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# ON FUZZY DIFFERENTIAL EQUATIONS

# *SOBRE EQUAÇÕES DIFERENCIAIS FUZZY*

**CAMPINAS**
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Universidade Estadual de Campinas
Instituto de Matemática, Estatística
e Computação Científica

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Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Applied Mathematics.

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**Supervisor/*Orientador*: Laécio Carvalho de Barros**
**Co-supervisor/*Co-orientador*: Barnabas Bede**

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## Abstract

From the definition of fuzzy derivative and integral via Zadeh's extension of the derivative and integral for classical functions we obtain a fundamental theorem of calculus and develop a new theory for fuzzy differential equations (FDEs). Different from the previous concepts of fuzzy derivatives (Hukuhara and generalized derivatives) and integrals, defined for fuzzy-set-valued functions, the approach we propose deals with fuzzy bunches of functions (fuzzy subsets of spaces of functions). Under reasonable conditions, the new operations are equivalent to differentiating (or integrating) the classical functions of the levels.

We present the most known previous approaches of FDEs. Comparisons with the new theory we propose are carried out calculating fuzzy attainable sets of the solutions. Under certain conditions, the solutions via strongly generalized derivative coincide with solutions using our approach. The same happens with solutions to fuzzy differential inclusions and Zadeh's extension of the crisp solution. Although these two methods do not treat FDEs, they are widespread for making use of classical functions (similarly to what is proposed in this thesis) and for preserving properties of classical dynamical systems. These are advantageous features since it shows that the new theory presents desirable properties of the other two mentioned theories (allowing for instance periodicity and stability of solutions), besides treating FDEs.

The theory is illustrated by applying it on biological models and commenting the results.

**Keywords:** Fuzzy differential equations; Zadeh's extension; Fuzzy derivative; Fuzzy integral; Analysis.

## Resumo

A partir da proposta das definições de derivada e integral fuzzy via extensão de Zadeh dos respectivos operadores para funções clássicas, obtemos uma versão do teorema fundamental do cálculo e desenvolvemos uma nova teoria de equações diferenciais fuzzy (EDFs). Diferentemente dos conceitos anteriores de derivadas (Hukuhara e generalizadas) e integrais para funções fuzzy, em que as funções assumem valores em conjuntos fuzzy, a abordagem aqui proposta lida com tubos fuzzy de funções (subconjuntos fuzzy de espaços de funções). Sob condições razoáveis, as novas operações equivalem a diferenciar (ou integrar) as funções clássicas dos níveis.

Apresentamos as abordagens anteriores de EDFs mais conhecidas e, para realizar comparações com a nova teoria, calculamos os conjuntos atingíveis fuzzy das soluções. Provamos que algumas soluções da teoria proposta equivalem às via derivada fortemente generalizada. Também demonstramos a equivalência, sob determinadas condições, com as soluções via inclusões diferenciais fuzzy e extensão de Zadeh da solução clássica. Apesar destas duas abordagens não tratarem de EDFs, elas são largamente difundidas por utilizarem derivadas de funções clássicas (de modo similar ao aqui proposto) e de preservarem características das soluções de sistemas dinâmicos clássicos. Esses são fatos vantajosos, pois mostram que a teoria proposta, além de tratar de EDFs, possui propriedades desejáveis das outras duas mencionadas, permitindo a ocorrência de estabilidade e periodicidade de soluções, por exemplo.

A teoria é ilustrada através de sua aplicação em modelos biológicos e análise dos resultados.

**Palavras-chave:** Equações diferenciais fuzzy; Extensão de Zadeh; Derivada Fuzzy; Integral fuzzy; Análise.

# Contents

*The time has come*
*For closing books*
*And long last looks must end*
*And as I leave*
*I know that I am leaving my best friend*
*A friend who taught me right from wrong*
*And weak from strong*
*That's hard to learn.*
*(...)*
*To Sir, with Love.*

Don Black and Mark London

*So much things to say.*

Bob Marley

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# List of Symbols

$a, b, f, g, x, y$, etc. (lower case): non-fuzzy elements.
$A, B, F, G, X, Y$, etc. (capital letters): (fuzzy or non-fuzzy) subsets.
$\mathbb{U}, \mathbb{V}$: general non-fuzzy universes.
$\mathcal{K}^n$ the family of all nonempty compact subsets of $\mathbb{R}^n$.
$\mathcal{K}^n_{\mathcal{C}}$ the family of all nonempty compact and convex subsets of $\mathbb{R}^n$.
$\mathcal{F}(\mathbb{U})$: the family of all fuzzy subsets of $\mathbb{U}$.
$\mathcal{F}_{\mathcal{K}}(\mathbb{U})$: the family of all fuzzy subsets of $\mathbb{U}$ with nonempty compact $\alpha$-cuts.
$\mathcal{F}_{\mathcal{C}}(\mathbb{U})$: the family of all fuzzy subsets of $\mathbb{U}$ with nonempty compact and convex $\alpha$-cuts.
$[A]_\alpha$: $\alpha$-cut of the fuzzy subset $A$.
$a^\alpha_-, a^\alpha_+$: lower and upper endpoints of the $\alpha$-cut of the fuzzy number $A$, that is, $[A]_\alpha = [a^\alpha_-, a^\alpha_+]$.
$E([a, b]; \mathbb{U})$: general space of functions from $[a, b]$ to $\mathbb{U}$.
$E([a, b]; \mathcal{F}(\mathbb{U}))$: general space of functions from $[a, b]$ to $\mathcal{F}(\mathbb{U})$.
$C([a, b]; \mathbb{R}^n)$: space of continuous functions from $[a, b]$ to $\mathbb{R}^n$.
$\mathcal{A}C([a, b]; \mathbb{R}^n)$: space of absolutely continuous functions from $[a, b]$ to $\mathbb{R}^n$ (see Appendix).
$L^p([a, b]; \mathbb{R}^n)$: $L^p$ space of functions from $[a, b]$ to $\mathbb{R}^n$ (see Appendix).
$\widehat{f}$: Zadeh's extension of function $f$ (see Section 2.2).
$\widehat{D}$: Zadeh's extension of the derivative operator $D$ (see Subsection 3.2.2).
$\widehat{\int}$: Zadeh's extension of the integral operator $\int$ (see Subsection 3.2.1).
$a \wedge b$: $a \wedge b = \min\{a, b\}$.
$d_\infty$: Pompeiu-Hausdorff metric for spaces of fuzzy subsets (see Section 2.4).
$d_E$: Endographic metric for spaces of fuzzy subsets (see Section 2.4).
$d_p$: $L^p$ type metric for spaces of fuzzy subsets (see Section 2.4).
$\overline{B}(X, q)$: closed ball $\overline{B}(X, q) = \{A \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}) : d_\infty(X, A) \leq q\}$.
$F'_H(x)$: Hukuhara derivative of the fuzzy function $F$ at $x$ (see Subsection 3.1.2).
$F'_G(x)$: strongly generalized derivative of the fuzzy function $F$ at $x$ (see Subsection 3.1.2).
$F'_{gH}(x)$: generalized Hukuhara derivative of the fuzzy function $F$ at $x$ (see Subsection 3.1.2).
$F'_g(x)$: fuzzy generalized derivative of the fuzzy function $F$ at $x$ (see Subsection 3.1.2).

# Chapter 1

# Introduction

The first example Zadeh describes in the first publication on fuzzy sets (Zadeh, 1965) is about taxonomy (classification of organisms). More specifically, it illustrates the lack of sharpness in the classification of beings into animals. A dog, a horse and a bird are animals; a rock and a plant are clearly not animals; a starfish and a bacteria, though, are not so clearly in or out the group of animals. This shows that human beings' classification is subjective. In the beginning this argument was not enough to convince the scientific community of the importance of the proposed theory and, after initially being the object of much skepticism and derision (see Zadeh (2008)), fuzzy set theory is now widespread in many different fields of knowledge and human reasoning modeling.

The modeling of any phenomena by humans is also subjected to the limitation of the human being in understanding, collecting data, interpreting and concluding. Besides, it is very common that the phenomena itself have some kind of inherent uncertainty (noise), reinforcing the need of a theory that deals with this type of concept.

## Fuzzy Sets and Mathematical Biology

Mathematical biology (see e.g. Edelstein-Keshet (1988) and Murray (2004)) consists in employing mathematical tools to model biological phenomena, such as epidemiology problems, population dynamics, ecological systems and genetics. As cited before, uncertainties are present in the process of modeling, as well as in the behavior such as in viruses and populations in general. To deal with uncertainties there are mainly two kinds of approaches: the probabilistic one and the fuzzy one. We are interested in the last one.

The employment of fuzzy sets theory is present in many studies in biological problems:

- In epidemiology the SI (suscetiptible-infectious) system has been explored in various research articles (Barros et al., 2003; Cabral, 2011; Cabral and Barros, 2012; Santo Pedro and Barros, 2013); communicable diseases such as HIV (Jafelice, 2003; Jafelice et al., 2004) and dengue (Gomes and Barros, 2009; Silveira and Barros, 2013) have also been modeled and studied. Other studies of interest are Barros and Bassanezi (2010); Massad et al. (2008); Ortega et al. (2003).

- In population dynamics exponential and logistic growth, prey-predator models, among others have been explored (Barros and Bassanezi, 2010; Cecconello, 2010; Křivan and Colombo, 1998; Simões, 2013; Silva et al., 2013).
- In diagnosis and other applications in medicine (Barros and Bassanezi, 2010; Castanho et al., 2013; Majumder and Majumdar, 2004; Massad et al., 2008; Silveira et al., 2008; Fialho et al., 2012).

The probabilistic approach deals with random (or stochastic) processes, that is, it has to do with the measure of the chance of occurrence of events. Fuzzy approaches were created to model entities whose sets have nonsharp boundaries in which the elements may have *partial* membership to it, not being considered completely in or completely out of the given set. But it also plays important role in the theory of possibility which uses fuzzy sets theory and deals with partial knowledge and the related uncertainties. What we see in general is the presence of both kinds of uncertainties in the biological phenomena, though the modeling is complex and few dared to explore and apply this joint approach (see Gomes and Barros (2009); Missio (2008); Silveira and Barros (2013); Fialho et al. (2012)).

Modeling of various phenomena frequently makes use of differential equations. In order to include the imprecision, the fuzzy approach is often used. In particular, differential inclusions and, more recently, fuzzy differential equations, or even fuzzy differential inclusions have been used.

In population dynamics, for example, Křivan (1995) recalls that individuals may exhibit some preferences or strategies, that is, inside a group they do not behave all in the same manner. Environmental or demographic noise also is another source of uncertainty. Via standard theory of differential equations, it is not possible to take these factors into account.

Křivan (1995) and Křivan and Colombo (1998) claim for the use of fuzzy differential inclusions in population dynamics. According to these studies, the stochastic approach, via the use of white noise (a linear term in the differential equation) to model the uncertainty of the dynamics, is not the most appropriate one. The probabilistic approach would be suitable for the "hard sciences", such as physics and electronics, not for a "soft science" as biology. The white noise would emphasize short time scales and would lead to mathematically tractable models, hence it was used to treat many problems, but there are many others that demand for a different approach. The alternative would be the deterministic noise, including what they call the unknown-but-bounded-noise, i.e., the imprecision enters the dynamics via a parameter whose only assumption is that its values belong to a bounded set $U$, which may depend on time or the state variable. This approach leads to differential inclusions (see Aubin and Cellina (1984)) and considering some kind of "preference" of some parameter(s) in $U$, determines a higher membership degree of a more suitable solution characterizing differential inclusions that are fuzzy.

It is also possible to define fuzzy derivatives and consider the function as fuzzy as it has been done by many authors (Puri and Ralescu, 1983; Kaleva, 1987; Seikkala, 1987; Bede and Gal, 2005; Stefanini and Bede, 2009; Chalco-Cano et al., 2011b; Barros et al., 2013). The first proposal, based on the Hukuhara derivative for interval-valued functions, has been criticized for presenting nondecreasing fuzziness. In other words, a dynamical system whose initial uncertainty is different from zero does not evolve to nonfuzzy states. This situation is not consistent with population (or nuclear) decay model.

## Initial Value Problems

This thesis treats fuzzy initial value problems (FIVP). An initial value problem (IVP) is a system of an ordinary differential equation together with a value called initial condition:

$$\begin{cases} x'(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{cases} . \tag{1.1}$$

In what we call the "classical case", the solution is usually defined as a real-valued continuous function $x(\cdot)$ that satisfies the initial condition and the differential equation at every $t$ in a given domain. The function $x(\cdot)$ is interpreted as a curve such that the direction to be followed is determined by the function $f$, at each real value $t$. In this case, the solution is a real-valued function with real-valued argument. We call the IVP a "classical IVP" and the solution a "classical solution".

This approach is widely used to model physical, biological, chemical phenomena. In a biological interpretation, $x$ can be the number of individuals of a population (ants, fishes, predators, humans, viruses, infected people), $t$ is time and $f$ is the *rate* with which the population changes in quantity. The ability of modeling various phenomena, as well as theorems regarding existence of solution and practical techniques to find it (analitically or numerically), justifies wide use of IVPs.

Fuzzy set theory treats of sets in a universe such that the elements have partial membership degree. That is, it is admissible that an element is not completely in or completely out of the set, but presents an intermediate degree. The success of fuzzy set theory, specially in modeling some control problems, has generated interest in many fields. Several concepts of the "non-fuzzy" theory were extended to the fuzzy case. This is no different in differential equations theory.

### Fuzzy Initial Value Problem

Problem (1.1) becomes a new problem if any parameter presents fuzziness and it is called fuzzy initial value problem. The first time the term *Fuzzy Differential Equation* (*FDE*) was used, was Kandel and Byatt (1980) and only in 1987 did FDE took on characteristic of the way it is used nowadays (Kaleva, 1987; Seikkala, 1987). Kaleva (1987) made use of the Hukuhara derivative for fuzzy-set-valued functions and Seikkala (1987) used an equivalent definition. In both studies, the FIVP was defined using a fuzzy differential equation and a fuzzy initial value:

$$\begin{cases} X'(t) &= F(t, X(t)) \\ X(0) &= X_0 \end{cases} . \tag{1.2}$$

The function $F$ is a fuzzy-set-valued function, that is, its values are fuzzy sets. Hence, the derivative $X'$ of the unknown function $X$ is also fuzzy. It means that the direction to be followed by the solution is a fuzzy set and the solution, at each $t$, is also a fuzzy set.

There is also an integral equation associated with (1.2) as in the classical case. It involves a fuzzy integral and the Minkowski sum and one realizes that the solution to this kind of equation always has nondecreasing diameter. That is, the *fuzziness* (or, according to the interpretation, the *uncertainty*) does not decrease with time. As it will be fully explained in Section 4.2, this is considered a shortcoming since this is not expected from phenomena such as decay in population dynamics.

Generalizations of the Hukuhara derivative fixed this defect (see next Historical Overview section nd Section 4.3), but before that, other interesting approaches emerged and are still being intensively studied, namely the fuzzy differential inclusions (FDIs) and Zadeh's extension of the classical (or crisp) solution. These other approaches are based on a completely different view of fuzzy differential equations. Though they receive this classification, they are not really FDEs. There is no equality between the derivative of a fuzzy function and the function that determines the direction of the dynamic since there is no derivative of fuzzy function. The derivative is that of classical functions. At each crisp pair $(t, x)$, there are different possible values of the function $f$, each one with a membership degree to the set "fuzzy direction field". In the FDIs case, the memberhip degree of the initial solution (to the set "fuzzy initial condition") and the direction field establish the membership degree of a non-fuzzy function (or its attainable set) to the solution of the FIVP. The common approach of Zadeh's extension of the classical solution solves classical differential equations. The initial condition and fuzzy parameters determine the solution, which is a fuzzy-set-valued function.

A novel idea has been recently developed and connects both mentioned interpretations for FIVPs (Barros et al., 2013; Gomes and Barros, 2012, 2013). This is the subject of this thesis. A fuzzy derivative is proposed, defined via Zadeh's extension of the derivative operator, denoted by $\widehat{D}$, and it turns out to be based on differentiating classical functions. Moreover, a fuzzy differential equation has to be satisfied. Comparisons of the results between this and the other approaches are inevitable and, in fact, the new derivative leads to the same solutions produced by the other methods, provided some conditions are satisfied (see Section 3.3).

In summary, a solution to IVP (1.1) is a function $x(\cdot) \in E([a, b]; \mathbb{R}^n)$, where $E([a, b]; \mathbb{R}^n)$ is a space of functions from $[a, b]$ to $\mathbb{R}^n$ and $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ (see Figure 1.1). In the novel approach developed in this thesis, a solution to FIVP (1.2) is a fuzzy set of a space of functions, that is, $X(\cdot) \in \mathcal{F}(E([a, b]; \mathbb{R}^n))$, where $\mathcal{F}(\mathbb{X})$ denotes all the fuzzy sets of the universe $\mathbb{X}$. In words, each crisp function has membership degree to the fuzzy set "solution". Moreover, $F : [a, b] \times \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$, as illustrated in Figure 1.2. A solution to FDIs is also of type $X(\cdot) \in \mathcal{F}(E([a, b]; \mathbb{R}^n))$, but the domain of the right-hand-side function is crisp, that is, $F : [a, b] \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$. The space of the solutions of the approach of Hukuhara derivative (or H-derivative) and the strongly generalized derivative (or GH-derivative) is another one: $X(\cdot) \in E([a, b]; \mathcal{F}(\mathbb{R}^n))$. In words, $X(\cdot)$ is a function that maps real values into fuzzy values. The right-hand-side function is of type $F : [a, b] \times \mathcal{F}(\mathbb{R}^n) \to \mathcal{F}(\mathbb{R}^n)$. The illustrations of these two approaches are displayed in Figures 1.3 and 1.4. Finally, Zadeh's extension of the crisp solution solves classical differential equations and extends the solution at each $t \in [a, b]$ such that $X(\cdot) \in E([a, b]; \mathcal{F}(\mathbb{R}^n))$.

The fuzziness in the solution enriches the theory of differential equations since the solutions are not composed of single points, but of sets of points associated with membership degrees. In the words of Cellina (2005), writing about multi-valued functions (which is a particular case of fuzzy-set-valued functions), "while to describe the behaviour of a point valued function is easy (a point can only displace itself), a set, besides displacing can be larger or smaller, can be convex or not. All these different facts are relevant to the problem of the existence of solutions and to their properties."

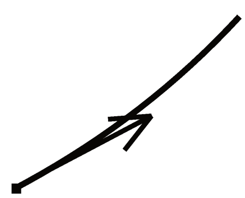

Figure 1.1: Classical IVP: the solution is a crisp function $x(\cdot) \in E([a,b];\mathbb{R})$ and $f$ is a crisp function such that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$.

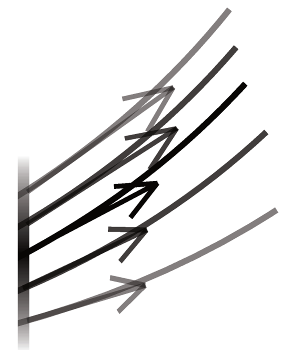

Figure 1.2: FIVP using $\widehat{D}$-derivative: the solution is a fuzzy bunch of functions $X(\cdot) \in \mathcal{F}(E([a,b];\mathbb{R}))$ and $F$ is a fuzzy-set-valued function such that $F : [a,b] \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$.

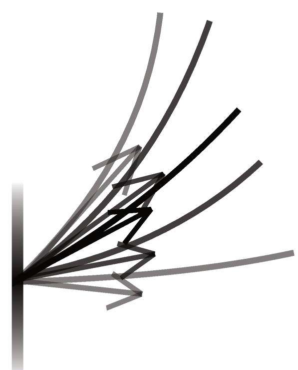

Figure 1.3: FIVP via FDIs: the solution is a fuzzy bunch of functions $X(\cdot) \in \mathcal{F}(E([a,b];\mathbb{R}))$ and $F$ is a fuzzy-set-valued function such that $F : [a,b] \times \mathbb{R} \to \mathcal{F}(\mathbb{R})$.

Figure 1.4: FIVP using H and GH-derivatives: the solution is a fuzzy-set-valued function $X(\cdot) \in E([a,b];\mathcal{F}(\mathbb{R}))$ and $F$ is such that $F : [a,b] \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$.

## Historical Overview

Kandel and Byatt (1980) first used the term "fuzzy differential equations", with a completely different meaning from nowadays. They used and extended Zadeh's definition of the probability of a fuzzy event and solved differential equations involving the membership function of a given fuzzy set (which was not the unknown variable).

Later on, Puri and Ralescu (1983) defined the derivative for fuzzy functions based on the concept of Hukuhara derivative for set-valued functions. The first theorem of existence using this derivative was proposed by Kaleva (1987), where the Lipschitz condition was used to assure existence and uniqueness of solution to a fuzzy initial value problem (FIVP). With an equivalent derivative, in the same year Seikkala published similar result. Both explored the equivalence of the FIVP with a fuzzy integral equation using Aumann integral for fuzzy-set-valued functions proposed by Puri and Ralescu (1986) (a generalization of the Aumann integral for set-valued functions). A version of Peano theorem of existence of solution was published by (Kaleva, 1990). Kaleva proved that the continuity of the function $F$ in (1.2) and local compactness of the domain and the codomain of the state variable assured existence of solution to the FIVP.

Many other studies regarding solutions to FIVPs using the Hukuhara derivative were published. Wu et al. (1996) proposed an existence and uniqueness theorem based on approximation by successive iterations. Lupulescu (2009b) established local and global existence and uniqueness results for functional (or delay) differential equations. See also Bede (2008); Lakshmikantham and Mohapatra (2003); Lupulescu (2009a); Nieto (1999).

The concept of Hukuhara derivative to solve FDE, in spite their use in many research articles, is considered to be defective since the differentiable functions have non-decreasing diameters. Therefore the solutions to differential equations cannot have decreasing diameter and, consequently, no periodic behavior can be modeled (nor can contractie nehavior), except in the crisp case.

Based on the theory of differential inclusions for set-valued functions (interested readers can find good review and list of references by Cellina (2005) and the main results by Aubin and Cellina (1984) and Filippov (1963)), Aubin (1990) and Baidosov (1990) proposed to solve fuzzy differential inclusions. The idea is to solve differential inclusions considering the membership degrees for initial conditions, right-hand-side functions and solutions.

Hüllermeier (1997) suggested solving differential inclusions for each level of the right-hand side fuzzy function (a multi-valued function). Using this interpretation, Diamond (1999) proved an existence theorem for fuzzy differential inclusions with fuzzy initial condition. It stated that if some hypotheses are met, the solutions of the differential inclusions produce a fuzzy bunch of functions, that is, a fuzzy set of a classical space of functions. Moreover, its attainable sets are fuzzy numbers.

The solution of a fuzzy differential inclusion may present decreasing diameter, overcoming the Hukuhara defect. This advantage and the richness of the fuzzy and the multi-valued functions led many other authors to study this theory (see Barros et al. (2004); Křivan and Colombo (1998); Lakshmikantham and Mohapatra (2003); Mizukoshi et al. (2007); Zhu and Rao (2000)).

This approach seems attractive on the one hand since it has no fuzzy derivatives. Hence it avoids the problem of solving equations with fuzzy sets, which looks much more complicated (minimization and maximization problems have to be frequently solved). On the other hand, solving differential inclusions is not an easy task as well. Zadeh's extension of the solution of the crisp initial value problem is intuitive and easier to solve (see Buckley and Feuring (2000); Mizukoshi et al. (2007); Oberguggenberger and Pittschmann (1999)). It also preserves the main properties of the crisp case. Mizukoshi (2004) and Mizukoshi et al. (2009) proved that solutions that are stable for classical models are also stable for the fuzzy case. It does not use any fuzzy derivative, as in fuzzy differential inclusions. In fact, under certain conditions, these two approaches are very similar and produces the same solutions for FIVPs.

Bede and Gal (2005) proposed and developed the theory for the first generalizations of the Hukuhara derivative. The strongly generalized and the weakly generalized differentiabilities are derivatives for fuzzy functions that differentiate all Hukuhara differentiable functions and others, including a class of functions with decreasing diameter. They also proved an existence theorem, assuring two solutions to FIVPs, one for models of nonincreasing processes and the other one with nondecreasing diameter, using the strongly generalized differentiability. A characterization result was also obtained (Bede, 2008) stating that the fuzzy differential equations are equivalent to classical differential equations systems. That is, it is possible to solve many FIVPs by using only classical theory.

Stefanini and Bede (2009) more recently suggested other more general derivatives, the generalized Hukuhara derivative and the most general so far, the fuzzy generalized derivative. These generalized derivatives have been extensively studied (see Bede and Gal (2005); Bede and Stefanini (2013); Chalco-Cano et al. (2011b); Stefanini and Bede (2009)). Some interesting behaviors of solutions to FIVPs via generalized derivatives are novel in the field of differential equations. Phenomena such as "switch points", in which other solutions arise at determined points of the dynamics (even in very well-behaved dynamics), do not exist in classical theory.

Another derivative, namely the $\pi$-derivative, was extended from the case in which the functions are set-valued to the fuzzy-set-valued ones (see Chalco-Cano et al. (2011b)). The $\pi$-derivative is based on the embedding of the family of nonempty compact sets of $\mathbb{R}^n$ in a real normed linear space.

Barros et al. (2010) suggested the use of Zadeh's extension to define the derivative and integral operators. Barros et al. (2013) further investigated these concepts and stated an existence theorem for solutions to FIVPs. The proof is based on the theorem of existence of solutions of fuzzy differential inclusion, revealing a connection between these two approaches. The mentioned theorem is part of this thesis, as well as other results connecting Zadeh's extension of the derivative operator and via generalized derivatives. It will be clear that all approaches mentioned here have some similarities. Some of these results have already been published (Barros et al., 2013; Gomes and Barros, 2012, 2013).

Chapter 2 contains some necessary basic concepts that are introduced and discussed such as the definition of fuzzy sets, Zadeh's extension principle, fuzzy arithmetics and fuzzy functions. The different approaches to fuzzy derivatives and fuzzy integrals commented previously are presented in Chapter 3, as well as results connecting them. In Chapter 4 the various interpretations of FIVPs and results of existence of solutions are discussed, followed by final comments in Chapter 5.

# Chapter 2

# Basic Concepts

Some basic concepts are presented in this chapter. For a deeper understanding, the reader can refer to Bede (2013), Diamond and Kloeden (1994), Klir and Yuan (1995), Nguyen and Walker (2005), Pedrycz and Gomide (2007) and Barros and Bassanezi (2010) (in Portuguese) and the papers cited in this thesis.

## 2.1 Fuzzy Subsets

**Definition 2.1.1** *A fuzzy subset $A$ of a universe $\mathbb{U}$ is characterized by a function*

$$\mu_A : \mathbb{U} \to [0,1] \tag{2.1}$$

*called membership function.*

If

$$\mu_A : \mathbb{U} \to \{0,1\} \tag{2.2}$$

the subset $A$ is said to be crisp.

The crisp case (2.2), $\mu_A$ is called the *characteristic function* (or *indicator function*) and it is often denoted by $\chi_A$. If $\chi_A(x) = 0$, then $x$ does not belong to $A$, whereas if $\chi_A(x) = 1$, then $x$ belongs to $A$. The fuzzy subset is a generalization in which an element of $\mathbb{U}$ has partial membership to $A$ characterized by a degree in the interval $[0,1]$. Hence the assignment $\mu_A(x) = 0$ means that $x$ does not belong to $A$, while the closer $\mu_A(x)$ is to 1, the more $x$ is considered in $A$.

Some important classical subsets related to fuzzy subsets are defined in what follows.

**Definition 2.1.2** *Given a fuzzy subset $A$ of a universe $\mathbb{U}$, its $\alpha$-cuts (or $\alpha$-levels) are the subsets*

$$[A]_\alpha = \begin{cases} \{x \in \mathbb{U} : \mu_A(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ cl\{x \in \mathbb{U} : \mu_A(x) > 0\}, & \text{if } \alpha = 0 \end{cases}$$

*where $cl\, Z$ denotes the closure of the classical subset $Z$.*

*The support is*

$$supp\, A = \{x \in \mathbb{U} : \mu_A(x) > 0\}.$$

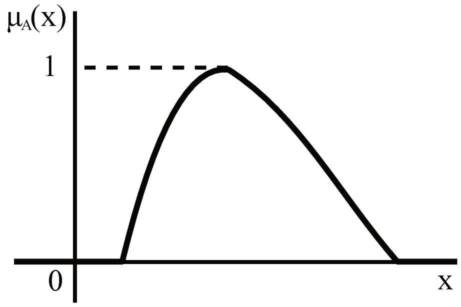


Figure 2.1: A fuzzy subset of $\mathbb{R}$ that is a fuzzy number.

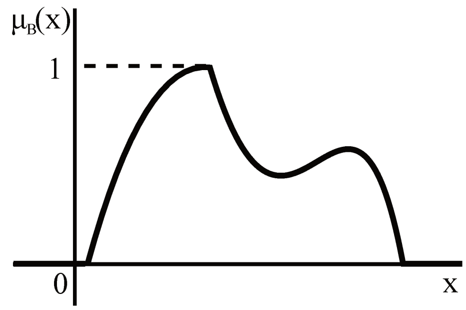


Figure 2.2: A fuzzy subset of $\mathbb{R}$ that is not a fuzzy number.

*The core is*

$$core\, A = \{x \in \mathbb{U} : \mu_A(x) = 1\}.$$

Two fuzzy subsets $A$ and $B$ of $\mathbb{U}$ are said to be equal if their membership functions are the same for all element in $\mathbb{U}$, (i.e., $\mu_A(x) = \mu_B(x), \forall x \in \mathbb{U}$). Or, equivalently, if all $\alpha$-cuts coincide ($[A]_\alpha = [B]_\alpha$, for all $\alpha \in [0, 1]$).

We denote by

- $\mathcal{K}^n$ the family of all nonempty compact subsets of $\mathbb{R}^n$;
- $\mathcal{K}^n_{\mathcal{C}}$ the family of all nonempty compact and convex subsets of $\mathbb{R}^n$;
- $\mathcal{P}(\mathbb{U})$ the family of all subsets of $\mathbb{U}$;
- $\mathcal{F}(\mathbb{U})$ the family of all fuzzy sets of $\mathbb{U}$;
- $\mathcal{F}_{\mathcal{K}}(\mathbb{U})$ the family of fuzzy sets of $\mathbb{U}$ whose $\alpha$-cuts are nonempty compact subsets of $\mathbb{U}$;
- $\mathcal{F}_{\mathcal{C}}(\mathbb{U})$ the family of fuzzy sets of $\mathbb{U}$ whose $\alpha$-cuts are nonempty compact and convex subsets of $\mathbb{U}$.
- $E([a, b]; \mathbb{R}^n)$ a space of function from $[a, b]$ to $\mathbb{R}^n$, $a \leq b$, $a, b \in \mathbb{R}$. For instance, $C([a, b]; \mathbb{R}^n)$ is the space of continuous functions. Further examples appear in Appendix.

If the core of a fuzzy subset is nonempty, the fuzzy subset is called *normal*. A fuzzy subset $A$ of a vector space $\mathbb{U}$ is said to be *fuzzy convex* if $\mu_A(\lambda x + (1 - \lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$ for every $x, y \in \mathbb{U}$, $\lambda \in [0, 1]$, that is, if its membership function is quasiconcave. If $\mathbb{U} = \mathbb{R}$, this condition assures that the $\alpha$-cuts are intervals (convex subsets).

**Definition 2.1.3** *A fuzzy number is a fuzzy convex and normal fuzzy subset of* $\mathbb{R}$ *with upper semicontinuous membership function and compact support.*

The family of the fuzzy numbers coincides with $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ (see Figures 2.1 and 2.2 for examples of fuzzy sets in $\mathbb{R}$, one that is a fuzzy number and another that is not). Theorem 2.1.4 assures that all $\alpha$-cuts of a fuzzy number are nonempty closed and bounded intervals with some properties whereas Theorem 2.1.5 is its converse, that is, if a family of nonempty closed intervals has some properties, they are the $\alpha$-cuts of a unique fuzzy number. Hence, when dealing with fuzzy numbers it suffices to operate with their $\alpha$-cuts; it is equivalent to operating with the fuzzy number itself.

**Theorem 2.1.4** *(Stacking Theorem, Negoita and Ralescu (1975)) A fuzzy number $A$ satisfies the following conditions:*
*(i) its $\alpha$-cuts are nonempty closed intervals, for all $\alpha \in [0,1]$;*
*(ii) if $0 \le \alpha_1 \le \alpha_2 \le 1$ then $[A]_{\alpha_2} \subseteq [A]_{\alpha_1}$;*
*(iii) for any nondecreasing sequence $(\alpha_n)$ in $[0,1]$ converging to $\alpha \in (0,1]$ we have*

$$\bigcap_{n=1}^{\infty} [A]_{\alpha_n} = [A]_\alpha;$$

*and*
*(iv) for any nonincreasing sequence $(\alpha_n)$ in $[0,1]$ converging to zero we have*

$$cl\left(\bigcup_{n=1}^{\infty} [A]_{\alpha_n}\right) = [A]_0.$$

**Theorem 2.1.5** *(Characterization Theorem, Negoita and Ralescu (1975)) If $\{A_\alpha : \alpha \in [0,1]\}$ is a family of subets of $\mathbb{R}$ such that*
*(i) $A_\alpha$ are nonempty closed intervals, for all $\alpha \in [0,1]$;*
*(ii) if $0 \le \alpha_1 \le \alpha_2 \le 1$ then $A_{\alpha_2} \subseteq A_{\alpha_1}$;*
*(iii) for any nondecreasing sequence $(\alpha_n)$ in $[0,1]$ converging to $\alpha \in (0,1]$ we have*

$$\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha;$$

*and*
*(iv) for any nonincreasing sequence $(\alpha_n)$ in $[0,1]$ converging to zero we have*

$$cl\left(\bigcup_{n=1}^{\infty} A_{\alpha_n}\right) = A_0,$$

*then there exists a unique fuzzy number $A$ such that $\{A_\alpha : \alpha \in [0,1]\}$ are its $\alpha$-cuts.*

If $A$ is a fuzzy number we denote its $\alpha$-cuts by $A_\alpha = [a_\alpha^-, a_\alpha^+]$ where $a_\alpha^-$ and $a_\alpha^+$ are the lower and upper endpoints of the closed interval $[A]_\alpha$.

A particular kind of fuzzy numbers is the *triangular fuzzy number*. If $A$ is a triangular fuzzy number with support $[a,c]$ and core $\{b\}$, it is denoted by $(a;b;c)$ and its membership function is

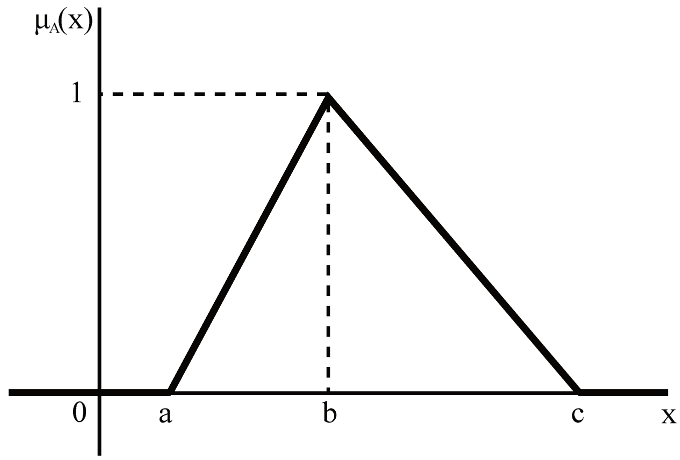


Figure 2.3: Triangular fuzzy number.

given by

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, & \text{if } \ x \in [a,b] \\ \dfrac{-x+c}{c-b}, & \text{if } \ x \in (b,c] \\ 0, & \text{if } \ x \notin [a,c] \end{cases},$$

where $a < b < c$ (see Figure 2.3 for an example).

A particular set of fuzzy numbers is

$$\mathcal{F}_{\mathcal{C}}^{0}(\mathbb{R}) = \{A \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}) : [A]_\alpha = [a_\alpha^-, a_\alpha^+], a_\alpha^-, a_\alpha^+ \in C([0,1];\mathbb{R})\},$$

that is, the lower and upper endpoints of the level set functions of each fuzzy number are continuous. Some studies have been done using this particular class of fuzzy numbers (see e.g. Bede (2013); Bede and Stefanini (2013)) and it will play an important role in the connection to the Hukuhara and generalized differentiabilities and the $\widehat{D}$-derivative (see Chapters 3 and 4).

The definition of t-norm will be explored in the next section, to define a particular case of fuzzy arithmetic.

**Definition 2.1.6** *A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ that satisfies the following properties, for all $x, y, u, v \in [0,1]$:*

*(i) Neutral element: $T(x,1) = x$.*

*(ii) Commutativity: $T(x,y) = T(y,x)$.*

*(iii) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$.*

*(iv) Monotonicity: if $x \leq u$ and $y \leq v$ then $T(x,y) \leq T(u,v)$.*

In order to deal with fuzzy subsets of general spaces, we present a generalization of Theorems 2.1.4 and 2.1.5 from the space $\mathbb{R}$ to more general topological spaces. Cecconello (2010) has proved it and, since we will deal not only with fuzzy subsets of $\mathbb{R}$, but with fuzzy subsets of spaces of functions as well, we state these results in what follows.

**Theorem 2.1.7** *(Cecconello, 2010) Let $\mathbb{X}$ be a topological space. A fuzzy subset $A \in \mathcal{F}_{\mathcal{K}}(\mathbb{X})$ satisfies the following conditions:*
*(i) if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $[A]_{\alpha_2} \subseteq [A]_{\alpha_1}$;*
*(ii) for any nondecreasing sequence $(\alpha_n)$ in $[0,1]$ converging to $\alpha \in (0,1]$ we have*

$$\cap_{n=1}^{\infty}[A]_{\alpha_n} = [A]_{\alpha};$$

*and*
*(iii) for any nonincreasing sequence $(\alpha_n)$ in $[0,1]$ converging to zero we have*

$$cl\,(\cup_{n=1}^{\infty}[A]_{\alpha_n}) = [A]_0.$$

**Theorem 2.1.8** *(Cecconello, 2010) Let $\mathbb{X}$ be a topological space. If $\{A_\alpha : \alpha \in [0,1]\}$ is a family of subets of $\mathbb{X}$ such that*
*(i) $A_\alpha$ are nonempty compact subsets, for all $\alpha \in [0,1]$;*
*(ii) if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then $A_{\alpha_2} \subseteq A_{\alpha_1}$;*
*(iii) for any nondecreasing sequence $(\alpha_n)$ in $[0,1]$ converging to $\alpha \in (0,1]$ we have*

$$\cap_{n=1}^{\infty}A_{\alpha_n} = A_{\alpha};$$

*and*
*(iv) for any nonincreasing sequence $(\alpha_n)$ in $[0,1]$ converging to zero we have*

$$cl\,(\cup_{n=1}^{\infty}A_{\alpha_n}) = A_0,$$

*then there exists a unique fuzzy subset $A \in \mathcal{F}_{\mathcal{K}}(\mathbb{X})$ such that $\{A_\alpha : \alpha \in [0,1]\}$ are its $\alpha$-cuts.*

We finish this section by presenting the linear structure in $\mathcal{F}_{\mathcal{K}}(\mathbb{R}^n)$ used in the literature.
Consider the fuzzy subsets $A, B \in \mathcal{F}_{\mathcal{K}}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$,

$$\mu_{A+B}(z) = \sup_{x+y=z} \min\{\mu_A(x), \mu_B(y)\}$$

and

$$\mu_{\lambda A}(z) = \begin{cases} \mu_A(z/\lambda), & \text{if } \lambda \neq 0 \\ \chi_0(z), & \text{if } \lambda = 0 \end{cases}.$$

From the theory presented in the next section, it can be proved that

$$[A+B]_\alpha = [A]_\alpha + [B]_\alpha \quad \text{and} \quad [\lambda A]_\alpha = \lambda [A]_\alpha,$$

where

$$[A+B]_\alpha = \{a+b : a \in [A]_\alpha, b \in [B]_\alpha\}$$

and

$$[\lambda A]_\alpha = \{\lambda a : a \in [A]_\alpha\}.$$

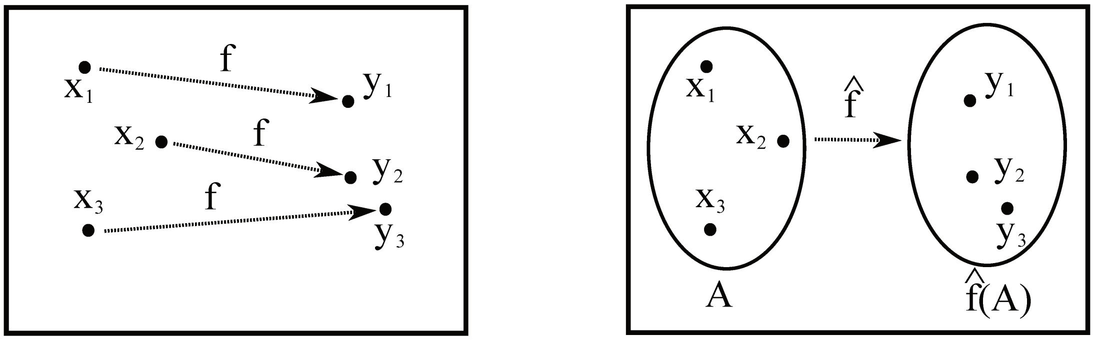


Figure 2.4: United extension of a function $f$ on a classical subset $A$: $f$ evaluated at each element of $A$ defines the united extension of $f$ at $A$.

## 2.2 Zadeh's Extension

Let $\mathbb{U}$ and $\mathbb{V}$ be two nonempty universes and $f : \mathbb{U} \to \mathbb{V}$ be a single-valued map. Given a subset $A$ of $\mathbb{U}$, the function $\widehat{f} : \mathcal{P}(\mathbb{U}) \to \mathcal{P}(\mathbb{V})$ such that $\widehat{f}(A) = \{y : y = f(x)\}$ is called united extension of $f$.

Zadeh's extension (see Zadeh (1975); Nguyen (1978)) is a case of united extension (according to Lodwick (2008)), i.e., it extends a function $f$, where the elements of the domain are single points, to one that has fuzzy subsets as inputs. In the classical case, the idea of the united extension is that, given a classical function $f$ and a classical subset $A$ as input, the subset $\widehat{f}(A)$ is defined as the union of the images of all elements of $A$ (see Figure 2.4). In the fuzzy case, it is intuitive that if a membership degree $\mu_A(x)$ is assigned to one element $x$ in a subset $A$, the image $y = f(x)$ of this element by means of an injective function has the same membership degree $\mu_{\widehat{f}(A)}(y) = \mu_A(x)$ (see Figure 2.5). If an element has more than one preimage, the membership degre is given by the supremum of the membership degrees of all possible preimages.

**Definition 2.2.1** *(Zadeh, 1975; Nguyen, 1978) Let $\mathbb{U}$ and $\mathbb{V}$ be two classical spaces and $f : \mathbb{U} \to \mathbb{V}$ a classical function. For each $A \in \mathcal{F}(\mathbb{U})$ we define Zadeh's extension of $f$ as $\widehat{f}(A) \in \mathcal{F}(\mathbb{V})$ such that*

$$\mu_{\widehat{f}(A)}(y) = \begin{cases} \sup_{s \in f^{-1}(y)} \mu_A(s), & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{if } f^{-1}(y) = \emptyset \end{cases}, \tag{2.3}$$

*for all $y \in \mathbb{V}$, where $f^{-1}(y) = \{x \in \mathbb{U} : f(x) = y\}$.*

**Remark 2.2.2** *We use $\widehat{f}$ to denote the fuzzy extension of Zadeh (2.3).*

**Example 2.2.3** *Let*

$$f(x) = ax + b$$

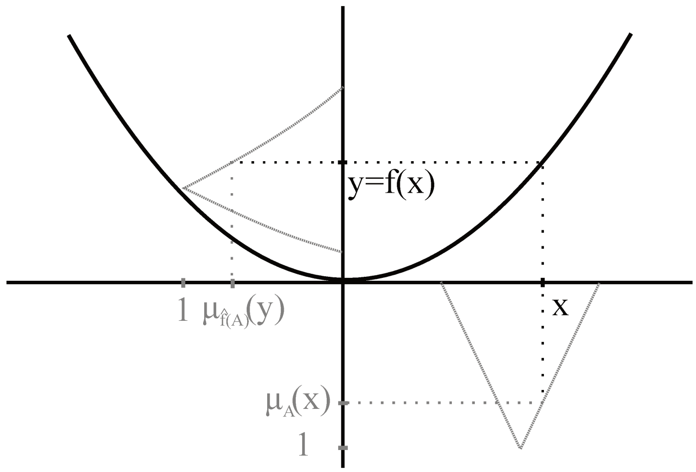


Figure 2.5: Zadeh's extension of a function $f$ on a fuzzy subset $A$: $f$ evaluated at each element of $A$, together with its memberhip degree to $A$, defines the Zadeh's extension $\widehat{f}(A)$.

*with $a, b \in \mathbb{R}$, $a \neq 0$. Since $f^{-1}(y) = a^{-1}(y - b)$, Zadeh's extension of $f$ is the fuzzy function $\widehat{f}$ such that, given $X \in \mathcal{F}(\mathbb{R})$,*

$$\mu_{\widehat{f}(X)}(y) = \sup_{x=a^{-1}(y-b)} \mu_X(x) = \mu_X(a^{-1}(y-b))$$

*for all $y \in \mathbb{R}$. Or*

$$\mu_{\widehat{f}(X)}(ax + b) \;\; = \;\; \mu_X(x)$$

*that is,*

$$\widehat{f}(X) = aX + b.$$

The next theorem allows us to determine Zadeh's extension of continuous and/or surjective functions more easily. It states that it suffices to calculate the image of the function on each element of the $\alpha$-cut of the argument, in order to obtain the $\alpha$-cut of the image.

**Theorem 2.2.4** *(Nguyen, 1978; Barros et al., 1997) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function.*

*(a) If $f$ is surjective, then a necessary and sufficient condition for*

$$[\widehat{f}(A)]_\alpha = f([A]_\alpha)$$

*to hold is that $\sup\{A(s) : s \in f^{-1}(x)\}$ be attained for all $x \in \mathbb{R}^n$.*

*(b) If $f$ is continuous, then $\widehat{f} : \mathcal{F}_\mathcal{K}(\mathbb{R}^n) \to \mathcal{F}_\mathcal{K}(\mathbb{R}^n)$ is well-defined and*

$$[\widehat{f}(A)]_\alpha = f([A]_\alpha)$$

*for all $\alpha \in [0, 1]$.*

A generalization of item (b) and its proof can be found in Cecconello (2010) (in Portuguese).

**Theorem 2.2.5** *(Cecconello, 2010) Let $\mathbb{U}$ and $\mathbb{V}$ be two Hausdorff spaces and $f : \mathbb{U} \to \mathbb{V}$ be a function. If $f$ is continuous, then $\widehat{f} : \mathcal{F}_{\mathcal{K}}(\mathbb{U}) \to \mathcal{F}_{\mathcal{K}}(\mathbb{V})$ is well-defined and*

$$[\widehat{f}(A)]_\alpha = f([A]_\alpha) \tag{2.4}$$

*for all $\alpha \in [0,1]$.*

**Proof.** (Adapted from Cecconello (2010)) From Definition 2.3, $\widehat{f}(A)$ is a fuzzy subset in $\mathbb{V}$. To prove that $\widehat{f} : \mathcal{F}_{\mathcal{K}}(\mathbb{U}) \to \mathcal{F}_{\mathcal{K}}(\mathbb{V})$, it is needed to prove that the $\alpha$-cuts $[\widehat{f}(A)]_\alpha$ are nonempty compact subsets of $\mathbb{V}$. Since $f$ is continuous, it assigns compact subsets to compact subsets, hence it suffices to prove Equation $[\widehat{f}(A)]_\alpha = f([A]_\alpha)$.

Equation (2.4) will be proved. First we show that

*(i)* $f([A]_\alpha) \subseteq [\widehat{f}(A)]_\alpha$. Consider $A \in \mathcal{F}_{\mathcal{K}}(\mathbb{U})$ and $y \in f([A]_\alpha)$. Then there exists $x \in [A]_\alpha$ such that $y = f(x)$. From Definition 2.3 (Zadeh's extension), $\mu_{\widehat{f}(A)}(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \geq \alpha$. Hence $y \in [\widehat{f}(A)]_\alpha$ and the conclusion is $f([A]_\alpha) \subseteq [\widehat{f}(A)]_\alpha$. Now we show that

*(ii)* $[\widehat{f}(A)]_\alpha \subseteq f([A]_\alpha)$. Let us remark first that, since $\mathbb{U}$ and $\mathbb{V}$ are Hausdorff spaces, a single point $y \in \mathbb{V}$ is closed. Moreover, the continuity of $f$ implies that $f^{-1}(y)$ is closed. Since $[A]_0$ is compact, $f^{-1}(y) \cap [A]_0$ is also compact. For $\alpha > 0$, consider $y \in [\widehat{f}(A)]_\alpha$. Then $\mu_{\widehat{f}(A)}(y) = \sup_{x \in f^{-1}(y)} \mu_A(x) \geq \alpha > 0$ and, therefore, there exist $x \in f^{-1}(y)$ such that $f^{-1}(y) \cap [A]_0 \neq \emptyset$.

Also, since $\mu_A(x)$ is upper semicontinuous and $f^{-1}(y) \cap [A]_0$ is compact, the supremum is attained, that is, there exists $x \in f^{-1}(y) \cap [A]_0$ with $\mu_{\widehat{f}(A)}(y) = \mu_A(x) \geq \alpha$. Hence $y = f(x)$ for some $x \in [A]_\alpha$, that is, $y \in f([A]_\alpha)$.

For $\alpha = 0$, the results obtained yields

$$\bigcup_{\alpha \in (0,1]} [\widehat{f}(A)]_\alpha = \bigcup_{\alpha \in (0,1]} f([A]_\alpha) \subseteq f([A]_0).$$

Since $f([A]_0)$ is closed,

$$[\widehat{f}(A)]_0 = cl\left(\bigcup_{\alpha \in (0,1]} [\widehat{f}(A)]_\alpha\right) = cl\left(\bigcup_{\alpha \in (0,1]} f([A]_\alpha)\right) \subseteq f([A]_0)$$

Hence $[\widehat{f}(A)]_\alpha \subseteq f([A]_\alpha)$ for all $\alpha \in [0,1]$ and (2.4) follows from *(i)* and *(ii)*. ∎

A fuzzy-set-valued function whose domain is not fuzzy is extended in the following way using the following definition. It is a more general case than Definition 2.2.1 and it has a wider application.

**Definition 2.2.6** *(Zadeh, 1975; Nguyen, 1978) Let $\mathbb{U}$ and $\mathbb{V}$ be two topological spaces and $F : \mathbb{U} \to \mathcal{F}(\mathbb{V})$ a function. For each $A \in \mathcal{F}(\mathbb{U})$ we define Zadeh's extension of $F$ as $\widehat{F}(A) \in \mathcal{F}(\mathbb{V})$ where its (unique) membership function is given by*

$$\mu_{\widehat{F}(A)}(y) = \sup_{x \in \mathbb{U}} \{\mu_{F(x)}(y) \wedge \mu_A(x)\}. \tag{2.5}$$

*for all $y \in \mathbb{V}$.*

## 2.3 Fuzzy Arithmetics for Fuzzy Numbers

The $\alpha$-cuts of fuzzy numbers are closed intervals so it is inevitable the influence of the concepts of the interval arithmetic on the arithmetic of fuzzy numbers. The first fuzzy arithmetic approach presented in this study is equivalent to the interval arithmetic with $\alpha$-cuts of fuzzy numbers.

### 2.3.1 Standard Interval Arithmetic (SIA) and Zadeh's Extension

The standard interval arithmetic (SIA) (Moore, 1966) can be regarded as the united extension of the operators addition (+), subtraction (−), multiplication (·) and division (÷) between real numbers. For instance, the addition of two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$, is defined by applying the operation "addition" on every single pair $(a, b) \in A \times B$, that is,

$$A + B = \{a + b : a \in A, b \in B\}$$

The other three operations are defined likewise, i.e,

$$\begin{aligned} A - B &= \{a - b : a \in A, b \in B\} \\ A \cdot B &= \{a \cdot b : a \in A, b \in B\} \\ A \div B &= \{a \div b : a \in A, b \in B, 0 \notin B\}. \end{aligned}$$

It is obvious from the definition of SIA that the arithmetic for real numbers is a particular case.

The fuzzy arithmetic based on SIA is the application of SIA on the $\alpha$-cuts of two fuzzy numbers. It is equivalent to Mizumoto and Tanaka's proposal Mizumoto and Tanaka (1976) of Zadeh's extension of the arithmetic operators, defined for real numbers. Given a arithmetic operator $\odot \in \{+, -, \cdot, \div\}$ and two fuzzy numbers $A$ and $B$, Zadeh's extension is given by

$$\mu_{A\odot B}(c) = \sup_{a\odot b=c} \min\{\mu_A(a), \mu_B(b)\} \tag{2.6}$$

Since the arithmetic operators are continuous functions (except division by zero), it is equivalent to operating on the elements of the $\alpha$-cuts.

Consider two fuzzy numbers $A$ and $B$ with $\alpha$-cuts $[A]_\alpha = [a^-_\alpha, a^+_\alpha]$ and $[B]_\alpha = [b^-_\alpha, b^+_\alpha]$. Using Zadeh's extension (Definition 2.2.1), levelwise the sum is equivalent to

$$[A + B]_\alpha = [a^-_\alpha + b^-_\alpha, a^+_\alpha + b^+_\alpha],$$

the subtraction is

$$[A - B]_\alpha = [a^-_\alpha - b^+_\alpha, a^+_\alpha - b^-_\alpha],$$

the product is

$$[A \cdot B]_\alpha = \left[\min_{s,r\in\{-,+\}} a^s_\alpha \cdot b^r_\alpha, \max_{s,r\in\{-,+\}} a^s_\alpha \cdot b^r_\alpha\right],$$

and the division is

$$[A \div B]_\alpha = \left[\min_{s,r\in\{-,+\}} \left\{\frac{a^s_\alpha}{b^r_\alpha}\right\}, \max_{s,r\in\{-,+\}} \left\{\frac{a^s_\alpha}{b^r_\alpha}\right\}\right],\ 0 \notin \operatorname{supp} B.$$

As mentioned above, this is the same as interval arithmetic on $\alpha$-cuts.

Note that the difference between two identical nonzero width intervals is never the crisp number zero. That is, there is no additive inverse (there is no multiplicative inverse either). The difference in the limit

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$$

of a constant non-crisp function $F$ is a constant noncrisp fuzzy number. The division by a variable tending towards zero is not defined. Therefore, to define the derivative of a fuzzy-number-valued function with the above arithmetic leads to a serious shortcoming.

This problem in defining the derivative happens due to the fact that this arithmetic takes into account every possible result. The sum is the same as Minkowski sum, in which all elements of a subset are added to all elements of the other subset, generating a the largest possible subset as result. The same happens to the subtraction. There are some approaches to overcome this, considering some kind of dependency between the variables.

### 2.3.2 Interactive Arithmetic

Dubois and Prade (1981) proposed the addition of interactive fuzzy numbers using the generalization of Zadeh's extension via t-norms. Fullér and Keresztfalvi (1992) explored it arguing that it provides a means of controlling the growth of uncertainty in calculations, differently from the arithmetic via traditional Zadeh's extension. To define interactivity, the concept of joint membership (analogous to joint possibility distribution, from the possibility theory) is needed.

**Definition 2.3.1** *If $A_1$ and $A_2$ are two fuzzy numbers, $C$ is said to be their joint membership if*

$$\mu_{A_i}(a_i) = \max_{a_j \in \mathbb{R}, j \neq i} \mu_C(a_1, a_2).$$

Two fuzzy numbers $A_1$ and $A_2$ are said to be non-interactive if their joint membership satisfies

$$\mu_C(a_1, a_2) = \min\{\mu_{A_1}(a_1), \mu_{A_2}(a_2)\}.$$

In words, the joint membership is given by the t-norm of minimum. Otherwise, they are said to be interactive.

The generalization in Dubois and Prade (1981) admits that any t-norm $T$ can replace the min operator in (2.6). In Carlsson et al. (2004) a generalization of the t-norms case can be found, using joint membership:

$$\mu_{\widehat{f(A_1,A_2)}}(y) = \begin{cases} \sup_{(a_1,a_2) \in f^{-1}(c)} \mu_C(a_1, a_2), & \text{if } f^{-1}(c) \neq \emptyset \\ 0, & \text{if } f^{-1}(c) = \emptyset \end{cases}. \tag{2.7}$$

In Fullér (1998) addition, subtraction, multiplication and division are obtained extending the respective crisp operators via Formula (2.7) where the joint membership are t-norms. In Carlsson et al. (2004) a particular case of joint membership is used to define addition and subtraction of interactive fuzzy numbers. It is based on completely correlated fuzzy numbers, i.e., given two fuzzy numbers $A_1$ and $A_2$, their joint membership is

$$\mu_C(a_1, a_2) = \mu_{A_1} \cdot \chi_{\{qa_1+r=a_2\}}(a_1, a_2) = \mu_{A_2} \cdot \chi_{\{qa_1+r=a_2\}}(a_1, a_2),$$

where $\chi_{\{qa_1+r=a_2\}}$ is the characteristic function of the line

$$\{(a_1, a_2) \in R^2 | qa_1 + r = a_2\}.$$

### 2.3.3 Constraint Interval Arithmetic (CIA)

Lodwick (1999) proposed the constraint interval arithmetic (CIA), which deals with dependencies. He redefined intervals as single-valued functions. That is, an interval $[a^-, a^+]$ is given by the function $A^I(a^-, a^+, \lambda_A) = \{a : a = (1 - \lambda_A)a^- + \lambda_A a^+, 0 \leq \lambda_A \leq 1\}$.

Addition, multiplication, subtraction and division between two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ are given by the formula

$$A \circ B = \{z : [(1 - \lambda_A)a^- + \lambda_A a^+] \circ [(1 - \lambda_B)b^- + \lambda_A a^+], 0 \leq \lambda_A \leq 1, 0 \leq \lambda_B \leq 1\}$$

where $\circ$ stands for any of the four arithmetic operations. In the case in which the two variables are the same,

$$A \circ A = \{z : [(1 - \lambda_A)a^- + \lambda_A a^+] \circ [(1 - \lambda_A)a^- + \lambda_A a^+], 0 \leq \lambda_A \leq 1\}$$

where, unlike SIA, $A - A = [0, 0]$ and $A \div A = [1, 1]$ (see Lodwick and Jenkins (2013)).

Lodwick and Untiedt (2008) extended this idea to the fuzzy case, based on the affirmation that arithmetic of fuzzy numbers is arithmetic of intervals in each $\alpha$-cut. They also have demonstrated that this arithmetic for fuzzy numbers is the same as gradual number arithmetic (see Dubois and Prade (2008)).

This approach can be interpreted as a particular case of the arithmetic presented in Subsection 2.3.2, where $A$ is completely correlated to $A$ with $q = 1$ and $A$ and $B$ are noninteractive.

### 2.3.4 Hukuhara and Generalized Differences

In order to overcome the difficulty of not having additive inverse, Hukuhara (1967) defined the Hukuhara difference for intervals and Puri and Ralescu (1983) used it to define the Hukuhara difference for elements of $\mathcal{F}_\mathcal{C}(\mathbb{R}^n)$.

**Definition 2.3.2** *(Puri and Ralescu, 1983) Given two fuzzy numbers, $A, B \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ the Hukuhara difference (H-difference for short) $A \ominus_H B = C$ is the fuzzy number $C$ such that $A = B + C$, if it exists.*

The Hukuhara difference has the property $A \ominus_H A = \{0\}$. However, this difference is not defined for pairs of fuzzy numbers such that the support of a fuzzy number has bigger diameter than the one that is subtracted. Stefanini and Bede (2009) and Stefanini (2010) proposed two new definitions for difference of fuzzy numbers, which generalizes the Hukuhara difference.

**Definition 2.3.3** *(Stefanini, 2010; Stefanini and Bede, 2009) Given two fuzzy numbers $A, B \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, the generalized Hukuhara difference (gH-difference for short) $A \ominus_{gH} B = C$ is the fuzzy number $C$, if it exists, such that*

$$\begin{cases} (i) & A = B + C \quad or \\ (ii) & B = A - C. \end{cases}$$

**Definition 2.3.4** *(Stefanini, 2010; Bede and Stefanini, 2013) Given two fuzzy numbers $A, B \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, the generalized difference (g-difference for short) $A \ominus_g B = C$ is the fuzzy number $C$, if it exists, with $\alpha$-cuts*

$$[A \ominus_g B]_\alpha = cl \bigcup_{\beta \geq \alpha} ([A]_\beta \ominus_{gH} [B]_\beta), \forall \alpha \in [0, 1],$$

*where the gH-difference $\ominus_{gH}$ is with interval operands $[A]_\beta$ and $[B]_\beta$.*

**Example 2.3.5** *The fuzzy numbers $A$ and $B$ with membership functions defined by*

$$\mu_A(x) = \begin{cases} x + 1, & \text{if } x \in [-1, 0], \\ -x + 1, & \text{if } x \in (0, 1], \\ 0, & \text{otherwise} \end{cases} \qquad \mu_B(x) = \begin{cases} 1, & \text{if } x \in [-1, 1], \\ 0, & \text{otherwise} \end{cases}$$

*have, as gH-difference levelwise,*

$$[A \ominus_{gH} B]_\alpha = [-\alpha, \alpha],$$

*for all $\alpha \in [0, 1]$. This is not a fuzzy number. But for the g-difference we have*

$$\begin{aligned} [A \ominus_g B]_\alpha &= cl \bigcup_{\beta \geq \alpha} [-\beta, \beta] \\ &= [-1, 1] \end{aligned}$$

*for all $\alpha \in [0, 1]$ and this is a fuzzy number.*

Example 2.3.5 illustrates that the Definition 2.3.4 is more general than 2.3.3, that is, it is defined for more pairs of fuzzy numbers. This means that whenever the gH-difference exists the g-difference exists and it is the same. In terms of $\alpha$-cuts we have

$$[A \ominus_{gH} B]_\alpha = [\min\{a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+\}, \max\{a_\alpha^- - b_\alpha^-, a_\alpha^+ - b_\alpha^+\}]$$

and

$$[A \ominus_g B]_\alpha = \left[ \inf_{\beta \geq \alpha} \min\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\}, \sup_{\beta \geq \alpha} \max\{a_\beta^- - b_\beta^-, a_\beta^+ - b_\beta^+\} \right].$$

for all $\alpha \in [0, 1]$.

The g-difference is not defined for every pair of fuzzy numbers, though. But among the differences that generalize the H-difference, it is the most general proposed so far. This possibility of non-existence of the g-difference is illustrated in the next example.

**Example 2.3.6** *Consider the fuzzy numbers A and B with membership functions defined by*

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in [2,3], \\ 0.5, & \text{if } x \in [0,2) \cup (3,5], \\ 0, & \text{otherwise} \end{cases} \qquad \mu_B(x) = \begin{cases} 1, & \text{if } x \in [2,3], \\ 0.5, & \text{if } x \in [-1,2) \cup (3,4], \\ 0, & \text{otherwise} \end{cases}$$

*The gH-difference levelwise is*

$$[A \ominus_{gH} B]_\alpha = \begin{cases} \{0\}, & \text{if } 0.5 < \alpha \leq 1, \\ \{1\}, & \text{if } 0 \leq \alpha \leq 0.5. \end{cases}$$

*Hence we have the g-difference levelwise*

$$[A \ominus_{g} B]_\alpha = \begin{cases} \{0\}, & \text{if } 0.5 < \alpha \leq 1, \\ \{0\} \cup \{1\}, & \text{if } 0 \leq \alpha \leq 0.5, \end{cases}$$

*which is not a fuzzy number.*

## 2.4 Fuzzy Metric Spaces

This section reviews some important definitions and results regarding fuzzy metric spaces. They can be found, together with proofs, in several references, e.g. Puri and Ralescu (1983); Diamond and Kloeden (1994); Bede (2013).

The most used metric for fuzzy numbers is the Pompeiu-Hausdorff, based on Pompeiu-Hausdorff distance for compact convex subsets of a metric space $\mathbb{U}$. It is in turn based on the concept of Hausdorff separation.

**Definition 2.4.1** *Let A and B be two nonempty compact subsets of a metric space $\mathbb{U}$. The pseudometric*

$$\rho(A, B) = \sup_{a \in A} d(a, B),$$

*where*

$$d(a, B) = \inf_{b \in B} ||a - b||$$

*is called Hausdorff separation.*

**Definition 2.4.2** *Let A and B be two nonempty compact subsets of a metric space $\mathbb{U}$. The Pompeiu-Hausdorff metric $d_H$ is given by*

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

For the space $\mathcal{F}_{\mathcal{K}}(\mathbb{U})$ (recall that the space of fuzzy numbers $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ is a particular case where $\mathbb{U} = \mathbb{R}$), the Pompeiu-Hausdorff metric is defined as follows.

**Definition 2.4.3** *Let $A$ and $B$ be subsets of $\mathcal{F}_{\mathcal{K}}(\mathbb{U})$, where $\mathbb{U}$ is a metric space. The Pompeiu-Hausdorff metric $d_\infty$ is defined as*

$$d_\infty(A,B) = \sup_{\alpha\in[0,1]} d_H([A]_\alpha,[B]_\alpha).$$

In the case of fuzzy numbers, that is, $A, B \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, $d_\infty(A,B)$ can be rewritten as

$$d_\infty(A,B) = \sup_{\alpha\in[0,1]} \max\{|a_\alpha^- - b_\alpha^-|, |a_\alpha^+ - b_\alpha^+|\}.$$

Another known metrics are the endographic and the $L^p$-type distances.

**Definition 2.4.4** *Let $A$ and $B$ be subsets of $\mathcal{F}_{\mathcal{K}}(\mathbb{U})$, where $\mathbb{U}$ is a metric space. The endographic metric $d_E$ is defined as*

$$d_E(A,B) = d_H(\mathrm{send}(A), \mathrm{send}(B)),$$

*where*

$$\mathrm{send}(A) = ([A]_0 \times [0,1]) \cap \mathrm{end}(A)$$

*with*

$$\mathrm{end}(A) = \{(x,\alpha) \in \mathbb{R}^n \times [0,1] : \mu_A(x) \geq \alpha)\}.$$

**Definition 2.4.5** *Let $A$ and $B$ be subsets of $\mathcal{F}_{\mathcal{K}}(\mathbb{U})$, where $\mathbb{U}$ is a metric space. The $d_p$ distance is defined as*

$$d_p(A,B) = \left(\int_0^1 d_H([A]_\alpha,[B]_\alpha)^p d\alpha\right)^{1/p}.$$

We denote by $\overline{B}(X,q)$ the closed ball

$$\overline{B}(X,q) = \{A \in \mathcal{F}_{\mathcal{C}}(\mathbb{U}) : d_\infty(X,A) \leq q\}.$$

The following theorem is a well-known result.

**Theorem 2.4.6** *(Puri and Ralescu, 1986) The space of fuzzy numbers endowed with the $d_\infty$ metric $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), d_\infty)$ is a complete metric space.*

Note that $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), d_\infty)$ is not separable. There exist other metrics that make the space of fuzzy numbers separable, but not complete (e.g. $d^p$ with $1 \leq p < \infty$ (Bede, 2013) and $d_E$ (Barros et al., 1997; Kloeden, 1980)).

Another important result is the Embedding Theorem. It connects the space of fuzzy numbers to a subset of pairs of crisp functions, that define the endpoints of the $\alpha$-cuts of fuzzy numbers. In other words, it allows us to use a well-known theory and tools for real functions, instead of operating with fuzzy numbers, which is more complicated. A general version of the theorem is for the space $\mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ and is as follows.

**Theorem 2.4.7** *(Embedding Theorem, Puri and Ralescu (1983); Kaleva (1990); Ma (1993)) There exists a real Banach space $\mathbb{X}$ such that the metric space $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}^n), d_\infty)$ can be embedded isometrically into $\mathbb{X}$.*

Another application of the $d_\infty$ metric is a result analogous to Theorem 2.2.4 (b), stated in Huang and Wu (2011). It regards continuity of fuzzy-number valued functions and Zadeh's extension. The concept of fuzzy function will be further explored in the next section.

**Theorem 2.4.8** *(Huang and Wu, 2011) Let $F : \mathbb{R} \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ be a $d_\infty$-continuous function. Then Zadeh's extension $\widehat{F} : \mathcal{F}_\mathcal{C}(\mathbb{R}) \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ is well-defined, is $d_\infty$-continuous and*

$$[\widehat{F}(A)]_\alpha = \bigcup_{a \in [A]_\alpha} [F(a)]_\alpha$$

*for all $\alpha \in [0,1]$.*

**Example 2.4.9** *Let*

$$F(x) = \Lambda x$$

*with $\Lambda \in \mathcal{F}_\mathcal{C}(\mathbb{R})$. Then $F$ is a $d_\infty$-continuous function and $F : \mathbb{R} \to \mathcal{F}_\mathcal{C}(\mathbb{R})$. Applying Theorem 2.4.8,*

$$[\widehat{F}(X)]_\alpha = \bigcup_{x \in [X]_\alpha} [\Lambda x]_\alpha = \bigcup_{x \in [X]_\alpha} x[\Lambda]_\alpha = [X]_\alpha [\Lambda]_\alpha = [\Lambda X]_\alpha$$

*where multiplication between intervals and multiplication between fuzzy numbers is the one defined in Subsection 2.3.1.*

*As result,*

$$\widehat{F}(X) = \Lambda X.$$

Since the mentioned results are important in this study, whenever we treat limits of sequences of fuzzy subsets or continuity of fuzzy-set valued functions, it will assumed to be with respect to the $d_\infty$ metric, unless another distance is specified.

## 2.5 Fuzzy Functions

Dubois and Prade (1980) called fuzzy functions both the fuzzy bunches of functions and the fuzzy-set-valued functions. The same is done in this text. In the literature in general, fuzzy calculus deals only with mappings from a crisp space to a fuzzy space, although a mapping from a fuzzy space to another fuzzy space is a more general *fuzzy-set-valued function*. The fuzzy-set-valued functions that will be treated will take $\mathbb{R}$ to $\mathcal{F}(\mathbb{R}^n)$. A *fuzzy-number-valued function* is a more restricted case, since its images are fuzzy numbers. Fuzzy-set-valued functions are generalizations of *set-valued functions*. A set-valued function on $I$ is a mapping $G : I \to \mathcal{P}(\mathbb{R}^n)$ such that $G(t) \neq \emptyset$ for all $t \in I$, where $I$ is an interval.

A *fuzzy bunch of functions* (or fuzzy bunch, for short) is a fuzzy subset of a crisp space of functions. To be precise, it is not a function, but it is used to define solutions to fuzzy initial value problems and to each fuzzy bunch of functions corresponds a fuzzy-set-valued function, via *attainable fuzzy sets*. For each fuzzy bunch $F \in \mathcal{F}(E([0,T];\mathbb{R}^n))$, the *attainable fuzzy sets at $t$*, $F(t)$, are the fuzzy sets of $\mathbb{R}^n$

$$[F(t)]^\alpha = [F]^\alpha(t) = \{f(t) : f \in [F]^\alpha\}.$$

**Example 2.5.1** *The mapping $F(x) = Ax$, where $A = [-1, 1]$, is a set-valued function whose images are intervals.*

**Example 2.5.2** *The mapping $F(x) = Ax$, where $A = (-1; 0; 1)$ is a fuzzy-set-valued function whose images are triangular fuzzy numbers.*

**Example 2.5.3** *Consider $f_1, f_2$ and $f_3$ continuous functions on an interval $[a, b]$. The fuzzy subset $F \in \mathcal{F}(C([a, b]; \mathbb{R}))$ such that*

$$\mu_F(f) = \begin{cases} \alpha, & \text{if } f = f_1 + \alpha(f_2 - f_1) \\ \alpha, & \text{if } f = f_3 + \alpha(f_2 - f_3) \\ 0, & \text{otherwise} \end{cases},$$

*has triangular fuzzy numbers as attainable fuzzy sets. This is defined in Gasilov et al. (2012) and is a particular kind of fuzzy bunch of functions, called* triangular fuzzy function*.*

*The fuzzy-number-valued function of the previous example can be constructed by considering the triangular fuzzy function with $f_1(x) = -x$, $f_2(x) = 0$ and $f_3(x) = x$, $x \in \mathbb{R}$ and calculating its attainable sets.*

Using the definition of attainable sets, to each fuzzy bunch there corresponds only one fuzzy-set-valued function. But the converse is not true, as the next examples illustrate.

**Example 2.5.4** *Barros et al. (2010) define the fuzzy bunches in $\mathcal{F}_\mathcal{K}(\mathcal{AC}([0, 2]; \mathbb{R}))$ (see Appendix for definition of $\mathcal{AC}([a, b]; \mathbb{R})$) such that*

$$\begin{aligned} F_1 = &\ \{x(\cdot) : x(t) = a, a \in [0, 2]\} \\ F_2 = &\ F_1 \cup \{y(\cdot) : y(t) = 2 - t\} \end{aligned} \tag{2.8}$$

*where $x, y : [0, 2] \to [0, 2]$.*

*We have $[F_1]_\alpha = [F_2]_\alpha = [0, 2]$, for all $\alpha \in [0, 1]$, though $F_1 \neq F_2$.*

**Example 2.5.5** *The fuzzy bunches of functions $F_1, F_2 \in \mathcal{F}(C([-1, 1]; \mathbb{R}))$*

$$\mu_{F_1}(f) = \begin{cases} \alpha, & \text{if } f : f(x) = -x(1 - \alpha) \\ \alpha, & \text{if } f : f(x) = x(1 - \alpha) \\ 0, & \text{otherwise} \end{cases},$$

*and*

$$\mu_{F_2}(f) = \begin{cases} \alpha, & \text{if } f : f(x) = -|x|(1 - \alpha) \\ \alpha, & \text{if } f : f(x) = |x|(1 - \alpha) \\ 0, & \text{otherwise} \end{cases},$$

*are not equal, though their attainable sets are the same: they are the images of the function in Example 2.5.1. Each function in $F_1$ is a straight line on $[-1, 1]$, different from the functions in $F_2$, as Figures 2.6 and 2.7 exemplify.*

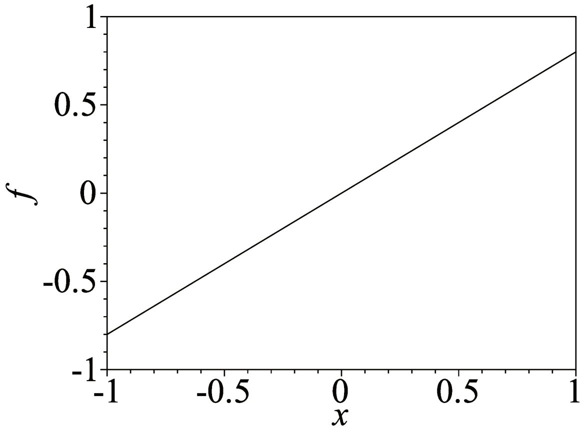


Figure 2.6: A function of the support of the fuzzy bunch of functions $F_1$ of Example 2.5.5.

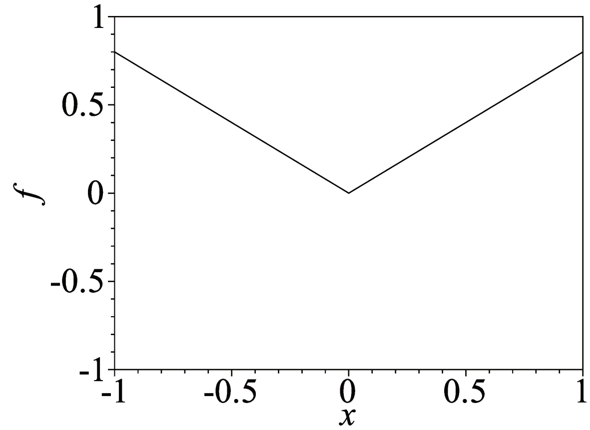


Figure 2.7: A function of the support of the fuzzy bunch of functions $F_2$ of Example 2.5.5.

The level set function that define the $\alpha$-cuts of $F : x \mapsto F(x)$ will always be denoted $f_\alpha^-(x)$ and $f_\alpha^+(x)$ for this text. That is,

$$[F(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)].$$

We are interested in defining fuzzy bunches of functions from fuzzy-number-valued functions such that the former preserves the main properties of the latter. The main property is the equivalence of its attainable sets and the fuzzy-number-valued function.

Given a fuzzy-number-valued function $F : [a, b] \to \mathcal{F}_\mathcal{C}(\mathbb{R})$, the idea (in order to a fuzzy bunch be similar in its properties to that of the fuzzy-set-valued function that generated it) is to consider the convex combinations of the functions $f_\alpha^-$ and $f_\alpha^+$ (see Figures 2.8 (a)-(c) and 2.9 (a)-(c)). This assures that the attainable sets of the $\alpha$-cuts of the fuzzy bunches have no "holes", that is, they are convex subsets of $\mathbb{R}$. Convexity also preserves properties such as continuity and differentiability, which is a very important point. A possible problem is that an $\alpha$-cut may not contain another $\alpha$-cut with smaller value of $\alpha$. The solution is to take all convex combinations of the upper $\alpha$-cuts (see Figure 2.9 (d)-(f)). Finally, to make compactness (a desirable property) more likely to occur, the closure of each $\alpha$-cut is a property that we will require.

Based on these observations, two types of fuzzy bunches are created. The first one is simpler and the second kind has more elements (contains the elements in the first kind, see Figure 2.10). The reason why we define both kinds of fuzzy bunches has to do with differentiability which will be explained in Section 3.2.2.

**Definition 2.5.6** *Consider $F : [a, b] \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ where $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are continuous functions with respect to $x$.*

*Define the subsets of functions*

$$A_\alpha = cl\left( \bigcup_{\beta \geq \alpha} \bigcup_{0 \leq \lambda \leq 1} f_\beta^\lambda(\cdot) \right), \quad \alpha \in [0, 1]$$

*where* $f_\beta^\lambda(\cdot) = (1-\lambda) f_\beta^-(\cdot) + \lambda f_\beta^+(\cdot)$ *and*

$$B_\alpha = cl\left(\bigcup_{\beta_1,\beta_2 \geq \alpha} \bigcup_{0\leq\lambda\leq 1} f_{\beta_1,\beta_2}^\lambda(\cdot)\right), \quad \alpha \in [0,1]$$

*where* $f_{\beta_1,\beta_2}^\lambda(\cdot) = (1-\lambda) f_{\beta_1}^-(\cdot) + \lambda f_{\beta_2}^+(\cdot)$.

*If the families* $\{A_\alpha : \alpha \in [0,1]\}$ *and* $\{B_\alpha : \alpha \in [0,1]\}$ *each define a fuzzy bunch of functions, we call them* representative affine fuzzy bunch of functions of first kind *(or representative bunch of first kind for short) and* representative affine fuzzy bunch of functions of second kind *(or representative bunch of second kind for short), respectively.*

It is important to remark that whenever the symbol $\tilde{F}(x)$ is used where $\tilde{F}$ is a fuzzy bunch of functions, it refers to the attainable fuzzy sets of $\tilde{F}$ at $x$.

**Example 2.5.7** *Consider the triangular fuzzy functions of Gasilov et al. (2012) (see Example 2.5.3) with* $f_2 - f_1 = f_3 - f_2$*, that is, the attainable sets are symmetrical triangular fuzzy numbers, and* $f_1(x) \neq f_3(x)$ *for all* $x$*. This is an example of representative bunches of first kind.*

**Example 2.5.8** *A function* $F : [a,b] \to \mathcal{F}_{\mathcal{C}}^0(\mathbb{R})$*, where* $f_\alpha^\pm(x)$ *are continuous, defines representative bunches of first and second kinds in* $C([a,b];\mathbb{R})$*. In order to prove this statement, it suffices to demonstrate that the subsets* $A_\alpha$ *and* $B_\alpha$ *of Definition 2.5.6 satisfy conditions (i), (ii), (iii) and (iv) of Theorem 2.1.8. We prove this with respect to* $A_\alpha$*. For* $B_\alpha$ *the reasoning is analogous.*

*Let us first prove (i), that is,* $A_\alpha$ *are nonempty compact sets, for all* $\alpha \in [0,1]$*. Since* $A_\alpha = cl\left(\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda(\cdot)\right)$ *contains* $f_\alpha^\pm(\cdot)$*, it is nonempty. Note that the continuity in* $\alpha$ *implies*

$$f_{\alpha_n}^\pm(x) \to f_\alpha^\pm(x) \quad \textit{if} \quad \alpha_n \to \alpha$$

*for* $\alpha_n, \alpha \in I \subset [0,1]$*, for all* $x \in [a,b]$*. According to Dini's Theorem (Bartle and Sherbert, 2011), pointwise convergence implies uniform convergence if the pointwise limits define a continuous function, the sequence of functions is monotonic and each function is defined on a compact set. Since this is the case, the convergence is uniform in* $x$*. Hence*

$$f_{\alpha_n}^\pm(\cdot) \to f_\alpha^\pm(\cdot) \quad \textit{if} \quad \alpha_n \to \alpha.$$

*Similarly,*

$$f_\alpha^{\lambda_n}(\cdot) \to f_\alpha^\lambda(\cdot) \quad \textit{if} \quad \lambda_n \to \lambda.$$

*As a consequence,*

$$f_{\alpha_n}^{\lambda_n}(\cdot) \to f_\alpha^\lambda(\cdot) \quad \textit{if} \quad \alpha_n \to \alpha \textit{ and } \lambda_n \to \lambda$$

*for* $\alpha_n, \alpha \in I \subset [0,1]$ *and* $\lambda_n, \lambda \in [0,1]$.

*This means that* $\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda(\cdot)$ *is sequentially compact. Since* $C([a,b];\mathbb{R})$ *is a metric space, sequentially compactness is equivalent to compactness. Hence* $\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda(\cdot)$ *is closed and equals* $A_\alpha$.

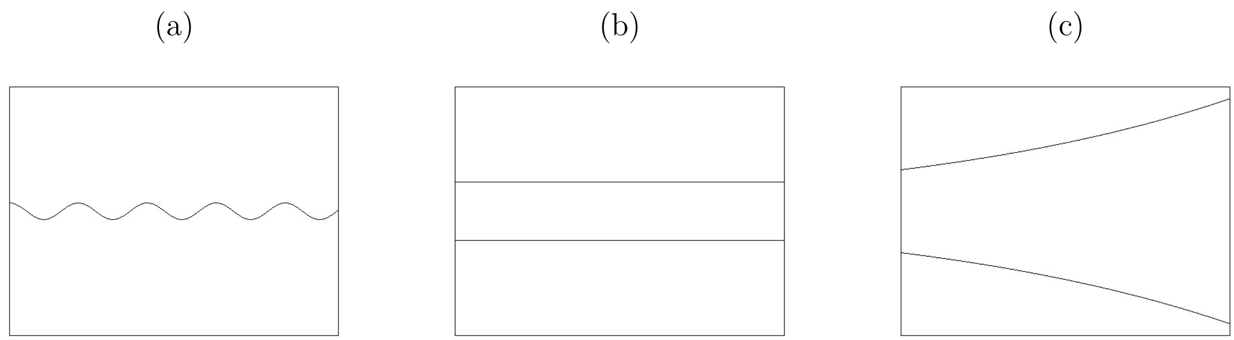


Figure 2.8: Level set functions of the (a) 1-cut, (b) 0.6-cut and (c) 0.2-cut of a fuzzy-number-valued function, supposed to have only these three $\alpha$-cuts different from each other.

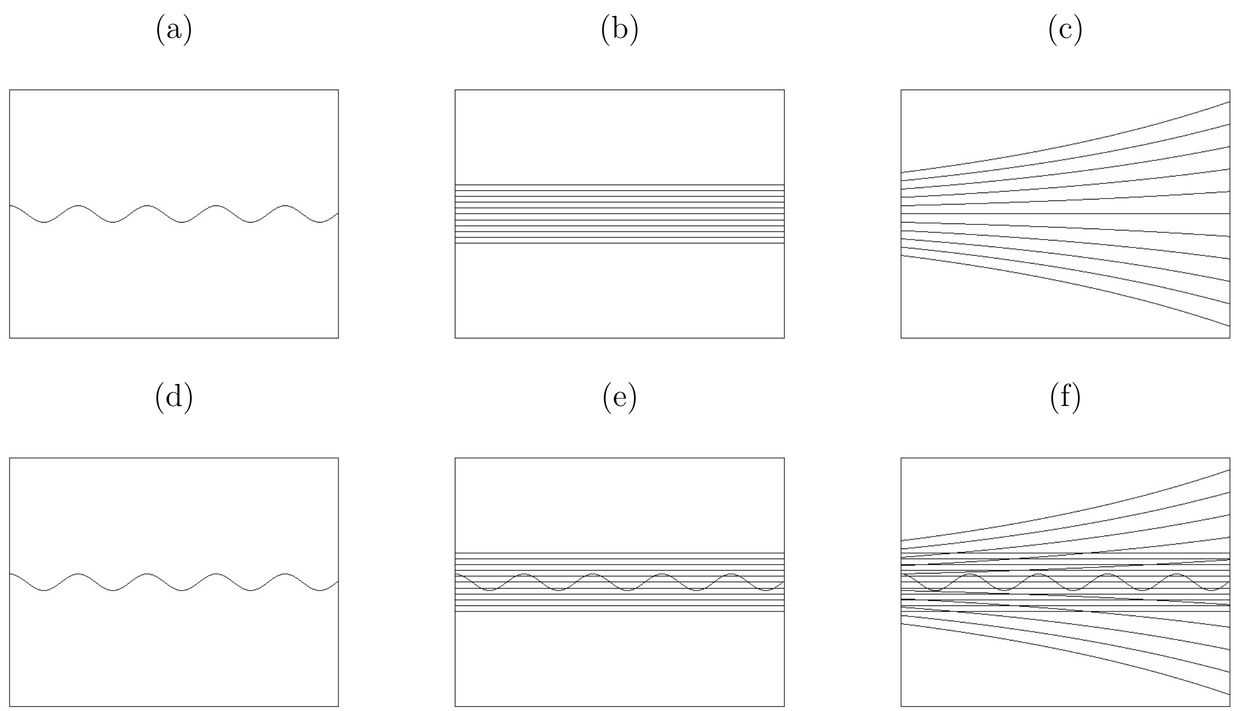


Figure 2.9: Convex combinations of the level set functions of the (a) 1-cut, (b) 0.6-cut and (c) 0.2-cut of a fuzzy-number-valued function and construction of the $\alpha$-cuts of the representative bunch of first kind. These $\alpha$-cuts of the representative bunch of first kind are defined as the union of the convex combinations corresponding to the $\alpha$-cut and the $\alpha$-cuts above: (d) 1-cut, (e) 0.6-cut and (f) 0.2-cut.

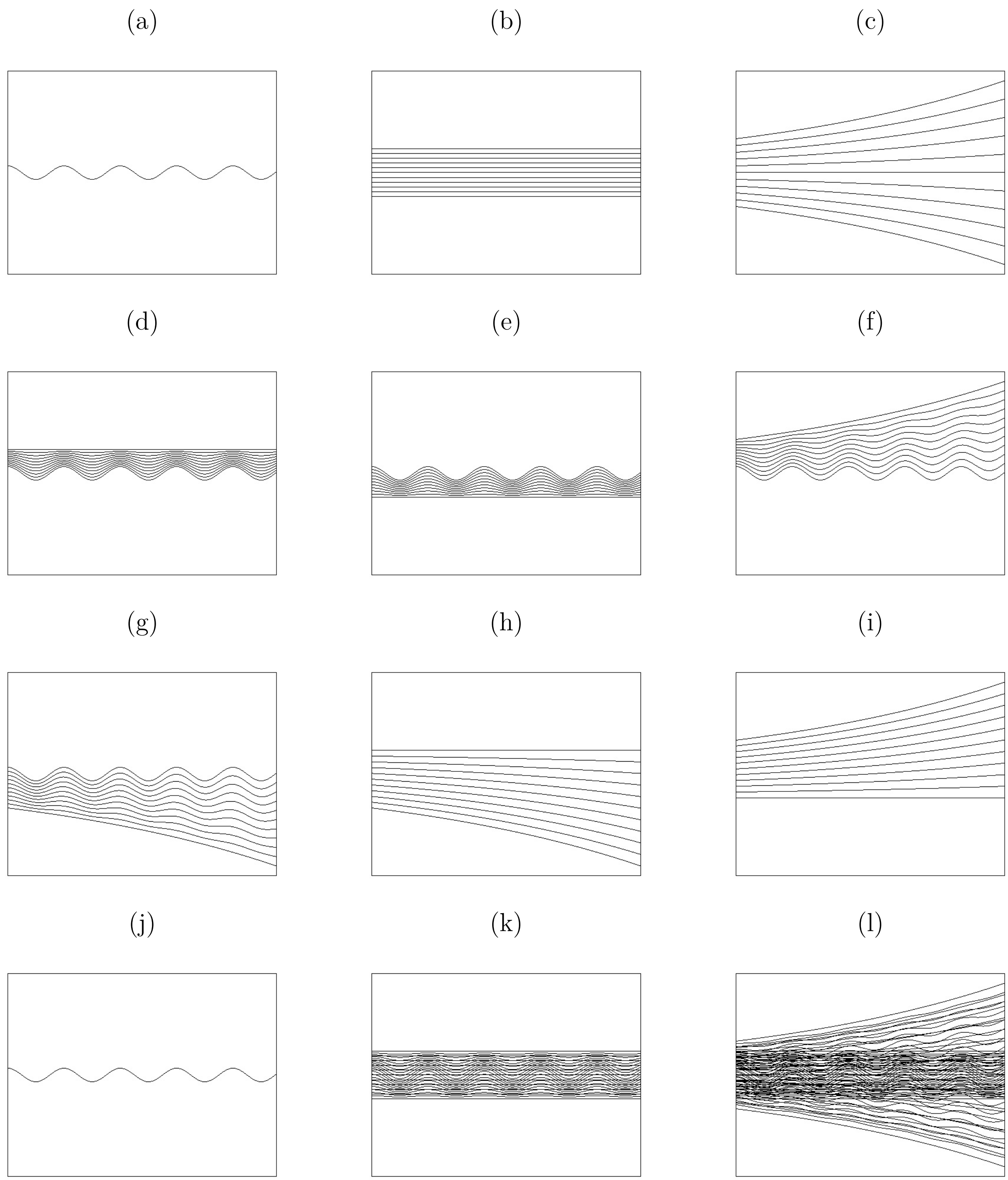


Figure 2.10: Convex combinations of the level set functions of the (a) 1-cut, (b) 0.6-cut, (c) 0.2-cut, (d)-(e) 1-cut with 0.6-cut, (f)-(g) 1-cut with 0.2-cut and (h)-(i) 0.2-cut with 0.6-cut of a fuzzy-number-valued function and construction of the $\alpha$-cuts of the representative bunch of second kind. These $\alpha$-cuts of the representative bunch of second kind are defined as the union of the convex combinations corresponding to the $\alpha$-cut and the $\alpha$-cuts above: (j) 1-cut, (k) 0.6-cut and (l) 0.2-cut.

*Condition (ii) states that if* $0 \le \alpha_1 \le \alpha_2 \le 1$ *then* $A_{\alpha_2} \subseteq A_{\alpha_1}$. *Indeed,*

$$A_{\alpha_2} = \bigcup_{\beta \ge \alpha_2} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \subseteq \left( \bigcup_{\alpha_1 \le \beta < \alpha_2} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right) \bigcup \left( \bigcup_{\beta \ge \alpha_2} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right) = A_{\alpha_1}.$$

*We now prove condition (iii), that is, for any nondecreasing sequence* $(\alpha_n)$ *in* $[0,1]$ *converging to* $\alpha \in (0,1]$ *we have* $\cap_{n=1}^\infty A_{\alpha_n} = A_\alpha$. *From condition (ii) we have* $A_\alpha \subseteq \cap_{n=1}^\infty A_{\alpha_n}$. *To prove* $A_\alpha \supseteq \cap_{n=1}^\infty A_{\alpha_n}$ *consider* $f \in \cap_{n=1}^\infty A_{\alpha_n}$. *The function* $f$ *is in each* $A_{\alpha_n}$ *and it can be written as* $f = f_{\beta_n}^{\overline{\lambda}_n}$, *with* $\beta_n \in [\alpha_n, 1]$, $\overline{\lambda}_n \in [0,1]$ *(it is the same function* $f$ *but written differently, according to the set* $A_\alpha$ *in which it is). Hence* $\beta_n$ *admits subsequence converging to* $\beta \in [\alpha, 1]$ *and* $\overline{\lambda}_n$ *admits subsequence converging to* $\overline{\lambda} \in [0,1]$, *so that it defines* $f_\beta^{\overline{\lambda}} \in A_\alpha$. *Therefore,* $A_\alpha \supseteq \cap_{n=1}^\infty A_{\alpha_n}$ *and condition (iii) is proved.*

*The last condition is proved if for any nonincreasing sequence* $(\alpha_n)$ *in* $[0,1]$ *converging to zero we have* $cl\,(\cup_{n=1}^\infty A_{\alpha_n}) \subseteq A_0$ *and* $cl\,(\cup_{n=1}^\infty A_{\alpha_n}) \supseteq A_0$. *We first simplify the expression*

$$cl\left( \bigcup_{n=1}^\infty A_{\alpha_n} \right) = cl\left( \bigcup_{n=1}^\infty \bigcup_{\beta \ge \alpha_n} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right) = cl\left( \bigcup_{\beta > 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right).$$

*Note that*

$$cl\left( \bigcup_{\beta > 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right) \subseteq cl\left( \bigcup_{\beta \ge 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right) = A_0$$

*which is the first inclusion. To prove the second one we need to prove that* $f \in A_0 = \bigcup_{\beta \ge 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot)$ *implies* $f \in cl\left( \bigcup_{\beta > 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right)$. *There are two possibilities for* $f \in A_0$: *(a)* $f \in \bigcup_{\beta > 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot)$ *or (b)* $f \in \bigcup_{\beta = 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) = \bigcup_{0 \le \lambda \le 1} f_0^\lambda(\cdot)$. *We only need to prove case (b). We use the fact that* $F(x)$ *is a fuzzy number and therefore satisfies*

$$cl\left( \bigcup_{n=1}^\infty [f_{\alpha_n}^-(x), f_{\alpha_n}^+(x)] \right) = [f_0^-(x), f_0^+(x)]$$

*for* $(\alpha_n)$ *a nonincreasing sequence converging to zero. Hence*

$$f_{\alpha_n}^\pm(x) \to f_0^\pm(x) \quad \textit{and} \quad f_{\alpha_n}^{\lambda_n}(x) \to f_0^\lambda(x)$$

*for* $\alpha_n \searrow 0$ *and* $\lambda_n \to \lambda$, $\alpha_n \in [0,1]$ *and* $\lambda_n, \lambda \in [0,1]$. *Using the same arguments as before we have*

$$f_{\alpha_n}^{\lambda_n}(\cdot) \to f_0^\lambda(\cdot)$$

*uniformly and hence* $f_0^\lambda(\cdot)$ *is a point of closure. This means that* $f_0^\lambda(\cdot) \in cl\left( \bigcup_{\beta > 0} \bigcup_{0 \le \lambda \le 1} f_\beta^\lambda(\cdot) \right)$. *That is, the second inclusion is also satisfied and we have obtained the equality of condition (iv).*

*Having proved (i), (ii), (iii) and (iv), it follows that* $A_\alpha$ *are* $\alpha$*-cuts of the representative bunch of first kind of* $F$.

**Example 2.5.9** *Consider the fuzzy-number-valued function $F : [-1,1] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ with $\alpha$-cuts*

$$[F(x)]_\alpha = \begin{cases} [10x^2 - 12, 10x^2 + 2]\,, & \text{if} \quad 0 \le \alpha \le 0.5 \\ [-1, 1]\,, & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}.$$

*The representative bunch of first kind is given by the $\alpha$-cuts*

$$[\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{2} \bigcup_{0 \le \lambda \le 1} y_i^\lambda(\cdot), & \text{if} \quad 0 \le \alpha \le 0.5 \\ \bigcup_{0 \le \lambda \le 1} y_1^\lambda(\cdot), & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}$$

*and the representative bunch of second kind is defined by*

$$[\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{4} \bigcup_{0 \le \lambda \le 1} y_i^\lambda(\cdot), & \text{if} \quad 0 \le \alpha \le 0.5 \\ \bigcup_{0 \le \lambda \le 1} y_1^\lambda(\cdot), & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}$$

*where*

$$\begin{cases} y_1^\lambda(\cdot) : y_1^\lambda(x) = (1-\lambda)(10x^2 - 12) + \lambda(10x^2 + 2), \\ y_2^\lambda(\cdot) : y_2^\lambda(x) = (1-\lambda)(-1) + \lambda, \\ y_3^\lambda(\cdot) : y_3^\lambda(x) = (1-\lambda)(-1) + \lambda(10x^2 + 2), \\ y_4^\lambda(\cdot) : y_4^\lambda(x) = (1-\lambda)(10x^2 - 12) + \lambda, \end{cases}$$

*for all $\lambda \in [0,1]$.*

*Figures 2.11 and 2.12 present some of the elements of the supports of $\tilde{F}_1(\cdot)$ and $\tilde{F}_2(\cdot)$. It is remarkable that $\tilde{F}_2(\cdot)$ has more elements in its support than in the support of $\tilde{F}_1(\cdot)$ (in fact, $\tilde{F}_2(\cdot)$ contains $\tilde{F}_1(\cdot)$). Moreover, some elements in $\tilde{F}_2(\cdot)$ have different behavior than those in $\tilde{F}_1(\cdot)$, though both fuzzy bunches have the same attainable sets.*

**Example 2.5.10** *Consider the fuzzy-number-valued function $F : [0, 0.5] \to \mathcal{F}_{\mathcal{C}}^0(\mathbb{R})$ with $\alpha$-cuts*

$$[F(x)]_\alpha = \begin{cases} [x^2 - 3 + \alpha, (1 - 2\alpha)x^2 - 2\alpha + 2]\,, & \text{if} \quad 0 \le \alpha \le 0.5 \\ [x^2 - 3 + \alpha, (2\alpha - 1)x^2 - 6\alpha + 4]\,, & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}.$$

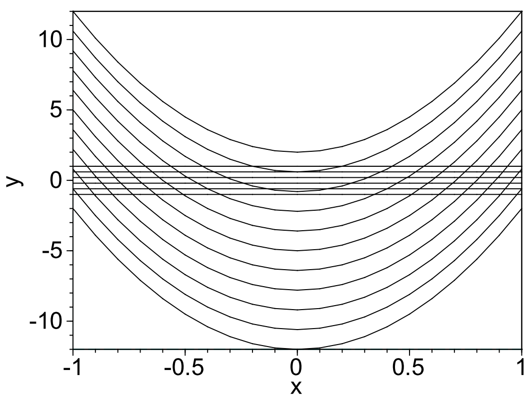


Figure 2.11: Some elements of the support of the representative bunch of first kind of Example 2.5.9. The real-valued functions are the convex combinations of a level set function (of a fuzzy number-valued function) with the opposite level set function in the same $\alpha$-cut.

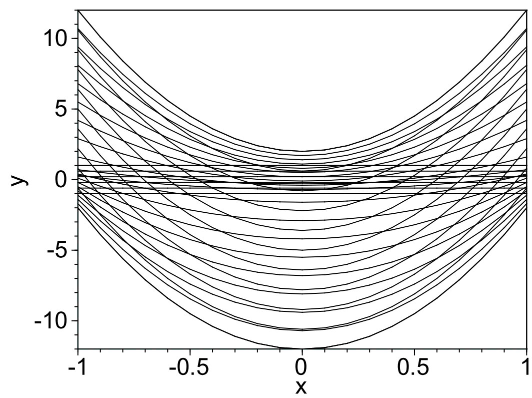


Figure 2.12: Some elements of the support of the representative bunch of second kind of Example 2.5.9. The real-valued functions are the convex combinations of a level set function (of a fuzzy number-valued function) with opposite level set function that may belong to different $\alpha$-cuts.

*The representative bunch of first kind is given by the $\alpha$-cuts*

$$
\begin{aligned}
[\tilde{F}_1(\cdot)]_\alpha &= \begin{cases} cl\left\{\left(\bigcup\limits_{\beta>0.5}\bigcup\limits_{0\le\lambda\le1} f_\beta^\lambda\right)\bigcup\left(\bigcup\limits_{\alpha\le\beta\le0.5}\bigcup\limits_{0\le\lambda\le1} g_\beta^\lambda\right)\right\}, & if \quad 0\le\alpha\le0.5 \\ \bigcup\limits_{\beta\ge\alpha}\bigcup\limits_{0\le\lambda\le1} f_\beta^\lambda, & if \quad 0.5<\alpha\le1 \end{cases} \\
&= \begin{cases} \left(\bigcup\limits_{\beta\ge0.5}\bigcup\limits_{0\le\lambda\le1} f_\beta^\lambda\right)\bigcup\left(\bigcup\limits_{\alpha\le\beta\le0.5}\bigcup\limits_{0\le\lambda\le1} g_\beta^\lambda\right), & if \quad 0\le\alpha\le0.5 \\ \bigcup\limits_{\beta\ge\alpha}\bigcup\limits_{0\le\lambda\le1} f_\beta^\lambda, & if \quad 0.5<\alpha\le1 \end{cases}
\end{aligned}
$$

*where*

$$
\begin{cases} f_\beta^\lambda(\cdot) : f_\beta^\lambda(x) = (1-\lambda)(x^2-3+\beta)+\lambda((2\beta-1)x^2-6\beta+4), \\ g_\beta^\lambda(\cdot) : g_\beta^\lambda(x) = (1-\lambda)(x^2-3+\beta)+\lambda((1-2\beta)x^2-2\beta+2), \end{cases}
$$

*for all $\lambda\in[0,1]$ and $\beta\in[0,1]$.*

We define semicontinuity of set-valued functions as follows.

**Definition 2.5.11** *(Diamond and Kloeden, 1994) A set-valued function $F:\Omega\to\mathcal{P}(\mathbb{R}^n)$, $\Omega\in\mathbb{R}^m$, is* upper semicontinuous (usc) *at $t_0\in\Omega$ if for every $\epsilon>0$ there exists a $\delta=\delta(t_0,\epsilon)>0$ such that*

$$\rho(F(t),F(t_0))<\epsilon$$

*if* $||t - t_0|| < \delta$, $t \in \Omega$.

**Definition 2.5.12** *(Diamond and Kloeden, 1994) A set-valued function* $F : \Omega \to \mathcal{P}(\mathbb{R}^n)$, $\Omega \in \mathbb{R}^m$, *is* lower semicontinuous (lsc) *at* $t_0 \in \Omega$ *if for every* $\epsilon > 0$ *there exists a* $\delta = \delta(t_0, \epsilon) > 0$ *such that*

$$\rho(F(t_0), F(t)) < \epsilon$$

*if* $||t - t_0|| < \delta$, $t \in \Omega$.

The set-valued function $F$ is said to be usc (lsc) if it is usc (lsc) at every $t \in \Omega$. If it is both usc and lsc, the function is continuous.

**Example 2.5.13** *The set-valued functions* $F, G : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *such that*

$$F(x) = \begin{cases} [-1,1], & \text{if } x = 0 \\ \{0\}, & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} [-1,1], & \text{if } x \neq 0 \\ \{0\}, & \text{if } x = 0 \end{cases}$$

*are usc and lsc, respectively.*

The concept of semicontinuity of fuzzy-set-valued functions is similar.

**Definition 2.5.14** *(Diamond and Kloeden, 1994) A fuzzy-set-valued function* $F : \Omega \to \mathcal{F}_{\mathcal{K}}(\mathbb{R}^n)$, $\Omega \in \mathbb{R}^m$, *is* upper semicontinuous (usc) *at* $t_0 \in \Omega$ *if for every* $\epsilon > 0$ *there exists a* $\delta = \delta(t_0, \epsilon) > 0$ *such that*

$$\rho([F(t_0)]^\alpha, [F(t)]^\alpha) < \epsilon$$

*if* $||t - t_0|| < \delta$, $t \in \Omega$, *for all* $\alpha \in [0,1]$.

**Definition 2.5.15** *(Diamond and Kloeden, 1994) A fuzzy-set-valued function* $F : \Omega \to \mathcal{F}_{\mathcal{K}}(\mathbb{R}^n)$, $\Omega \in R^m$, *is* lower semicontinuous (lsc) *at* $t_0 \in \Omega$ *if for every* $\epsilon > 0$ *there exists a* $\delta = \delta(t_0, \epsilon) > 0$ *such that*

$$\rho([F(t)]^\alpha, [F(t_0)]^\alpha) < \epsilon$$

*if* $||t - t_0|| < \delta$, $t \in \Omega$, *for all* $\alpha \in [0,1]$.

The fuzzy-set-valued function $F$ is said to be usc (lsc) if it is usc (lsc) at every $t \in \Omega$. If the fuzzy-set-valued function $F$ is usc (lsc), the set-valued functions $[F]^\alpha : \Omega \to \mathcal{K}^n$ are clearly usc (lsc). The converse implication is not necessarily true, unless $[F]^\alpha$ are uniformly usc (lsc) in $\alpha \in [0,1]$. As a result of these definitions, a fuzzy-set-valued function $F$ is $d_\infty$-continuous if and only if it is usc and lsc. In this text if a function is $d_\infty$-continuous it will be said that it is continuous. If another metric is used we will specify it.

Let us denote by $C([a,b]; \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n))$ the space of continuous fuzzy-set-valued functions from $[a,b]$ to $\mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ endowed with the metric $H(F,G) = \sup_{x \in [a,b]} d_\infty(F(x), G(x))$ for $F, G \in C([a,b]; \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n))$. The next result is important to assure existence of solution of FDEs.

**Theorem 2.5.16** *(Bede, 2013) The space* $C([a,b]; \mathcal{F}_{\mathcal{C}}(\mathbb{R}))$ *is a complete metric space.*

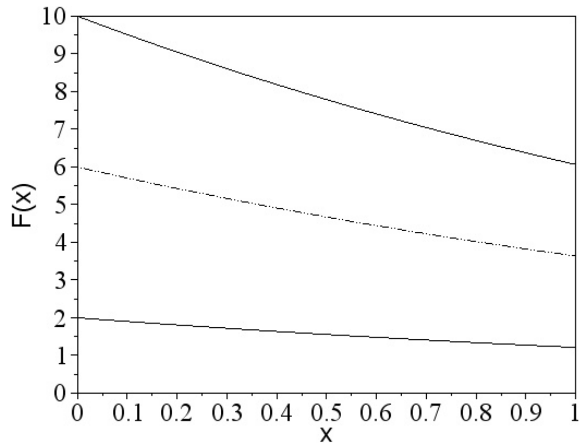


Figure 2.13: Real-valued functions delimiting the 0-cut (continuous line) and defining the core (dashed-dotted line) of a fuzzy function.)

We finish this section by presenting different ways to graphically illustrate fuzzy functions. Consider $F : [a, b] \to \mathcal{F}(\mathbb{R})$ a fuzzy-set-valued function and $\tilde{F} \in \mathcal{F}(E([a, b], \mathbb{R}))$.

1. Fuzzy-set-valued function (or fuzzy attainable sets):
   (a) level set function plot: presents real-valued functions that define two important levels of the fuzzy function, the 0-cut and the core (Figure 2.13).
   (b) gray scale plot: for different values of $x$ we associate a degree of gray to the possible values of $F(x)$, so that the darker the color, the higher the membership degree (Figure 2.14).
2. Fuzzy bunch of functions:
   (a) gray scale plot: for different crisp functions we associate a degree of gray, so that the darker the color, the higher the membership degree to $\tilde{F}$ (Figure 2.15).

## 2.6 Summary

Basic concepts such as the definition of fuzzy sets, fuzzy arithmetic and fuzzy metric spaces were reviewed in this chapter. We recalled Zadeh's extension principle and some of its results which will play important role in the definition of derivative and integral used in this thesis. We also defined and pointed out differences and connections between fuzzy-set-valued functions and fuzzy bunches of functions. Two special kinds of fuzzy bunch of functions were defined to connect with fuzzy-number-valued functions. All the presented concepts will be extensively used to develop the fuzzy calculus and the FDE theory proposed in this thesis.

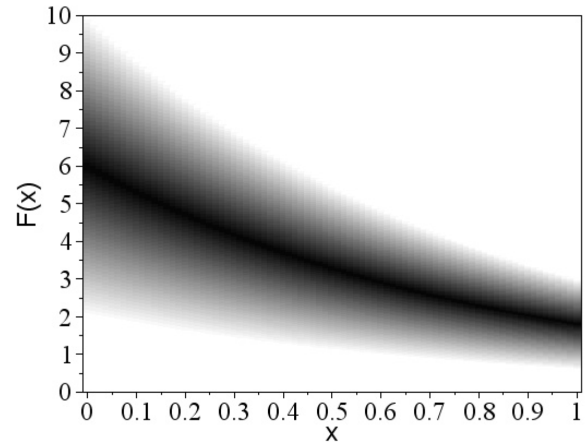


Figure 2.14: Fuzzy function $F$ represented by gray scale: at each $x$ the fuzzy number $F(x)$ is displayed such that the darker the color, the higher the membership degree of each point to $F(x)$.

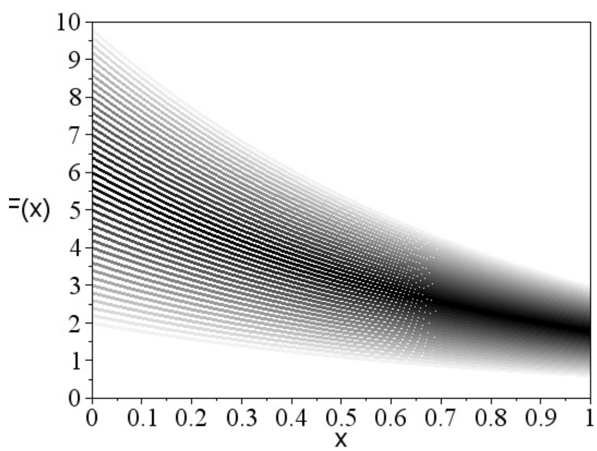


Figure 2.15: Fuzzy bunch of functions $\tilde{F}$ with membership degree represented by gray scale: the darker the color, the higher the membership degree of each real-valued function to $\tilde{F}$.

# Chapter 3

# Fuzzy Calculus

Arithmetic for fuzzy numbers can be seen as interval arithmetic on its $\alpha$-cuts. Therefore, fuzzy calculus for fuzzy-number-valued function is equivalent to interval calculus on $\alpha$-cuts of these functions (interval-valued functions). This interpretation is valid for some approaches, for example in the Hukuhara derivative and its generalizations and in the Aumann, Henstock and Riemann integrals that will be presented in Section 3.1. Fuzzy calculus has been extensively investigated (see e.g. Puri and Ralescu (1983, 1986); Kaleva (1987); Seikkala (1987); Wu and Gong (2001); Bede and Gal (2005); Stefanini and Bede (2009)).

The definitions of derivative and integral proposed in this study have already been introduced in Barros et al. (2013), Gomes and Barros (2012) and Gomes and Barros (2013). They are not connected with interval calculus, since they operate on fuzzy bunches of functions. Some comparisons between these operators and those for fuzzy-set-valued functions are subjects of this thesis. In Section 3.2 these results are explored.

We first review known approaches (see Puri and Ralescu (1983, 1986); Seikkala (1987); Gal (2000); Bede and Gal (2005, 2004b); Stefanini and Bede (2009); Wu and Gong (2001)), defined for fuzzy-set-valued functions. The interested reader can also refer to some other approaches in the literature (e.g. Chang and Zadeh (1972); Dubois and Prade (1980); Goetschel Jr. and Voxman (1984); Chalco-Cano et al. (2011b)).

## 3.1 Fuzzy Calculus for Fuzzy-Set-Valued Functions

We review in this section some known approaches for integrals (Aumann, Riemann and Henstock integrals) and derivatives (Hukuhara and generalized derivatives) for fuzzy-set-valued functions. We also present results connecting these fuzzy integrals and derivatives.

### 3.1.1 Integrals

The first integral proposed for fuzzy-number-valued functions is based on Aumann integral for multivalued functions (Aumann, 1965). Kaleva defined it in Kaleva (1987). The same idea can be found in Puri and Ralescu (1986) who were studying the topic of fuzzy random variables.

Denote by $S(G)$ the subset of all integrable selections of a set-valued function $G : I \to \mathcal{P}(\mathbb{R}^n)$, i.e.,

$$S(G) = \{g : I \to \mathbb{R}^n : g \text{ is integrable and } g(t) \in G(t), \forall t \in I\}.$$

**Definition 3.1.1** *(Puri and Ralescu, 1986; Kaleva, 1987) The Aumann integral of a fuzzy-set-valued function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ over $[a, b]$ is defined levelwise*

$$\begin{aligned}\left[(A)\int_a^b F(x)\,dx\right]_\alpha &= \int_a^b [F]_\alpha\,dx \\ &= \left\{\int_a^b g(x)\,dx : g \in S([F(x)]_\alpha)\right\}\end{aligned}$$

*for all $\alpha \in [0, 1]$.*

The function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ is said to be Aumann integrable over $[a, b]$ if $(A)\int_a^b F(x)\,dx \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$.

The following integrals have been defined for functions $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$.

**Definition 3.1.2** *(Gal, 2000; Wu and Gong, 2001) The Riemann integral of a fuzzy-number-valued function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ over $[a, b]$ is the fuzzy number $A$ such that for every $\epsilon > 0$ there exist $\delta > 0$ such that for any division $d : a = x_0 < x_1 < ... < x_n = b$ with $x_i - x_{i-1} < \delta$, $i = 1, ..., n$, and $\xi_i \in [x_i - x_{i-1}]$*

$$d_\infty\left(\sum_{i=1}^{n-1} F(\xi_i)(x_i - x_{i-1}), A\right) < \epsilon.$$

The function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ is said to be Riemann integrable over $[a, b]$ if $A \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$. We denote $(R)\int_a^b F(x)\,dx = A$

**Definition 3.1.3** *(Bede and Gal, 2004b; Wu and Gong, 2001) Consider $\delta_n : a = x_0 < x_1 < ... < x_n = b$ a partition of the interval $[a, b]$, $\xi_i \in [x_i - x_{i-1}]$, $i = 1, ..., n$, a sequence $\xi$ in $\delta_n$ and $\delta(x) > 0$ a real-valued function over $[a, b]$. The division $P(\delta_n, \xi)$ is considered to be $\delta$-fine if*

$$[x_{i-1}, x_i] \subseteq (\xi_{i-1} - \delta(\xi_{i-1}), \xi_{i-1} + \delta(\xi_{i-1}))$$

*The Henstock integral of a fuzzy-number-valued function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ over $[a, b]$ is the fuzzy number $A$ such that for every $\epsilon > 0$ there exist a real-valued function $\delta$ such that for any $\delta$-fine division $P(\delta_n, \xi)$,*

$$d_\infty\left(\sum_{i=1}^{n-1} F(\xi_i)(x_i - x_{i-1}), A\right) < \epsilon.$$

The function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ is said to be Henstock integrable over $[a, b]$ if $A \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$. We denote $(H)\int_a^b F(x)\,dx = A$.

Henstock integral is more general than Riemann, i.e., whenever a function is Riemann integrable, it is Henstock integrable as well.

**Remark 3.1.4** *If we write that a function is* integrable, *without specifying whether it is Aumann, Riemann or Henstock, we mean it is integrable in all these three senses.*

**Corollary 3.1.5** *(Bede, 2013; Kaleva, 1987; Wu and Gong, 2001) If a function $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ is continuous then it is integrable. Moreover,*

$$\left[\int F\right]_\alpha = \left[\int f_\alpha^-, \int f_\alpha^+\right]$$

*for all $\alpha \in [0, 1]$.*

**Theorem 3.1.6** *(Bede, 2013; Kaleva, 1987; Wu and Gong, 2001) Let $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ be integrable and $a \leq x_1 \leq x_2 \leq x_3 \leq b$. Then*

$$\int_{x_1}^{x_3} F = \int_{x_1}^{x_2} F + \int_{x_2}^{x_3} F.$$

**Theorem 3.1.7** *(Bede, 2013; Kaleva, 1987; Wu and Gong, 2001) Let $F, G : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ be integrable, then*

*(i)* $\int (F + G) = \int F + \int G$;

*(ii)* $\int (\lambda F) = \lambda \int F$, *for any* $\lambda \in \mathbb{R}$;

*(iii)* $d_\infty(F, G)$ *is integrable;*

*(iv)* $d_\infty(\int F, \int G) \leq \int d_\infty(F, G)$.

### 3.1.2 Derivatives

The Hukuhara differentiability for fuzzy functions is based on the concept of Hukuhara differentiability for interval-valued functions (Hukuhara, 1967).

**Definition 3.1.8** *(Puri and Ralescu, 1983) Let $F : (a, b) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$. If the limits*

$$\lim_{h \to 0^+} \frac{F(x_0 + h) \ominus_H F(x_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(x_0) \ominus_H F(x_0 - h)}{h}$$

*exist and equal some element $F'_H(x_0) \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$, then $F$ is Hukuhara differentiable (H-differentiable for short) at $x_0$ and $F'_H(x_0)$ is its Hukuhara derivative (H-derivative for short) at $x_0$.*

**Example 3.1.9** *The fuzzy-number-valued function of Example 2.5.2, $F(x) = Ax$ with $A = (-1; 0; 1)$, is an H-differentiable function for $x \geq 0$ and*

$$F'_H(x) = A.$$

*For $x < 0$, $F$ is not H-differentiable since $F(x + h) \ominus_H F(x)$ is not defined. Considering $x > 0$, $F$ is a particular case of Example 8.30 in Bede (2013), which shows that any function $G(x) = Bg(x)$ with $g(x) > 0$, $g'(x) > 0$ and $B$ a fuzzy number is H-differentiable. Moreover,*

$$G'_H(x) = Bg'(x).$$

An H-differentiable fuzzy function has H-differentiable $\alpha$-cuts (that is, its $\alpha$-cuts are interval-valued H-differentiable functions). The converse, however, is not true, unless its $\alpha$-cuts are uniformly H-differentiable (see Kaleva (1987)).

**Definition 3.1.10** *(Seikkala, 1987) Let $F:[a,b]\to\mathcal{F}_{\mathcal{C}}(\mathbb{R})$. If*

$$[(f_\alpha^-)'(x_0),(f_\alpha^+)'(x_0)]$$

*exists for all $\alpha\in[0,1]$ and defines the $\alpha$-cuts of a fuzzy number $F_S'(x_0)$, then $F$ is Seikkala differentiable at $x_0$ and $F_S'(x_0)$ is the Seikkala derivative of $F$ at $x_0$.*

Kaleva (1987) proved that if $F:[a,b]\to\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ is H-differentiable, then $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable and

$$[F'(x_0)]_\alpha=[(f_\alpha^-)'(x_0),(f_\alpha^+)'(x_0)],$$

that is, if $F$ is H-differentiable, it is Seikkala differentiable and the derivatives are the same.

**Theorem 3.1.11** *(Kaleva, 1987) Let $F:[a,b]\to\mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ be a H-differentiable function. Then it is continuous.*

**Theorem 3.1.12** *(Kaleva, 1987) Let $F,G:[a,b]\to\mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ be H-differentiable functions and $\lambda\in\mathbb{R}$. Then $(F+G)_H'=F_H'+G_H'$ and $(\lambda F)_H'=\lambda F_H'$.*

It is easy to see that if $F$ is Seikkala (or Hukuhara) differentiable, $(f_\alpha^-)'(x)\leq(f_\alpha^+)'(x)$, hence the function $\operatorname{diam}[F(x)]_\alpha=f_\alpha^+(x)-f_\alpha^-(x)$ is nondecreasing on $[a,b]$. It means that the function has nondecreasing fuzziness. As will be clear in Chapter 4, this is considered a shortcoming since an H-differentiable function can not represent a function with decreasing fuzziness or periodicity. In order to overcome this, the generalized differentiability concepts were created. They generalize the H-differentiability, that is, they are defined for more cases of fuzzy-number-valued functions and whenever the H-derivative of a function exists, its generalization exist and has the same value.

**Definition 3.1.13** *(Bede and Gal, 2004a, 2005) Let $F:(a,b)\to\mathcal{F}_{\mathcal{C}}(\mathbb{R})$. If the limits of some pair*

*(i)* $\displaystyle\lim_{h\to0^+}\frac{F(x_0+h)\ominus_H F(x_0)}{h}$ *and* $\displaystyle\lim_{h\to0^+}\frac{F(x_0)\ominus_H F(x_0-h)}{h}$ *or*

*(ii)* $\displaystyle\lim_{h\to0^+}\frac{F(x_0)\ominus_H F(x_0+h)}{-h}$ *and* $\displaystyle\lim_{h\to0^+}\frac{F(x_0-h)\ominus_H F(x_0)}{-h}$ *or*

*(iii)* $\displaystyle\lim_{h\to0^+}\frac{F(x_0+h)\ominus_H F(x_0)}{h}$ *and* $\displaystyle\lim_{h\to0^+}\frac{F(x_0-h)\ominus_H F(x_0)}{-h}$ *or*

*(iv)* $\displaystyle\lim_{h\to0^+}\frac{F(x_0)\ominus_H F(x_0+h)}{-h}$ *and* $\displaystyle\lim_{h\to0^+}\frac{F(x_0)\ominus_H F(x_0-h)}{h}$

*exist and are equal to some element $F'_G(x_0)$ of $\mathcal{F}_\mathcal{C}(\mathbb{R})$, then $F$ is strongly generalized differentiable (or GH-differentiable) at $x_0$ and $F'_G(x_0)$ is the strongly generealized derivative (GH-derivative for short) of $F$ at $x_0$.*

An (i)-strongly generalized differentiable function presents nondecreasing diameter, since it is the definition of the H-differentiability. (ii)-strongly generalized differentiability (we call (ii)-differentiability, for short) on the other hand implies in nonincreasing diameter. The (iii) and (iv)-differentiability cases correspond to points where the function changes its behavior with respect to the diameter. It means that a strongly differentiable non-crisp function may present periodical behaviour, as well as convergence to a single point.

In case $F$ is defined on a closed interval, that is, $F : [a, b] \to \mathcal{F}_\mathcal{C}(\mathbb{R})$, we define the derivative at $a$ using the limit from the right and at $b$ using the limit from the left.

**Example 3.1.14** *The fuzzy-number-valued function of Example 2.5.2, $F(x) = Ax$ with $A = (-1; 0; 1)$, is a GH-differentiable function for $x \in \mathbb{R}$ and*

$$F'_{gH}(x) = A.$$

*Different from the H-derivative case, the GH-derivative of $F$ is defined for $x < 0$. According to Example 8.35 in Bede (2013), any function $G(x) = Bg(x)$ with $B$ a fuzzy number and $g : (a, b) \to \mathbb{R}$ differentiable with at most a finite number of roots in $(a, b)$ is GH-differentiable. Moreover,*

$$G'_H(x) = Bg'(x).$$

Example 3.1.14 illustrates that, different from the H-derivative, GH-differentiable functions can have decreasing diameter.

**Definition 3.1.15** *(Bede and Gal, 2005) Let $F : (a, b) \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ and $x_0 \in (a, b)$. For a nonincreasing sequence $h_n \to 0$ and $n_0 \in \mathbb{N}$ we denote*

$$A^{(1)}_{n_0} = \left\{ n \geq n_0; \exists E^{(1)}_n := F(x_0 + h_n) \ominus_H F(x_0) \right\},$$

$$A^{(2)}_{n_0} = \left\{ n \geq n_0; \exists E^{(2)}_n := F(x_0) \ominus_H F(x_0 + h_n) \right\},$$

$$A^{(3)}_{n_0} = \left\{ n \geq n_0; \exists E^{(3)}_n := F(x_0) \ominus_H F(x_0 - h_n) \right\},$$

$$A^{(4)}_{n_0} = \left\{ n \geq n_0; \exists E^{(4)}_n := F(x_0 - h_n) \ominus_H F(x_0) \right\}.$$

*The function $F$ is said to be weakly generalized differentiable at $x_0$ if for any nonincreasing sequence $h_n \to 0$ there exists $n_0 \in \mathbb{N}$, such that*

$$A^{(1)}_{n_0} \cup A^{(2)}_{n_0} \cup A^{(3)}_{n_0} \cup A^{(4)}_{n_0} = \{n \in \mathbb{N}; n \geq n_0\}$$

*and moreover, there exists an element in $\mathcal{F}_C(\mathbb{R})$, such that if for some $j \in \{1, 2, 3, 4\}$ we have $card(A^{(j)}_{n_0}) = +\infty$, then*

$$\lim_{h_n \searrow 0, n\to\infty, n\in A^{(j)}_{n_0}} d_\infty \left( \frac{E^{(j)}_n}{(-1)^{j+1} h_n}, F'(x_0) \right) = 0.$$

Definition 3.1.15 is more general than Definition 3.1.13, that is, it is defined for more cases of fuzzy-number-valued functions and whenever the latter exists, the former also exists and has the same value.

The next definition is equivalent to Definition 3.1.15 (see Bede and Stefanini (2013)).

**Definition 3.1.16** *(Stefanini and Bede, 2009; Bede and Stefanini, 2013) Let $F : (a, b) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$. If the limit*

$$\lim_{h\to 0} \frac{F(x_0 + h) \ominus_{gH} F(x_0)}{h}$$

*exist and belongs to $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ then $F$ is generalized Hukuhara differentiable (gH-differentiable for short) at $x_0$ and $F'_{gH}(x_0)$ is the generearlized Hukuhara derivative (gH-derivative for short) of $F$ at $x_0$.*

**Theorem 3.1.17** *(Bede and Stefanini, 2013) Let $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ be a gH-differentiable function at $x_0$. Then it is levelwise continuous at $x_0$.*

**Theorem 3.1.18** *(Bede and Stefanini, 2013) Let $F : [a, b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ be such that the functions $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are real-valued functions, differentiable with respect to $x$, uniformly in $\alpha \in [0, 1]$. Then the function $F(x)$ is gH-differentiable at a fixed $x \in [a, b]$ if and only if one of the following two cases holds:*

*a) $(f_\alpha^-)'(x)$ is increasing, $(f_\alpha^+)'(x)$ is decreasing as functions of $\alpha$, and $\left(f_1^-\right)'(x) \le \left(f_1^+\right)'(x)$, or*

*b) $(f_\alpha^-)'(x)$ is decreasing, $(f_\alpha^+)'(x)$ is increasing as functions of $\alpha$, and $\left(f_1^+\right)'(x) \le \left(f_1^-\right)'(x)$. Moreover,*

$$\left[F'_{gH}(x)\right]_\alpha = [\min\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}, \max\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}],$$

*for all $\alpha \in [0, 1]$.*

The next concept further extends the gH-differentiability.

**Definition 3.1.19** *(Stefanini and Bede, 2009) Let $F : (a, b) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$. If the limit*

$$\lim_{h\to 0} \frac{F(x_0 + h) \ominus_{g} F(x_0)}{h}$$

*exist and belongs to $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ then $F$ is generalized differentiable (g-differentiable for short) at $x_0$ and $F'_g(x_0)$ is the fuzzy generearlized derivative (g-derivative for short) of $F$ at $x_0$.*

**Example 3.1.20** *Recall the fuzzy-number-valued function of Example 2.5.10, $F : [0, 0.5] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ with $\alpha$-cuts*

$$[F(x)]_\alpha = \begin{cases} [x^2 - 3 + \alpha, (1 - 2\alpha)x^2 - 2\alpha + 2], & \text{if} \quad 0 \le \alpha \le 0.5 \\ [x^2 - 3 + \alpha, (2\alpha - 1)x^2 - 6\alpha + 4], & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}.$$

*We try to calculate the gH and the g-derivative of $F$.*

*First we try to find the limit*

$$\lim_{h\to 0} \frac{F(x+h)\ominus_{gH} F(x)}{h}$$

*that defines the gH-differentiability. We make use of*

$$\begin{aligned}[F(x+h)\ominus_{gH} F(x)]_\alpha &= [\min\{f_\alpha^-(x+h)-f_\alpha^-(x), f_\alpha^+(x+h)-f_\alpha^+(x)\},\\ &\quad \max\{f_\alpha^-(x+h)-f_\alpha^-(x), f_\alpha^+(x+h)-f_\alpha^+(x)\}]\end{aligned}$$

*for all $\alpha\in[0,1]$.*

*For $\alpha\in[0,0.5]$ we obtain*

$$[F(x+h)\ominus_{gH} F(x)]_\alpha = [(1-2\alpha)(2xh+h^2), 2xh+h^2].$$

*Thus*

$$\lim_{h\to 0}\frac{[F(x+h)\ominus_{gH} F(x)]_\alpha}{h} = [(1-2\alpha)2x, 2x]$$

*and as consequence*

$$\lim_{h\to 0}\frac{[F(x+h)\ominus_{gH} F(x)]_0}{h} = \{2x\}$$

*and*

$$\lim_{h\to 0}\frac{[F(x+h)\ominus_{gH} F(x)]_{0.25}}{h} = [x, 2x].$$

*The condition*

$$\alpha<\beta \;\Rightarrow \lim_{h\to 0}\frac{[F(x+h)\ominus_{gH} F(x)]_\beta}{h} \subset \lim_{h\to 0}\frac{[F(x+h)\ominus_{gH} F(x)]_\alpha}{h}$$

*does not hold, hence* $\lim_{h\to 0}\frac{[F(x+h)\ominus_{gH} F(x)]_\alpha}{h}$ *can not be a fuzzy number and the gH-derivative is not defined for this function.*

*We now calculate the limit*

$$\lim_{h\to 0}\frac{F(x+h)\ominus_g F(x)}{h}$$

*that defines the g-differentiability. We use*

$$\begin{aligned}[F(x+h)\ominus_g F(x)]_\alpha &= \Big[\inf_{\beta\geq\alpha}\min\{f_\beta^-(x+h)-f_\beta^-(x), f_\beta^+(x+h)-f_\beta^+(x)\}\,,\\ &\quad \sup_{\beta\geq\alpha}\max\{f_\beta^-(x+h)-f_\beta^-(x), f_\beta^+(x+h)-f_\beta^+(x)\}\Big]\end{aligned}$$

*which, in this case, holds true for all $\alpha\in[0,1]$. Since $f_\beta^-(x+h)-f_\beta^-(x)=2xh+h^2$ and $f_\beta^+(x+h)-f_\beta^+(x)=(1-2\alpha)2xh+h^2$ for $\beta\leq 0.5$ and $f_\beta^-(x+h)-f_\beta^-(x)=2xh+h^2$ and $f_\beta^+(x+h)-f_\beta^+(x)=(2\alpha-1)2xh+h^2$ for $\beta>0.5$, we obtain for $\alpha>0.5$:*

$$\lim_{h\to 0}\frac{[F(x+h)\ominus_g F(x)]_\alpha}{h} = cl\bigcup_{\beta\geq\alpha>0.5}[(2\beta-1)2x, 2x] = [(2\alpha-1)2x, 2x].$$

*For $\alpha \leq 0.5$,*

$$\lim_{h\to 0}\frac{[F(x+h)\ominus_g F(x)]_\alpha}{h} = cl\left(\bigcup_{0.5\geq\beta\geq\alpha\geq 0}[(1-2\beta)2x,2x]\right)\bigcup\left(\bigcup_{\beta>0.5}[(2\beta-1)2x,2x]\right) = [0,2x].$$

*We have obtained the fuzzy number $F_g' : [0,0.5] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ with $\alpha$-cuts*

$$[F_g'(x)]_\alpha = \begin{cases} [0,2x]\,, & \textit{if} \quad 0 \leq \alpha \leq 0.5 \\ [(2\alpha-1)2x,2x]\,, & \textit{if} \quad 0.5 < \alpha \leq 1 \end{cases}.$$

*as the g-derivative.*

The g-difference is not defined for all pairs of fuzzy numbers, as we showed in Example 2.3.6. The same happens to the g-derivative, that is, it is not always well-defined.

**Example 3.1.21** *The Definition of the g-derivative of the fuzzy-number-valued function of Example 2.5.9 leads to*

$$[F_g'(x)]_\alpha = \begin{cases} \{20x\}\bigcup\{0\}, & \textit{if} \quad 0 \leq \alpha \leq 0.5 \\ \{0\}, & \textit{if} \quad 0.5 < \alpha \leq 1 \end{cases}.$$

*That is, it is not a fuzzy-number-valued function. Hence $F$ is not g-differentiable.*

The function $F$ in Example 3.1.21 has $f_\alpha^-(x)$ and $f_\alpha^+(x)$ differentiable real-valued functions with respect to $x$, uniformly with respect to $\alpha \in [0,1]$, but it is not g-differentiable. In the case a function is g-differentiable and satisfy the just mentioned hypothesis, it has a formula that has been proved by Bede and Stefanini (2013).

**Theorem 3.1.22** *Let $F : [a,b] \to \mathbb{R}_{\mathcal{F}}$ with $f_\alpha^-(x)$ and $f_\alpha^+(x)$ differentiable real-valued functions with respect to $x$, uniformly with respect to $\alpha \in [0,1]$. Then*

$$\left[F_g'(x)\right]_\alpha$$

$$= \left[\inf_{\beta\geq\alpha}\min\{\left(f_\beta^-\right)'(x),\left(f_\beta^+\right)'(x)\},\sup_{\beta\geq\alpha}\max\{\left(f_\beta^-\right)',\left(f_\beta^+\right)'(x)\}\right]$$

*whenever $F$ is g-differentiable.*

**Proof.** See Bede and Stefanini (2013).

■

### 3.1.3 Fundamental Theorem of Calculus

**Theorem 3.1.23** *(Kaleva, 1987) Let $F : [a,b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ be continuous, then $G(x) = \int_a^x F(s)ds$ is H-differentiable and*

$$G'_H(x) = F(x).$$

**Theorem 3.1.24** *(Kaleva, 1987) Let $F : [a,b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ be H-differentiability and the H-derivative $F'_H$ be integrable over $[a,b]$. Then*

$$F(x) = F(a) + \int_a^x F'_H(s)ds,$$

*for each $x \in [a,b]$.*

The H-differentiable is equivalent to strongly generalized differentiability (i) in Definition 3.1.13. For the case (ii) in the same definition, Bede and Gal have proved the following theorem.

**Theorem 3.1.25** *(Bede and Gal, 2010) Let $F : [a,b] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ be (ii)-differentiable. Then the derivative $F'_G$ is integrable over $[a,b]$ and*

$$F(x) = F(b) - \int_x^b F'_G(s)ds,$$

*for each $x \in [a,b]$.*

## 3.2 Fuzzy Calculus for Fuzzy Bunches of Functions

Recently, Barros et al. (2013) and Gomes and Barros (2012) elaborated the fuzzy calculus for fuzzy bunches of functions, based on the definitions of derivative and integral via Zadeh's extension of the correspondent operators for classical functions. This original theory is reviewed and further developed in the present section.

### 3.2.1 Integral

The integral operator will be represented by $\int$, i.e.,

$$\begin{array}{rccc} \int : & L^1([a,b];\mathbb{R}^n) & \to & \mathcal{A}C([a,b];\mathbb{R}^n) \\ & f & \mapsto & \int_a^t f \end{array}$$

$t \in [a,b]$ (see Appendix for definitions of spaces of functions).

**Definition 3.2.1** *(Barros et al., 2010; Gomes and Barros, 2012) Let $F \in \mathcal{F}(L^1([a,b];\mathbb{R}^n))$. The integral of $F$ is given by $\widehat{\int} F$, whose membership function is*

$$\mu_{\widehat{\int} F}(y) = \begin{cases} \sup_{f \in \int^{-1} y} \mu_F(f), & \text{if } \int^{-1} y \neq \emptyset \\ 0, & \text{if } \int^{-1} y = \emptyset \end{cases}, \tag{3.1}$$

*for all $y \in \mathcal{A}C([a,b];\mathbb{R}^n)$. In words, $\widehat{\int}$ is Zadeh's extension of the operator $\int$.*

The next theorem is a consequence of Theorem 2.2.5.

**Theorem 3.2.2** *If* $F \in \mathcal{F}(L^1([a,b];\mathbb{R}^n))$,

$$\begin{aligned}\left[\widehat{\int} F\right]_\alpha &= \int [F]_\alpha \\ &= \left\{\int f : f \in [F]_\alpha \subset L^1([a,b];\mathbb{R}^n)\right\},\end{aligned}$$

*for all* $\alpha \in [0,1]$.

**Proof.** Since the integral is a continuous operator, the result follows directly from Theorem 2.2.5.

∎

We next define a linear structure in $\mathcal{F}(L^1([a,b];\mathbb{R}^n))$. Given two fuzzy bunches of functions $F$ and $G$ and $\lambda \in \mathbb{R}$,

$$\mu_{F+G}(h) = \sup_{f+g=h} \min\{\mu_F(f), \mu_G(g)\},$$

$$\mu_{\lambda F}(f) = \begin{cases} \mu_F(h/\lambda) & \text{if } \lambda \neq 0 \\ \chi_0(f) & \text{if } \lambda = 0 \end{cases}.$$

Since these operations are Zadeh's extension of addition and multiplication by scalar, which are continuous (except multiplication by zero), Theorem 2.2.5 assures that given $F, G \in \mathcal{F}_\mathcal{K}(L^1([a,b];\mathbb{R}^n))$ and $\lambda \in \mathbb{R}$,

$$F + G \in \mathcal{F}_\mathcal{K}(L^1([a,b];\mathbb{R}^n)) \quad \text{and} \quad [F+G]_\alpha = [F]_\alpha + [G]_\alpha$$

and

$$\lambda F \in \mathcal{F}_\mathcal{K}(L^1([a,b];\mathbb{R}^n)) \quad \text{and} \quad [\lambda F]_\alpha = \lambda [F]_\alpha$$

for all $\alpha \in [0,1]$.

**Theorem 3.2.3** *Let* $F, G \in \mathcal{F}_\mathcal{K}(L^1([a,b];\mathbb{R}^n))$, *then*

*(i)* $\widehat{\int}(F+G) = \widehat{\int} F + \widehat{\int} G$;

*(ii)* $\widehat{\int} \lambda F = \lambda \widehat{\int} F$, *for any* $\lambda \in \mathbb{R}$.

**Proof.** From Theorem 2.2.5 and the linearity of the integral operator,

$$\begin{aligned}[\widehat{\int}(F+G)]_\alpha &= \int [F+G]_\alpha \\ &= \int \{h : h = f+g, f \in [F]_\alpha, g \in [G]_\alpha\} \\ &= \{\int (f+g), f \in [F]_\alpha, g \in [G]_\alpha\} \\ &= \{\int f + \int g, f \in [F]_\alpha, g \in [G]_\alpha\} \\ &= \{\int f, f \in [F]_\alpha\} + \{\int g, g \in [G]_\alpha\} \\ &= \int [F]_\alpha + \int [G]_\alpha \\ &= [\widehat{\int} F]_\alpha + [\widehat{\int} G]_\alpha\end{aligned}$$

and

$$\begin{aligned}
[\widehat{\int}\lambda F]_\alpha &= \int[\lambda F]_\alpha \\
&= \{\int \lambda f : f \in [F]_\alpha\} \\
&= \{\lambda \int f : f \in [F]_\alpha\} \\
&= \lambda\{\int f : f \in [F]_\alpha\} \\
&= \lambda \int [F]_\alpha \\
&= \lambda[\widehat{\int} F]_\alpha
\end{aligned}$$

for all $\alpha \in [0, 1]$.

■

**Example 3.2.4** *Let $A$ be the symmetrical triangular fuzzy number with support $[-a, a]$, $a > 0$. The fuzzy function $F(\cdot) \in \mathcal{F}(L^1([0, T]; \mathbb{R}))$ such that*

$$[F(\cdot)]_\alpha = \{f(\cdot) : f(t) = \gamma t, \gamma \in [A]^\alpha\}$$

*where $f(\cdot) : [0, T] \to \mathbb{R}$, for each $\alpha \in [0, 1]$, has attainable sets*

$$F(t) = At. \tag{3.2}$$

*To determine the integral of $F$ using Definition 3.2.1, we need to explicit the membership function of $A$ and $F$:*

$$\mu_A(\gamma) = \begin{cases} \dfrac{\gamma}{a} + 1, & \textit{if } -a \leq \gamma < 0 \\ -\dfrac{\gamma}{a} + 1, & \textit{if } 0 \leq \gamma < a \\ 0, & \textit{otherwise} \end{cases} \tag{3.3}$$

*and*

$$\mu_F(f) = \begin{cases} \dfrac{\gamma}{a} + 1, & \textit{if } f(t) = \gamma t \textit{ with } -a \leq \gamma < 0 \\ -\dfrac{\gamma}{a} + 1, & \textit{if } f(t) = \gamma t \textit{ with } 0 \leq \gamma < a \\ 0, & \textit{otherwise} \end{cases} \tag{3.4}$$

*Formula (3.1) states that $\mu_{\widehat{\int} F}(y) \neq 0$ only if there exists $f$ such that $\int f = y$ and $\mu_F(f) \neq 0$. In this example, it happens only if $f(t) = \gamma t$ with $\gamma \in [A]^\alpha$, that is, $y = \gamma t^2/2$.*

$$\begin{aligned}
\mu_{\widehat{\int} F}(\gamma t^2/2) &= \sup\nolimits_{\int f = \gamma t^2/2} \mu_F(f) \\
&= \sup\nolimits_{\int (\gamma t) = \gamma t^2/2} \mu_F(\gamma t) \\
&= \mu_F(\gamma t) \\
&= \begin{cases} \frac{\gamma}{a} + 1, & \textit{if } -a \leq \gamma < 0 \\ -\frac{\gamma}{a} + 1, & \textit{if } 0 \leq \gamma < a \\ 0, & \textit{otherwise} \end{cases} \\
&= \mu_A(\gamma).
\end{aligned} \tag{3.5}$$

*Hence*

$$\mu_{\widehat{\int} F}(f) = \begin{cases} \dfrac{\gamma}{a} + 1, & \text{if } f(t) = \gamma t^2/2 \text{ with } -a \le \gamma < 0 \\ -\dfrac{\gamma}{a} + 1, & \text{if } f(t) = \gamma t^2/2 \text{ with } 0 \le \gamma < a \\ 0, & \text{otherwise} \end{cases} \tag{3.6}$$

*or*

$$[F(\cdot)]_\alpha = \{f(\cdot) : f(t) = \gamma t^2/2, \gamma \in [A]_\alpha\}.$$

*For each* $\alpha \in [0,1]$*, its attainable sets are*

$$F(t) = At^2/2.$$

*The Aumann integral of (3.2) can be calculated levelwise and we obtain the same attainable sets as obtained with* $\widehat{\int}$*:*

$$\begin{aligned} [\textstyle\int F(t)]_\alpha &= [\textstyle\int f_\alpha^-, \int f_\alpha^-] \\ &= [-at^2/2, at^2/2] \\ &= [A]_\alpha t^2/2. \end{aligned}$$

The next section introduces the derivative operator for fuzzy bunches of functions. It is defined for more restricted spaces than the integral since they are extensions of the classical case. Also, different from the integral case, we explore the derivative on different spaces (Example 3.2.12) due to the fact that it is not a continuous operator (in general). We are more interested, though, in differentiating fuzzy bunches of the space of absolutely continuous functions (see Appendix), since we can differentiate more elements in this space than in the space of differentiable functions. Furthermore, it is used and has been explored in the differential inclusions theory, which, as already mentioned, has important connections with the theory we propose to develop.

### 3.2.2 Derivative

The derivative operator in the sense of distributions (see Aubin and Cellina (1984)) will be represented by $D$, that is,

$$\begin{aligned} D: \ \mathcal{AC}([a,b];\mathbb{R}^n) &\to L^1([a,b];\mathbb{R}^n) \\ f &\mapsto Df \end{aligned}$$

Thus, there exists $Df(t)$ *a.e.*, in $[a,b]$.

**Definition 3.2.5** *Let* $F \in \mathcal{F}(\mathcal{AC}([a,b];\mathbb{R}^n))$*. The derivative of* $F$ *is given by* $\widehat{D}F$*, whose membership function is*

$$\mu_{\widehat{D}F}(y) = \begin{cases} \sup_{f \in D^{-1}y} \mu_F(f), & \text{if } D^{-1}y \neq \emptyset \\ 0, & \text{if } D^{-1}y = \emptyset \end{cases} . \tag{3.7}$$

*for all* $y \in L^1([a,b];\mathbb{R}^n)$*. In words,* $\widehat{D}$ *is Zadeh's extension of operator* $D$*.*

**Example 3.2.6** *Let $F(\cdot)$ be the same fuzzy bunch as in Example 3.2.4. We note that $F(\cdot) \in \mathcal{F}(\mathcal{AC}([a,b];\mathbb{R}))$.*

*Following the same reasoning as Example 3.2.4,*

$$\begin{aligned}
\mu_{\widehat{D}F}(\gamma) &= \sup_{Df=\gamma} \mu_F(f) \\
&= \sup_{D(\gamma t)=\gamma} \mu_F(\gamma t) \\
&= \mu_F(\gamma t) \\
&= \begin{cases} \frac{\gamma}{a}+1, & \text{if } -a \le \gamma < 0 \\ -\frac{\gamma}{a}+1, & \text{if } 0 \le \gamma < a \\ 0, & \text{otherwise} \end{cases} \\
&= \mu_A(\gamma).
\end{aligned} \tag{3.8}$$

*It means that the support of $\widehat{D}F(\cdot)$ is composed of constant functions such that, at each instant $t$, the derivative of $F(\cdot)$ is always the fuzzy number $A$.*

**Lemma 3.2.7** *For $D$ defined as above, the preimage $D^{-1}g$ is a closed nonempty subset of the space of functions $\mathcal{AC}([a,b];\mathbb{R}^n)$ with respect to the uniform norm for each $g \in L^1([a,b];\mathbb{R}^n)$.*

**Proof.** $D^{-1}g$ is a finite dimensional subspace of $\mathcal{AC}([a,b];\mathbb{R}^n)$ since $D^{-1}g = \{f+k : k \in \mathbb{R}^n\}$ for $f \in \mathcal{AC}([a,b];\mathbb{R}^n)$ such that $f = \int_a^x g$. Hence $D^{-1}g$ is closed.

■

**Theorem 3.2.8** *(Barros et al., 2013) Let $F \in \mathcal{F}_{\mathcal{K}}(\mathcal{AC}([a,b];\mathbb{R}^n))$. Then*

$$[\widehat{D}F]_\alpha = D[F]_\alpha.$$

**Proof.** $D^{-1}(g) \neq \emptyset$ and it is closed, according to Lemma 3.2.7. Thus, $[F]_0 \cap D^{-1}(g)$ is compact, since it is a closed subset of the compact set $[F]_0$.

We show inclusion $[\widehat{D}(F)]_\alpha \subset D([F]_\alpha)$ considering two cases: $\alpha \in (0,1]$ and later $\alpha = 0$.
(i) For $\alpha \in (0,1]$, let $g \in [\widehat{D}(F)]_\alpha$, then

$$\alpha \le \widehat{D}(F)(g) = \sup_{h \in D^{-1}(g)} F(h) = \sup_{h \in [F]_0 \cap D^{-1}(g)} F(h) = F(f)$$

for some $f$, since $F$ is an upper semicontinuous function (that is, the membership of $F$ is usc) and $[F]_0 \cap D^{-1}(g)$ is compact. So, $F(f) \ge \alpha$. That is, $f \in [F]_\alpha \cap D^{-1}(g)$. Hence $g \in D([F]_\alpha)$.
(ii) For $\alpha = 0$,

$$\cup_{\alpha \in (0,1]} [\widehat{D}(F)]_\alpha \subset \cup_{\alpha \in (0,1]} D([F]_\alpha) \subseteq D([F]_0).$$

Consequently,

$$[\widehat{D}(F)]_0 = \overline{\cup_{\alpha \in (0,1]} [\widehat{D}(F)]_\alpha} \subset \overline{\cup_{\alpha \in (0,1]} D([F]_\alpha)} \subseteq \overline{D([F]_0)} = D([F]_0).$$

The last equality holds because $D$ is a closed operator.

Now, the inclusion $D([F]_\alpha) \subset [\widehat{D}(F)]_\alpha$ is considered next. If $g \in D([F]_\alpha)$, there exists $f \in [F]_\alpha$ such that $D(g) = f$. Thus,

$$\widehat{D}(F)(g) = \sup_{h \in D^{-1}(g)} F(h) \ge F(f) \ge \alpha \Rightarrow g \in [\widehat{D}(F)]_\alpha$$

for all $\alpha \in [0,1]$. Thus, the theorem is proved.

■

**Example 3.2.9** *Consider $g : [a,b] \to \mathbb{R}$ a differentiable and positive function, $A = (c;d;e)$ a triangular fuzzy number and the fuzzy-number-valued function*

$$F(x) = Ag(x).$$

*We have*

$$[F(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$$

*with*

$$f_\alpha^-(x) = [a + \alpha(b-a)]g(x) \quad \text{and} \quad f_\alpha^+(x) = [e - \alpha(e-d)]g(x)$$

*differentiable with respect to $x$ and continuous with respect to $\alpha$.*

*The continuity in $\alpha$ means that $F(x) \in \mathcal{F}_C^0(\mathbb{R})$. It will be proved in Theorem 3.3.3 that the representative bunch of first kind of this function has compact $\alpha$-cuts in $\mathcal{AC}([a,b];\mathbb{R})$, since it satisfies the hypotheses of the theorem.*

*The derivative of the representative bunch of first kind has $\alpha$-cuts*

$$\begin{aligned}
[\widehat{D}\tilde{F}]_\alpha &= \bigcup_{\beta \geq \alpha} \bigcup_{0 \leq \lambda \leq 1} (f_\beta^\lambda)' \\
&= (1-\lambda)[a + \alpha(b-a)]g' + \lambda[e - \alpha(e-d)]g' \\
&= \{a \cdot g', a \in [A]_\alpha\}
\end{aligned}$$

*for all $\alpha \in [0,1]$, that is,*

$$\widehat{D}\tilde{F} = Ag'.$$

*It is a similar result as in Example 3.1.14 for GH-derivative, in terms of attainable sets.*

**Example 3.2.10** *Let*

$$f(x) = Be^{cx} \tag{3.9}$$

*be a fuzzy-set-valued function where $c$ is a real constant and $B$ is a fuzzy subset of $\mathbb{R}$ such that $B(1) = 1$, $B(0.5) = 0.5$ and $B(x) = 0$ everywhere else. Hence $f(x)$ is not differentiable using Hukuhara or any generalized derivatives since it is not a fuzzy-number-valued function. On the other hand, the fuzzy bunch of functions with $\alpha$-levels*

$$[\tilde{f}(\cdot)]^\alpha = \begin{cases} \{y_1(\cdot), y_2(\cdot)\}, & \text{if } 0 \leq \alpha \leq 0.5 \\ \{y_1(\cdot)\}, & \text{if } 0.5 < \alpha \leq 1 \end{cases},$$

*where $y_1(x) = e^{cx}$ and $y_2(x) = 0.5e^{cx}$, has (3.9) as attainable fuzzy sets and is $\widehat{D}$-differentiable. Since this $\alpha$-levels are compact subsets of $\mathcal{AC}([a,b];\mathbb{R})$, we apply Theorem 3.2.8 and obtain*

$$[\widehat{D}f(\cdot)]^\alpha = \begin{cases} \{z_1(\cdot), z_2(\cdot)\}, & \text{if } 0 \leq \alpha \leq 0.5 \\ \{z_1(\cdot)\}, & \text{if } 0.5 < \alpha \leq 1 \end{cases},$$

*where $z_1(x) = ce^{cx}$ and $z_2(x) = 0.5ce^{cx}$. Its attainable sets are*

$$\widehat{D}f(x) = cBe^{cx}.$$

**Remark 3.2.11** *We cannot use Hukuhara of the generalized derivatives to differentiate fuzzy-set-valued functions whose images are not fuzzy numbers, as the function in Example 3.2.10. On the other hand, we can use $\widehat{D}$ on its fuzzy bunches of functions and assume its attainable fuzzy sets as derivative.*

**Example 3.2.12** *(Barros et al., 2013) The operator $\widehat{D} : \mathcal{F}_{\mathcal{K}}(C^1([a,b];\mathbb{R}^n)) \to \mathcal{F}_{\mathcal{K}}(C([a,b];\mathbb{R}^n))$ is well-defined and for each $F \in \mathcal{F}_{\mathcal{K}}(C^1([a,b];\mathbb{R}^n))$ we have*

$$[\widehat{D}F]^\alpha = D[F]^\alpha$$

*for all $\alpha \in [0,1]$, if $C^1([a,b];\mathbb{R}^n)$ is endowed with the norm $\| x \|_1 = \sup_{0\leq t\leq T}\{|x(t)| + |x'(t)|\}$ and $C([a,b];\mathbb{R}^n)$ is endowed with the the usual supremum norm. The result follows from Theorem 2.2.5 since $D$ is a continuous function for these spaces.*

Another possibility of $D$ being a continuous operator is as follows.

**Theorem 3.2.13** *(Barros et al., 2013) Consider the subset of $\mathcal{AC}([0,T];\mathbb{R}^n)$:*

$$Z_T(\mathbb{R}^n) = \{x(\cdot) \in C([0,T];\mathbb{R}^n) : \exists\ x'(\cdot) \in L^\infty([0,T];\mathbb{R}^n)\},$$

*with $Z_T(\mathbb{R}^n)$ having the uniform norm topology and $L^\infty([0,T];\mathbb{R}^n)$ with the weak*-topology. Thus,*

$$\widehat{D} : \mathcal{F}_{\mathcal{K}}(Z_T(\mathbb{R}^n)) \to \mathcal{F}_{\mathcal{K}}(L^\infty([0,T];\mathbb{R}^n)),$$

*where $\widehat{D}$ is Zadeh's extension of the derivative $D$, is well defined, that is, for each $F \in \mathcal{F}_{\mathcal{K}}(Z_T(\mathbb{R}^n))$, the $\alpha$-level $[\widehat{D}F]^\alpha$ is a compact subset of $\mathcal{F}_{\mathcal{K}}(L^\infty([0,T];\ \mathbb{R}^n))$ and $[\widehat{D}F]^\alpha = D[F]^\alpha$.*

**Proof.** The result follows from the Theorem 2.2.5 because

$$D : Z_T(\mathbb{R}^n) \to L^\infty([0,T];\mathbb{R}^n)$$

is a continuous linear operator (see Aubin and Cellina (1984) - page 104).

■

**Theorem 3.2.14** *Let $F, G \in \mathcal{F}_{\mathcal{K}}(\mathcal{AC}([a,b];\mathbb{R}^n))$, then*

*(i)* $\widehat{D}(F+G) = \widehat{D}F + \widehat{D}G$;

*(ii)* $\widehat{D}\lambda F = \lambda\widehat{D}F$, *for any* $\lambda \in \mathbb{R}$.

**Proof.** This proof is completely analogous to the one of Theorem 3.2.14, due to the linearity of the derivative operator.

$$\begin{aligned}
[\widehat{D}(F+G)]_\alpha &= D[F+G]_\alpha \\
&= \{D(f+g), f \in [F]_\alpha, g \in [G]_\alpha\} \\
&= \{Df + \textstyle\int g, f \in [F]_\alpha, g \in [G]_\alpha\} \\
&= \{Df, f \in [F]_\alpha\} + \{Dg, g \in [G]_\alpha\} \\
&= D[F]_\alpha + D[G]_\alpha \\
&= [\widehat{D}F]_\alpha + [\widehat{D}G]_\alpha
\end{aligned}$$

and

$$\begin{aligned}[\widehat{D}\lambda F]_\alpha &= D[\lambda F]_\alpha \\ &= \{D\lambda f : f \in [F]_\alpha\} \\ &= \lambda\{Df : f \in [F]_\alpha\} \\ &= \lambda D[F]_\alpha \\ &= \lambda[\widehat{D}F]_\alpha\end{aligned}$$

for all $\alpha \in [0,1]$. Thus using Theorem 2.1.8 we obtain the desired result.

■

### 3.2.3 Fundamental Theorem of Calculus

We also obtain a result connecting the concepts of derivative and integral for fuzzy bunches of functions as in the classical case and in the fuzzy-set-valued function case.

**Theorem 3.2.15** *Let $F \in \mathcal{F}_{\mathcal{K}}(L^1([a,b];\mathbb{R}^n))$. Hence*

$$\widehat{D}\left(\widehat{\int} F\right) = F,$$

*that is,*

$$\left[\widehat{D}\left(\widehat{\int} F\right)\right]^\alpha = [F]^\alpha.$$

*for all $\alpha \in [0,1]$.*

**Proof.** Since Theorem 3.2.2 holds,

$$\begin{aligned}[\widehat{\int} F]_\alpha &= \int [F]_\alpha \\ &= \{\int f : f \in [F]_\alpha\}\end{aligned}$$

for all $\alpha \in [0,1]$ and $\widehat{\int} F \in \mathcal{F}_{\mathcal{K}}(\mathcal{A}C([0,T];\mathbb{R}^n))$. Then Theorem 3.2.8 holds and,

$$\begin{aligned}[\widehat{D}\widehat{\int} F]_\alpha &= D[\widehat{\int} F]_\alpha \\ &= \{D\int f : f \in [F]_\alpha\} \\ &= [F]_\alpha\end{aligned}$$

for all $\alpha \in [0,1]$ .

■

## 3.3 Comparison

We can associate more than one fuzzy bunch of functions with the same attainable fuzzy sets. Choosing the suitable fuzzy bunch may lead to equivalence of $\widehat{D}$ with derivatives for fuzzy-set-valued functions and equivalence of $\widehat{\int}$ with integrals for fuzzy-set-valued functions (in terms of attainable sets). In this section the similarities of the proposed theory with other approaches are demonstrated.

The motivation for this comparison and the definition of the two different fuzzy bunches of functions of Definition 2.5.6 is what happens with the fuzzy-number-valued functions of Examples 3.1.20 and 3.1.21. In the former the gH-derivative does not exist whereas the g-derivative does and in the latter both do not exist. We calculate the $\widehat{D}$-derivative of the corresponding fuzzy bunches of the fuzzy-valued functions in Examples 3.1.20 and 3.1.21 next. The fuzzy-number-valued functions do not meet the conditions of the theorems to be stated, revealing the importance of the hypotheses of these theorems.

**Example 3.3.1** *Recall Examples 2.5.10 and 3.1.20 where the representative bunch of first kind is given by the $\alpha$-cuts*

$$[\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \left(\bigcup_{\beta\geq 0.5}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda\right)\bigcup\left(\bigcup_{\alpha\leq\beta\leq 0.5}\bigcup_{0\leq\lambda\leq 1} g_\beta^\lambda\right), & \textit{if} \quad 0\leq\alpha\leq 0.5 \\ \bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda, & \textit{if} \quad 0.5<\alpha\leq 1 \end{cases}$$

*where*

$$\begin{cases} f_\beta^\lambda(\cdot) : f_\beta^\lambda(x) = (1-\lambda)(x^2-3+\beta)+\lambda((2\beta-1)x^2-6\beta+4), \\ g_\beta^\lambda(\cdot) : g_\beta^\lambda(x) = (1-\lambda)(x^2-3+\beta)+\lambda((1-2\beta)x^2-2\beta+2), \end{cases}$$

*for all $\lambda\in[0,1]$. Since*

$$\begin{cases} (f_\beta^\lambda)'(\cdot) : f_\beta^\lambda(x) = (1-2\lambda+2\beta\lambda)2x, \\ (g_\beta^\lambda)'(\cdot) : g_\beta^\lambda(x) = (1-2\beta\lambda)2x, \end{cases}$$

*using Theorem 3.2.8 to calculate $\widehat{D}\tilde{F}_1(\cdot)$ we obtain*

$$[\widehat{D}\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \left(\bigcup_{\beta\geq 0.5}\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)'\right)\bigcup\left(\bigcup_{\alpha\leq\beta\leq 0.5}\bigcup_{0\leq\lambda\leq 1} (g_\beta^\lambda)'\right), & \textit{if} \quad 0\leq\alpha\leq 0.5 \\ \bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)', & \textit{if} \quad 0.5<\alpha\leq 1 \end{cases}$$

*At $x\in[0,0.8]$*

$$[\widehat{D}\tilde{F}_1(x)]_\alpha = [m, M]$$

*with*

$$m = \begin{cases} \min\left\{ \left( \bigcup_{\beta \geq 0.5} \bigcup_{0 \leq \lambda \leq 1} (f_\beta^\lambda)'(x) \right) \bigcup \left( \bigcup_{\alpha \leq \beta \leq 0.5} \bigcup_{0 \leq \lambda \leq 1} (g_\beta^\lambda)'(x) \right) \right\}, & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ \min\left\{ \bigcup_{\beta \geq \alpha} \bigcup_{0 \leq \lambda \leq 1} (f_\beta^\lambda)'(x) \right\}, & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}$$

$$= \begin{cases} 0, & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ (2\alpha - 1)2x, & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}$$

*and*

$$M = \begin{cases} \max\left\{ \left( \bigcup_{\beta \geq 0.5} \bigcup_{0 \leq \lambda \leq 1} (f_\beta^\lambda)'(x) \right) \bigcup \left( \bigcup_{\alpha \leq \beta \leq 0.5} \bigcup_{0 \leq \lambda \leq 1} (g_\beta^\lambda)'(x) \right) \right\}, & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ \max\left\{ \bigcup_{\beta \geq \alpha} \bigcup_{0 \leq \lambda \leq 1} (f_\beta^\lambda)'(x) \right\}, & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}$$

$$= \begin{cases} 2x, & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ 2x, & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}$$

*Hence the attainable sets of the $\widehat{D}$-derivative are*

$$[\widehat{D}\tilde{F}_1(x)]_\alpha = \begin{cases} [0, 2x], & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ [(2\alpha - 1)2x, 2x], & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}$$

*that is, the same as the g-derivative of the fuzzy-number-valued function $F$.*

**Example 3.3.2** *Recall Examples 2.5.9 and 3.1.21 where the representative bunch of first kind is given by the $\alpha$-cuts*

$$[F(x)]_\alpha = \begin{cases} [10x^2 - 12, 10x^2 + 2], & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ [-1, 1], & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}.$$

*and the representative bunch of second kind is defined by*

$$[\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{2} \bigcup_{0 \leq \lambda \leq 1} y_i^\lambda(\cdot), & \text{if} \quad 0 \leq \alpha \leq 0.5 \\ \bigcup_{0 \leq \lambda \leq 1} y_1^\lambda(\cdot), & \text{if} \quad 0.5 < \alpha \leq 1 \end{cases}$$

*where*

$$\begin{cases} y_1^\lambda(\cdot) : y_1^\lambda(x) = (1-\lambda)(10x^2-12) + \lambda(10x^2+2), \\ y_2^\lambda(\cdot) : y_2^\lambda(x) = (1-\lambda)(-1) + \lambda, \\ y_3^\lambda(\cdot) : y_3^\lambda(x) = (1-\lambda)(-1) + \lambda(10x^2+2), \\ y_4^\lambda(\cdot) : y_4^\lambda(x) = (1-\lambda)(10x^2-12) + \lambda, \end{cases}$$

*for all* $\lambda \in [0,1]$.

*The derivatives of the representative bunch of first kind is given by the* $\alpha$*-cuts*

$$[\widehat{D}\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{2} \bigcup_{0\le\lambda\le 1} (y_i^\lambda)'(\cdot), & \text{if} \quad 0 \le \alpha \le 0.5 \\ \bigcup_{0\le\lambda\le 1} y_1^\lambda(\cdot), & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}$$

*and the representative bunch of second kind is defined by*

$$[\widehat{D}\tilde{F}_2(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{4} \bigcup_{0\le\lambda\le 1} (y_i^\lambda)'(\cdot), & \text{if} \quad 0 \le \alpha \le 0.5 \\ \{(y_1)'(\cdot)\} \bigcup_{0\le\lambda\le 1} y_1^\lambda(\cdot), & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}$$

*where*

$$\begin{cases} (y_1)'(\cdot) : (y_1)'(x) = 20x, \\ (y_2)'(\cdot) : (y_2)'(x) = 0, \\ (y_3)'(\cdot) : (y_3)'(x) = \lambda 20x, \\ (y_4)'(\cdot) : (y_4)'(x) = (1-\lambda)20x, \end{cases}$$

*for all* $\lambda \in [0,1]$.

*In terms of attainable sets, the derivative of the representative bunch of first kind has attainanble sets*

$$[\widehat{D}\tilde{F}_1(x)]_\alpha = \begin{cases} \{0\} \bigcup \{20x\}, & \text{if} \quad 0 \le \alpha \le 0.5 \\ \{20x\}, & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}.$$

*The derivative of the representative bunch of second kind for* $x \in [0,1]$ *has attainanble sets*

$$[\widehat{D}\tilde{F}_2(x)]_\alpha = \begin{cases} [0, 20x], & \text{if} \quad 0 \le \alpha \le 0.5 \\ \{20x\}, & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}$$

*and for* $x \in [-1,0]$,

$$[\widehat{D}\tilde{F}_1(x)]_\alpha = \begin{cases} [20x, 0], & \text{if} \quad 0 \le \alpha \le 0.5 \\ \{20x\}, & \text{if} \quad 0.5 < \alpha \le 1 \end{cases}.$$

*Hence the derivative of the representative bunch of first kind at each* $x \in [-1,1]$ *does not define fuzzy numbers while the derivative of the representative bunch of second kind does.*

Example 3.3.1 illustrates that the $\widehat{D}$-derivative of the fuzzy bunch of first kind of the given fuzzy-number-valued function $F$ exists but its attainable sets are not fuzzy numbers (while the gH-derivative of the fuzzy-number-valued function does not exist). The result that we state next regards the necessary conditions for equivalence between the gH-derivative of a fuzzy-number-valued function and the $\widehat{D}$-derivative of the corresponding fuzzy bunch of first kind. The result we state later is connected with Example 3.3.2, that is, it is necessary that the g-derivative exist for the equivalence with the derivative of the representative bunch of second kind. The $\widehat{D}$ derivative in this last case provided a fuzzy-number-valued function, which no derivative for fuzzy-number-valued functions that we presented can do.

**Theorem 3.3.3** *Let* $F : [a,b] \to \mathcal{F}^0_{\mathcal{C}}(\mathbb{R})$ *be such that the functions* $f^-_\alpha(x)$ *and* $f^+_\alpha(x)$ *are real-valued functions, differentiable with respect to* $x$*, uniformly in* $\alpha \in [0,1]$*. Suppose also that one of the following two cases holds:*

*a)* $(f^-_\alpha)'(x)$ *is increasing,* $(f^+_\alpha)'(x)$ *is decreasing as functions of* $\alpha$*, and*

$$\left(f^-_1\right)'(x) \le \left(f^+_1\right)'(x),$$

*or*

*b)* $(f^-_\alpha)'(x)$ *is decreasing,* $(f^+_\alpha)'(x)$ *is increasing as functions of* $\alpha$*, and*

$$\left(f^+_1\right)'(x) \le \left(f^-_1\right)'(x).$$

*Then* $F$ *generates a representative bunch of first kind* $\tilde{F}(\cdot)$ *with compact* $\alpha$*-levels and whose* $\widehat{D}$*-derivative has attainable sets*

$$\left[\widehat{D}\tilde{F}(x)\right]_\alpha = [\min\{\left(f^-_\alpha\right)'(x), \left(f^+_\alpha\right)'(x)\}, \max\{\left(f^-_\alpha\right)'(x), \left(f^+_\alpha\right)'(x)\}].$$

*In words, the* $\widehat{D}$*-derivative coincides with the gH-derivative at each* $x$*.*

**Proof.**

We prove that the sets $A_\alpha$ in Definition 2.5.6 are $\alpha$-cuts of a fuzzy set in $\mathcal{AC}([a,b];\mathbb{R})$ using the same arguments as in Example 2.5.8. The only difference is to demonstrate compactness, which we do next. Note that any sequence $(f^{\lambda_i}_{\alpha_i})$ in $\bigcup_{\beta \ge \alpha} \bigcup_{0 \le \lambda \le 1} f^\lambda_\beta(\cdot)$ has a convergent subsequence whose limit belongs to $\bigcup_{\beta \ge \alpha} \bigcup_{0 \le \lambda \le 1} f^\lambda_\beta(\cdot)$, due to the continuity of $f^\lambda_\beta(\cdot)$ as function of the real parameters $\lambda$ and $\beta$ defined on closed intervals (compact subsets) $[0,1]$ and $[\alpha, 1]$, respectively. And since $f^\pm_\beta$

are differentiable, so are $f_\beta^\lambda$. According to Frink Jr (1935), the differentiability with respect to $x$, uniformly in $\alpha \in [0,1]$, assures that if a sequence of functions converges to a function $f$, the sequence of its derivatives converges to $f'$. Since $f$ is differentiable, it belongs to $\mathcal{AC}([a,b];\mathbb{R})$. As a result, $\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda$ is compact in $\mathcal{AC}([a,b];\mathbb{R})$ and it is equal to its closure and hence to $A_\alpha$.

We next make use of Theorem 3.2.8 since $\tilde{F} \in \mathcal{F}_{\mathcal{K}}(\mathcal{AC}([a,b];\mathbb{R}))$:

$$\begin{aligned}[\widehat{D}\tilde{F}]_\alpha &= D[\tilde{F}]_\alpha \\ &= \bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)'\end{aligned}$$

for all $\alpha \in [0,1]$. And we observe that for case a)

$$\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)'(x) = [(f_\beta^-)'(x), (f_\beta^+)'(x)]$$

and

$$(f_\alpha^-)'(x) \leq (f_\beta^-)'(x) \leq (f_1^-)'(x) \leq (f_1^+)'(x) \leq (f_\beta^+)'(x) \leq (f_\alpha^+)'(x)$$

for $0 \leq \alpha \leq \beta \leq 1$,

$$[(f_\beta^-)'(x), (f_\beta^+)'(x)] \subseteq [(f_\alpha^-)'(x), (f_\alpha^+)'(x)].$$

Hence

$$\begin{aligned}[\widehat{D}\tilde{F}(x)]_\alpha &= \bigcup_{\beta\geq\alpha} [(f_\beta^-)'(x), (f_\beta^+)'(x)] \\ &= [(f_\alpha^-)'(x), (f_\alpha^+)'(x)]\end{aligned}$$

for all $\alpha \in [0,1]$.

Similarly, case b) leads to

$$[\widehat{D}\tilde{F}(x)]_\alpha = [(f_\alpha^+)'(x), (f_\alpha^-)'(x)].$$

As result we obtain the desired expression,

$$\left[\widehat{D}\tilde{F}(x)\right]_\alpha = \left[\min\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}, \max\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}\right],$$

for all $\alpha \in [0,1]$, which the same as stated in Theorem 3.1.18 for the gH-derivative.

∎

A similar result for connecting $\widehat{D}$-derivative and g-derivative is presented in what follows.

**Theorem 3.3.4** *Let $F \in [a,b] \to \mathcal{F}_C^0(\mathbb{R})$ be a function such that $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable real-valued functions with respect to $x$, uniformly with respect to $\alpha \in [0,1]$. Then $F$ generates a representative bunch of second kind $\tilde{F}(\cdot)$ with compact $\alpha$-levels and whose $\widehat{D}$-derivative has attainable sets*

$$[\widehat{D}\tilde{F}(x)]_\alpha = \left[\inf_{\beta\geq\alpha} \min\left\{(f_\beta^-)'(x), (f_\beta^+)'(x)\right\}, \sup_{\beta\geq\alpha} \max\left\{(f_\beta^-)'(x), (f_\beta^+)'(x)\right\}\right]$$

*It means that the values of the g-derivative of $F(x)$ and the attainable sets of the $\widehat{D}$-derivative of $\tilde{F}(\cdot)$ coincide in every $x \in [a,b]$, whenever the g-derivative exists.*

**Proof.** Using the same argument of the previous proof, it follows that the resultant $B_\alpha$ in Definition 2.5.6 are compact sets in $\mathcal{AC}([a,b];\mathbb{R})$ and are the $\alpha$-cuts of the representative bunch of second kind of $F$, $\tilde{F}$. We use Theorem 3.2.8 and obtain

$$\begin{aligned}[\widehat{D}\tilde{F}]_\alpha &= D[\tilde{F}]_\alpha \\ &= \bigcup_{\beta,\gamma\geq\alpha}\bigcup_{0\leq\lambda\leq 1}(f^\lambda_{\beta,\gamma})'\end{aligned}$$

We will prove that $L = \inf_{\beta,\gamma\geq\alpha}\left\{(f^\lambda_{\beta,\gamma})'(x)\right\}$ is attained, that is, that there exists a triple $(\overline{\lambda},\overline{\beta},\overline{\gamma})$ such that $(f^{\overline{\lambda}}_{\overline{\beta},\overline{\gamma}})'(x) = L$ with $\overline{\beta},\overline{\gamma}\in[\alpha,1]$, $\overline{\lambda}\in[0,1]$. From the definition of infimum, $y \geq L$ if $y \in \bigcup\limits_{\beta,\gamma\geq\alpha}\bigcup\limits_{0\leq\lambda\leq 1}(f^\lambda_{\beta,\gamma})'(x)$ and there exists a sequence $(y_n)$, $y_n = (f^{\lambda_n}_{\beta_n,\gamma_n})'(x)$ such that

$$(f^{\lambda_n}_{\beta_n,\gamma_n})'(x)\to L, \quad L\leq(f^{\lambda_n}_{\beta_n,\gamma_n})'(x).$$

To the sequence $(y_n)$ in $\mathbb{R}$ there corresponds a sequence $(g_n(\cdot))$ of functions such that $g_n(\cdot) = (f^{\lambda_n}_{\beta_n,\gamma_n})'(\cdot)$. This sequence of functions has a convergent subsequence, since the set is sequentially compact (where we use the same result in Frink Jr (1935) as previously used). This subsequence of functions defines a subsequence in $(y_n)$, $y_{n_k} = g_{n_k}(x)$. The subsequence $(y_{n_k})$ also converges to $L$. The limit of $g_{n_k}(\cdot)$ is attained for some triple $(\overline{\lambda},\overline{\beta},\overline{\gamma})$ and its value in $x$ is

$$(f^{\overline{\lambda}}_{\overline{\beta},\overline{\gamma}})'(x) = \lim g_{n_k}(x) = \lim y_{n_k} = L.$$

Similarly we prove that the supremum $M$ is also attained. Now we prove that

$$L = \inf_{\beta\geq\alpha}\min\left\{(f^-_\beta)'(x),(f^+_\beta)'(x)\right\}.$$

For any $(f^\lambda_{\beta,\gamma})'(x)$, we have

$$(f^-_\beta)'\leq(f^\lambda_{\beta,\gamma})'(x)\leq(f^+_\gamma)' \quad \text{or} \quad (f^+_\gamma)'\leq(f^\lambda_{\beta,\gamma})'(x)\leq(f^+_\beta)'.$$

Hence

$$\inf_{\beta\geq\alpha}\min\left\{(f^-_\beta)'(x),(f^+_\beta)'(x)\right\}\leq\inf_{\beta,\gamma\geq\alpha}\left\{(f^\lambda_{\beta,\gamma})'(x)\right\}.$$

Since

$$\bigcup_{\beta\geq\alpha}\left\{(f^-_\beta)'(x),(f^+_\beta)'(x)\right\}\subset\bigcup_{\beta,\gamma\geq\alpha}\left\{(f^\lambda_{\beta,\gamma})'(x)\right\}$$

the equality of the infimum holds.

Hence the value $L = \inf_{\beta\geq\alpha}\min\left\{(f^-_\beta)'(x),(f^+_\beta)'(x)\right\}$ is attained by $(f^-_\beta)'(x)$ or $(f^+_\beta)'(x)$, for some $\beta\geq\alpha$. The same happens to $M = \sup_{\beta\geq\alpha}\max\left\{(f^-_\beta)'(x),(f^+_\beta)'(x)\right\}$. As a consequence, there are four possible cases:

1) $L = (f^-_{\beta_1})'(x)$ and $M = (f^+_{\beta_2})'(x)$ and any value between $L$ and $M$ is attained by $(f^\lambda_{\beta_1,\beta_2})'(x)$ for some $\lambda\in[0,1]$;

2) $L = (f^+_{\beta_1})'(x)$ and $M = (f^-_{\beta_2})'(x)$ and any value between $L$ and $M$ is attained by $(f^\lambda_{\beta_2,\beta_1})'(x)$ for some $\lambda \in [0,1]$;

3) $L = (f^-_{\beta_1})'(x)$ and $M = (f^-_{\beta_2})'(x)$ and any value between $L$ and $M$ is attained by $(f^\lambda_{\beta_1,\beta_1})'(x)$ or $(f^\lambda_{\beta_2,\beta_1})'(x)$ for some $\lambda \in [0,1]$.

4) $L = (f^+_{\beta_1})'(x)$ and $M = (f^+_{\beta_2})'(x)$ and any value between $L$ and $M$ is attained by $(f^\lambda_{\beta_1,\beta_1})'(x)$ or $(f^\lambda_{\beta_1,\beta_2})'(x)$ for some $\lambda \in [0,1]$.

It proves that all values in

$$\left[\inf_{\beta\geq\alpha} \min\left\{(f^-_\beta)'(x), (f^+_\beta)'(x)\right\}, \sup_{\beta\geq\alpha} \max\left\{(f^-_\beta)'(x), (f^+_\beta)'(x)\right\}\right]$$

are attained.

Then the same expression as in Theorem 3.1.22 for g-differentiable functions is found and the desired result is proved.

■

The attainable sets of the $\widehat{\int}$-integral of certain bunches of functions also coincide with integrals for fuzzy-set-valued functions, as it will be stated in Theorem 3.3.5.

**Theorem 3.3.5** *Let $F : [a,b] \to \mathcal{F}^0_{\mathcal{C}}(\mathbb{R})$ be continuous. Then the $\widehat{\int}$-integral of the representative bunch of first kind has attainable fuzzy sets*

$$\left[\widehat{\int_a^x} \tilde{F}\right]_\alpha = \left[\int_a^x f^-_\alpha, \int_a^x f^+_\alpha\right]$$

*for all $\alpha \in [0,1]$.*

*In words, the $\widehat{\int}$-integral coincides with the integrals for fuzzy-set-valued functions at each $x$.*

**Proof.** It is not hard to prove the compacity of $A_\alpha$ (Definition 2.5.6) in $L^1([a,b];\mathbb{R})$. This is assured by the arguments that were used in proving compacity in $\mathcal{AC}([a,b];\mathbb{R})$. Following the reasoning of the previous results one demonstrate that $A_\alpha$ are the $\alpha$-cuts of a fuzzy subset in $L^1([a,b];\mathbb{R})$.

We observe that $\int_a^x f^\lambda_\beta$ is well-defined and that

$$\int_a^x f^-_\alpha \leq \int_a^x f^\lambda_\beta \quad \text{and} \quad \int_a^x f^\lambda_\beta \leq \int_a^x f^+_\alpha$$

for all $\lambda \in [0,1]$ and $0 \leq \alpha \leq \beta \leq 1$. Hence we obtain, for all $\alpha \in [0,1]$,

$$\begin{aligned} [\widehat{\int} \tilde{F}]_\alpha &= \bigcup_{\beta\geq\alpha} \bigcup_{\lambda\in[0,1]} \int_a^x f^\lambda_\beta \\ &= [\int f^-_\alpha, \int f^+_\alpha] \end{aligned}$$

where the last identity holds due to the continuity of $\int_a^x f^\lambda_\beta(x)$ on $\lambda$, $\beta$ and $x$. Thus, we have proved that the attainable sets of the $\widehat{\int}$-integral of $\tilde{F}$ have the same expression of the integrals for fuzzy-set-valued functions at each $x$.

■

## 3.4 Summary

This chapter reviewed fuzzy calculus for fuzzy-set-valued functions and presented the new fuzzy calculus using fuzzy bunches of functions. The concepts and results here displayed are essential for the development of the various approaches of FDEs, to be presented in the next chapter.

# Chapter 4

# Fuzzy Differential Equations

We review some approaches of FDEs in this chapter and propose and explore a new theory of FDEs using the $\widehat{D}$-derivative introduced in Subsection 3.2.2. Two theorems of existence of solutions to FIVPs are proved and we compare the theory with other approaches, exemplifying with biological models.

## 4.1 Approaches of FIVPs

Fuzzy differential equations have been extensively studied after Kandel and Byatt first used this expression in 1980. However, the treated problems were not FDEs, strictly, since they did not explicitly use fuzzy sets. Only after the definition of Hukuhara derivative in 1983 did Kaleva (1987) developed a theory for FDEs proposing an existence and uniqueness theorem for solutions to fuzzy initial value problems. Simultaneously, Seikkala built up a similar theory for fuzzy-number-valued functions. The proposal of this chapter is to solve

$$\begin{cases} X'(t) &= F(t, X(t)) \\ X(0) &= X_0 \end{cases} \tag{4.1}$$

where $F$ is a function that indicates the direction of the state variable $X$ at a given instant $t$.

The function $F$ is real-valued in the crisp initial value problem (IVP) as well as the initial condition and the solution. This can be interpretated as a crisp alternative (unique, given some conditions) for the direction to the state variable to follow, at each instant $t$.

The fuzzy function $F$ indicates a fuzzy direction to be followed in the fuzzy initial value problem (FIVP). In this case, there are two interpretations. One can fill this trajectory with different crisp solutions, attaching a membership degree to each of them, or one can fill this trajectory with a function that assigns to each instant $t$ a fuzzy subset (that is, the state variable is fuzzy). In the first approach the solution is a fuzzy bunch of functions while in the second one obtains a fuzzy-set-valued function.

Kaleva and Seikkala's theory of FDEs were made for fuzzy-set-valued functions. The existence theorem for these fuzzy-valued functions is found in Section 4.3 is for this kind of functions. The theory for FDEs using fuzzy bunches is what is original research and is presented in Section 4.6.

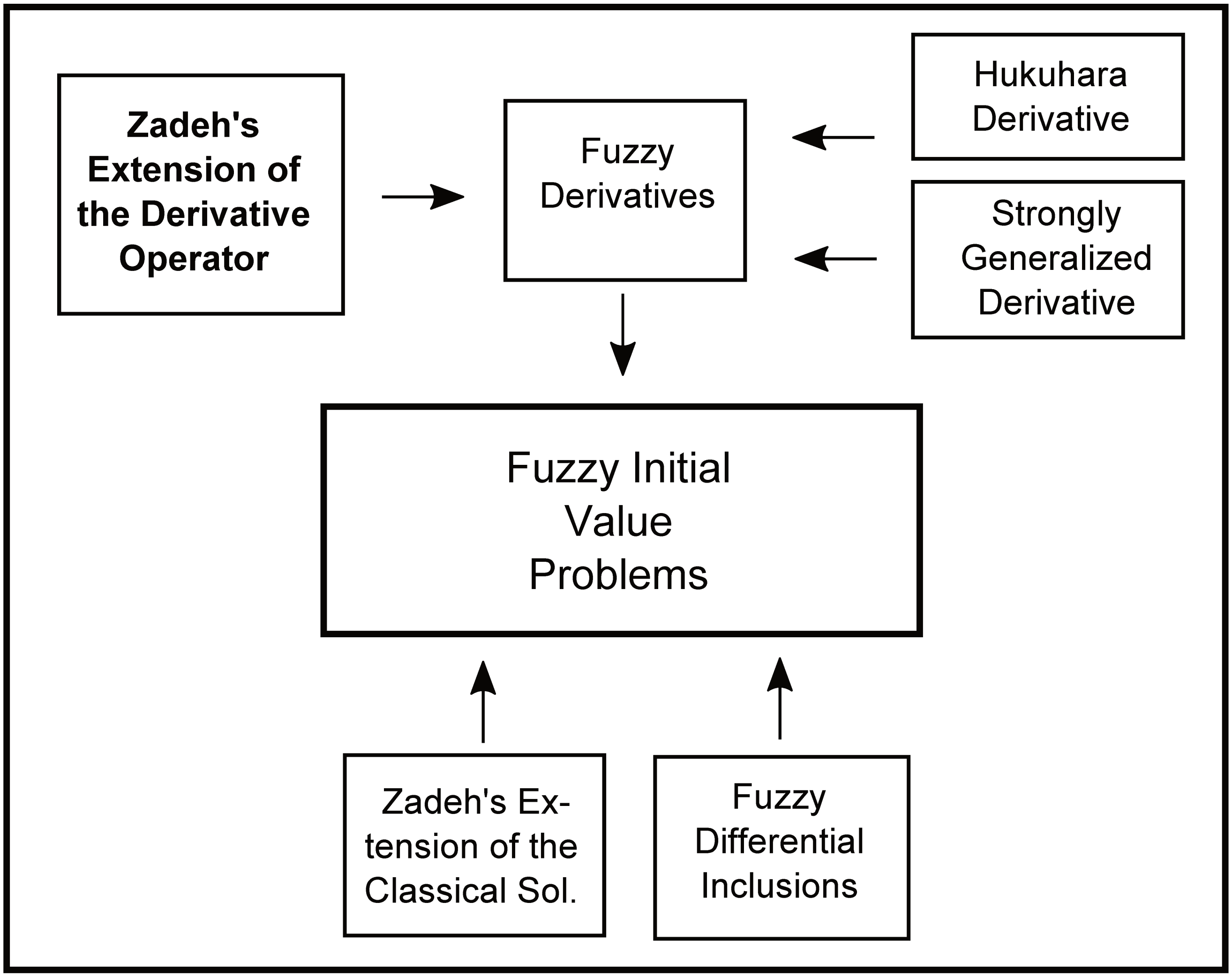


Figure 4.1: Approaches of FIVPs.

Other approaches that are not strictly FDEs are the extension of the crisp solution and fuzzy differential inclusions. The latter makes use of fuzzy bunches of functions and both are based on solving crisp differential equations. Though they do not explicitly use equality of fuzzy sets, they are explored in this chapter since they have similarities with the approach in Section 4.6.

It is important to make it clear what “solution” means in each method. Solutions of different approaches may lie in distinct spaces of fuzzy functions, which makes them uncomparable, *a priori*. In this case, we will compare the respective attainable sets. In what follows we will briefly explore this in order to make the comparisons clearer, but it will be further explained in the next sections. The various approaches of FIVPs that we treat are displayed in Figure 4.1, where we stress the use of derivatives by three of them.

**Fuzzy Differential Equations with Fuzzy Derivatives**

Consider the FIVP

$$\left\{\begin{array}{rcl} X'(t) & = & F(t, X(t)) \\ X(0) & = & X_0 \end{array}\right. , \tag{4.2}$$

where $F : [0, T] \times \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ and $X_0 \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$. A solution, if the derivative is *Hukuhara derivative*, is a continuous fuzzy-set-valued function $X : [0, T] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ that satisfies $X'(t) = F(t, X(t))$, for all $t \in [0, T]$, and the initial condition $X(0) = X_0$. In case the derivative is the *strongly generalized derivative*, the FIVP is defined only for case $n = 1$ and the solution is a continuous fuzzy-number-valued function $X : [0, T] \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ that satisfies $X'(t) = F(t, X(t))$, for all $t \in [0, T]$, and the initial condition $X(0) = X_0$. If the derivative is $\widehat{D}$*-derivative*, the solution is a fuzzy bunch of functions $X(\cdot) \in \mathcal{F}_{\mathcal{K}}(\mathcal{AC}([0, T]; \mathbb{R}^n))$ that satisfies $\widehat{D}X(t) = F(t, X(t))$ a.e. in $[0, T]$ and the initial condition. That is, given a solution $X(\cdot)$, its derivative $\widehat{D}X(\cdot)$ calculated in $t$ (attainable set in $t$) must be equal to $F(t, X(t))$, a.e. in $[0, T]$. Moreover, the attainable set at $t = 0$, $X(0)$ must satisfy the initial condition.

**Fuzzy Differential Inclusions**

Fuzzy differential inclusions are defined levelwise

$$\begin{array}{rcl} x'(t) & \in & [F(t, x(t))]_\alpha \\ x(0) & \in & [X_0]_\alpha \end{array} \tag{4.3}$$

for all $\alpha \in [0, 1]$, where $[F]_\alpha : [0, T] \times \mathbb{R}^n \to \mathcal{K}^n_{\mathcal{C}}$ and $[X_0]_\alpha \in \mathcal{K}^n_{\mathcal{C}}$. The solution to 4.3 is a fuzzy bunch of functions $X(\cdot) \in \mathcal{F}_{\mathcal{K}}(\mathcal{AC}([0, T]; \mathbb{R}^n))$ whose elements (functions) of its $\alpha$-cuts satisfies the differential inclusions (4.3) a.e. in $[0, T]$.

Note that there is no fuzzy derivative. We use the classical derivative for real-valued functions and there is no equality between fuzzy sets, hence we do not have a *fuzzy* differential equation).

**Zadeh's Extension of the Classical Solution**

Consider the classical IVP

$$\left\{\begin{array}{rcl} x'(t) & = & f(t, x(t), w) \\ x(0) & = & x_0 \end{array}\right. , \tag{4.4}$$

where $f : [0, T] \times \mathbb{R}^{n+p} \to \mathbb{R}^n$, with $w$ a parameter in $\mathbb{R}^p$, is continuous and $x_0 \in \mathbb{R}^n$. If the parameter $w$ and/or the initial condition $x_0$ are now fuzzy subsets ($W$ and $X_0$), the solution via Zadeh's extension of the classical solution is a fuzzy-set-valued function $X : [0, T] \to \mathcal{F}(\mathbb{R}^n)$ obtained from use of Zadeh's extension on the solution of (4.4) at each $t \in [0, T]$, $x(t, x_0, w)$, that depends on $x_0$ and $w$. In other words, $X$ is a solution to the fuzzy initial value problem if

$$X(t) = \widehat{x}(t, X_0, W).$$

As in the previous case, since there is no fuzzy derivative, it is not a *fuzzy* differential equation.

Note that in each case, the right-hand-side term of the differential equation belongs to a different space or is defined over a different space. However, to compare all the approaches we need to analyse equivalent FIVPs, in some sense.

First, given the function of the right-hand-side term of the differential equation of one approach, we want to be able to find the corresponding function for the other approaches. Second, we want to compare the five different kinds of FIVPs when they are modeling the same phenomenon. For instance, consider $\lambda \in \mathbb{R}$, $x \in \mathbb{R}$ and

$$f(x) = \lambda x.$$

It is intuitive to compare IVP (4.4) with this $f$ to FIVP (4.2) having

$$F(X) = \lambda X,$$

$X \in \mathcal{F}(\mathbb{R})$, since the idea of both is that the rate of change of the variable $x$ (or $X$) is proportional do the variable itself. In this case, the FDI (4.3) is also well defined when $H(x) = \lambda x$.

However, when $F$ is given, it is not always possible to find a corresponding expression for $f$, as, for example, in the case $[F(X)]_\alpha = [x_0^-, x_\alpha^+]$ for $[X]_\alpha = [x_\alpha^-, x_\alpha^+]$. Hence, a good alternative is to consider Zadeh's extension of $f$. But, in this case, given $f$, we cannot always find explicitly the expression of $F$. If $f = x(1-x)$, Zadeh's extension of $f$ is not $F(X) = X(1-X)$, unless the multiplication here carries some kind of interactivity (see Subsections 2.3.2 and 2.3.3). But the Hukuhara and strongly generalized differentiability approaches use other arithmetic to define the derivatives, hence the use of different arithmetics in each side of the equation could be criticized for not being the same.

In conclusion, we will first take into consideration the interpretation of the FIVP. We will compare the different approaches trying to model the same biological phenomenon.

The solution may belong to different spaces as well. To compare the solutions to the different approaches we do the same as when comparing the derivatives in Section 3.3. That is, in case the solution is a fuzzy bunch of functions (via $\widehat{D}$-derivative and FDIs), we compare the approaches to the fuzzy attainable sets defined in Section 2.5.

This comparison will be carried out for the exponential decay and the logistic models. The classical models are the following IVPs

$$x'(t) = -\lambda x(t),\ x(0) = x_0 \quad \text{(exponential decay)} \tag{4.5}$$

and

$$x'(t) = ax(t)(k - x(t)),\ x(0) = x_0 \quad \text{(logistic equation)} \tag{4.6}$$

with $\lambda, a, k, x_0 > 0$.

We are interested in these models for mathematical and biological reasons. Mathematically, these models are interesting for their equilibrium points: $\overline{x} = 0$ in (4.5) and $\overline{x} = 0$ and $\overline{\overline{x}} = k$ in (4.6). Moreover, $\overline{x} = 0$ in (4.5) and $\overline{\overline{x}} = k$ in (4.6) are asymptotically stable, which means that, given an initial condition in a certain neighborhood, the solution will tend to these equilibrium points. Indeed, any positive initial condition will lead to this behaviour. The biological meaning of this is that a population that is represented by equation (4.5) tends to disappear. This is consistent with the model, since the mathematical equation means that the rate of change is negative and proportional to the existing population. A population whose which is modeled by (4.6) will tend to the constant $k$, called carrying capacity (the amount of the population that the environment can support), since if $x < k$ the rate of change is positive and if $x > k$ it is negative.

It is interesting to examine the fuzzy case of these two models because the first one has Zadeh's extension given by $-\lambda X$, as has already been mentioned. Hence the formula is similar to the classical case. And, as it has been discussed earlier, Zadeh's extension is a good criteria to decide which FIVPs to compare. The expression of Zadeh's extension of the logistic model, on the other hand, does not have the same representation with the standard arithmetic for fuzzy numbers. But this does not mean that the fuzzy logistic model cannot be written as $X'(t) = AX(t)(K - X(t))$, where all the involved terms are fuzzy. The examples involving the logistic case will use parameters based on the research of Gause (1969), presented as an example by Edelstein-Keshet (1988). Gause carried out an experiment of cultivation of the yeast *Scrhizosaccharomyces kephir* and found out that, begining with the amount of $x_0 = 0.45$ the population of this yeast clearly satisfied the logistic equation with parameters $k = 5.8$ and $a = 0.01$ for a total time of 160 hours.

## 4.2 Hukuhara Derivative

Kaleva (1987) stated conditions for existence of solution to differential equations in which the involved functions were fuzzy-set-valued functions and the derivative was also fuzzy (namely, the Hukuhara derivative). Seikkala (1987) independently reached similar results using an equivalent fuzzy derivative. Both proposed a theorem for existence and uniqueness of solutions to FIVPs, that is, a Picard-Lindelöf type theorem. Kaleva (1990) also proved that the Peano Theorem does not hold because the metric space $(\mathcal{F}_\mathcal{C}(\mathbb{R}^n), d_\infty)$ generally is not locally compact. Later on, Nieto succeeded in proving a Peano type Theorem, adding a condition (boundedness) to its classical version. By solution we mean a fuzzy-set-valued function $X : [0,T] \to \mathcal{F}_\mathcal{C}(\mathbb{R}^n)$ that satisfies the differential equation for each $t \in [0,T]$ and the initial condition in (1.2).

We are interested in studying

$$\begin{cases} X'_H(t) &= F(t, X(t)) \\ X(0) &= X_0 \end{cases}, \tag{4.7}$$

where $F : [0,T] \times \mathcal{F}_\mathcal{C}(\mathbb{R}) \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ is continuous and $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R}^n)$. First, we need the following lemma.

**Lemma 4.2.1** *(Kaleva, 1987) A fuzzy-set-valued function $x : [0,T] \to \mathcal{F}_\mathcal{C}(\mathbb{R}^n)$ is a solution to FIVP (4.7) if and only if it is continuous and satisfies*

$$X(t) = X_0 + \int_0^t F(s, X(s))ds$$

*for all $t \in [0,T]$.*

Kaleva remarked that it is not possible to extend Lemma 4.2.1 for $t < 0$, due to the property of nondecreasing diameter of the $\alpha$-levels.

**Theorem 4.2.2** *(Nieto, 1999) Consider $F : [0,T] \times \mathcal{F}_\mathcal{C}(\mathbb{R}^n) \to \mathcal{F}_\mathcal{C}(\mathbb{R}^n)$ continuous and bounded. Then there is at least one solution to FIVP (1.2) on $[0,T]$.*

The following result is close to Picard-Lindelöf type theorem, since it establishes continuity and the Lipschitz condition as sufficient for existence and uniqueness of soltution.

**Theorem 4.2.3** *(Kaleva, 1987) Consider a continuous function* $F : [0,T] \times \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$ *satisfying the Lipschitz conditon in the second argument, that is, there exists* $k > 0$ *such that*

$$d_\infty(F(t,X), F(t,Y)) \leq k d_\infty(X,Y)$$

*for all* $t \in [0,T], X, Y \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}^n)$. *Then there is a unique solution to FIVP (1.2) on* $[0,T]$.

A characterization theorem, stated in Bede (2008) simplifies the calculations. The result assures that it suffices to calculate a system of crisp ODEs.

**Theorem 4.2.4** *(Bede, 2008, 2013) Consider a continuous function* $F : R_0 \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, $R_0 = [0,T] \times \overline{B}(X_0, q)$, $q > 0$, $X_0 \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, *such that*

$$[F(t,x)]_\alpha = [f_\alpha^-(t, x_\alpha^-, x_\alpha^+), f_\alpha^+(t, x_\alpha^-, x_\alpha^+)], \quad \alpha \in [0,1]$$

*with* $f_\alpha^-(t, x_\alpha^-, x_\alpha^+)$ *and* $f_\alpha^+(t, x_\alpha^-, x_\alpha^+)$ *equicontinuous and uniformly Lipschitz in the second and third arguments, that is, there exists* $L > 0$ *such that*

$$\left| f_\alpha^-(t, x_\alpha^-, x_\alpha^+) - f_\alpha^+(t, x_\alpha^-, x_\alpha^+) \right| \leq L(|x_\alpha^- - y_\alpha^-)| + |x_\alpha^+ - y_\alpha^+)|),$$

*for any* $(t,x), (t,y) \in R_0$ *and for any* $\alpha \in [0,1]$. *Then the FIVP (1.2) has a unique solution in an interval* $[0,k]$, *for some* $k > 0$, *characterized levelwise by the system of crisp ODEs*

$$\left\{ \begin{array}{rcl} (x_\alpha^-)'(t) & = & f_\alpha^-(t, x_\alpha^-(t), x_\alpha^+(t)) \\ (x_\alpha^+)'(t) & = & f_\alpha^+(t, x_\alpha^-(t), x_\alpha^+(t)) \\ x_\alpha^-(0) & = & (x_0^-)_\alpha \\ x_\alpha^+(0) & = & (x_0^+)_\alpha \end{array} \right. , \tag{4.8}$$

$\alpha \in [0,1]$.

Some examples with biological interpretation will illustrate the use of Hukuhara derivative in FIVPs.

**Example 4.2.5** *Consider the decay model*

$$\left\{ \begin{array}{rcl} X_H'(t) & = & -\lambda X(t) \\ X(0) & = & X_0 \end{array} \right. , \tag{4.9}$$

*where* $\lambda \in \mathbb{R}^+$ *and* $X_0 \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, *supp*$(X_0) \subset \mathbb{R}^+$. *From now on we denote* $[X_0]_\alpha = [(x_0)_\alpha^-, (x_0)_\alpha^+]$.

*The crisp case associated to System (4.9) is frequently used to model population growth (or decay, depending on the sign of* $\lambda$*) or nuclear decay. It is a simple model, yet a reasonable approximation for a short period of observation of the phenomenon. The interpretation of FIVP (4.9) is the decay model with non-fuzzy coefficient and fuzzy initial condition. One explanation is that* $X_0$

*can be a label such as "high" or "small" and each real number in $\mathbb{R}$ has a membership degree to this subset. Another interpretation is that there is a partial knowledge of the initial condition, and the most likely values have membership degrees close to one.*

*Levelwise, solving (4.9) is equivalent to solving*

$$\begin{cases} [X'_H(t)]_\alpha &= [-\lambda X(t)]_\alpha \\ [X(0)]_\alpha &= [X_0]_\alpha \end{cases}$$

*for all* $\alpha \in [0,1]$.

*Hence,*

$$\begin{cases} [x_\alpha^-(t), x_\alpha^+(t)] &= [-\lambda x_\alpha^+(t), -\lambda x_\alpha^-(t)] \\ [x_\alpha^-(0), x_\alpha^+(0)] &= [x_{0\alpha}^-, x_{0\alpha}^+] \end{cases},$$

*that is,*

$$\begin{cases} (x_\alpha^-(t))' &= -\lambda x_\alpha^+(t) \\ (x_\alpha^+(t))' &= -\lambda x_\alpha^-(t) \\ x_\alpha^-(0) &= x_{0\alpha}^- \\ x_\alpha^+(0) &= x_{0\alpha}^+ \end{cases}. \tag{4.10}$$

*The solution one obtains is:*

$$\begin{cases} x_\alpha^-(t) &= c_\alpha^{(1)} e^{\lambda t} + c_\alpha^{(2)} e^{-\lambda t} \\ x_\alpha^+(t) &= -c_\alpha^{(1)} e^{\lambda t} + c_\alpha^{(2)} e^{-\lambda t} \end{cases}$$

*with*

$$c_\alpha^- = \frac{x_{0\alpha}^- - x_{0\alpha}^+}{2} \quad and \quad c_\alpha^+ = \frac{x_{0\alpha}^- + x_{0\alpha}^+}{2}.$$

*for all* $\alpha \in [0,1]$.

*Since it models population or nuclear particles, there is no meaning in the solution when it assumes negative values. This is why it is omitted in Figures 4.2 and 4.3. The result of nonzero membership degree to negative values is a defect resulting of the Hukuhara derivative. Hence, from* $t \approx 40$ *on, the solution has no biological meaning anymore, since in the calculations negative values for the state variable are used. Furthermore, in a population that is decreasing proportionaly to its quantity it is expected that it tends towards zero, no matter its initial value, uncertainty or its membership to a determined subset ("large", "medium" or "small", for instance). It is expected, actually, that the fuzziness goes to zero. Hence the increasing fuzziness (or diameter) is not considered a good modeling of the decay phenomenon.*

**Example 4.2.6** *Consider the coefficient of* $X$ *also fuzzy in the decay model:*

$$\begin{cases} X'_H(t) &= -\Lambda X(t) \\ X(0) &= X_0 \end{cases}, \tag{4.11}$$

*where* $\Lambda \in \mathcal{F}_\mathcal{C}(\mathbb{R})$, *supp*$(\Lambda) \subset \mathbb{R}^+$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ *and supp*$(X_0) \subset \mathbb{R}^+$. *From now on we always consider* $[\Lambda]_\alpha = [\lambda_\alpha^-, \lambda_\alpha^+]$.

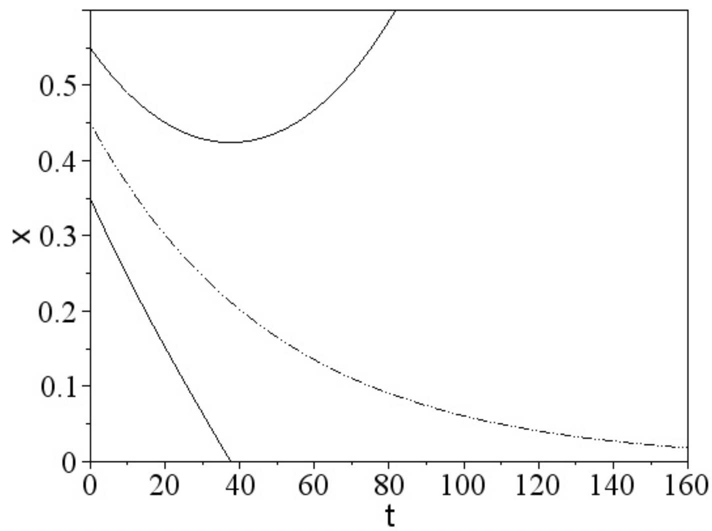


Figure 4.2: The 0-level (continuous line) and the core (dashed-dotted line) of solution to the decay model via Hukuhara derivative in Example 4.2.5. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\lambda = 0.02$.

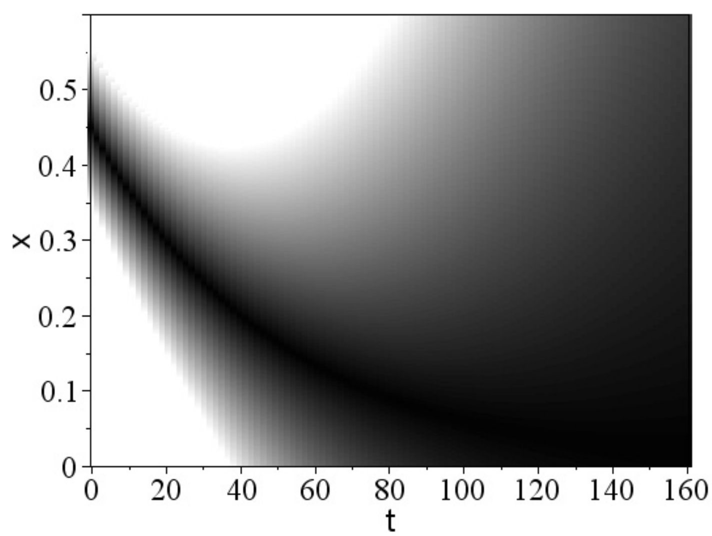


Figure 4.3: Attainable fuzzy sets of solution to the decay model via Hukuhara derivative in Example 4.2.5. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\lambda = 0.02$.

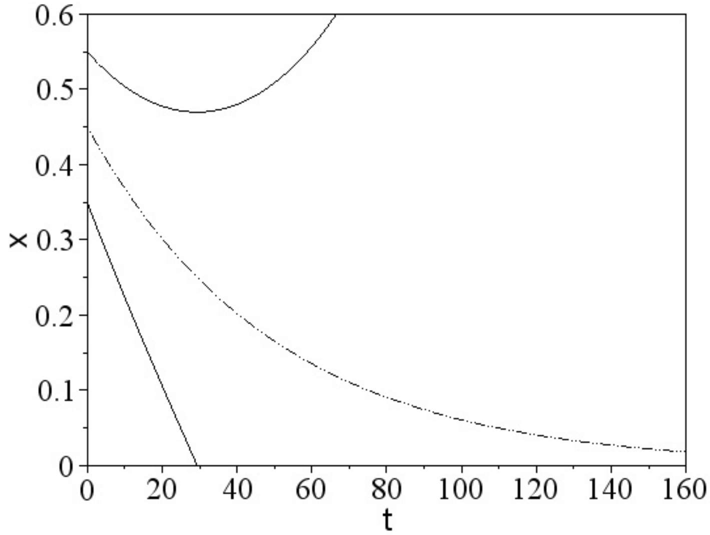


Figure 4.4: The 0-level (continuous line) and the core (dashed-dotted line) of solution to the decay model via Hukuhara derivative in Example 4.2.6. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\Lambda = (0.016; 0.020; 0.024)$.

*The biological meaning of this model is the same as the previous one. The sole difference is in the parameter* $\Lambda$*, which is fuzzy in the present example.*

*Since* $[-\Lambda X(t)]_\alpha = [-\lambda_\alpha^+ x_\alpha^+(t), -\lambda_\alpha^- x_\alpha^-(t)]$*, the FDE in levels is equivalent to the following system of differential equations*

$$\left\{\begin{array}{ccc} (x_\alpha^-(t))' & = & -\lambda_\alpha^+ x_\alpha^+(t) \\ (x_\alpha^+(t))' & = & -\lambda_\alpha^- x_\alpha^-(t) \\ x_\alpha^-(0) & = & x_{0\alpha}^- \\ x_\alpha^+(0) & = & x_{0\alpha}^+ \end{array}\right. \tag{4.12}$$

*The solution is*

$$\left\{\begin{array}{ccc} x_\alpha^-(t) & = & c_\alpha^{(1)} e^{\sqrt{\lambda_\alpha^- \lambda_\alpha^+}\ t} + c_\alpha^{(2)} e^{-\sqrt{\lambda_\alpha^- \lambda_\alpha^+}\ t} \\ x_\alpha^+(t) & = & -\sqrt{\frac{\lambda_\alpha^-}{\lambda_\alpha^+}} c_\alpha^{(1)} e^{\sqrt{\lambda_\alpha^- \lambda_\alpha^+}\ t} + \sqrt{\frac{\lambda_\alpha^-}{\lambda_\alpha^+}} c_\alpha^{(2)} e^{-\sqrt{\lambda_\alpha^- \lambda_\alpha^+}\ t} \end{array}\right.$$

*with*

$$c_\alpha^{(1)} = \frac{x_{0\alpha}^- - \sqrt{\lambda_\alpha^+/\lambda_\alpha^-}\ x_{0\alpha}^+}{2} \quad \text{and} \quad c_\alpha^{(2)} = \frac{x_{0\alpha}^- + \sqrt{\lambda_\alpha^+/\lambda_\alpha^-}\ x_{0\alpha}^+}{2}.$$

*for all* $\alpha \in [0,1]$*.*

*As in the previous case, no matter the fuzziness of the initial condition or the parameter, it is not expected to increase the diameter of the solution, though it always happens when employing the Hukuhara derivative.*

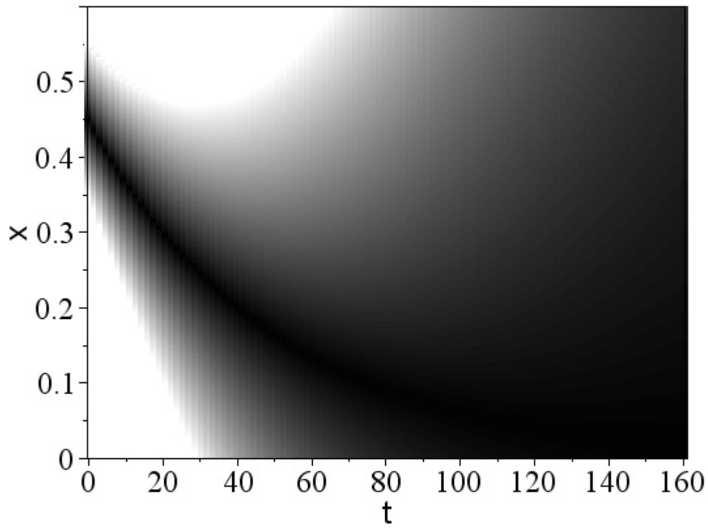


Figure 4.5: Attainable fuzzy sets of solution to the decay model via Hukuhara derivative in Example 4.2.6. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\Lambda = (0.016; 0.020; 0.024)$.

**Example 4.2.7** *Another well known biological model is the logistic growth*

$$\begin{cases} X'_H(t) &= aX(t)(k - X(t)) \\ X(0) &= X_0 \end{cases} . \tag{4.13}$$

*Here we assume fuzziness only in the initial condition, that is, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ and $supp(X_0) \subset \mathbb{R}^+$. The other parameters are non-fuzzy, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}^+$.*

*The model takes into account that the environment has a limited number of individuals it can support, in a given population. This is characterized by the parameter $k$, called carrying capacity, here considered constant. Parameter $a$ has to do with the reproduction. The term $akX(t)$ corresponds to the growth rate, controled by the term $-aX(t)^2$, which corresponds to intraspecific competition. If the population is modeled by a crisp variable, note that if $X(t) \approx 0$, $X'_H(t) \approx akX(t)$, that is, there is no obstruction for the population to grow. The change rate is positive while $X(t) < k$, but it tends towards zero while $X(t)$ tends to $k$. For $X(t) > k$, that is, above the carrying capacity, the change of rate is negative.*

*The Hukuhra difference is not defined for $k \ominus_H X(t)$ if $X(t)$ is fuzzy. Therefore we use the difference based on SIA (or gH-difference, which gives us the same result for this case). Note that $0 < u < v$ implies*

$$\min\{au(k-u), au(k-v), av(k-u), av(k-v)\} = au(k-v)$$

*and*

$$\max\{au(k-u), au(k-v), av(k-u), av(k-v)\} = av(k-u).$$

*Hence*

$$[aX(t)(k - X(t))]_\alpha = [ax^-_\alpha(t)(k - x^+_\alpha(t)), ax^+_\alpha(t)(k - x^-_\alpha(t))]$$

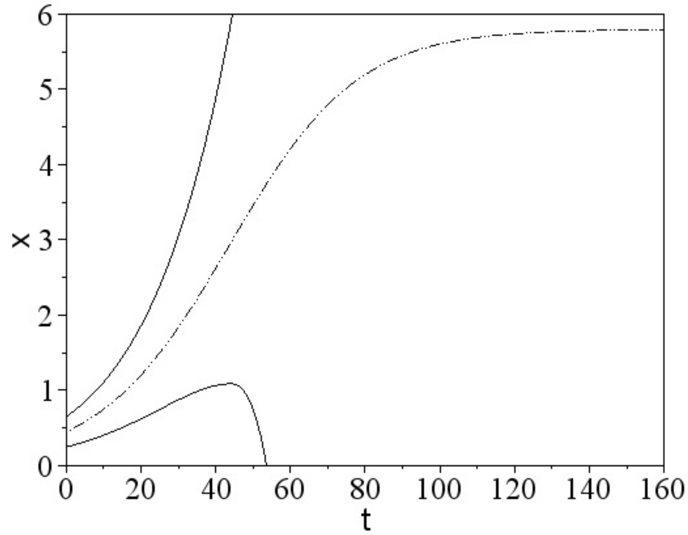


Figure 4.6: The 0-level (continuous line) and the core (dashed-dotted line) of solution to the logistic model via Hukuhara derivative in Example 4.2.7. Initial condition (0.35; 0.45; 0.55) below carrying support $k = 5.8$ and growth parameter $a = 0.01$.

*where we used the fact that* $0 < x^-_\alpha(t) < x^+_\alpha(t)$.

*We solve*

$$\left\{\begin{array}{rcl} (x^-_\alpha(t))' &=& a x^-_\alpha(t)(k - x^+_\alpha(t)) \\ (x^+_\alpha(t))' &=& a x^+_\alpha(t)(k - x^-_\alpha(t)) \\ x^-_\alpha(0) &=& x^-_{0\alpha} \\ x^+_\alpha(0) &=& x^+_{0\alpha} \end{array}\right. \tag{4.14}$$

*numerically, by applying first-order Euler method. That is, we approximate* $x^-_\alpha(t)$, $x^+_\alpha(t)$, $x^-_\alpha(t+h)$ *and* $x^+_\alpha(t+h)$ *by* $u^{(i)}_\alpha$, $v^{(i)}_\alpha$, $u^{(i+1)}_\alpha$ *and* $v^{(i+1)}_\alpha$ *such that*

$$u^{(i+1)}_\alpha = u^{(i)}_\alpha + h \cdot a\, u^{(i)}_\alpha (k - v^{(i)}_\alpha)$$

*and*

$$v^{(i+1)}_\alpha = v^{(i)}_\alpha + h \cdot a\, v^{(i)}_\alpha (k - u^{(i)}_\alpha),$$

*where* $i = 1, 2, ..., n$, $n$ *is the number of divisions of* $[0, T]$ *and* $h = T/(n-1)$ *is the size of each subinterval of* $[0, T]$.

*The results are illustrated in Figures 4.6, 4.7, 4.8 and 4.9. As expected, the solution has increasing diameter. Since the core of the initial condition is just one point, the core of the solution is the same as the solution of the crisp case, with initial condition* $x^-_{01} = x^+_{01}$. *But as in the decay model, it is expected, no matter the initial condition, from"very small" to "very large", that the population goes to a determined value (*$k$*, in this case) as* $t$ *increases.*

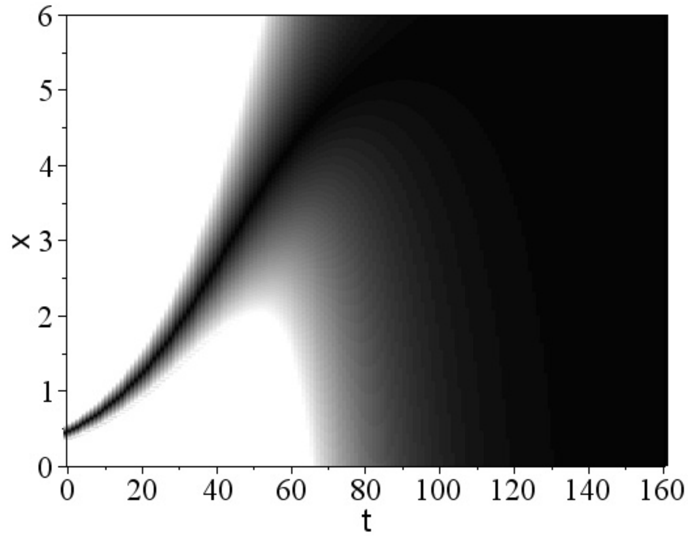


Figure 4.7: Attainable fuzzy sets of solution to the decay model via Hukuhara derivative in Example 4.2.7. Initial condition $(0.35; 0.45; 0.55)$ below carrying support $k = 5.8$ and growth parameter $a = 0.01$.

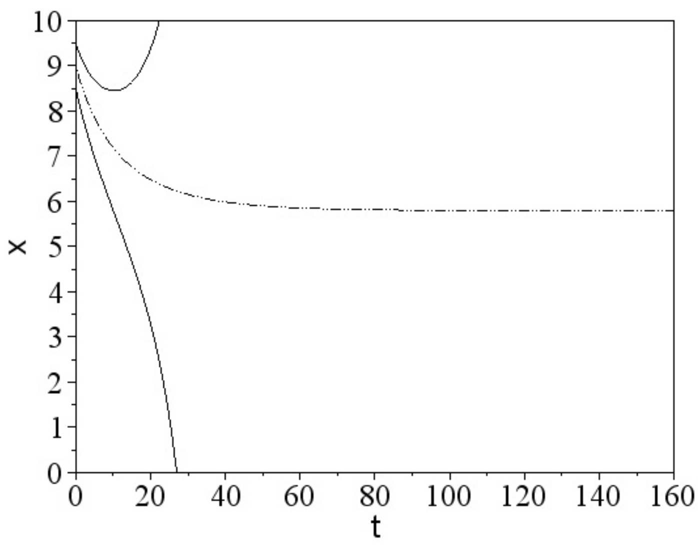


Figure 4.8: The 0-level (continuous line) and the core (dashed-dotted line) of solution to the logistic model via Hukuhara derivative in Example 4.2.7. Initial condition $(8.5; 9.0; 9.5)$ above carrying support $k = 5.8$ and growth parameter $a = 0.01$.

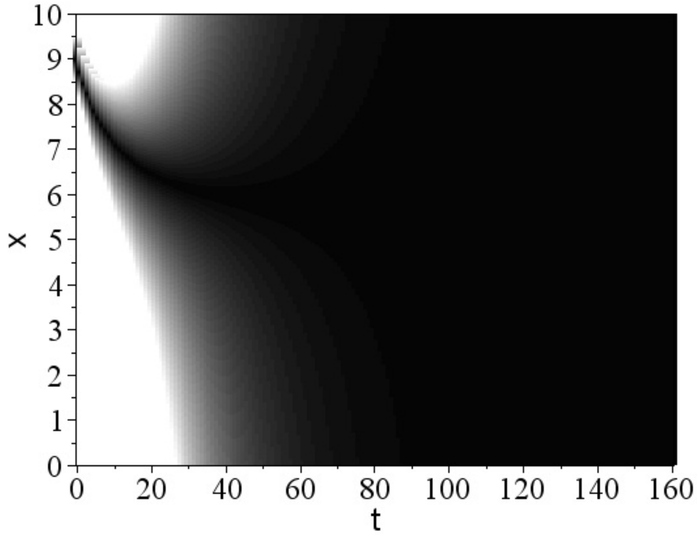


Figure 4.9: Attainable fuzzy sets of solution to the decay model via Hukuhara derivative in Example 4.2.7. Initial condition $(8.5; 9.0; 9.5)$ above carrying support $k = 5.8$ and growth parameter $a = 0.01$.

*We now try another approach. As mentioned at the beginning of this Section, other arithmetics can be used. Though interactivity is not used in the definition of the Hukuhara derivative, let us see what happens if we admit the CIA to calculate the differential field $aX(k - X)$:*

$$[aX(k-X)]_\alpha = [\min_{x \in [X]_\alpha} \{ax(k-x)\}, \max_{x \in [X]_\alpha} \{ax(k-x)\}].$$

*We use again the Euler method, calculating at each step the minimum and the maximum in the last equation and solving*

$$[(x_\alpha^-)'(t), (x_\alpha^+)'(t)] = \left[\min_{x \in [X(t)]_\alpha} \{ax(t)(k-x(t))\}, \max_{x \in [X(t)]_\alpha} \{ax(t)(k-x(t))\}\right].$$

*Since $f(x) = ax(k-x)$ is a parabola with maximum value at $x = k/2$, it is not hard to to determine the minimum and the maximum at each step:*

- *If $x_\alpha^-(t) < k/2 < x_\alpha^+(t)$, the maximum of $f(x)$ is attained by $x = k/2$. And the minimum is attained at $\min\{ax_\alpha^-(k - x_\alpha^-), ax_\alpha^+(k - x_\alpha^+)\}$.*

- *If $x_\alpha^+(t) < k/2$, $f(x)$ is increasing with respect to $x \in [x_\alpha^-, x_\alpha^+]$, hence the minimum is $f(x_\alpha^-)$ and the maximum is $f(x_\alpha^+)$.*

- *If $x_\alpha^+(t) < k/2$, $f(x)$ is decreasing with respect to $x \in [x_\alpha^-, x_\alpha^+]$, hence the minimum is $f(x_\alpha^+)$ and the maximum is $f(x_\alpha^-)$.*

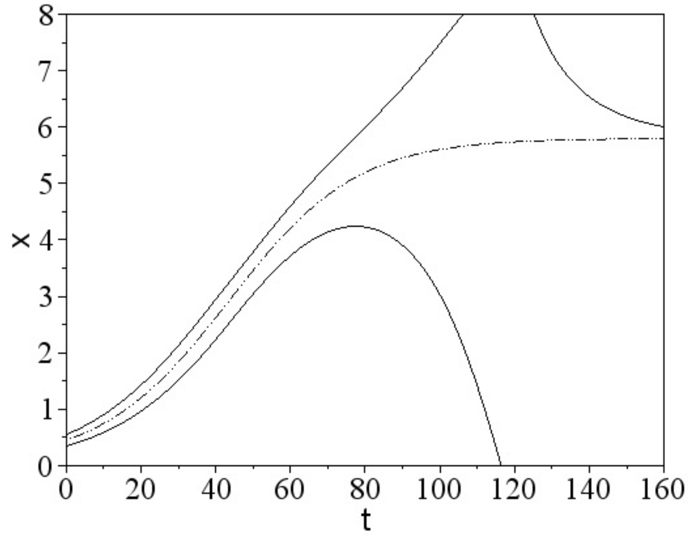


Figure 4.10: The 0-level (continuous line) and the core (dashed-dotted line) of solution to the logistic model with CIA and Hukuhara derivative in Example 4.2.7. Initial condition $(0.35; 0.45; 0.55)$ below carrying support $k = 5.8$ and growth parameter $a = 0.01$.

*The results are displayed in Figure 4.10. It is different from the result employing noninteractive arithmetic, especially when the* 0*-level start to assume negative values. It is mathematically interesting, though biologically it is meaningless from* $t \approx 125$*, since negative values are used to calculate the upper 0-level set function and negative values do not make sense as number of individuals.*

*Nevertheless, as mentioned before, it seems incoherent to use different arithmetics to define the derivative and to operate with the right-hand-side function. One would expect thence to define the derivative via CIA.*

We remark that the examples we solved are in accordance with the hypotheses of Theorem 4.2.4, that is, each $\alpha$-cut of the function $F$ can be written as function of $x_\alpha^-$ and $x_\alpha^+$. However, this is not always true and solving the system may become more complicated if we drop this condition. The reader can refer to Chalco-Cano and Román-Flores (2009) for further information about this subject. To briefly illustrate this case, consider the next example.

**Example 4.2.8** *Consider FIVP (4.7) with $F$ such that*

$$[F(t, X(t))]_\alpha = [x_0^-(t), x_\alpha^+(t)]$$

*which is equivalent to*

$$\begin{cases} (x_\alpha^-)'(t) = x_0^-(t), & x_\alpha^-(0) = (x_0^-)_\alpha \\ (x_\alpha^+)'(t) = x_\alpha^+(t), & x_\alpha^+(0) = (x_0^+)_\alpha \end{cases}.$$

*The second equation can be solved directly:* $x_\alpha^+(t) = (x_0^+)_\alpha e^t$. *The first one needs two steps and we begin with* $\alpha = 0$*:*

$$(x_0^-)'(t) = x_0^-(t), \quad x_0^-(0) = (x_0^-)_0$$

*which leads to* $x_0^-(t) = (x_0^-)_0 e^t$ *and*

$$(x_\alpha^-)'(t) = (x_0^-)_0 e^t, \quad x_\alpha^-(0) = (x_0^-)_\alpha$$

*The result is*

$$\begin{cases} x_\alpha^-(t) &=& (x_0^-)_0 e^t + (x_0^-)_\alpha - (x_0^-)_0 \\ x_\alpha^+(t) &=& (x_0^+)_\alpha e^t \end{cases} .$$

Example 4.2.8 illustrated the fact that the function $F$ in FIVP (4.7) may not be written directly as function of $x_\alpha^-$ and $x_\alpha^+$ (in this example, $F$ is function of $x_0^-$ and $x_\alpha^+$). The process of solving may become more difficult but since $F$ satisfies the existence Theorem 4.2.3, there is a solution (and we managed to find it).

## 4.3 Strongly Generalized Derivative

As stated before, the strongly generalized derivative (see Definition 3.1.13) "fixes" the defect of nondecreasing length of the support of a H-differentiable fuzzy function. In this Section we present the existence and uniqueness of two solution theorem, first stated by Bede and Gal (2005). As in the Hukuhara case, Bede and Gal prove the equivalence of the solution to an FIVP with integral equations, in such manner that, provided some conditions, there is always a solution with increasing diameter (strongly generalized differentiability of type (i)) and other with decreasing diameter (strongly generalized differentiability of type (ii)). The possibility of change of type of differentiability (i)-(iv) characterizes interesting phenomena called *switch points*.

We restate the FIVP we will study:

$$\begin{cases} X_G'(t) &=& F(t, X(t)) \\ X(0) &=& X_0 \end{cases} , \tag{4.15}$$

where $F : \mathbb{R} \times \mathcal{F}_\mathcal{C}(\mathbb{R}) \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ is continuous and $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$. As in Hukuhara derivative case, there is a result connecting FDEs with fuzzy integral equations.

**Theorem 4.3.1** *(Bede and Gal, 2005) The FIVP (4.15) is equivalent to the integral equation*

$$X(t) = X_0 + \int_0^t F(s, X(s))ds,$$

*if the derivative considered is type (i), or to the integral equation*

$$X_0 = X(t) + (-1)\int_0^t F(s, X(s))ds,$$

*if the derivative considered is type (ii), on some interval* $[t_1, t_2] \subset [0, T]$.

Based on the next lemma, Bede (2013) proves the existence and uniqueness of two solutions (Theorem 4.3.3).

**Lemma 4.3.2** *(Bede, 2013) Let $X \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ be such that $[X]_\alpha = [x_\alpha^-, x_\alpha^+]$, $\alpha \in [0,1]$, $x_\alpha^-$ and $x_\alpha^+$ differentiable, with $x^-$ strictly increasing on $[0,1]$, such that there exist the constants $c_1 > 0$, $c_2 < 0$ satisfying $(x_\alpha^-)' \geq c_1$ and $(x_\alpha^+)' \leq c_2$ for all $\alpha \in [0,1]$.*

*Let $F : [a,b] \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ be continuous with respect to $t$, having the level sets $[F(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$ with bounded partial derivatives $\frac{\partial f_\alpha^-(t)}{\partial \alpha}$ and $\frac{\partial f_\alpha^+(t)}{\partial \alpha}$, for all $t \in [a,b]$.*

*If one of the following two cases occur*

*a) $x_1^- < x_1^+$ or*

*b) $x_1^- = x_1^+$ and the core $[F(s)]_1$ consists of exactly one element for any $s \in [a,b]$,*

*then there exists $h > a$ such that the H-difference*

$$X \ominus \int_a^t F(s)ds$$

*exists for any $t \in [a,h]$.*

**Theorem 4.3.3** *(Bede, 2013) Let $R_0 = [0,T] \times \overline{B}(X_0, q)$, $q > 0$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ and $F : R_0 \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ be continuous such that the following assumptions hold:*

*(i) There exists a constant $L > 0$ such that*

$$d_\infty(F(t,X), F(t,Y)) \leq L d_\infty(X,Y)$$

*for all $(t,X), (t,Y) \in R_0$.*

*(ii) Let $[F(t,X)]_\alpha = [f_\alpha^-(t,X), f_\alpha^+(t,X)]$ be the level set representation of $F$, then $f_\alpha^-$, $f_\alpha^+$ : $R_0 \to \mathbb{R}$ have bounded partial derivatives with respect to $\alpha \in [0,1]$, the bounds being independent of $(t,X) \in R_0$ and $\alpha \in [0,1]$.*

*(iii) The functions $x_0^-$ and $x_0^+$ are differentiable (as functions of $\alpha$), existing $c_1 > 0$ with $\left(x_0^-\right)_\alpha' \geq c_1$, and $c_2 < 0$ with $\left(x_0^+\right)_\alpha' \leq c_2$, for all $\alpha \in [0,1]$, and we have the following possibilities*

*a) $(x_0)_1^- < (x_0)_1^+$*

*or*

*b) if $(x_0)_1^- = (x_0)_1^+$ then the core $[F(t,X)]_1$ consists in exactly one element for any $(t,X) \in R_0$, whenever $[X]_1$ consists in exactly one element.*

*Then the FIVP (1.2) has exactly two solutions on some interval $[0,k], k > 0$.*

The solution of FIVPs using strongly generalized differentiability, as in Hukuhara case, can also be obtained by solving systems of crisp ODEs to find the level set functions of the solution.

**Theorem 4.3.4** *(Bede, 2013) Let $R_0 = [0,T] \times \overline{B}(X_0, q)$, $q > 0$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ and $F : R_0 \to \mathcal{F}_\mathcal{C}(\mathbb{R})$ be such that*

$$[F(t,X)]_\alpha = [f_\alpha^-(t, x_\alpha^-, x_\alpha^+), f_\alpha^+(t, x_\alpha^-, x_\alpha^+)], \ \forall \alpha \in [0,1]$$

*and the following assumptions hold:*

*(i)* $f_\alpha^{\pm}(t, x_\alpha^-, x_\alpha^+)$ *are equicontinuous, uniformly Lipschitz in their second and third arguments, that is, there exists a constant* $L > 0$ *such that*

$$|f_\alpha^{\pm}(t, x_\alpha^-, x_\alpha^+) - f_\alpha^{\pm}(t, y_\alpha^-, y_\alpha^+)| \leq L(|x_\alpha^- - y_\alpha^-| + |x_\alpha^+ - y_\alpha^+|),$$

$\forall (t, X), (t, Y) \in R_0, \alpha \in [0, 1]$.

*(ii)* $f_\alpha^-$, $f_\alpha^+ : R_0 \to \mathbb{R}$ *have bounded partial derivatives with respect to* $\alpha \in [0, 1]$*, the bounds being independent of* $(t, X) \in R_0$ *and* $\alpha \in [0, 1]$.

*(iii) The functions* $x_0^-$ *and* $x_0^+$ *are differentiable, existing* $c_1 > 0$ *with* $\left(x_0^-\right)'_\alpha \geq c_1$*, and* $c_2 < 0$ *with* $\left(x_0^+\right)'_\alpha \leq c_2$*, for all* $\alpha \in [0, 1]$*, and we have the following possibilities*

*a)* $(x_0)_1^- < (x_0)_1^+$

*or*

*b) if* $(x_0)_1^- = (x_0)_1^+$ *then the core* $[F(t, X)]_1$ *consists in exactly one element for any* $(t, X) \in R_0$*, whenever* $[X]_1$ *consists in exactly one element.*

*Then the FIVP (1.2) is equivalent on some interval* $[t_0, t_0 + k]$ *with the union of the following two ODEs:*

$$\begin{cases} \left(x_\alpha^-\right)'(t) = f_\alpha^-(t, x_\alpha^-(t), x_\alpha^+(t)) \\ \left(x_\alpha^+\right)'(t) = f_\alpha^+(t, x_\alpha^-(t), x_\alpha^+(t)) \\ x_\alpha^-(t_0) = (x_0)_\alpha^-, x_\alpha^+(t_0) = (x_0)_\alpha^+ \end{cases}, \ \alpha \in [0, 1] \tag{4.16}$$

$$\begin{cases} \left(x_\alpha^-\right)'(t) = f_\alpha^+(t, x_\alpha^-(t), x_\alpha^+(t)) \\ \left(x_\alpha^+\right)'(t) = f_\alpha^-(t, x_\alpha^-(t), x_\alpha^+(t)) \\ x_\alpha^-(t_0) = (x_0)_\alpha^-, x_\alpha^+(t_0) = (x_0)_\alpha^+ \end{cases}, \ \alpha \in [0, 1]. \tag{4.17}$$

**Example 4.3.5** *Consider the decay model*

$$\begin{cases} X_G'(t) &= -\lambda X(t) \\ X(0) &= X_0 \end{cases}, \tag{4.18}$$

*where* $\lambda \in \mathbb{R}^+$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ *and supp*$(X_0) \subset \mathbb{R}^+$.

*The (i)-differentiable solution is Hukuhara differentiable solution, the same as in Example 4.2.5. The other solution is the one obtained solving*

$$\begin{cases} (x_\alpha^-(t))' &= -\lambda x_\alpha^-(t) \\ (x_\alpha^+(t))' &= -\lambda x_\alpha^+(t) \\ x_\alpha^-(0) &= x_{0\alpha}^- \\ x_\alpha^+(0) &= x_{0\alpha}^+ \end{cases}. \tag{4.19}$$

*The solution is*

$$\begin{cases} x_\alpha^-(t) &= x_{0\alpha}^- e^{-\lambda t} \\ x_\alpha^+(t) &= x_{0\alpha}^+ e^{-\lambda t} \end{cases}$$

*for all* $\alpha \in [0, 1]$.

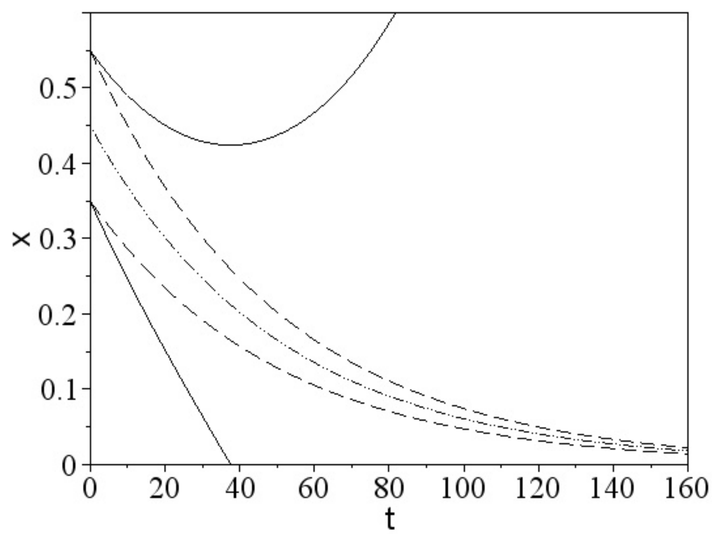


Figure 4.11: Solutions to the decay model via strongly generalized derivative in Example 4.3.5: the 0-level (continuous line) of the (i)-differentiable solution (in the strongly generalized sense), the 0-level (dashed line) of the (ii)-differentiable solution and the core (dashed-dotted line) of both. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\lambda = 0.02$.

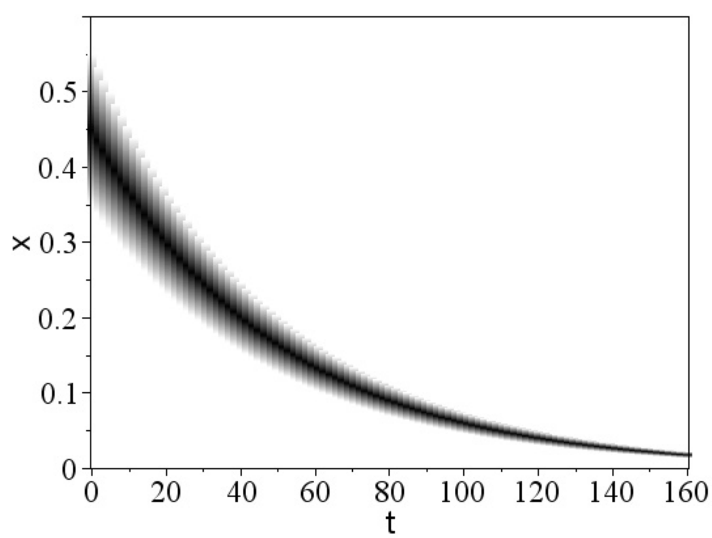


Figure 4.12: Attainable fuzzy sets of the (ii)-differentiable solution to the decay model via strongly generalized derivative in Example 4.3.5. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\lambda = 0.02$.

The (ii)-differentiable solution is a good alternative to the Hukuhara differentiable solution, since it "fixes" the deffect of increasing diameter of the solution of the decay model. It is biologically more meaningful in Example 4.3.5 since it is expected that the uncertainty vanishes as the population gets close to zero.

**Example 4.3.6** *As in Example 4.2.6, consider the coefficient of $X$ also fuzzy in the decay model:*

$$\begin{cases} X'_G(t) &= -\Lambda X(t) \\ X(0) &= X_0 \end{cases}, \tag{4.20}$$

*where $\Lambda \in \mathcal{F_C}(\mathbb{R})$, $supp(\Lambda) \subset \mathbb{R}^+$, $X_0 \in \mathcal{F_C}(\mathbb{R})$ and $supp(X_0) \subset \mathbb{R}^+$.*

*The (i)-differentiable solution is Hukuhara differentiable solution, the same as in Example 4.2.6. The other solution is obtained by solving*

$$\begin{cases} (x^-_\alpha(t))' &= -\lambda^-_\alpha x^-_\alpha(t) \\ (x^+_\alpha(t))' &= -\lambda^+_\alpha x^+_\alpha(t) \\ x^-_\alpha(0) &= x^-_{0\alpha} \\ x^+_\alpha(0) &= x^+_{0\alpha} \end{cases}. \tag{4.21}$$

*The solution is*

$$\begin{cases} x^-_\alpha(t) &= x^-_{0\alpha} e^{-\lambda^-_\alpha t} \\ x^+_\alpha(t) &= x^+_{0\alpha} e^{-\lambda^+_\alpha t} \end{cases}. \tag{4.22}$$

*This solution is defined while $x^-_\alpha(t) < x^+_\alpha(t)$, that is, for*

$$t < T_m = \frac{1}{\lambda^+_\alpha - \lambda^-_\alpha} \ln\left(\frac{x^+_{0\alpha}}{x^-_{0\alpha}}\right).$$

*The solution also has to satisfy $x^-_\alpha(t) \le x^-_\beta(t)$ and $x^+_\alpha(t) \ge x^+_\beta(t)$ for $0 \le \alpha \le \beta \le 1$. For the same values as Example 4.2.6, that is, $X_0 = (0.35; 0.45; 0.55)$ and $\Lambda = (0.016; 0.020; 0.024)$, numerically we find $T_m \approx 48$. The solution is displayed in Figures 4.13 and 4.14. It is clear that the solution does not exist from $T_m \approx 44$ on.*

*The condition $x^-_\alpha(t) < x^+_\alpha(t)$ restricts the domain for which the solution is defined. This is a particular problem of the solution with decreasing diameter, that is, the Hukuhara differentiable solution does not degenerates and hence its domain is bigger. On the other hand, since this particular problem is an application in population modeling, it does not make sense (biologically) that $x^-_\alpha(t) < 0$, what limits the Hukuhara solution as well.*

*Note also that if the initial condition is crisp with $x^-_\alpha(t) = x^+_\alpha(t)$, we have $\ln\left(\frac{x^+_{0\alpha}}{x^-_{0\alpha}}\right) = 0$ and hence there is no domain for the solution. This is in accordance with the existence Theorem 4.3.3.*

**Example 4.3.7** *The FIVP*

$$\begin{cases} X'(t) &= aX(t)(k - X(t)) \\ X(0) &= X_0 \end{cases}. \tag{4.23}$$

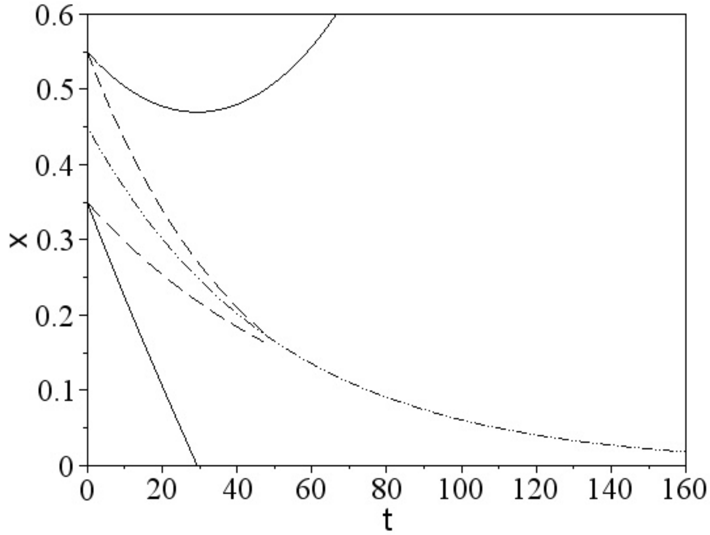


Figure 4.13: Solutions to the decay model via strongly generalized derivative in Example 4.3.6: the 0-level (continuous line) of the (i)-differentiable solution (in the strongly generalized sense), the 0-level (dashed line) of the (ii)-differentiable solution, defined for $t < T_m$, $T_m \approx 48$, and the core (dashed-dotted line) of both (for the (ii)-differentiable solution, it is defined for $t < T_m$). Initial condition $(0.35; 0.45; 0.55)$ and parameter $\Lambda = (0.016; 0.020; 0.024)$.

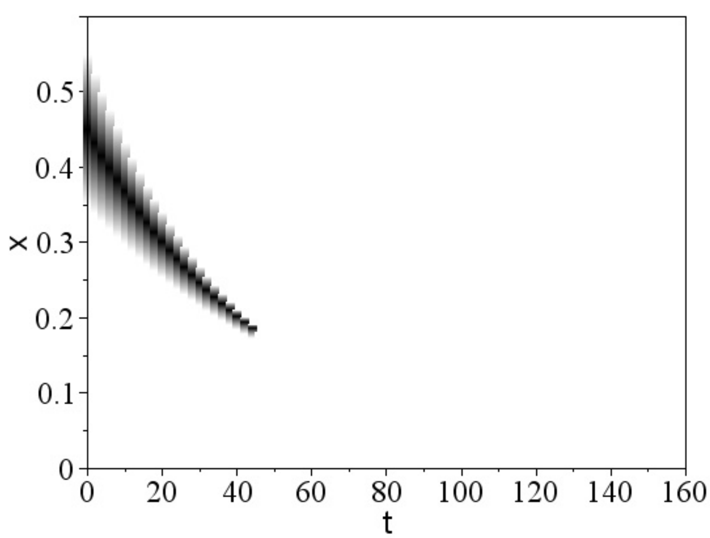


Figure 4.14: Attainable fuzzy sets of the (ii)-differentiable solution to the decay model via strongly generalized derivative in Example 4.3.6, which is defined for $t < T_m$, $T_m \approx 48$. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\Lambda = (0.016; 0.020; 0.024)$.

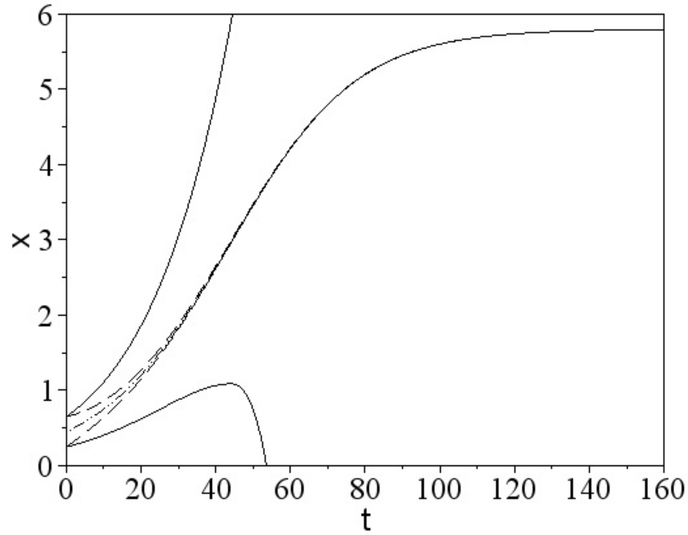


Figure 4.15: Solutions to the logistic model via strongly generalized derivative in Example 4.3.7: the 0-level (continuous line) of the (i)-differentiable solution (in the strongly generalized sense), the 0-level (dashed line) of the (ii)-differentiable solution and the core (dashed-dotted line) of both. Initial condition $(0.35; 0.45; 0.55)$ below carrying support $k = 5.8$ and growth parameter $a = 0.01$.

*where* $X_0 \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ *and* $supp(X_0) \subset \mathbb{R}^+$, $a \in \mathbb{R}^+$ *and* $k \in \mathbb{R}^+$ *was evaluated numerically in Example 4.2.7 using the Hukuhara derivative (which is equal to (i)-differentiability).*

*The (ii)-differentiable solution is obtained considering*

$$\left\{ \begin{array}{rcl} (x_\alpha^-(t))' & = & a x_\alpha^+(t)(k - x_\alpha^-(t)) \\ (x_\alpha^+(t))' & = & a x_\alpha^-(t)(k - x_\alpha^+(t)) \\ x_\alpha^-(0) & = & x_{0\alpha}^- \\ x_\alpha^+(0) & = & x_{0\alpha}^+ \end{array} \right. . \tag{4.24}$$

*In order to solve it numerically, we approximate* $x_\alpha^-(t)$, $x_\alpha^+(t)$, $x_\alpha^-(t+h)$ *and* $x_\alpha^+(t+h)$ *by* $u_\alpha^{(i)}$, $v_\alpha^{(i)}$, $u_\alpha^{(i+1)}$ *and* $v_\alpha^{(i+1)}$ *such that*

$$u_\alpha^{(i+1)} = u_\alpha^{(i)} + ha\, v_\alpha^{(i)}(k - u_\alpha^{(i)})$$

*and*

$$v_\alpha^{(i+1)} = v_\alpha^{(i)} + ha\, u_\alpha^{(i)}(k - v_\alpha^{(i)}),$$

*where* $i = 1, 2, ..., n$, $n$ *is the number of divisions of* $[0, T]$ *and* $h = T/(n-1)$ *is the size of each subinterval of* $[0, T]$.

The reader may observe in Figure 4.15 of the previous example that the diameter decreases and the function tends to $k$. As it was earlier observed, no matter the positive initial condition, the trajectory of the classical case always tends towards $k$, and that is what is expected from Equation (4.23), considering the phenomenon that originated it. Hence it does not matter the initial fuzziness as well, it is *certain* that, after a certain time, the state varible will be very close to the carrying capacity. With this point of view the (ii)-differentiable solution is more appropriate.

**Example 4.3.8** *(Bede and Gal, 2010; Bede, 2013) This example is similar to the one presented in Bede and Gal (2010) and in Bede (2013). The authors propose to numerically solve*

$$\begin{cases} X'_H(t) & = & AX(t)(K \ominus_{gH} X(t)) \\ X(0) & = & x_0 \end{cases}, \tag{4.25}$$

*where* $x_0 \in \mathbb{R}^0$, $K \in \mathcal{F}_\mathcal{C}(\mathbb{R})$, *supp*$(K) \subset \mathbb{R}^+$, $A \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ *and supp*$(A) \subset \mathbb{R}^+$. *The values of the parameters are different from Bede and Gal (2010); Bede (2013). We set* $[K]_\alpha = [k_\alpha^-, k_\alpha^+]$ *and* $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$

*The crisp initial value does not meet condition (iii) of Theorem 4.3.3, that guarantees two solutions. In fact, only the Hukuhara differentiable (or (i)-differentiable) solution is admitted, since the initial condition is not fuzzy.*

*We obtain it by solving*

$$\begin{cases} (x_\alpha^-)'(t) = f_\alpha^-(t, x_\alpha^-(t), x_\alpha^+(t)) \\ (x_\alpha^+)'(t) = f_\alpha^+(t, x_\alpha^-(t), x_\alpha^+(t)) \quad , \alpha \in [0,1] \\ x_\alpha^-(t_0) = (x_0)_\alpha^-, x_\alpha^+(t_0) = (x_0)_\alpha^+ \end{cases} \tag{4.26}$$

*To obtain the expression of* $f_\alpha^-$ *and* $f_\alpha^+$, *note that the supports of* $A$ *and* $X$ *are positive, in order to preserve the biological meaning. Hence*

$$[AX]_\alpha = [A]_\alpha [X]_\alpha = [a_\alpha^- x_\alpha^-, a_\alpha^+ x_\alpha^+].$$

*Also,*

$$[K \ominus_{gH} X]_\alpha = [\min\{k_\alpha^- - x_\alpha^-, k_\alpha^+ - x_\alpha^+\}, \max\{k_\alpha^- - x_\alpha^-, k_\alpha^+ - x_\alpha^+\}],$$

*provided* $K \ominus_{gH} X$ *defines a fuzzy number. As a result,*

$$\begin{cases} f_\alpha^-(t, x_\alpha^-, x_\alpha^+) = \min_{s,p \in \{-,+\}}\{a_\alpha^s x_\alpha^s (k_\alpha^p - x_\alpha^p)\} \\ f_\alpha^+(t, x_\alpha^-, x_\alpha^+) = \max_{s,p \in \{-,+\}}\{a_\alpha^s x_\alpha^s (k_\alpha^p - x_\alpha^p)\} \end{cases}, \alpha \in [0,1]. \tag{4.27}$$

*We solve it using the Euler method, calculating at each step the minimum and maximum needed to determine* $f_\alpha^-$ *and* $f_\alpha^+$. *If at some point* $(x_\alpha^-)'(t) = (x_\alpha^+)'(t)$, *that is,* $f_\alpha^- = f_\alpha^+$, *then the solution would be (iv)-differentiable, according to Definition 3.1.13. Hence, considering this point as new initial condition, from this point on it would have decreasing diameter or increasing diameter, that is, be (i)-differentiable or (ii)-differentiable. The new problem would meet all conditions of Theorem 4.3.3 and we would be able to find these two solutions.*

*This point at which there are two options of solutions, each one with a different differentiability, is called switch point. With the parameters used in this example, there is no such point. Hence, there exists only the Huuhara differentiable solution.*

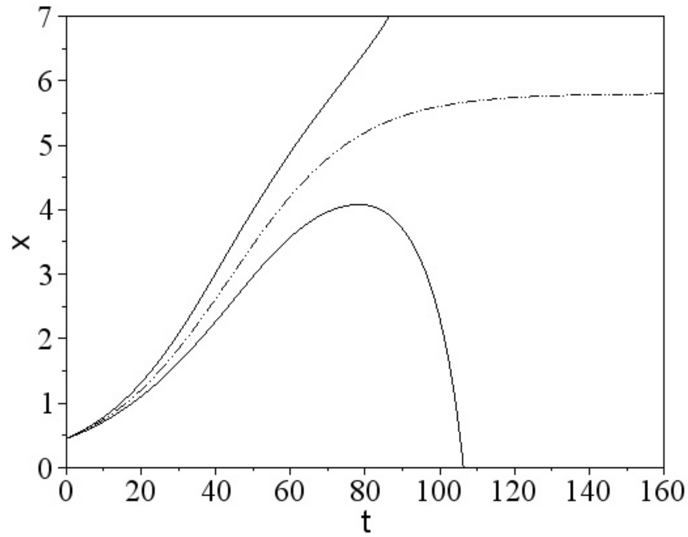


Figure 4.16: The 0-level (continuous line) and the core (dashed-dotted line) of the (i)-differentiable solution to the logistic model via strongly generalized derivative in Example 4.3.8. Initial condition 0.45 below carrying support $K = (5.3; 5.8; 6.3)$ and growth parameter $A = (0.005; 0.010; 0.015)$.

## 4.4 Fuzzy Differential Inclusions

The theory of differential inclusions was developed to deal with some kinds of uncertainties not described by classical dynamical systems. These uncertainties are due, for instance, to partial knowledge arisen from the impossibility of total understanding of a phenomenon or to the ignorance of laws related to the control of the system. Control can be direction, accelaration, fuel, temperature, weight or other variables that may affect the system.

The mathematical model involves a family of differential equations

$$x'(t) = f(x(t), u(t)), \quad u(t) \in U((x(t)), \tag{4.28}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control and $U$ is the subset of admissible controls. Together with $x$, $u$ defines the velocity of the system.

Defining the set-valued map $H : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ as $H(x) = f(x, U(x)) = \{f(x, u)\}_{u \in U(x)}$, the equation in (4.28) can be rewritten as

$$x'(t) \in H(x(t), u(t)). \tag{4.29}$$

We have the following problem: finding a solution $x$, that is, an everywhere differentiable function that satisfies (4.28) and a given initial condition. This problem is said to be *parametrizable*, that is, there exists a single-valued function $f$ of two variables $x, u$ such that, for every $x$, $H(x) = f(x, U)$. The initial condition can also assume values in a given set of $\mathbb{R}^n$. Being parametrizable is an important property, since $f$ continuous implies that, for every fixed $u^0 \in U$, $x \mapsto f(x, u^0)$

is a continuous selection. The existence of a continuous selection guarantees at least one solution to the Problem (4.29), that is, an everywhere differentiable function that satisfies the inclusion, obtained by solving the differential equation (4.28) (and given an initial condition $x(0) = x_0$).

The differential inclusion can take a more general form:

$$\begin{cases} x'(t) & \in & F(t, x(t)) \\ x(0) & \in & \Gamma \end{cases} \tag{4.30}$$

where $F : \mathbb{R} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ and $\Gamma \subset \mathbb{R}^n$. In this case, which we call the time dependent case, the selections of $F$ that are measurable with respect to the time variable are considered, and hence the concept of solution is also more general. It is an absolutely continuous (see Appendix) function that satisfies the inclusion a.e. in (4.30), obtained by solving the differential equation $x'(t) = f(t, x(t)), x(0) = x_0 \in \Gamma$, where $f$ is a selection of $F$.

The absolutely continuous functions are the weakest acceptable solutions, according to Aubin and Cellina (1984), since they are continuous and, moreover, they are differentiable except on a set of measure zero. This allows solutions with discontinuities in its derivatives at some points and at the same time avoids some bizarre cases (such as a function that has derivative zero a.e. but is strictly monotonic).

A fuzzy differential inclusion is a generalization of a differential inclusion and was first proposed by Aubin (1990) and Baidosov (1990). It is symbolically written as

$$\begin{cases} x'(t) & \in & F(t, x(t)) \\ x(0) & \in & X_0 \end{cases} \tag{4.31}$$

and, as Hüllermeier (1997) proposed, is interpreted levelwise as the family of differential inclusions

$$\begin{cases} x'(t) & \in & [F(t, x(t))]_\alpha \\ x(0) & \in & [X_0]_\alpha \end{cases} \tag{4.32}$$

for all $\alpha \in [0, 1]$, where $[F]_\alpha : [0, T] \times \mathbb{R}^n \to \mathcal{K}^n_{\mathcal{C}}$ and $[X_0]_\alpha \in \mathcal{K}^n_{\mathcal{C}}$.

A solution to Problem (4.32) is an absolutely continuous function $x : [0, T] \to \mathbb{R}^n$ that satisfies the inclusion a.e. in $[0, T]$ and $x(0) = x_0 \in [X_0]_\alpha$. The set of all solutions of (4.32) is denoted by $\Sigma_\alpha(x_0, T)$ and the attainable set at $t \in [0, T]$ by $\mathcal{A}_\alpha(x_0, t)$. Diamond (1999) proved that the sets $\Sigma_\alpha(x_0, T)$ are the $\alpha$-cuts of the fuzzy solution $\Sigma(x_0, T)$ of (4.31), a fuzzy subset of $Z_T(\mathbb{R}^n)$ (see Appendix), that is, $\Sigma(x_0, T) \in \mathcal{F}(Z_T(\mathbb{R}^n))$.

In the fuzzy differential inclusions some trajectories may have more "preference" than the others which is characterized by the value of its membership degree. This discrimination does not exist in the traditional differential inclusions.

The following assumption assures that all the absolutely continuous solutions to Problem (4.32) are defined on the same interval of existence.

Let $\Omega$ be an open subset of $\mathbb{R} \times \mathbb{R}^n$ such that $(0, x_0) \in \Omega$ and $H$ a mapping from $\Omega$ into the compact and convex subsets of $\mathbb{R}^n$. If there exist $b, T, M > 0$ such that:

- the set $Q = [0, T] \times (x_0 + (b + MT)B^n) \subset \Omega$, where $B^n$ is the unit ball of $\mathbb{R}^n$;

- $H$ maps $Q$ into the ball of radius $M$

then it is said that the *boundedness assumption* holds (see Diamond (1999) and Aubin and Cellina (1984)).

**Existence Theorem**

Diamond (1999) proved existence of solutions to FDIs for the time-dependent case. It does not requires continuity of the differential field, but needs upper semicontinuity and boundedness assumption.

**Theorem 4.4.1** *(Diamond, 1999) Suppose that $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R}^n)$, let $\Omega$ be an open set in $\mathbb{R} \times \mathbb{R}^n$ containing $\{0\} \times supp(X_0)$ and let $F : \Omega \times \mathcal{F}_\mathcal{C}(\mathbb{R}^n) \to \mathcal{F}_\mathcal{C}(\mathbb{R}^n)$ be usc. Suppose that the boundedness assumption holds for all $x_0 \in supp(X_0)$ and*

$$x' \in [F(t,x)]_0, \quad x(0) \in supp(X_0).$$

*The families $\Sigma_\alpha(X_0,T)$ of all solutions to (4.32) are compact subsets in $Z_T(\mathbb{R}^n)$ for all $\alpha \in [0,1]$. Moreover, these subsets are $\alpha$-cuts of a fuzzy subset of $Z_T(\mathbb{R}^n)$, $\Sigma(X_0,T) \in \mathcal{F}_\mathcal{K}(Z_T(\mathbb{R}^n))$, which is the solution to (4.31). The attainable sets $\mathcal{A}_\alpha(X_0,t)$ of $\Sigma_\alpha(X_0,T)$ define the fuzzy subset $\mathcal{A}(X_0,t) \in \mathcal{F}_\mathcal{K}(\mathbb{R}^n)$.*

The selection method is considered in order to search for solutions. It consists in finding a selection $f(t,x)$ of the set-valued function $[F(t,x)]_\alpha$ and solving the classical IVP

$$\begin{cases} x'(t) &= f(t,x(t)) \\ x(0) &= x_0 \end{cases}, \tag{4.33}$$

where $x_0 \in [X_0]_\alpha$. If $f$ is continuous and bounded, for instance, there will be a solution to (4.33) and hence a solution to the differential inclusion.

**Example 4.4.2** *Let us solve the fuzzy differential inclusion associated to the family of problems*

$$\begin{cases} x'(t) &\in [-\lambda x(t)]_\alpha \\ x(0) &= x_0 \in [X_0]_\alpha \end{cases}, \tag{4.34}$$

*where $\lambda, x_0 \in \mathbb{R}^+$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$ and $supp(X_0) \subset \mathbb{R}^+$. The function $[-\lambda x(t)]_\alpha$ is a singleton, therefore (4.34) is equivalent to*

$$\begin{cases} x'(t) &= -\lambda x(t) \\ x(0) &= x_0 \in [X_0]_\alpha \end{cases}, \tag{4.35}$$

*The set of all solutions is*

$$\Sigma_\alpha(X_0,T) = \{x : x(t) = x_0 e^{-\lambda t}, x_0 \in [X_0]_\alpha\}.$$

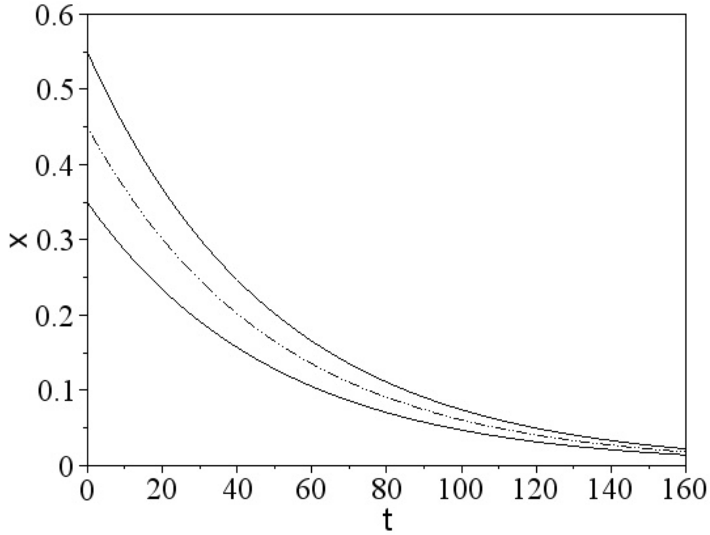


Figure 4.17: Attainable sets of the 0-level (continuous line) and the core (dashed-dotted line) of solution to the decay model via FDIs in Example 4.4.2. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\lambda = 0.02$.

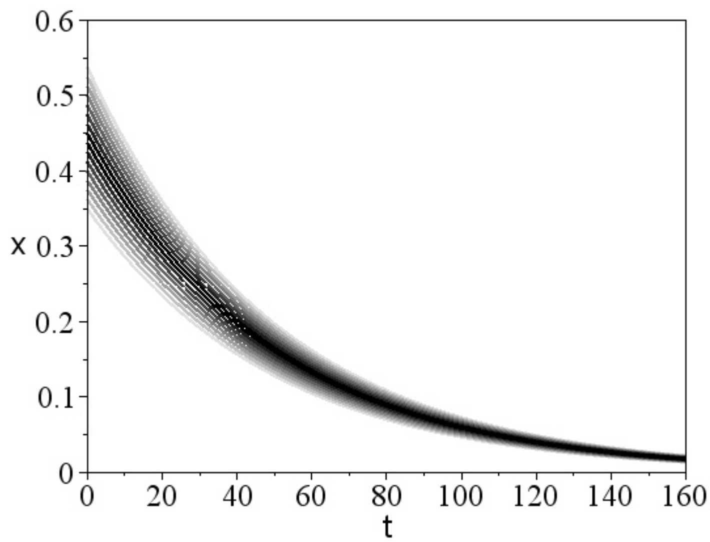


Figure 4.18: Solution to the decay model via FDIs in Example 4.4.2, that is, a fuzzy bunch of functions whose membership of each function to the solution is represented by the scale of gray: the darker the color the higher the membership degree. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\lambda = 0.02$.

*These are the $\alpha$-cuts of the fuzzy bunch of functions that is the solution to the FDI. Its attainable sets are*

$$\mathcal{A}(X_0, t) = e^{-\lambda t} X_0.$$

*Note that the FDI (4.34) is comparable with the decay model with initial condition in Example 4.3.5, since $F(t, X) = \lambda X$ is Zadeh's extension of $f(t, x) = \lambda x$. Indeed, the attainable sets $\mathcal{A}(X_0, t)$ are the same as the solution at $t$ via (ii)-differentiability, the one that presents decreasing diameter. And it is obviously different from the H-differentiable solution, that has increasing diameter.*

**Example 4.4.3** *Now we also consider the parameter as a fuzzy number:*

$$\begin{cases} x'(t) & \in & [-\Lambda x(t)]_\alpha \\ x(0) & = & x_0 \in [X_0]_\alpha \end{cases}, \tag{4.36}$$

*where $\Lambda \in\in \mathcal{F_C}(\mathbb{R})$, supp$(\Lambda) \subset \mathbb{R}^+$, $x_0 \in \mathbb{R}^+$, $X_0 \in \mathcal{F_C}(\mathbb{R})$ and supp$(X_0) \subset \mathbb{R}^+$.*

*All solutions to (4.36) are confined between two values, for each $t$:*

$$\begin{cases} x^-(t) & = & \min\{x(t) : x'(t) = \lambda(t)x(t),\, \lambda(t) \in [-\Lambda]_\alpha,\, x(0) \in [X_0]_\alpha\} \\ x^+(t) & = & \max\{x(t) : x'(t) = \lambda(t)x(t),\, \lambda(t) \in [-\Lambda]_\alpha,\, x(0) \in [X_0]_\alpha\} \end{cases}.$$

*That is,*

$$\begin{cases} x^-(t) & = & x_{0\alpha}^- e^{-\lambda_\alpha^+ t} \\ x^+(t) & = & x_{0\alpha}^+ e^{-\lambda_\alpha^- t} \end{cases}.$$

*The set of all solutions has the attainable sets*

$$\mathcal{A}_\alpha(X_0, t) = [x_{0\alpha}^- e^{-\lambda_\alpha^+ t}, x_{0\alpha}^+ e^{-\lambda_\alpha^- t}].$$

*The FDI that has been just solved is also comparable to one FDE in the previous section, namely System (4.20). The present solution has decreasing diameter, hence it is clearly different from H-differentiable solution. It also has different attainable sets from the (ii)-differentiable solution. We showed in Example 4.3.6 that the (ii)-differentiable solution colapses at a certain value of $t$. The solution via differential inclusions is defined for all $t > 0$, preserving the property of assintotic solution of the classical case.*

**Example 4.4.4** *Let us solve the fuzzy differential inclusion associated to the family of problems*

$$\begin{cases} x'(t) & \in & [a(k - x(t))]_\alpha \\ x(0) & = & x_0 \in [X_0]_\alpha \end{cases}, \tag{4.37}$$

*where $a, k, x_0 \in \mathbb{R}^+$, and $X_0 \in \mathcal{F_C}(\mathbb{R})$ and supp$(X_0) \subset \mathbb{R}^+$. As in Example 4.4.2, the function $[a(k - x(t))]_\alpha$ is a singleton. The set of all solutions of (4.37) is obtained by solving the classical case and varying the value of the initial condition (as in Example 4.4.2):*

$$\Sigma_\alpha(X_0, T) = \left\{ x : x(t) = \frac{k x_0 e^{akt}}{k + x_0(e^{akt} - 1)}, x_0 \in [X_0]_\alpha \right\}. \tag{4.38}$$

*The fuzzy solution $\Sigma(X_0, T)$ has $\alpha$-cuts given by (4.38).*

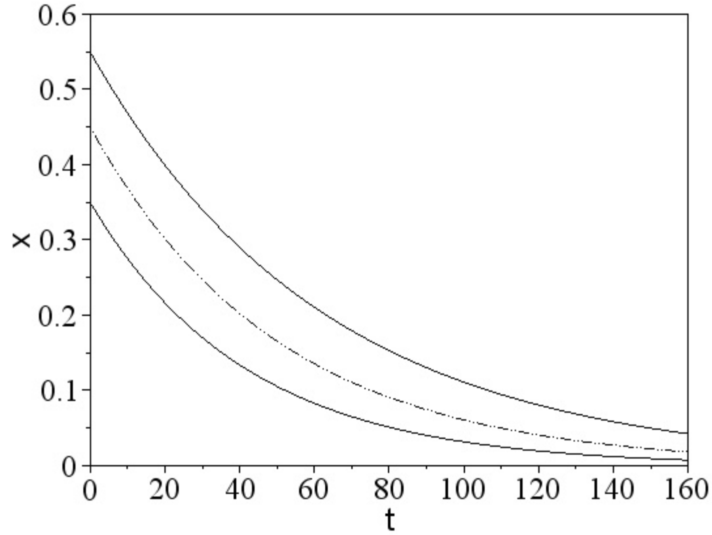


Figure 4.19: Attainable sets of the 0-level (continuous line) and the core (dashed-dotted line) of solution to the decay model via FDIs in Example 4.4.3. Initial condition $(0.35; 0.45; 0.55)$ and parameter $\Lambda = (0.016; 0.020; 0.024)$.

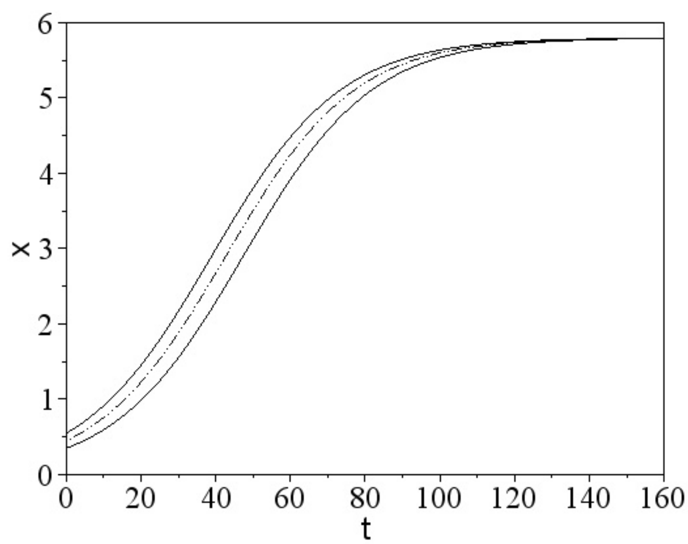


Figure 4.20: Attainable sets of the 0-level (continuous line) and the core (dashed-dotted line) of solution to the logistic model via FDIs in Example 4.4.4. Initial condition $(0.35; 0.45; 0.55)$ and parameters $k = 5.8$ and $a = 0.01$.

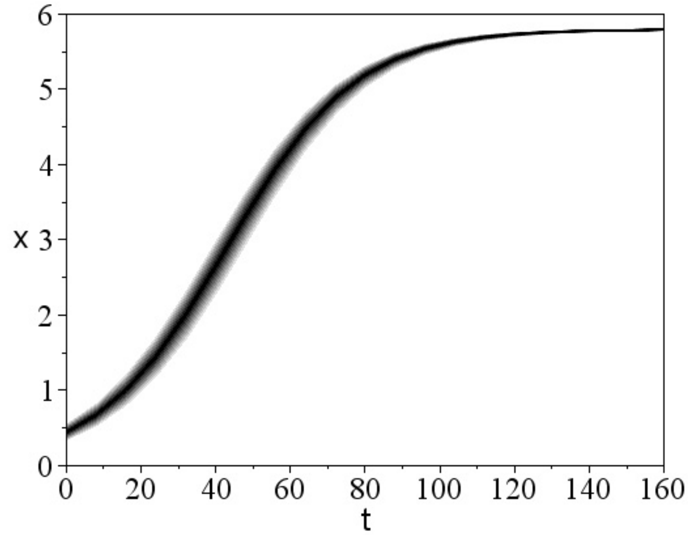


Figure 4.21: Solution to the logistic model via FDIs in Example 4.4.4, that is, a fuzzy bunch of functions whose membership of each function to the solution is represented by the scale of gray: the darker the color the higher the membership degree. Initial condition $(0.35; 0.45; 0.55)$ and parameters $k = 5.8$ and $a = 0.01$.

The classical functions that constitute the fuzzy solution in the Example 4.4.4 are solutions of the associated classical just as in Example 4.4.2 and hence preserves the properties of the classical case. In these two examples the method for finding solutions is the same as the one that will be presented in the next section, hence the solutions are the same.

## 4.5 Extension of the Crisp Solution

Zadeh's extension of the crisp solution is a very intuitive method. It consists in solving a crisp ODE and extending the crisp solution according to a fuzzy parameter (that is, the initial condition or some other parameter is considered to be fuzzy). As it will be clear, besides intuititive, the fuzzy solution preserves properties from the crisp solution. Hence the function may present decreasing length of its support, periodicity and other behaviors that may be inherent of the phenomena being modeled.

Buckley and Feuring (2000) and Oberguggenberger and Pittschmann (1999) presented this method for solving first-order FIVPs. Many other authors developed the same idea in the following years (see for instance Mizukoshi et al. (2007); Cecconello (2010); Gasilov et al. (2012)), though the authors not always use the specific term *Zadeh's extension of the solution*.

It should be clear that this method does not solve a fuzzy differential equation and has no fuzzy derivatives involved. Solutions are obtained by extending the operator that associates each crisp ODE and its parameter to a crisp solution, *in each value in the domain*. The result of this operation is a fuzzy subset of $\mathbb{R}^n$. Buckley and Feuring (2000) claim that it is equivalent to the united extension of the operator that associates each crisp ODE and its parameter to a crisp solution (obtaining a fuzzy subset of a space of functions) and calculating its attainable sets. This is a connection to the approach via fuzzification of the derivative operator (see Section 4.6).

We consider the crisp IVP

$$\begin{cases} x'(t) & = & f(t, x(t), w) \\ x(0) & = & x_0 \end{cases}, \tag{4.39}$$

where $x_0 \in \mathbb{R}^n$, $w \in \mathbb{R}^k$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. A solution is a continuous function that satisfies the initial condition and the differential equation for all $t$. For each pair of parameter and initial condition there is a solution associated with (4.39) that we denote by $x(\cdot, x_0, w)$.

### Autonomous FIVP with Fuzzy Initial Condition

Let us first consider that a phenomena is modeled by system (4.39) and that only its initial condition is a fuzzy subset. Each element $x_0$ of this fuzzy subset $X_0$ leads to a different solution $x(\cdot, x_0)$. It is natural to consider that, given an initial value, there is a solution and, given a set of initial values, there is a set of solutions. *Each classical solution evaluated at $t$ is associated with the membership of the correspondent initial value via Zadeh's extension.* This fuzzy subset is the solution to the FIVP at $t$. In other words, given $x(\cdot, x_0)$ a solution to (4.39), if the initial condition is the fuzzy subset $X_0$, the solution to the FIVP is based on Zadeh's extension of $x(t, x_0)$, that is, $\widehat{x}(t, X_0)$, $X_0 \in \mathcal{F}_{\mathcal{C}}(R^n)$.

Mizukoshi et al. (2007) considered the autonomous system:

$$\begin{cases} x'(t) & = & f(x(t)) \\ x(0) & = & x_0 \end{cases}, \tag{4.40}$$

with fuzzy initial value. They have proved existence and uniqueness of the solution via Zadeh's extension of the classical solution and also demonstrated the equivalence with the fuzzy differential inclusion

$$\begin{cases} x'(t) & = & f(x(t)) \\ x(0) & \in & X_0 \end{cases}, \tag{4.41}$$

where $X_0 \in \mathcal{F}_{\mathcal{C}}(R^n)$.

**Theorem 4.5.1** *(Mizukoshi et al., 2007) Consider $U$ an open set in $\mathbb{R}^n$ and suppose that the IVP (4.40) admits only one solution $x(\cdot, x_0)$ for each $x_0 \in U$, with $f$ continuous. Suppose also that $x(t, \cdot)$ depends continuously on the initial condition. Given $X_0 \in \mathcal{F}_{\mathcal{C}}(U)$, Zadeh's extension $\widehat{x}(t, X_0)$ of $x(t, X_0)$, that is, the solution of the FIVP with fuzzy initial condition correspondent to (4.40), is well-defined. Moreover, the solution coincides with the (attainable sets of the) solution of the FDI (4.41).*

**Remark 4.5.2** *We can assure existence of the solution by using Lipschitz condition. That is, if there exists a constant $L > 0$ such that*

$$||f(x) - f(y)|| \leq L||x - y||$$

*then there is only one solution $x(\cdot, x_0)$, for each $x_0 \in \mathbb{R}^n$, to IVP (4.40).*

**Remark 4.5.3** *Theorem 4.5.1 establishes that, given some conditions, the attainable sets of the solution via FDIs are the same as the solution calculated at each $t \in$ via Zadeh's extension of the classical solution.*

**Example 4.5.4** *Let us consider the classical decay model*

$$\left\{ \begin{array}{rcl} x'(t) & = & -\lambda x(t) \\ x(0) & = & x_0 \end{array} \right. , \tag{4.42}$$

*where $\lambda \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^+$. The solution is known to be*

$$x(t) = x_0 e^{-\lambda t}.$$

*According to the method of the present section, if the initial condition $x_0$ is considered to be fuzzy, that is, $x_0 = X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R})$, the solution to the new problem is Zadeh's extension of the solution $x(t) = x(t, x_0)$ with respect to the initial condition. Levelwise we obtain*

$$[X(t)]_\alpha = \widehat{x}(t, X_0) = \{x_0 e^{-\lambda t}, x_0 \in [X_0]_\alpha\}.$$

**Example 4.5.5** *The logistic model*

$$\left\{ \begin{array}{rcl} x'(t) & = & ax(t)(k - x(t)) \\ x(0) & = & x_0 \end{array} \right. , \tag{4.43}$$

*where $a, k, x_0 \in \mathbb{R}^+$ is known to have the solution*

$$x(t) = \frac{kx_0 e^{akt}}{k + x_0(e^{akt} - 1)}.$$

*Considering the initial condition $x_0$ a fuzzy number $X_0$, the solution to the new problem is Zadeh's extension of the solution $x(t) = x(t, x_0)$:*

$$[X(t)]_\alpha = \widehat{x}(t, X_0) = \left\{ \frac{kx_0 e^{akt}}{k + x_0(e^{akt} - 1)}, x_0 \in [X_0]_\alpha \right\}.$$

**FIVP with Fuzzy Initial Condition and Fuzzy Parameter**

The general case, in which the FIVP is not autonomous and there are other fuzzy parameters influencing the differential equation, is treated by Bede (2013). If the parameter $w$ is also fuzzy, the solution to the FIVP is Zadeh's extension of $x(t, x_0, w)$, that is, $\widehat{x}(t, X_0, W)$, $X_0 \in \mathcal{F}_\mathcal{C}(R^n)$, $W \in \mathcal{F}_\mathcal{C}(R^k)$. According to result in Perko (200), Lipschitz condition on the second and third variables of $f$ assures existence, uniqueness and continuity of the solution, with respect to the parameter $w$ and the initial condition. Bede (2013) uses this result to state the existence and uniqueness theorem for the FIVP.

**Remark 4.5.6** *Many authors write $X'(t) = \widehat{F}(t, X(t), W), X(0) = X_0$, which is just a notation. As we said, this is not a fuzzy diffential equation since there is no fuzzy derivative, hence $X'(t)$ does not make sense.*

**Theorem 4.5.7** *(Bede, 2013) Let $f : [t_0, t_0 + p] \times [x_0 - q, x_0 + q] \times \overline{B}(w, r)$ and assume that $F$ is Lipschitz in its second and third variables, that is, there exist constants $L_1 > 0$ and $L_2 > 0$ such that*

$$||f(t, x, w) - f(t, y, w)|| \leq L_1||x - y||$$

*and*

$$||f(t, x, w) - f(t, x, z)|| \leq L_2||w - z||$$

*Then the solution to (4.39) with fuzzy parameter, defined as Zadeh's extension $\widehat{x}(t, X_0, W)$, where $x(\cdot, x_0, w)$ is solution to (4.39) in its classical form, is well defined, unique and continuous. Moreover, it is can be defined levelwise:*

$$[X(t)]_\alpha = [\widehat{x}(t, X_0, W)]_\alpha = x(t, [X_0]_\alpha, [W]_\alpha).$$

**Example 4.5.8** *Now in the decay model we also consider the parameter $\lambda$ as a fuzzy number. The solution is Zadeh's extension of the solution $x(t) = x(t, x_0, \lambda)$ with respect to the $x_0$ and $\lambda$. Levelwise we obtain*

$$[X(t)]_\alpha = \widehat{x}(t, X_0, \Lambda) = \{x_0 e^{-\lambda t}, x_0 \in [X_0]_\alpha, \lambda \in [\Lambda]_\alpha\}.$$

*Hence*

$$[X(t)]_\alpha = [x_{0\alpha}^- e^{-\lambda_\alpha^+ t}, x_{0\alpha}^+ e^{-\lambda_\alpha^- t}].$$

*In this case, the solution is the same as via FDIs (see Example 4.4.3), but it does not happen in general (see Allahviranloo et al. (2009)).*

**Example 4.5.9** *Now consider the parameter $k$ and $a$ in the logistic model as fuzzy numbers $K$ and $A$ with supp$(K) \subset \mathbb{R}^+$ and supp$(A) \subset \mathbb{R}^+$ and set $Z = A \times K$. The solution is Zadeh's extension of the solution $x(t) = x(t, x_0, z)$ with respect to $x_0$ and $z = k \times a$. Levelwise we obtain*

$$[X(t)]_\alpha = \widehat{x}(t, X_0, Z) = \left\{ \frac{k x_0 e^{akt}}{k + x_0(e^{akt} - 1)}, x_0 \in [X_0]_\alpha, a \in [A]_\alpha, k \in [K]_\alpha \right\}.$$

*To obtain the expression of the level set functions we need to find the minimum and the maximum of the expression above. We calculate numerically the 0-levels and cores of the solution at each $t \in [0, T]$ and display it in Figure 4.22.*

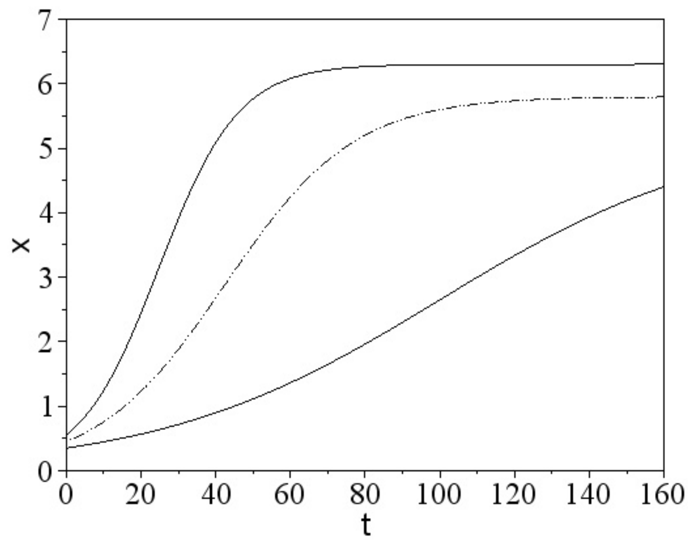


Figure 4.22: The 0-level (continuous line) and the core (dashed-dotted line) of solution to the logistic model via Zadeh's extension of the classical solution in Example 4.5.9. Initial condition $(0.35; 0.45; 0.55)$ below carrying support $K = (5.3; 5.8; 6.3)$ and growth parameter $A = (0.005; 0.010; 0.015)$.

Zadeh's extension of the classical solution is an intuitive method. However, calculating the solution at each $t$ in the domain is not obvious. It demands minimization and maximization for each $t$ and each $\alpha$-cut and, most of the times, it cannot be done analitically. The previous approaches present the same difficulty. The next approach does not solve this problem, since operating with fuzzy subsets is complex, which also makes the theory more challenging. On the contrary, the next approach unifies all the others presented so far.

## 4.6 Extension of the Derivative Operator

We next study FDEs using the $\widehat{D}$-derivative. This derivative operates on fuzzy subsets of functions, but the equality in the fuzzy differential equation is evaluated for each $t$, that is, on the attainable sets.

Consider the FIVP

$$\begin{cases} \widehat{D}X(t) &= F(t, X(t)) \\ X(0) &= X_0 \end{cases}, \tag{4.44}$$

where $X_0 \in \mathcal{F}(\mathbb{R}^n)$ and $\widehat{D}X(t) \in \mathcal{F}(\mathbb{R}^n)$ is the attainable set of the $\widehat{D}$-derivative (see Subsection 3.2.2) of the fuzzy bunch of functions $X(\cdot)$ at $t$.

This approach is not equivalent to any other considered, but has many similarities with them. Some points should be highlighted:

(i) $\widehat{D}$ is a fuzzy derivative and so are Hukuhara and strongly generalized derivatives. Fuzzy differential inclusions and Zadeh's extension of the classical solution do not use fuzzy derivatives.

(ii) $\widehat{D}$-derivative does not differentiate fuzzy-set-valued functions (differently from Hukuhara and strongly generalized Hukuhara derivatives). It operates on fuzzy bunches of functions (fuzzy subsets of spaces of classical functions).

(iii) Provided some conditions, $\widehat{D}$-derivative operates by differentiating classical functions (as fuzzy differential inclusions and Zadeh's extension of the classical solution).

(iv) The FIVP demands equality between the fuzzy subset of the left-hand-side and the fuzzy subset of the right-hand-side, as with Hukuhara and strongly generalized Hukuhara derivatives.

In what follows we will prove that

(a) Provided some hypotheses hold, the solution of the FIVP via FDIs is one solution via Zadeh's extension of the derivative operator.

(b) Provided some hypotheses hold, the solution of the FIVP via Zadeh's extension of the classical solution is one solution via Zadeh's extension of the derivative operator (in the sense of attainable sets).

(c) Provided some conditions hold, the solutions of the FIVP via strongly generalized Hukuhara derivative (and particularly, via Hukuhara derivative) are solutions via Zadeh's extension of the derivative operator (in the sense of attainable sets).

Items (a) and (b) will be briefly illustrated in the next example.

**Example 4.6.1** *We solved the decay model with nonfuzzy coefficient using other methods. Now consider*

$$\begin{cases} \widehat{D}X(t) &= -\lambda X(t) \\ X(0) &= X_0 \end{cases}, \tag{4.45}$$

*where* $\lambda \in \mathbb{R}^+$, $X_0 \in \mathcal{F}_{\mathcal{C}}(\mathbb{R}$ *and* $supp(X_0) \subset \mathbb{R}^+$.

*The solution via FDIs,*

$$\Sigma_\alpha(X_0, T) = \{x : x(t) = x_0 e^{-\lambda t}, x_0 \in [X_0]_\alpha\}.$$

*is solution to (4.45), since* $X(\cdot) = \Sigma(X_0, T)$ *satisfies the hypothesis of Theorem 3.2.8 and hence*

$$\begin{aligned} [\widehat{D}X(\cdot)]_\alpha &= D[X(\cdot)]_\alpha \\ &= \{Dx(\cdot) : x(t) = x_0 e^{-\lambda t}, x_0 \in [X_0]_\alpha\} \\ &= \{\lambda x(\cdot) : x(t) = x_0 e^{-\lambda t}, x_0 \in [X_0]_\alpha\} \\ &= [\lambda X(\cdot)]_\alpha \end{aligned}$$

*for all* $\alpha \in [0, 1]$.

*Since it has already been shown that the attainable sets of the cited solution is solution via Zadeh's extension of the crisp solution, the FIVP via Zadeh's extension of the crisp solution and FDIs have the same solution, which is one solution via* $\widehat{D}$*-derivative.*

The fact that the solutions via FDIs and via $\widehat{D}$-derivative are the same is not a mere coincidence. In what follows we make this connection between these two theories. Let us first recall the FIVP modeled by FDIs:

$$\begin{cases} x'(t) & \in & F(t, x(t)) \\ x(0) & \in & X_0 \end{cases} \tag{4.46}$$

Levelwise it is equivalent to

$$\begin{cases} x'_\alpha(t) & \in & [F(t, x_\alpha(t))]_\alpha \\ x_\alpha(0) & \in & [X_0]_\alpha \end{cases}. \tag{4.47}$$

Taking the union of all functions $x_\alpha(\cdot)$

$$\begin{cases} \bigcup x'_\alpha(t) & \subseteq & \bigcup [F(t, x_\alpha(t))]_\alpha \\ \bigcup x_\alpha(0) & \subseteq & [X_0]_\alpha \end{cases}. \tag{4.48}$$

The union of all solutions $x_\alpha(\cdot)$ of the differential inclusion (4.47) defines the $\alpha$-cut of the solution $X(\cdot)$ of problem (4.46). We can rewrite last system as

$$\begin{cases} D\,[X(t)]_\alpha & \subseteq & \bigcup [F(t, x_\alpha(t))]_\alpha \\ [X(0)]_\alpha & \subseteq & [X_0]_\alpha \end{cases},$$

where $[X(\cdot)]_\alpha = \{x_\alpha(\cdot) : x_\alpha(\cdot) \text{ is solution to (4.47) }\}$.

We have $[X(0)]_\alpha = [X_0]_\alpha$ by the construction of the solution. We are interested in finding conditions for

$$\left[\widehat{D}X(t)\right]_\alpha = [F(t, X(t))]_\alpha \tag{4.49}$$

to hold, that is,

$$\left[\widehat{D}X(t)\right]_\alpha \subseteq [F(t, X(t))]_\alpha \tag{4.50}$$

and

$$\left[\widehat{D}X(t)\right]_\alpha \supseteq [F(t, X(t))]_\alpha\,. \tag{4.51}$$

If $D\,[X(t)]_\alpha = \left[\widehat{D}X(t)\right]_\alpha$, the condition

$$[F(t, X(t))]_\alpha = \bigcup [F(t, x_\alpha(t))]_\alpha \tag{4.52}$$

guarantees (4.50) Since $X(\cdot)$ is a solution via FDIs, it has compact $\alpha$-cuts in $Z_T(\mathbb{R})$ and hence Theorem 3.2.8 can be used.

Hence the solution via FDIs is a good candidate for being a solution to (4.44), provided condition (4.52) holds. We need only to prove (4.51).

**Example 4.6.2** *The function $F(t,X(t)) = \lambda X(t)$, where $\lambda \in \mathbb{R}$, satisfies condition (4.52), that is,*

$$\cup_{x_\alpha \in [X]^\alpha} [\lambda x_\alpha(t)]^\alpha = [\lambda X(t))]^\alpha.$$

*If the parameter $\Lambda$ is fuzzy, we also have that:*

$$\cup_{x_\alpha \in [X]^\alpha} [\Lambda x_\alpha(t)]^\alpha = [\Lambda X(t))]^\alpha.$$

*Note that $\lambda X$ is Zadeh's extension of $\lambda x$ with respect to $x$ according to Definition 2.2.1. And according to Definition 2.2.6, $\Lambda X$ is Zadeh's extension of $\Lambda x$. On the other hand, $(1-X)X$ is not Zadeh's extension of $(1-x)x$ and*

$$\cup_{x_\alpha \in [X]^\alpha} [(1 - x_\alpha(t))x_\alpha(t)]^\alpha \subset [(1 - X(t))X(t)]^\alpha.$$

To prove the existence Theorem 4.6.4 we will need some results.

**Theorem 4.6.3** *(Michael's Selection Theorem, see e.g. Aubin and Cellina (1984)) Let $\mathbb{X}$ be a metric space, $\mathbb{Y}$ a Banach space and $G$ a map from $\mathbb{X}$ to convex and closed subsets of $\mathbb{Y}$. If $G$ is lower semicontinuous then there exists a continuous selection $f : \mathbb{X} \to \mathbb{Y}$ of $G$.*

The following statement is a result of Michael's Selection Theorem according to Aubin and Cellina (1984), page 83: if $\mathbb{X}$ is a paracompact space ($\mathbb{R}^n$ is paracompact), for any $y_0 \in G(x_0)$ the set-valued map $G_0$ defined by

$$G_0(x_0) = \{y_0\}, \quad G_0(x) = G(x) \quad \forall x \neq x_0$$

is also lsc with convex values and hence there exists a continuous selection $g_0$ of $G_0$. In other words, *for every $y_0$ in $G(x_0)$ there passes a continuous selection of $G$.*

Now we are able to prove the existence Theorem 4.6.4, one of the main results of this study, already published (Barros et al., 2013).

**Theorem 4.6.4** *(Barros et al., 2013) Let $X_0 \in \mathcal{F}_C(\mathbb{R}^n)$ and $\Omega$ be an open set in $\mathbb{R}\times\mathbb{R}^n$ containing $\{0\} \times suppX_0$ and $F : \mathbb{R}\times\mathcal{F}_\mathcal{K}(\mathbb{R}^n) \to \mathcal{F}_\mathcal{K}(\mathbb{R}^n)$ a fuzzy-set-valued function such that $F(t,x) = F|_\Omega$ is continuous with $[F(t,x)]^\alpha$ compact, convex and $[F(t,X)]^\alpha = \bigcup_{x\in[X]^\alpha}[F(t,x)]^\alpha$. Also, suppose that the boundedness assumption holds. Then, there exists a solution $X(\cdot) \in \mathcal{F}_\mathcal{K}(Z_T(\mathbb{R}^n))$ for problem (1.2). Moreover, $[X(t)]^\alpha$ are compact and connected in $\mathbb{R}^n$, for all $\alpha \in [0,1]$.*

**Proof.** The stated hypotheses are stronger than those in Theorem 4.4.1, including condition (4.52). Hence the solution $X(\cdot)$ to (4.46), whose existence is guaranteed, satisfies (4.50). In what follows we prove that $X(\cdot)$ also satisfies (4.51).

We want to prove that given $z \in [F(s,x)]_\alpha$ there exists $x(\cdot) \in [X(\cdot)]_\alpha$ such that $x'(s) = z$, or $z \in D[X]_\alpha$. Note that, given $z \in [F(s,x)]_\alpha$, there exists $y(\cdot) \in [X(\cdot)]_\alpha$ such that $y(s) = x$, hence $z \in [F(s,y(s))]_\alpha$. Michael's Selection Theorem applied to $[F(t,y(t))]_\alpha$ guarantees that there exists a continuous selection $z(\cdot)$ such that $z(t) \in [F(t,y(t))]_\alpha$, for all $t \in [0,T]$.

We use this selection $z(\cdot)$ to define

$$x(t) = x_0 + \int_0^t z(\tau)d\tau$$

for some $x_0 \in [X_0]_\alpha$.

This function $x(\cdot)$ belongs to $[X(\cdot)]_\alpha$, since $x'(t) = z(t)$ for all $t \in [0, T]$. Hence we have proved (4.51).

■

**Remark 4.6.5** *Theorem 4.6.4 establishes that, provied some conditions hold, the solution via FDIs is the same as via $\widehat{D}$-derivative, that is, the fuzzy bunches of functions that satisfy the respective FIVPs are the same. Consequently, the attainable sets obtained from the two methods are the same.*

FIVPs using Zadeh's extension of real-valued functions has a rich literature (see Kaleva (2006); Cecconello (2010); Mizukoshi et al. (2007); Chalco-Cano et al. (2011a)). The generalized case, though, is not much mentioned. In what follows we use the second case, that is, $f : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$, which has a wider application.

**Corollary 4.6.6** *Let $X_0 \in \mathcal{F}_C(R)$, $\Omega$ be an open set in $\mathbb{R} \times \mathbb{R}^n$ containing $\{0\} \times suppX_0$, $f : \mathbb{R} \times \mathbb{R} \to \mathcal{F}_C(\mathbb{R})$ be $d_\infty$-continuous and $\widehat{f} : \mathbb{R} \times \mathcal{F}_C(\mathbb{R}) \to \mathcal{F}_C(\mathbb{R})$ be Zadeh's extension of $f$. Also, let the boundedness assumption hold. Then the FIVP (4.44), with right-hand-side function $\widehat{f}$, has a solution.*

**Proof.** From Theorem 2.4.8 $\widehat{f}$ is $d_\infty$-continuous and $[\widehat{f}(t, X)]^\alpha = \bigcup_{x \in [X]^\alpha} [f(t, x)]^\alpha$. Thus the conditions of Theorem 4.6.4 are satisfied and therefore the FIVP (4.44) has a solution.

■

**Example 4.6.7** *The FIVP*

$$\begin{cases} \widehat{D}X(t) &= -\Lambda X(t) \\ X(0) &= X_0 \end{cases}, \tag{4.53}$$

$\lambda \in \mathbb{R}^+$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R}$ *and* $supp(X_0) \subset \mathbb{R}^+$.

*where* $\Lambda \in \mathcal{F}_\mathcal{C}(\mathbb{R})$, $supp(\Lambda) \subset \mathbb{R}^+$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R}$ *and* $supp(X_0) \subset \mathbb{R}^+$, *has right-hand-side function given by Zadeh's extension, according to Definition 2.2.6. Indeed,* $F(X) = -\Lambda X$ *is Zadeh's extension of* $G(x) = \Lambda x$.

*We apply Theorem 4.6.4 and the solution via fuzzy differential inclusions of Example 4.4.3 is solution to (4.53). Hence the attainable sets of the solution are*

$$X(t) = [x_{0\alpha}^- e^{-\lambda_\alpha^+ t}, x_{0\alpha}^+ e^{-\lambda_\alpha^- t}].$$

Mizukoshi et al. (2007) have proved that the solution of an IVP with fuzzy initial condition via Zadeh's extension of the classical solution is the same as the solution of the associated FDI (in terms of attainable sets), provided the function $f$ is continuous and the IVP has a unique solution. As a result of Theorem 4.6.4 and Corollary 4.6.6, if the right-hand-side function $F$ is Zadeh's extension of a continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, the solution of the FDI is a solution to the associated FIVP with the derivative via Zadeh's extension. Hence, if the associated IVP has a unique solution for each initial condition, we have the same attainable sets of the fuzzy solutions for the three mentioned approaches.

We next prove the following autonomous case: if the differential field $F$ is Zadeh's extension of $f : \mathbb{R}^n \to \mathbb{R}^n$, the solutions via FDI, Zadeh's extension of the classical solution of

$$\begin{cases} x'(t) &= f(x(t)) \\ x(0) &= x_0 \end{cases}, \tag{4.54}$$

and $\widehat{D}$-derivative are the same (in terms of attainable sets).

**Theorem 4.6.8** *Consider the FIVP*

$$\begin{cases} \widehat{D}X(t) &= \widehat{f}(X(t)) \\ X(0) &= X_0 \end{cases}, \tag{4.55}$$

*where $X_0 \in \mathcal{F}_C(\mathbb{R}^n)$, $\widehat{f}$ is Zadeh's extension of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that (4.54) has only one solution. Then the solution of (4.55) is given by Zadeh's extension of the solution of (4.54), $X(\cdot) = \widehat{x}(\cdot, x_0)$.*

**Proof.** Note that Zadeh's extension of the solution was previously (Section 4.5) defined at each $t \in [0, T]$. However, it is also possible to define it as a fuzzy bunch of functions, considering the solution not at each $t$, but as the whole function. Gasilov et al. (2013) and Buckley and Feuring (2000) claim that both approaches are equivalent in terms of attainable sets. The approach we adopt here is the one that deals with fuzzy bunch of functions.

Consider $x_\alpha(\cdot, x_0)$ an element of the $\alpha$-cut of Zadeh's extension $X(\cdot) = \widehat{x}(\cdot, x_0)$ of the solution $x(\cdot, x_0)$ to (4.54).

$$\begin{aligned} Dx_\alpha(\cdot) &= f(x_\alpha(\cdot)) \\ \textstyle\bigcup_{x_\alpha \in [X(\cdot)]_\alpha} Dx_\alpha(\cdot) &= \textstyle\bigcup_{x_\alpha \in [X(\cdot)]_\alpha} f(x_\alpha(\cdot)) \\ D \textstyle\bigcup_{x_\alpha \in [X(\cdot)]_\alpha} x_\alpha(\cdot) &= f(\textstyle\bigcup_{x_\alpha \in [X(\cdot)]_\alpha} x_\alpha(\cdot)) \\ D[X(\cdot)]_\alpha &= f([X(\cdot)]_\alpha) \end{aligned}$$

Since $X(\cdot)$ is also solution (fuzzy bunch of functions) via FDIs, we have $X(\cdot) \in \mathcal{F}_{\mathcal{K}}(Z_T(\mathbb{R}^n))$ and Theorem 3.2.8 is valid. Also, since $f$ is continuous, Theorem 2.2.4 holds. Hence

$$\left[\widehat{D}X(\cdot)\right]_\alpha = D[X(\cdot)]_\alpha = f([X(\cdot)]_\alpha) = [\widehat{f}(X(\cdot))]_\alpha$$

for all $\alpha \in [0, 1]$. Thus

$$\widehat{D}X(\cdot) = \widehat{f}(X(\cdot))$$

and, in particular,

$$\widehat{D}X(t) = \widehat{f}(X(t))$$

for all $t \in [0, T]$.

$\blacksquare$

**Remark 4.6.9** *Theorem 4.6.8 establishes that, under some conditions, the fuzzy bunches of functions of the solution via Zadeh's extension of the classical solution is the same as via $\widehat{D}$-derivative. Consequently, the attainable sets obtained from the two methods are the same.*

**Example 4.6.10** *The solution of the logistic model*

$$\left\{ \begin{array}{rcl} \widehat{D}X(t) & = & \widehat{f}(t, X(t))) \\ X(0) & = & X_0 \end{array} \right. , \tag{4.56}$$

*where $f(t,x) = ax(k-x)$, $a, k \in \mathbb{R}^+$, $X_0 \in \mathcal{F}_\mathcal{C}(\mathbb{R}$ and $supp(X_0) \subset \mathbb{R}^+$ is based on the solution of Example 4.5.5, that is, the same problem solved via Zadeh's extension of the crisp solution. In that example, we solved the classical differential equation. Instead of extending the solution at each $t$ of the domain, here we extend the solutions as elements in the space of functions. Hence we consider the solution $x(\dot{})$ such that*

$$x(t) = \frac{kx_0 e^{akt}}{k + x_0(e^{akt} - 1)}.$$

*Applying Theorem 4.56, a solution to (4.56) is Zadeh's extension of $x(\cdot) = x(\cdot, x_0)$,*

$$X(\cdot) = \widehat{x}(\cdot, X_0).$$

*As in Example 4.6.1, it can be verified that the solution satisfies (4.56) by applying Theorem 3.2.8:*

$$\begin{array}{rcl} [\widehat{D}X]_\alpha & = & D[X]_\alpha \\ & = & \left\{ Dx(\cdot) : x(t) = \dfrac{kx_0 e^{akt}}{k + x_0(e^{akt} - 1)}, x_0 \in [X_0]_\alpha \right\} \\ & = & \left\{ ax(\cdot)(k - x(\cdot)) : x(t) = \dfrac{kx_0 e^{akt}}{k + x_0(e^{akt} - 1)}, x_0 \in [X_0]_\alpha \right\} \\ & = & [\widehat{f}(t, X(t))]_\alpha \end{array}.$$

*Note that we do not have the expression for $\widehat{f}$ in terms of the standard arithmetic. As it has been already mentioned, $\widehat{f}(t, X(t))) \not\subseteq aX(t)(k - X(t))$, hence*

$$\left\{ \begin{array}{rcl} \widehat{D}X(t) & = & aX(t)(k - X(t)) \\ X(0) & = & X_0 \end{array} \right. , \tag{4.57}$$

*is a different problem, which will be solved using other method (see Theorem 4.6.11 and Example 4.6.12).*

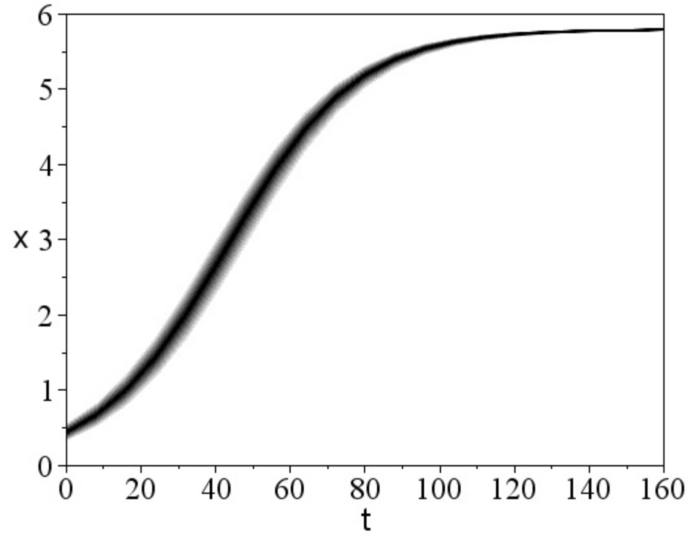


Figure 4.23: Solution to the logistic model via $\widehat{D}$-derivative in Example 4.6.10, that is, a fuzzy bunch of functions whose membership of each function to the solution is represented by the scale of gray: the darker the color the higher the membership degree. Initial condition $(0.35; 0.45; 0.55)$ and parameters $k = 5.8$ and $a = 0.01$.

In Section 3.3 a connection between the generalized differentiabilities and the $\widehat{D}$-derivative was established. Since they coincide for the representative bunches of functions of a special class of fuzzy-number-valued numbers (see Theorems 3.3.3 and 3.3.4), it is natural to wonder if the solution to an FIVP also coincides in both approaches. Indeed, provided some conditions hold, they do: the hypotheses of the characterization Theorem 4.3.4 for strongly generalized differentiability assure us that we have two solutions (fuzzy-number-valued functions) that generate two differet fuzzy bunches of functions that are solutions to the corresponding FIVP with $\widehat{D}$-derivative.

**Theorem 4.6.11** *Assume the hypotheses of Theorem 4.3.4 hold true for $F$ and $X_0$ in FIVP (4.44) and that the solutions obtained from (4.16) and (4.17) belong to $\mathcal{F}^0_{\mathcal{C}}(\mathbb{R})$. Then FIVP (4.44) has at least two solutions.*

**Proof.** Theorem 4.3.4 assures two solutions via strongly generalized differentiability. We will prove that if the solutions assume values in $\mathcal{F}^0_{\mathcal{C}}(\mathbb{R})$, they satisfy Theorem 3.3.3 which provides us with two representative bunches of functions whose derivative is the same as those via generalized Hukuhara differentiability.

Let $X$ be the solution to the FIVP via strongly generalized differentiability obtained by solving (4.16). It is obvious that $X$ is continuous and $x^-_\alpha$ and $x^+_\alpha$ are differentiable with respect to $t$. We will prove that this differentiability is uniform with respect to $\alpha \in [0, 1]$. Since $X$ is strongly generalized differentiable, it is gH-differentiable (the latter is more general than the former) and, by Theorem 3.1.18, it satisfies a) or b) of Theorem 3.3.3.

First we will show uniform differentiability, that is, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{x_\alpha^\pm(t+h) - x_\alpha^\pm(t)}{h} - (x_\alpha^\pm)'(t)\right| < \epsilon$$

if $|h| < \delta$, for all $\alpha \in [0,1]$. In what follows we will use the fact that $x_\alpha^-$ and $x_\alpha^+$ are solutions to (4.16) and thus satisfy

$$x_\alpha^\pm(t+h) = x_\alpha^\pm(t) + \int_t^{t+h} f_\alpha^\pm(s, x_\alpha^-(s), x_\alpha^+(s))ds.$$

Therefore,

$$\begin{aligned}
&\left|\frac{x_\alpha^\pm(t+h) - x_\alpha^\pm(t)}{h} - (x_\alpha^\pm)'(t)\right| = \\
&\quad = \left|\frac{x_\alpha^\pm(t+h) - x_\alpha^\pm(t)}{h} - f_\alpha^\pm(t, x_\alpha^-(t), x_\alpha^+(t))\right| \\
&\quad = \left|\frac{x_\alpha^\pm(t+h) - x_\alpha^\pm(t) - hf_\alpha^\pm(t, x_\alpha^-(t), x_\alpha^+(t))}{h}\right| \\
&\quad = \left|\frac{x_\alpha^\pm(t) + \int_t^{t+h} f_\alpha^\pm(s, x_\alpha^-(s), x_\alpha^+(s))ds - x_\alpha^\pm(t) - \int_t^{t+h} f_\alpha^\pm(t, x_\alpha^-(t), x_\alpha^+(t))ds}{h}\right| \\
&\qquad \le \frac{\int_t^{t+h} \left|f_\alpha^\pm(s, x_\alpha^-(s), x_\alpha^+(s)) - f_\alpha^\pm(t, x_\alpha^-(t), x_\alpha^+(t))\right| ds}{|h|}
\end{aligned}$$

The hypothesis of equicontinuity ensures that there is a $\delta > 0$ such that

$$\left|f_\alpha^\pm(s, x_\alpha^-(s), x_\alpha^+(s)) - f_\alpha^\pm(t, x_\alpha^-(t), x_\alpha^+(t))\right| < \epsilon \text{ if } |s-t| < \delta,$$

that is,

$$\int_t^{t+h} \left|f_\alpha^\pm(s, x_\alpha^-(s), x_\alpha^+(s)) - f_\alpha^\pm(t, x_\alpha^-(t), x_\alpha^+(t))ds\right| < \epsilon|h|.$$

This is means that

$$\left|\frac{x_\alpha^\pm(t+h) - x_\alpha^\pm(t)}{h} - (x_\alpha^\pm)'(t)\right| < \epsilon.$$

if $|h| < \delta$, that is, $x_\alpha^-$ and $x_\alpha^+$ are differentiable real-valued functions with respect to $t$, uniformly with respect to $\alpha$.

We have proved that the solution $X$ obtained by solving (4.16) satisfies the conditions of Theorem 3.3.3. Hence the representative bunch of first kind, $\tilde{X}(\cdot)$ has $\widehat{D}$-derivative equal to generalized Hukuhara derivative (in terms of attainable sets), which is the same as strongly generalized differentiability, since both exist. Thus

$$F(t, x(t)) = X'_G(t) = X'_{gH}(t) = \widehat{D}\tilde{X}(t)$$

and it is proved that $\tilde{X}$ is a solution to FIVP (4.44).

Following the same reasoning one obtains that the solution to (4.17) leads to other representative bunch of first kind which is also solution to (4.44).

■

**Example 4.6.12** *All examples in Section 4.3, that is, Examples 4.3.5, 4.3.6, 4.3.7 and 4.3.8 have solutions $X$ with $X(t) \in \mathcal{F}^0_{\mathcal{C}}(\mathbb{R})$. Hence, the representative bunches of first kind are solutions of the respective problems using the $\widehat{D}$-derivative.*

The last example of this section regards a system in which the state variable has values in $\mathbb{R}^2$. The solution we look for belongs to $\mathcal{F}_K(\mathcal{AC}([0,T];\mathbb{R}^2))$.

**Example 4.6.13** *Consider*

$$\begin{cases} \widehat{D}X(t) &= F(X(t))) \\ X(0) &= X_0 \end{cases}, \tag{4.58}$$

*where $X_0 \in \mathcal{F}(\mathbb{R}^2)$ and $F : \mathcal{F}(\mathbb{R}^2) \to \mathcal{F}(\mathbb{R}^2)$ such that*

$$\mu_{F(X)}(z,y) = \mu_X(y,z)$$

*We will try the solution obtained via Zadeh's extension of the following associated problem:*

$$\begin{cases} y'(t) = z(t), & y(0) = y_0 \\ z'(t) = y(t), & z(0) = z_0 \end{cases}, \tag{4.59}$$

*whose solution is*

$$x(t,x_0) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \frac{y_0}{2}\begin{pmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{pmatrix} + \frac{z_0}{2}\begin{pmatrix} e^t - e^{-t} \\ e^t + e^{-t} \end{pmatrix} \tag{4.60}$$

*where $x_0 = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$.*

*The function $x(t,x_0)$ is continuous with respect to the initial condition. Hence Zadeh's extension of this solution can be defined levelwise*

$$[\widehat{x}(t,X_0)]_\alpha = \widehat{x}(t,[X_0]_\alpha) = \{x(t,x_0), x_0 \in [X_0]_\alpha\}$$

*Hence*

$$\begin{aligned} \left[\widehat{D}\widehat{x}(\cdot,X_0)\right]_\alpha &= D\{x(\cdot) : x(t) = x(t,x_0), x_0 \in [X_0]_\alpha\} \\ &= \{x'(\cdot) : x(t) = x(t,x_0), x_0 \in [X_0]_\alpha\} \end{aligned}$$

*where*

$$\begin{aligned} x'(t,x_0) &= \begin{pmatrix} y'(t,y_0,z_0) \\ z'(t,y_0,z_0) \end{pmatrix} = \tfrac{y_0}{2}\begin{pmatrix} e^t - e^{-t} \\ e^t + e^{-t} \end{pmatrix} + \tfrac{z_0}{2}\begin{pmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} z(t,y_0,z_0) \\ y(t,y_0,z_0) \end{pmatrix} \end{aligned}$$

*Thus,*

$$\begin{aligned} \left[\widehat{D}\widehat{x}(t,X_0)\right]_\alpha &= \{(z(t),y(t)) : x(t,x_0) = (y(t),z(t)), x_0 \in [X_0]_\alpha\} \\ &= [F(\widehat{x}(t,X_0))]_\alpha . \end{aligned}$$

The method that has been used to solve the last example is based on Zadeh's extension of the classical solution. We have also shown that other techniques to find solutions to FIVPs involving $\widehat{D}$-derivative are solving the FIVP via strongly generalized derivative and taking the representative bunches as solutions via $\widehat{D}$-derivative; and using fuzzy differential inclusions.

## 4.7 Summary

We reviewed the most known approaches for FDEs: using Hukuhara and strongly generalized derivatives; via FDIs and via Zadeh's extension of the crisp solution. We developed new theory using $\widehat{D}$-derivative and demonstrated two existence theorems. We compared this new theory with the previous ones and illustrated with examples.

# Chapter 5

# Conclusions

The proposal of Barros et al. (2010) of fuzzifying the derivative and integral operators via Zadeh's extension has been developed in this thesis and published in Barros et al. (2013); Gomes and Barros (2012, 2013). It was verified the need of operating with fuzzy bunches of functions, which is not used in the traditional calculus of fuzzy functions, that is, calculus for fuzzy-set-valued functions. We have exposed both kinds of calculus and compared them. Under suitable conditions, they coincide in the sense of attainable sets. Other results are a Fundamental Theorem of Calculus and a theory of FDEs. It is worth remarking that the calculus via fuzzification the derivative and integral operators is equivalent to differentiating and integrating the crisp functions of the levels, provided some conditions hold.

The most known approaches of FDEs (Hukuhara and strongly generalized differentiabilities, FDIs and Zadeh's extension of the crisp solution) were presented and compared to the one via fuzzifying the derivative operator. We have proved theorems of existence of solutions to first-order FIVPs. The first theorem was based on the differentiation of crisp functions, similarly to FDIs and Zadeh's extension of the solution. Indeed, this similarity led to the result: a solution to certain FIVPs via FDIs are the same as via fuzzification of the derivative.

The most recent results regards equivalence (in the sense of attainable sets) of certain fuzzy bunches of functions and the generalized derivatives. We proved that the solution to an FIVP via strongly generalized derivative produces a fuzzy bunch of functions with same attainable sets and with derivative with the same attainable sets as the fuzzy-set-valued functions differentiated.

### Future Work

As presented in Subsection 2.3, arithmetics of fuzzy numbers may admit interactivity between elements. Comparisons with new theories of fuzzy calculus and FDEs considering interactive fuzzy numbers should also be carried out in the future.

The equivalence of the FIVP

$$\begin{cases} \widehat{D}X(t) &= F(t, X(t)) \\ X(0) &= X_0 \end{cases}$$

(where $F : [0, T] \times \mathcal{F}(R^n) \to \mathcal{F}(R^n)$ is a continuous function) with the integral equation

$$X(t) = X_0 + \widehat{\int_0^t} F(s, X(s))ds \tag{5.1}$$

has also been proposed (Gomes and Barros, 2012). Summation as occurs in this equation, however, is not the same as in the Minkowski sum:

$$X(t) = X_0 \widehat{+} \widehat{\int_0^t} F(s, X(s))ds$$

is interpreted as the fuzzy set with $\alpha$-cuts defined as the union of elements

$$\{x_0 + \int_0^t f(s, x(s))ds : x_0 = x(0) \in [X_0]^\alpha,\ f(\cdot, x(\cdot)) \in [F(\cdot, X(\cdot))]^\alpha\}$$

where $x_0 + \int_0^t f(s, x(s))ds$ is a solution to the classical associated IVP

$$\begin{cases} Dx(t) &= f(t, x(t)) \\ x(0) &= x_0 \end{cases} .$$

The use of constraint interval arithmetic, the interactive arithmetic and an arithmetic that explicitly models dependency among the terms involved in the equation (in this case, crisp functions) should be studied. What this interactivity means and how to operate with it is an interesting subject for further research.

Other future research should look for application and develop a theory for the appropriateness of the derivative and integral via Zadeh's extension in modeling the derivative and the integral of fuzzy variables in real problems. Certainly numerical methods for solving fuzzy differential equations will also be required.

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# Appendix A

# Prerequisites

The following concepts may be found in Ash (1972); Aubin and Cellina (1984); Hönig (1977); Rudin (1973).

### Continuity and semicontinuity

**Definition A.0.1** *A function $f : X \to \mathbb{R}$ is said to be upper (lower) semi-continuous at $x_0$ if for any $\epsilon > 0$ there exist a $\delta > 0$ such that $f(x) < f(x_0) + \epsilon$ ($f(x) > f(x_0) - \epsilon$) whenever $|x - x_0| < \delta$.*

**Definition A.0.2** *A family of real-valued functions $\{f_\alpha\}_\alpha$ is equicontinuous if given $\epsilon > 0$ and $x_0$ there exists $\delta > 0$ such that*

$$|f_\alpha(x) - f_\alpha(x_0)| < \epsilon$$

*whenever $|x - x_0| < \delta$, for all $f_\alpha$.*

### Spaces of functions

Denote by $L^0(\Omega)$ the space of all Lebesgue measurable functions on $\Omega \subseteq \mathbb{R}^n$. The $L^p$ spaces, to be defined in what follows, are contained in $L^0$.

**Definition A.0.3** *Let $\Omega$ be a measurable set, $f \in L^0(\Omega)$ and*

$$||f||_p = \begin{cases} \left(\int_\Omega |f(t)|^p dt\right)^{1/p}, & \text{if } 0 < p < \infty \\ ess\ sup\,|f|, & \text{if } p = \infty \end{cases} . \tag{A.1}$$

*The space $L^p(\Omega)$ is the collection of the equivalence classes of all Lebesgue measurable functions such that*

$$||f||_p < \infty$$

*and equivalence means $f \sim g$ iff $f = g$ a.e.*

The *ess sup* is the *essential supremum*,

$$\text{ess sup}\, f = \inf\{c \in \overline{\mathbb{R}} : \mu\{\omega : f(\omega) > c\} = 0\}$$

that is, the smallest value $c$ such that $f \leq c$ a.e. Hence $||f||_\infty < \infty$ means that $f$ is bounded except on a set of measure zero.

If $\Omega$ is a measurable set, we say that $f$ is *Lebesgue integrable* if

$$\int_\Omega |f(t)|dt < \infty.$$

**Definition A.0.4** *(See e.g. Aubin and Cellina (1984)) A function $f : [a,b] \to \mathbb{R}$ is called* absolutely continuous *if, given $\epsilon > 0$, there exists $\delta > 0$ such that for every countable collection of disjoint subintervals $[a_k, b_k]$ of $[a,b]$ such that*

$$\sum (b_k - a_k) < \delta$$

*we have*

$$\sum (f(b_k) - f(a_k)) < \epsilon.$$

Equivalently, we can say that a function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if it has a derivative almost everywhere (that is, except on a set of measure zero) and

$$f(x) - f(a) = \int_a^x f'(s)\, ds.$$

This result is stated in the Lebesgue Theorem:

**Theorem A.0.5** *(See e.g. Hönig (1977)) a) Let $g \in L^1([a,b])$ and*

$$f(x) = \int_a^x g(t)dt, \quad x \in [a,b].$$

*Then $f$ is absolutely continuous and $f' = g$ a.e.*

*b) Let $f : [a,b] \in \mathbb{R}$ be an absolutely continuous function. Then $f'$ is integrable in $[a,b]$, that is, $f' \in L^1([a,b])$, and*

$$f(x) - f(a) = \int_a^x f'(t)\, dt, \quad x \in [a,b].$$

The set of all absolutely continuous functions $f : [a,b] \to \mathbb{R}^n$ is denoted by $\mathcal{AC}([a,b];\mathbb{R}^n)$. The notation $\mathcal{AC}([a,b],\mathbb{R}^n)$ and $L^p([a,b];\mathbb{R}^n)$ stand for the generalization of these spaces from codomain $\mathbb{R}$ to $\mathbb{R}^n$. Derivative and integral are calculated term-by-term on the n-dimensional vector. A subset of $\mathcal{AC}([a,b],\mathbb{R}^n)$ that will be used is

$$Z([a,b],\mathbb{R}^n) = \{x(\cdot) \in C([a,b];\mathbb{R}^n) : x'(\cdot) \in L^\infty([0,T];\mathbb{R}^n)\}$$

and will be denoted $Z_T(\mathbb{R}^n)$ when $[a,b] = [0,T]$.

ERRATA nos 2 exemplares da tese da aluna Luciana Takata Gomes:

Em folha de Assinaturas da Banca (página v):

| Onde se lê: | Leia-se: |
|---|---|
| ROSANA SUELI DA MOTA JAFELICE | ROSANA SUELI DA MOTTA JAFELICE |

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