = \mathrm{span}\{v_1, ..., v_d\} \cup \{w_{st} + u_{st} : d+1 \leq s \leq l,\ 1 \leq t \leq d\}$, where the matrices w_{ij}, u_{ij}, v_j are defined in (1.2.2). These subspaces are not equivalent. According to Proposition 3.2.1, every invariant metric A with respect to the inner product (1.2.1) has the form

$$A|_{U_1} = \gamma I_{U_1},\ A|_{(V_1)_1} = \rho I_{(V_1)_1},\ A|_{(V_1)_2} = \mu I_{(V_1)_2}. \tag{4.2.8}$$

Proposition 4.2.5. *a) If $d \neq 2$, $\mathbb{F}_{\Sigma-\{\alpha_d\}}$ has at most four invariant Einstein metrics up to homotheties. In this case, a necessary condition for $\mathbb{F}_{\Sigma-\{\alpha_d\}}$ to have an Einstein metric is that the positive integers l, d satisfy*

$$l^2(l-2)^2 - 2(d-1)^2(d-2)(2l-d) > 0. \tag{4.2.9}$$

b) If $d = 2$, $\mathbb{F}_{\Sigma-\{\alpha_d\}}$ has at most three invariant Einstein metrics up to homotheties.

Proof. Let A be an invariant metric on $\mathbb{F}_{\Sigma-\{\alpha_d\}}$ as in (4.2.8) and consider the $A-$orthonormal basis

$$Y_{st} = \frac{u_{st}}{\sqrt{\gamma}},\ 1 \leq t < s \leq d,\ F_{st} = \frac{w_{st} - u_{st}}{\sqrt{2\rho}},\ 1 \leq t \leq d,\ d+1 \leq s \leq l,$$

$$Z_j = \frac{v_j}{\sqrt{\mu}},\ 1 \leq j \leq d,\ G_{st} = \frac{w_{st} + u_{st}}{\sqrt{2\mu}},\ 1 \leq t \leq d,\ d+1 \leq s \leq l.$$

The non-zero bracket relations of theses vectors are given by

$$[Y_{st}, F_{is}]_{\mathfrak{m}_\Theta} = \frac{F_{it}}{\sqrt{\gamma}}, \qquad [Y_{st}, F_{it}]_{\mathfrak{m}_\Theta} = -\frac{F_{is}}{\sqrt{\gamma}}, \quad [Y_{st}, G_{is}]_{\mathfrak{m}_\Theta} = -\frac{G_{it}}{\sqrt{\gamma}}, \quad [Y_{st}, G_{it}]_{\mathfrak{m}_\Theta} = \frac{G_{is}}{\sqrt{\gamma}}$$

$$[Y_{st}, Z_s]_{\mathfrak{m}_\Theta} = -\frac{Z_t}{\sqrt{\gamma}}, \qquad [Y_{st}, Z_t]_{\mathfrak{m}_\Theta} = \frac{Z_s}{\sqrt{\gamma}}, \qquad [F_{st}, F_{sj}] = \frac{\sqrt{\gamma}}{\rho}Y_{sj}, \quad [G_{st}, G_{sj}] = \frac{\sqrt{\gamma}}{\mu}Y_{sj},$$

$$[Z_s, Z_j]_{\mathfrak{m}_\Theta} = -\frac{\sqrt{\gamma}}{\mu}Y_{sj}, \quad \text{where } Y_{sj} = -Y_{js} \text{ if } \quad s \leq j.$$

By (4.2.2) we have that

$$S(A) = \frac{d(d-1)(d-2)}{\gamma} + d(l-d)\left(\frac{2(l-2)}{\rho} - \frac{(d-1)\gamma}{2\rho^2}\right) + d(l-d+1)\left(\frac{2(l-1)}{\mu} - \frac{(d-1)\gamma}{2\mu^2}\right).$$

Suppose that A has volume 1 and let $u = \frac{\rho}{\gamma}$, $v = \frac{\mu}{\gamma}$, then

$$\frac{S(A)}{v_0^{\frac{2}{N}}} = \frac{S(u,v)}{v_0^{\frac{2}{N}}} = d(d-1)(d-2)u^{\frac{d(l-d)}{N}}v^{\frac{d(l-d+1)}{N}} + 2d(l-d)(l-2)u^{\frac{d(l-d)}{N}-1}v^{\frac{d(l-d+1)}{N}}$$

$$- \frac{d(l-d)(d-1)}{2}u^{\frac{d(l-d)}{N}-2}v^{\frac{d(l-d+1)}{N}} + 2d(l-d+1)(l-1)u^{\frac{d(l-d)}{N}}v^{\frac{d(l-d+1)}{N}-1}$$

$$- \frac{d(l-d+1)(d-1)}{2}u^{\frac{d(l-d)}{N}}v^{\frac{d(l-d+1)}{N}-2},$$

where $N = \frac{d(d-1)}{2} + d(l-d) + d(l-d+1)$ and $v_0 = Vol(\mathbb{F}_\Theta, (\cdot,\cdot)|_{\mathfrak{m}_\Theta \times \mathfrak{m}_\Theta})$. The partial derivatives of S satisfy

$$f_1 := \frac{v_0^{-\frac{2}{N}}Nu^{-\frac{d(l-d)}{N}+3}v^{-\frac{d(l-d+1)}{N}+2}}{d^2(l-d)}\frac{\partial S}{\partial u} = (d-1)(d-2)u^2v^2 - 2(l-d)(l-2)\left(\frac{N}{d(l-d)} - 1\right)uv^2$$

$$+ \frac{(l-d)(d-1)}{2}\left(\frac{2N}{d(l-d)} - 1\right)v^2$$

$$+ 2(l-d+1)(l-1)u^2v - \frac{(l-d+1)(d-1)}{2}u^2,$$

$$f_2 := \frac{v_0^{-\frac{2}{N}}Nu^{-\frac{d(l-d)}{N}+2}v^{-\frac{d(l-d+1)}{N}+3}}{d^2(l-d+1)}\frac{\partial S}{\partial v} = (d-1)(d-2)u^2v^2 + 2(l-d)(l-2)uv^2$$

$$- \frac{(l-d)(d-1)}{2}v^2$$

$$- 2(l-d+1)(l-1)\left(\frac{N}{d(l-d+1)} - 1\right)u^2v$$

$$+\frac{(l-d+1)(d-1)}{2}\left(\frac{2N}{d(l-d+1)}-1\right)u^2.$$

If A is an Einstein metric, then $f_1 = f_2 = 0$ and, therefore,

$$\left(\frac{2N}{d(l-d+1)}-1\right)f_1 + f_2 = 0 \Longleftrightarrow C_1u^2v + C_2uv + C_3v + C_4u^2 = 0$$

where

$$C_1 = \frac{2N(d-1)(d-2)}{d(l-d+1)},\ C_2 = -\frac{2N(l-2)l}{d(l-d+1)},\ C_3 = \frac{N(d-1)(2l-d)}{d(l-d+1)},\ C_4 = \frac{2N(l-1)}{d}.$$

If $C_1u^2 + C_2u + C_3 = 0$ then

$$C_1u^2v + C_2uv + C_3v + C_4u^2 = 0 \Longrightarrow C_4u^2 = 0 \Longrightarrow u = 0,$$

a contradiction. Thus $C_1u^2 + C_2u + C_3 \neq 0$ and

$$v = \frac{-C_4u^2}{C_1u^2 + C_2u + C_3}. \tag{4.2.10}$$

Suppose that $d \neq 2$. Since $v > 0$ and $-C_4u^2 < 0$, we have that $C_1u^2 + C_2u + C_3 < 0$, so, the quadratic polynomial $C_1x^2 + C_2x + C_3$ must have two distinct real roots (otherwise, it would always be non-negative since $C_1 > 0$), but this occurs when

$$C_2^2 - 4C_1C_3 > 0 \Longleftrightarrow \frac{4N^2(l-2)^2l^2}{d^2(l-d+1)^2} - \frac{8N^2(d-1)^2(d-2)(2l-d)}{d^2(l-d+1)^2} > 0$$

$$\Longleftrightarrow l^2(l-2)^2 - 2(d-1)^2(d-2)(2l-d) > 0.$$

Now, substituting (4.2.10) in $f_1 = 0$ and multiplying it by $(C_1u^2 + C_2u + C_3)^2$ we obtain

$$\begin{aligned}0 = (C_1u^2 + C_2u + C_3)^2 f_1 =& (d-1)(d-2)C_4^2u^4 - 2(l-d)(l-2)\left(\frac{N}{d(l-d)}-1\right)C_4^2u^3\\ &+ \frac{(l-d)(d-1)}{2}\left(\frac{2N}{d(l-d)}-1\right)C_4^2u^2\\ &+ 2(l-d+1)(l-1)C_4u^2(C_1u^2 + C_2u + C_3) - \frac{(l-d+1)(d-1)}{2}\\ &=: g(u),\end{aligned}$$

where g is a fourth-degree polynomial. Hence, the set of invariant Einstein metrics $(1, u, v)$ of volume 1 is contained in the set

$$\{1\} \times \left\{(u, v) \in (\mathbb{R}^+)^2 : g(u) = 0 \text{ and } v = \frac{-C_4u^2}{C_1u^2 + C_2u + C_3}\right\}$$

which has at most four elements. This concludes the proof of item a). Now, when $d = 2$. Then, item b) follows from the fact that $C_1 = 0$ and, therefore, g is third-degree polynomial. □

4.2.5 $U(l)/(O(d) \times U(l-d)),\ l \geq 3,\ 2 \leq d \leq l-1$

In this section, we consider $\mathfrak{g} = C_l$, $l \geq 3$ and $\Theta = \Sigma - \{\alpha_d\}$, where $2 \leq d \leq l-1$. The associated flag manifold is diffeomorphic to $U(l)/(O(d)\times U(l-d))$. Fix the $\mathrm{Ad}(U(l))-$invariant product (1.3.1) and the basis (1.3.2). The subspace $\mathfrak{m}_\Theta$ is decomposed into the non-equivalent submodules $M_{21} = \mathrm{span}\{w_{st}, u_{st} : 1 \leq t \leq d,\ d+1 \leq s \leq l\}$, $U_1 = \mathrm{span}\{u_{jj} - u_{j+1,j+1} : j = 1, ..., d-1\} \cup \{u_{st} : 1 \leq t < s \leq d\}$, $V_1 = \mathrm{span}\{u_{11} + ... + u_{dd}\}$. Every invariant metric A is given by positive numbers μ_0, μ_1, μ_{21} such that

$$A|_{V_1} = \mu_0 I_{V_1},\ A|_{U_1} = \mu_1 I_{U_1},\ A|_{M_{21}} = \mu_{21} I_{M_{21}} \tag{4.2.11}$$

Proposition 4.2.6. *The flag $U(l)/(O(d) \times U(l-d))$, $l \geq 3$, and $2 \leq d \leq l-1$, has at most two invariant Einstein metrics up to homotheties.*

Proof. Let A be an invariant metric on $U(l)/(O(d) \times U(l-d))$ as in (4.2.11). Consider the $A-$orthonormal basis of $\mathfrak{m}_\Theta$ given by

$$Z_1 = \sqrt{\frac{2}{d\mu_0}}\,(u_{11} + ... + u_{dd})\,,\ T_j = \sqrt{\frac{2j}{(j+1)\mu_1}}\left(\frac{1}{j}(u_{11} + ... + u_{jj}) - u_{j+1,j+1}\right),$$

$$Y_{st} = \frac{u_{st}}{\sqrt{\mu_1}},\ 1 \leq t < s \leq d,\ X_{st} = \frac{w_{st}}{\sqrt{\mu_{21}}},\ Y_{st} = \frac{u_{st}}{\sqrt{\mu_{21}}},\ 1 \leq t \leq d,\ d+1 \leq s \leq l.$$

Then, we have the following bracket relations:

$$[Z_1, X_{st}]_{\mathfrak{m}_\Theta} = -\sqrt{\frac{2}{d\mu_0}}Y_{st},\ [Z_1, Y_{st}]_{\mathfrak{m}_\Theta} = \sqrt{\frac{2}{d\mu_0}}X_{st},\ 1 \leq t \leq d < s \leq l,$$

$$[T_j, X_{st}]_{\mathfrak{m}_\Theta} = -\sqrt{\frac{2}{(j+1)j\mu_1}}Y_{st},\ [T_{t-1}, X_{st}]_{\mathfrak{m}_\Theta} = \sqrt{\frac{2(t-1)}{t\mu_1}}Y_{st},\ 1 \leq t \leq j \leq d < s \leq l,$$

$$[T_j, Y_{st}]_{\mathfrak{m}_\Theta} = \sqrt{\frac{2}{(j+1)j\mu_1}}X_{st},\ [T_{t-1}, Y_{st}]_{\mathfrak{m}_\Theta} = -\sqrt{\frac{2(t-1)}{t\mu_1}}X_{st},\ 1 \leq t \leq j \leq d < s \leq l,$$

$$[Y_{ij}, X_{si}]_{\mathfrak{m}_\Theta} = -\frac{Y_{sj}}{\sqrt{\mu_1}},\ [Y_{ij}, X_{sj}]_{\mathfrak{m}_\Theta} = -\frac{Y_{si}}{\sqrt{\mu_1}},\ 1 \leq j < i \leq d < s \leq l,$$

$$[Y_{ij}, Y_{si}]_{\mathfrak{m}_\Theta} = \frac{X_{sj}}{\sqrt{\mu_1}},\ [Y_{ij}, Y_{sj}]_{\mathfrak{m}_\Theta} = \frac{X_{si}}{\sqrt{\mu_1}},\ 1 \leq j < i \leq d < s \leq l,$$

$$[X_{sj}, Y_{st}]_{\mathfrak{m}_\Theta} = -\sqrt{\frac{\mu_1}{\mu_{21}^2}}Y_{jt},\ 1 \leq j \neq t \leq d < s \leq l \qquad (Y_{jt} = -Y_{tj} \text{ if } j < t),$$

$$[X_{st}, Y_{st}]_{\mathfrak{m}_\Theta} = -\sqrt{\frac{2\mu_0}{d\mu_{21}^2}}Z_1 + \sqrt{\frac{2(t-1)\mu_1}{t\mu_{21}^2}}T_{t-1} - \sum_{j=t}^{d-1}\sqrt{\frac{2\mu_1}{j(j+1)\mu_{21}^2}}T_j,\ 1 \leq t \leq d < s \leq l,$$

where $T_0 = 0$ and $\sum_{j=t}^{d-1} \sqrt{\frac{2\mu_1}{j(j+1)\mu_{21}^2}} T_j = 0$ if $t = d$. Computing the scalar curvature we obtain

$$S(A) = \frac{d(d-1)(d+2)}{\mu_1} - \frac{(l-d)\mu_0}{\mu_{21}^2} - \frac{(l-d)(d-1)(d+2)\mu_1}{2\mu_{21}^2} + \frac{4ld(l-d)}{\mu_{21}}.$$

Assume that A has volume 1, that is,

$$\mu_0 \mu_1^{\frac{(d-1)(d+2)}{2}} \mu_{21}^{2d(l-d)} = \frac{1}{v_0^2},$$

where $v_0 = Vol(U(l)/(O(d) \times U(l-d)), (\cdot,\cdot)|_{\mathfrak{m}_\Theta \times \mathfrak{m}_\Theta})$. Let $u = \frac{\mu_1}{\mu_0}$, $v = \frac{\mu_{21}}{\mu_0}$, then

$$\begin{aligned} v_0^{-\frac{2}{N}} S(A) = v_0^{-\frac{2}{N}} S(u,v) = {} & d(d-1)(d+2) u^{\frac{(d-1)(d+2)}{2N}-1} v^{\frac{2d(l-d)}{N}} - (l-d) u^{\frac{(d-1)(d+2)}{2N}} v^{\frac{2d(l-d)}{N}-2} \\ & - \frac{(l-d)(d-1)(d+2)}{2} u^{\frac{(d-1)(d+2)}{2N}+1} v^{\frac{2d(l-d)}{N}-2} \\ & + 4ld(l-d) u^{\frac{(d-1)(d+2)}{2N}} v^{\frac{2d(l-d)}{N}-1}, \end{aligned}$$

where $N = \frac{(d-1)(d+2)}{2} + 2d(l-d) + 1$ is the dimension of the flag. Therefore,

$$\begin{aligned} g_1 := \frac{2N u^{-\frac{(d-1)(d+2)}{2N}+2} v^{-\frac{2d(l-d)}{N}+2} v_0^{-\frac{2}{N}}}{(d-1)(d+2)} \frac{\partial S}{\partial u} = & - d(d-1)(d+2) \left(\frac{2N}{(d-1)(d+2)} - 1 \right) v^2 \\ & - \frac{(l-d)(d-1)(d+2)}{2} \left(\frac{2N}{(d-1)(d+2)} + 1 \right) u^2 \\ & + 4ld(l-d)uv - (l-d)u, \end{aligned}$$

$$\begin{aligned} g_2 := \frac{N u^{-\frac{(d-1)(d+2)}{2N}+1} v^{-\frac{2d(l-d)}{N}+3} v_0^{-\frac{2}{N}}}{2d(l-d)} \frac{\partial S}{\partial v} = & \, d(d-1)(d+2)v^2 + (l-d) \left(\frac{N}{d(l-d)} - 1 \right) u \\ & + \frac{(l-d)(d-1)(d+2)}{2} \left(\frac{N}{d(l-d)} - 1 \right) u^2 \\ & - 4ld(l-d) \left(\frac{N}{2d(l-d)} - 1 \right) uv. \end{aligned}$$

By Proposition 4.2.2, if A is an Einstein metric then $g_1 = g_2 = 0$, thus

$$g_1 + \left(\frac{2N}{(d-1)(d+2)} - 1 \right) g_2 = 0 \Longleftrightarrow C_1 u + C_2 + C_3 v = 0,$$

where

$$C_1 = \frac{N}{d},\ C_2 = \frac{2N(d(l-d)+1)}{d(d-1)(d+2)},\ C_3 = -\frac{4lN}{(d-1)(d+2)},$$

so we have that

$$v = D_1 u + D_2, \tag{4.2.12}$$

where $D_1 = -\frac{C_1}{C_3}$ and $D_2 = -\frac{C_2}{C_3}$. Substituting (4.2.12) in $g_1 = 0$ we obtain

$$\begin{aligned} 0 = g_1 = &\ -d(d-1)(d+2)\left(\frac{2N}{(d-1)(d+2)} - 1\right)(D_1 u + D_2)^2 - (l-d)u \\ &- \frac{(l-d)(d-1)(d+2)}{2}\left(\frac{2N}{(d-1)(d+2)} + 1\right)u^2 + 4ld(l-d)u(D_1 u + D_2) \\ =: &\ h(u), \end{aligned}$$

where h is a two-degree polynomial. The result follows from the fact that h has at most two positive roots and formula (4.2.12). □

4.2.6 $(SO(l) \times SO(l))/S(O(l-1) \times O(1))$

Let us consider $\mathfrak{g} = D_l$, $l \geq 4$, $\Theta = \{\alpha_1, ..., \alpha_{l-2}\}$, $(\cdot,\cdot)$ as in (1.4.1) and the $(\cdot,\cdot)-$orthonormal basis (1.4.2). In this case, K is diffeomorphic to $SO(l) \times SO(l)$ and K_Θ is diffemorphic to $S(O(l-1) \times O(1))$. The isotropy representation of K_Θ on $\mathfrak{m}_\Theta$ decomposes into the irreducible submodule $U_1 = \text{span}\{u_{ij} : 1 \leq t < s \leq l-1\}$, which is not equivalent to any other submodule, and the equivalent irreducible sumodules $W_{21} = \text{span}\{w_{lj} : 1 \leq j \leq l-1\}$ and $U_{21} = \text{span}\{u_{lj} : 1 \leq j \leq l-1\}$. As a particular case of Propostition 2.4.2 we have that every invariant metric A is given by

$$A\,|_{U_1} = \gamma I_{U_1},\ Aw_{lj} = \lambda_1 w_{lj} + b u_{lj},\ Au_{lj} = b w_{lj} + \lambda_2 u_{lj},\ j = 1, ..., l-1. \tag{4.2.13}$$

for some $\mu, \lambda_1, \lambda_2 > 0$ and $b \in \mathbb{R}$.

Proposition 4.2.7. *Let A be an invariant metric on the flag $\mathbb{F}_{\{\alpha_1,\ldots,\alpha_{l-2}\}}$ written as in (4.2.13). Then A is an Einstein metric if and only if A satisfies one of the following conditions:*

(F1) $b = 0,\ \lambda_1 = \left(1 - \frac{\sqrt{l^2-5l+4}}{2(l-1)}\right)\gamma$ *and* $\lambda_2 = \left(1 + \frac{\sqrt{l^2-5l+4}}{2(l-1)}\right)\gamma$

(F2) $b = 0,\ \lambda_1 = \left(1 + \frac{\sqrt{l^2-5l+4}}{2(l-1)}\right)\gamma$ *and* $\lambda_2 = \left(1 - \frac{\sqrt{l^2-5l+4}}{2(l-1)}\right)\gamma$

(F3) $b > 0,\ b = \lambda_1 = \frac{\lambda_2}{3} = \frac{\gamma}{2}$

(F4) $b > 0,\ b = \lambda_2 = \frac{\lambda_1}{3} = \frac{\gamma}{2}$

(F5) $b < 0,\ -b = \lambda_1 = \frac{\lambda_2}{3} = \frac{\gamma}{2}$

(F6) $b < 0,\ -b = \lambda_2 = \frac{\lambda_1}{3} = \frac{\gamma}{2}$

Proof. The proof is analogous to the proof of Proposition 4.2.1. As before, we consider two cases:

• A diagonal ($b = 0$):

In this case, we have the $A-$orthonormal basis of $\mathfrak{m}_\Theta$ given by the vectors

$$Y_{ij} = \frac{u_{ij}}{\sqrt{\gamma}},\ 1 \leq j < i \leq l-1,\ X_{lj} = \frac{w_{lj}}{\sqrt{\lambda_1}},\ Y_{lj} = \frac{u_{lj}}{\sqrt{\lambda_2}},\ j = 1, ..., l-1, \tag{4.2.14}$$

which satisfy the following bracket relations

$$\begin{aligned}
[Y_{ij}, X_{li}]_{\mathfrak{m}_\Theta} &= -\sqrt{\frac{\lambda_2}{\gamma\lambda_1}} Y_{lj}, \qquad [Y_{ij}, X_{lj}]_{\mathfrak{m}_\Theta} = \sqrt{\frac{\lambda_2}{\gamma\lambda_1}} Y_{li},\\
[Y_{ij}, Y_{li}]_{\mathfrak{m}_\Theta} &= -\sqrt{\frac{\lambda_1}{\gamma\lambda_2}} X_{lj}, \qquad [Y_{ij}, Y_{lj}]_{\mathfrak{m}_\Theta} = \sqrt{\frac{\lambda_1}{\gamma\lambda_2}} X_{li},\\
[X_{lt}, Y_{ls}]_{\mathfrak{m}_\Theta} &= \sqrt{\frac{\gamma}{\lambda_1\lambda_2}} Y_{st}, \qquad \text{for } s \neq t,
\end{aligned}$$

where, $Y_{st} = -Y_{ts}$ if $s < t$. Observe that $[X, Y]$ is $A-$orthogonal to X and Y, for all X, Y in the basis (4.2.14), thus, $Z = \sum U(X, X) = 0$, where the sum extends over the basis (4.2.14). By formula (4.0.2), we obtain that

$$\begin{aligned}
r_0 &:= \mathrm{Ric}(Y_{ij}, Y_{ij}) = \frac{2(l-2)}{\gamma} + \frac{\gamma}{\lambda_1\lambda_2} - \frac{\lambda_1}{\gamma\lambda_2} - \frac{\lambda_2}{\gamma\lambda_1},\ 1 \leq j < i \leq l-1,\\
r_1 &:= \mathrm{Ric}(X_{lj}, X_{lj}) = (l-2)\left(\frac{2}{\lambda_1} + \frac{\lambda_1}{2\gamma\lambda_2} - \frac{\lambda_2}{2\gamma\lambda_1} - \frac{\gamma}{2\lambda_1\lambda_2}\right),\ 1 \leq j \leq l-1,\\
r_2 &:= \mathrm{Ric}(Y_{lj}, Y_{lj}) = (l-2)\left(\frac{2}{\lambda_2} + \frac{\lambda_2}{2\gamma\lambda_1} - \frac{\lambda_1}{2\gamma\lambda_2} - \frac{\gamma}{2\lambda_1\lambda_2}\right),\ 1 \leq j \leq l-1.
\end{aligned}$$

Therefore, A is an Einstein metric if and only if $r_0 = r_1 = r_2$, i.e.,

$$\begin{cases} \lambda_1 = \left(1 - \dfrac{\sqrt{l^2 - 5l + 4}}{2(l-1)}\right)\gamma \\[2ex] \lambda_2 = \left(1 + \dfrac{\sqrt{l^2 - 5l + 4}}{2(l-1)}\right)\gamma \end{cases} \quad \text{or} \quad \begin{cases} \lambda_1 = \left(1 + \dfrac{\sqrt{l^2 - 5l + 4}}{2(l-1)}\right)\gamma \\[2ex] \lambda_2 = \left(1 - \dfrac{\sqrt{l^2 - 5l + 4}}{2(l-1)}\right)\gamma \end{cases}.$$

• A non-diagonal ($b \neq 0$):

The eigenvalues of A are given by

$$\xi_0 = \gamma,\ \xi_1 = \frac{1}{2}\left(\lambda_1 + \lambda_2 - \sqrt{4b^2 + (\lambda_2 - \lambda_1)^2}\right) \text{ and } \xi_2 = \frac{1}{2}\left(\lambda_1 + \lambda_2 + \sqrt{4b^2 + (\lambda_2 - \lambda_1)^2}\right)$$

and satisfy

$$\xi_1 + \xi_2 = \lambda_1 + \lambda_2, \quad \xi_1\xi_2 = \lambda_1\lambda_2 - b^2,$$

$$\begin{aligned} b^2 &= (\xi_2 - \lambda_2)(\xi_2 - \lambda_1) = (\lambda_1 - \xi_1)(\xi_2 - \lambda_1) \\ &= (\lambda_2 - \xi_1)(\lambda_1 - \xi_1) = (\lambda_2 - \xi_1)(\xi_2 - \lambda_2). \end{aligned}$$

Since A is a positive operator, then $\xi_1, \xi_2 > 0$. An $A-$orthonormal basis of $\mathfrak{m}_\Theta$ is given by

$$Y_{ij} = \frac{u_{ij}}{\sqrt{\xi_0}},\ 1 \le j < i \le l-1,$$

$$X_{lj} = \frac{(\xi_1 - \lambda_2)w_{lj} + bu_{lj}}{\sqrt{c_1}},\ Y_{lj} = \frac{(\xi_2 - \lambda_2)w_{lj} + bu_{lj}}{\sqrt{c_2}},\ 1 \le j \le l-1,$$

where $c_1 = \xi_1(\xi_1 - \lambda_2)(\xi_1 - \xi_2)$ and $c_2 = \xi_2(\xi_2 - \lambda_2)(\xi_2 - \xi_1)$. These vectors satisfy the relations

$$[X_{lt}, X_{ls}]_{\mathfrak{m}_\Theta} = \tfrac{2b\sqrt{\xi_0}}{\xi_1(\xi_1-\xi_2)}Y_{st}, \quad [X_{lt}, Y_{ls}]_{\mathfrak{m}_\Theta} = \tfrac{b(\lambda_2-\lambda_1)}{|b|(\xi_1-\xi_2)}\sqrt{\tfrac{\xi_0}{\xi_1\xi_2}}Y_{st}, \quad [Y_{lt}, Y_{ls}]_{\mathfrak{m}_\Theta} = \tfrac{2b\sqrt{\xi_0}}{\xi_2(\xi_2-\xi_1)}Y_{st},\ s \ne t,$$

$$[Y_{ij}, X_{lj}]_{\mathfrak{m}_\Theta} = \frac{b}{\xi_1 - \xi_2}\left(\frac{2}{\sqrt{\xi_0}}X_{li} + \sqrt{\frac{\xi_2}{\xi_0\xi_1}}\left(\frac{\lambda_2 - \lambda_1}{|b|}\right)Y_{li}\right),$$

$$[Y_{ij}, Y_{li}]_{\mathfrak{m}_\Theta} = \frac{b}{\xi_1 - \xi_2}\left(\frac{2}{\sqrt{\xi_0}}Y_{lj} + \sqrt{\frac{\xi_1}{\xi_0\xi_2}}\left(\frac{\lambda_1 - \lambda_2}{|b|}\right)X_{lj}\right),$$

$$[Y_{ij}, Y_{lj}]_{\mathfrak{m}_\Theta} = \frac{b}{\xi_2 - \xi_1}\left(\frac{2}{\sqrt{\xi_0}}Y_{lj} + \sqrt{\frac{\xi_1}{\xi_0\xi_2}}\left(\frac{\lambda_1 - \lambda_2}{|b|}\right)X_{lj}\right),$$

$$[Y_{ij}, X_{li}]_{\mathfrak{m}_\Theta} = \frac{b}{\xi_2 - \xi_1}\left(\frac{2}{\sqrt{\xi_0}}X_{lj} + \sqrt{\frac{\xi_2}{\xi_0\xi_1}}\left(\frac{\lambda_2 - \lambda_1}{|b|}\right)Y_{lj}\right),\ 1 \le j < i \le l-1.$$

We may use formula (4.0.2) to obtain

$$\mathrm{Ric}(X_{lj}, Y_{lj}) = \frac{(l-2)|b|(\lambda_1 - \lambda_2)(\xi_0^2 - 2\xi_1\xi_2)}{(\xi_2 - \xi_1)\xi_0(\xi_1\xi_2)^{\frac{3}{2}}}, \text{ for all } j \in \{1, ..., l-1\}.$$

Thus, if A is an Einstein metric then $\mathrm{Ric}(X_{lj}, Y_{lj}) = cg(X_{lj}, Y_{lj}) = 0$ (for some $c \in \mathbb{R}$), i.e., $\lambda_1 = \lambda_2$ or $\xi_0 = \sqrt{2\xi_1\xi_2}$.

Case 1. $\lambda_1 = \lambda_2$.

When $\lambda_1 = \lambda_2$, the Ricci components are guven by

$$\begin{aligned} r_0 &:= \mathrm{Ric}(Y_{ij}, Y_{ij}) &=& \frac{2(l-3)}{\xi_0} + \frac{\xi_0}{2}\left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right),\ 1 \leq j < i \leq l-1, \\ r_1 &:= \mathrm{Ric}(X_{lj}, X_{lj}) &=& (l-2)\left(\frac{2}{\xi_1} - \frac{\xi_0}{2\xi_1^2}\right),\ 1 \leq j \leq l-1, \\ r_2 &:= \mathrm{Ric}(Y_{lj}, Y_{lj}) &=& (l-2)\left(\frac{2}{\xi_2} - \frac{\xi_0}{2\xi_2^2}\right),\ 1 \leq j \leq l-1. \end{aligned}$$

We shall show that the sytem of equations

$$\begin{cases} r_0 = r_1 \\ r_1 = r_2 \end{cases} \tag{4.2.15}$$

has not positive solutions. In fact,

$$\begin{aligned} r_1 = r_2 &\Longleftrightarrow \frac{2}{\xi_1} - \frac{\xi_0}{2\xi_1^2} = \frac{2}{\xi_2} - \frac{\xi_0}{2\xi_2^2} \\ &\Longleftrightarrow 4\xi_1\xi_2^2 - \xi_0\xi_2^2 = 4\xi_1^2\xi_2 - \xi_0\xi_1^2 \\ &\Longleftrightarrow 4\xi_1\xi_2(\xi_2 - \xi_1) - \xi_0(\xi_2 - \xi_1)(\xi_2 + \xi_1) = 0 \\ &\Longleftrightarrow (\xi_2 - \xi_1)(4\xi_1\xi_2 - \xi_0(\xi_1 + \xi_2)) = 0 \\ &\Longleftrightarrow \xi_1 = \xi_2 \text{ or } \xi_0 = \frac{4\xi_1\xi_2}{\xi_1 + \xi_2}. \end{aligned}$$

Since $b \neq 0$ then $\xi_1 \neq \xi_2$. Assuming $\xi_0 = \dfrac{4\xi_1\xi_2}{\xi_1 + \xi_2}$ we have that

$$r_0 = \frac{(l+1)\xi_1^2 + (l+1)\xi_2^2 + 2(l-3)\xi_1\xi_2}{2\xi_1\xi_2(\xi_1 + \xi_2)} \text{ and } r_1 = r_2 = \frac{4(l-2)\xi_1\xi_2}{2\xi_1\xi_2(\xi_1 + \xi_2)},$$

therefore

$$\begin{aligned} r_0 - r_1 = 0 &\Longleftrightarrow \frac{(l+1)\xi_1^2 + (l+1)\xi_2^2 - 2(l-1)\xi_1\xi_2}{2\xi_1\xi_2(\xi_1 + \xi_2)} = 0 \\ &\Longleftrightarrow (l+1)(\xi_1 - \xi_2)^2 + 4\xi_1\xi_2 = 0 \\ &\Longrightarrow \xi_1 = \xi_2 = 0. \end{aligned}$$

Hence, system (4.2.15) has no solutions ξ_1, ξ_2 with $\xi_1, \xi_2 > 0$.

Case 2. $\xi_0 = \sqrt{2\xi_1\xi_2}$.

In this case we have

$$\begin{aligned}
r_0 &:= \mathrm{Ric}(Y_{ij}, Y_{ij}) &= \frac{2(l-2)}{\xi_0} + \frac{8b^2 - (\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2},\\
r_1 &:= \mathrm{Ric}(X_{lj}, X_{lj}) &= (l-2)\left(\frac{2}{\xi_1} - \frac{2b^2\xi_0}{\xi_1^2(\xi_2-\xi_1)^2} + \frac{(\lambda_2-\lambda_1)^2}{2(\xi_2-\xi_1)^2}\left(\frac{\xi_1^2-\xi_2^2-\xi_0^2}{\xi_0\xi_1\xi_2}\right)\right),\\
r_2 &:= \mathrm{Ric}(Y_{lj}, Y_{lj}) &= (l-2)\left(\frac{2}{\xi_2} - \frac{2b^2\xi_0}{\xi_2^2(\xi_2-\xi_1)^2} + \frac{(\lambda_2-\lambda_1)^2}{2(\xi_2-\xi_1)^2}\left(\frac{\xi_2^2-\xi_1^2-\xi_0^2}{\xi_0\xi_1\xi_2}\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
r_1 - r_2 &= (l-2)\left(2\left(\frac{1}{\xi_1}-\frac{1}{\xi_2}\right) - \frac{2b^2\xi_0}{(\xi_2-\xi_1)^2}\left(\frac{1}{\xi_1^2}-\frac{1}{\xi_2^2}\right) + \frac{(\lambda_2-\lambda_1)^2}{\xi_0(\xi_2-\xi_1)^2}\left(\frac{\xi_1}{\xi_2}-\frac{\xi_2}{\xi_1}\right)\right)\\
&= (l-2)\left(2\left(\frac{\xi_2-\xi_1}{\xi_1\xi_2}\right) - \frac{2b^2\xi_0}{(\xi_2-\xi_1)^2}\left(\frac{\xi_2^2-\xi_1^2}{\xi_1^2\xi_2^2}\right) + \frac{(\lambda_2-\lambda_1)^2}{(\xi_2-\xi_1)^2}\left(\frac{\xi_1^2-\xi_2^2}{\xi_0\xi_1\xi_2}\right)\right)\\
&= (l-2)\left(2\left(\frac{\xi_2-\xi_1}{\xi_1\xi_2}\right) - \frac{4b^2}{(\xi_2-\xi_1)^2}\left(\frac{\xi_2^2-\xi_1^2}{\xi_0\xi_1\xi_2}\right) + \frac{(\lambda_2-\lambda_1)^2}{(\xi_2-\xi_1)^2}\left(\frac{\xi_1^2-\xi_2^2}{\xi_0\xi_1\xi_2}\right)\right)\\
&= (l-2)\left(2\left(\frac{\xi_2-\xi_1}{\xi_1\xi_2}\right) - \frac{\xi_2^2-\xi_1^2}{(\xi_2-\xi_1)^2}\left(\frac{4b^2+(\lambda_2-\lambda_1)^2}{\xi_0\xi_1\xi_2}\right)\right)\\
&= (l-2)\left(2\left(\frac{\xi_2-\xi_1}{\xi_1\xi_2}\right) - \frac{\xi_2^2-\xi_1^2}{\xi_0\xi_1\xi_2}\right)\\
&= \frac{(l-2)(\xi_2-\xi_1)(2\xi_0-(\xi_1+\xi_2))}{\xi_0\xi_1\xi_2}\\
&= \frac{(l-2)(\xi_2-\xi_1)(2\sqrt{2\xi_1\xi_2}-(\xi_1+\xi_2))}{\sqrt{2}(\xi_1\xi_2)^{\frac{3}{2}}}.
\end{aligned}$$

Observe that $\xi_2 - \xi_1 \neq 0$ (since $b \neq 0$), therefore, an Einstein metric A satisfiying $b \neq 0$ and $\xi_0 = \sqrt{2\xi_1\xi_2}$ also satisfies $2\sqrt{2\xi_1\xi_2} = \xi_1 + \xi_2$. In this situation we have

$$\begin{aligned}
r_0 - \frac{r_1+r_2}{2} &= \frac{2(l-2)}{\xi_0} + \frac{8b^2-(\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2}\\
&\quad - \left(\frac{l-2}{2}\right)\left(2\left(\frac{1}{\xi_1}+\frac{1}{\xi_2}\right) - \frac{2b^2\xi_0}{(\xi_2-\xi_1)^2}\left(\frac{1}{\xi_1^2}+\frac{1}{\xi_2^2}\right) - \frac{(\lambda_2-\lambda_1)^2}{(\xi_2-\xi_1)^2}\left(\frac{\xi_0}{\xi_1\xi_2}\right)\right)
\end{aligned}$$

$$= 2(l-2)\left(\frac{1}{\xi_0} - \frac{1}{2\xi_1} - \frac{1}{2\xi_2}\right) + \frac{8b^2 - (\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2} + \frac{2b^2(l-2)}{(\xi_2-\xi_1)^2}\left(\frac{\xi_1^2+\xi_2^2}{\xi_0\xi_1\xi_2}\right)$$

$$+ \frac{(\lambda_2-\lambda_1)^2(l-2)}{(\xi_2-\xi_1)^2}\left(\frac{\xi_0}{2\xi_1\xi_2}\right)$$

$$= \frac{(l-2)(2\xi_1\xi_2 - \xi_0(\xi_1+\xi_2))}{\xi_0\xi_1\xi_2} + \frac{8b^2-(\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2}$$

$$+ \left(\frac{l-2}{(\xi_2-\xi_1)^2}\right)\left(\frac{2b^2(\xi_1^2+\xi_2^2) + (\lambda_2-\lambda_1)^2\xi_1\xi_2}{\xi_0\xi_1\xi_2}\right)$$

$$= \frac{(l-2)(2\xi_1\xi_2 - 2\xi_0^2)}{\xi_0\xi_1\xi_2} + \frac{8b^2-(\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2}$$

$$+ \left(\frac{l-2}{(\xi_2-\xi_1)^2}\right)\left(\frac{2b^2(\xi_1^2+\xi_2^2) + (\xi_2-\xi_1)^2\xi_1\xi_2 - 4b^2\xi_1\xi_2}{\xi_0\xi_1\xi_2}\right)$$

$$= \frac{(l-2)(-2\xi_1\xi_2)}{\xi_0\xi_1\xi_2} + \frac{8b^2-(\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2}$$

$$+ \left(\frac{l-2}{(\xi_2-\xi_1)^2}\right)\left(\frac{2b^2(\xi_2-\xi_1)^2 + (\xi_2-\xi_1)^2\xi_1\xi_2}{\xi_0\xi_1\xi_2}\right)$$

$$= \frac{(l-2)(-2\xi_1\xi_2)}{\xi_0\xi_1\xi_2} + \frac{8b^2-(\xi_2-\xi_1)^2}{\xi_0\xi_1\xi_2} + \frac{2b^2(l-2) + \xi_1\xi_2(l-2)}{\xi_0\xi_1\xi_2}$$

$$= \frac{2b^2(l+2) + (4-l)\xi_1\xi_2 - \xi_1^2 - \xi_2^2}{\sqrt{2}(\xi_1\xi_2)^{\frac{3}{2}}}$$

$$= \frac{2b^2(l+2) - (l+2)\xi_1\xi_2}{\sqrt{2}(\xi_1\xi_2)^{\frac{3}{2}}} \qquad (\text{since } \xi_1+\xi_2 = 2\sqrt{2\xi_1\xi_2} \Longrightarrow \xi_1^2+\xi_2^2 = 6\xi_1\xi_2).$$

Then

$$r_0 - \frac{r_1+r_2}{2} = 0 \Longleftrightarrow \xi_1\xi_2 = 2b^2,$$

by solving $\xi_1+\xi_2 = 2\sqrt{2\xi_1\xi_2}$ and $\xi_1\xi_2 = 2b^2$ for $\xi_1 < \xi_2$, we obtain

$$\begin{cases} \xi_1 = (-2+\sqrt{2})b \\ \xi_2 = (-2-\sqrt{2})b \end{cases}, \ b<0, \qquad \begin{cases} \xi_1 = (2-\sqrt{2})b \\ \xi_2 = (2+\sqrt{2})b \end{cases}, \ b>0,$$

and $\xi_0 = 2|b|$. Hence, an Einstein metric satisfying $\xi_0 = \sqrt{2\xi_1\xi_2}$ must satisfy one of the conditions (F3), (F4), (F5) or (F6). Conversely, it is easy to verify that any invariant metric satisfying one of the conditions (F3), (F4), (F5),(F6) is in fact an Einstein metric. □

For $\Theta = \{\alpha_2, ..., \alpha_{l-1}\}$ or $\{\alpha_2, ..., \alpha_{l-2}, \alpha_l\}$, the isotropy representation of $\mathbb{F}_\Theta$ also decomposes into three irreducible $K_\Theta-$invariant subspaces. Consider the automorphisms ρ and η of $\mathfrak{so}(l) \oplus \mathfrak{so}(l)$ given by

$$\rho(w_{ij}) = w_{l-j+1,l-i+1},\ \rho(u_{ij}) = u_{l-j+1,l-i+1},\ 1 \leq j < i \leq l,$$
$$\eta(w_{ij}) = w_{ij},\ \eta(u_{ij}) = u_{ij},\ 1 \leq j < i \leq l-1,$$
$$\eta(w_{lj}) = u_{lj},\ \eta(u_{lj}) = w_{lj},\ 1 \leq j \leq l-1.$$

We have that

$$\rho(\mathfrak{m}_{\{\alpha_1,...,\alpha_{l-2}\}}) = \mathfrak{m}_{\{\alpha_2,...,\alpha_{l-1}\}},\ \eta(\mathfrak{m}_{\{\alpha_2,...,\alpha_{l-1}\}}) = \mathfrak{m}_{\{\alpha_2,...,\alpha_{l-2},\alpha_l\}},$$

and ρ, η, take an invariant inner product to an invariant inner product. Consequently, the components of the Ricci tensor of invariant metrics for $\{\alpha_2, ..., \alpha_{l-1}\}$ or $\{\alpha_2, ..., \alpha_{l-2}, \alpha_l\}$ are the same as in the case of $\{\alpha_1, ..., \alpha_{l-2}\}$. Therefore, Einstein invariant metrics on $\mathbb{F}_{\{\alpha_2,...,\alpha_{l-1}\}}$ and $\mathbb{F}_{\{\alpha_2,...,\alpha_{l-2},\alpha_l\}}$ have the form $(\rho^{-1})^*g$ and $(\rho^{-1} \circ \eta^{-1})^*g$, respectively, where g is an Einstein invariant metric on $\mathbb{F}_{\{\alpha_1,...,\alpha_{l-2}\}}$.

4.3 Equivalent metrics

In this section, we shall decide which of the Einstein metrics found in the previous sections are equivalent in the sense of the following definition:

Definition 4.3.1. *Let g_1, g_2 be Riemannian metrics on a manifold M. We say that g_1 and g_2 are equivalent if there exists an isometry $F : (M, g_1) \longrightarrow (M, g_2)$.*

As observed in [21], the Einstein constant corresponding to an Einstein Riemannian metric g of volume 1 on a compact manifold M is equal to $S/dim(M)$, where S is the scalar curvature of g. Since any isometry preserves the scalar curvature, then two Einstein metrics of volume 1 with different Einstein constant cannot be equivalent. We denote by v_0 the volume of the flag with respect to the invariant inner product fixed in each case.

Let us consider the flag $(SO(l) \times SO(l+1))/SO(l)$, $l \geq 3$. By Propositions 4.1.3 and 4.2.4, this manifold has two invariant Einstein metrics $A_i = (\mu_i, \gamma_i)$, $i = 1, 2$, where $\mu_1 = \frac{\gamma_1}{2}$ and $\mu_2 = \left(\frac{l}{2l-4}\right)\gamma_2$. When $\gamma_1 = v_0^{-\frac{4}{l(l+1)}} 2^{\frac{2}{l+1}}$ and $\gamma_2 = v_0^{-\frac{4}{l(l+1)}} \left(\frac{2l-4}{l}\right)^{\frac{2}{l+1}}$ we have volume 1 and the corresponding Einstein constants are given by

$$c_1 = \frac{(l-1)v_0^{\frac{4}{l(l+1)}}}{2^{\frac{2}{l+1}-1}} \text{ and } c_2 = \frac{(l-1)v_0^{\frac{4}{l(l+1)}}}{2^{\frac{2}{l+1}-1}} \left(\frac{l^{\frac{2}{l+1}-1}(l+2)}{(l-2)^{\frac{2}{l+1}-1}}\right).$$

So, $c_1 = c_2$ if and only if $\frac{l^{\frac{2}{l+1}-1}(l+2)}{(l-2)^{\frac{2}{l+1}-1}} = 1$ or, equivalently, $\left(\frac{l}{l-2}\right)^{\frac{2}{l+1}} = \frac{l}{(l-2)(l+2)}$, which is not possible since $\frac{l}{(l-2)(l+2)} < 1 < \frac{l}{l-2}$. Hence, A_1, A_2 are not equivalent.

For $SO(4)/S(O(2)\times O(1)\times O(1))$, denote by A and $\tilde{A}$ the the invariant metrics satisfying *(E2)* and *(E1)* respectively (see Proposition 4.2.1), their corresponding volumes are given by $2\sqrt{2}b^{\frac{5}{2}}$ and $\frac{9\mu_0^{\frac{5}{2}}}{16}$, and their corresponding Einstein constants are $\frac{1}{b}$ and $\frac{16}{9\mu_0}$. When $b=2^{-\frac{3}{5}}$ and $\mu_0=\left(\frac{4}{3}\right)^{\frac{4}{5}}$ we have volume 1, but $2^{\frac{3}{5}}\neq\left(\frac{4}{3}\right)^{\frac{6}{5}}$, so *(E1)* and *(E2)* cannot be equivalent. Now, if we consider the diffeomorphisms

$$\psi_i:\mathbb{F}_{\{\alpha_1\}}\longrightarrow\mathbb{F}_{\{\alpha_1\}};\quad \psi_i\left(kK_{\{\alpha_1\}}\right)=s_iks_i^TK_{\{\alpha_1\}},\quad i=3,4,5;$$

where

$$s_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\ s_4=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \text{and } s_5=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

Then it is easy to verify that the invariant metric ψ_i^*A satisfies *(Ei)*. This shows that *(E2), (E3), (E4)* and *(E5)* are equivalent. We can use a similar argument to show that metrics *(F1), (F2)* are equivalent as well as metrics *(F3), (F4), (F5), (F6)* and that *(F1)* is not equivalent to *(F3)* (see Proposition 4.2.7).

Chapter 5

Conclusions and future work

In this thesis, we addressed the problem of studying invariant Riemannian Geometry of real flag manifolds associated to classical Lie algebras. We can see that there exist remarkable differences between real and complex flag manifolds. For example, the set of invariant Riemannian metrics in a complex flag manifold can be identified with $(\mathbb{R}^{+})^{s}$ for some $s \in \mathbb{N}$. As seen in Chapter 2, this does not hold for the real case since the existence of equivalent irreducible submodules of the isotropy representation leads to non-diagonal invariant metrics. Another big difference is the fact that in real flag manifolds we can find a large family of non-trivial examples of g.o spaces (see Chapter 3), in contrast with the complex case, where the only flag manifolds of a simple Lie group which admit an invariant non-normal metric with homogeneous geodesics are the manifolds $SO(2l+1)/U(l)$ and $Sp(l)/(U(1) \cdot Sp(l-1))$.

One of the main contributions of this work is the description of invariant Riemannian metrics in real flag manifolds of classical Lie algebras. This description is important to study several geometric issues. Two of them were treated here: the classification of homogeneous spaces with geodesic orbits and the solution of the Einstein equation.

Another interesting problem is the study of geometric flows on homogeneous spaces: given a homogeneous manifold $M = G/H$ with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of its Lie algebra. We can consider a geometric flow on M of the form

$$\frac{\partial}{\partial t}\gamma(t) = q(\gamma(t)), \tag{5.0.1}$$

where $\{\gamma(t)\}$ is a one-parameter family of tensor fields on M and q assigns to each tensor field another tensor field of the same type. Assuming $\gamma(t)$ is $G-$invariant for all t, equation (5.0.1) becomes equivalent to an ODE for a one-parameter family $\gamma(t)$ of $\mathrm{Ad}(H)-$invariant tensors on $\mathfrak{m}$ of the form

$$\frac{d}{dt}\gamma(t) = q(\gamma(t)). \tag{5.0.2}$$

In the case of a real flag manifold, we can use the parametrization of the invariant Riemannian metrics obtained in Chapter 2 to study the behavior of the solutions of equation (5.0.2) when $\{\gamma(t)\}$ is a family of homogeneous metrics.

There is a list of specific goals that are expected to be achieved in the medium term by using the results and ideas of this work:

- Parametrize the set of invariant Riemannian metrics on real flag manifolds of exceptional Lie groups ($E_6,\ E_7,\ E_8,\ F_4$, and G_2).
- Classify the real flag manifolds of exceptional Lie groups which are g.o. spaces.
- Study the Einstein equation for real flag manifolds of exceptional Lie groups.
- Describe the invariant Riemannian metrics in other families of homogeneous spaces having equivalent isotropy summands.
- Study the Ricci flow equation for a homogeneous spaces whose isotropy representation splits into a low number of irreducible (possibly equivalent) submodules.

Bibliography

[1] D. Alekseevsky and A. Arvanitoyeorgos. “Metrics with homogeneous geodesics on flag manifolds”. In: *Commentationes Mathematicae Universitatis Carolinae* 43.2 (2002), pp. 189–199.

[2] D. Alekseevsky and A. Arvanitoyeorgos. “Riemannian Flag Manifolds with Homogeneous Geodesics”. In: *Transactions of the American Mathematical Society* 359.8 (2007), pp. 3769–3789. ISSN: 00029947.

[3] D. Alekseevsky, I. Dotti, and C. Ferraris. “Homogeneous Ricci positive 5-manifolds”. In: *Pacific Journal of Mathematics* 175.1 (1996).

[4] A. Arvanitoyeorgos. “Geometry of flag manifolds”. In: *International Journal of Geometric Methods in Modern Physics* 3 (2006), pp. 957–974.

[5] A. Arvanitoyeorgos. “New invariant Einstein metrics on generalized flag manifolds”. In: *Transactions of the American Mathematical Society* 337.2 (1993).

[6] A. Arvanitoyeorgos and I. Chrysikos. “Invariant Einstein metrics on flag manifolds with four isotropy summands”. In: *Annals of Global Analysis and Geometry* 37.2 (2010), pp. 185–219.

[7] A. Arvanitoyeorgos and I. Chrysikos. “Invariant Einstein metrics on generalized flag manifolds with two isotropy summands”. In: *Journal of the Australian Mathematical Society* 90.2 (2011), pp. 237–251.

[8] V. del Barco and Luiz A. B. San Martin. “De Rham 2-Cohomology of Real Flag Manifolds”. In: *Symmetry, Integrability and Geometry: Methods and Applications* 15 (2019).

[9] L.A. Besse. *Einstein manifolds. Reprint of the 1987 edition.* Classics in Mathematics, 2008.

[10] C. Böhm and M. Kerr. “Low-Dimensional Homogeneous Einstein Manifolds”. In: *Transactions of the American Mathematical Society* 358.4 (2006), pp. 1455–1468. ISSN: 00029947.

[11] C. Böhm, M. Wang, and W. Ziller. “A variational approach for compact homogeneous Einstein manifolds”. In: *Geometric and Functional Analysis* 14 (2004), pp. 681–733.

[12] N. Cohen, L. Grama, and Caio J. C. Negreiros. “Equigeodesics on flag manifolds”. In: *Houston journal of mathematics* 37 (Apr. 2011), pp. 113–125.

[13] A. P. C. Freitas, V. del Barco, and Luiz A. B. San Martin. “Invariant almost complex structures on real flag manifolds”. In: *Annali di Matematica Pura ed Applicata (1923 -)* 197 (2018), pp. 1821–1844.

[14] Carolyn S. Gordon. “Homogeneous Riemannian Manifolds Whose Geodesies Are Orbits”. In: *Topics in Geometry: In Memory of Joseph D’Atri*. Ed. by Simon Gindikin. Boston, MA: Birkhäuser Boston, 1996, pp. 155–174.

[15] L. Grama and R. M. Martins. “The Ricci flow of left-invariant metrics on full flag manifold $SU(3)/T$ from a dynamical systems point of view”. In: *Bulletin des Sciences Mathématiques* 133.5 (2009), pp. 463–469. ISSN: 0007-4497.

[16] L. Grama and Caio J. C. Negreiros. “Equigeodesics on Generalized Flag Manifolds with Two Isotropy Summands”. In: *Results in Mathematics* 60 (2011). ISSN: 1420-9012.

[17] A. Kaplan. “On the Geometry of Groups of Heisenberg Type”. In: *Bulletin of the London Mathematical Society* 15.1 (1983), pp. 35–42.

[18] M. Kimura. “Homogeneous Einstein Metrics On Certain Kähler C-Spaces”. In: *Recent Topics in Differential and Analytic Geometry* (1990), pp. 303–320.

[19] O. Kowalski and L. Vanhecke. “Riemannian manifolds with homogeneous geodesics”. In: *Bolletino U.M.I.* 7 (5-B) (1991), pp. 189–246.

[20] J. M. Lee. *Riemannian Manifolds: An Introduction to Curvature*. Graduate Texts in Mathematics. Springer New York, 1997. ISBN: 9780387982717.

[21] Y. Muto. “On Einstein metrics”. In: *Journal of Differential Geometry* 9.4 (1974), pp. 521–530.

[22] M. Patrão and Luiz A. B. San Martin. “The Isotropy Representation of a Real Flag Manifold: Split Real Forms”. In: *Indagationes Matematicae* 26 (2015), pp. 547–579.

[23] M. Patrão, Luiz A. B. San Martin; Laércio J. dos Santos, and L. Seco. “Orientability of vector bundles over real flag manifolds”. In: *Topology and its Applications* 159.10 (2012), pp. 2774–2786. ISSN: 0166-8641.

[24] R. do Prado and L. Grama. “Variational aspects of homogeneous geodesics on generalized flag manifolds and applications”. In: *Annals of Global Analysis and Geometry* 55.3 (2019), pp. 451–477. ISSN: 1572-9060.

[25] L. Rabelo and Luiz A. B. San Martin. “Cellular homology of real flag manifolds”. In: *Indagationes Mathematicae* 30.5 (2019), pp. 745–772. ISSN: 0019-3577.

[26] Y. Sakane. “Homogeneous Einstein metrics on flag manifolds”. In: *Lobachevskii Journal of Mathematics* 4 (1999), pp. 71–87.

[27] E. dos Santos and Caio J. C. Negreiros. “Einstein metrics on flag manifolds”. In: *Revista de la Unión Matemática Argentina* 47.2 (2006), pp. 77–84.

[28] N. Souris. “Geodesic orbit metrics in compact homogeneous manifolds with equivalent isotropy submodules”. In: *Transformation Groups* 23 (2018), pp. 1149–1165.

[29] M. Wang and W. Ziller. “Einstein metrics on principal torus bundles”. In: *Journal of Differential Geometry* 31.1 (1990), pp. 215–248.

[30] M. Wang and W. Ziller. “Existence and non-existence of homogeneous Einstein metrics”. In: *Inventiones mathematicae* 84 (1986), pp. 177–194.

[31] M. Wiggerman. “The fundamental group of a real flag manifold”. In: *Indagationes Mathematicae* 9.1 (1998), pp. 141–153. ISSN: 0019-3577.