Alguns Resultados de Existência de Soluções para Equações Elípticas Quasilineares

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Conteúdo

Introdução

Muitos problemas em matemática podem ser postos na forma de equações funcionais do tipo

$$F(u) = 0, \tag{0.0.1}$$

com a possível solução pertencente a uma certa classe de funções admissíveis contida em algum espaço de Banach X.

Na tentativa de resolver a equação (0.0.1), a qual usualmente é não linear, vários métodos foram desenvolvidos: princípios de contração, métodos de ponto fixo, grau de Leray-Schauder [42, 52, 47], métodos de iterações monotônicas [4], método de Galerkin, operadores monotônicos [32, 13, 34], estudo das singularidades de funcionais definidos em espaços de Banach [19], teoria de Morse [52, 11, 16, 50], teorema da função implícita generalizado [5, 42], métodos variacionais, etc. Neste trabalho fazemos uso dos métodos variacionais.

É dito que se emprega o método variacional para resolver o problema (0.0.1) quando este possui a assim chamada estrutura variacional, ou seja, o operador F pode ser visto como a derivada de algum funcional $J : X \longrightarrow \mathbb{R}^N$, isto é,

$$F(u) = \nabla J(u). \tag{0.0.2}$$

Portanto, a equação (0.0.1) reduz-se ao problema de encontrar os pontos críticos do funcional J.

A pré-história dos métodos variacionais, se é que assim podemos chamar, tem início com o princípio do mínimo com Heron de Alexandria (aproximadamente 100 dC). Baseado no princípio aristotélico que diz que *a natureza nada faz de modo mais difícil*, Heron, em sua obra chamada Catóptrica (ou reflexão), provou por um argumento geométrico simples a igualdade dos ângulos de incidência e reflexão, isto é, se um raio de luz deve ir de uma fonte ao olho de um observador passando antes por um espelho, então o caminho mais curto possível é aquele em que os ângulos de incidência e reflexão são iguais.

Este princípio foi mais tarde generalizado por Pierre de Fermat por volta de 1650. Fermat escreveu nove artigos importantes sobre o método de máximos e mínimos. Os dois últimos desta série [24], *The analysis of refractions* e *The synthesis of refractions*, têm como conseqüência a lei da refração da luz, hoje conhecida como lei de Snell. Nestes trabalhos Fermat enuncia o seu princípio: *a natureza opera por meios e modos que são mais fáceis e mais rápidos*, o que na óptica geométrica significa que o caminho seguido por um raio de luz de um ponto A até um ponto

B é aquele que torna mínimo o tempo de percurso entre esses pontos. Ou seja, entre todos os caminhos possíveis, a natureza escolhe aquele que o raio de luz pode percorrer no menor tempo possível. Alguns autores (veja, por exemplo, Goldstine [29]) veêm nos trabalhos de Fermat o início do cálculo das variações, devido ao fato de ser a primeira contribuição real e que certamente serviu de inspiração para a solução do problema da braquistócrona por John Bernoulli em 1696/97 [9]. Como se sabe, Bernoulli substituiu a partícula movendo-se sob a ação da gravidade por um raio de luz passando através de uma série de meios ópticos com densidades diferentes. Fermat antecipou o cálculo diferencial estabelecendo uma condição necessária para o máximo ou mínimo de um polinômio, a qual é equivalente ao anulamento de sua derivada.

Em 1744, em seu livro sobre cálculo das variações, Euler [23], e posteriormente Lagrange com um tratamento mais analítico, fornece uma extensão da condição necessária de Fermat para o extremo de uma função real para o caso de um funcional, as assim chamadas Equações de Euler-Lagrange.

No século passado Dirichlet e Riemann desenvolveram a primeira idéia importante e sistemática de transformar um problema de equações diferenciais em uma questão do cálculo das variações. Surgem assim os métodos diretos do cálculo das variações, os quais consistem em estudar diretamente o funcional sem fazer qualquer uso de sua equação de Euler-Lagrange. Dirichlet e Riemann utilizaram esse procedimento para provar a existência de uma solução para o que hoje chamamos de Problema de Dirichlet para a equação de Laplace em uma região plana limitada,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{0.0.3}$$

e com u coincidindo com uma dada função sobre o bordo da região.

A equação (0.0.3) é a equação de Euler-Lagrange para o funcional:

$$I(u) = \iint_\Omega [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]\, dx\, dy, \tag{0.0.4}$$

o qual assume somente valores não negativos. Possivelmente por esse fato, Dirichlet e Riemann admitiram sem demonstração a existência de um mínimo para esse funcional, o que para a época era tida como um fato natural, obtendo assim a existência de uma solução de (0.0.3). Este argumento foi utilizado por Riemann em seus artigos sobre funções holomorfas, superfícies de Riemann e integrais abelianas. A ele se deve a nomenclatura *Princípio de Dirichlet* para denominar esse método. Mas, em 1870, a validade do Princípio de Dirichlet foi posta em dúvida com o contra-exemplo apresentado por Weierstrass que fazia distinção entre as noções de mínimo e ínfimo. Somente neste século é que o Princípio de Dirichlet foi posto em bases sólidas por Hilbert [30, 31].

No final dos anos vinte, duas teorias tidas como fundamentais, a primeira de Morse [39] e a segunda dos matemáticos russos Ljusternik e Schnirelman [38], marcam o nascimento dos métodos de minimax. Tais métodos estão principalmente relacionados com a existência de pontos críticos de funcionais distintos de mínimos e máximos. A elegância da ferramenta abstrata e a abrangência das aplicações em problemas considerados difíceis para aquela época, como, por exemplo, a existência de geodésicas fechadas em variedades compactas, fizeram de tais métodos uma próspera área de pesquisa. Um importante progresso da teoria de Ljusternik-Schnirelman foi marcado nos anos setenta pelos trabalhos de Browder [15], Krasnoselski [33], Palais [43], Schwartz [53] e Vainberg [60] que estenderam a teoria para variedades de dimensão infinita.

Um novo impulso no uso de métodos variacionais no estudo das equações diferenciais não lineares foi dado por Ambrosetti e Rabinowitz [6] com a formulação do Teorema do Passo da Montanha, o Teorema do Ponto de Sela de Rabinowitz [49] e generalizações envolvendo noção de enlace em dimensão infinita de Benci e Rabinowitz [10].

No presente trabalho estudamos a questão de existência e multiplicidade de soluções para alguns problemas elípticos não lineares fazendo uso do método variacional.

O primeiro capítulo é dedicado ao estudo do problema de Dirichlet para equações elípticas semilineares com a função não linearidade tendo crescimento exponencial. Um típico modelo é o seguinte problema:

$$\begin{cases} -\Delta_N u = \lambda e^{u^\alpha}, & \text{em } \Omega, \\ u = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{0.0.5}$$

sendo Ω um subconjunto limitado do $\mathbb{R}^N$, com fronteira $\partial\Omega$ suave e λ e α são parâmetros reais sujeitos a certas restrições que possibilitam a existência de soluções.

Para o caso em que $\alpha = 1$, o problema (0.0.5) é bem conhecido e estudado por diversos autores. Primeiramente considerado por Liouville [37], para $N = 1$, e posteriormente por Bratu [12], para $N = 2$, e Gelfand [27], para $N \geq 1$. Durante as três últimas décadas esse problema tem sido muito estudado (veja [17, 18, 26] e suas referências). Como observado em [26], o problema (0.0.5) tem a sua importância uma vez que aparece em modelos matemáticos associados a fenômenos astrofísicos e aos problemas de reações de combustão.

No segundo capítulo considera-se uma equação elíptica em $\mathbb{R}^N$, tendo como modelo o seguinte problema:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda a(x)|u|^{q-2}u, \text{ em } \mathbb{R}^N, \\ u \geq 0, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\nabla u|^p \, dx < \infty, \end{cases} \tag{0.0.6}$$

Como observado em [46], o problema (0.0.6), para o caso em que $p = 2$, é motivado pelo estudo de ondas estacionárias do tipo $\Psi(x,t) = \exp(-iEt/h)v(x)$ da equação de Schrödinger não linear em $\mathbb{R}^N$:

$$ih\frac{\partial \Psi}{\partial t} = -\frac{h^2}{2m}\Delta\Psi + V(x)\Psi - \gamma|\Psi|^{q-1}\Psi, \tag{0.0.7}$$

Como é fácil ver, uma função Ψ dessa forma satisfaz a equação (0.0.7) se, e somente se, a função v resolve a equação elíptica

$$-\frac{h^2}{2m}\Delta v + (V(x) - E)v = -|v|^{q-1}v, \tag{0.0.8}$$

que após mudança de variáveis pode ser reescrita como

$$-\Delta v + b(x)v = -|v|^{q-1}v, \tag{0.0.9}$$

na qual $b(x) = 2(V(x) - E)$.

Uma importante diferença entre os dois problemas abordados neste trabalho encontra-se na noção de criticalidade envolvida. Como usualmente acontece, tal noção está intimamente relacionada ao espaço de funções escolhido para a obtenção de soluções e nas relações deste com os espaços L^p. Desse modo, para o problema com crescimento exponencial, a noção de criticalidade é dada pelo valor de α e é motivada pela imersão de Trudinger-Moser, a qual relaciona espaços de Orlicz determinados pela função $\phi(t) = \exp(\beta|t|^{N/(N-1)})$, para qualquer $\beta > 0$, com o espaço L^1. Ao passo que no segundo problema, o significado de criticalidade é dado por p^* e é motivada imersão de Sobolev do espaço $W^{1,p}$ em L^s para s satisfazendo certas relações envolvendo p e N.

Como sabemos, a aplicabilidade do método variacional depende da geometria do funcional associado ao problema e de alguma condição de compacidade, por exemplo, a condição de Palais–Smale. É nessa última que encontramos algumas dificuldades, pois lidamos com, o que se convencionou chamar, problemas com perda de compacidade. Tal perda origina-se da falta de compacidade das imersões. Ao estudar no capítulo 1 problemas com crescimento exponencial crítico, a imersão do espaço $W_0^{1,N}(\Omega)$ no espaço de Orlicz determinado por $\phi(t) = \exp(\alpha_N|t|^{N/N-1})$, com $\alpha_N = N w_{N-1}^{1/N-1}$ e w_{N-1} é o volume da esfera unitária $(N-1)$-dimensional, não é compacta. Para os problemas do capítulo 2, a invariância por translações em $\mathbb{R}^N$ é a típica dificuldade no estudo de problemas elípticos em domínios ilimitados, pois causa perda de compacidade das imersões de Sobolev. Mais ainda, essa dificuldade não é uma particularidade de domínios ilimitados, pois, mesmo a versão local desses problemas apresenta perda de compacidade em vista da imersão de $W_0^{1,p}(\Omega)$ em $L^{p^*}(\Omega)$ não ser compacta.

O que unifica o estudo dos dois problemas é a verificação com auxílio de dois lemas de Lions [35] de um resultado abstrato que estabelece que toda seqüência limitada no espaço de Sobolev apropriado, cuja derivada do funcional nestes pontos converge para zero, possui uma subseqüência que converge fracamente para uma solução do problema. Isto nos permite obter pelo menos uma solução do problema (0.0.5). Para obter uma segunda solução de (0.0.5) ou uma solução não trivial para (0.0.6), assumimos unicidade de solução e empregamos um argumento semelhante ao utilizado por Brezis e Nirenberg no famoso artigo [14].

Finalmente, ainda sobre os capítulos, informamos que são independentes entre si, dado que resolvemos escrevê-los na forma de artigo a fim de serem submetidos à publicação.

Capítulo 1

Liouville-Gelfand type problems for the N-Laplacian on bounded domains of $\mathbb{R}^N$

1.1 Introduction

In this article, we study the existence and multiplicity of nonzero solutions for the following quasilinear elliptic problem

$$(P)_\lambda \qquad \begin{cases} -\Delta_N u = -\text{div}(|\nabla u|^{N-2}\nabla u) = \lambda f(x,u), \text{ in } \Omega, \\ \quad u \geq 0, \text{ in } \Omega, \\ \quad u = 0, \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N (N \geq 2)$ with boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, and the nonlinearity $f(x,s)$ satisfies

(f_1) $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $f(x,0) > 0$, for every $x \in \Omega$,

and the growth condition

$(f)_{\alpha_0}$ There exists $\alpha_0 \geq 0$ such that

$$\lim_{s\to\infty} \frac{|f(x,s)|}{\exp(\alpha s^{N/N-1})} = \begin{cases} 0, \ \forall\ \alpha > \alpha_0, \text{ unif. on } \bar{\Omega}, \\ +\infty, \ \forall\ \alpha < \alpha_0, \text{ unif. on } \bar{\Omega}. \end{cases}$$

In the literature [1, 21, 25], $f(x,s)$ is said to have subcritical or critical growth when $\alpha_0 = 0$ or $\alpha_0 > 0$, respectively. We note that such notion is motivated by Trudinger-Moser estimates [41, 59] which provide

$$\exp(\alpha|u|^{N/N-1}) \in L^1(\Omega), \ \forall\, u \in W_0^{1,N}(\Omega), \ \forall\, \alpha > 0, \tag{1.1.1}$$

and

$$\sup_{\|u\|_{W_0^{1,N}} \leq 1} \int_\Omega \exp(\alpha|u|^{N/N-1})\, dx \leq C(N) \in I\!R, \ \forall\, \alpha \leq \alpha_N = N w_{N-1}^{\frac{1}{N-1}}, \tag{1.1.2}$$

where w_k is the volume of S^k. We also observe that a typical and relevant case to be considered for problem $(P)_\lambda$ is given by $f(x,s) = \exp(\alpha_0 s^{N/N-1})$.

In our first result, we establish the existence of a solution for $(P)_\lambda$ when $\lambda > 0$ is sufficiently small,

Theorem 1.1.1 *Suppose $f(x,s)$ satisfies (f_1) and $(f)_{\alpha_0}$. Then, there exists $\overline{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least one solution for every $\lambda \in (0, \overline{\lambda})$.*

To obtain the existence of a second solution for problem $(P)_\lambda$ in the subcritical case, we assume that $f(x,s)$ satisfies

(f_2) There are constants $\theta > N$ and $R > 0$ such that

$$0 < \theta F(x,s) \leq s f(x,s), \ \forall\, x \in \bar{\Omega}, \ s \geq R.$$

Theorem 1.1.2 *(Second solution: Subcritical case) Suppose $f(x,s)$ satisfies (f_1), (f_2) and $(f)_{\alpha_0}$, with $\alpha_0 = 0$. Then, there exists $\overline{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least two solutions for every $\lambda \in (0, \overline{\lambda})$.*

Note that (f_2) is the version of the famous Ambrosetti-Rabinowitz condition [6] for the N-Laplacian. It implies, in particular, that $f(x,s)/s^N \to \infty$, as $s \to \infty$, uniformly on $\bar{\Omega}$.

In our next result, we provide the existence of two solutions for $(P)_\lambda$ when $f(x,s)$ has critical growth. In that case, we shall need to suppose a stronger version of condition (f_2),

$(\hat{f}_2)$ For every $\theta > N$, there exists $R(\theta) > 0$ such that

$$0 < \theta F(x,s) \leq s f(x,s), \forall\, x \in \bar{\Omega}, \ s \geq R(\theta).$$

Assuming the following further restriction on the growth of $f(x,s)$,

(f_3) There exists an open set $\hat{\Omega} \subset \Omega$ such that

$$\lim_{s\to\infty} \inf_{x\in\hat{\Omega}} \frac{f(x,s)}{\exp(\alpha_0 s^{N/N-1})} = \infty,$$

we obtain

Theorem 1.1.3 *(Second solution: Critical case) Suppose $f(x,s)$ satisfies (f_1), $(\hat{f}_2)$, (f_3) and $(f)_{\alpha_0}$, with $\alpha_0 > 0$. Then, there exists $\overline{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least two solutions for every $\lambda \in (0, \overline{\lambda})$.*

Exploiting the convexity of the primitive $F(x,s)$, in our final result we are able to consider a weaker version of (f_3), obtaining the same conclusion of Theorem 1.1.3. More specifically, we suppose

$(\hat{f}_3)$ There exist an open set $\hat{\Omega} \subset \Omega$

$$\lim_{s\to\infty} \inf_{x\in\hat{\Omega}} \frac{f(x,s)s^\beta}{\exp(\alpha_0 s^{N/N-1})} = \infty,$$

where $\beta = \frac{1}{2(N-1)}$ if $N = 3$, and $\beta = \frac{1}{N-1}$ otherwise.

(f_4) $F(x,.)$ is convex on $[0,\infty)$ for every $x \in \hat{\Omega} \subset \Omega$, $\hat{\Omega}$ given by $(\hat{f}_3)$,

Theorem 1.1.4 *(Second solution: Convex Critical case) Suppose $f(x,s)$ satisfies (f_1), $(\hat{f}_2)$, $(\hat{f}_3)$, (f_4) and $(f)_{\alpha_0}$, with $\alpha_0 > 0$. Then, there exists $\overline{\lambda} > 0$ such that problem $(P)_\lambda$ possesses at least two solutions for every $\lambda \in (0, \overline{\lambda})$.*

We observe that Theorem 1.1.4 establishes the existence of two solutions of $(P)_\lambda$ for $\lambda > 0$ sufficiently small when $f(x,s) = \exp(\alpha_0 s^{N/N-1})$.

As it is well known, the classical Liouville-Gelfand problem is given by

$$(LG)_\lambda \qquad \begin{cases} -\Delta u = \lambda e^u, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $I\!R^N (N \geq 1)$ with boundary $\partial\Omega$, and $\lambda > 0$ is a real parameter. First considered by Liouville [37], for the case $N = 1$, and afterwards by Bratu [12], for $N = 2$, and Gelfand [27], for $N \geq 1$, this problem has been extensively studied during the last three decades (See [17, 18, 26] and references therein). As observed in [26], problem $(LG)_\lambda$ is of great relevance since it appears in mathematical models associated with astrophysical phenomena and to problems in combustion reactions.

In [18], Crandall and Rabinowitz used bifurcation theory to establish the existence of one solution for problem $(LG)_\lambda$, for $\lambda > 0$ sufficiently small, and a nonlinearity $f(x, s)$ replacing e^s. In [18] no growth restriction on $f(x, s)$ is assumed. To obtain such result, those authors assume $f(x, s) \in C^3(\Omega \times I\!R, I\!R)$, $f_s(x, 0) > 0$ and $f_{ss}(x, s) > 0$, for every $x \in \Omega$ and $s > 0$ Supposing that $f(x, s)$ has a subcritical growth, they show that this solution is a local minimum for the associated functional. Then, using critical point theory, they are able to prove the existence of a second solution. We note that Theorems 1.1.3 and 1.1.4 improve the last mentioned result of [18] when $N = 2$ since they allow $f(x, s)$ to have critical growth. In particular, we may consider $f(x, s) = e^{s^2}$.

In [26], Garcia Azorero and Peral Alonso proved the existence of solutions for $(LG)_\lambda$, with $\lambda > 0$ sufficiently small, when the Laplacian is replaced by a p-Laplacian operator. The nonexistence of solutions for $(LG)_\lambda$ for this more general class of operators, when $\lambda > 0$ is sufficiently large, was also established in [26]. We should also mention the article by Clément, Figueiredo and Mitidieri [17], where the exact number of solutions for an operator more general that the p-Laplacian is established when Ω is an open ball of $I\!R^N$. In [17], it is not assumed any growth restriction on $f(x, s)$.

We note that the solutions mentioned in Theorems 1.1.1–1.1.4 are weak solutions of $(P)_\lambda$ (See [49]). We also observe that in this article, we use minimax methods to derive such solutions.

To prove Theorem 1.1.1, we first provide an abstract result that establishes the existence of a critical point for a functional of class C^1 defined on a real Banach space assuming a version of the famous Palais-Smale condition for the weak topology (See Definitions and Proposition 1.2.2 in Section 1.2). Motivated by the argument used in [45], we prove that the associated functional satisfies such condition under hypotheses (f_1) and $(f)_{\alpha_0}$. Taking $\lambda > 0$ sufficiently small, we are able to apply the mentioned abstract result. In our proof of Theorem 1.1.2, we use condition (f_2) to verify that the associated functional satisfies the Palais-Smale condition. As in [18], this provides the existence of a second solution for $(P)_\lambda$ via the Mountain Pass Theorem [6].

In the proofs of Theorems 1.1.3 and 1.1.4, we argue by contradiction, assuming that Theorem 1.1.1 provides the only possible solution of $(P)_\lambda$. This assumption and condition $(\hat{f}_2)$ allow us to use the argument of Brezis and Nirenberg [14] and a result of Lions [35] to verify that the associated functional satisfies the Palais-Smale condition on a given interval of the real line.We use conditions (f_3) and $(\hat{f}_3)$, respectively, to establish that the level associated with the Mountain Pass Theorem belongs to this interval. As in the proof of Theorem 1.1.2, that implies the existence of a second solution.

Finally, we should mention that the existence of a nonzero solution for $(P)_\lambda$ when $f(x,0) \equiv 0$ has been intensively studied in recent years (See [1, 2, 21, 25] and references therein) . As it is shown in [1] (See also [21]), when $f(x,s) \geq 0$, for $s \geq 0$, a weaker version of (f_3) may be considered. We also observe that our method may be used to improve such results since in those articles a stronger version of $(\hat{f}_2)$ is assumed. Condition (f_3) can also be used in that setting to study the case where $f(x,s)$ may assume negative values.

The article is organized in the following way: In Section 1.2, we introduce the notion of Palais-Smale condition for the weak topology and establish two abstract results which are used to prove our results. There, we also recall the variational framework associated with $(P)_\lambda$ and state a version of Trudinger-Moser inequality (1.1.2) for $W^{1,N}(\Omega)$ when Ω is an open ball in $\mathbb{R}^N$. In section 1.2, we also state a result by Lions [35] that will be used, via contradiction, to verify $(PS)_c$, for c below a given level, when condition $(f)_{\alpha_0}$ holds with $\alpha_0 > 0$. In Section 1.3, we prove the weak version of Palais-Smale condition for the associated functional. In Section 1.4, we prove Theorems 1.1.1 and 1.1.2. In Section 1.5 we establish the estimates that are used to prove Theorem 1.1.3. In Section 1.6, we prove Theorem 1.1.3. In Section 1.7, we establish the estimates for the associated functional when conditions $(\hat{f}_3)$ and (f_4) are assumed. There, we also present the proof of Theorem 1.1.4. In Appendix A, we prove the Trundinger-Moser inequality mentioned in Section 1.2. Finally, in Appendix B, we prove an inequality for vector fields on $\mathbb{R}^N$, used in Section 1.7 to establish the necessary estimates.

1.2 Preliminaries

Given E a real Banach space and Φ a functional of class C^1 on E, we recall that Φ satisfies Palais-Smale condition at level $c \in \mathbb{R}$ [Denoted $(PS)_c$] on an open set $\mathcal{O} \subset E$ if every sequence $(u_n) \subset \mathcal{O}$ for which (i) $\Phi(u_n) \to c$ and (ii) $\Phi'(u_n) \to 0$, as $n \to \infty$, possesses a converging subsequence. We also observe that Φ satisfies $(PS)_c$

if it satisfies $(\mathrm{PS})_c$ on E, and we say that Φ satisfies (PS) when it satisfies $(PS)_c$ for every $c \in I\!R$. Finally, we note that every sequence $(u_n) \subset E$ satisfying (i) and (ii) is called a Palais-Smale [(PS)] sequence.

To establish the existence of a critical point when the functional is bounded from below on a closed convex subsets of E, we introduce a version of the Palais-Smale condition for the weak topology.

Definition 1.2.1 *Given $c \in I\!R$, we say that $\Phi \in C^1(E, I\!R)$ satisfies the $(wPS)_c$ on $A \subset E$ if every sequence $(u_n) \subset A$ for which $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a critical point of Φ. We say that Φ satisfies (wPS) on A if Φ satisfies $(wPS)_c$ on A, for every $c \in I\!R$. When Φ satisfies (wPS) on E, we simply say that Φ satisfies (wPS).*

Assuming

(Φ_1) There exist a closed bounded set $A \subset E$, constants $\gamma \leq b \in I\!R$, and $u_0 \in \overset{\circ}{A}$ such that

(i) $\Phi(u) \geq \gamma, \ \forall\, u \in A,$
(ii) $\Phi(u) \geq b \geq \Phi(u_0), \ \forall\, u \in \partial A,$

we define

$$c_1 = \inf_{u \in A} \Phi(u). \tag{1.2.3}$$

The following abstract result provides a critical point for Φ under conditions (Φ_1) and (wPS).

Proposition 1.2.2 *Let E be a real Banach space. Suppose $\Phi \in C^1(E, I\!R)$ satisfies (Φ_1), with A a closed bounded convex subset of E. Then, Φ possesses a critical point $u \in A$ provided it satisfies $(wPS)_{c_1}$ on A.*

Proof: Arguing by contradiction, we suppose that Φ does not have a critical point $u \in A$. Under this assumption, we claim that Φ satisfies $(PS)_{c_1}$ on $\overset{\circ}{A}$. Effectively, given a sequence $(u_n) \subset \overset{\circ}{A}$ such that $\Phi(u_n) \to c_1$ and $\Phi'(u_n) \to 0$, as $n \to \infty$, by $(wPS)_{c_1}$, (u_n) possesses a subsequence converging weakly to a critical point u. Furthermore, $u \in A$ since A is a closed convex subset of E. This contradicts our assumption and proves the claim.

We note that $\gamma \leq c_1 \leq \Phi(u_0)$. If $c_1 = \Phi(u_0)$, the conclusion is immediate. Thus, we may assume $c_1 < \Phi(u_0) \leq b$. In this case, we take $0 < \overline{\varepsilon} < \Phi(u_0) - c_1$. Then, we argue as in Proposition 2.7 of [49], using a local version of the Deformation Lemma [54], to obtain a contradiction with the definition of c_1. Proposition 1.2.2 is proved. ■

Remark 1.2.3 *When Φ satisfies $(PS)_{c_1}$ on $\overset{\circ}{A}$, the second part of the proof of Proposition 1.2.2 shows that actually Φ possesses a local minimum $u \in \overset{\circ}{A}$ such that $\Phi(u) = c_1$.*

Taking $b \in I\!R$ and A, given by (Φ_1), we consider

(Φ_2) There exists $e \in E \setminus A$ such that

$$\Phi(e) \leq b \leq \Phi(u), \ \forall\, u \in \partial A,$$

and we define

$$c_2 = \inf_{g \in \Gamma} \max_{u \in g} \Phi(u) \geq b,$$

where

$$\Gamma = \{g \in C([0,1], E); \quad g(0) = u_0, \quad g(1) = e\}.$$

As a consequence of Proposition 1.2.2, Remark 1.2.3 and the argument employed in [54], we obtain the following version of the Mountain Pass Theorem [6].

Proposition 1.2.4 *Let E be a real Banach space. Suppose $\Phi \in C^1(E, I\!R)$ satisfies (Φ_1), with A closed and convex subset of E, and (Φ_2). Then, Φ possesses at least two critical points provided it satisfies $(PS)_c$, for every $c \leq c_2$.*

Proof: By Proposition 1.2.2 and Remark 1.2.3, Φ possesses a local minimum $u_1 \in \overset{\circ}{A}$ such that $\Phi(u_1) = c_1$. Furthermore, if Φ does not have any critical point on ∂A, we may invoke the local version of the Deformation Lemma [54] one more time to obtain a neighbourhood V of u_0 and $\epsilon > 0$ such that $u_0 \in V$, $e \in V$ and

$$c_1 \leq \max\{\Phi(u_0), \Phi(e)\} < \inf_{u \in \partial A} \Phi(u) + \epsilon \ \leq \inf_{u \in \partial V} \Phi(u) \leq c_2.$$

Consequently, by the Mountain Pass Theorem [6], c_2 is a critical value of Φ. The proposition is proved. ■

Observe that when $c_1 = c_2$, by the above proof, Φ must have a critical point $u \in \partial A$ such that $\Phi(u) = c_1$.

Now, we recall the variational framework associated with problem $(P)_\lambda$. Considering the Sobolev space $W_0^{1,N}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^N \, dx\right)^{1/N}, \ \forall\, u \in W_0^{1,N}(\Omega),$$

the functional associated with $(P)_\lambda$ $I_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ is given by

$$I_\lambda(u) = \frac{1}{N}\int_\Omega |\nabla u|^N \, dx - \lambda \int_\Omega F(x,u)\, dx, \ \forall\, u \in W_0^{1,N}(\Omega), \tag{1.2.4}$$

where we assume $f(x,s) = f(x,0)$, for every $x \in \bar{\Omega}$, $s < 0$, and we take $F(x,s) = \int_0^s f(x,t)\, dt$, for $x \in \bar{\Omega}$, $s \in \mathbb{R}$. Under the hypothesis $(f)_{\alpha_0}$, the functional I_λ is well defined and belongs to $C^1(W_0^{1,N}(\Omega), \mathbb{R})$ (See [1, 21]). Furthermore,

$$I'_\lambda(u)v = \int_\Omega |\nabla u|^{N-2}\nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u)v\, dx, \ \forall\, u, v \in W_0^{1,N}(\Omega).$$

Thus, every critical point of I_λ is a weak solution of $(P)_\lambda$.

We also remark that if $f(x,s)$ satisfies conditions (f_1) and $(f)_{\alpha_0}$, then, for every $\beta > \alpha_0$, there exists $C = C(\beta) > 0$ such that

$$\max\{|f(x,s)|, |F(x,s)|\} \leq C \exp(\beta |s|^{\frac{N}{N-1}}), \ \forall\, x \in \bar{\Omega}, \ s \geq 0. \tag{1.2.5}$$

As a direct consequence of (1.1.1) and (1.2.5), we obtain that $F(x,u(x)) \in L^1(\Omega)$ and $f(x,u(x)) \in L^q(\Omega)$, for every $q \geq 1$, whenever $u \in W_0^{1,N}(\Omega)$.

The following lemma establishes a version of Trudinger-Moser inequality (1.1.2) for $W^{1.N}(\Omega)$ when Ω is an open ball in $\mathbb{R}^N$.

Lemma 1.2.5 *Let $B(x_0, R)$ be an open ball in $\mathbb{R}^N$ with radius $R > 0$ and center $x_0 \in \mathbb{R}^N$. Then, there exist constants $\hat{\alpha} = \hat{\alpha}(N) > 0$ and $C(N,R) > 0$ such that*

$$\int_{B(x_0,R)} \exp(\hat{\alpha}|u|^{N/N-1})\, dx \leq C(N,R),$$

for every $u \in W^{1,N}(B(x_0,R))$ such that $\|u\|_{W^{1,N}(B(x_0,R))} \leq 1$.

Proof: For the sake of completeness, we present the proof of Lemma 1.2.5 in Appendix A. ∎

Finally, we state a theorem due to Lions [35] which will be essential to verify, via contradiction, that the functional I_λ satisfies $(PS)_c$, for c below a given level, when $f(x,s)$ satisfies the critical growth condition.

Theorem 1.2.6 *Let $\{u_n \in W_0^{1,N}(\Omega) \mid \|u_n\| = 1\}$ be a sequence in $W_0^{1,N}(\Omega)$ converging weakly to a nonzero function u. Then, for every $0 < p < (1 - \|u\|^N)^{\frac{-1}{N-1}}$, we have*

$$\sup_{n \in \mathbb{N}} \int_\Omega \exp\left(p\alpha_N |u_n|^{\frac{N}{N-1}}\right) dx < \infty.$$

1.3 (wPS) condition

In this section, we shall prove a technical result that will be used to establish (wPS) condition for the functional $I_\lambda(u)$, defined by (1.2.4), when the nonlinearity $f(x,s)$ satisfies the critical growth condition,

(f_5) There exist $\alpha, C > 0$ such that

$$|f(x,s)| \leq C \exp\left(\alpha |s|^{\frac{N}{N-1}}\right), \ \forall\, x \in \bar{\Omega}, \ \ s \in \mathbb{R}.$$

Our objective is to verify that any bounded sequence $(u_n) \subset W_0^{1,N}(\Omega)$ such that $I'_\lambda(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a solution of $(P)_\lambda$. Such result provides (wPS) condition for the functional I_λ.

Considering that next result is independent of the parameter $\lambda > 0$, we denote by (P) and I the problem $(P)_\lambda$ and the functional I_λ, respectively.

The proof of the following proposition is based on the argument used in [45] for the Neumann problem (See also [21]).

Proposition 1.3.1 *Let Ω be a bounded smooth domain in $\mathbb{R}^N$. Suppose $f(x,s) \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies (f_5). Then, any bounded sequence $(u_n) \subset W_0^{1,N}(\Omega)$ such that $I'(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a solution of (P).*

Remark 1.3.2 *(i) Note that Proposition 1.3.1 generalizes to the N-Laplacian a well known fact for the Laplacian operator on $\Omega \subset \mathbb{R}^N$, $N > 2$, when the nonlinearity*

$f(x,s)$ satisfies the polynomial critical growth condition. (ii) We also observe that in Proposition 1.3.1 it is not assumed that (u_n) is a Palais-Smale sequence since $I(u_n)$ may be unbounded. (iii) Finally, we note that in [55], we prove a similar result for the p-Laplacian on $\Omega = \mathbb{R}^N$.

The proof of Proposition 1.3.1 will be carried out in a series of steps. First, by the Sobolev Embedding Theorem, Banach-Alaoglu Theorem and the characterization of $C(\bar{\Omega})^*$, given by the Riesz Representation Theorem [51], we may suppose that there exist $u \in W_0^{1,N}(\Omega)$ and $\mu \in \mathcal{M}(\bar{\Omega})$, the space of regular Borel measure on $\bar{\Omega}$, such that

$$\begin{cases} u_n \rightharpoonup u, \text{ weakly in } W_0^{1,N}(\Omega), \\ |\nabla u_n|^N \rightharpoonup \mu, \text{ weakly}* \text{ in } \mathcal{M}(\bar{\Omega}), \\ u_n \to u, \text{ strongly in } L^p(\Omega),\ 1 \le p < \infty, \\ u_n(x) \to u(x), \text{ a. e. in } \Omega, \\ |u_n(x)| \le h_p(x), \text{ a. e. in } \Omega, \text{ where } h_p \in L^p(\Omega),\ 1 \le p < \infty. \end{cases} \tag{1.3.6}$$

Now, we fix $0 < \sigma < \infty$ such that $\alpha\sigma^{\frac{N}{N-1}} < \hat{\alpha}$, with $\hat{\alpha}$ given by Lemma 1.2.5. Setting $\Omega_\sigma = \{x \in \Omega \mid \mu(x) \ge \sigma\}$, we have that Ω_σ is a finite set since μ is a bounded nonnegative measure on $\bar{\Omega}$. Furthermore,

Lemma 1.3.3 *Let $K \subset (\Omega \setminus \Omega_\sigma)$ be a compact set. Then, there exist $q > 1$ and $M = M(K) > 0$ such that*

$$\int_K |f(x, u_n(x))|^q \, dx \le M, \ \forall\, n \in \mathbb{N}.$$

<u>Proof:</u> To prove such result, we take $q > 1$ such that $\alpha q\sigma^{\frac{N}{N-1}} < \hat{\alpha}$ and consider $r_1 = \operatorname{dist}(K, \partial\Omega \cup \Omega_\sigma) > 0$, the distance between K and $\partial\Omega \cup \Omega_\sigma$. For every $x \in K$, there exists $0 < r_x < r_1$ such that

$$\mu(B(x, 2r_x)) + \|u\|^N_{L^N(B(x,2r_x))} < \sigma^N. \tag{1.3.7}$$

Using the compactness of K, we find $j \in \mathbb{N}$ so that

$$K \subset \bigcup_{i=1}^{j} B(x_i, r_{x_i}) \equiv \bigcup_{i=1}^{j} B_i. \tag{1.3.8}$$

Applying (1.3.6) and (1.3.7), we find $n_0 \in \mathbb{N}$ such that

$$\|u_n\|^N_{W^{1,N}(B_i)} \le \sigma^N, \ \forall\, n \ge n_0, \ 1 \le i \le j.$$

Consequently, from Lemma 1.2.5, (f_5), (1.3.8) and our choice of q, there exists $M > 0$ such that

$$\int_K |f(x,u_n(x))|^q dx \le \sum_{i=1}^{N} \int_{B_i} \exp\left(\hat{\alpha}\left(\frac{|u_n(x)|}{\|u_n\|_{W^{1,N}(B_i)}}\right)^{\frac{N}{N-1}}\right) dx \le M,$$

for every $n \ge n_0$. This proves the lemma. ■

Lemma 1.3.4 *Let $K \subset (\Omega \setminus \Omega_\sigma)$ be a compact set. Then, $\nabla u_n \to \nabla u$, strongly in $(L^N(K))^N$, as $n \to \infty$.*

Proof: Taking $\psi \in C_0^\infty(\Omega \setminus \Omega_\sigma)$ such that $\psi \equiv 1$, on K, and $0 \le \psi \le 1$, and considering that

$$(|a|^{N-2}a - |b|^{N-2}b) \cdot (a-b) \ge 2^{2-N}|a-b|^N, \ \forall\, a,b \in I\!R^N, \tag{1.3.9}$$

we obtain

$$\begin{aligned} &2^{2-N}\|\nabla u_n - \nabla u\|^N_{L^N(K)} \le \\ &\le \textstyle\int_\Omega \left[(|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u).(\nabla u_n - \nabla u)\right]\psi\, dx = \\ &= \textstyle\int_\Omega \left[|\nabla u_n|^N\psi - |\nabla u_n|^{N-2}(\nabla u_n.\nabla u)\psi - \right. \\ &\left. - |\nabla u|^{N-2}(\nabla u.\nabla(u_n - u))\psi\right] dx. \end{aligned} \tag{1.3.10}$$

As $I'(u_n) \to 0$, as $n \to \infty$, we have

$$\int_\Omega \left[|\nabla u_n|^{N-2}\left((\nabla u_n.\nabla u)\psi + (\nabla u_n.\nabla\psi)u\right) - \psi f(x,u_n)u\right] dx = o(1), \tag{1.3.11}$$

as $n \to \infty$. Moreover, since (ψu_n) is a bounded sequence in $W_0^{1,N}(\Omega)$, we also have

$$\int_\Omega \left[|\nabla u_n|^N\psi + |\nabla u_n|^{N-2}(\nabla u_n.\nabla\psi)u_n - \psi f(x,u_n)u_n\right] dx = o(1), \tag{1.3.12}$$

as $n \to \infty$. Combining (1.3.10)-(1.3.12), we obtain

$$\begin{aligned}&2^{2-N}\|\nabla u_n - \nabla u\|^N_{L^N(K)} \leq \\ &\leq \int_\Omega \psi f(x,u_n)(u_n - u)\,dx + \int_\Omega |\nabla u_n|^{N-2}(u-u_n)(\nabla u_n.\nabla\psi)\,dx + \\ &+ \int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla(u-u_n))\psi\,dx + o(1), \text{ as } n\to\infty.\end{aligned}$$

Applying Lemma 1.3.3, for the compact set $\text{supp}\psi \subset (\Omega\setminus\Omega_\sigma)$, and using Hölder's inequality, we get

$$\begin{aligned}&2^{2-N}\|\nabla u_n - \nabla u\|^N_{L^N(K)} \leq \\ &\leq \|psi\|_{L^\infty(\Omega)} M^{\frac{1}{q}} \|u_n - u\|_{L^{\frac{q}{q-1}}(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)}\|\nabla u_n\|^{N-1}_{L^N(\Omega)}\|u-u_n\|_{L^N(\Omega)} + \\ &+ \int_\Omega |\nabla u|^{N-2}\psi(\nabla u.\nabla(u-u_n))\,dx + o(1), \text{ as } n\to\infty.\end{aligned}$$

The hypothesis that $(u_n)\subset W_0^{1,N}(\Omega)$ is bounded and (1.3.6) show that $\nabla u_n \to \nabla u$, strongly in $(L^N(K))^N$, as desired. The lemma is proved. ■

As a direct consequence of Lemma 1.3.4, we have

Corollary 1.3.5 *The sequence $(u_n)\subset W_0^{1,N}(\Omega)$ possesses a subsequence (u_{n_i}) satisfying $\nabla u_{n_i}(x)\to\nabla u(x)$, for almost every $x\in\Omega$.*

The following Lemma shows that $I'(u)$ restricted to $W_0^{1,N}(\Omega\setminus\Omega_\sigma)$ is the null operator.

Lemma 1.3.6

$$(I'(u),\phi) = \int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla\phi)\,dx - \int_\Omega f(x,u)\phi\,dx = 0, \tag{1.3.13}$$

for every $\phi\in C_0^\infty(\Omega\setminus\Omega_\sigma)$.

Proof: Given $\phi\in C_0^\infty(\Omega\setminus\Omega_\sigma)$, by Hölder's inequality and the fact that $(u_n)\subset W_0^{1,N}(\Omega)$ is a bounded sequence, we have that $(|\nabla u_{n_i}|^{N-2}\nabla u_{n_i}.\nabla\phi)$ is a family of uniformly integrable functions in $L^1(\Omega)$. Thus, by Vitali's Theorem [51] and Corollary 1.3.5, we get

$$\int_\Omega |\nabla u_{n_i}|^{N-2}(\nabla u_{n_i}.\nabla\phi)\,dx \to \int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla\phi)\,dx, \text{ as } i\to\infty. \tag{1.3.14}$$

We also assert that

$$\int_\Omega f(x,u_{n_i})\phi\,dx \to \int_\Omega f(x,u)\phi\,dx, \text{ as } i\to\infty, \tag{1.3.15}$$

for every $\phi \in C_0^\infty(\Omega\backslash\Omega_\sigma)$. Effectively, by Lemma 1.3.3, there exist $q > 1$ and $M_1 > 0$ so that

$$\int_{K_2} |f(x,u_n)|^q \, dx \leq M_1, \tag{1.3.16}$$

where $K_2 = \text{supp}\phi$. Given $\epsilon > 0$, from (1.3.6) and Egoroff's Theorem, there exists $E \subset \Omega$ such that $|E| < \epsilon$ and $u_n(x) \to u(x)$, uniformly on $(\Omega \setminus E)$. Using Hölder's inequality, (1.3.16), and (f_5), we get $M_2 > 0$ such that

$$\begin{aligned} &|\textstyle\int_\Omega (f(x,u_n) - f(x,u))\phi \, dx| \leq \\ &\leq \textstyle\int_{\Omega\backslash E} |f(x,u_n) - f(x,u)||\phi| \, dx + M_2 \epsilon^{\frac{q-1}{q}}. \end{aligned}$$

As $\epsilon > 0$ can be chosen arbitrarily small and $f(x,u_n(x)) \to f(x,u(x))$, uniformly on $\Omega\backslash E$, we derive (1.3.15). Now, we use (1.3.14), (1.3.15) and the fact that $I'(u_n) \to 0$, as $n \to \infty$, to verify that (1.3.13) holds. ■

In the following, we conclude the proof of Proposition 1.3.1. In view of (1.1.1), (f_5) and the density of $C_0^\infty(\Omega)$ in $W_0^{1,N}(\Omega)$, it suffices to show that relation (1.3.13) holds for every $\phi \in C_0^\infty(\Omega)$.

Given $\phi \in C_0^\infty(\Omega)$ such that $\text{supp}\phi \cap \Omega_\sigma = \emptyset$, we take $K = \text{supp}\phi$, $\hat{\Omega}_\sigma = \Omega_\sigma \cap K = \{y_1, \ldots, y_l\}$, $1 \leq l \leq j$, and $r_1 > 0$ such that $2r_1 < |y_i - y_m|$, $i = m$, and $2r_1 < \text{dist}(K, \partial\Omega)$. We consider, $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$, on $[0,1]$, and $\psi \equiv 0$, on $[2,\infty)$, and we define

$$\psi_{i,r}(x) = \psi(\frac{|x - y_i|}{r}), \ \forall \, x \in \Omega, \ 1 \leq i \leq l, \ 0 < r < r_1.$$

We also set $\psi_{l+1,r}(x) = 1 - \sum_{i=1}^{l} \psi_{i,r}(x)$, for every $x \in \Omega$. Hence, $\phi(x) \equiv \sum_{i=1}^{l+1} \phi\psi_{i,r}(x)$ and $\phi\psi_{l+1,r} \in C_0^\infty(\Omega \setminus \Omega_\sigma)$. From Lemma 1.3.6, we have

$$\begin{aligned} (I'(u),\phi) = &\textstyle\sum_{i=1}^{l} \int_\Omega |\nabla u|^{N-2}(\nabla u, \nabla(\phi\psi_{i,r})) \, dx - \\ &- \textstyle\sum_{i=1}^{l} \int_\Omega f(x,u)\phi\psi_{i,r} \, dx, \ \forall \, 0 < r < r_1. \end{aligned} \tag{1.3.17}$$

Applying Hölder's inequality, for every $1 \leq i \leq l$, we get

$$\begin{aligned} &|\textstyle\int_\Omega |\nabla u|^{N-2}(\nabla u . \nabla(\phi\psi_{i,r})) \, dx| \leq \\ &\left[\textstyle\int_{B_i} |\nabla u|^N \, dx\right]^{\frac{N-1}{N}} \left[\|\nabla\phi\|_{L^\infty(\Omega)} \|\psi_{i,r}\|_{L^N(B_i)} + \right. \\ &\left. + \|\phi\|_{L^\infty(\Omega)} \|\nabla\psi_{i,r}\|_{L^N(B_i)}\right], \end{aligned} \tag{1.3.18}$$

where $B_i \equiv B(y_i, 2r)$, $0 < r < r_1$. On the other hand, from the first Trudinger-Moser inequality (1.1.1) and (f_5), we find $M_3 > 0$ such that, for every $1 \leq i \leq l$,

$$|\int_{\Omega} f(x,u)\phi\psi_{i,r}\,dx| \leq M_3\|\phi\|_{L^\infty(\Omega)}|B_i|^{\frac{N-1}{2N}}\|\psi_{i,r}\|_{L^N(\Omega)}. \tag{1.3.19}$$

We use our definition of $\psi_{i,r}$ to get $M_4 > 0$ so that

$$\|\psi_{i,r}\|_{W^{1,N}(B_i)} \leq M_4,\ \forall\, 0 < r < r_1,\ 1 \leq i \leq l.$$

Consequently, given $\epsilon > 0$, by Lebesgue's Dominated Convergence Theorem, (1.3.18) and (1.3.19), we find $0 < r_2 < r_1$ so that

$$\begin{cases} |\int_\Omega |\nabla u|^{N-2}(\nabla u.\nabla(\phi\psi_{i,r}))\,dx| < \epsilon, \\ |\int_\Omega f(x,u)\phi\psi_{i,r}\,dx| < \epsilon,\ \forall\, 1 \leq i \leq l,\ 0 < r < r_2. \end{cases} \tag{1.3.20}$$

for every $0 < r < r_2$, $1 \leq i \leq l$. From (1.3.17), (1.3.20) and the fact that $\epsilon > 0$ can be chosen arbitrarily small, we obtain that (1.3.13) holds for every $\phi \in C_0^\infty(\Omega)$. This concludes the proof of Proposition 1.3.1. ■

As a direct consequence of Proposition 1.3.1, we have the following results:

Corollary 1.3.7 *Let Ω be a bounded smooth domain in $I\!R^N$. Suppose that $f(x,s) \in C(\bar{\Omega} \times I\!R, I\!R)$ satisfies (f_5). Then, I satisfies (wPS) on A, for every bounded set $A \subset W_0^{1,N}(\Omega)$.*

Corollary 1.3.8 *Let Ω be a bounded smooth domain in $I\!R^N$. Suppose that $f(x,s) \in C(\bar{\Omega} \times I\!R, I\!R)$ satisfies (f_5). Then, I satisfies (wPS) provided every (PS) sequence associated with I possesses a bounded subsequence.*

1.4 Theorems 1.1.1 and 1.1.2

In this section, we apply the abstract results described in Section 1.2 to prove Theorems 1.1.1 and 1.1.2.

Proof of Theorem 1.1.1: The weak solution of problem (P_λ) will be established with the aid of Proposition 1.2.2. For this, it suffices to verify that I_λ, for $\lambda > 0$ sufficiently small, satisfies (Φ_1) and $(wPS)_{c_1}$ on the closure of $B(0,\rho)$, denoted by $B[0,\rho]$, for some appropriate value of $\rho > 0$.

Given $\beta > \alpha_0$, we take $\rho \in \left(0, \left(\frac{\alpha_N}{\beta}\right)^{\frac{N-1}{N}}\right)$ and use (1.2.5) to obtain $C_1 > 0$ such that

$$I_\lambda(u) \geq \tfrac{1}{N}||u||^N - \lambda C_1 \textstyle\int_\Omega \exp(\beta|u|^{\frac{N}{N-1}})\, dx =$$
$$= \tfrac{1}{N}||u||^N - \lambda C_1 \textstyle\int_\Omega \exp\Big(\beta||u||^{\frac{N}{N-1}}\Big(\tfrac{|u|}{||u||}\Big)^{\frac{N}{N-1}}\Big)\, dx,$$

for every $u \in W_0^{1,N}(\Omega)$ such that $\|u\| \leq \rho$. Hence, by Trudinger-Moser inequality (1.1.2), we find $C_2(N) > 0$ such that

$$I_\lambda(u) \geq \frac{1}{N}\|u\|^N - \lambda C_2(N),$$

for every $u \in B[0, \rho]$.

Taking $\overline{\lambda} = N^{-1}C_2(N)^{-1}\rho^N$, $u_0 = 0$, $\gamma = -\overline{\lambda}C_2(N)$, $b = 0$, and considering $c_\lambda = c_1$, c_1 given by (1.2.3), we have that I_λ satisfies condition (Φ_1), for every $0 < \lambda < \overline{\lambda}$.

Finally, we observe that conditions (f_1), $(f)_{\alpha_0}$ and Corollary 1.3.7 imply that I_λ satisfies (wPS) condition on $B[0, \rho]$. Theorem 1.1.1 is proved. ■

Before proving Theorem 1.1.2, we note that, from (f_1) and (f_2), there exists a constant $C > 0$ such that

$$F(x, s) \geq C|s|^\theta - C, \ \forall\, x \in \bar{\Omega}, \ s \geq 0. \tag{1.4.21}$$

Proof of Theorem 1.1.2: Considering $\overline{\lambda} > 0$, given in the proof of Theorem 1.1.1, we have that the functional I_λ satisfies (Φ_1), for every $\lambda \in (0, \overline{\lambda})$. Thus, by Proposition 1.2.4 , it suffices to verify that I_λ satisfies (Φ_2) and (PS) for such values of λ.

Choosing $u \in W_0^{1,N}(\Omega) \setminus \{0\}$ such that $u(x) > 0$, for every $x \in \Omega$, from (1.4.21), we obtain

$$I_\lambda(tu) \leq \frac{t^N}{N}||u||^N - \lambda C t^\theta \int_\Omega u^\theta\, dx + C|\Omega|.$$

Therefore, $I_\lambda(tu) \to -\infty$, as $t \to +\infty$, since $C > 0$ and $\theta > N$. Consequently, I_λ satisfies Φ_2.

Now, we shall verify that I_λ satisfies (PS). Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a sequence such that $(I_\lambda(u_n)) \subset I\!R$ is bounded, and $I'_\lambda(u_n) \to 0$, as $n \to \infty$, i.e,

$$|\frac{1}{N}\int_\Omega |\nabla u_n|^N\, dx - \lambda \int_\Omega F(x, u_n)\, dx| \leq C < \infty\ , \ \forall\, n \in I\!N, \tag{1.4.22}$$

and

$$|\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla v\, dx - \lambda \int_\Omega f(x,u_n)v\, dx| \le \varepsilon_n ||v||, \qquad (1.4.23)$$

for every $v \in W_0^{1,N}(\Omega)$, where $\varepsilon_n \to 0$, as $n \to \infty$. Taking $\theta > N$, given by (f_2), we use (1.4.22) and (1.4.23) to get

$$\int_\Omega |\nabla u_n|^N\, dx - \lambda \int_\Omega (\theta F(x,u_n) - f(x,u_n)u_n)\, dx \le C + \varepsilon_n ||u_n||.$$

From this inequality, (f_2), and our definition of $f(x,s)$ for $s \le 0$, we conclude that (u_n) is a bounded sequence in $W_0^{1,N}(\Omega)$. Consequently, we may assume that

$$u_n \rightharpoonup u \ \text{ weakly in } \ W_0^{1,N}(\Omega), \quad u_n \to u \ \text{ strongly in } \ L^q(\Omega),\ \forall\, q > 1.$$

From (1.4.23), with $v = u_n - u$, we have

$$\lim_{n\to\infty}\{\int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla(u_n - u)\, dx - \lambda \int_\Omega f(x,u_n)(u_n-u)\, dx\} = 0. \qquad (1.4.24)$$

Using Hölder's inequality, we may estimate the second integral in the above equation,

$$|\int_\Omega f(x,u_n)(u_n-u)\, dx| \le (\int_\Omega |f(x,u_n)|^p\, dx)^{1/p} ||u_n - u||_{L^q},$$

where $p,q > 1$ are fixed with $\frac{1}{q} + \frac{1}{p} = 1$. Noting that (u_n) is a bounded sequence, we may find $\beta > \alpha_0 = 0$ such that $\beta p ||u_n||^{N/N-1} < \alpha_N$, for every $n \in I\!N$. Hence, by (1.2.5), we have

$$\int_\Omega |f(x,u_n)(u_n-u)| dx \le C\Big\{\int_\Omega \exp\Big(p\beta ||u_n||^{\frac{N}{N-1}} (\frac{|u_n|}{||u_n||})^{\frac{N}{N-1}}\Big)\Big\}^{\frac{1}{p}} ||u_n - u||_{L^q}.$$

Thus, by Trudinger-Moser inequality (1.1.2), we obtain $C_2 > 0$ such that

$$|\int_\Omega f(x,u_n)(u_n-u)\, dx| \le C_2 ||u_n - u||_{L^q}.$$

Since $u_n \to u$ strongly in $L^q(\Omega)$, from (1.4.24) and the above inequality, we have

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{N-2}\nabla u_n \nabla(u_n - u)\, dx = 0.$$

On the other hand,

$$\lim_{n\to\infty}\int_\Omega |\nabla u|^{N-2}\nabla u \nabla(u_n - u)\,dx = 0,$$

because $u_n \rightharpoonup u$ weakly in $W_0^{1,N}(\Omega)$. Consequently,

$$\lim_{n\to\infty}\int (|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u)(\nabla u_n - \nabla u)\,dx = 0.$$

Thus, by inequality (1.3.9), we have

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^N\,dx = 0.$$

This implies that I_λ satisfies (PS) condition. Theorem 1.1.2 is proved. ∎

Remark 1.4.1 *As it is shown in [20] (See also [28].), any solution of $(P)_\lambda$ is in $C^{1,\alpha}(\Omega)$, for $N \geq 3$, and in $C^{2,\alpha}(\Omega)$, for $N = 2$.*

1.5 Estimates

We start this section with the definition of Moser functions (See [41]). Let $x_0 \in \Omega$ and $R > 0$ be such that the ball $B(x_0, R)$ of radius R centered at x_0 is contained in Ω. The Moser functions are defined for $0 < r < R$ by

$$M_r(x) = \frac{1}{w_{N-1}^{1/N}}\begin{cases} (\log\frac{R}{r})^{\frac{N-1}{N}}, & if \quad 0 \leq |x - x_0| \leq r, \\ \frac{\log(\frac{R}{|x-x_0|})}{(\log\frac{R}{r})^{1/N}}, & if \quad r \leq |x - x_0| \leq R, \\ 0, & if \quad |x - x_0| \geq R. \end{cases}$$

Then, $M_r \in W_0^{1,N}(\Omega), ||M_r|| = 1$ and supp (M_r) is contained in $B(x_0, R)$. Considering $\hat{\Omega}$ given by (f_3), we take $x_0 \in \hat{\Omega}$ and consider the Moser sequence $M_n(x) = M_{\frac{R_n}{n}}(x)$ where $R_n = (\log n)^{\frac{1-N}{N}}$, for every $n \in \mathbb{N}$. Without loss of generality, we may suppose that supp$(M_n) \subset \hat{\Omega}$, for every $n \in \mathbb{N}$.

Taking $\bar{\lambda} > 0$ and u_λ, for $\lambda \in (0, \bar{\lambda})$, given in the proof of Theorem 1.1.1, we have

Proposition 1.5.1 *Suppose $f(x,s)$ satisfies $(f_1), (f)_{\alpha_0}$, with $\alpha_0 > 0$, and (f_3). Then, for every $\lambda \in (0, \bar{\lambda})$, there exists $n \in \mathbb{N}$ such that*

$$\max\{I_\lambda(u_\lambda + tM_n) \mid t \geq 0\} < I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

The proof of Proposition 1.5.1 will be carried out through the verification of several steps. First, we suppose by contradiction that, for every n, we have

$$\max\{I_\lambda(u_\lambda + tM_n) \mid t \geq 0\} \geq I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}. \qquad (1.5.25)$$

Now, we apply the argument employed in the proof of Theorem 1.1.2 to conclude that $I_\lambda(u_\lambda + tM_n) \to -\infty$, as $t \to \infty$, for every $n \in \mathbb{N}$. Thus, there exists $t_n > 0$ such that

$$I_\lambda(u_\lambda + t_n M_n) = \max\{I_\lambda(u_\lambda + tM_n) \mid t \geq 0\}. \qquad (1.5.26)$$

The following lemmas provide estimates for the value of t_n.

Lemma 1.5.2 *The sequence $(t_n) \subset \mathbb{R}$ is bounded.*

Proof: Since $\frac{d}{dt}[I_\lambda(u_\lambda + tM_n)] = 0$ for $t = t_n$, it follows that

$$\int_\Omega |\nabla(u_\lambda + t_n M_n)|^{N-2}\nabla(u_\lambda + t_n M_n) \cdot \nabla M_n \, dx = \lambda \int_\Omega f(x, u_\lambda + t_n M_n) M_n \, dx.$$

Invoking Holder's inequality, we obtain

$$\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda \int_\Omega f(x, u_\lambda + t_n M_n) M_n \, dx. \qquad (1.5.27)$$

We observe that given $M > 0$, from (f_3), there exists a positive constant C such that

$$f(x,s) \geq M \exp(\alpha_0 |s|^{N/N-1}) - C, \quad \forall\, s \geq 0, x \in \hat{\Omega}. \qquad (1.5.28)$$

Thus, from (1.5.27)-(1.5.28), the definition of the function M_n and the nonnegativity of u_λ, we have

$$\begin{aligned} \|u_\lambda + t_n M_n\|^{N-1} \;&\geq\; \lambda M \int_{B(x_0,R_n)} \exp(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}) M_n \, dx \\ &-\; \lambda C \int_{B(x_0,R_n)} M_n \, dx. \end{aligned}$$

Using the definition of the function M_n one more time, we find $\hat{C} > 0$ such that

$$\begin{aligned} \|u_\lambda + M_n\|^{N-1} &\geq \lambda M \int_{B(x_0, \frac{R_n}{n})} \exp(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}) M_n \, dx \\ - \lambda \hat{C} R_n^N &= \frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N} \exp\left[\left(\frac{\alpha_0}{\alpha_N} t_n^{\frac{N}{N-1}} - 1\right) N \log n\right] R_n^N (\log n)^{\frac{N-1}{N}} \\ & -\lambda \hat{C} R_n^N. \end{aligned}$$

Hence, from the definition of R_n, we get

$$\|u_\lambda + M_n\|^{N-1} \geq \frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N} \exp\left[\left(\frac{\alpha_0}{\alpha_N} t_n^{\frac{N}{N-1}} - 1\right) N \log n\right] - \lambda \hat{C} R_n^N. \tag{1.5.29}$$

Since $R_n \to 0$, as $n \to \infty$, from (1.5.29), we conclude that $(t_n) \subset \mathbb{R}$ is a bounded sequence. Lemma 1.5.2 is proved. ∎

Lemma 1.5.3 *There exist a positive constant $C = C(\lambda, \alpha_0, N)$ and $n_0 \in \mathbb{N}$ such that*

$$t_n^{N/N-1} \geq \frac{\alpha_N}{\alpha_0} - \frac{C R_n^N}{(\log n)^{1/N}}, \ \forall \ n \geq n_0.$$

Proof: From equation (1.5.25),

$$I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1} \leq \frac{1}{N} \int_\Omega |\nabla(u_\lambda + t_n M_n)|^N \, dx - \lambda \int_\Omega F(x, u_\lambda + t_n M_n) \, dx.$$

Hence,

$$\begin{aligned} \frac{1}{N}\left(\frac{\alpha_N}{\alpha_0}\right)^{N-1} &\leq \frac{t_n^N}{N} - \lambda \int_\Omega (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda)) \, dx + \\ &+ \frac{1}{N} \sum_{k=1}^{N-1} \binom{N}{k} t_n^k \int_\Omega |\nabla u_\lambda|^{N-k} |\nabla M_n|^k \, dx. \end{aligned}$$

Furthermore,

$$F(x, u_\lambda + t_n M_n) - F(x, u_\lambda) = \int_0^{t_n M_n} f(x, s + u_\lambda) \, ds \geq -m t_n M_n,$$

where $m \geq 0$ is given by (f_1) and $(f)_{\alpha_0}$. Consequently,

$$\begin{aligned}\frac{1}{N}\Big(\frac{\alpha_N}{\alpha_0}\Big)^{N-1} &\leq \frac{t_n^N}{N} + \lambda m t_n \int_\Omega M_n dx + \\ &+ \frac{1}{N}\sum_{k=1}^{N-1}\binom{N}{k} t_n^k \int_\Omega |\nabla u_\lambda|^{N-k}|\nabla M_n|^k\, dx. \qquad (1.5.30)\end{aligned}$$

On the other hand, from the definition of the sequence (M_n), we have

$$\int_\Omega M_n\, dx = \frac{R_n^N w_{N-1}^{\frac{N-1}{N}}}{N}\left\{\frac{2(\log n)^{\frac{N-1}{N}}}{n^N} + \frac{1}{N}(1-\frac{1}{n^N})\frac{1}{(\log n)^{1/N}}\right\} \quad (1.5.31)$$

$$\int_\Omega |\nabla u_\lambda|^{n-k}|\nabla M_n|^k\, dx \leq C(N,n,\lambda,k)\frac{1}{(\log n)^{k/N}}, \quad (1.5.32)$$

where $C(N,n,\lambda,k) = \frac{R_n^{Nk} w_{N-1}^{(1-\frac{k}{N})}}{(N-k)}(1-\frac{1}{n^{N-k}})||\nabla u_\lambda||_{L^\infty}^{N-k}$.

Using (1.5.30)-(1.5.32) and Lemma 1.5.2, we find a constant $C > 0$ such that

$$t_n^{N/N-1} \geq (\frac{\alpha_N}{\alpha_0})(1 - \frac{C(\frac{\alpha_N}{\alpha_0})^{N-1}R_n^N}{(\log N)^{1/n}})^{1/(N-1)}.$$

A direct application of Mean Value Theorem to the function $h(s) = (1-s)^{1/(N-1)}$ on the above relation provides the conclusion of Lemma 1.5.3. ■

Now, we shall use Lemmas 1.5.2 and 1.5.3 to derive the desired contradiction. From (1.5.29), Lemma 1.5.3 and the definition of R_n, we obtain

$$||u_\lambda + t_n M_n||^{N-1} \geq \frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N}\exp\left(-\frac{\alpha_0 C N}{\alpha_N}\right) - \lambda \hat{C} R_n^N.$$

Thus,

$$\frac{\lambda M w_{N-1}^{\frac{N-1}{N}}}{N}\exp\left(-\frac{\alpha_0 C N}{\alpha_N}\right) \leq (||u_\lambda|| + t_n)^{N-1} + \lambda \hat{C} R_n^N.$$

But, this contradicts Lemma 1.5.2, since M can be arbitrarily chosen and $R_n \to 0$, as $n \to \infty$. Proposition 1.5.1 is proved. ■

1.6 Theorem 1.1.3

In this section, after the verification of some preliminary results, we prove Theorem 1.1.3.

Lemma 1.6.1 *Suppose $f(x,s)$ satisfies (f_1), $(\hat{f}_2)$ and $(f)_{\alpha_0}$. Then, any (PS) sequence $(u_n) \subset W_0^{1,N}(\Omega)$ associated with I_λ possesses a subsequence (u_{n_i}) converging weakly in $W_0^{1,N}(\Omega)$ to a solution u of $(P)_\lambda$. Furthermore,*

$$\int_\Omega F(x, u_{n_i}(x))\,dx \to \int_\Omega F(x,u(x))\,dx, \quad \text{as } n \to \infty.$$

Remark 1.6.2 *We note that Lemma 1.6.1 also holds when $f(x,s)$ satisfies $(\hat{f}_2)$ and $(f)_{\alpha_0}$ for $s \le -R(\theta)$, and $s \le 0$, respectively.*

Proof: Consider a sequence $(u_n) \subset W_0^{1,N}(\Omega)$ such that

$$\begin{cases} I_\lambda(u_n) \to c, \\ I'_\lambda(u_n) \to 0, \text{ as } n \to \infty. \end{cases} \tag{1.6.33}$$

Arguing as in Section 1.4, we obtain that (u_n) is a bounded sequence. Therefore, by Proposition 1.3.1, there exists a subsequence, that we continue to denote by (u_n), converging weakly in $W_0^{1,N}(\Omega)$ to a solution u of $(P)_\lambda$. Moreover, we may assume that $u_n(x) \to u(x)$, for almost every $x \in \Omega$. From (1.6.33) and (f_1), we get

$$\|u_n^-\|^N \le \|u_n^-\|^N + \lambda \int_\Omega f(x,0)u_n^-(x)\,dx \le \|I'_\lambda(u_n)\| \|u_n^-\| \to 0, \tag{1.6.34}$$

as $n \to \infty$. Hence, $u_n(x) \to u(x) \ge 0$, as $n \to \infty$, for almost every $x \in \Omega$. Now, we fix $\theta_1 > N$, and we consider $R_1 = R(\theta_1) > 0$ given by $(\hat{f}_2)$. From (1.6.33) and $(\hat{f}_2)$, we find $M_1 > 0$ such that

$$\int_{\{|u_n(x)| \ge R_1\}} \left[\frac{1}{\theta_1} f(x,u_n(x))u_n(x) - F(x,u_n(x))\right] dx \le M_1. \tag{1.6.35}$$

Observing that $|\{x \in \Omega \mid u_n(x) \le -R_1\}| \to 0$, as $n \to \infty$, from (1.2.5), (1.6.34), (1.6.35) and Hölder's inequality, we have

$$\int_{\{u_n^+(x) \ge R_1\}} \left[\frac{1}{\theta_1} f(x,u_n^+(x))u_n^+(x) - F(x,u_n^+(x))\right] dx \le M_1. \tag{1.6.36}$$

Given $\epsilon > 0$, we take $\theta_2 > \theta_1$ such that $\frac{\theta_1 M_1}{\theta_2 - \theta_1} \leq \epsilon$ and $R_2 > \max\{R_1, R(\theta_2)\}$, $R(\theta_2)$ given by $(\hat{f}_2)$. Applying (1.6.36) and $(\hat{f}_2)$, we obtain

$$\int_{\{u_n^+(x) \geq R_2\}} |F(x, u_n^+(x))| \, dx \leq \epsilon. \tag{1.6.37}$$

Applying Egoroff's Theorem, we find $E \subset \Omega$ such that $|E| < \epsilon$ and $u_n(x) \to u(x)$, as $n \to \infty$, uniformly on $(\Omega \setminus E)$. Hence, from (1.2.5) and (1.6.34), we have

$$\begin{aligned} &| \textstyle\int_\Omega [F(x, u_n(x)) - F(x, u(x))] \, dx \leq \\ &\leq \textstyle\int_E |F(x, u_n^+(x))| \, dx + \int_E |F(x, u(x))| \, dx + o(1), \text{ as } n \to \infty. \end{aligned} \tag{1.6.38}$$

Fixed $q > 1$, we use (1.2.5) and Hölder's inequality to get $M_2 > 0$ such that

$$\int_E |F(x, u(x))| \, dx \leq M_2 \epsilon^{\frac{1}{q}}. \tag{1.6.39}$$

From (1.6.37), (1.6.39) and Lesbegue's Dominated Convergence Theorem, we have

$$\begin{aligned} &\textstyle\int_E |F(x, u_n^+(x))| \, dx \leq \epsilon + \int_{E \cap \{0 \leq u_n(x) \leq R_2\}} |F(x, u_n^+(x))| \, dx \leq \\ &\leq \epsilon + \textstyle\int_{E \cap \{0 \leq u_n(x) \leq R_2\}} |F(x, u(x))| \, dx + o(1) \leq \\ &\leq \epsilon + M_2 \epsilon^{\frac{1}{q}} + o(1), \text{ as } n \to \infty. \end{aligned}$$

The above inequality, (1.6.38), (1.6.39) and the fact that $\epsilon > 0$ can be chosen arbitrarily provide the conclusion of the proof of Lemma 1.6.1. ■

Considering $c_\lambda = I_\lambda(u_\lambda) + \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$, with u_λ given by the proof of Theorem 1.1.1, we shall verify that I_λ satisfies (PS) condition below the level c_λ, whenever we suppose that $u = u_\lambda$ is the only possible solution of $(P)_\lambda$.

Lemma 1.6.3 *Suppose $f(x, s)$ satisfies (f_1), (f_{α_0}), with $\alpha_0 > 0$, and $(\hat{f}_2)$. Assume that u_λ is the only possible solution of $(P)_\lambda$, for $0 < \lambda < \bar{\lambda}$. Then, I_λ satisfies $(PS)_c$, for every $c < c_\lambda = I_\lambda(u_\lambda) + \frac{1}{N} \left(\frac{\alpha_N}{\alpha_0} \right)^{N-1}$.*

Proof: Let $(u_n) \subset W_0^{1,N}(\Omega)$ be a sequence such that

$$\begin{cases} I_\lambda(u_n) \to c < c_\lambda, \\ I'_\lambda(u_n) \to 0, \text{ as, } n \to \infty. \end{cases} \tag{1.6.40}$$

Since u_λ is the only solution of $(P)_\lambda$, by Lemma 1.6.1, we may assume that $u_n \rightharpoonup u_\lambda$, as $n \to \infty$, weakly in $W_0^{1,N}(\Omega)$ and

$$\int_\Omega F(x,u_n(x))\,dx \to \int_\Omega F(x,u_\lambda(x))\,dx, \text{ as } n\to\infty. \tag{1.6.41}$$

From (1.6.40) and (1.6.41), we have

$$\|u_n\|^N \to N\left(c+\lambda\int_\Omega F(x,u_\lambda(x))\,dx\right), \text{ as } n\to\infty. \tag{1.6.42}$$

Taking $v_n = \frac{u_n}{\|u_n\|}$, we get that

$$v_n \rightharpoonup v = \frac{u_\lambda}{[N(c+d)]^{\frac{1}{N}}},$$

where $d = \lambda\int_\Omega F(x,u_\lambda(x))\,dx$. Considering $\beta > \alpha_0$ such that

$$c < I_\lambda(u_\lambda) + \frac{1}{N}\left(\frac{\alpha_N}{\beta}\right)^{N-1}, \tag{1.6.43}$$

by (1.2.5), we find $q > 1$ and $C > 0$ so that

$$|f(x,s)|^q \le C\exp\left(\beta|s|^{\frac{N}{N-1}}\right), \ \forall\, x\in\Omega,\ s\in I\!R.$$

Thus,

$$\int_\Omega |f(x,u_n(x))|^q\,dx \le C\int_\Omega \exp\left(\beta\|u_n\|^{\frac{N}{N-1}}|v_n(x)|^{\frac{N}{N-1}}\right)\,dx, \tag{1.6.44}$$

for every $n\in I\!N$. On the other hand, by (1.6.43),

$$1-\|v\|^N < (\frac{\alpha_N}{\beta})^{N-1}\frac{1}{N(c+d)}.$$

Consequently, from (1.6.42), there exists $p > 0$ such that

$$\frac{\beta}{\alpha_N}\|u_n\|^{\frac{N}{N-1}} < p < \left(1-\|v\|^N\right)^{\frac{-1}{N-1}}.$$

Hence, by Theorem 1.2.6 and (1.6.44), there exists $M > 0$ such that

$$\int_\Omega |f(x,u_n(x))|^q\,dx \le M, \ \forall\, n\in I\!N.$$

Applying Egoroff's Theorem, the above inequaltiy and the argument employed in the proof of Proposition 1.3.1, we obtain

$$\int_\Omega f(x,u_n(x))u_n(x)\,dx \to \int_\Omega f(x,u_\lambda(x))u_\lambda(x)\,dx, \text{ as } n\to\infty.$$

Therefore, by (1.6.40),

$$\|u_n\|^N \to \lambda\int_\Omega f(x,u_\lambda(x))u_\lambda(x)\,dx = \|u_\lambda\|^N.$$

The Lemma 1.6.3 is proved. ■

Now, we may conclude the proof of Theorem 1.1.3. Arguing by contradiction, we suppose that u_λ, for $0<\lambda<\bar{\lambda}$, is the only possible solution of $(P)_\lambda$. By Lemma 1.6.3, I_λ satisfies $(PS)_c$ for every $c<c_\lambda$. Furthermore, by the argument employed in the proof of Theorem 1.1.1, I_λ satisfies (Φ_1) on $B[0,\rho]$, for $\rho>0$ sufficiently small. Hence, Proposition 1.2.2 and Remark 1.2.3 imply $u_\lambda \in \overset{\circ}{B}[0,\rho]$. Invoking Propositions 1.5.1 and 1.2.4 and Lemma 1.6.3, we conclude that I_λ possesses at least two critical points. However, this contradicts the fact that u_λ is the only critical point of I_λ. Theorem 1.1.3 is proved. ■

1.7 Theorem 1.1.4

In this section we establish a proof of Theorem 1.1.4. The key ingredient is the verification of Proposition 1.5.1 under conditions $(\hat{f}_3)$ and (f_4). To obtain such result we exploit the convexity of the function $F(x,s)$ and the fact that u_λ, for $\lambda\in(0,\bar{\lambda})$, is a solution of $(P)_\lambda$.

First, we state a basic result that will be used in our estimates.

Lemma 1.7.1 *Let $a,b\in\mathbb{R}^N, N\geq 2$, and $\langle .,.\rangle$ the standard scalar product in $\mathbb{R}^N$. Then, there exists a nonnegative polynomial $p_N(x,y)$ ($p_2\equiv 0$) such that*

$$|a+b|^N \leq |a|^N + N|a|^{N-2}\langle a,b\rangle + |b|^N + p_N(|a|,|b|). \qquad (1.7.45)$$

Furthermore, the smallest exponent of the variable y of $p_N(x,y)$ is $3/2$ for $N=3$ and 2 for $N\geq 4$, and the greatest exponent of y is strictly smaller than N.

Proof: We present a proof of Lemma 1.7.1 in Appendix B. ■

Now, we are ready to establish the version of Proposition 1.5.1. Consider β, $\hat{\Omega}$ given by $(\hat{f}_3)$. Let $x_0\in\hat{\Omega}$ and the Moser sequence associated $M_n = M_{\frac{R_n}{n}}$, where $R_n = (\log n)^{-\frac{(N-1)(1-\beta)}{N^2}}$ if $N\geq 3$, and $R_n = R$ if $N=2$, where $R>0$ is chosen so that $B(x_0,R)\subset\hat{\Omega}$.

Proposition 1.7.2 *Suppose $f(x,s)$ satisfies $(f_1), (f)_{\alpha_0}$, with $\alpha_0 > 0$, $(\hat{f}_3)$, and (f_4). Then, for every $\lambda \in (0, \bar{\lambda})$, there exists $n \in I\!N$ such that*

$$\max\{I_\lambda(u_\lambda + tM_n) \mid t \geq 0\} < I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_N}{\alpha_0})^{N-1}.$$

Arguing as in the proof of Proposition 1.5.1, we suppose by contradiction that for every $n \in I\!N$, (1.5.25) holds. As before, there exists $t_n \in I\!R$ satisfying equation (1.5.26). The following two results are versions of Lemmas 1.5.2 and 1.5.3 for this new situation.

Lemma 1.7.3 *The sequence $(t_n) \subset I\!R$ is bounded.*

Proof: Arguing as in the proof of Lemma 1.5.2, we have that equation (1.5.27) must hold. By (f_1) and (f_4), for every $x \in \hat{\Omega}$, the function $f(x,.)$ is positive on $[0,\infty)$ and nondecreasing. Thus, from (1.5.27),

$$\|u_\lambda + t_n M_n\|^{N-1} \geq \lambda \int_{B(x_0, \frac{R_n}{n})} f(x, t_n M_n) M_n \, dx. \tag{1.7.46}$$

Now, by $(\hat{f}_3)$, given $M > 0$ there exists $R_M > 0$ such that

$$s^\beta f(x,s) \geq M \exp(\alpha_o s^{\frac{N}{N-1}}), \ \forall \ s \geq R_M, x \in \hat{\Omega}. \tag{1.7.47}$$

Consequently, by the definition of M_n, for n sufficiently large, we get

$$\begin{aligned} t_n \|u_\lambda + t_n M_n\|^{N-1} &\geq \lambda M w_{N-1}^{\frac{N-1}{N}} \exp\left[\left(\frac{\alpha_0}{\alpha_N} t_n^{\frac{N}{N-1}} - 1 + \frac{\log R_n^N}{N \log n} + \right.\right. \\ &+ \left.\left. \frac{(N-1)(1-\beta)\log(\log n)}{N \log n}\right) N \log n\right]. \end{aligned}$$

Now, from definition of R_n, we have

$$\frac{\log R_n^N}{N \log n} = \frac{(\beta - 1)(N-1)\log(\log n)}{N \log n} \to 0, \quad \text{as } n \to \infty.$$

Thus, we conclude that $(t_n) \subset I\!R$ is a bounded sequence. The lemma is proved. ■

Lemma 1.7.4 *There exist $n_0 > 0$, and a postive constant $C(\lambda, \alpha_0, N) \geq 0$ ($C(\lambda, \alpha_0, 2) = 0$) such that*

$$t_n^{N/N-1} \geq \frac{\alpha_N}{\alpha_0} - \frac{CR_n^N}{(\log n)^{\gamma/N}}, \quad \forall\, n \geq n_0,$$

where $\gamma = 3/2$ if $N = 3$, and $\gamma = 2$ if $N \geq 4$.

Proof: From equations (1.5.25)-(1.5.26), we have

$$I_\lambda(u_\lambda) + \frac{1}{N}(\frac{\alpha_0}{\alpha_N})^{N-1} \leq I_\lambda(u_\lambda + t_n M_n).$$

Consequently,

$$\begin{aligned} \frac{1}{N}(\frac{\alpha_0}{\alpha_N})^{N-1} \quad &\leq \quad \frac{1}{N}\int_\Omega (|\nabla u_\lambda + \nabla(t_n M_n)|^N - |\nabla u_\lambda|^N)\, dx \\ &- \quad \lambda \int_\Omega (F(x, u_\lambda + t_n M_n) - F(x, u_\lambda))\, dx. \end{aligned} \qquad (1.7.48)$$

Using Lemma 1.7.1 with $a = \nabla u_\lambda(x)$, and $b = \nabla(t_n M_n(x))$, we have

$$\int_\Omega \left(|\nabla u_\lambda + \nabla(t_n M_N)|^N - |\nabla u_\lambda|^N\right)\, dx \leq \qquad (1.7.49)$$

$$\int_\Omega \left(N|\nabla u_\lambda|^{N-2}\nabla u_\lambda \nabla(t_n M_n) + |\nabla(t_n M_n)|^N + p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|)\right)\, dx.$$

From (1.7.48), (1.7.49), $\|M_n\| = 1$, and the fact that u_λ is a solution of $(P)_\lambda$, we obtain

$$\begin{aligned} \frac{1}{N}(\frac{\alpha_0}{\alpha_N})^{N-1} \leq\ & \frac{t_n^N}{N} - \lambda \int_\Omega [(F(x, u_\lambda + t_n M_n) - F(x, u_\lambda) - f(x, u_\lambda) t_n M_n) \\ & + p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|)]\, dx. \end{aligned} \qquad (1.7.50)$$

Hence, from (1.7.50) and (f_4), we get

$$\frac{1}{N}(\frac{\alpha_0}{\alpha_N})^{N-1} \leq \frac{t_n^N}{N} + \int_\Omega p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|)\, dx. \qquad (1.7.51)$$

In the particular case $N = 2$, from Lemma 1.7.1, we have that $p_2 = 0$. From (1.7.51), we obtain

$$t_n^{\frac{N}{N-1}} \geq \frac{\alpha_0}{\alpha_N}.$$

Thus, it suffices to consider $N \geq 3$. Using the definition of the function M_n, we obtain the following estimates

$$\int_\Omega |\nabla u_\lambda|^l |\nabla M_n|^k \, dx \leq \|\nabla u_\lambda\|^l_{L^\infty(\Omega)} \frac{R_n^N w_{N-1}^{\frac{N-k}{N}}}{(N-k)(\log n)^{\frac{k}{N}}}, \ \forall l \geq 0, \ 1 \leq k < N. \quad (1.7.52)$$

Now, from (1.7.52), Lemma 1.7.3, and the definition of the polynomial $p_N(x,y)$ (See Lemma 1.7.1.), there exists a positive constant C such that

$$\int_\Omega p_N(|\nabla u_\lambda|, |\nabla(t_n M_n)|) \, dx \leq \frac{CR_n^N}{(\log n)^{\gamma/N}}, \quad (1.7.53)$$

where $\gamma = 3/2$, for $N = 3$, and $\gamma = 2$, for $N \geq 4$. Hence, from (1.7.51), (1.7.53), we have

$$\frac{t_n^N}{N} \geq \frac{1}{N}(\frac{\alpha_0}{\alpha_N})^{N-1} - \frac{CR_n^N}{(\log n)^{\frac{\gamma}{N}}}.$$

Arguing as in the proof of Lemma 1.5.2, we get the conclusion of Lemma 1.7.4. ■

To prove Proposition 1.7.2, we use Lemmas 1.7.3 and 1.7.4 to derive the desired contradiction. From (1.7.46) and (1.7.47), for n sufficiently large, we have

$$t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M \int_{B(x_0, \frac{R_n}{N})} \exp(\alpha_0 |t_n M_n|^{\frac{N}{N-1}}) M_n^{1-\beta} \, dx.$$

Using the definition of M_n and Lemma 1.7.4, we get

$$t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M w_{N-1}^{\frac{N-(1-\beta)}{N}} \exp\left(\frac{-NCR_n^N \log n}{(\log n)^{\gamma/N}}\right) R_n^N (\log n)^{\frac{(N-1)(1-\beta)}{N}}, \quad (1.7.54)$$

for $N \geq 3$, and

$$t_n^\beta \|u_\lambda + t_n M_n\| \geq \lambda M w_{N-1}^{\frac{N-1}{N}} R^N \exp[(\frac{\alpha_0}{\alpha_N} t_n^{\frac{N}{N-1}} - 1) N \log n] \geq \lambda M w_{N-1}^{\frac{N-1}{N}} R^N, \quad (1.7.55)$$

for $N = 2$.

From the definition of R_n, we obtain

$$R_n^N (\log n)^{\frac{(N-1)(1-\beta)}{N}} = 1, \quad \text{and} \quad \frac{R_n^N \log n}{(\log n)^{\gamma/N}} = 1. \quad (1.7.56)$$

From (1.7.54) or (1.7.55) and (1.7.56), we have a contradiction because the left hand sides of (1.7.54) and (1.7.55) are bounded and M can be chosen arbitrarily large. This proves Proposition 1.7.2. ■

Finally, we observe that the proof of Theorem 1.1.4 follows the same argument employed in the proof of Theorem 1.1.3, with Proposition 1.7.2 replacing Proposition 1.5.1.

1.8 Appendix A

In this Appendix, we prove Lemma 1.2.5. First, we note that, without loss of generality, we may suppose $B(x_0, R) = B(0, R) \equiv B_R$.

Setting $u_M = \frac{1}{B_R} \int_{B_R} u(x)\, dx$, we may apply Lemma 7.16 in [28] to find $C = C(N) > 0$ such that

$$|u(x) - u_M| \leq C(N) \int_{B_R} \frac{|\nabla u(y)|}{|x-y|^{N-1}}\, dy, \text{ a. e. in } B_R.$$

Taking $v(x) = u(x) - u_M$, $h \in L^p(B_R)$, $p > 1$, $q = \frac{p}{p-1}$, and we use Hölder's inequality, as in [59], to obtain

$$\int_{B_R} |h(x)||v(x)|\, dx \leq$$
$$\leq C(N) \left[\int\int_{B_R \times B_R} \frac{|h(x)|}{|x-y|^{N-\frac{1}{q}}}\, dxdy \right]^{\frac{N-1}{N}} \left[\int\int_{B_R \times B_R} \frac{|\nabla u(x)|^N |h(x)|}{|x-y|^{\frac{N-1}{q}}}\, dxdy \right]^{\frac{1}{N}}.$$

Observing that the diameter of B_R is equal to $2R$, we get a constant $C_1(N) > 0$ such that

$$\int\int_{B_R \times B_R} \frac{|h(x)|}{|x-y|^{N-\frac{1}{q}}}\, dxdy \leq C_1(N) q \|h\|_{L^p(B_R)} R^{\frac{N+1}{q}}.$$

Applying Hölder's inequality one more time, we find $C_2(N) > 0$ such that

$$\int_{B_R} \frac{|h(x)|}{|x-y|^{\frac{N-1}{q}}}\, dx \leq C_2(N) \|h\|_{L^p(B_R)} R^{\frac{1}{q}}.$$

Combining the above inequalities, we find $C_3(N) > 0$ such that

$$\int_{B_R} |h(x)||v(x)|\, dx \leq C_3(N) q^{\frac{N-1}{N}} R^{\frac{N}{q}} \|h\|_{L^p(B_R)} \|\nabla u\|_{L^N(B_R)},$$

for every $h \in L^p(B_R)$. Therefore,

$$\|v\|_{L^q(B_R)} \leq C_3(N) q^{\frac{N-1}{N}} R^{\frac{N}{q}} \|\nabla u\|_{L^N(B_R)},$$

for every $q > 1$. Consequently, there exists $C_4(N) > 0$ so that

$$\int_{B_R} |u - u_M|^{\frac{Nq}{N-1}}\, dx \leq C_4(N) q^q R^N,$$

whenever $u \in W^{1,N}(B_R)$, $\|u\|_{W^{1,N}(B_R)} \le 1$. Now, we use the power series expansion of $\psi(t) = e^t$ and the above inequality to derive

$$\begin{aligned}&\textstyle\int_{B_R} \exp(\alpha|u-u_M|^{\frac{N}{N-1}})\,dx \le \\ &\le |B_R| + \alpha \textstyle\int_{B_R} |u-u_M|^{\frac{N}{N-1}}\,dx + R^N \sum_{q=2}^{\infty} \frac{\alpha^q C_4(N)^q q^q}{q!},\end{aligned}$$

if $\|u\|_{W^{1,N}(B_R)} \le 1$. Hence, there exist $\hat{\alpha} = \hat{\alpha}(N) > 0$, and $C_5(N) > 0$ such that

$$\int_{B_R} \exp\left(\hat{\alpha} 2^{\frac{1}{N}} |u-u_M|^{\frac{N}{N-1}}\right) dx \le C_5(N) R^N + \hat{\alpha} 2^{\frac{1}{N-1}} \|u-u_M\|^{\frac{N}{N-1}}_{L^{\frac{N}{N-1}}(B_R)}.$$

Since

$$|u_M| \le C_6(N) R^{-1} \|u\|^{L^N(B_R)}, \ \forall\, u \in W^{1,N}(B_R),$$

for some $C_6(N) > 0$, we may use the convexity of the function $\psi(t) = t^{\frac{N}{N-1}}$ to obtain $C(N,R) > 0$ such that

$$\int_{B_R} \exp\left(\hat{\alpha}|u|^{\frac{N}{N-1}}\right) dx \le C(N,R),$$

for every $u \in W^{1,N}(B_R)$ satisfying $\|u\|_{W^{1,N}(B_R)} \le 1$. Lemma 1.2.5 is proved. ∎

1.9 Appendix B

In this Appendix we prove Lemma 1.7.1. First, we establish an inequality that will be necessary in the sequel.

Lemma 1.9.1 *Let x, y be real numbers with $x > 0$ and $x + y \ge 0$. Consider $k = \frac{N}{2}$, where $N \in I\!N$, and $N \ge 3$. Then, there exist nonnegative constants C_1, C_2 such that*

$$(x+y)^k \le x^k + kx^{k-1}y + |y|^k + C_1 x|y|^{k-1} + C_2 x^{k-2} y^2.$$

Furthermore, $C_1 = C_2 = 0$ if $N = 3$ and 4, and $C_1 = 0$ when $N = 5$.

Proof: Since $x > 0$ and $(x+y)^k = x^k(1+yx^{-1})^k$, it suffices to consider $(1+z)^k$ for every $z \ge -1$.

(i) Case $N = 3$. Let $g(z) = 1 + \frac{3}{2}z + |z|^{\frac{3}{2}} - (1+z)^{\frac{3}{2}}$, for every $z \ge -1$. We must show that the function g is nonnegative. Direct calculation shows that $g'(z) \ge 0$ for every $z \ge 0$, and $g(0) = 0$. When $z \in [-1, 0]$, we consider $r = |z|$. Thus,

$g(z) = h(r) = 1 - \frac{3}{2}r + r^{\frac{3}{2}} - (1-r)^{\frac{3}{2}}$, and $h'(r) \geq 0$, and $h(0) = 0$. Hence, $g(z) \geq 0$ for every $z \geq -1$.

(ii) Case $N = 4$. The proof is immediate.

(iii) Case $N = 5$. Consider the polynomial function:

$$P(z) = (1+z)^{\frac{5}{2}} - (1 + \frac{5}{2}z + |z|^{\frac{5}{2}}).$$

By L'Hospital's Theorem, we have

$$\lim_{|z| \to 0} \frac{P(z)}{z^2} = \frac{15}{8}.$$

Moreover, by Mean Value Theorem, we get

$$\lim_{|z| \to \infty} \frac{P(z)}{z^2} = 0.$$

Consequently, there exists a nonnegative constant C such that

$$(1+z)^k \leq 1 + \frac{5}{2}z + |z|^{\frac{5}{2}} + Cz^2, \ \forall\ z \in I\!R.$$

(iv) Case $N \geq 6$. Consider the polynomial function:

$$P_k(z) = (1+z)^k - (1 + kz + |z|^k), \text{ where } k = \frac{N}{2}, \text{ and } z \geq -1.$$

Arguing as above, we have

$$\lim_{|z| \to 0} \frac{P_k(z)}{z^2} = \frac{k(k-1)}{2},$$

and

$$\lim_{|z| \to \infty} \frac{P_k(z)}{|z|^{k-1}} = k.$$

Consequently, there exist nonegative constants C_1, C_2 such that

$$(1+z)^k \leq 1 + kz + |z|^k + C_1|z|^{k-1} + C_2 z^2, \ \forall\ z \in I\!R.$$

Lemma 1.7.3 is proved. ∎

Proof of Lemma 1.7.1: The proof is immediate when $N = 2$. Thus, it suffices to verify the lemma for $N \geq 3$. Writing

$$|a+b|^N = (|a+b|^2)^{N/2} = (|a|^2 + 2\langle a,b\rangle + |b|^2)^{N/2},$$

and using Lemma 1.9.1 with $x = |a|^2$, and $y = 2\langle a,b\rangle + |b|^2$, we have

$$\begin{aligned} |a+b|^N &\leq |a|^N + N|a|^{N-2}\langle a,b\rangle + \frac{N}{2}|a|^{N-2}|b|^2 + (2|a||b| + |b|^2)^{N/2} + \\ &+ 2^{(N-2)/2}C_1|a|^2(2|a||b|)^{(N-2)/2} + 2^{(N-2)/2}C_1|a|^2|b|^{N-2} + \\ &+ 4C_2|a|^{N-2}|b|^2 + 4C_2|a|^{N-3}|b|^3 + C_2|a|^{N-4}|b|^4. \end{aligned}$$

Applying Lemma 1.9.1 one more time, we obtain

$$|a+b|^N \leq |a|^N + N|a|^{N-2}\langle a,b\rangle + |b|^N + p_N(|a|,|b|),$$

where

$$\begin{aligned} p_N(|a|,|b|) &= (\frac{N}{2} + 4C_2)|a|^{N-2}|b|^2 + N2^{(N-4)/2}|a|^{(N-2)/2}|b|^{(N+2)/2} + \\ &+ 2^{N-2}C_1|a|^{(N+2)/2}|b|^{(N-2)/2} + 2^{(N-2)/2}C_1|a|^2|b|^{N-2} + \\ &+ 2^{N/2}|a|^{N/2}|b|^{N/2} + 2^{(N-4)/2}C_2|a|^{(N-4)/2}|b|^{(N+4)/2} + \\ &+ 2C_1|a||b|^{N-1} + 4C_2|a|^{N-3}|b|^3 + C_2|a|^{N-4}|b|^4. \end{aligned}$$

Finally, since $C_1 = C_2 = 0$ if $N = 3$ and 5, and $C_1 = 0$ when $N = 5$, from the definition of p_N we conclude that the smallest exponent of $|b|$ is $3/2$, for $N = 3$, and 2, for $N \geq 4$, and the greatest exponent of $|b|$ is strictly smaller than N. Lemma 1.7.1 is proved. ∎

Capítulo 2

Quasilinear Dirichlet problems in $I\!R^N$ with critical growth

2.1 Introduction

In this article, we use variational methods to study the following quasilinear problem:

$$(GP) \qquad \begin{cases} -\Delta_p u = u^{p^*-1} + \lambda f(x,u) \ \text{ in } I\!R^N, \\ \quad u \geq 0 \text{ in } I\!R^N, \\ \quad \int_{I\!R^N} |\nabla u|^p \, dx < \infty, \end{cases}$$

where $\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian of u, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $1 < p < N$, $\lambda > 0$ is a real parameter and $f : I\!R^N \times I\!R \longrightarrow I\!R$ satisfies the following conditions:

(f_1) $f \in C(I\!R^N \times I\!R, I\!R)$ and $f(x,0) \equiv 0$.

(f_2) Given $R > 0$ there exist $\theta_R \in [p, p^*)$ and positive constants $a_R, b_R > 0$ such that

$$|f(x,s)| \leq a_R s^{\theta_R - 1} + b_R, \quad \forall\, |x| \leq R, \ \forall\, s \geq 0.$$

(f_3) There exist $r_1, r_2, q \in (1, p^*)$, with $r_1 \leq q \leq r_2$, an open subset $\Omega_0 \subset I\!R^N, c_i \in L^{\frac{p^*}{p^*-r_i}}(I\!R^N)$, $i = 1, 2$, and a positive constant a such that

$$\begin{cases} f(x,s) \leq c_1(x) s^{r_1-1} + c_2(x) s^{r_2-1}, & \forall\, x \in I\!R^N, \ s \geq 0, \\ F(x,s) \geq a s^q, & \forall\, x \in \Omega_0, \ s \geq 0, \end{cases}$$

where $F(x,s) = \int_0^s f(x,t)\, dt$.

We also assume a version of the famous Ambrosetti–Rabinowitz condition [6],

(f_4) There exist $p < \tau < p^*$, $1 < \mu < p^*$, and $c_3 \in L^{\frac{p^*}{p^*-\mu}}(\mathbb{R}^N)$ such that

$$\frac{1}{\tau}f(x,s)s - F(x,s) \geq -c_3(x)s^{\mu}, \ \forall\, x \in \mathbb{R}^N, \ s \geq 0.$$

Observing that $u \equiv 0$ is a (trivial) solution of (GP), our objective in this article is to apply minimax methods to study the existence of nontrivial solutions for (GP). However, it should be pointed out that we may not apply directly such methods since, under conditions $(f_1) - (f_4)$, the associated functional is not well defined in general. We also note that we look for weak solution $u \in D^{1,p}(\mathbb{R}^N)$ in the sense of distributions (See definition in Section 2.2).

Our technique combines pertubation arguments, the concentration-compactness principle [35, 36], appropriate estimates for the levels associated with the Mountain Pass Theorem [6], and the argument employed by Brezis and Nirenberg [14] to study semilinear elliptic problems with critical growth.

Considering $q \in \mathbb{R}$ given by condition (f_3), in our first result we also suppose the following technical condition:

(H) $q \in (1, p^*)$ satisfies $\hat{p} = p^* - \frac{p}{p-1} < q$.

Note that $\hat{p} < p$, $\hat{p} = p$ and $\hat{p} > p$ for $p^2 < N$, $p^2 = N$ and $p^2 > N$, respectively. We can now state our main theorem on the existence of a nontrivial solution for (GP):

Theorem 2.1.1 *Suppose f satisfies $(f_1) - (f_4)$, with q, r_1 given by (f_3) and q satisfying condition (H). Then,*

1. *If $1 < r_1 \leq p$, there exists $\lambda^* > 0$ such that problem (GP) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$.*
2. *If $p < r_1 < p^*$, then problem (GP) possesses a nontrivial solution for every $\lambda > 0$.*

We observe that a particular and relevant case associated with problem (GP) is given by

$$(P) \qquad \begin{cases} -\Delta_p u = u^{p^*-1} + \lambda a(x) u^{q-1} \ \text{ in } \mathbb{R}^N, \\ \quad u \geq 0 \text{ in } \mathbb{R}^N, \\ \quad \int_{\mathbb{R}^N} |\nabla u|^p \, dx < \infty, \end{cases}$$

where $q \in (1, p^*)$ satisfies (H) and $a : I\!R^N \longrightarrow I\!R$ is a continuous function satisfying the condition

$$(a_0) \quad a^+ = max(a, 0) \in L^{\frac{p^*}{p^*-q}}(I\!R^N) \text{ and } \exists\, x_0 \in I\!R^N \text{ such that } a(x_0) > 0.$$

The following result is a direct consequence of Theorem 2.1.1,

Theorem 2.1.2 *Suppose q satisfies (H) and a satisfies (a_0). Then,*

1. *If $1 < q \le p$, there exists $\lambda^* > 0$ such that problem (P) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$.*

2. *If $p < q < p^*$, then problem (P) possesses a nontrivial solution for every $\lambda > 0$.*

We observe that $f(x,s) = a(x)s^{q-1}$ satisfies $(f_1) - (f_2)$, (f_3) with $r_1 = q = r_2$, $c_1 = a^+$ and $c_2 \equiv 0$, and (f_4) with $\tau = q = \mu$ and $c_3 \equiv 0$ if $q > p$, and $\tau \in (p, p^*)$, $\mu = q$ and $c_3 = (\frac{1}{q} - \frac{1}{\tau})a^+$ if $1 < q \le p$.

Assuming the positivity of the primitive of the nonlinearity, we do not need to consider condition (H). More specifically, supposing

(f_5) $F(x,s) = \int_0^s f(x,t)\,dt \ge 0 \quad \forall x \in I\!R^N, \ s \ge 0,$

we obtain

Theorem 2.1.3 *Suppose f satisfies $(f_1) - (f_5)$, with r_1 given by condition (f_3). Then,*

1. *If $1 < r_1 \le p$, there exists $\lambda^* > 0$ such that problem (GP) possesses a nontrivial solution for every $\lambda \in (0, \lambda^*)$.*

2. *If $p < r_1 < p^*$, then problem (GP) possesses a nontrivial solution for every $\lambda > 0$.*

It is worthwhile to mention that Theorem 2.1.3 provides a version of Theorem 2.1.2 when $a \ge 0$, without assuming that q satisfies condition (H).

Problems involving critical Sobolev exponents have been considered by several authors since the seminal work of Brezis and Nirenberg [14], mainly when the domain is bounded. In recent years, the related problem for unbounded domain has been intensively studied (See, e.g., [3, 7, 8, 40, 44, 61] and their references).

In [7], Ben-Naoum, Troestler and Willem proved the existence of a nontrivial solution for (P), defined on a domain $\Omega \subset I\!R^N$, by considering the problem:

$$(P') \qquad \begin{cases} \text{minimize} & E(u) = \int_\Omega (|\nabla u|^p + a(x)|u|^q)\,dx, \\ \text{on the constraint} & u \in D^{1,p}(\Omega),\ \int_\Omega |u|^{p^*}\,dx = 1, \end{cases}$$

where $a \in L^{\frac{p^*}{p^*-q}}(\Omega)$, $a < 0$ on some subset of Ω with positive measure and $q > p^* - \frac{p}{p-1}$ when $p^2 > N$.

A recent result by Alves and Gonçalves [3] (See also [44]) establishes the existence of a nontrivial solution for (P), with $h(x)$ replacing $\lambda a(x)$ and satisfying $h(x) \geq 0$ and $h \in L^{\frac{p^*}{p^*-q}}$. In [3], it is supposed that either $1 < q < p$ and h is small, or $p < q < p^*$.

In [8], Benci and Cerami considered the case $p = q = 2$ and proved that problem (P) has at least one solution if $a(x)$ is a negative function, strictly negative somewhere, having $L^{N/2}$ norm bounded and belonging to $L^p(\mathbb{R}^N)$, for every p in a suitable neighbourhood of $\frac{N}{2}$.

Our theorems may be seen as a complement for the above mentioned results. We observe that in Theorems 2.1.1 and 2.1.3 a more general class of nonlinearity is considered. We also note that condition (f_3) provides only a local growth restriction on $f^-(x,s) \equiv \max\{-f(x,s), 0\}$. For example, we do not assume $a^- \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N)$ in Theorem 2.1.2. Finally, we should mention that our argument also holds for quasilinear equations defined on bounded or unbounded domains $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary conditions.

To prove Theorems 2.1.1 and 2.1.3, we first provide a technical result that establishes the existence of a weak solution in the sense of distributions for a class of quasilinear problems which may not have the associated functional well defined. In this technical result we assume the existence of a bounded sequence in $D^{1,p}$ of almost critical points for a sequence of functionals of class C^1. The main tool for our proof of this result is the concentration-compactness principle [35, 36]. To apply such result, we modify the nonlinearity, obtaining a family of functionals. Employing conditions $(f_2) - (f_3)$, we show that these functionals satisfy the geometric hypotheses of the Mountain Pass Theorem in a uniform way. Using this fact, (f_4) and our technical result, we are able to verify the existence of a sequence in $D^{1,p}(\mathbb{R}^N)$ converging weakly to a solution of (GP). Finally, we argue by contradiction, assuming that (GP) possesses only the trivial solution. This allows us to employ an argument similar to the one used by Brezis and Nirenberg in [14], deriving a contradiction.

The article is organized in the following way: Section 2.2 contains some preliminary materials, including the version of the Mountain Pass Theorem used in this article. In Section 2.3, we establish the above mentioned technical result. In Section 2.4, the estimates for the geometric hypotheses of the Mountain Pass Theorem are verified. Section 2.5 is devoted to prove the estimates from above for the critical levels. In Section 2.6, we prove Theorem 2.1.1. In Section 2.7, we establish the estimates when conditions (f_3) and (f_5) are assumed. There, we also present a proof of Theorem 2.1.3.

2.2 Preliminaries

Motivated by the Sobolev embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, for $1 < p < N$, and $p^* = \frac{Np}{N-p}$, we define $D^{1,p} \equiv D^{1,p}(\mathbb{R}^N)$ as the closure of $D(\mathbb{R}^N)$, the space of C^∞-functions with compact support, with respect to norm given by

$$\|\phi\| = \left(\int_{\mathbb{R}^N} |\nabla \phi|^p \, dx\right)^{1/p}.$$

Inspired by the work of Brezis and Nirenberg, [14], we make use in our argument of the extremal functions associated with the above embedding. For this purpose, we denote by S the best Sobolev constant, that is,

$$S = \inf_{u \in D^{1,p}\setminus\{0\}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\left(\int_{\mathbb{R}^N} |u|^{p^*} \, dx\right)^{p/p^*}} \right\}. \tag{2.2.1}$$

The infimum in (2.2.1) is achieved by the functions (See Talenti [58], Egnell [22]),

$$w_\varepsilon(x) = \frac{\left\{N\varepsilon[(N-p)/(p-1)]^{p-1}\right\}^{\frac{N-p}{p^2}}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}, \quad \forall\, x \in \mathbb{R}^N, \ \varepsilon > 0, \tag{2.2.2}$$

with

$$\|w_\varepsilon\|^p = \|w_\varepsilon\|_{L^{p^*}}^{p^*} = S^{N/p}, \ \forall\, \varepsilon > 0.$$

By weak solution of (GP), we mean a function $u \in D^{1,p}$ such that $u \geq 0$ *a.e.* in $\mathbb{R}^N$ and the following identity holds:

$$\int_{\mathbb{R}^N} |\nabla u|^{N-2} \nabla u . \nabla \phi \, dx - \int_{\mathbb{R}^N} |u|^{p^*-1} \phi \, dx - \lambda \int_{\mathbb{R}^N} f(x,u)\phi \, dx = 0,$$

for every $\phi \in D(\mathbb{R}^N)$.

Following a well known device used to obtain a solution for (GP), we let $f(x,s) = f(x,0) = 0$, for every $x \in \mathbb{R}^N$ and $s < 0$.

To modify the nonlinearity, we choose $\phi \in D(I\!\!R^N)$ satisfying $0 \leq \phi(x) \leq 1$, $\phi \equiv 1$ on the ball $B(0,1)$, and $\phi \equiv 0$ on $I\!\!R^N \setminus B(0,2)$. Let $n \in I\!\!N$ and $\phi_n(x) = \phi(\frac{x}{n})$. Define $f_n(x,s) = \phi_n(x) f(x,s)$, and consider the sequence of problems:

$$(GP)_n \qquad \begin{cases} -\Delta_p u = u^{p^*-1} + \lambda f_n(x,u), \text{ in } I\!\!R^N, \\ u \geq 0, \quad u \in D^{1,p}. \end{cases}$$

We now recall the variational framework associated with problem $(GP)_n$. Considering $D^{1,p}$ endowed with norm $\|u\| = \|\nabla u\|_{L^p}$, the functional associated with $(GP)_n$ is given by

$$I_{\lambda,n}(u) = \frac{1}{p}\int_{I\!\!R^N} |\nabla u|^p \, dx - \frac{1}{p^*}\int_{I\!\!R^N} (u^+)^{p^*} \, dx - \lambda \int_{I\!\!R^N} F_n(x,u) \, dx,$$

where $u^+ = \max\{u, 0\}$ and $F_n(x,s) = \int_0^s f_n(x,t)\, dt$. By hypothesis (f_2) and our construction, the functional $I_{\lambda,n}$ is well defined and belongs to $C^1(D^{1,p}, I\!\!R)$ (See[49]). Furthermore,

$$I'_{\lambda,n}(u)\phi = \int_{I\!\!R^N} |\nabla u|^{p-2}\nabla u.\nabla\phi \, dx - \int_{I\!\!R^N} (u^+)^{p^*-1}\phi \, dx - \lambda\int_{I\!\!R^N} f_n(x,u)\phi \, dx.$$

for every u and $\phi \in D^{1,p}$.

Now, for the sake of completeness, we state a basic compactness result (See [7] for a proof),

Proposition 2.2.1 *Let Ω be a domain, not necessarily bounded, of $I\!\!R^N$, $1 \leq p < N$, $1 \leq q < p^*$, and $a \in L^{\frac{p^*}{p^*-q}}(\Omega)$. Then, the functional*

$$D^{1,p}(\Omega) \rightarrow I\!\!R : u \longmapsto \int_\Omega a|u|^q \, dx,$$

is well defined and weakly continuous.

Finally, we state the version of the Mountain Pass Theorem of Ambrosetti-Rabinowitz [6] used in this work. Given E a real Banach space, $\Phi \in C^1(E, I\!\!R)$ and $c \in I\!\!R$, we recall that $(u_n) \subset E$ is a Palais-Smale $(PS)_c$ sequence associated with functional Φ if $\Phi(u_n) \rightarrow c$, and $\Phi'(u_n) \rightarrow 0$, as $n \rightarrow \infty$.

Theorem 2.2.2 *Let E be a real Banach space and suppose $\Phi \in C^1(E, I\!\!R)$, with $\Phi(0) = 0$, satisfies*
(Φ_1) *There exist positive constants β, ρ such that $\Phi(u) \geq \beta$, $\|u\| = \rho$,*

(Φ_2) *There exists* $e \in E$, $\|e\| > \rho$, *such that* $\Phi(e) \leq 0$.
Then, for the constant

$$c = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} \Phi(u) \geq \beta,$$

where $\Gamma = \{\gamma \in C([0,1], E);\ \gamma(0) = 0, \gamma(1) = e\}$, *there exists a* $(PS)_c$ *sequence* (u_j) *in* E *associated with* Φ.

2.3 Technical result

In this section we study the existence of a weak solution in the sense of distributions for the p-Laplacian in $\mathbb{R}^N$. Consider $g(x,s) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying

(g_1) Given $R > 0$ there exist positive constants a_R, b_R such that for every $x \in \mathbb{R}^N$ with $|x| \leq R$, and $s \in \mathbb{R}$,

$$|g(x,s)| \leq a_R |s|^{p^*-1} + b_R.$$

The associated functional I in $D^{1,p}$ is defined by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx - \int_{\mathbb{R}^N} G(x,u)\, dx, \tag{2.3.3}$$

where $G(x,s) = \int_0^s g(x,t)\,dt$. It is clear that, under condition (g_1), I may assume the values $\pm\infty$. However, if we assume the following stronger version of condition (g_1),

(g_2) There exist $a > 0$, $b \in C_0(\mathbb{R}^N)$, the space of continuous functions with compact support in $\mathbb{R}^N$, such that, for every $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$,

$$|g(x,s)| \leq a|s|^{p^*-1} + b(x),$$

then, I belongs to $C^1(D^{1,p}, \mathbb{R})$ and critical points of I are weak solutions of the associated quasilinear equation in $\mathbb{R}^N$. To establish the existence of a solution for the associated equation when (g_2) does not hold, we suppose the existence of a sequence of functions $\{g_n\} \subset C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying (g_2) and converging to g. More specifically, we assume

(g_3) Given $n \in \mathbb{N}$ there exists $g_n \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfying (g_2) and

$$g(x,s) = g_n(x,s),\ \forall\ |x| \leq n,\ \ s \in \mathbb{R}^N.$$

Let I_n be the sequence of functionals in $D^{1,p}$ associated with g_n via (2.3.3). We can now state our main result in this section,

Proposition 2.3.1 *Suppose $g(x,s) \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies (g_1) and (g_3). Then, any bounded sequence $(u_n) \subset D^{1,p}$ such that $I_n'(u_n) \to 0$, as $n \to \infty$, possesses a subsequence converging weakly to a solution of*

$$\begin{cases} -\Delta_p u = g(x,u), & in \ \mathbb{R}^N, \\ u \in D^{1,p}. \end{cases}$$

Remark 2.3.2 *We observe that in [56], we prove a related result for the N-Laplacian on bounded domain of $\mathbb{R}^N$ when the nonlinearity possesses exponential growth. But, unlike what happens in [56], here the functional is not of class C^1.*

The proof of Proposition 2.3.1 will be carried out through a series of steps. First, by Sobolev embedding and the principle of concentration-compactness [35, 36], we may assume that there exist $u \in D^{1,p}$, a nonnegative measure ν on $\mathbb{R}^N$, and sequences $x_i \in \mathbb{R}^N, \nu_i > 0$ and Dirac measures δ_{x_i} such that

$$\begin{cases} u_n \rightharpoonup u, \text{ weakly in } D^{1,p}, \\ u_n \to u, \text{ strongly in } L^s_{loc}(\mathbb{R}^N), \ 1 \le s < p^*, \\ u_n(x) \to u(x), \text{ a.e. in } \mathbb{R}^N, \\ |u_n|^{p^*} \rightharpoonup \nu = |u|^{p^*} + \sum_i \nu_i \delta_{x_i}, \text{ weakly* in } \mathcal{M}(\mathbb{R}^N), \\ |\nabla u_n|^p \rightharpoonup \mu, \text{ weakly* in } \mathcal{M}(\mathbb{R}^N), \\ \sum_i \nu_i^{p/p^*} < \infty. \end{cases} \tag{2.3.4}$$

Lemma 2.3.3 *There exists at most a finite number of points x_i on bounded subsets of $\mathbb{R}^N$.*

<u>**Proof:**</u> First, we note that it suffices to prove that there exists at most a finite number of points x_i on $B(0,r)$ for every $r > 0$. From (2.2.1) and Lemma 1.2 in [35], we obtain

$$\mu(\{x_i\}) \ge S\nu_i^{\frac{p}{p^*}}. \tag{2.3.5}$$

Now, for every $\varepsilon > 0$, we set $\psi_\varepsilon(x) = \psi(\frac{x-x_i}{\varepsilon})$, $x \in \mathbb{R}^N$, where $\psi \in D(\mathbb{R}^N), 0 \le \psi(x) \le 1, \psi(x) \equiv 1$ on $B(0,1)$, and $\psi(x) \equiv 0$ on $\mathbb{R}^N \setminus B(0,2)$. Since $I_n'(u_n) \to 0$, as $n \to \infty$, and $(\psi_\varepsilon u_n)$ is a bounded sequence, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n . \nabla(\psi_\varepsilon u_n)\, dx = \int_{\mathbb{R}^N} g_n(x,u_n)\psi_\varepsilon u_n\, dx + o(1).$$

By conditions (g_1), with $R > 2r$, and (g_3), for n sufficiently large, we get

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n.\nabla(\psi_\varepsilon u_n)\,dx \le a_R \int_{\mathbb{R}^N} |u_n|^{p^*}\psi_\varepsilon\,dx + b_R \int_{\mathbb{R}^N} |u_n|\psi_\varepsilon\,dx + o(1).$$

Now, from (2.3.4), taking $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n.\nabla(\psi_\varepsilon u_n)\,dx \le a_R \int_{\mathbb{R}^N} \psi_\varepsilon\,d\nu + b_R \int_{\mathbb{R}^N} |u|\psi_\varepsilon\,dx.$$

Invoking Lemma 1.2 in [35] again and taking $\varepsilon \to 0$, we obtain

$$\mu(\{x_i\}) \le a_R \nu(\{x_i\}) = a_R \nu_i.$$

Thus, from (2.3.5), we get $a_R\nu_i \ge S\nu_i^{\frac{p}{p^*}}$ and, consequently, $\nu_i \ge \frac{S^{\frac{N}{p}}}{a_R}$. Since $\sum_i \nu_i^{\frac{p}{p^*}} < \infty$, we conclude the proof of Lemma 2.3.3. ■

Lemma 2.3.4 *Let $K \subset \mathbb{R}^N$ be a compact set. Then, there exist $n_0 \in \mathbb{N}$ and $M = M(K) > 0$ such that*

$$\int_K |g_n(x, u_n(x))|^{\frac{p^*}{p^*-1}}\,dx \le M, \quad \forall\, n \ge n_0.$$

Proof: Take $n_0 \in \mathbb{N}$ such that $K \subset B(0, n_0)$. From (g_3), we have $g_n(x, u_n(x)) = g(x, u_n(x))$, for every $x \in K$, and $n \ge n_0$. Now, by condition (g_1) with $R = n_0$,

$$\int_K |g_n(x, u_n(x))|^{\frac{p^*}{p^*-1}}\,dx \le (2a_{n_0})^{\frac{p^*}{p^*-1}} \|u_n\|_{L^{p^*}}^{p^*} + (2b_{n_0})^{\frac{p^*}{p^*-1}} |K|, \quad \forall\, n \ge n_0.$$

The lemma follows by the Sobolev embedding and the hypothesis that (u_n) is a bounded sequence. ■

Lemma 2.3.5 *Let $K \subset (\mathbb{R}^N \setminus \{x_i\})$ be a compact set. Then $u_n \to u$ strongly in $L^{p^*}(K)$, as $n \to \infty$.*

Proof: Let $r > 0$ such that $K \subset B(0, r)$. By Lemma 2.3.3, there exists at most a finite number of points x_i on $B(0, r)$. Since K is a compact set and $K \cap \{x_i\} = \emptyset$, $\delta = d(K, \{x_i\})$, the distance between K and $\{x_i\}$, with $x_i \in B(0, r)$, is positive. Let $0 < \varepsilon < \delta$ and define $A_\varepsilon = \{x \in B(0, r) \mid d(x, K) < \varepsilon\}$. Choose $\psi \in D(\mathbb{R}^N), 0 \le \psi(x) \le 1, \psi \equiv 1$ on $A_{\frac{\varepsilon}{2}}$, and $\psi \equiv 0$ on $\mathbb{R}^N \setminus A_\varepsilon$. By construction, we have

$$\int_K |u_n|^{p^*}\,dx \le \int_{A_\varepsilon} \psi |u_n|^{p^*}\,dx = \int_{\mathbb{R}^N} \psi |u_n|^{p^*}\,dx.$$

Since $supp(\psi) \subset A_\varepsilon$ and $A_\varepsilon \cap \{x_i\} = \emptyset$, with $x_i \in B(0,r)$, from (2.3.4), we obtain

$$\begin{aligned}\lim_{n\to\infty}\sup \int_K |u_n|^{p^*}\,dx &\leq \int_{\mathbb{R}^N} \psi\,d\nu = \int_{\mathbb{R}^N} \psi|u|^{p^*}\,dx = \\ &= \int_{A_\varepsilon} \psi|u|^{p^*}\,dx \leq \int_{A_\varepsilon} |u|^{p^*}\,dx.\end{aligned}$$

Now, taking $\varepsilon \to 0$ and applying the Lebesgue's Dominated Convergence Theorem, we get

$$\lim_{n\to\infty}\sup \int_K |u_n|^{p^*}\,dx \leq \int_K |u|^{p^*}\,dx.$$

On the other hand, since $u_n \rightharpoonup u$ weakly in $L^{p^*}(K)$, it follows that

$$\|u\|_{L^{p^*}(K)} \leq \lim_{n\to\infty}\inf \|u_n\|_{L^{p^*}(K)}.$$

Consequently, as $L^{p^*}(K)$ is uniformly convex, $u_n \to u$ strongly in $L^{p^*}(K)$. Lemma 2.3.5 is proved. ■

Lemma 2.3.6 *Let $K \subset \mathbb{R}^N \setminus \{x_i\}$ be a compact set. Then, $\nabla u_n \to \nabla u$ strongly in $(L^p(K))^N$, as $n \to \infty$.*

Proof: Let $\psi \in C_0^\infty(\mathbb{R}^N \setminus \{x_i\})$ such that $\psi = 1$ on K and $0 \leq \psi \leq 1$. Using that the function $h : \mathbb{R}^N \to \mathbb{R}$, $h(x) = |x|^p$ is strictly convex, we have

$$0 \leq \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u).$$

Consequently,

$$\begin{aligned}0 &\leq \int_K \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u)\,dx \leq \\ &\leq \int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u)\psi\,dx,\end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_K [(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u).(\nabla(u_n - u))]\,dx \leq \\ &\leq \textstyle\int_{\mathbb{R}^N} [|\nabla u_n|^p\psi - |\nabla u_n|^{p-2}(\nabla u_n.\nabla u)\psi - \\ &- |\nabla u|^{p-2}(\nabla u.\nabla(u_n - u))\psi]\,dx.\end{aligned} \tag{2.3.6}$$

On the other hand, since $I_n'(u_n) \to 0$, as $n \to \infty$, we also have

$$\int_{\mathbb{R}^N} \left[|\nabla u_n|^{p-2}\left((\nabla u_n \nabla u)\psi + (\nabla u_n \nabla\psi)u\right) - \psi g_n(x, u_n)u\right]\,dx = o(1), \tag{2.3.7}$$

as $n \to \infty$. Moreover, since (ψu_n) is a bounded sequence in $D^{1,p}$, we get

$$\int_{\mathbb{R}^N} \left[|\nabla u_n|^p \psi + |\nabla u_n|^{p-2} (\nabla u_n \nabla \psi) u_n - \psi g_n(x, u_n) u \right] dx = o(1), \tag{2.3.8}$$

as $n \to \infty$. Combining (2.3.6)-(2.3.8), we obtain

$$\begin{aligned} 0 &\leq \textstyle\int_K [(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u).\nabla(u_n - u)]\, dx \leq \\ &\leq \textstyle\int_{\mathbb{R}^N} \psi g_n(x, u_n)(u_n - u)\, dx + \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}(\nabla u_n.\nabla\psi)(u_n - u)\, dx + \\ &+ \textstyle\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u.\nabla(u - u_n)\psi\, dx + o(1), \text{ as } n \to \infty. \end{aligned}$$

Applying Lemma 2.3.4 for the compact set $\Omega = supp(\psi)$, and using Holder's inequality, we get

$$\begin{aligned} 0 &\leq \textstyle\int_K [(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u).\nabla(u_n - u)]\, dx \leq \\ &\leq M^{\frac{p^*}{p^*-1}} \|u_n - u\|_{L^{p^*}(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)} \|u_n\|^{p-1} \|u - u_n\|_{L^p(\Omega)} + \\ &+ \textstyle\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u(\nabla u_n - \nabla u)\psi\, dx + o(1), \text{ as } n \to \infty. \end{aligned}$$

Now, applying Lemma 2.3.5 for the compact set $\Omega = supp(\psi) \subset (\mathbb{R}^N \setminus \{x_i\})$, from (2.3.4) and boundedness of (u_n), we have

$$\int_K \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right).\nabla(u_n - u)\, dx \to 0, \text{ as } n \to \infty.$$

Considering that

$$(|a|^{p-2}a - |b|^{p-2}b, a - b) \geq \begin{cases} C_p |a - b|^p & \text{if } p \geq 2, \\ C_p \frac{|a-b|^2}{(|a|+|b|)^{2-p}} & \text{if } 1 < p < 2, \end{cases}$$

for every a, $b \in \mathbb{R}^N$ (See [57]), if $p \geq 2$, we get

$$\lim_{n\to\infty} C_p \int_K |\nabla u_n - \nabla u|^p\, dx = 0.$$

Furthermore, when $1 < p < 2$, we have

$$\lim_{n\to\infty} C_p \int_K \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u| + |\nabla u_n|)^{2-p}}\, dx = 0. \tag{2.3.9}$$

Thus, by Holder's inequality,

$$\begin{aligned} \textstyle\int_K |\nabla(u_n - u)|^p\, dx &= \textstyle\int_K \frac{|\nabla(u_n-u)|^p}{(|\nabla u_n|+|\nabla u|)^{\frac{p(p-2)}{2}}} (|\nabla u_n| + |\nabla u|)^{\frac{p(p-2)}{2}}\, dx \leq \\ &\leq \left(\textstyle\int_K \frac{|\nabla(u_n-u)|^2}{(|\nabla u_n|+|\nabla u|)^{2-p}}\, dx\right)^{\frac{p}{2}} \left(\int_K (|\nabla u_n| + |\nabla u|)^p\, dx\right)^{\frac{2-p}{2}}. \end{aligned}$$

Finally, from this last inequality, (2.3.9), and the boundedness of (u_n), we have

$$\lim_{n\to\infty}\int_K |\nabla u_n - \nabla u|^p\,dx = 0.$$

Lemma 2.3.6 is proved. ■

As a direct consequence of Lemma 2.3.6, we have

Corollary 2.3.7 *The sequence $(u_n) \subset D^{1,p}$ possesses a subsequence (u_{n_j}) satisfying $\nabla u_{n_j}(x) \to \nabla u(x)$, for almost every $x \in I\!R^N$.*

Finally, we conclude the proof of Proposition 2.3.1: Given $\phi \in D(I\!R^N)$, take $n_0 > 0$ such that $supp(\phi) \subset B(0, n_0)$. From (g_3), we have

$$g_n(x,s) = g(x,s),\ \forall\, x \in supp(\phi),\ \text{and } n \geq n_0. \tag{2.3.10}$$

Condition (g_1), with $R > n_0$, and (2.3.10) provide

$$|g_n(x,s)\phi(x)| \leq (a_R s^{p^*-1} + b_R)|\phi(x)|,\ \forall\, x \in supp(\phi),\ s \in I\!R^N,\ n \geq n_0. \tag{2.3.11}$$

Invoking (2.3.4), (2.3.11) and the fact that $(u_n) \subset D^{1,p}$ is a bounded sequence, it follows that $(g_n(x,u_n)\phi)$ and $(|\nabla u_n|^{p-2}\nabla u_n \nabla\phi)$ are uniformly integrable families in $L^1(I\!R^N)$. Thus, by Vitali's Theorem and Corollary 2.3.7, we get

$$\begin{cases} \lim_{n\to\infty}\int_{I\!R^N} g_n(x,u_n(x))\phi(x)\,dx = \int_{I\!R^N} g(x,u(x))\phi(x)\,dx,\ \forall\,\phi \in D(I\!R^N), \\ \lim_{n\to\infty}\int_{I\!R^N} |\nabla u_n|^{p-2}\nabla u_n\nabla\phi\,dx = \int_{I\!R^N}|\nabla u|^{p-2}\nabla u\nabla\phi\,dx,\ \forall\,\phi \in D(I\!R^N). \end{cases} \tag{2.3.12}$$

Consequently, from (2.3.12) and $I_n'(u_n) \to 0$, as $n \to \infty$, we have

$$\int_{I\!R^N} |\nabla u|^{p-2}\nabla u\nabla\phi\,dx - \int_{I\!R^N} g(x,u(x))\phi(x)\,dx = 0,\ \forall\,\phi \in D(I\!R^N).$$

Proposition 2.3.1 is proved. ■

2.4 Mountain pass geometry

In this section, we prove that the family of functionals $I_{\lambda,n}$ satisfies conditions (Φ_1) and (Φ_2) of Theorem 2.2.2 in a uniform way.

Lemma 2.4.1 *Suppose f satisfies (f_2) and (f_3). Then,*

1. *If $1 < r_1 \le p$, there exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, $I_{\lambda,n}$ satisfies (Φ_1), with β and ρ independent of n.*

2. *If $p < r_1 < p^*$, then for every $\lambda > 0$, $I_{\lambda,n}$ satisfies (Φ_1), with β and ρ independent of n.*

Proof: Let $u \in D^{1,p}$, and $u \neq 0$. Using Holder's inequality with exponents $\frac{p^*}{p^*-r_i}$ and $\frac{p^*}{r_i}$, $i = 1, 2$, we have

$$\int_{\mathbb{R}^N} c_i(x)(u^+)^{r_i}\,dx \le \|c_i\|_{L^{\frac{p^*}{p^*-r_i}}} \|u^+\|^{r_i}_{L^{p^*}}. \tag{2.4.13}$$

Now, from the definition of ϕ_n, (f_3), (2.2.1) and (2.4.13), we get

$$\begin{aligned} I_{\lambda,n}(u) &= \frac{1}{p}\|u\|^p - \frac{1}{p^*}\|u\|^{p^*}_{L^{p^*}} - \lambda\int_{\mathbb{R}^N} F_n(x,u)\,dx \ge \\ &\ge \frac{1}{p}\|u\|^p - \frac{1}{p^*}\|u\|^{p^*}_{L^{p^*}} - \lambda\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}}\|u\|^{r_1}_{L^{p^*}} - \lambda\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}}\|u\|^{r_2}_{L^{p^*}} \ge \\ &\ge \frac{1}{p}\|u\|^p - \frac{1}{p^*S^{p^*/p}}\|u\|^{p^*} - \frac{\lambda}{r_1S^{r_1/p}}\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}}\|u\|^{r_1} - \\ &- \frac{\lambda}{r_2S^{r_2/p}}\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}}\|u\|^{r_2}. \end{aligned}$$

Case 1: $1 < r_1 \le p$. We have

$$I_{\lambda,n}(u) \ge \|u\|^p\Big(\frac{1}{p} - \frac{1}{p^*S^{p^*/p}}\|u\|^{p^*-p}\Big) - \lambda\Big(\frac{\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}}}{r_1S^{r_1/p}}\|u\|^{r_1} + \frac{\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}}}{r_2S^{r_2/p}}\|u\|^{r_2}\Big).$$

Consider

$$Q(t) = \frac{1}{p^*S^{p^*/p}}t^{p^*-p} \quad\text{and}\quad R(t) = \frac{\|c_1\|_{L^{\frac{p^*}{p^*-r_1}}}}{r_1S^{r_1/p}}t^{r_1} + \frac{\|c_2\|_{L^{\frac{p^*}{p^*-r_2}}}}{r_2S^{r_2/p}}t^{r_2},$$

Since $Q(t) \to 0$, as $t \to 0$, there exists $\rho > 0$ such that

$$\frac{1}{p} - Q(\rho) > 0.$$

Now, we choose $\lambda^* > 0$ such that

$$\frac{1}{p} - Q(\rho) - \lambda^* R(\rho) > 0.$$

Consequently, there exist ρ and $\beta > 0$, with ρ and β independent of n, such that

$$I_{\lambda,n}(u) \geq \beta, \quad \|u\| = \rho.$$

Case 2: $p < r_1 < p^*$. We have

$$I_{\lambda,n}(u) \geq \|u\|^p \Big(\frac{1}{p} - \frac{1}{p^* S^{p^*/p}} \|u\|^{p^*-p} - \frac{\lambda}{r_1 S^{r_1/p}} \|c_1\|_{L^{\frac{p^*}{p^*-r_1}}} \|u\|^{r_1-p} - \frac{\lambda}{r_2 S^{r_2/p}} \|c_2\|_{L^{\frac{p^*}{p^*-r_2}}} \|u\|^{r_2-p} \Big).$$

Considering

$$Q(t) = \frac{1}{p^* S^{p^*/p}} t^{p^*-p} + \frac{\lambda}{r_1 S^{r_1/p}} \|c_1\|_{L^{\frac{p^*}{p^*-r_1}}} t^{r_1-p} + \frac{\lambda}{r_2 S^{r_2/p}} \|c_2\|_{L^{\frac{p^*}{p^*-r_2}}} t^{r_2-p},$$

we note that $Q(t) \to 0$, as $t \to 0$, since $p < r_1 \leq r_2$. Hence, there exists $\rho > 0$ such that

$$\frac{1}{p} - Q(\rho) > 0.$$

Consequently, we get ρ and $\beta > 0$, with ρ and $\beta > 0$ independent of n, such that

$$I_{\lambda,n}(u) \geq \beta, \quad \|u\| = \rho.$$

Lemma 2.4.1 is proved. ■

Lemma 2.4.2 *Suppose f satisfies (f_2) and (f_3). Then, for every $\lambda > 0$ and $n \in \mathbb{N}$, $I_{\lambda,n}$ satisfies (Φ_2).*

Proof: Consider Ω_0 given by (f_3) and $\phi \in D(\mathbb{R}^N)$, a positive function with $supp(\phi) \subset \Omega_0$. For every $t > 0$, we have

$$I_{\lambda,n}(t\phi) \leq \frac{t^p}{p} \int_{\mathbb{R}^N} |\nabla \phi|^p \, dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |\phi|^{p^*} \, dx - \lambda a t^q \int_{\mathbb{R}^N} |\phi|^q \, dx.$$

Since $p^* > p$, there exists $t > 0$ sufficiently large such that $I_{\lambda,n}(t\phi) < 0$ and $\|t\phi\| > \rho$, with ρ given by Lemma 2.4.1. This proves the lemma. ■

2.5 Estimates

Considering Ω_0 given by (f_3), we take $x_0 \in \Omega_0$ and $r_0 > 0$ such that $B(x_0, 2r_0) \subset \Omega_0$. Now, let $n_0 \in \mathbb{N}$ be such that $B(x_0, 2r_0) \subset B(0, n_0)$. Choose $\phi \in D(\mathbb{R}^N)$ satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ on the ball $B(x_0, r_0)$, and $\phi \equiv 0$ on $\mathbb{R}^N \setminus B(x_0, 2r_0)$. Given $\varepsilon > 0$ and w_ε defined in Section 2.2, set

$$v_\varepsilon = \frac{\phi w_\varepsilon}{\|\phi w_\varepsilon\|_{L^{p^*}}}.$$

Then, v_ε satisfies (See, e.g., [14], [40])

$$X_\varepsilon \equiv \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^p \, dx \leq S + O(\varepsilon^{\frac{N-p}{p}}), \text{ as } \varepsilon \to 0. \tag{2.5.14}$$

Proposition 2.5.1 *Suppose f satisfies (f_2) and (f_3), with q satisfying condition (H). Then, for every $\lambda > 0$, there exist $\varepsilon > 0$, $n_0 \in \mathbb{N}$ and $d_\lambda > 0$ such that, for every $n \geq n_0$,*

$$\max\{I_{\lambda,n}(tv_\varepsilon) \mid t \geq 0\} \leq d_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

Proof: From (f_3) and the definitions of ϕ_n and v_ε, we have

$$\begin{aligned} I_{\lambda,n}(tv_\varepsilon) &= \frac{t^p}{p} X_\varepsilon - \frac{t^{p^*}}{p^*} - \lambda \int_{\mathbb{R}^N} F_n(x, tv_\varepsilon) \, dx \leq \\ &\leq \frac{t^p}{p} X_\varepsilon - \frac{t^{p^*}}{p^*} - \lambda t^q \int_{\mathbb{R}^N} a |v_\varepsilon|^q \, dx \equiv J_\lambda(tv_\varepsilon). \end{aligned}$$

Thus, to prove the proposition, it suffices to obtain $\varepsilon > 0$ and $d_\lambda > 0$ such that

$$\max\{J_\lambda(tv_\varepsilon) \mid t \geq 0\} \leq d_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

We argue as in the proof of Lemma 2.4.2. Given $\varepsilon > 0$, there exists some $t_\varepsilon > 0$ such that

$$\max_{t \geq 0} J_\lambda(tv_\varepsilon) = J_\lambda(t_\varepsilon v_\varepsilon) \text{ and } \frac{d}{dt} J_\lambda(tv_\varepsilon) = 0 \text{ in } t = t_\varepsilon.$$

This implies

$$0 < t_\varepsilon \leq X_\varepsilon^{\frac{1}{p^*-p}}.$$

On the other hand, from Lemma 2.4.1 and (2.5.14), we have

$$0 < \beta \le J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{t_\varepsilon^p}{p}(S + O(\varepsilon^{\frac{N-p}{p}})).$$

Hence, there exists $\alpha_0 > 0$ such that

$$\alpha_0 \le t_\varepsilon \le X_\varepsilon^{\frac{1}{p^*-p}}, \quad \forall\, \varepsilon > 0.$$

Since the function $h(s) = \frac{s^p}{p}X_\varepsilon - \frac{s^{p^*}}{p^*}$ is increasing on the interval $(0, X_\varepsilon^{\frac{1}{p^*-p}})$, we obtain

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N}X_\varepsilon^{\frac{N}{p}} - \lambda a {\alpha_0}^q \int_{\mathbb{R}^N} |v_\varepsilon|^q\, dx.$$

From (2.5.14) and using the inequality

$$(b+c)^\alpha \le b^\alpha + \alpha(b+c)^{\alpha-1}c \qquad \forall\, b,\ c \ge 0,\ \forall\, \alpha > 1,$$

with $b = S$, $c = O(\varepsilon^{\frac{N-p}{p}})$, and $\alpha = \frac{N}{p}$, we get

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N}S^{N/p} + O(\varepsilon^{\frac{N-p}{p}}) - \lambda\alpha_0^q \int_{\mathbb{R}^N} a|v_\varepsilon|^q\, dx.$$

Thus, there exists $M > 0$ such that

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N}S^{N/p} + \varepsilon^{\frac{N-p}{p}}\left(M - \frac{\lambda\alpha_0^q}{\varepsilon^{\frac{N-p}{p}}}\int_{\mathbb{R}^N} a|v_\varepsilon|^q\, dx\right) \le$$

$$\le \frac{1}{N}S^{N/p} + \varepsilon^{\frac{N-p}{p}}\left(M - \frac{\lambda a\alpha_0^q}{\varepsilon^{\frac{N-p}{p}}}\int_{B(0,1)} \frac{\varepsilon^{\frac{(N-p)q}{p^2}}}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{(N-p)q}{p}}}\, dx\right).$$

By changing variables, we obtain

$$J_\lambda(t_\varepsilon v_\varepsilon) \le \frac{1}{N}S^{N/p} + \varepsilon^{\frac{N-p}{p}}\Bigg(M -$$

$$-\lambda a w_{N-1}\alpha_0^q \varepsilon^{[(\frac{(N-p)}{p^2} - \frac{N-p}{p})q + \frac{(p-1)N}{p} + \frac{p-N}{p}]} \int_0^{\varepsilon^{\frac{1-p}{p}}} \frac{s^{N-1}}{(1 + s^{p/(p-1)})^{\frac{(N-p)q}{p}}}\, ds\Bigg).$$

Furthermore, for $\varepsilon > 0$ sufficiently small, we have

$$\int_0^{\varepsilon^{\frac{1-p}{p}}} \frac{s^{N-1}}{(1 + s^{p/(p-1)})^{\frac{(N-p)q}{p}}}\, ds \ge \int_0^1 \frac{s^{N-1}}{(1 + s^{p/(p-1)})^{\frac{(N-p)q}{p}}}\, ds \ge \frac{2^{\frac{(p-N)q}{p}}}{N}$$

because $g(s) = (1 + s^{p/(p-1)})^{-1} \geq g(1) = 2^{-1}$ for $s \in [0,1]$. Consequently, there exists a positive constant C, such that

$$J_\lambda(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} S^{N/p} + \varepsilon^{\frac{N-p}{p}} \Big(M - \lambda C \varepsilon^{[(\frac{(N-p)}{p^2} - \frac{N-p}{p})q + \frac{(p-1)N}{p} + \frac{p-N}{p}]} \Big).$$

Since $(\frac{(N-p)}{p^2} - \frac{(N-p)}{p})q + \frac{(p-1)N}{p} + \frac{p-N}{p}$ is negative, when q satisfies condition (H), we find $\varepsilon_0 > 0$ such that

$$\begin{aligned} J_\lambda(t_{\varepsilon_0} v_{\varepsilon_0}) &< d_\lambda \equiv \frac{1}{N} S^{N/p} + \varepsilon_0^{\frac{N-p}{p}} \Big(M - \lambda C \varepsilon_0^{[(\frac{(N-p)}{p^2} - \frac{N-p}{p})q + \frac{(p-1)N}{p} + \frac{p-N}{p}]} \Big) \\ &< \frac{1}{N} S^{N/p}. \end{aligned}$$

Proposition 2.5.1 is proved. ∎

2.6 Theorem 2.1.1

In view of Lemmas 2.4.1 and 2.4.2, we may apply Theorem 2.2.2 to the sequence of functionals $I_{\lambda,n}$, obtaining a positive level $c_{\lambda,n}$, and a $(PS)_{c_{\lambda,n}}$ sequence $(u_j^{(n)})_j$ in $D^{1,p}$, i.e.,

$$I_{\lambda,n}(u_j^{(n)}) \to c_{\lambda,n} \quad \text{and} \quad I'_{\lambda,n}(u_j^{(n)}) \to 0 \quad \text{as} \quad j \to \infty.$$

Moreover, from Lemma 2.4.1 and Proposition 2.5.1 , we have

$$0 < \beta \leq c_{\lambda,n} = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} I_{\lambda,n}(u) \leq d_\lambda < \frac{1}{N} S^{N/p}.$$

Taking a subsequence if necessary, we find $c_\lambda \in [\beta, d_\lambda]$ such that

$$c_\lambda = \lim_{n \to \infty} c_{\lambda,n}.$$

Thus, given $0 < \epsilon < \min\{c_\lambda, \frac{1}{N} S^{N/p}\}$, there exists $n_0 > 0$ such that $c_{\lambda,n} \in (c_\lambda - \epsilon\, , \, c_\lambda + \epsilon)$ for every $n \geq n_0$. Now, for each $n \geq n_0$, there exists $u_n = u_{j_n}^{(n)}$ satisfying

$$c_\lambda - \epsilon < I_{\lambda,n}(u_n) < c_\lambda + \epsilon, \tag{2.6.15}$$

and

$$\|I'_{\lambda,n}(u_n)\| \leq \frac{1}{n}. \tag{2.6.16}$$

Lemma 2.6.1 *The sequence (u_n) is bounded in $D^{1,p}$.*

Proof: From (f_4), there exist $\tau \in (p, p^*)$ and $\mu \in (1, p^*)$ such that

$$\begin{aligned} I_{\lambda,n}(u_n) &- \frac{1}{\tau} I'_{\lambda,n}(u_n)u_n = (\frac{1}{p} - \frac{1}{\tau})\|u_n\|^p + (\frac{1}{\tau} - \frac{1}{p^*})\|u_n^+\|^{p^*}_{L^{p^*}} + \\ &+ \lambda \int_{\mathbb{R}^N} \left(\frac{1}{\tau} f_n(x, u_n)u_n - F_n(x, u_n)\right) dx \geq \\ &\geq (\frac{1}{p} - \frac{1}{\tau})\|u_n\|^p + (\frac{1}{\tau} - \frac{1}{p*})\|u_n^+\|^{p^*}_{L^{p^*}} - \lambda \int_{\mathbb{R}^N} c_3(x)|u_n^+|^\mu \, dx. \\ &\geq (\frac{1}{p} - \frac{1}{\tau})\|u_n\|^p + (\frac{1}{\tau} - \frac{1}{p^*})\|u_n^+\|^{p^*}_{L^{p^*}} - \lambda \|c_3\|_{L^{\frac{p^*}{p^*-\mu}}} \|u_n^+\|^\mu_{L^{p^*}}. \end{aligned} \tag{2.6.17}$$

On the other hand, from (2.6.15) and (2.6.16), we have

$$I_{n,\lambda}(u_n) - \frac{1}{\tau} I'_{n,\lambda}(u_n)u_n \leq C + \frac{1}{\tau}\|u_n\|. \tag{2.6.18}$$

Denoting $h(t) = (\frac{1}{\tau} - \frac{1}{p^*})t^{p^*} - \lambda\|c_3\|t^\mu$, for $t \geq 0$, from (2.6.17), (2.6.18), and using that $h(t)$ is bounded from below, we conclude that the sequence (u_n) is bounded in $D^{1,p}$. Lemma 2.6.1 is proved. ■

Applying Proposition 2.3.1 to the diagonal sequence (u_n), we obtain a weak solution u for problem (GP). The final step is the verification that u is nontrivial. First of all, we note that $u^- = 0$. Effectively

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(u_n^-)|^p \, dx &\leq \int_{\mathbb{R}^N} \left(|\nabla(u_n^+)|^2 + |\nabla(u_n^-)|^2\right)^{\frac{p}{2}} dx = \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^p \, dx. \end{aligned}$$

Thus, the sequence (u_n^-) is bounded in $D^{1,p}$. Consequently, $I'_{\lambda,n}(u_n)(u_n^-) \to 0$, as $n \to \infty$. Since $I'_{\lambda,n}(u_n)(u_n^-) = \frac{1}{p}\|u_n^-\|^p$, it follows that $u_n^- \to 0$ in $D^{1,p}$, as $n \to \infty$.

Now, we assume by contradiction that $u \equiv 0$ is the only possible solution of (GP). Let

$$l = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n^+|^{p^*} \, dx.$$

From (f_3) and (2.6.16), we have

$$0 = \lim_{n \to \infty} I'_{\lambda,n}(u_n)u_n \geq \lim_{n \to \infty} \int_{\mathbb{R}^N} \left(|\nabla u_n|^p - |u_n^+|^{p^*} - \lambda c_1|u_n|^{r_1} - \lambda c_2|u_n|^{r_2}\right) dx.$$

Consequently, by Proposition 2.2.1, we have

$$\lim_{n\to\infty}\int_{I\!R^N} |\nabla u_n|^p\,dx \le l. \tag{2.6.19}$$

We claim that $l > 0$. Effectively, arguing by contradiction, we suppose that $l = 0$. Under this assumption, from (2.6.19) and (f_3), $I_{\lambda,n}(u_n) \to 0$, as $n \to \infty$. But this is impossible in view of (2.6.15). The claim is proved.

Invoking (2.2.1), we have

$$\|\nabla u_n\|_p^p \ge \|\nabla(u_n^+)\|_p^p \ge S\Big(\int_{I\!R^N} |u_n^+|^{p^*}\Big)^{\frac{p}{p^*}}. \tag{2.6.20}$$

As a direct consequence of (2.6.20), and (2.6.19), we obtain

$$l \ge S^{\frac{N}{p}}. \tag{2.6.21}$$

By (f_4), (2.6.15) and (2.6.16), we get

$$\begin{aligned}
\frac{1}{N}S^{N/p} &> d_\lambda \ge c_\lambda = \lim_{n\to\infty} I_{\lambda,n}(u_n) = \lim_{n\to\infty}\Big[I_{\lambda,n}(u_n) - \frac{1}{\tau}I'_{\lambda,n}(u_n)u_n\Big] = \\
&= \lim_{n\to\infty}\Big[(\frac{1}{p}-\frac{1}{\tau})\|\nabla u_n\|^p + (\frac{1}{\tau}-\frac{1}{p^*})\|u_n^+\|_{L^{p^*}}^{p^*} - \int_{I\!R^N} c_3 u_n^\mu\,dx\Big] \ge \\
&\ge \lim_{n\to\infty}\Big[(\frac{1}{p}-\frac{1}{\tau})S\|u_n\|_{L^{p^*}}^p + (\frac{1}{\tau}-\frac{1}{p^*})\|u_n^+\|_{L^{p^*}}^{p^*} - \int_{I\!R^N} c_3 u_n^\mu\,dx\Big].
\end{aligned}$$

Consequently, from Proposition 2.2.1 and (2.6.21), we have

$$\begin{aligned}
\frac{1}{N}S^{N/p} &> (\frac{1}{p}-\frac{1}{\tau})Sl^{p/p^*} + (\frac{1}{\tau}-\frac{1}{p^*})l \ge \\
&\ge (\frac{1}{p}-\frac{1}{\tau})S^{1+\frac{N}{p^*}} + (\frac{1}{\tau}-\frac{1}{p^*})S^{N/p} = \frac{1}{N}S^{N/p}.
\end{aligned}$$

This concludes the proof of Theorem 2.1.1. ■

2.7 Theorem 2.1.3

In this section we establish a proof of Theorem 2.1.3. The key ingredient is the verification of Proposition 2.5.1 under conditions (f_3) and (f_5). To obtain such result we exploit the positivity of the function $F(x,s)$. Considering the extremal functions w_ε defined by (2.2.2), we have

Proposition 2.7.1 *Suppose f satisfies $(f_2), (f_3)$, and (f_5). Then, for every $\lambda > 0$, there exist $\varepsilon > 0$, $n_0 \in I\!N$ and $d_\lambda > 0$ such that, for every $n \geq n_0$*

$$\max\{I_{\lambda,n}(tw_\varepsilon) \mid t \geq 0\} \leq d_\lambda < \frac{1}{N} S^{\frac{N}{p}}.$$

Proof: Let $n_0 \in I\!N$ such that $\hat{\Omega}_0 \equiv B(0, n_0) \cap \Omega_0 \neq \emptyset$. From $(f_2), (f_3), (f_5)$ and our definition of $I_{\lambda,n}$, for every $n \geq n_0$, we have

$$I_{\lambda,n}(tw_\varepsilon) = (\frac{t^p}{p} - \frac{t^{p^*}}{p^*})S^{N/p} - \lambda \int_{I\!R^N} \phi_n(x) F(x, tw_\varepsilon)\, dx \leq$$

$$\leq (\frac{t^p}{p} - \frac{t^{p^*}}{p^*})S^{N/p} - \lambda \int_{\hat{\Omega}_0} \phi_n(x) F(x, tw_\varepsilon)\, dx \leq$$

$$\leq (\frac{t^p}{p} - \frac{t^{p^*}}{p^*})S^{N/p} - \lambda a t^q \int_{\hat{\Omega}_0} |w_\varepsilon|^q\, dx \equiv J_\lambda(tw_\varepsilon).$$

Thus, it suffices to obtain $\varepsilon > 0$ and $d_\lambda > 0$ such that

$$\max\{J_\lambda(tw_\varepsilon) \mid t \geq 0\} \leq d_\lambda < \frac{1}{N} S^{N/p}.$$

To prove such result we follow the argument employed in [3]. By (2.2.2), the sequence w_ε is bounded in $L^{\frac{p^*}{q}}(I\!R^N)$ and $w_\varepsilon(x) \to 0$ a.e. in $I\!R^N$, as $\varepsilon \to 0$. Thus, $w_\varepsilon \rightharpoonup 0$ weakly in $L^{\frac{p^*}{q}}(I\!R^N)$, as $\varepsilon \to 0$. On the other hand, the restriction $w_\varepsilon \mid_{\hat{\Omega}_0}$ belongs to $W^{1,p}(\hat{\Omega}_0)$. Hence, by the Sobolev Embedding Theorem, $w_\varepsilon \to 0$ strongly in $L^r(B(0, 2n))$, for every $1 \leq r < p^*$. Consequently,

$$\lim_{\varepsilon \to 0} \int_{\hat{\Omega}_0} |w_\varepsilon|^q\, dx = 0.$$

Therefore, there exists $\varepsilon_0 > 0$ such that

$$0 < J_\lambda(w_{\varepsilon_0}) < \frac{1}{N} S^{N/p}. \tag{2.7.22}$$

Now, arguing as in the proof of Lemma 2.4.2 , we take $t_{\varepsilon_0} > 0$ such that

$$J_\lambda(t_{\varepsilon_0} w_{\varepsilon_0}) = \max\{J_\lambda(tw_\varepsilon) \mid t \geq 0\}.$$

Since $\frac{d}{dt}I_{\lambda,n}(tw_{\varepsilon_0}) = 0$, for $t = t_{\varepsilon_0}$, we get

$$(t_{\varepsilon_0}^{p-1} - t_{\varepsilon_0}^{p^*-1})S^{N/p} - \lambda a t_{\varepsilon_0}^{q-1} \int_{\hat{\Omega}_0} |w_{\varepsilon_0}|^q \, dx = 0. \tag{2.7.23}$$

From (2.7.22) and (2.7.23), we have

$$0 < t_{\varepsilon_0} < 1.$$

Observing that the function $h(t) = \frac{t^p}{p} - \frac{t^{p^*}}{p^*}$ achieves its maximum at $t = 1$, we get

$$J_\lambda(t_{\varepsilon_0} w_{\varepsilon_0}) \leq d_\lambda \equiv J_\lambda(w_{\varepsilon_0}) < \frac{1}{N} S^{N/p}.$$

Proposition 2.7.1 is proved. ■

Finally, we observe that the proof of Theorem 2.1.3 follows by the same argument employed in the proof of Theorem 2.1.1, with Proposition 2.7.1 replacing Proposition 2.5.1.

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