Alguns resultados em partições e teoria de códigos

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Alguns resultados em teoria de partições e teoria de códigos

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Resumo

Esta tese é uma coletânea de trabalhos feitos pelo candidato. Importantes ferramentas combinatórias são utilizadas, dentre as quais: funções geradoras, q-cálculo, várias propriedades de seqüencias de números inteiros, etc; todas direcionadas para a teoria aditiva dos números (teoria de partições) e teoria de códigos.

A tese consiste de seis trabalhos: três deles tratam de aspectos combinatóriais (interpretações em termos de partições) de identidades do tipo Rogers-Ramanujan e onde várias seqüências de números inteiros aparecem.

Um trabalho onde uma conjectura sobre transformação de Hankel e seqüencias de Catalan e Fibonacci foi provada.

Um trabalho onde uma construção combinatória de uma classe de low-density parity-check códigos é apresentada. Neste trabalho demonstra-se também uma interessante conexão entre uma seqüencia de números inteiros, definida por Odlyzko e Stanley, e esta classe de códigos.

O último trabalho trata o problema de determinar a capacidade de canal de um sistema óptico usando um método numérico.

Abstract

This thesis consists of the publications done by the candidate. In these publications we have used many combinatorial tools including: generating functions, q-calculus, various properties of sequences of integer numbers etc. were used in the theory of partitions and the coding theory.

The thesis consists of six papers: three of them take into consideration combinatorial aspects (interpretations in terms of different classes of partitions) of identities of the Rogers-Ramanujan type, are explored and where different sequences of integer numbers naturally appear.

The fourth paper deals with Catalan sequence, discrete Hankel transform and Fibonacci sequence. A conjecture by Layman is proved.

In the fifth paper a construction of a class of Low-Density Parity-Check codes is proposed. An interesting connection between this class of codes and a sequence examined by Odlyzko and Stanley is also shown.

The last paper deals with the problem of determining Shannon capacity of an optical system by a numerical method.

Conteúdo

Introdução

Esta tese é uma coletânea de trabalhos feitos pelo candidato. Consiste de seis trabalhos: três já publicados, um aceito para publicação e dois submetidos.

Nestes trabalhos as ferramentas combinatórias: funções geradoras, q-cálculo, varias propriedades de seqüencias dos números inteiros, etc. foram usadas na teoria aditiva dos números (teoria de partições) e teoria da codificação.

Os trabalhos estão agrupados pelos assuntos tratados e não estão em ordem cronológica. Em princípio podem ser lidos independentemente um do outro. No que segue apresentamos uma pequena descrição de cada um deles.

1. "Fibonacci Numbers and Partitions" publicado no Fibonacci Quarterly, Vol. 41.3 2003. Este é o primeiro trabalho que o candidato fez com seu orientador. O trabalho é baseado em duas conjecturas obtidas por Santos em sua tese de doutorado . Ferramentas de q-cálculo foram aplicadas em identidades do tipo Rogers-Ramanujan para se obter três interpretações combinatórias para os números de Fibonacci em termos de partições restritas e uma em termos de caminhos reticulados. Novas formulas para números de Fibonacci foram obtidas também. A interpretação combinatoria em termos de caminhos reticulados sai usando estas formulas.

2. "Colored partitions and the Fibonacci sequence", submetido para a revista TEMA. Este pequeno trabalho pode ser visto como uma continuação de trabalho anterior. Aqui, temos uma nova interpretação dos números de Fibonacci, desta vez em termos de partições coloridas, mais uma nova fórmula para números de Fibonacci. Neste trabalho o candidato escreveu um pouco sobre historia e onipresença dos números de Fibonacci na natureza.

3. "Polynomial generalizations of the Pell sequence and the Fibonacci sequence" aceito para publicação na revista Fibonacci Quarterly. Este é o trabalho mais longo nesta tese. Neste trabalho as mesmas ferramentas foram usadas para se obter generalizações polinomiais de seqüencias de Pell e Fibonacci. As interpretações combinatórias foram apresentadas, junto com uma prova bijetiva de equivalência de duas classes de partições. Uma destes classes aparece na identidade sobre partições de Göllnitz-Gordon, um resultado clássico na teoria de partições.

4. "Catalan Numbers, the Hankel Transform, and Fibonacci Numbers", publicado no Journal of Integer Sequences, Vol. 5.1, 2002. Uma conjectura sobre transformação discreta de Hankel foi provada neste artigo. O trabalho despertou muita atenção. Vale mencionar que o candidato recebeu várias questões de diversos matemáticos

perguntando sobre detalhes e possíveis generalizações. O método aplicado usa teoria de polinômios ortogonais junto com funções geradoras e varias propriedades dos números de Fibonacci.

5. "High-rate girth-eight low-density parity-check codes on rectangular integer lattices", publicado em IEEE Transactions on Communications, Vol.:52.8, 2004. A construção combinatorial de uma classe de códigos low-density parity-check é apresentada neste trabalho. Estes códigos possuem grande girth -a propriedade desejada em caso de decodificação iterativa. Uma interessante e não esperada conexão entre uma seqüencia de números inteiros, definida por Odlyzko e Stanley, e esta classe de códigos foi provada no Teorema 3.1. Isso é a primeira aplicação dessa seqüencia na teoria de codificação.

6. "Achievable Information Rates for High-Speed Long-Haul Optical Transmission", submetido ao Photonics Technology Letters. Este trabalho foi realizado durante o período em que o candidato esteve na University of Arizona, convidado pelos co-autores do trabalho anterior. Este trabalho apresenta uma excursão fora de área de seqüencias de números inteiros e combinatória. O problema de determinar a capacidade de canal de um sistema óptico é tratado usando um método numérico. O assunto tratado é difícil e existe grande interesse nele. Espera-se que este seja o primeiro numa serie de trabalhos relacionados com este tópico.

Fibonacci Numbers and Partitions

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Abstract

In this paper we present four different combinatorial interpretations for the Fibonacci numbers. Three in terms of restricted partitions and one in terms of lattice paths.

1 Introduction

In a series of two papers ([6] and [7]) Slater gave a list of 130 identities of the Rogers-Ramanujan type. In [2] Andrews has introduced a two variable generalization in order to look for combinatorial interpretations for those identities. In [5] one of us, Santos, gave conjectures for explicit formulas for families of polynomial that can be obtained using Andrews method for 74 identities of the Slater's list.

In this paper we are going to prove the conjectures given by Santos in [5] for identities 94 and 99.

We show, also, that the family of polynomials $P_n(q)$ related to identity 94 given by

$$\begin{aligned} &P_0(q) = 1, \quad P_1(q) = 1 + q + q^2 \\ &P_n(q) = (1 + q + q^{2n})P_{n-1}(q) - qP_{n-2}(q) \end{aligned} \tag{1.1}$$

is the generating function for partitions into at most n parts in which every even smaller than the largest part appears at least once and that the family $T_n(q)$ related to identity 99 given by

$$\begin{aligned} &T_0(q) = 1, \;\; T_1(q) = 1 + q^2 \\ &T_n(q) = (1 + q + q^{2n})T_{n-1}(q) - qT_{n-2}(q) \end{aligned} \tag{1.2}$$

is the generating function for partitions into at most n parts in which the largest part is even and every even smaller than the largest appears at least once.

In what follows we denote the Fibonacci numbers by F_n where $F_0 = 0$; $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 2$, and use the standard notation

$$(A;q)_n = (1-A)(1-Aq)\dots(1-Aq^{n-1})$$

and

$$(A;q)_\infty = \prod_{n=0}^{\infty}(1-Aq^n), \quad |q| < 1.$$

We need also the following identities for the Gaussian polynomials

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m}\begin{bmatrix} n-1 \\ m-1 \end{bmatrix} \tag{1.4}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m}\begin{bmatrix} n-1 \\ m \end{bmatrix} \tag{1.5}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{array}{ll} \dfrac{(q;q)_m}{(q;q)_m(q;q)_{n-m}}, & \text{for} \quad 0 \leq m \leq n, \\ 0 \quad \text{otherwise} & \end{array} \tag{1.6}$$

2 The first family of polynomials

We consider now the two variable function associated to identity 94 of Slater [7] which is:

$$f_{94}(q,t)=\sum_{n=0}^{\infty}\frac{t^n q^{n^2+n}}{(t;q^2)_{n+1}(tq;q^2)_{n+1}} \tag{2.1}$$

From this we have that

$$(1-t)(1-tq)f_{94}(q,t)=1+tq^2 f_{94}(q;tq^2)$$

and in order to obtain a recurrence relation from this functional equation we make the following substitution

$$f_{94}(q,t)=\sum_{n=0}^{\infty}P_n t^n.$$

Now we have:

$$(1-t)(1-tq)\sum_{n=0}^{\infty}P_n t^n=1+tq^2\sum_{n=0}^{\infty}P_n(tq^2)^n$$

which implies

$$\sum_{n=0}^{\infty}P_n t^n-\sum_{n=1}^{\infty}P_{n-1}t^n-\sum_{n=1}^{\infty}qP_{n-1}t^n+\sum_{n=2}^{\infty}qP_{n-2}t^n=1+\sum_{n=1}^{\infty}q^{2n}P_{n-1}t^n.$$

From this last equation it is easy to see that

$$\begin{aligned} &P_0(q)=1\ ;\ P_1(q)=1+q+q^2\\ &P_n(q)=(1+q+q^{2n})P_{n-1}(q)-qP_{n-2}(q). \end{aligned} \tag{2.2}$$

Santos gave in [5] a conjecture $C_n(q)$, for an explicit formula for this family of polynomials:

$$C_n(q)=\sum_{j=-\infty}^{\infty}q^{15j^2+4j}\begin{bmatrix}2n+1\\ n-5j\end{bmatrix}-\sum q^{15j^2+14j+3}\begin{bmatrix}2n+1\\ n-5j-2\end{bmatrix}. \tag{2.3}$$

In our next theorem we prove that this conjecture is true.

Theorem 2.1. The family $P_n(q)$ given in (2.2) is equal to $C_n(q)$ given in (2.3).

Proof. Considering that $C_0(q)=1$ and $C_1(q)=1+q+q^2$ we have to show that

$$C_n(q)=(1+q+q^{2n})C_{n-1}(q)-qC_{n-2}(q) \quad \text{that is:}$$

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\begin{bmatrix} 2n+1 \\ n-5j-2 \end{bmatrix}$$
$$=(1+q+q^{2n})\left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix}\right)$$
$$-q\left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}\right) \qquad (2.4)$$

If we apply (1.4) in each expression on the left side of (2.4) we get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\begin{bmatrix} 2n \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1}\begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix}$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\begin{bmatrix} 2n \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n}\begin{bmatrix} 2n \\ n-5j-3 \end{bmatrix}$$

Applying now (1.5) to each sum in the expression above and replacing it in (2.4) we get after some cancellations

$$\sum_{j=-\infty}^{\infty} q^{15j^2+j+n}\begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+4}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n}\begin{bmatrix} 2n-1 \\ n-5j-4 \end{bmatrix}$$
$$=\sum_{j=-\infty}^{\infty} q^{15j^2+4j+1}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4}\begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix}$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2+4j+1}\begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4}\begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}. \qquad (2.5)$$

Considering the right side of the last expression and applying (1.4) on the first two sums we get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+1+n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+4j+1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+4} \begin{bmatrix} 2n-3 \\ n-5j-4 \end{bmatrix}$$

Applying now (1.5) on the first and third sums on this last expression and making some cancellations we have that the right side of (2.5) is equal to:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-j+n} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+9j+1+n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+9j+j+1+n} \begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+19j+6+n} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$

If we take now the left side of (2.5) and apply (1.4) to all sums we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-j+n} \begin{bmatrix} 2n-2 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$
$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+9j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+14j+n+6} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+24j+2n+9} \begin{bmatrix} 2n-2 \\ n-5j-5 \end{bmatrix} \quad (2.6)$$

Applying now (1.5) on the first and fifth sums of this last expression and making cancellations with the sums from the right side given in (2.6) we are left with:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-6j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+\sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2}\left[\begin{array}{c}2n-2\\ n-5j-3\end{array}\right]-\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1}\left[\begin{array}{c}2n-3\\ n-5j-2\end{array}\right]$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2+14j+2n+2}\left[\begin{array}{c}2n-2\\ n-5j-3\end{array}\right]-\sum_{j=-\infty}^{\infty} q^{15j^2+24j+2n+9}\left[\begin{array}{c}2n-2\\ n-5j-5\end{array}\right]$$

Observing that the third sum cancels the fifth and replacing j by $j+1$ in the last sum we get after using (1.4)

$$\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n+1}\left[\begin{array}{c}2n-2\\ n-5j-1\end{array}\right]-\sum_{j=-\infty}^{\infty} q^{15j^2+4j+2n-1}\left[\begin{array}{c}2n-3\\ n-5j-2\end{array}\right]$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2-j+3n-2}\left[\begin{array}{c}2n-3\\ n-5j-1\end{array}\right]$$

which is identically zero by (1.5) completing the proof. □

Next we make a few observations regarding the combinatorics of $P_N(q)$ given in (2.2). Knowing that $P_N(q)$ is the coefficient of t^N in (2.1) that is:

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(1-t)(tq^2;q^2)_n(tq;q^2)_{n+1}}$$

and considering that $n^2+n = 2+4+\cdots+2n$ we can see that the coefficient of t^N in

$$\frac{t^n q^{n^2+n}}{(tq^2;q^2)_n(tq;q^2)_{n+1}}$$

is the generating function for partitions into exactly N parts in which every even smaller than the largest part appears at least once. Because of the factor $(1-t)$ in the denominator we have proved the following theorem:

Theorem 2.2: $P_N(q)$ is the generating function for partitions into at most N parts in which every even smaller than the largest part appears at least once.

To see, now, the connection between the family of polinomials $P_N(q)$ and the Fibonacci numbers we observe first that if we replace q by 1 in (2.2) we have

$$P_0(1) = 1;\ P_1(1) = 3$$
$$P_n(1) = 3P_{n-1}(1) - P_{n-2}(1)$$

and that for the Fibonacci sequence F_n we have also that $F_2 = 1;\ F_4 = 3$ and

$$F_{2n+2} = 3F_{2n} - F_{2n-2}$$

which allow us to conclude that

$$C_n(1) = P_n(1) = F_{2n+2}$$

and from these considerations we have proved the following:

Theorem 2.3: The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once is equal to F_{2N+2}.

The family given in (2.2) has also an interesting property at $q = -1$. At this point we have

$$P_0(-1) = 1;\ P_1(-1) = 1$$
$$P_n(-1) = P_{n-1}(-1) + P_{n-2}(-1)$$

which tell us that for $q = -1$ we have all the Fibonacci numbers, i.e. $P_n(-1) = F_{n+1}$. In order to be able to see what happens combinatorially at -1 we have to observe that when we change q by $-q$ in (2.1) the only term that changes is $(tq; q^2)_{n+1}$ and that now the coefficient of t^N is going to be just the number of partitions as described in Theorem 2.3 having an even number of odd parts minus the number of partitions of that type with and odd number of odd parts. We state this in our next theorem.

Theorem 2.4. The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once and having and even number of odd parts minus the number of those with an odd number of odd parts is equal to F_{N+1}.

In the table (2.1) we present, for a few values of n, all the results proved so far. The first column has n, the second the partitions described in theorem 2.4 with and even number of odd parts and the third column those with an odd number of odd parts. The fourth column has F_{2n+2} which is the total number of partitions in columns 2 and 3 and the fifth column has the difference between the number of partitions on the second and third column which is F_{n+1}.

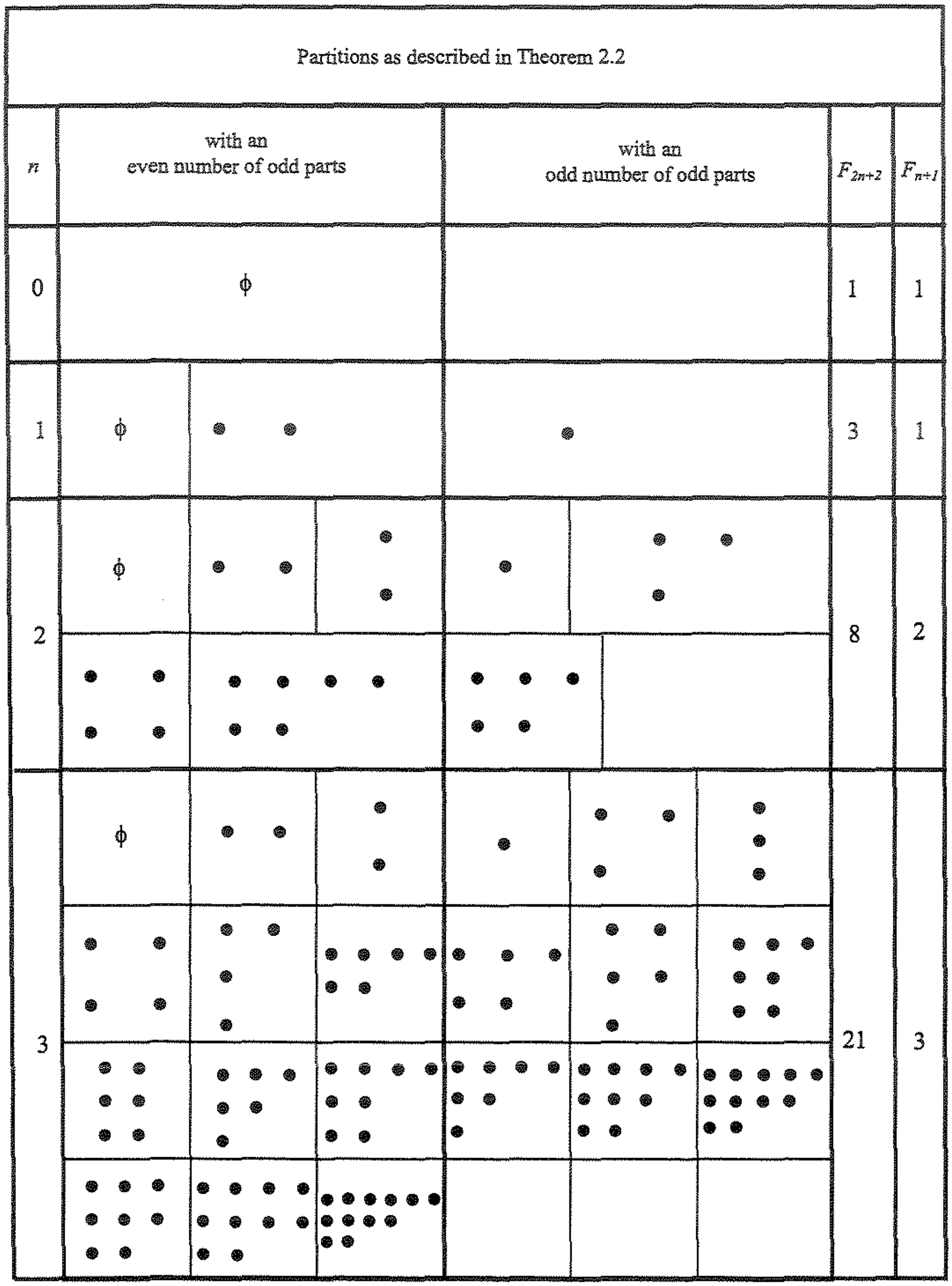
Partitions as described in Theorem 2.2
n
with an even number of odd parts
with an odd number of odd parts
F_{2n+2}
F_{n+1}
0
ϕ
1
1
1
ϕ
3
1
2
ϕ
8
2
3
ϕ
21
3

Table 2.1

3 The second family of polynomials

Now we consider the two variable function given in Santos [5] associated to identity 99 of Slater [7] which is:

$$f_{99}(q,t)=\sum_{n=0}^{\infty}\frac{t^n q^{n^2+n}}{(t;q^2)_{n+1}(tq;q^2)_n} \tag{3.1}$$

From this we can get

$$(1-t)(1-tq)f_{99}(q,t)=1-tq+tq^2 f_{99}(q,tq^2)$$

from which we obtain in a way similar to the one used to get (2.2) the following family of polynomials

$$\begin{aligned} &T_0(q)=1;\ T_1(q)=1+q^2 \\ &T_n(q)=(1+q+q^{2n})T_{n-1}(q)-qT_{n-2}(q) \end{aligned} \tag{3.2}$$

As for the family (2.2) Santos gave in [5] a conjecture for an explicity formula for (3.2) which is

$$B_n(q)=\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix} \tag{3.3}$$

The proof for this conjecture is given in the next theorem.

Theorem 3.1. The family $T_n(q)$ given in (3.2) is equal to $B_n(q)$ given in (3.3)

Proof. Considering that $B_0(q)=1$ and $B_1(q)=1+q^2$ we have to show that
$B_n(q)=(1+q+q^{2n})B_{n-1}(q)-qB_{n-2}(q)$ which is:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n+1 \\ n-5j \end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n+1 \\ n-5j-1 \end{bmatrix}$$

$$=(1+q+q^{2n})\left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}-\sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}\right)$$

$$-q\left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix}\right) \tag{3.4}$$

We apply (1.4) on each sum on the left to get

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+2}\begin{bmatrix} 2n \\ n-5j-2 \end{bmatrix}$$

Applying, now, (1.5) in all sums we obtain:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2-3j}\begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix}$$
$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+2}\begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$

Replacing this in (3.4) and making cancellations we are left with:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n}\begin{bmatrix} 2n-1 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+3}\begin{bmatrix} 2n-1 \\ n-5j-3 \end{bmatrix}$$
$$= \sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix} 2n-1 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix} 2n-1 \\ n-5j-2 \end{bmatrix} \tag{3.5}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+2j+1}\begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2}\begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix}$$

Applying (1.4) on the first two sums on the right side of this last expression we get for that side:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j+1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+2j+1} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2} \begin{bmatrix} 2n-3 \\ n-5j-3 \end{bmatrix}$$

Using (1.5) on the first and third sums we get after cancellations

$$\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+n+1} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+2+n} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix}$$

Applying (1.4) in all sums on the left side of (3.5) and making cancellations with the corresponding sums on the right we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-2 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$
$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+12j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+18j+2n+4} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix}$$
$$= \sum_{j=-\infty}^{\infty} q^{15j^2-3j+n} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+3j+n} \begin{bmatrix} 2n-3 \\ n-5j-2 \end{bmatrix}$$

Using (1.5) on the first and fourth sums on the LHS we get:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-8j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+2n-1} \begin{bmatrix} 2n-2 \\ n-5j-1 \end{bmatrix}$$

$$+\sum_{j=-\infty}^{\infty} q^{15j^2+12j+2n+2} \begin{bmatrix} 2n-2 \\ n-5j-3 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2-2j+2n-1} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} - \sum_{j=-\infty}^{\infty} q^{15j^2+18j+5} \begin{bmatrix} 2n-2 \\ n-5j-4 \end{bmatrix} = 0$$

Replacing j by $j-1$ in the last sum and using (1.3) that sum cancels with the third.

If we replace j by $-j$ in the fourth sum using (1.3) and subtract from the second by (1.4) we get finally:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-8j+2n} \begin{bmatrix} 2n-3 \\ n-5j \end{bmatrix} + \sum_{j=-\infty}^{\infty} q^{15j^2-3j+3n-2} \begin{bmatrix} 2n-3 \\ n-5j-1 \end{bmatrix}$$
$$-\sum_{j=-\infty}^{\infty} q^{15j^2+8j+2n} \begin{bmatrix} 2n-2 \\ n-5j-2 \end{bmatrix} = 0$$

To see that this expression is, in fact, identically zero we apply (1.4) on the first two sums replacing j by $-j$ and using (1.3) on the result which completes the proof.

Considering that $T_N(q)$ is the coefficient of t^N in the sum

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2+n}}{(1-t)(tq^2;q^2)_n(tq;q^2)_n}$$

and observing again that $n^2+n = 2+4+\cdots+2n$ we see that the coefficient of t^N in

$$\frac{t^n q^{n^2+n}}{(tq^2;q^2)_n(tq;q^2)_n}$$

is the generating function for partitions into exactly N parts in which the largest part is even and every even smaller the largest part appears at least once. From the presence of the factor $(1-t)$ in the denominator we have proved the following theorem:

Theorem 3.2. $T_n(q)$ is the generating function for partitions into at most N parts in which the largest part is even and every even smaller than the largest appears at least once.

Replacing now q by 1 in (3.2) we get

$$T_0(1) = 1;\ T_1(1) = 2$$
$$T_n(1) = 3T_{n-1}(1) - T_{n-2}(1)$$

But for F_n we have

$$F_1 = 1;\ F_3 = 2$$
$$F_{2n+1} = 3F_{2n-1} - F_{2n-3}$$

which allow us to conclude that

$$B_n(1) = T_n(1) = F_{2n+1}$$

and by these results we have proved

Theorem 3.3. The total number of partitions into at most N parts in which the largest part is even and every even smaller than the largest part appears at least once is equal to F_{2n+1}.

For family (3.2) we have also that, at $q = -1$, we get all the Fibonacci numbers $F_n,\ n \geq 2$.

$$T_0(-1) = 1;\ T_1(-1) = 2$$
$$T_n(-1) = T_{n-1}(-1) + T_{n-2}(-1)$$

i.e., $T_n(-1) = F_{n+2},\ n \geq 0$.

If we make the same observation that have made for the first family of polynomials regarding the combinatorial interpretation at $q = -1$ we have proved the following result:

Theorem 3.4. The total number of partitions into at most N parts in which the largest part is even and every even smaller than the largest part appears at least once and having an even number of odd parts minus the number of those with an odd number of odd parts is equal to F_{N+2}.

In the table (3.1) we present, for a few values of n, all the results proved in this section. The first column has n, the second the partitions described in theorem 3.3 with and even number of odd parts and the third column those with an odd number of odd parts. The fourth column has F_{2n+1} which is the total number of partitions in columns 2 and 3 and the fifth column has the difference between the number of partitions on the second and third column which is F_{n+2}.

Partitions as described in Theorem 3.3							
n	with an even number of odd parts			with an odd number of odd parts		F_{2n+1}	F_{n+2}
0	ϕ					1	1
1	ϕ	• •				2	2
2	ϕ	• •	• • • •	• • •		5	3
	• • • • • •						
3	ϕ	• •	• • • •	• • •	• • • • •	13	5
	• • • •	• • • • • •	• • • • • •	• • • • • • •	• • • • • • • • •		
	• • • • • • • •	• • • • • • • • • •	• • • • • • • • • • • •				

Table 3.1

4 A formula for F_n

Using the fact that the Gaussian polynomias given in (1.6) are q-analogue of the binomial coefficient, i.e., that

$$\lim_{q\to 1}\left[\begin{array}{c} n \\ m \end{array}\right] = \binom{n}{m}$$

we may take the limits as q approaches 1 in (2.3) and (3.3) to get

$$\begin{aligned}\lim_{q\to 1} C_n(q) &= \lim_{q\to 1}\left(\sum_{j=-\infty}^{\infty} q^{15j^2+4j}\left[\begin{array}{c} 2n+1 \\ n-5j \end{array}\right] - \sum_{j=-\infty}^{\infty} q^{15j^2+14j+3}\left[\begin{array}{c} 2n+1 \\ n-5j-2 \end{array}\right]\right) \\ &= \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2}\right] = C_n(1)\end{aligned}$$

and

$$\begin{aligned}\lim_{q\to 1} B_n(q) &= \lim_{q\to 1}\left(\sum_{j=-\infty}^{\infty} q^{15j^2+2j}\left[\begin{array}{c} 2n+1 \\ n-5j \end{array}\right] - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+1}\left[\begin{array}{c} 2n+1 \\ n-5j-1 \end{array}\right]\right) \\ &= \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1}\right] = B_n(1)\end{aligned}$$

But as we have observed

$$C_n(1) = F_{2n+2} \qquad \text{and} \qquad B_n(1) = F_{2n+1}$$

which tell us that

$$F_{2n+2} = \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2}\right] \tag{4.1}$$

and

$$F_{2n+1} = \sum_{j=-\infty}^{\infty}\left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1}\right] \tag{4.2}$$

5 Lattice path and Fibonacci

In this section we are going to show how to express the Fibonacci numbers in terms of lattice path.

In Narayana [4], lemma 4A one can find the following formula

$$|L(m,n;t,s)| = \sum_{j=-\infty}^{\infty} \left[\binom{m+n}{m-k(t+s)} - \binom{m+n}{n+k(t+s)+t} \right] \quad (4.3)$$

which give the total number of lattice paths from the origin to (m,n) not touching the lines $y = x - t$ and $y = x + s$.

But considering that we can write (4.1) and (4.2) as follows

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{n+(n+1)}{n-j(2+3)} - \binom{n+(n+1)}{n+1+j(2+3)+2} \right] \quad (4.4)$$

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{n+(n+1)}{n-j(1+4)} - \binom{n+(n+1)}{(n+1)+j(1+4)+1} \right] \quad (4.5)$$

we can conclude just by comparing (4.4) and (4.5) with (4.3) that the following theorem holds:

Theorem 5.1. F_{2n+i} is the number of lattice paths from the origin to $(n, n+1)$ not touching the line $y = x - i$ and $y = x + 5 - i$, where $i = 1, 2$.

References

[1] G.E. Andrews, "The Theory of Partitions", *Encyclopedia of Math. and Its Applications* (G.-C. Rota, ed.), Addison-Wesley, Reading, 1976, vol. 2 (Reprinted: Cambridge University Press, Cambridge, 1984).

[2] G.E. Andrews, "*Combinatorics and Ramanujan's "lost" notebook*", London, 1985, pp. 1-23.

[3] Mondek, P. "*Identidades de Slater: Novas identidades e interpretações combinatórias*", Tese de Doutorado, IMECC-UNICAMP, 1997.

[4] T.V. Narayana, "*Lattice path combinatorics with statistical applications*", Mathematical Expositions nº 23, University of Totonto Press, 1976.

[5] J.P.O. Santos, "*Computer Algebra and Identities of the Rogers-Ramanujan Type*", Ph.D. thesis, Pennsylvania State University, 1991.

[6] L.J. Slater, "A new proof of Rogers' transformations of infinite series", *Proc. London Math. Soc.* **53**(2)(1951), 460-475.

[7] L.J. Slater, "Further identities of the Rogers-Ramanujan type", *Proc. London Math. Soc.* **54**(2)(1952), 147-167.

Colored partitions and the Fibonacci sequence

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Abstract

We present interesting combinatorial interpretations for the Fibonacci numbers in terms of colored partitions obtained by using finite versions of two identities of the Rogers-Ramanujan type. New formula for the Fibonacci numbers is also given.
Key words: Partitions, Fibonacci numbers, Rogers-Ramanujan identities.

1 Introduction

Considered for the first time in a modest example for the facility of calculation in positional number system (*Liber abaci* 1202.), the Fibonacci numbers showed to be intrinsic in nature (*phyllotaxis*), and omnipresent in arts (poetry, architecture, etc.).

Many properties of these numbers are known. They appear in numerical mathematics (Adby[1]), game theory (Tosić[14]), as well as in combinatorics and partition theory, where there are interpretations in terms of compositions, for example:

The number of compositions of n in which no 1's appear is F_{n-1} (Andrews[2]).

Or in terms of partitions, one interpretation obtained by the authors is:

The total number of partitions into at most N parts in which every even smaller than the largest part appears at least once is equal to F_{2N+2} (Santos&Ivković[10]).

In this paper we present two new combinatorial interpretations in terms of colored partitions based on identities number 63 and 62 in the Slater's list of identities of the Rogers-Ramanujan type [13]. We use a method introduced by Andrews[5], and used by Santos[9] to obtain finite versions of Rogers-Ramanujan type identities. Basic tools are presented in the following section.

2 Basic definitions

We start with *Gaussian polynomials* (see Andrews[2]), that are q-analog of the binomial coefficients:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_n}{(q)_m(q)_{n-m}} & \text{if} \quad 0 \leq m \leq n \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where

$$(a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$$

n a nonnegative integer.

When dealing with the expression

$$(1+x+x^2)^n \tag{1.2}$$

we call the coefficients of x^j in the expanded form of (1.2) *the trinomial coefficients.*

It is easy to show that if

$$(1+x+x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{j+n} \tag{1.3}$$

then

$$\binom{n}{j}_2 \quad = \quad \sum_{h\geq 0} \frac{n!}{h!(h+j)!(n-j-2h)!} \tag{1.4}$$

$$= \sum_{h\geq 0}(-1)^h \binom{n}{h}\binom{2n-2h}{n-j-h} \tag{1.5}$$

also

$$\binom{n}{j}_2 = \binom{n}{-j}_2 \tag{1.6}$$

and

$$\binom{n}{j}_2 = \binom{n-1}{j-1}_2 + \binom{n-1}{j}_2 + \binom{n-1}{j+1}_2 \tag{1.7}$$

The following expressions (Andrews & Baxter[7]) are q-analogs of the trinomial coefficient in the same way that the Gaussian polynomial is a q-analog of the binomial coefficient, that is, the limit of each one of them when q approaches 1 is equal to the trinomial coefficient given by (1.4) and (1.5).

$$T_0(m,A,q) = \sum_{j=0}^{m}(-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}, \tag{1.8}$$

$$T_1(m,A,q) = \sum_{j=0}^{m}(-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{1.9}$$

There are the following Pascal-triangle type relations:

$$\begin{aligned} T_1(m,A,q) &= T_1(m-1,A,q) + q^{m+A}T_0(m-1,A+1,q) \\ &\quad + q^{m-A}T_0(m-1,A-1,q) \end{aligned} \tag{1.10}$$

$$\begin{aligned} T_0(m,A,q) &= T_0(m-1,A-1,q) + q^{m+A}T_1(m-1,A,q) \\ &\quad + q^{2m+2A}T_0(m-1,A+1,q) \end{aligned} \tag{1.11}$$

It is also valid (Andrews & Baxter[7]):

$$\begin{aligned} &T_1(m,A,q) - q^{m-A}T_0(m,A,q) - T_1(m,A+1,q) \\ &+ q^{m+A+1}T_0(m,A+1,q) = 0 \end{aligned} \tag{1.12}$$

¿From (1.8) and (1.9) we can see that

$$T_0(m,A,q) = T_0(m,-A,q) \tag{1.13}$$

$$T_1(m,A,q) = T_1(m,-A,q) \tag{1.14}$$

where we have used the following property of the Gaussian polynomials which follows from the definition (1.1):

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ n-m \end{bmatrix}. \tag{1.15}$$

3 The Main theorem

3.1 Identity number 63

We start this section by considering identity number 63 in the Slater's list [13] that is:

$$(q;q^5)_\infty(q^4;q^5)_\infty(q;q^{10})_\infty(q^9;q^{10})_\infty(-q;q)_\infty = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\frac{3}{2}n(n+1)}}{(q;q)_{2n}} \tag{2.1}$$

On the left side we introduce a new variable "t" in the following way:

$$\begin{aligned} f_{63}(t,q) &= \textstyle\sum_{n=0}^{\infty} \frac{(-t;q)_{n+1} t^{3n} q^{\frac{3}{2}n(n+1)}}{(t^2;q)_{2n+1}} && (2.2)\\ &= \textstyle\sum_{n=0}^{\infty} \frac{(-tq;q)_n t^{3n} q^{\frac{3}{2}n(n+1)}}{(1-t)(t^2q;q)_{2n+1}} && (2.3) \end{aligned}$$

in this sum the factor:

$$(1+tq)\dots(1+tq^n)t^n q^{1+2+\dots+n} \tag{2.4}$$

is clearly the generating function for partitions where every integer less then or equal to the largest part appears at least once and at most twice. We shall call them the *green parts.*

The factor

$$\frac{t^{2n} q^{2+4+\dots+2n}}{(1-t)(1-t^2q)\dots(1-t^2q^{2n+1})} \tag{2.5}$$

is the generating function for partitions where every even part less than or equal to the largest part appears at least once. These are the *yellow parts.*

Notice that the largest yellow part that is even is equal to twice the largest green part.

Now we return to (2.2). It is valid:

$$(1-t)(1-t^2q)f_{63}(q,t) = 1 + t^3q^3f_{63}(q,tq) \tag{2.6}$$

Taking $f_{63}(t,q) = \sum_{n=0}^{\infty} P_n(q)t^n$ from (2.6) we have:

$$P_0 = P_1 = 1; P_2 = 1+q$$
$$P_n = P_{n-1} + qP_{n-2} - (q-q^n)P_{n-3} \tag{2.7}$$
$$\tag{2.8}$$

Santos in [8] conjectured:

$$P_n = \sum_{j=-\infty}^{\infty} q^{\frac{15}{2}j^2+\frac{7}{2}j} T_1(n, 5j+1, q^{\frac{1}{2}}) - \sum_{j=-\infty}^{\infty} q^{\frac{15}{2}j^2+\frac{13}{2}j+1} T_1(n, 5j+2, q^{\frac{1}{2}}) \tag{2.9}$$

We prove this in the appendix.

Once having proved this theorem, we can make a connection between the green-yellow partitions and the Fibonacci numbers. Taking $q \to 1$ in (2.7) recurrent relation for the Fibonacci numbers appears. Thus, it is valid:

Theorem 1. *The total number of green-yellow partitions in which the number of green parts plus twice the number of yellow parts is smaller then or equal to N is equal to F_{n+1}.*

Here, of course F_n is the n-th Fibonacci number. Considering that T_1 is a q-analog of the trinomial coefficients we may take the limit in (2.9) when q approaches 1, getting a formula for the Fibonacci numbers:

$$F_N = \sum_{j=-\infty}^{\infty} \left(\binom{N}{5j+1}_2 - \binom{N}{5j+2}_2 \right) \tag{2.10}$$

3.2 Identity number 62

Identity number 62 in the Slater's list [13] is very similar:

$$(q^2;q^5)_\infty (q^3;q^5)_\infty (q^3;q^{10})_\infty (q^7;q^{10})_\infty (-q;q)_\infty = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{\frac{1}{2}n(3n+1)}}{(q;q)_{2n}} \tag{2.11}$$

The new parameter "t" is introduced in the similar way:

$$f_{62}(t,q) = \sum_{n=0}^{\infty} \frac{(-t;q)_{n+1} t^{3n} q^{\frac{1}{2}n(3n+1)}}{(t^2;q^2)_{2n+1}}. \tag{2.12}$$

The only difference is that a generating function for the yellow parts now is:

$$\frac{t^{2n} q^{1+3+\ldots+2n-1}}{(1-t)(1-t^2q)\ldots(1-t^2q^{2n+1})} \tag{2.13}$$

i.e., generating function for the partitions where every odd integer less than or equal to the largest part appears at least once.

Bijection between the partitions of the yellow-green type connected with identity 63 and those connected with identity 62 is direct.

Not surprisingly, when writing f_{62} in a form $\sum_{n=-\infty}^{\infty} Q_n t^n$ polynomials Q_n satisfy very similar recurrent relation as P_n:

$$\begin{aligned} Q_0 = Q_1 = 1; Q_2 = 1+q \\ Q_n = Q_{n-1} + qQ_{n-2} - (q - q^{n-1})Q_{n-3} \end{aligned} \tag{2.14}$$

Explicit formula for this family is

$$Q_n = \sum_{j=-\infty}^{\infty} q^{\frac{15}{2}j^2+\frac{1}{2}j} Q_1(n,5j,q^{\frac{1}{2}}) - \sum_{j=-\infty}^{\infty} q^{\frac{15}{2}j^2+\frac{11}{2}j+1} Q_1(n,5j+2,q^{\frac{1}{2}}) \tag{2.15}$$

Giving, at the end, the same formula (2.10) for the Fibonacci numbers.

Observation: Identities 62 and 63 from the Slater's list are equal to the identities number 46 and 44 respectively.

4 Appendix

Theorem 2. *The family of polynomials defined in (2.9) satisfies recurrent relation (2.7).*

Proof: To make calculation easier the base is changed by taking $q \to q^2$. We need to prove:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+7j}T_1(n,5j+1,q) - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+1}T_1(n,5j+2,q)$$

$$= \sum_{j=-\infty}^{\infty} q^{15j^2+7j}T_1(n-1,5j+1,q) - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+1}T_1(n-1,5j+2,q) \tag{A.1}$$

$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+7j+1}T_1(n-2,5j+1,q) - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+2}T_1(n-2,5j+2,q)$$

$$+(q^{2n}-q^2)\left(\sum_{j=-\infty}^{\infty} q^{15j^2+7j}T_1(n-3,5j+1,q) - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+1}T_1(n-3,5j+2,q)\right)$$

¿From now on all the polynomials are in base "q", so we omit to explicitly state it, i.e., $T_0(n-1,5j+2) = T_0(n-1,5j+2,q)$. After applying (1.10) on both sums on the left and some cancellations, the expression is:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+12j+n+1}T_0(n-1,5j+2) + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+n-1}T_0(n-1,5j)$$

$$- \sum_{j=-\infty}^{\infty} q^{15j^2+18j+4+n}T_0(n-1,5j+3) - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+n}T_0(n-1,5j+1)$$

$$= \sum_{j=-\infty}^{\infty} q^{15j^2+7j+1}T_1(n-2,5j+1) - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+2}T_1(n-2,5j+2)$$

$$+(q^{2n}-q^2)\left(\sum_{j=-\infty}^{\infty} q^{15j^2+7j}T_1(n-3,5j+1) - \sum_{j=-\infty}^{\infty} q^{15j^2+13j+1}T_1(n-3,5j+2)\right)$$

By making "$j \to -j$" and "$j \to j+1$" at the first sum on the left that sum cancels with third by (1.13). The two sums left on left side we transform by (1.12) so they are written using T_1 polynomials. That side becomes:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+3j}T_1(n-1,5j) - \sum_{j=-\infty}^{\infty} q^{15j^2+3j}T_1(n-1,5j) \tag{A.2}$$

Now we apply (1.10) on this two sums. Four resulting sums cancel by (1.12) leaving:

$$\sum_{j=-\infty}^{\infty} q^{15j^2-2j+n}T_0(n-2,5j-1) - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+n}T_0(n-2,5j+2) \quad \text{(A.3)}$$

By (1.11) this two sums can be written as:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+2j+n-1}T_0(n-3,5j) + \sum_{j=-\infty}^{\infty} q^{15j^2+7j+2n}T_1(n-3,5j+1)$$
$$+ \sum_{j=-\infty}^{\infty} q^{15j^2+12j+3n-3}T_0(n-3,5j+3) - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+n}T_0(n-3,5j+1)$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+13j+2n}T_1(n-3,5j+2) - \sum_{j=-\infty}^{\infty} q^{15j^2+2j+n-1}T_0(n-3,5j+3)$$

The second and the fourth sum from this expression cancel with corresponding sums on the right side of (A.1). The third and the sixth cancel between themselves.

Nw we return to the right side. By applying (1.10) on the first two sides of (A.1) and some cancellations that side becomes:

$$\sum_{j=-\infty}^{\infty} q^{15j^2+12j+n+1}T_0(n-3,5j+2) + \sum_{j=-\infty}^{\infty} q^{15j^2+2j+n-1}T_0(n-3,5j)$$
$$- \sum_{j=-\infty}^{\infty} q^{15j^2+18j+n+4}T_0(n-3,5j+3) - \sum_{j=-\infty}^{\infty} q^{15j^2+8j+n}T_0(n-3,5j+1)$$

The first sum cancels with the third. The second and the fourth cancel with the remaining sums from the left side, thus proving the identity.

References

[1] Adby, P.Rr, Dempster M.A.H., "*Introduction to Optimization Methods*", Chapman&Hall, London, 1974.

[12] Santos, J.P.O., Sills, A.V., "*q-Pell Sequences and Two Identities of V.S Lebesgue*", Discrete Mathematics, (1) 257 (2002), pp. 125-142.

[13] Slater, L.J., "*Further identities of the Rogers-Ramanujan type*", Proc. London Math. Soc. (2) 54 (1952), 147-167.

[14] R. Tosić, Kombinatorika (in serbian)
Univerzitet u Novom Sadu (1999), Novi Sad.

Polynomial Generalizations of the Pell sequence and the Fibonacci sequence

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1 Introduction

In this paper, in order to find polynomial generalizations and combinatorial interpretations for the Pell sequence, we consider the identities 36 and 34 of Slater[16] that are respectively:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^5;q^8)_\infty(q^3;q^8)_\infty(q^8;q^8)_\infty \quad (1.1)$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(q^7;q^8)_\infty(q^1;q^8)_\infty(q^8;q^8)_\infty \quad (1.2)$$

where

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$$

n a nonnegative integer.

These identities are the analytic counterparts of the Göllnitz-Gordon partition identities first found by Göllnitz[8] and then rediscovered by Gordon[7].

We start by following Andrews[2], to provide a polynomial generalization for the Pell sequence and a combinatorial interpretation for this sequence. We offer a bijection between the class of partitions that appear in the Göllnitz-Gordon identities and another class of partitions that can be obtained from the left side of (1.1). New formulas for the two related sequences are given. Details are in section 3.

A third polynomial generalization including a nice relation between the Pell numbers and the Fibonacci numbers is given in section 4.

In section 5, by making use of the Rogers-Ramanujan identities, two new combinatorial interpretations for the Fibonacci numbers are given.

2 Some definitions and results

The Pell numbers 1, 2, 5, 12,... defined by $a_0 = 1; a_1 = 2; a_n = 2a_{n-1} + a_{n-2}$ are the denominators of the sequence of rational numbers:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots \tag{2.1}$$

that are the continued fraction convergent to $\sqrt{2}$.

The Gaussian polynomials are defined as follows:

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} \frac{(q)_n}{(q)_m (q)_{n-m}} & \text{if } \ 0 \le m \le n \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

For more details see Andrews[1].

When dealing with the expression

$$(1 + x + x^2)^n \tag{2.3}$$

we call the coefficients of x^j in the expanded form of (2.3) the trinomial coefficients.

It is easy to show that if

$$(1 + x + x^2)^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^{j+n} \tag{2.4}$$

then

$$\binom{n}{j}_2 = \sum_{h \ge 0} (-1)^h \binom{n}{h} \binom{2n - 2h}{n - j - h} \tag{2.5}$$

The following expression (Andrews&Baxter[6]) is a q-analog of the trinomial coefficient in the same way that the Gaussian polynomial is a q-analog of the binomial coefficient, that is, its limit, when q approaches 1, is equal to the trinomial coefficient given by (2.6).

$$T_0(m, A, q) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix} \tag{2.6}$$

In order to condense the notation the following expression is defined

$$U(m, A, q) = T_0(m, A, q) + T_0(m, A+1, q) \tag{2.7}$$

3 Polynomial generalizations and combinatorial interpretations for the Pell sequence

3.1 A first polynomial generalization for the Pell sequence

We mentioned that, following Andrews[2] one can introduce a parameter t in the left hand side of (1.1),

$$f(q,t) = \sum_{n=0}^{\infty} \frac{(-tq;q^2)_n t^n q^{n^2}}{(t;q^2)_{n+1}} \tag{3.1}$$

From here a functional equation can be obtained:

$$(1-t)f(q,t) = 1 + (1+tq)tqf(q,tq^2). \tag{3.2}$$

Knowing that the coefficient of t^n in the expansion of (3.1) is a polynomial in q, i.e., that

$$f(q,t) = \sum_{n=0}^{\infty} P_n(q)t^n \tag{3.3}$$

it is easy to see that

$$\begin{aligned} &P_0(q) = 1 \\ &P_1(q) = 1+q \qquad \text{and} \\ &P_n(q) = (1+q^{2n-1})P_{n-1}(q) + q^{2n-2}P_{n-2} \end{aligned} \tag{3.4}$$

The whole procedure described above can be easily done with A. Sills' RRtools Maple package [15].

The family of polynomials (3.4) appears in Gordon[7]. $P_m(q)$ is interpreted as a generating function for partitions of the form $n = n_1 + n_2 \ldots + n_k$ where $n_1 \leq 2m-1$, $n_i \geq n_{i+1}+2$ and $n_i \geq n_{i+1}+3$ if n_i is even.

Here we concentrate on the alternative combinatorial interpretation. The equation (3.1) written in the following form:

$$f(q,t) = \frac{1}{1-t}\sum_{n=0}^{\infty}\frac{(1+tq)(1+tq^3)\ldots(1+tq^{2n-1})t^n q^{1+3+5+\cdots+2n-1}}{(1-tq^2)(1-tq^4)\ldots(1-tq^{2n})}$$

tells us that in this sum the coefficient of $t^N q^M$ is the total number of partitions of M into exactly N parts in which every odd less than or equal to the largest part appears at least once and at most twice.

By taking into consideration the factor $\frac{1}{(1-t)}$ we have proved the following theorem:

Theorem 3.1. $P_n(q)$ is the generating function for partitions into at most n parts in which every odd less than or equal to the largest part appears at least once and at most twice.

To see what is the relation between this family of polynomials and the Pell sequence we may replace q by 1 in (3.4). By doing this we get

$$\begin{aligned} &P_0(1) = 1 \\ &P_1(1) = 2 \\ &P_n(1) = 2P_{n-1}(1) + P_{n-2}(1) \end{aligned} \tag{3.5}$$

which is the Pell sequence given at the beginning of section 2.

From this observation we get the following combinatorial interpretation for the Pell sequence which we state as a theorem.

Corollary 3.2. The total number of partitions into at most n parts in which every odd less that or equal to the largest part appears at least once and at most twice is equal to the Pell number $P_n(1)$.

In [5] Andrews proved the following explicit formula for the family of polynomials given by (3.4):

$$P_n(q) = \sum_{j=-\infty}^{\infty} q^{16j^2+2j}U(n,8j) - \sum_{j=-\infty}^{\infty} q^{16j^2-14j+3}U(n,3-8j) \tag{3.6}$$

where $U(n, A) = U(n, A, q)$ given by (2.7).

Knowing that $P_n(1)$ is the Pell sequence, as we have seen in (3.5), we can get an explicit formula for this sequence by making use of formula (3.6)

$$P_n(1) = \lim_{q\to 1} P_n(q) =$$

$$\lim_{q\to 1} \sum_{j=-\infty}^{\infty} q^{16j^2+2j} U(n, 8j) - \sum_{j=-\infty}^{\infty} q^{16j^2-14j+3} U(n, 3-8j)$$

$$= \sum_{j=-\infty}^{\infty} \lim_{q\to 1} (T_0(n, 8j, q) + T_0(n, 8j+1, q) -$$

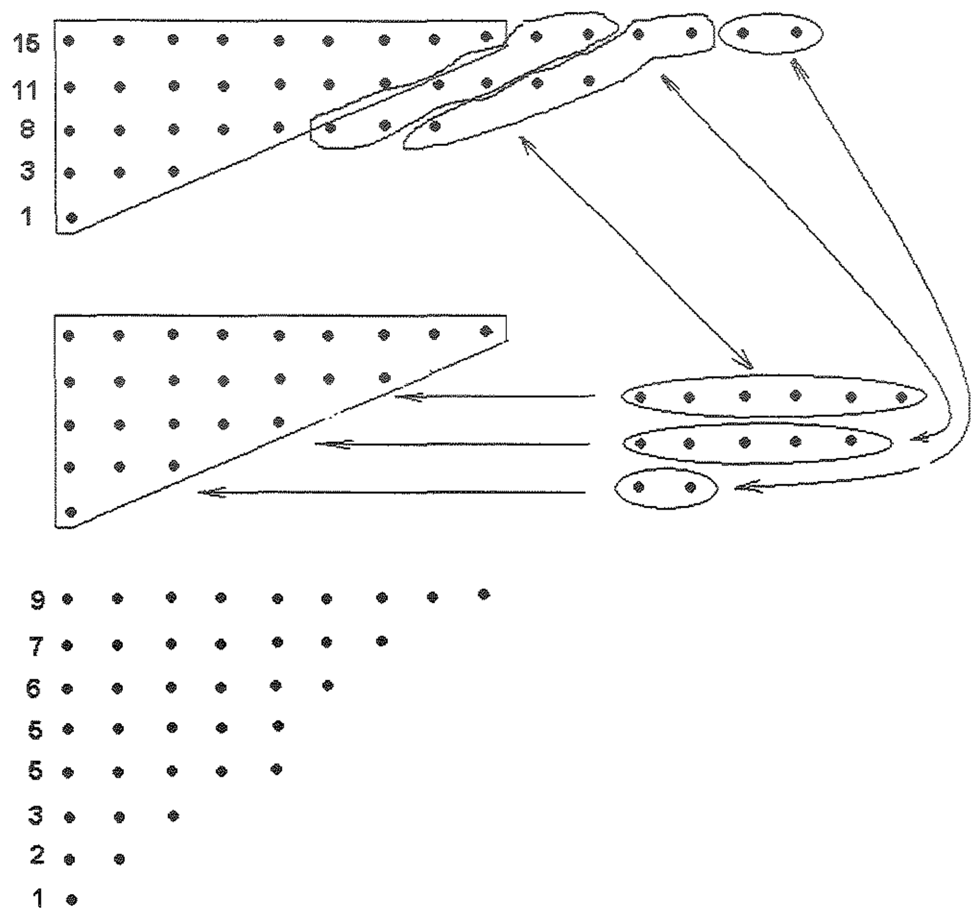
$$T_0(n, 3-8j, q) - T_0(n, 4-8j, q))$$

$$= \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j}_2 + \binom{n}{8j+1}_2 - \binom{n}{3-8j}_2 - \binom{n}{4-8j}_2 \right]$$

$$= \sum_{j=-\infty}^{\infty} \left[\binom{n+1}{8j+1}_2 - \binom{n+1}{8j+3}_2 \right]$$

It is natural to look for a bijection between the class of partitions defined by Gordon and the class appearing in theorem 3.1. It is sufficient to define a bijection that takes a partition of n where the biggest part is $2m-1$ or $2m-2$, and satisfies the conditions defined by Gordon into a partition of n in exactly m parts in which every odd less then or equal to the largest part appears at least once and at most twice. We give one as follows:

Take a partition from the class defined by Gordon. Let the number of parts of that partition be k. Let the biggest part be $2m-2$ or $2m-1$. Fix the number $2k-1$ from the biggest part, $2k-3$ from the second biggest part and proceed in the same manner until fixing one from the smallest part. Note that this is possible since $n_i \geq n_{i+1} + 2$ and that, for this reason and $n_i \geq n_{i+1}$ if n_i is even, if nothing or just one is left out from the biggest part, then nothing is left out from the second biggest part and similarly for the other parts. If one is left out we take it as a new part. See the illustration on the next page.



If the biggest part is $\geq 2k+1$ take two from the part of it that was not fixed, two from the second biggest part, and so on, until there is a part from which only one (or nothing) can be taken. If there is one, we take it. From the "taken" twos and possible one we make a new part for the new partition being formed. If there are still some parts that are left out we form another part in the same way. The partition obtained satisfies the conditions given in the theorem 3.1. We observe the following:

1. A partition with m parts is obtained. To see this note that the number left out after fixing $2k-1$ in the biggest part is minimally $2(m-k)-1$, so we are actually adding $m-k$ parts.

2. The biggest new-formed part is smaller then or equal to $2k$-so there is no need to add new odd parts.

3. By construction the new parts created form a non-increasing sequence, odd number can be formed only once -forming it twice would mean taking two ones from the same original part, which is impossible by construction. This argument also proves that the mapping defined by this procedure is injective.

The inversion mapping is defined similarly: Take a partition defined by conditions in the theorem 3.1. that has m parts. We let one copy of all odd parts fixed. Let the biggest of the fixed parts be $2k-1$. Now, each of the remaining parts is transformed in the following manner: The biggest remaining part is divided in a way that two is added to the biggest fixed part, two to the second biggest part, etc. If the number divided is odd we add one to some fixed part at the end. Note that according to the partition definition the biggest remaining part cannot be bigger than $2k$, so it can be divided among the fixed parts. The second biggest part (and all remaining) is divided in the same manner, always starting by adding two's to the biggest fixed part.

It is obvious that the resulting partition satisfies Gordon's conditions. Its biggest part is, by construction, smaller than or equal to $2k-1+2(m-k)$ and bigger than or equal to $2k-1+2(m-k)-1=2m-2$, the last -1 on the left side corresponding to the case when $m-k=1$ and the collected part is one.

3.2 A second polynomial generalization for the Pell sequence

By considering, now, a two variable function $f_{34}(q,t)$ associated with equation 34 of Slater[16] given in (1.2)

$$f_{34}(q,t) = \sum_{n=0}^{\infty} \frac{(-tq;q^2)_n t^n q^{n^2+2n}}{(t;q^2)_{n+1}} \tag{3.7}$$

and following the same steps used to get (3.2) we may obtain the functional equation

$$(1-t)f_{34}(q,t) = 1 + (1+tq)tq^3 f_{34}(q,tq^2) \tag{3.8}$$

and by replacing $f_{34}(q,t)$ with

$$\sum_{n=0}^{\infty} D_n(q)t^n$$

in (3.7) we can get

$$\begin{aligned} &D_0(q) = 1; \\ &D_1(q) = 1 + q^3 \\ &D_n(q) = (1+q^{2n+1})D_{n-1}(q) + q^{2n}D_{n-2}(q) \end{aligned} \tag{3.9}$$

To find a combinatorial interpretation for this family of polynomials we can write (3.7) in the following form:

$$\sum_{n=0}^{\infty} \frac{(-tq;q^2)_n t^n q^{n^2+2n}}{(t;q^2)_{n+1}} =$$

$$\frac{1}{(1-t)} \sum_{n=0}^{\infty} \frac{(1+tq)(1+tq^3)\dots(1+tq^{2n-1})t^n q^{(2+1)+(4+1)\dots(2n-2+1)+(2n+1)}}{(1-tq^2)(q-tq^4)\dots(1-tq^{2n})}$$

which tells us that in this sum the coefficient of $t^N q^M$ is the total number of partitions of M into exactly N parts in which the largest part is odd, appearing only once, and every odd smaller than the largest part and greater than or equal to 3 appears at least once and at most twice.

By considering the factor $1/(1-t)$ and that for $q = 1$

$$\begin{aligned} &D_0(1) = 1 \\ &D_1(1) = 2 \\ &D_n(1) = 2D_{n-1}(1) + D_{n-2}(1) \end{aligned} \tag{3.10}$$

which is the Pell sequence, we have proved the following:

Theorem 3.3. The total number of partitions into at most n parts in which the largest part is odd, greater than or equal to 3, appearing only once, and every odd smaller than the largest part and greater than or equal to 3 appears at least once and at most twice is equal to the Pell number $D_n(1)$.

For the family of polynomials (3.9) an explicit formula in terms of the q-trinomial coefficients can be also found in Andrews[5]:

$$D_n(q) = \sum_{j=-\infty}^{\infty} q^{16j^2+6j} U(n, 8j+1) - \sum_{j=-\infty}^{\infty} q^{16j^2-10j+1} U(n, -8j+2) \quad (3.11)$$

From which we have, again, the following formula for the Pell numbers:

$$\sum_{j=-\infty}^{\infty} \left[\binom{n}{8j+1}_2 + \binom{n}{8j+2}_2 - \binom{n}{-8j+2}_2 - \binom{n}{-8j+3}_2 \right]$$
$$= \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j+1}_2 - \binom{n}{8j+3}_2 \right] \quad (3.12)$$

This formula was proved in [14] by making use of an identity of Lebesgue that is also included in Slater's list.

It is an easy matter to give a bijection between the two distinct interpretations for the Pell sequence given in the corollary 3.2 and the theorem 3.3.

In the class of partitions described in the theorem 3.3 every odd part larger than or equal to 3 appears at least once and at most twice. We can subtract 2 from just one copy of each of those odd parts. By doing this we obtain a partition in the class described in theorem 3.2 It is clear that this procedure can be inverted. Note that now the largest part obtained is necessarily odd.

We illustrate this in the table 2 where in the first column we have the partitions as described by the corollary 3.2, and in the second one those described by the theorem 3.3.

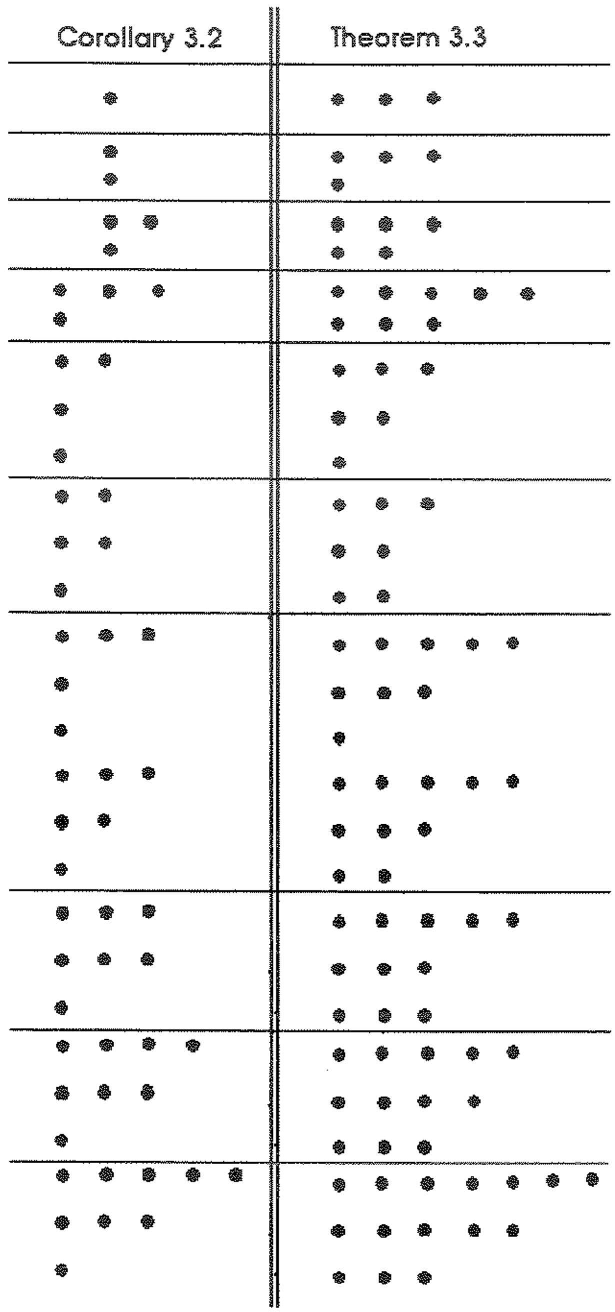

Corollary 3.2
Theorem 3.3

Table 2

We observe that to get the partitions in column one we took $n = 3$ in (3.4) getting

$$P_3(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + q^6 + q^7 + q^8 + q^9$$

and the ones in column two by taking $n = 3$ in (3.9) getting:

$$D_3(q) = 1 + q^3 + 2q^4 + q^5 + q^6 + q^7 + q^8 + q^9 + q^{10} + q^{11} + q^{12} + q^{15}$$

3.3 New formulas for two related sequences

Motivated by this formula it was possible to find and prove, by induction, the following two formulas: the first one gives us the sequence S_n of the partial sums for the Pell numbers and the second the numerators N_m of the sequence of rational numbers given in 2.1.

$$S_n = \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j+2}_2 - \binom{n}{8j+4}_2 \right] \tag{3.13}$$

where

$$S_1 = 1, \quad S_2 = 3, \quad S_n = 2S_{n-1} + S_{n-2} + 1; \quad n \geq 3 \tag{3.14}$$

and

$$N_n = \sum_{j=-\infty}^{\infty} \left[\binom{n}{8j}_2 - \binom{n}{8j+4}_2 \right] \tag{3.15}$$

where

$$N_1 = 1, \quad N_2 = 3, \quad N_n = 2N_{n-1} + N_{n-2}; \quad n \geq 3 \tag{3.16}$$

4 Fibonacci numbers and a third polynomial generalization for the Pell sequence

The following identity can be found in Göllnitz[8]:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{8n-1})(1-q^{8n-5})(1-q^{8n-6})} \tag{4.1}$$

Here we note that, by introducing the variable t in the following manner (again by use of Sills[15]),:

$$F(q,t) = \sum_{n=0}^{\infty} \frac{(-tq;q^2)_n t^n q^{n(n+1)}}{(t;q^2)_{n+1}} \tag{4.2}$$

recurrent relation can be obtained:

$$\begin{aligned} T_0(q) &= 1 \\ T_1(q) &= 1 + q^2 \\ T_n(q) &= (1 + q^{2n})T_{n-1} + q^{2n-1}T_{n-2}(q) \end{aligned}$$

where we are taking

$$F(q,t) = \sum_{n=0}^{\infty} T_n(q)t^n$$

Therefore we have a theorem involving Pell numbers:

Theorem 4.1. The total number of partitions into at most n parts in which the largest part is even, each even smaller than the largest part appears at least once and the odd's are distinct is equal to the Pell number $T_n(1)$.

What is nice about this theorem is the fact that for the Fibonacci numbers F_n defined by $F_0 = 0; F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ we have proved in [12] (Theorem 3.3) that:

"The total numbers of partitions into at most n parts in which the largest part is even and every even smaller than the largest part appears at least once is equal to F_{2n+1}". This tells us that by adding the restriction "distinct odd's" we move from Fibonacci with odd index F_{2n+1} to the Pell numbers $T_n(1)$.

5 Fibonacci numbers from the Rogers-Ramanujan identities

The following polynomial generalization of the Fibonacci sequence has been used by Schur[17] to prove the Rogers-Ramanujan identities (see also Andrews[2]).

$$F_0(q) = 1$$

$$\begin{aligned} F_1(q) &= 1 \\ F_n(q) &= F_{n-1} + q^n F_{n-2}(q) \end{aligned} \tag{5.1}$$

which can obtained from the second of the Rogers-Ramanujan identities (Rogers[9]), that is:

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-2})(1-q^{5n-3})},$$

by defining the following two variable function:

$$f(q,t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n(n+1)}}{(t;q)_{n+1}} \tag{5.2}$$

Considering that for $q = 1$ (5.1) is the Fibonacci sequence and that (5.2) can be written in the form:

$$\sum_{n=0}^{\infty} \frac{t^{2n} q^{1+1+2+2+\ldots+n+n}}{(1-t)(tq;q)_n} \tag{5.3}$$

it is easy to see that we get a new combinatorial interpretation for the Fibonacci numbers that is stated in the following theorem:

Theorem 5.1. The total number of partitions into at most n parts in which every integer less than or equal to the largest part appears at least twice is equal to the Fibonacci number F_n.

In [10] we find, also, a two variable function similar to (5.2) related to the first Rogers-Ramanujan identity (Rogers[9]) given by:

$$f_1(q,t) = \sum_{n=0}^{\infty} \frac{t^{2n} q^{n^2}}{(t;q)_{n+1}} = \sum_{n=0}^{\infty} P_n(q) t^n \tag{5.4}$$

from which we get

$$(1-t) f_1(q,t) = 1 + t^2 q f_1(q, tq)$$

Knowing this functional equation one can get the following recurrence relation for $P_n(q)$:

$$\begin{aligned} P_0(q) &= 1 \\ P_1(q) &= 1 \\ P_n(q) &= P_{n-1} + q^{n-1} P_{n-2}(q) \end{aligned} \tag{5.5}$$

To get one more combinatorial interpretation for the Fibonacci numbers we observe that, for $q = 1$, (5.5) is the Fibonacci sequence and that from the first sum in (5.4) we have the following nice result:

Theorem 5.2. The total number of partitions in which the side of the Durfee squares equals to the largest part and the largest part plus the number of parts is at most n is equal to F_n.

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References

[1] Andrews, G.E. "The theory of partitions", Encyclopedia of Mathematics and Its Applications (Rota, Editor), Vol. 2, G.-C, Addison-Wesley, Reading, 1976. (Reissued: Cambridge University Press, London and New York, 1985).

[2] __________, "q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra", *CBMS Regional Conf. Ser. in Math.*, nº 66, Amer. Math. Soc., Providence, R.I., 1986.

[3] __________, Euler's "*Exemplum Memorabile Inductionis Fallacis* and q-trinomial coefficients", *Dep. of Math. Penn State*, Report # PM 65, May 1989.

[4] __________, "Combinatorics and Ramanujan's "lost" notebook", *London Math. Soc. Lecture Note Series*, Nº 103, Cambridge Univ. Press, London, 1985, pp. 1-23.

[5] __________, "q-Trinomial Coefficients and Rogers-Ramanujan Type Identities", *Analytic Number Theory* edited by B. Berndt, Boston, 1990.

[6] Andrews, G.E., R.J. Baxter. "Lattice gas generalization of the hard hexagon model. III. q-trinomial coefficients", *J. Stat. Phys.,* 47 (1987), pp. 297-330.

[7] Gordon, B. "Some Continued Fractions of the Rogers-Ramanujan Type," *Duke Math. J.*, 31 (1965), pp. 741-748.

[8] Göllnitz, "Partionen mit Differenzenbedingungen", *J. Reine Agnew. Math.* 225 (1967), pp. 154-190.

[9] Rogers, L.J., "Second memoir on the expansion of certain infinite products", *Proceedings of the London Mathematical Society*, 1894.

[10] Santos, J.P.O., "Computer algebra and identities of the Rogers-Ramanujan type". Ph.D. Thesis, Pennsylvania State University, 1991.

[11] Santos, J.P.O., "On the combinatorics of polynomial generalizations of Rogers-Ramanujan type identities" *Discrete Mathematics*, (1-3) 254 (2002), pp.497-511.

[12] Santos, J.P.O., Ivković, M., "*Fibonacci Numbers and Partitions*" *The Fibonacci Quarterly*, 41.3 (2003), pp. 263-278.

[13] Santos, J.P.O., Mondek, P.,"Extending Theorems of Göllnitz, A New Family of Partition Identities", *The Ramanujan Journal*, 3(1999), pp. 359-365.

[14] Santos, J.P.O., Sills, A.V., " q-Pell Sequences and Two Identities of V.A Lebesgue", *Discrete Mathematics*, (1) 257 (2002), pp. 125-142.

[15] Sills, A.V., "RRtools-a Maple Package for aiding the discovery and proof of finite Rogers-Ramanujan type identities," *Journal of Symbolic Computation*/to appear.
Package available at http://www.math.rutgers.edu/ asills/papers.html

[16] Slater, L.J., " Further identities of the Rogers-Ramanujan type", *Proc. London Math. Soc.* (2) 54 (1952), pp. 147-167.

[17] Schur I., "Ein Beitrag zur aditiven Zahlentheorie und zur Theorie der Kettenbrüche", *Sitzungsberichte der Berliner Academie* 1917, pp. 302-321.

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Catalan Numbers, the Hankel Transform, and Fibonacci Numbers

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Abstract

We prove that the Hankel transformation of a sequence whose elements are the sums of two adjacent Catalan numbers is a subsequence of the Fibonacci numbers. This is done by finding the explicit form for the coefficients in the three-term recurrence relation that the corresponding orthogonal polynomials satisfy.

1 Introduction

Let $A = \{a_0, a_1, a_2, ...\}$ be a sequence of real numbers. The Hankel matrix generated by A is the infinite matrix $H = [h_{i,j}]$, where $h_{i,j} = a_{i+j-2}$, i.e.,

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_1 & a_2 & a_3 & a_4 & \dots \\ a_2 & a_3 & a_4 & a_5 & \dots \\ a_3 & a_4 & a_5 & a_6 & \dots \\ a_4 & a_5 & a_6 & a_7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The *Hankel matrix* H_n *of order* n is the upper-left $n \times n$ submatrix of H and the *Hankel determinant of order* n of A, denoted by h_n, is the determinant of the corresponding Hankel matrix.

For a given sequence $A = \{a_0, a_1, a_2, ...\}$, the *Hankel transform* of A is the corresponding sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ (see Layman [5]).

The elements of the sequence in which we are interested A005807 of the On-Line Encyclopedia of Integer Sequences (EIS) [10], also INRIA [3]) are the sums of two adjacent Catalan numbers:

$$\begin{aligned} a_n &= c(n) + c(n+1) = \frac{1}{n+1}\binom{2n}{n} + \frac{1}{n+2}\binom{2n+2}{n+1} \\ &= \frac{(2n)!(5n+4)}{n!(n+2)!} \qquad (n = 0, 1, 2, \dots). \end{aligned}$$

This sequence starts as follows:

$$2, \quad 3, \quad 7, \quad 19, \quad 56, \quad 174 \dots$$

In a comment stored with sequence A001906 Layman conjectured that the Hankel transformation of $\{a_n\}_{n\geq 0}$ equals the sequence A001906 i.e., the bisection of Fibonacci sequence. In this paper we shall prove a slight generalization of Layman's conjecture.

The generating function $G(x)$ for the sequence $\{a_n\}_{n\geq 0}$ is given by

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{x}\left(\frac{(1-\sqrt{1-4x})(1+x)}{2x} - 1\right) \tag{1}$$

It is known (for example, see Krattenthaler [4]) that the Hankel determinant h_n of order n of the sequence $\{a_n\}_{n\geq 0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1}, \tag{2}$$

where $\{\beta_n\}_{n\geq 1}$ is the sequence given by:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \cfrac{a_0}{1 + \alpha_0 x - \cfrac{\beta_1 x^2}{1 + \alpha_1 x - \cfrac{\beta_2 x^2}{1 + \alpha_2 x - \cdots}}} \tag{3}$$

The sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ are the coefficients in the recurrence relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_n P_{n-1}(x)$$

where $\{P_n(x)\}_{n\geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional L determined by

$$L[x^n] = a_n \quad (n = 0, 1, 2, \ldots). \tag{4}$$

In the next section this functional is constructed and a theorem concerning the polynomials $\{P_n(x)\}_{n\geq 0}$ and the sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 1}$ is proved.

2 Main Theorem

We would like to express $L[f]$ in the form:

$$L[f(x)] = \int_R f(x) d\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$.

Denote by $F(z)$ the function

$$F(z) = \sum_{k=0}^{\infty} a_k z^{-k-1},$$

From the generating function (1), we have:

$$F(z) = z^{-1}\, G(z^{-1}) = \frac{1}{2}\left\{ z - 1 - (z+1)\sqrt{1 - \frac{4}{z}} \right\}. \tag{5}$$

with

$$\alpha_n = 2 - \frac{1}{F_{2n+1}F_{2n+3}}, \quad \beta_n = 1 + \frac{1}{F_{2n+1}^2}, \qquad k \geq 0 \tag{11}$$

where F_i *is the i-th Fibonacci number.*

Example 1. The first members of this sequence are:

$$\begin{aligned} P_0(x) &= 1; \\ P_1(x) &= x - \frac{3}{2}; \\ P_2(x) &= x^2 - \frac{17}{5}x + \frac{8}{5}; \\ P_3(x) &= x^3 - \frac{70}{13}x^2 + \frac{95}{13}x - \frac{21}{13}; \\ P_4(x) &= x^4 - \frac{251}{34}x^3 + \frac{290}{17}x^2 - \frac{435}{34}x + \frac{55}{34}. \end{aligned}$$

Notice that $P_n(0) = (-1)^n F_{2n+2}/F_{2n+1}$.

Proof of Theorem 1. Denoting by $W_n(x) = P_n^{(1/2,-1/2)}(x)$ $(n \geq 0)$ a special Jacobi polynomial, which is also known as *the Chebyshev polynomial of the fourth kind.*

The sequence of these polynomials is orthogonal with respect to $p^{(1/2,-1/2)}(x) = (1-x)^{1/2}(1+x)^{-1/2}$ on the interval $(-1,1)$. These polynomials can be expressed (Szegö [9]) by

$$W_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{2^n \sin\frac{1}{2}\theta}.$$

and satisfy the three-term recurrence relation (Chihara [1]):

$$W_{n+1}(x) = (x - \alpha_n^*)\, W_n(x) - \beta_n^* W_{n-1}(x) \quad (n = 0, 1, \ldots),$$
$$W_{-1}(x) = 0, \quad W_0(x) = 1,$$

where

$$\alpha_0^* = -\frac{1}{2}, \quad \alpha_n^* = 0, \qquad \beta_0^* = \pi, \quad \beta_n^* = \frac{1}{4} \qquad (n \geq 1).$$

If we use the weight function $\hat{p}(t) = (t-c)p^{(1/2,-1/2)}(t)$, then the corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows (see, for example, Gautschi [2])

$$\hat{\alpha}_n = c - \frac{W_{n+1}(c)}{W_n(c)} - \beta_{n+1}^* \frac{W_n(c)}{W_{n+1}(c)}, \tag{12}$$

$$\hat{\beta}_n = \beta_n^* \frac{W_{n-1}(c)W_{n+1}(c)}{W_n^2(c)}, \quad n \in \mathbb{N}. \tag{13}$$

Here, we use $c = -3/2$ and $\hat{p}(x) = (x+3/2)(1-x)^{1/2}(1+x)^{-1/2}$.

If we write $\lambda_n = W_n(-3/2)$ then, using the three-term recurrence relation for $W_n(x)$, we have

$$4\lambda_{n+1} + 6\lambda_n + \lambda_{n-1} = 0,$$

with initial values $\lambda_0 = 1, \quad \lambda_1 = -1$.

So, we find

$$\lambda_n = W_n(-3/2) = \frac{(-1)^n}{2\sqrt{5}\ 4^n}\left\{(\sqrt{5}+1)(3+\sqrt{5})^n + (\sqrt{5}-1)(3-\sqrt{5})^n\right\}.$$

Denoting by

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\phi} = \frac{1-\sqrt{5}}{2} \tag{14}$$

the golden section numbers, we can write:

$$\lambda_n = W_n(-3/2) = \frac{(-1)^n}{\sqrt{5}\ 2^n}(\phi^{2n+1} - \overline{\phi}^{2n+1}) = \frac{(-1)^n}{2^n}F_{2n+1}. \tag{15}$$

In order to simplify further algebraic manipulations we shall use

$$F_{2n-1}F_{2n+3} = F_{2n+1}^2 + 1 \tag{16}$$

This formula is a special case of the identity (Vajda [12]):

$$G(n+i)H(n+k) - G(n)H(n+i-k) = (-1)^n(G(i)H(k) - G(0)H(i+k)) \tag{17}$$

where G and H are sequences that satisfy the same recurrence relation as the Fibonacci numbers with possibly different initial conditions. However, we take both G and H to be the Fibonacci numbers and $n \to 2n+1$, $i = 2, k = -2$.

Now

$$\begin{aligned} \hat{\beta}_n &= \frac{1}{4}\frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2} = \frac{1}{4}\frac{F_{2n-1}F_{2n+3}}{F_{2n+1}^2} \\ &= \frac{1}{4}\left\{1 + \frac{1}{F_{2n+1}^2}\right\} \end{aligned} \tag{18}$$

and

$$\begin{aligned}\hat{\alpha}_n &= -\frac{3}{2} - \frac{\lambda_{n+1}}{\lambda_n} - \frac{1}{4}\frac{\lambda_n}{\lambda_{n+1}} \\ &= \frac{-3F_{2n+1}F_{2n+3} + F_{2n+3}^2 + F_{2n+1}^2}{2F_{2n+1}F_{2n+3}} \\ &= \frac{F_{2n+2}^2 - F_{2n+1}F_{2n+3}}{2F_{2n+1}F_{2n+3}} \\ &= -\frac{1}{2F_{2n+1}F_{2n+3}}.\end{aligned}$$

If a new weight function $p(x)$ is introduced by

$$p(x) = \hat{p}(ax+b)$$

then we have

$$\alpha_n = \frac{\hat{\alpha}_n - b}{a}, \qquad \beta_n = \frac{\hat{\beta}_n}{a^2} \qquad (n \geq 0).$$

Now, by using $x \mapsto x/2 - 1$, i.e., $a = 1/2$ and $b = -1$, we have the wanted weight function

$$w(x) = \hat{p}(\frac{x}{2} - 1) = \frac{1}{2}(x+1)\sqrt{\frac{4-x}{x}}.$$

Thus

$$\alpha_n = 2 - \frac{5}{(\phi^{2n+1} - \overline{\phi}^{2n+1})(\phi^{2n+3} - \overline{\phi}^{2n+3})} = 2 - \frac{1}{F_{2n+1}F_{2n+3}} \tag{19}$$

and

$$\beta_n = 1 + \frac{5}{(\phi^{2n+1} - \overline{\phi}^{2n+1})^2} = 1 + \frac{1}{F_{2n+1}^2} \tag{20}$$

finishing the proof of (1) .□

3 Layman's conjecture

By making use of (2) we have that:

$$h_n = a_0^n \left(1 + \frac{1}{F_3^2}\right)^{n-1} \left(1 + \frac{1}{F_5^2}\right)^{n-2} \cdots \left(1 + \frac{1}{F_{2n-1}^2}\right) \tag{21}$$

Using (16) we can write (21) as:

$$h_n = a_0^n \left(\frac{F_1F_5}{F_3^2}\right)^{n-1}\left(\frac{F_3F_7}{F_5^2}\right)^{n-2}\left(\frac{F_5F_9}{F_7^2}\right)^{n-3}\cdots\frac{F_{2n-3}F_{2n+1}}{F_{2n-1}^2} \qquad (22)$$

Since $a_0 = 2 = F_3$ the corresponding factors cancel, therefore:

$$h_n = F_{2n+1} \qquad (n \geq 0),$$

thus proving that Hankel transform of A005807equals A001519 -sequence of Fibonacci numbers with odd indices.

As we have mentioned in the introduction, Layman observed that the Hankel transform of A005807 equals A001906 -sequence of Fibonacci numbers with even indices. This sequence is obtained if we start the Hankel matrix from $a_1 = 3$, i.e., determinants will have a_1 on the position $(1, 1)$.

The proof of this fact is almost identical with the proof presented here, and so we do not include it. Notice that now we construct $L[x^n] = a_{n+1}$ and that $a_1 = 3 = F_4$; in (17) we take $n \to 2n$. We also use the easily provable fact $P_n(0) = (-1)^n F_{2n+2}/F_{2n+1}$ (see Example 1).

Finally we mention that, following Layman [5], it is known that the Hankel transform is invariant with the respect to the Binomial and Invert transform, so all the sequences obtained from A005807 using these two transformations have the Hankel transform shown here.

References

[1] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.

[2] W. Gautschi, Orthogonal polynomials: applications and computations, in *Acta Numerica, 1996*, Cambridge University Press, 1996, pp. 45–119.

[3] INRIA Algorithms Project, Encyclopedia of Combinatorial Structures, at http://algo.inria.fr/bin/encyclopedia?Search=ESCnb&argseach=431.

[4] C. Krattenthaler, Advanced Determinant Calculus, at http://www.mat.univie.ac.at/People/kratt/artikel/detsurv.html.

[5] J. W. Layman, The Hankel Transform and Some of its Properties, Journal of Integer Sequences, Article 01.1.5, Volume 4, 2001.

[6] P. Peart and W. J. Woan, Generating functions via Hankel and Stieltjes matrices, Journal of Integer Sequences, Article 00.2.1, Issue 2, Volume 3, 2000.

[7] W. J. Woan, Hankel Matrices and Lattice Paths, Journal of Integer Sequences, Article 01.1.2, Volume 4, 2001.

[8] J. P. O. Santos and M. Ivković, Fibonacci numbers and partitions, *Fibonacci Quarterly*, to appear.

[9] G. Szegö, *Orthogonal Polynomials*, AMS, 4th. ed., Vol. 23 (Colloquium publications), 1975.

[10] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Published electronically at
http://www.research.att.com/~njas/sequences.

[11] R. Tosić, Kombinatorika (in Serbian) Univerzitet u Novom Sadu (1999), Novi Sad.

[12] S. Vajda, *Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications*, Halsted Press, 1989.

[13] D. Zeilberger and T. Amdeberhan, DODGSON: A Maple package for conjecturing and proving determinant identities by Dodgson's condensation method,
at http://astro.ocis.temple.edu/ doron/programs.html.

High-Rate Girth-Eight Low-Density Parity Check Codes On Rectangular Integer Lattices

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Abstract -**This paper introduces a combinatorial construction of girth-eight high-rate low-density parity check codes based on integer lattices. The parity check matrix of a code is defined as a point-line incidence matrix of a 1-configuration based on a rectangular integer lattice, and the girth-eight property is achieved by a judicious selection of sets of parallel lines included in a configuration. A class of codes with a wide range of lengths and column weights is obtained. The resulting matrix of parity checks is an array of circulant matrices.**

Index Terms-Error control coding, iterative decoding, low-density parity check codes, combinatorial designs, finite geometries, graph girth.

I. Introduction

Codes on graphs, especially low-density parity check (LDPC) codes, is a research area of great current interest. The theory of codes on graphs has not only yielded capacity-approaching codes, it has also opened new research avenues for investigating alternative optimal and sub-optimal decoding schemes based on belief propagation. Applied on a Tanner graph of a linear block code [12] [6], belief propagation algorithm gives an exact a posteriori probability mass function for a given a probability density function of the observed variables, but only if the factor graph is cycle free. Extensive simulation results of MacKay and Neal [14] showed that message-passing algorithm also performs well in graphs with cycles. However, the presence of short cycles hurts the performance.

In this paper we address the problem of finding codes with good cycle properties. We are interested in "deterministic" sparse parity check matrices as opposed to the common "random code" assumption that has been widely used in recent research [13]. One of the first attempts to design a deterministic LDPC for iterative decoding is due to Kou, Lin and Fossorier [3] and it is based on projective and Euclidean geometries. The codes given in [3] are one-step majority logic

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decodable, and therefore the girth of the associated Tanner graph [6] is six. Lucas *et. al* showed that the such LDPC codes can be efficiently decoded by belief propagation algorithms [15]. A first attempt to construct deterministic codes with large girth is due to Margulis [18] who introduced an explicit construction of LDPC codes using k-regular graphs obtained as Caley graphs of $SL_2(F_q)$, a special linear group, and $PGL_2(F_q)$, a projective general linear group, of dimension two over F_q, the finite field with q elements (q power of a prime). By careful selection of transformation matrices, the author was able to achieve good girth properties. This idea was further developed by Rosenthal and Vontobel [18]. They were able to construct a short code (of length less than 5000) with girth 12. Recently, the explicit construction of families of LDPC with girth at least six has been discussed in Jon-Lark Kim *et. al* [17]. The authors extended Lazebnik and Ustimenko's [19] method for explicit construction of graphs with arbitrary large girth based on regular graphs.

The code construction presented in this paper is based on balanced incomplete block designs (BIBD) [1]. More specifically, the codes are based on sub-designs of a 2-(v,k,1) design, where v is a number of parity bits, k is the column weight of a parity check-matrix. The parity-check matrix is a point-block incidence matrix of the design *(V,B),* where V is a set of points and B is a set of blocks of size k. As we have shown in [7], the removal of certain blocks from a design can result in eliminating Pasch and generalized Pasch configurations and, consequently, in increasing minimum distance of a code. In this paper we exploit the idea that a judicious selection of disregarded blocks can also increase the girth of a design. It is desirable property of a bipartite graph to have a large girth, because in the message passing decoding algorithm [10] on such graphs it takes more iterations until extrinsic information originating from different nodes in the bipartite graph becomes correlated. The construction of designs with high girths appears to be a very difficult problem in general [8]. However, the designs based on rectangular integer lattices introduced in [7] allow for a simple algorithm for finding a girth-eight sub-design. In [23] a condition for absence of cycles of lengths smaller than a given constant was given for array codes, but no explicit construction is given for girths larger than six. In this paper we give an explicit construction for k=3 using arithmetically constrained sequences.

In this paper we are interested in very high rate codes ($R \geq 3/4$) for which the girth-eight property is much "rarer" than in low rate codes. We present a construction based on sets of parallel lines on a rectangular integer lattice, which is conceptually simple, and gives a large family of codes. The number of parity bits is equal to $v=m\cdot k$, $m>k$, and the blocks are defined as lines of different slopes connecting points of an $m\times k$ integer lattice.

Section II introduces some definitions necessary for dealing with BIBDs. Section III introduces a construction of low-density parity check codes using rectangular integer lattices and construction of girth-eight codes. It also gives the bit error rate (BER) performance of these codes in additive white Gaussian (AWGN) channel obtained by computer simulations.

II. Balanced Incomplete Block Designs and LDPC Codes

In this section we introduce some definitions. A *balanced incomplete block design* (BIBD) is a pair (V, B), where V is a v-element set and B is a collection of b k-subsets of V, called blocks, such that each element of V is contained in exactly r blocks and any 2-subset of V is contained in exactly λ blocks. The parameter r is called the *replication number*. The notation 2-(v, k, λ) design is used for a BIBD on v *points, block size* k, and index λ. In this paper we consider slightly different class of combinatorial designs called λ-configurations. A λ-configuration is an incidence structure of v points and b blocks such that i) each lock contains k points, ii) each point is incident with r blocks, and iii) two different points are contained in at most λ blocks. A λ-configuration can be obtained from a 2-(v,k,λ) design by removing some of its blocks. Two blocks in a design are referred to as *parallel* if they are disjoint. A design is called *resolvable* if there exists a partition of its block set B into parallel classes, each of which partitions the set V. As we will show, the lines on a lattice introduced in [7] and analyzed in [19] and [20] form a resolvable 1-configuration.

We define the point-block incidence matrix of a (V,B) as a $v\times b$ matrix $H=(h_{ij})$, in which $h_{ij}=1$ if the *i-th* element of V occurs in the j-th block of B, and $h_{ij}=0$ otherwise. It is easy to see that H is a matrix of parity checks of a Gallager code [2]. The row weight is r, column weight is k, and the code rate is $R\geq(b\text{-min}(v,b))/b$. In this paper we are interested in designs in which no more than one block contains the same pair of points. Such codes are one-step majority logic decodable, or equivalently, there are no cycles of length four in a bipartite graph [6]. The main idea of this paper is that a 1-configuration with large girth can be constructed by removing whole classes of parallel blocks rather than removing individual blocks.

III. Lattice Construction of 2-$(v,k,1)$ Girth-Eight 1-Configurations

In this section we address the problem of construction of resolvable 1-configurations with a wide range of block sizes. 2-$(v,k,1)$ designs naturally come as girth-six designs because no pair of points occurs in more than one block. In other words, if $|B|=b=\binom{v}{2}/\binom{k}{2}$ (i.e., design has maximum possible number of blocks, then the girth is $g(V,B)$=6. As we will show, for every design (V,B), there exists a 1-configuration (V,B'), $B'\subset B$, such that $g(V,B')\geq 8$. In a 1-configuration, there exist a pair of points that are disconnected, i.e. there is no line incident with both of them. This is why the girth of a 1-configuration can be larger than six. Our construction is based on the integer lattice construction given in [7], and briefly summarized as follows.

We define a class of 1-configurations as sets of lines connecting the points of a rectangular integer lattice. Consider a rectangular integer lattice $L=\{(x,y):0\leq x\leq k-1, 0\leq y\leq m-1\}$, where m is a prime. The construction can be readily generalized to the case when m is a prime power (i.e., $m=p^l$). Let $l{:}L\rightarrow V$ be an one-to-one mapping of the lattice L to the point set V. An example of such mapping is a simple linear mapping $l(x,y)=m\cdot x+y+1$. The numbers $l(x,y)$ are referred to as lattice point labels.

A *line with slope s, $0\leq s\leq m-1$, starting at* the point $(0,a)$, Is the m set of points $\{(x, a+sx \bmod m):0\leq x\leq k-1\}$, where $0\leq a\leq m-1$. We are concerned with a 1-configuration which is an incidence structure comprised of points on the integer lattice and all lines of slopes s, $0\leq s\leq m-1$. As mentioned earlier, two lines are referred to as parallel if they do not have any common points. There are, therefore, m classes of parallel lines in our 1-configuration corresponding to m different slopes. Each class of parallel lines comprises m lines.

Example 3.1:

Figure 1 depicts the rectangular integer lattice with m=5 and k=3. It also shows two classes of parallel lines (with slopes $s=1$ and $s=2$).

In our example, the lines of slope 1 are the points {1,7,13}, {2,8,14}, {3,9,15} etc. We assume that the lattice labels are periodic in vertical (y) dimension, and therefore the line comprising points {4,10,11} also has the slope 1. The examples of lines with slope two are {1,8,15} and {2,9,11}. The slopes: 3,4,…,m-1 can be defined analogously. Notice that no line of infinite slopes belongs to the design. Each column in TABLE I gives a set of parallel lines with slope s. A set of parallel lines defines a resolvability class.

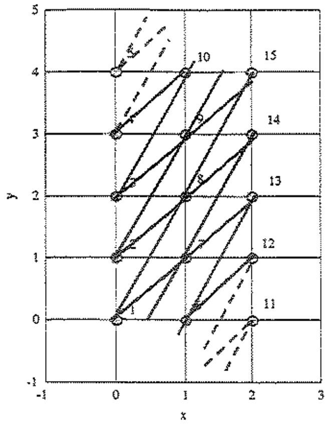

Figure 1: An example of the rectangular grid for m=5 and k=3

TABLE I
Resolvability classes of a lattice design in Figure 1

s=0	s=1	S=2	s=3	s=4
{1, 6, 11}	{1, 7, 13}	{1, 8, 15}	{1, 9, 12}	{1, 10, 14}
{2, 7, 12}	{2, 8, 14}	{2, 9, 11}	{2, 10, 13}	{2, 6, 15}
{3, 8, 13}	{3, 9, 15}	{3, 10, 12}	{3, 6, 14}	{3, 7, 11}
{4, 9, 14}	{4, 10, 11}	{4, 6, 13}	{4, 7, 15}	{4, 8, 12}
{5, 10, 15}	{5, 6, 12}	{5, 7, 14}	{5, 8, 11}	{5, 9, 13}

Remark 3.1:

Notice that in general there are m parallel classes of blocks (lines), each corresponding to the different slope.

Lemma 3.1:

A set B of all m k-element sets of V obtained by taking the labels of the points along the lines with slopes s, $0 \leq s \leq m-1$ is a 1-configuration.

Proof: The design B containing all m k-element sets of points in V obtained by taking labels of points along the lines with slopes s, $0 \leq s \leq m-1$ is a 1-configuration. It can be readily verified by noticing that because m is a prime, for each lattice point (x,y) there cannot be more than one line with slope s that passes through (x,y).

Q.E.D.

Remark 3.2:

The generalization to the case when m is a power of prime is straightforward.

Remark 3.3: The block size is k, number of blocks is $b=m^2$ and each point in the design occurs in exactly m blocks. The matrix of parity checks of a lattice code can be written in the form

$$H = \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,m} \\ H_{2,1} & H_{2,2} & \cdots & H_{2,m} \\ \vdots & \vdots & & \vdots \\ H_{k,1} & H_{k,2} & \cdots & H_{k,m} \end{bmatrix}$$

wherein each sub-matrix $H_{i,j}$ is a circulant with column weight equal to one. H is a line-point incidence matrix of a 1-configuration defined by a integer lattice defined above.

The position of the only nonzero position in the first column of $H_{i,j}$ can be found by using $c_i^{(j-1)}$, the i-th element of the first base block in the class of blocks corresponding to the j-th slope (see [7]) . The construction can be readily extended to the case when m is a p, power of prime.

Remark 3.4:
Notice a similarity of the structure of the above parity-check matrix with that obtained in [17]. The codes denoted by *LU(2,q)* in [17] have square matrix of parity checks, while our codes have rectangular matrices of parity checks. It is not surprising because it was shown in [17] that *LU*(2;4) and *LU*(2;8) are equivalent to Euclidean geometry codes [3], while a square lattice design (which includes the lines with infinity slope) is equivalent to the Euclidean plane. Notice also similarity with a parity check matrix of array codes [23].

Remark 3.5:
The ensemble of integer-lattice codes defined by matrices of parity checks obtained by random selection of slopes and starting points has well defined asymptotic distance distribution. Litsyn and Shevelev [21] showed that such an ensemble (they called it "Ensemble A") has superior distance distribution compared to other ensembles they considered in [21].

Denote by $B(s)$ a resolvability class corresponding to slope s, and B' a set of blocks of a sub-design composed of resolvability classes corresponding to the slopes from the set S, i.e., $B' = \bigcup_{s \in S} B(s)$. We are interested in the following problem: find a maximum cardinality slope set S, such that B' is a girth-eight design. We are concerned with finding a set of slopes of maximal cardinality because the slope set cardinality directly influences the code rate, i.e. the larger slope set cardinality, the higher code rate.

The problem of finding a set of maximal cardinality for given integers m and k, is generally difficult, i.e. the complexity of the algorithm for finding such a set of slopes is exponential in $m \cdot k$. Instead of solving hard problem of finding maximal slope set, we give a polynomial algorithm that constructs a set of slopes S resulting in a girth-eight 1-configuration (V,B'). The algorithm is based on select, check and disregard procedure.

```
s=0, S={s}, B'= B(s), S'={1,...,m-1}.
while S'≠∅
s=s+1
if g(V, B'∪B(s))≥8
          S=S∪{s}
          S'=S'\{s}
           B'=B'∪B(s)
 else
           S'=S'\{s}
 end
end
```

The function $g:(V,B)\rightarrow Z_+$ gives the girth of a graph for a (V,B) design, and can be computed in time $O((v+b)\cdot vk)$ time (e.g. Dijkstra or Bellman-Ford algorithm-for more details see [16]).

Clearly, the algorithm is greedy in the sense that a slope s does not satisfy the girth-eight criterion, it checks for the immediate next slope $s+1$. Although this method does not result in a maximal slope set, the simulation results of the different codes obtained through this method have shown to yield good performance (discussed later).

In [23] Fan gave a condition for absence of cycles of an arbitrary length in a Tanner graph of an array code in terms of a relation among powers of a permutation matrix used as blocks in H. Notice however that this condition still requires a search for finding a desired set of powers, and is equivalent to a "triangle" condition in this paper (see Appendix).

Note that finding a set of slopes for $k=2$ is trivial, because $S=\{1,\ldots,v/2-1\}$is always a solution. However, for $k=3$, we deduce a simple way of generating a slope set S, such that $B'=\bigcup_{s\in S} B(s)$ is a girth-eight design. The simplification stems from the interesting relationship between the elements of S with the "arithmetic constrained" sequences defined by Odlyzko and Stanley [22].

Definition 3.1:

Given a fixed positive integer q, we define the arithmetically constrained sequence as the sequence $Y(q)$ of positive integers $0= a_0 < a_1 < a_2 < \ldots$ by the conditions:

i. $a_1 = q$

ii. having chosen $a_0, a_1, a_2 \ldots a_n$ $(n > 0)$, let a_{n+1} be the least integer such that $a_{n+1} > a_n$ and such that the sequence $a_0, a_1, a_2 \ldots a_n, a_{n+1}$ contains no three terms (not necessarily consecutive) in an arithmetic progression.

Furthermore, we define "earliest" sequence A: a_0, a_1, a_2 ...a_n,... such that $a_0 = 0, a_{2n} = 3a_n, a_{2n+1} = a_{2n} + 1$. It can be verified [23] that A is an *arithmetically constrained* sequence with the property that the ternary expansion of $a_n = \sum t_i 3^i$ has t_i =0 or 1 [Theorem 2, 23] ("Earliest" corresponds to "greedy" when it comes to choosing the terms, for example: for the set {1, 2}, since 3 cannot be considered in the sequence, the sequence considers the first possible number i.e., 4).

Theorem 3.1:

For m arbitrary and k=3, $S = \{a : a \in A \wedge a \le m/2\}$ results in a girth-eight 1-configuration (V,B'), where A is the earliest sequence.

Proof:

Given in Appendix.

Theorem 3.1 is consequence of the fact that sequence of the set of slopes obtained from the algorithm is greedy and so is the earliest sequence. As we will in the next section, the codes obtained by the proposed slope selection algorithm or the above straight forward implication (k=3), have performance slightly worse than codes with a slope set with maximum cardinality.

The lower bound on a minimum distance of a code with girth g, column weight k code is given by the following formula [2]

$$d_{\min} \ge \begin{cases} 1+\dfrac{k}{k-2}\cdot\left((k-1)^{\lfloor (g-2)/4 \rfloor}-1\right) & \text{if } g/2 \text{ odd} \\ 1+\dfrac{k}{k-2}\cdot\left((k-1)^{\lfloor (g-2)/4 \rfloor}-1\right)+(k-1)^{\lfloor (g-2)/4 \rfloor} & \text{if } g/2 \text{ even} \end{cases}$$

The above bound can be aptly applied for the above girth-eight codes with g =8.

IV. SIMULATION RESULTS

The first experiment deals with the performance of codes obtained from the slope set generated using the above select and discard procedure. (Note that we do not consider high rate codes for the reason that we need to find a maximal slope set, which is a tough to obtain for high rate codes as it involves large block length). For the specific case of k=3 & m=53, either with the proposed algorithm (or using the theorem 3.1), we obtain a slope set S= {0, 1, 3, 4, 9, 10, 12, 13} resulting a code rate of R= 0.67 (and b = 424). On the other hand, with extensive computer search for a maximal slope set, we obtained S_{max} ={7,8,14,16,19,33,35,41,42,45} resulting in a code rate of R=0.7 (and b= 530). As we can see from Figure 2, the code constructed from the maximal set of slopes is 2.8dB away from respective Shannon limit, while code constructed using polynomial time algorithm is 3.4dB away from respective Shannon limit, which is not bad for such short codes. In this particular case the code constructed from the set with maximal cardinality slope set has better performance. However, we do not have enough evidence to state this as a general conclusion.

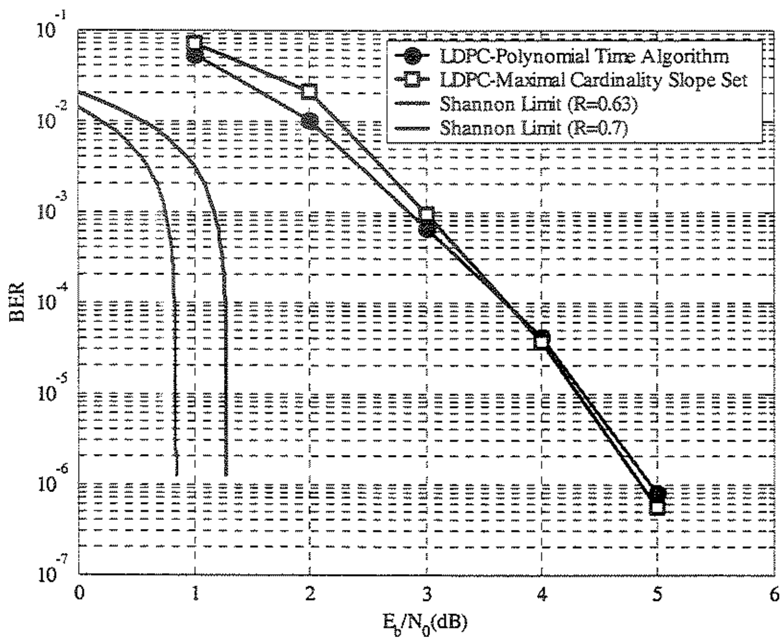

Figure 2
Performance comparison of LDPC codes with maximal set slopes and the linear time method

The next simulation result gives the BER performance of girth-eight codes obtained from rectangular integer lattices in AWGN channel. Figure 3 show the comparison of girth-eight codes with randomly constructed codes with column weights three and four respectively. Girth-eight LDPC code with column weight three (k=3), code rate R=0.81 with set of slopes S={0,1,3,4,9,10,12,13,27,28,30,31,36,37,39,40} and girth eight code with parameters k=4 and R=0.76 are used. The result shows that girth eight codes constructed using above algorithm perform quite close to random codes. Random codes are generated such that cycles of length four are omitted.

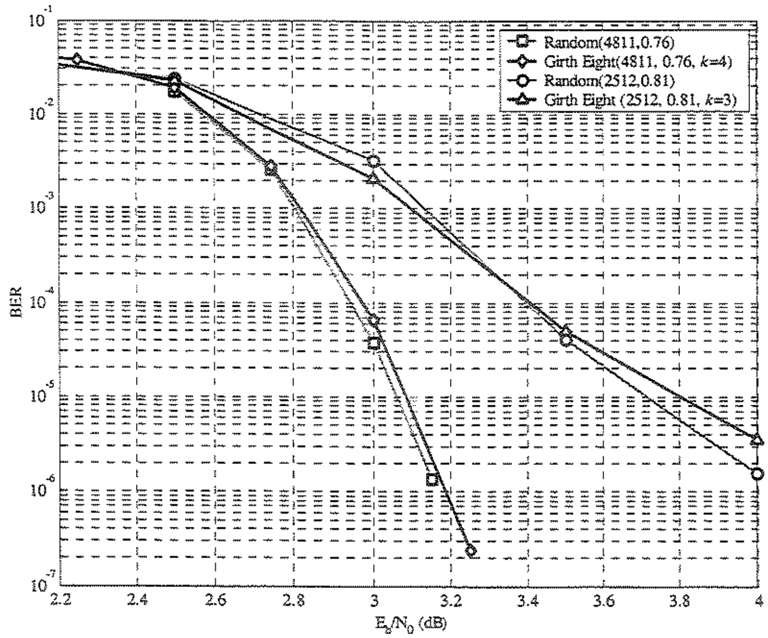

Figure 3
Performance of girth-eight codes in AWGN channel

CONCLUSION

We have given a simple construction of girth-eight codes using integer lattices. The construction is based on a judicious selection of sets of parallel lines in the lattice sub-geometry. The construction gives a set of codes with a wide range of lengths and rates. The algorithm used to create girth-eight codes can be readily generalized to higher girths. In the $k=3$ we have an explicit construction using arithmetic sequences.

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APPENDIX

PROOF OF THEOREM 3.1

Proof:

John L. Fan [23] derived the condition for occurrence of a cycle in a particular class of Tanner graphs whose parity check matrix H has the structure defined as in remark 3.1. For the specific case of eliminating cycles of length six with k =3, the condition in [25] reduces to the following triangle condition.

$$\alpha(s_2 - s_1) + \beta(s_3 - s_1) \neq 0 (\bmod m) \quad (1)$$

Where: $\alpha = i_1 - i_3$ and $\beta = i_2 - i_3$ (the indices i_h are associated with the sub matrices H_{i_h, j_h} such that

$$\alpha \neq 0, \beta \neq 0 \text{ and } \alpha \neq \beta \quad (2)$$

For k=3, since $i_h \in \{1,2,3\}$, we have the obvious inequality,

$$-k< \alpha,\ \beta <k. \quad (3)$$

From (2) and (3), we are left with the possibility that α and β take values, with the opposite sign and with a different absolute value, from the set {-2, -1, 1, 2} or both to be equal 1 or -1. Thus, the set of slopes S must be a subset of the earliest sequence A defined above. An arithmetic progression, where, for example, $s_2 = \frac{s_1 + s_3}{2}$ corresponds to the case $\alpha = 2, \beta = -1$.

Now we shall prove that in S cannot be values bigger that $m/2$:

Let s_3 be the potential slope considered to be included in S. s_1 and s_2 are two elements already in S. From now on, all numbers, if not specifically stated differently, are considered in ternary expansion.

Number $d = m - s_3$ is smaller than $m/2$. If d is has 1´s and 0´s as digits only it is already in S, in that case we take s_2=d, s_1=0 and $\alpha = 1, \beta = 1$ eliminating potential s_3. If that is not the case we can add to d a number s_1 made out 0´s and 1´s as digits in that way that sum is a number that is made out 0´s and 2´s. We do that simply by taking a digit of s_1 to be 1 when a corresponding digit of d is 1. We have $d + s_1 < m$. This result can be seen as double of the number that has only 1's and 0's as digits and is smaller than $m/2$. We constructed s_2, thus eliminating s_3. This construction corresponds to $\alpha = 2, \beta = -1$.

REFERENCES

[1] L. M. Batten, *Combinatorics of finite geometries*, London, Cambridge University Press 1997.

[2] R. G. Gallager, *Low-Density Parity-Check Codes*. Cambridge, MA: MIT Press, 1963.

[3] Y. Kou, S. Lin, and M. P.C. Fossorier, "Low density parity check codes based on finite geometries: A rediscovery and new results", to appear in *IEEE Trans. Inform. Theory*, 2001.

[4] D. MacKay, M. Davey, " Evaluation of Gallager codes for short block length and high rate applications," http://www.cs.toronto.edu/~mackay/CodesRegular.html.

[5] J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. San Mateo, CA: Morgan Kaufmann, 1988.

[6] R. M. Tanner, "A recursive approach to low complexity codes," *IEEE Trans. Inform. Theory*, vol. IT-27, pp. 533–547, Sept. 1981.

[7] B. Vasic, "Combinatorial constructions of structured low-density parity-check codes for iterative decoding," in Proc. *2001 IEEE Information Theory Workshop*, p. 134, 2001, also submitted to *IEEE Trans. Inform. Theory*

[8] R. A. Beezer, "The girth of a design," to appear, Journal of Combinatorial Mathematics and Combinatorial Computing.

[9] F. R. Kschischang and B. J. Frey, "Iterative decoding of compound codes by probability propagation in graphical models," *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 219-230, February 1998.

[10] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger," Factor graphs and the sum-product algorithm," *IEEE Trans. on Inform. Theory*, vol. 47, no. 2, pp. 498 –519, Feb. 2001

[11] G.D. Forney, Jr. "Codes on graphs: normal realizations," *IEEE Trans. on Inform. Theory*, vol. 47, no. 2, pp. 520 –548, Feb. 2001.

[12] N. Wiberg, H.-A. Loeliger, and R. Kötter, "Codes and iterative decoding on general graphs," Eur. Trans. Telecommun., vol. 6, pp. 513–525, Sept./Oct. 995.

[13] D. J. C. MacKay, "Good error-correcting codes based on very sparse matrices," IEEE Trans. Inform. Theory, vol. 45, pp. 399-431, March 1999.

[14] D. J. C. MacKay and R. M. Neal, "Good codes based on very sparse matrices," in Cryptography and Coding, 5th IMA Conference, in Lecture Notes in Computer Science, C. Boyd, Ed., 1995, vol. 1025, pp. 110–111.

[15] R. Lucas, M. Fossorier, Y. Kou and S. Lin, "Iterative decoding of one-step majority logic decodable codes based on belief propagation", *IEEE Trans. Commun.* vol. 48, no. 6, pp. 931 -937 June 2000.

[16] T. H. Cormen, C. E. Leiserdon, R. L. Rivest, *Introduction to algorithms:* Cambridge, MA, McGraw-Hill, 1989.

[17] Jon-Lark Kim, Uri N. Peled, Irina Perepelitsa, and Vera Pless, "Explicit construction of families of LDPC codes with girth at least six", presented at 40th Annual Allerton conference on communications, control and computing, October 2002

[18] I J. Rosenthal and P. O. Vontobel, "Construction of LDPC codes using Ramanujan graphs and ideas from Margulis," in Proc. 2001 IEEE International Symposium on Information Theory, p. 4, 2001.

[19] F. Lazebnik and V. A. Ustimenko, "Explicit construction of graphs with arbitrary large girth and of large size," Discrete Applied Math., vol. 60, pp. 275–284, 1997.

[20] B. Vasic, "Combinatorial construction of low-density parity-check codes," Proceedings. 2002 *IEEE International Symposium on Information Theory*, p. 312, June 30 - July 5, 2002, Lausanne, Switzerland.

[21] S. Litsyn and V. Shevelev, "On Ensembles of Low-Density Parity-Check Codes: Asymptotic Distance Distributions," *IEEE Trans. Inform. Theory*, vol. 48, no. 4, pp. 887-908, April 2002.

[22] M. Odlyzko and R. P. Stanley, Some curious sequences constructed with the greedy algorithm, Bell Labs Internal Memorandum, 1978.

[23] J.L. Fan, "Array codes as Low-Density Parity Check Codes" *2nd International Symposium on Turbo Codes and Related Topics* (Brest, France), September 2000.

[24] M. A. Margulis, "Explicit group-theoretic constructions for combinatorial designs with applications to expanders and concentrators," *Problemy Peredachi Informatsii*, vol. 24, no. 1, pp. 51-60, 1988.

Achievable Information Rates for High-Speed Long-Haul Optical Transmission

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Abstract—**Achievable information rates for high-speed optical transmission (40 Gb/s and above) are calculated using finite state machine approach. Combined effect of all major effects including: amplified spontaneous emission noise, Kerr nonlinearity (self-phase modulation, intrachannel four-wave mixing, intrachannel cross-phase modulation), stimulated Raman scattering, chromatic dispersion (group velocity dispersion (GVD), second order GVD) and (optical/electrical) filtering on the Shannon capacity is considered. The lower bound for the channel capacity is determined for various dispersion maps and number of spans, and shown to depend strongly on the dispersion map and average launched power. The user bit rate maximizing the spectral efficiency is found to be around 25Gb/s..**

Index Terms—**Achievable information rates, Shannon capacity, finite state machine, fiber nonlinearities, optical communications, long-haul transmission**

I. INTRODUCTION

Increase of optical bandwidth and improvement of spectral efficiency are common ways to increase the throughput of a dense wavelength-division-multiplexed (DWDM) system. Recent progress in DWDM transmission technology has lead to a fundamental and very difficult question: "What is the Shannon's capacity of a nonlinear fiber optics communication channel?" There have been numerous attempts to tackle this problem [1-13]. In all of them it is assumed that the amplified spontaneous emission (ASE) from inline amplifiers is a predominant effect, while the effect of nonlinearities is taken into an account in an approximate fashion.

In this paper we report the achievable information rates for high-speed long-haul optical transmission with independent uniformly distributed (i.u.d.) source and the combined effect of ASE noise, Kerr nonlinearity (self-phase modulation, intrachannel four-wave mixing, intrachannel cross-phase modulation), stimulated Raman scattering, chromatic dispersion (group velocity dispersion (GVD), second order GVD) and (optical/electrical) filtering. The method employs finite state machine approach outlined in [17-20], and was initially proposed for magnetic channels, which are essentially nonlinear as well. The propagation of the signal is modeled by NLSE, solved using the Fourier-split-step algorithm [14]. The ASE noise-signal interaction during transmission is taken into account, and the fiber channel is modeled as a channel with memory. Our approach is applicable to any transmission system and allows the usage of the achievable information rate as figure of merit for a long-haul transmission system similarly as Q-factor, eye opening or bit-error rate.

The paper is organized as follows. The Section II gives a brief glance on recent capacity calculation attempts, and underlines their limitations. Sections III and IV explain the channel model and our method, while Section V contains numerical results and conclusions.

II. An Overview of Prior Work

In the existing literature the fiber nonlinearities are considered either as (i) the perturbation of a linear case [4], [12], as (ii) the multiplicative noise [6], or (iii) the dispersion was ignored [7-8]. Although the perturbative methods in general may yield to reliable results in the domain of validity [4], they are applicable in the regime of relatively small nonlinearities (which is not the case at 40Gb/s or higher). Mitra and Stark [6] treat a nonlinear noisy channel as a linear one with an effective nonlinear noise. However, as indicated in [2], such an approximation needs to be carefully justified for each particular transmission system. In dispersion free transmission [7-8] the nonlinear Schrödinger equation (NLSE) [14], (see also Eq. (1) in Section II), can be solved analytically, but such a result is only of academic interest [7]. In [3] Tang determined the channel capacity of a multispan DWMD system employing dispersive nonlinear optical fibers and an ideal coherent optical receiver. The results obtained in [3] are based on solving the NLSE by Voltera series expansion up to the first order. Such method is valid in systems for which the maximum non-linear rotation is small compared to 2π [3]. The channel capacity is determined using the Pinsker's formula [15], which may lead to wrong conclusions, as it was shown in [2], especially when the signal-to-noise ratio tends to infinity. The statistics of optical transmission in a noisy nonlinear channel with weak dispersion management and zero average dispersion is considered in [2], and the lower bounds for channel capacity are determined (although numerical results are not reported). However, this method is applicable only for weak dispersion management systems with zero average dispersion, as described in [16]. The spectral efficiency limits in DWDM systems with coherent detection in nonlinear regime limited by cross-phase modulation or four-wave mixing are reported in [1]. However, calculating the channel capacity when combined effects of ASE noise, Kerr nonlinearities, dispersion and filtering effects are taken into account is still an open problem.

III. Channel Model

The signal channel transmission at high bit rate (40 Gb/s and above) is considered. The carrier-suppressed RZ (CSRZ) modulator employed is composed of a laser diode, two MZ intensity modulators (the first serving as modulator, the second as a NRZ to RZ converter), and a PRBS generator. Erbium-doped fiber amplifiers (EDFA) and dispersion compensating fibers (DCF) are deployed periodically to compensate the loss and accumulated dispersion of the standard single mode fiber (SMF). The direct detection receiver observed is composed of an optical filter, a PIN photodiode, an electrical filter, and a sampler followed by a decision circuit. An EDFA is used as a pre-amplifier. (Polarizatoion mode dispersion and polarization dispersion loss effects are ignored.)

The propagation of a signal through the transmission media is modeled by NLSE [14],

$$\frac{\partial A}{\partial z} = -\frac{\alpha}{2}A - \frac{i}{2}\beta_2 \frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} + i\gamma\left(|A|^2 - T_R \frac{\partial\ |A|^2}{\partial T}\right)A, \tag{1}$$

where z is the propagation distance along the fiber, relative time $T = t - z/v_g$ gives a frame of reference moving at the group velocity v_g, $A(z,T)$ is the complex field amplitude of the pulse, α is the attenuation coefficient of the fiber, β_2 is the group velocity dispersion (GVD) coefficient, β_3 is the second-order GVD, γ is the nonlinearity coefficient giving rise to Kerr effect nonlinearities: self-phase modulation (SPM), intra-channel cross-phase modulation (IXPM) and intrachannel four-wave mixing (IFWM) and T_R is the Raman coefficient describing the stimulated Raman scattering (SRS).

IV. Calculation of Achievable Information Rates Using Finite State Machine Approach

The method used in determining of a lower bound on the achievable information rate is to estimate the mutual information between the input random process X and output process Y by modeling the channel as a finite state machine [17-20]. The finite state machine is described by the input alphabet X, output alphabet Y, finite set of states S, and by the conditional probability density function determined from nonlinear channel modeled by NLSE (1), $p(Y_n,s|s')$, $s=(X_{n-m}, X_{n-m+1},\dots, X_n, X_{n+1},\dots, X_{n+m})$ ($X_i\in\{0,1\}$), where $2m+1$ is the channel memory. It is assumed that m previous and m next bits influence the observed bit, and the state s is determined by a sequence of $2m+1$ input bits. Given the previous state s', received sample Y_n the "probability" $p(Y^n)$, $Y^n=(Y_1,Y_2,\dots,Y_n)$, c by the BCJR algorithm [21] to the probability of the next state s at instance n, $\alpha_n(s)$ can be determined:

$$\alpha_n(s)=\sum_{s'}\gamma_n(s',s)\alpha_{n-1}(s'), \qquad (2)$$

where $\gamma_n(s',s)$ is given by $\gamma_n(s',s)=p(s,Y_n\mid s')$.

The problem of computing channel capacity involves the maximization of mutual information $I(X, Y)$ over all possible input distributions $p(X)$

$$C_{2m+1}=\lim_{n\to\infty}\frac{1}{n}\max_{p(X)} I_n(X,Y), \qquad (3)$$

where $2m+1$ denotes the memory and mutual information can be calculated by

$$I_n(X,Y)=H_n(Y)-H(Y\mid X), \qquad (4)$$

with $H_n(Y)=-\frac{1}{n}\log p(Y^n)$ being the output process entropy rate. According to the Shannon-McMillan-Breimann theorem [18-19], for a stationary ergodic finite-state Markov process Y,

$$H_n(Y)=-\frac{1}{n}\log p(Y^n)\to H(Y) \qquad (5)$$

as $n\to\infty$, with $H(Y)$ being the output process entropy. The conditional entropy $H(Y\mid X)$ can be calculated in a similar fashion.

As already mentioned, we reduce our attention to the uniform-input information rate (the case commonly considered in practical systems), also known as the achievable information rate [19].

V. Numerical Results and Conclusions

Consider a single channel transmission system at 40Gb/s bit-rate located at 1552.524 nm (193.1 THz). The dispersion map is composed of N spans of length L (96 or 48 km), consisting of $2L/3$ km of D+ fiber followed by $L/3$ km of D- fiber, with pre-compensation of –320 ps/nm and corresponding post-compensation. The D+ fiber parameters are as follows; dispersion of 20 ps/(nm·km), dispersion slope of 0.06 ps/(nm^2·km), effective cross-sectional area equal to 110 μm^2 and loss equal to 0.19 dB/km. The corresponding D- fiber parameters are: -40 ps/(nm·km), -0.12 ps/(nm^2·km), 50 μm^2 and 0.25 dB/km, respectively. The nonlinear Kerr coefficient is set to 2.6 x 10^{-20} m^2/W, and a carrier-suppressed RZ signal format is assumed. The influence of optical and electrical filters is taken into account as well. To compensate the fiber loss EDFAs with the noise figure of 6 dB are located after every fiber section.

The results given in Figs. 1-3. Two different span lengths, 96 km and 48 km, are observed. As expected, for larger channel memory and longer sequence of bits better lower bounds for the information rate are obtained. The channel capacity is monotonically decreasing function of the average launched power (Fig. 2). Also the lower bound on the i.i.d. capacity varies with the dispersion map. It is also interesting (see Fig. 3) that there exists the optimum bit rate (at around 25Gb/s), which means that it is best to use an error correction code having the rate of approximately 2/3 (=25/40).

It is important to notice that the finite state machine can efficiently capture all nonlinear effect one would like to include in the model as long as they have local character, i.e., the energy transfer between distant symbols is limited, which is the case in practical systems.

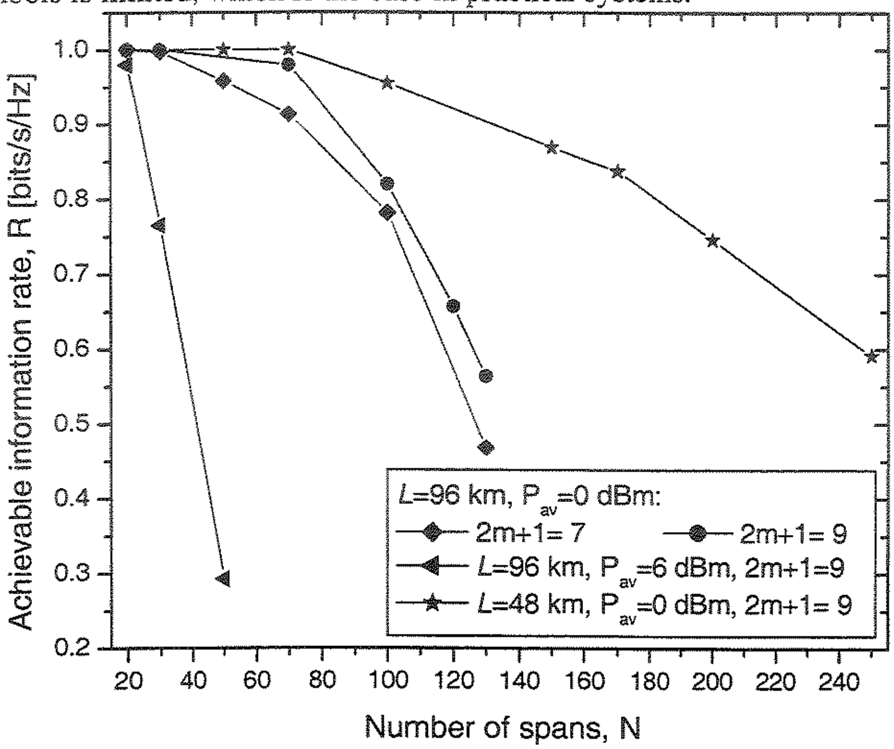

Fig. 1 Achievable information rate versus number of spans for two different map strengths and two different span lengths. Bit rate is 40 Gb/s.

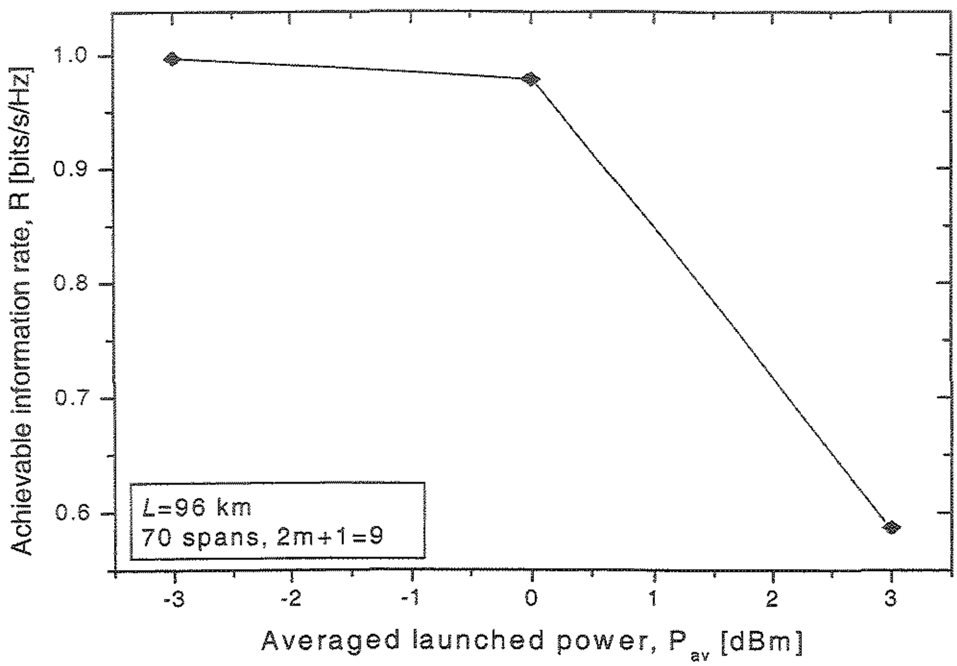

Fig. 2 Achievable information rate versus averaged launch power. Bit rate is 40Gb/s.

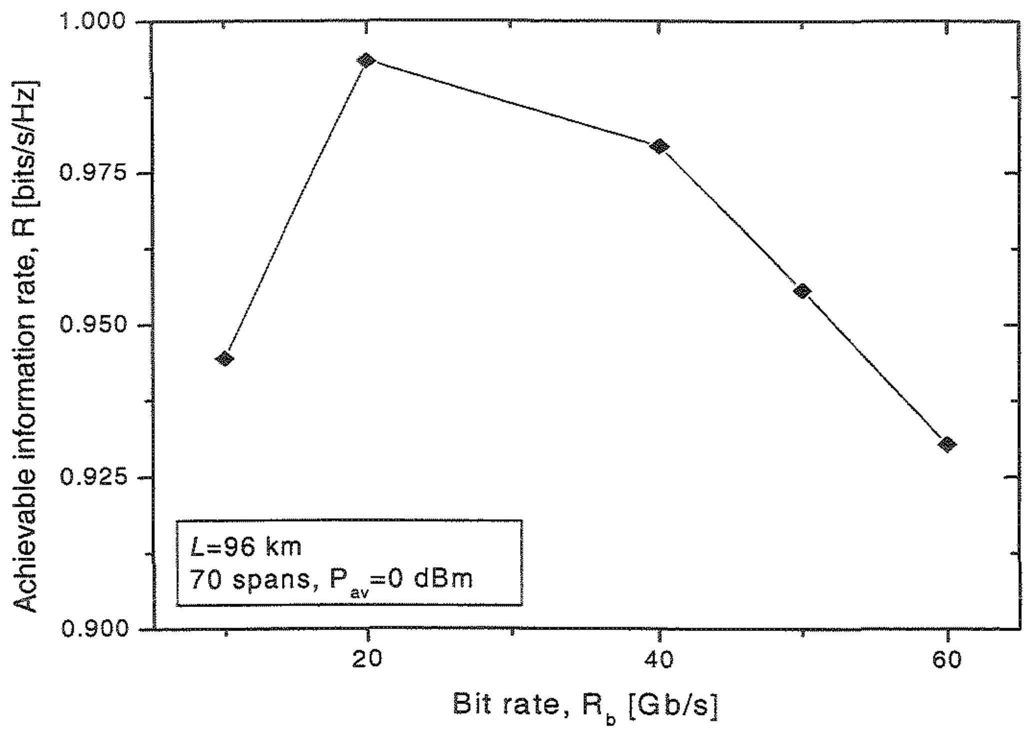

Fig. 3 Achievable information rate versus bit-rate

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References

[1] J. M. Kahn, K.-P. Ho, "Spectral efficiency limits and modulation/detection techniques for DWDM systems," *IEEE Selected Topics Quantum Electron.*, vol. 10, pp. 259-272, March/April 2004.

[2] K. S. Turitsyn, S. A. Derevyanko, I. V. Yurkevich, and S. K. Turitsyn, "Information capacity of optical fiber channels with zero average dispersion," *Phys. Rev. Lett.*, vol. 91, no. 20, p. 203 901, Nov. 2003.

[3] J. Tang, "The channel capacity of a multispan DWDM system employing dispersive nonlinear optical fibers and an ideal coherent optical receiver," *J. Lightwave Technol.*, vol. 20, pp. 1095–1101, July 2002.

[4] E. E. Narimanov and P. Mitra, "The channel capacity of a fiber optics communication system: perturbation theory," *J. Lightwave Technol.*, vol. 20, pp. 530–537, March 2002.

[5] A. Mecozzi and M. Shtaif, "On the capacity of intensity modulated systems using optical amplifiers," *IEEE Photon. Technol. Lett.*, vol. 13, pp. 1029–1031, Sep. 2001.

[6] P. P. Mitra and J. B. Stark, "Nonlinear limits to the information capacity of optical fiber communications," *Nature*, vol. 411, pp. 1027–1030, June 2001.

[7] J. Tang, "The Shannon channel capacity of dispersion-free nonlinear optical fiber transmission," *J. Lightwave Technol.*, vol. 19, pp. 1104–1109, Aug. 2001.

[8] J. Tang, "The multispan effects of Kerr nonlinearity and amplifier noises on Shannon channel capacity for a dispersion-free nonlinear optical fiber," *J. Lightwave Technol.*, vol. 19, pp. 1110–1115, Aug. 2001.

[9] K.-P. Ho and J. M. Kahn, "Channel capacity of WDM systems using constant-intensity modulation formats," in *Opt. Fiber Comm. Conf. (OFC '02)*, 2002, paper ThGG85.

[10] J. M. Kahn, K.-P. Ho, "Ultimate spectral efficiency limits in DWDM systems," *Optoelectronics Commun. Conf.*, Yokohama, Japan, July 8-12, 2002.

[11] I. B. Djordjevic, B. Vasic, "Approaching Shannon's capacity limits of fiber optics communications channels using short LDPC codes," *CLEO/IQEC 2004*, paper CWA7.

[12] E. Narimanov, P. Patel, "Channel capacity of fiber optics communications systems: WDM vs. TDM," in Proc. *Conf. on Lasers and Electro-Optics (CLEO '03)*, 2003, pp. 1666 – 1668.

[13] J. Li, "On the achievable information rate of asymmetric optical fiber channels with amplifier spontaneous emission noise," in Proc. *IEEE Military Comm. Conf. (MILCOM '03)*, Boston, MA, Oct. 2003.

[14] G. P. Agrawal, *Nonlinear Fiber Optics*. San Diego, CA: Academic Press, 2001.

[15] M. S. Pinsker, *Information and Information Stability of Random Variables and Processes*. San Francisco, CA: Holden Day, 1964.

[16] S. K. Turritsyn et. al., "Averaged model and integrable limits in nonlinear double-periodic Hamiltonian systems," *Phys. Rev. E*, vol. 61, pp. 3127-3132, Mar. 2000.

[17] D. Arnold, A. Kavcic, H.-A. Loeliger, P. O. Vontobel, W. Zeng, "Simulation-based computation of information rates: upper and lower bounds," in Proc. *IEEE Intern. Symp. Inform. Theory (ISIT 2003)*, 2003, p. 119.

[18] D. Arnold, H.-A. Loeliger, "On the information rate of binary-input channels with memory," in Proc. *2001 Int. Conf. Communications*, Helsinki, Finland, June 11-14, 2001, pp. 2692-2695.

[19] H. D. Pfitser, J. B. Soriaga, P. H. Siegel, "On the achievable information rates of finite state ISI channels," in Proc. *Globecom 2001*, San Antonio, TX, Nov. 25-29, 2001, pp. 2992-2996.

[20] V. Sharma, S. K. Singh, "Entropy and channel capacity in the regenerative setup with applications to Markov channels," in Proc. *IEEE Int. Symp. Information Theory*, Washington, DC, June 24-29, 2001, p. 283.

[21] L. R. Bahl, J. Cocke, F. Jelinek, J. Raviv, "Optimal decoding of linear codes for minimizing symbol error rate," *IEEE Trans. Information Theory,* pp. 284-287 March 1978.

Considerações finais

Existem várias formas de se dar continuidade aos trabalhos apresentados. Por exemplo, Prof. Dr. J. Cigler da Universität Wien deu uma generalização de nosso trabalho "Catalan Numbers, the Hankel Transform, and Fibonacci Numbers" no artigo "Some Relations Between Generalized Fibonacci and Catalan Numbers" publicado em Sitzungsber Abt. II (2002) 211, de Österreichische Akademie der Disenchantment. O trabalho dele, por sua vez, pode ser generalizado e mais explorado.

É bem conhecido que identidades de MacDonald podem ser vistas como generalizações multivariaveis de identidades de Rogers-Ramanujan. O candidato julga possível a obtenção de interpretações combinatórias, mais provavelmente em termos de partições, para estes identidades.

Para finalizar, teoria de códigos é uma área de grande interesse no momento. Possíveis caminhos são inúmeros. Uma possibilidade é estudar códigos especializados para canais ópticos. Demanda para transmissão de dados em grandes velocidades está motivando o desenvolvimento de técnicas que tratam os limites físicos do canal. Isto representa uma oportunidade para se fazer um trabalho mais aplicado, mas com uma fundamentação matemática mais ligada à combinatória.