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Tese de Doutorado

Singularidades Quânticas

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SINGULARIDADES QUÂNTICAS.

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Abstract

Classically singular spacetimes will be studied from a quantum mechanical point of view. The use of quantum mechanics will be handled in two different ways. The first consists in finding the wave function of the universe by solving the Wheeler-DeWitt equation for the canonical variables of spacetime. The second is through the conformal coupling of scalar and spinorial fields with the gravitational field, where we will study the behavior of wave packets in this curved spacetime.

Resumo

Espaços-tempo classicamente singulares serão estudados de um ponto de vista quântico. A utilização da mecânica quântica será feita de duas maneiras. A primeira consiste em encontrar a função de onda do Universo, resolvendo a equação de Wheeler-DeWitt para as variáveis canônicas do espaço-tempo. A segunda consiste em acoplar conformemente campos escalares e spinoriais ao campo gravitacional, estudando o comportamento de pacotes de ondas neste espaço-tempo curvo.

Sumário

Capítulo 1

Introdução

Singularidades do espaço-tempo são um fardo da relatividade geral. Elas são indicadas por geodésicas incompletas e/ou curvas incompletas de aceleração limitada [1]. Nos pontos onde essas curvas se encerram, as noções clássicas de espaço-tempo deixam de ser válidas, assim como as leis da física, que são formuladas tendo como pano de fundo uma variedade diferenciável. Existem três tipos de singularidades [2,3]: as singularidades quase regulares, onde nenhum observador experencia quantidades físicas divergirem, exceto no momento em que sua trajetória atinge a singularidade (por exemplo, a singularidade no espaço-tempo de uma corda cósmica); as singularidades escalares de curvatura, onde todo observador perto da singularidade sente quantidades físicas divergirem [por exemplo, a singularidade no espaço-tempo de Schwarzchild ou a singularidade do big-bang na cosmologia de Friedmann-Robertson-Walker (FRW)]; as singularidades não-escalares de curvatura, onde existem algumas curvas nas quais os observadores[1] experimentam forças de maré não limitadas (cosmologias do tipo whimper são um bom exemplo).

É sabido que, sob condições bem razoáveis (as condições de energia [1]), as singularidades estão sempre presentes em modelos cosmológicos. Já que a relatividade geral não pode escapar deste fardo, há um consenso geral de que uma teoria completa da gravitação quântica resolverá este problema, nos ensinando como lidar com as singularidades ou mesmo excluindo-as. Enquanto tal teoria não existe, existem muitas tentativas de incorporar a mecânica quântica na relatividade geral. Uma das primeiras tentativas foi a cosmologia

[1]Consideramos aqui como observadores objetos com certa extensão, a fim de que possam experienciar forças de maré.

quântica [4, 5]. Um dos maiores problemas da cosmologia quântica é a ausência de uma variável temporal natural, já que a equação de Wheeler-DeWitt é uma equação funcional diferencial de segunda ordem no super-espaço e não existe uma derivada funcional de primeira ordem pra fazer o papel do tempo. Este problema pode ser contornado com a introdução de campos de matéria, já que a evolução de algum parâmetro da matéria pode nos dar uma noção de tempo. Com a introdução de um fluido perfeito no formalismo Hamiltoniano de Schutz [6, 7], podemos recuperar uma equação do tipo Schrödringer, onde o momento associado ao grau de liberdade do fluido aparece linearmente na equação de Wheeler-DeWitt.

Outro problema associado à cosmologia quântica é a interpretação da função de onda do universo. Já que tudo está incluso no universo, a interpretação probabilística se torna inviável, já que ela assume que existe um processo fundamental de medida fora do mundo quântico, num domínio clássico [8]. Duas das interpretações alternativas mais comuns para a função de onda em cosmologia quântica são as interpretações de Varios-Mundos [9] e a de de Broglie-Bohm [10]. Na primeira, todas as possibilidades de ramificação são de fato realizadas e existe um observador em cada ramo com o conhecimento do auto-valor correspondente, estes ramos nao se interferem. Toda vez que um experimento é realizado, todos os resultados são obtidos, cada um num mundo diferente. Já na segunda, é suposta existir uma trajetória no espaço de fases para cada variável dinâmica, independentemente de qualquer medida. Numa medida, a trajetória entra num dos ramos, dependendo das condições iniciais, que são desconhecidas.

Outra forma de introduzir a mecânica quântica em relatividade geral é através do estudo de equações de campos num espaço-tempo não trivial, solução das equações de Einstein. Pode-se então estudar a quantização destes campos no background curvo, ou a descrição de uma partícula (problemática por motivos já conhecidos da teoria quântica de campos usual), que é o que faremos na segunda parte deste trabalho. Seguindo esta descrição, para um espaço-tempo estático, existe uma definição bem razoável de singularidades quânticas. Como introduzido por Horowitz e Marolf [11], um espaço-tempo estático é quanticamente singular se a evolução do pacote de ondas de uma partícula não é unicamente determinado pelo pacote inicial. Da mesma forma que em relatividade geral, as condições iniciais para uma partícula num background singular não são suficientes para determinar seu futuro se sua linha de mundo atinge uma singularidade (uma condição extra sob a singularidade deve ser imposta), num espaço-tempo quanticamente singular, perdemos da mesma forma a condição de prever o futuro da partícula dada sua configuração inicial e uma condição extra deve ser imposta (condições de contorno como veremos mais pra frente).

Este trabalho será organizado da seguinte maneira: no capítulo 2 apresentaremos a teoria de hipersuperfícies do tipo-espaço. Neste capítulo introduziremos os importantes conceitos de primeira e segunda formas fundamentais de uma hipersuperfície Σ, discutiremos rapidamente o problema de valor inicial em relatividade geral e começaremos a divisão do espaço-tempo em espaço+tempo (formalismo ADM), onde encontraremos uma expressão para a açao em relatividade geral em função da curvatura extrínsica da hipersuperfície Σ, de suas primeira e segunda forma fundamentais e da função lapso, que nos diz como a hipersuperfície evolui no tempo. No capítulo 3 apresentaremos a teoria de dirac [12] de quantização de sitemas com vínculos. No capítulo 4 discutiremos o formalismo Hamiltoniano da relatividade geral, onde mostraremos que esta se trata de uma teoria com vínculos e iniciaremos a quantização do minisuperespaço de FRW, utilizando a teoria desenvolvida no capítulo 3. No capítulo 5 faremos uma breve introdução à interpretação de de Broglie-Bohm da mecânica quântica e no capítulo 6 introduziremos o formalismo de Schutz para fluidos perfeitos, onde terminaremos a quantização do minisuperespaço de FRW, tendo como direção para o tempo o grau de liberdade do fluido. No capítulo 7 daremos as ferramentas matemáticas necessárias para o entendimento do conceito de singularidades quânticas em espaços-tempos estáticos. Os apêndices reproduzem artigos que representam os resultados deste trabalho. Finalmente no capítulo 8, concluiremos o trabalho.

Capítulo 2

Hipersuperfícies do tipo-espaço

Neste capítulo estudaremos as geometrias intrínsica e extrínsica de hipersuperfícies do tipo-espaço Σ mergulhadas num espaço-tempo $\mathcal{M}$, de assinatura Lorentziana $(-1, +1, +1, +1)$. Este estudo será baseado na referência [13], que possui um tratado bem completo sobre este assunto, incluindo o estudo de hipersuperfícies do tipo-tempo e do tipo-luz. Os principais conceitos introduzidos neste capítulo serão os conceitos de derivada intrínsica, que é a derivada covariante compatível com a métrica induzida na hipersuperfície, e o conceito de curvatura extrínsica, que mede como a hipersuperfície se curva em relação à variedade que a engloba. A partir da equação de Gauss-Codazzi encontraremos uma forma mais conveniente para a ação da relatividade geral em função das curvaturas intrínsica e extrínsica da hipersuperfície e discutiremos rapidamente o problema de valor inicial em relatividade geral. Realizado este estudo poderemos iniciar uma pesquisa sobre o formalismo ADM, que a partir da divisão do espaço-tempo em uma família de hipersuperfícies do tipo-espaço parametrizadas pelo tempo, desenvolve a teoria Hamiltoniana para a relatividade geral.

2.1 Equações Básicas

No espaço-tempo quadri-dimensional $\mathcal{M}$, uma hipersuperfície Σ[1] é uma subvariedade tri-dimensional que pode ser do tipo-tempo, tipo-espaço ou tipo-luz. A hipersuperfície pode ser criada a partir de uma restrição nas coordenadas, isto é,

$$\Phi(x^\alpha) = 0, \tag{2.1}$$

ou a partir de equações paramétricas da forma

$$x^\alpha = x^\alpha(y^a), \tag{2.2}$$

onde y^a (a=1,2,3) são as coordenadas intrínsicas da hipersuperfície.

A partir da equação $\Phi(x^\alpha) = 0$, podemos extrair um vetor normal à hipersuperfície da forma

$$n_\alpha = -\frac{\Phi_{,\alpha}}{|g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}|^{1/2}}, \tag{2.3}$$

onde o sinal negativo é convenção para hipersuperfícies do tipo-espaço[2] e nos garante que $n^\alpha \Phi_{,\alpha} > 0$, já que $\Phi_{,\alpha}$ é um vetor do tipo-tempo ortogonal à hipersuperfície Σ.

A métrica induzida em Σ é obtida restringindo o elemento de linha ds^2 a deslocamentos restritos à hipersuperfície. Localmente, toda hipersuperfície pode ser parametrizada da forma (2.2). Desta forma obtemos os vetores

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}, \tag{2.4}$$

[1] Neste capítulo, assumiremos que a variedade $\mathcal{M}$ pode ser folheada por hipersuperfícies 3-dimensionais. As condições para existência de tais folheações serão satisfeitas trivialmente para as variedades que serão estrudadas nos próximos capítulos. Para mais detalhes ver [14].

[2] Como este capítulo é referência para o capítulo 3, que se utiliza da divisão do espaço-tempo numa família de um parâmetro de superfícies do tipo-espaço a fim de obter a teoria Hamiltoniana da relatividade geral, escolhemos estudar neste capítulo apenas hipersuperfícies do tipo-espaço mergulhadas em uma variedade. O estudo de hipersuperfícies do tipo-tempo requer poucas mudanças, principalmente relacionadas à convenções de sinais, enquanto que o estudo de hipersuperfícies do tipo-luz possui algumas sutilezas que de nada serviriam para o restante do trabalho.

tangentes a curvas em Σ. Então, restrito à deslocamentos em Σ, temos

$$ds^2_\Sigma = g_{\alpha\beta}dx^\alpha dx^\beta = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b}dy^b\right) = h_{ab}dy^a dy^b, \tag{2.5}$$

onde

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b \tag{2.6}$$

é chamada métrica induzida ou primeira forma fundamental de Σ. Note que h_{ab} é um escalar com relação à mudança de coordenadas $x^\alpha \to x^{\alpha'}$, mas se transforma como um três-tensor com relação à mudança de coordenadas $y^a \to y^{a'}$ intrínsicas da hipersuperfície.

A seguir um resultado importante para o desenvolvimento da teoria de hipersuperfícies do tipo-espaço.

Teorema 1. *A métrica $g_{\alpha\beta}$ do espaço-tempo se relaciona com a primeira forma fundamental h_{ab} de Σ da forma*

$$g^{\alpha\beta} = -n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b, \tag{2.7}$$

onde h^{ab} é a inversa de h_{ab}, isto é, $h^{ac}h_{cb} = \delta^a_b$.

Demonstração. Para demonstrarmos a identidade acima, vamos calcular os produtos escalares entre os elementos da basa $\{n^\alpha, e^\alpha_a\}$.

- $(-n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b)n_\alpha n_\beta = -\underbrace{n^\alpha n^\beta n_\alpha n_\beta}_{=1} + h^{ab}\underbrace{e^\alpha_a e^\beta_b n_\alpha n_\beta}_{=0} = -1 = g_{\alpha\beta}n^\alpha n^\beta$;
- $(-n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b)n_\alpha e_{c\beta} = -\underbrace{n^\alpha n^\beta n_\alpha e_{c\beta}}_{=0} + h^{ab}\underbrace{e^\alpha_a e^\beta_b n_\alpha e_{c\beta}}_{=0} = 0 = g_{\alpha\beta}n^\alpha e^\beta_c$;
- $(-n^\alpha n^\beta + h^{ab}e^\alpha_a e^\beta_b)e_{c\alpha}e_{d\beta} = -\underbrace{n^\alpha n^\beta e_{c\alpha}e_{d\beta}}_{=0} + h^{ab}e^\alpha_a e_{c\alpha}e^\beta_b e_{d\beta} = h^{ab}h_{ac}h_{bd} = h_{cd} = g_{\alpha\beta}e^\alpha_c e^\beta_d$.

□

Na demonstração acima ultilizamos o fato que $n^\alpha n_\alpha = -1$ (ver eq. (2.3)) e que e^α_a e n^α são perpendiculares, isto é, $e^\alpha_a n_\alpha = 0$.

2.2 Vetores tangentes e derivada intrínsica

Quando estamos trabalhando com uma hipersuperfície Σ, é comum termos objetos geométricos $A^{\alpha\beta\cdots}$ definidos apenas em Σ e que são tangentes à hipersuperfície. Tais objetos possuem a seguinte decomposição

$$A^{\alpha\beta\cdots} = A^{ab\cdots} e^{\alpha}_{a} e^{\beta}_{b} \dots . \tag{2.8}$$

Note que a equação acima implica que $A^{\alpha\beta\cdots} n_\alpha = A^{\alpha\beta\cdots} n_\beta = \cdots = 0$, o que nos diz que $A^{\alpha\beta\cdots}$ é realmente tangente à Σ. Um tensor qualquer $T^{\mu\nu\cdots}$ pode ser projetado na hipersuperfície de forma que somente sua componente tangencial sobrevive. Tal projetor é dado por $h^{\alpha\beta} = h^{ab} e^{\alpha}_{a} e^{\beta}_{b}$ e a projeção de $T^{\mu\nu\cdots}$ em Σ é dada por

$$h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \dots T^{\mu\nu\dots}. \tag{2.9}$$

Note que $h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \dots T^{\mu\nu\cdots} n_\alpha = \underbrace{h^{\alpha}{}_{\mu} n_\alpha}_{=0} h^{\beta}{}_{\nu} \dots T^{\mu\nu\cdots}$, o que nos mostra que $h^{\alpha}{}_{\mu} h^{\beta}{}_{\nu} \dots T^{\mu\nu\cdots}$ é tangente à Σ.

A projeção

$$A_{\alpha\beta\dots} e^{\alpha}_{a} e^{\beta}_{b} = A_{ab\dots} \equiv h_{am} h_{bn} \dots A^{mn\dots} \tag{2.10}$$

nos dá o três-tensor $A_{ab\dots}$ associados ao tensor tangente $A^{\alpha\beta\cdots}$. As equações (2.8) e (2.10) nos mostram como obtermos um quadri-tensor tangente $A^{\alpha\beta\cdots}$ a partir de seu três-tensor correspondente $A^{ab\cdots}$ e vice-versa. O três-tensor $A_{ab\dots}$ se transforma como um tensor pela mudança de coordenadas intrínsicas $y^a \to y^{a'}$, enquanto que é um escalar pela mudança de coordenadas do espaço-tempo $x^\alpha \to x^{\alpha'}$.

Vamos agora defininir a derivada covariante intrínsica à Σ de um campo vetorial tangente A^α tal que

$$A^\alpha n_\alpha = 0; \quad A^\alpha = A^a e^{\alpha}_{a}; \quad A_a = A_\alpha e^{\alpha}_{a}. \tag{2.11}$$

Definimos a derivada covariante intrínsica de A_a como sendo a projeção de $A_{\alpha;\beta}$ na hipersuperfície Σ, isto é,

$$A_{a|b} = A_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b}. \tag{2.12}$$

Vamos mostrar agora que a derivada covariante de A_a definida desta maneira nada mais é do que a derivada covariante de A_a definida da maneira usual com a conexão Γ^a_{bc} compatível com h_{ab}. Para isto, comecemos com

$$
\begin{aligned}
A_{\alpha;\beta}e^\alpha_a e^\beta_b &= (A_\alpha e^\alpha_a)_{;\beta}e^\beta_b - A_\alpha e^\alpha_{a;\beta}e^\beta_b \\
&= A_{a,\beta}e^\beta_b - A^\alpha e_{a\alpha;\beta}e^\beta_b \\
&= \frac{\partial A_a}{\partial x^\beta}\frac{\partial x^\beta}{\partial y^b} - A^c e^\alpha_c e_{a\alpha;\beta}e^\beta_b \\
&= A_{a,b} - A^c\Gamma_{cab} \\
&= A_{a,b} - A_c\Gamma^c_{ab},
\end{aligned}
\tag{2.13}
$$

onde

$$
\Gamma_{cab} = e^\alpha_c e_{a\alpha;\beta}e^\beta_b. \tag{2.14}
$$

Vamos mostrar que a conexão acima é compatível com a métrica γ induzida em Σ, isto é, vamos mostrar que

$$
\Gamma_{cab} = \frac{1}{2}(h_{ca,b} + h_{cb,a} - h_{ab,c}). \tag{2.15}
$$

Para isto basta mostrarmos que a derivada covariante itrínsica definida à pouco é compatível com a métrica, ou seja, $h_{ab|c} = 0$. Tomemos então

$$
\begin{aligned}
h_{ab|c} = h_{\alpha\beta;\gamma}e^\alpha_a e^\beta_b e^\gamma_c &= (g_{\alpha\beta} + n_\alpha n_\beta)_{;\gamma}e^\alpha_a e^\beta_b e^\gamma_c \\
&= (n_\alpha n_{\beta;\gamma} + n_{\alpha;\gamma}n_\beta)e^\alpha_a e^\beta_b e^\gamma_c \\
&= 0.
\end{aligned}
\tag{2.16}
$$

Note que na demonstração acima utilizamos o fato que $g_{\alpha\beta;\gamma} = 0$ e $n_\alpha e^\alpha_a = 0$. Mostramos assim que a derivada covariante intrínsica a Σ como definida em (2.12) nada mais é do que a derivada covariante usual compatível com a métrica induzida h em Σ.

2.3 Curvatura extrínsica

A quantidade $A_{a|b}$ é a componente tangencial do vetor $A^{\alpha}{}_{;\beta}e^{\beta}_{b}$. Nos perguntamos agora se este vetor possui uma componente normal à hipersuperfície.

$$\begin{aligned} A^{\alpha}{}_{;\beta}e^{\beta}_{b} &= g^{\alpha}_{\mu}A^{\mu}{}_{;\beta}e^{\beta}_{b} = (-n^{\alpha}n_{\mu} + h^{am}e^{\alpha}_{a}e_{m\mu})A^{\mu}{}_{;\beta}e^{\beta}_{b} \\ &= -(n_{\mu}A^{\mu}{}_{;\beta}e^{\beta}_{b})n^{\alpha} + h^{am}(A_{\mu;\beta}e^{\mu}_{m}e^{\beta}_{b})e^{\alpha}_{a} \\ &= (n_{\mu;\beta}A^{\mu}e^{\beta}_{b})n^{\alpha} + h^{am}A_{m|b}e^{\alpha}_{a} \\ &= A^{a}{}_{|b}e^{\alpha}_{a} + A^{a}\underbrace{(n_{\mu;\beta}e^{\mu}_{a}e^{\beta}_{b})}_{\equiv K_{ab}}n^{\alpha}. \end{aligned} \tag{2.17}$$

Note que da segunda para a terceira linha da equação acima ultilizamos o fato que A^{α} é perpendicular à n^{α} e na última equação definimos o que chamaremos de curvatura extrínsica ou segunda forma fundamental

$$K_{ab} \equiv (n_{\mu;\beta}e^{\mu}_{a}e^{\beta}_{b}), \tag{2.18}$$

que nada mais é do que a projeção na hipersuperfície Σ da derivada covariante do vetor normal n^{μ}.

Temos assim a seguinte decomposição do vetor $A^{\alpha}{}_{;\beta}e^{\beta}_{b}$:

$$A^{\alpha}{}_{;\beta}e^{\beta}_{b} = A^{a}{}_{|b}e^{\alpha}_{a} + A^{a}K_{ab}n^{\alpha}. \tag{2.19}$$

A componente normal só se anula se a curvatura extrínsica é nula. Se no lugar de A^{α} colocarmos e^{α}_{a}, temos $e^{\alpha}_{a} = \delta^{c}_{a}e^{\alpha}_{c}$, logo

$$\begin{aligned} e^{\alpha}_{a;\beta}e^{\beta}_{b} &= \delta^{c}_{a|b}e^{\alpha}_{c} + \delta^{c}_{a}K_{cb}n^{\alpha} \\ &= \Gamma^{c}_{ab}e^{\alpha}_{c} + K_{ab}n^{\alpha}, \end{aligned} \tag{2.20}$$

que é conhecida como equação de Gaus-Weingarten.

Utilizando o fato que e^{α}_{a} e n^{α} são ortogonais e que $\{e^{\alpha}_{a}\}$ é uma base coordenada, o que implica que $e^{\alpha}_{a;\beta}e^{\beta}_{b} = e^{\alpha}_{b;\beta}e^{\beta}_{a}$, temos

$$K_{ab} = n_{\alpha;\beta}e^{\alpha}_{a}e^{\beta}_{b} = -n_{\alpha}e^{\alpha}_{a;\beta}e^{\beta}_{b} = -n_{\alpha}e^{\alpha}_{b;\beta}e^{\beta}_{a} = n_{\alpha;\beta}e^{\alpha}_{b}e^{\beta}_{a} = K_{ba}. \tag{2.21}$$

Logo, o três-tensor curvatura extrínsica K_{ab} é simétrico, portanto

$$K_{ab} = n_{(\alpha;\beta)} e^\alpha_a e^\beta_b = (\mathcal{L}_n g_{\alpha\beta}) e^\alpha_a e^\beta_b, \tag{2.22}$$

onde $\mathcal{L}_n$ denota a derivada de Lie na direção do vetor n.

O tensor de curvatura extrínsica é a projeção na hipersuperfície Σ da derivada de Lie da métrica $g_{\alpha\beta}$ com relação ao vetor normal n^μ. O traço de K_{ab} é obtido da seguinte maneira

$$K \equiv K^a{}_a = n_{\alpha;\beta} h^{ab} e^\alpha_a e^\beta_b = (g^{\alpha\beta} + n^\alpha n^\beta) n_{\alpha;\beta} = g^{\alpha\beta} n_{\alpha;\beta} = n^\alpha{}_{;\alpha}, \tag{2.23}$$

onde utilizamos na equação acima que $n^\alpha n_{\alpha;\beta} = 0$ pois n^α é um vetor unitário.

2.4 Equações de Gauss-Codazzi

A partir da métrica h_{ab} induzida em Σ, podemos extrair o tensor de curvatura da hipersuperfície da forma

$$R^c{}_{dab} = \Gamma^c_{db,a} - \Gamma^c_{da,b} + \Gamma^c_{ma}\Gamma^m_{db} - \Gamma^c_{mb}\Gamma^m_{da}. \tag{2.24}$$

Este tensor satisfaz a identidade

$$A^c{}_{|ab} - A^c{}_{|ba} = R^c{}_{dab} A^d. \tag{2.25}$$

Vamos agora relacionar os tensores de curvatura do espaço-tempo quadri-dimensional $\mathcal{M}$, $R^\alpha{}_{\beta\gamma\delta}$, e da hipersuperfície Σ, $R^a{}_{bcd}$. Para isto, comecemos com a equação de Gaus-Weingarten (2.20)

$$(e^\alpha_{a;\beta} e^\beta_b)_{;\gamma} e^\gamma_c = (\Gamma^d_{ab} e^\alpha_d + K_{ab} n^\alpha)_{;\gamma} e^\gamma_c. \tag{2.26}$$

Vamos analisar cada um dos lados da equação acima separadamente:

- Lado esquerdo:

$$\begin{aligned}(e^\alpha_{a;\beta}e^\beta_b)_{;\gamma}e^\gamma_c &= e^\alpha_{a;\beta\gamma}e^\beta_b e^\gamma_c + e^\alpha_{a;\beta}e^\beta_{b;\gamma}e^\gamma_c \\ &= e^\alpha_{a;\beta\gamma}e^\beta_b e^\gamma_c + e^\alpha_{a;\beta}(\Gamma^d_{bc}e^\beta_d + K_{bc}n^\beta) \\ &= e^\alpha_{a;\beta\gamma}e^\beta_b e^\gamma_c + \Gamma^d_{bc}(\Gamma^e_{ad}e^\alpha_e + K_{ad}n^\alpha) + K_{bc}e^\alpha_{a;\beta}n^\beta.\end{aligned} \tag{2.27}$$

- Lado direito:

$$\begin{aligned}(\Gamma^d_{ab}e^\alpha_d + K_{ab}n^\alpha)_{;\gamma}e^\gamma_c &= \Gamma^d_{ab,c}e^\alpha_d + \Gamma^d_{ab}e^\alpha_{d;\gamma}e^\gamma_c + K_{ab,c}n^\alpha + K_{ab}n^\alpha{}_{;\gamma}e^\gamma_c \\ &= \Gamma^d_{ab,c}e^\alpha_d + \Gamma^d_{ab}(\Gamma^e_{dc}e^\alpha_e + K_{dc}n^\alpha) + K_{ab,c}n^\alpha + K_{ab}n^\alpha{}_{;\gamma}e^\gamma_c.\end{aligned} \tag{2.28}$$

Resolvendo para $(e^\alpha_{a;\beta\gamma}e^\beta_b e^\gamma_c)$, temos

$$\begin{aligned}(e^\alpha_{a;\beta\gamma}e^\beta_b e^\gamma_c) =&\Gamma^d_{ab,c}e^\alpha_d + \Gamma^d_{ab}(\Gamma^e_{dc}e^\alpha_e + K_{dc}n^\alpha) \\ &+ K_{ab,c}n^\alpha + K_{ab}n^\alpha{}_{;\gamma}e^\gamma_c - K_{bc}e^\alpha_{a;\beta}n^\beta \\ &- \Gamma^d_{bc}(\Gamma^e_{ad}e^\alpha_e + K_{ab}n^\alpha).\end{aligned} \tag{2.29}$$

Subtraindo $e^\alpha_{a;\gamma\beta}e^\gamma_c e^\beta_b$, temos

$$R^\mu{}_{\alpha\beta\gamma}e^\alpha_a e^\beta_b e^\gamma_c = R^m{}_{abc}e^\mu_m - (K_{ab|c} - K_{ac|b})n^\mu - K_{ab}n^\mu{}_{;\gamma}e^\gamma_c + K_{ac}n^\mu{}_{;\beta}e^\beta_b. \tag{2.30}$$

Projetando a equação acima com respeito a $e_{d\mu}$ obtemos

$$R_{\alpha\beta\gamma\delta}e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d = R_{abcd} - (K_{ad}K_{bc} - K_{ac}K_{bd}) \tag{2.31}$$

e vemos que a curvatura intrínsica à superfície não é apenas a projeção da curvatura da variedade devido a um termo de curvatura extrínsica. Projetando agora com relação à n^μ temos

$$R_{\mu\alpha\beta\gamma}n^\mu e^\alpha_a e^\beta_b e^\gamma_c = K_{ab|c} - K_{ac|b}. \tag{2.32}$$

As duas equações acima são conhecidas como equações de Gauss-Codazzi. Elas dizem que as componentes da curvatura da três-geometria concordam com as componentes correspondentes da curvatura da quatro-geometria nos pontos do mergulho da hipersuperfície que são livres de curvatura extrínsica. A situação oposta é ilustrada pelo exemplo da esfera mergulhada no $\mathbb{R}^3$. Neste exemplo o lado esquerdo da equação (2.31) é nulo e as curvaturas intrínsica e extrínsicas se cancelam exatamente.

Vamos agora encontrar como o escalar de Ricci se relaciona com as quantidades encontradas até agora ao longo deste capítulo. Para isto vejamos

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu}R_{\mu\alpha\nu\beta} = (-n^{\mu}n^{\nu} + h^{mn}e^{\mu}_{m}e^{\nu}_{n})R_{\mu\alpha\nu\beta} \\ &= -R_{\mu\alpha\nu\beta}n^{\mu}n^{\nu} + h^{mn}R_{\mu\alpha\nu\beta}e^{\mu}_{m}e^{\nu}_{n}. \end{aligned} \tag{2.33}$$

Temos assim

$$\begin{aligned} R = g^{\alpha\beta}R_{\alpha\beta} &= (-n^{\alpha}n^{\beta} + h^{ab}e^{\alpha}_{a}e^{\beta}_{b})(-R_{\mu\alpha\nu\beta}n^{\mu}n^{\nu} + h^{mn}R_{\mu\alpha\nu\beta}e^{\mu}_{m}e^{\nu}_{n}) \\ &= -2h^{ab}R_{\mu\alpha\nu\beta}n^{\mu}e^{\alpha}_{a}n^{\nu}e^{\beta}_{b} + h^{ab}h^{mn}R_{\mu\alpha\nu\beta}e^{\mu}_{m}e^{\alpha}_{a}e^{\nu}_{n}e^{\beta}_{b}, \end{aligned} \tag{2.34}$$

onde usamos acima que $R_{\mu\alpha\nu\beta}n^{\mu}n^{\alpha}n^{\nu}n^{\beta} = 0$. Vejamos agora termo a termo do lado direito da equação acima.

- Primeiro termo:

$$\begin{aligned} -2h^{ab}R_{\mu\alpha\nu\beta}n^{\mu}e^{\alpha}_{a}n^{\nu}e^{\beta}_{b} &= -2g^{\alpha\beta}R_{\mu\alpha\nu\beta}n^{\mu}n^{\nu} = -2R_{\mu\nu}n^{\mu}n^{\nu} \\ &= -2(-n^{\mu}{}_{;\mu\nu}n^{\nu} + n^{\mu}{}_{;\nu\mu}n^{\nu}) \\ &= -2\left[-(n^{\mu}{}_{;\mu}n^{\nu})_{;\nu} + n^{\mu}{}_{;\mu}n^{\nu}{}_{;\nu} + (n^{\mu}{}_{;\nu}n^{\nu})_{;\mu} - n^{\mu}{}_{;\nu}n^{\nu}{}_{;\mu}\right]. \end{aligned} \tag{2.35}$$

O segundo termo da equação acima nada mais é que $-2K^2$, onde $K = K^{a}{}_{a}$, já o ultimo termo é

$$\begin{aligned} 2n^{\mu}{}_{;\nu}n^{\nu}{}_{;\mu} &= 2g^{\beta\mu}g^{\alpha\nu}n_{\alpha;\beta}n_{\mu;\nu} \\ &= 2(n^{\beta}n^{\mu} + h^{\beta\mu})(n^{\alpha}n^{\nu} + h^{\alpha\nu})n_{\alpha;\beta}n_{\mu;\nu} \\ &= 2h^{\beta\mu}h^{\alpha\nu}n_{\alpha;\beta}n_{\mu;\nu} \\ &= 2h^{bm}h^{an}n_{\alpha;\beta}e^{\alpha}_{a}e^{\beta}_{b}n_{\mu;\nu}e^{\mu}_{m}e^{\nu}_{n} \\ &= 2h^{bm}h^{an}K_{ab}K_{mn} \\ &= 2K^{ab}K_{ab}. \end{aligned} \tag{2.36}$$

Temos assim

$$-2h^{ab}R_{\mu\alpha\nu\beta}n^{\mu}e^{\alpha}_{a}n^{\nu}e^{\beta}_{b} = -2(n^{\nu}n^{\mu}{}_{;\nu} - n^{\mu}n^{\nu}{}_{;\nu})_{;\mu} - 2K^2 + 2K^{ab}K_{ab}. \tag{2.37}$$

- Segundo termo: Pela equação de Gauss-Codazzi

$$h^{ab}h^{mn}R_{\mu\alpha\nu\beta}e^{\mu}_{m}e^{\alpha}_{a}e^{\nu}_{n}e^{\beta}_{b} = h^{ab}h^{mn}[R_{manb} - (K_{mb}K_{an} - K_{mn}K_{ab})] = {}^{(3)}R - K_{ab}K^{ab} + K^2. \tag{2.38}$$

Portanto o escalar de Ricci é expresso como

$${}^{(4)}R = {}^{(3)}R - K^2 + K_{ab}K^{ab} - 2(n^{\alpha}{}_{;\beta}n^{\beta} - n^{\alpha}n^{\beta}{}_{;\beta})_{;\alpha}. \tag{2.39}$$

2.5 Problema de valor inicial

Para discutirmos o problema de valor inicial em relatividade geral, vamos primeiramente tomar o exemplo do eletromagnetismo (referências [15] e [16]). As equações de Maxwell para o potencial vetor A_μ no espaço-tempo de Minkowski tem a forma

$$\partial^{\mu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) = 0. \tag{2.40}$$

O teorema que garante a solução de um problema de valor inicial é o teorema de Cauchy-Kowalewski [15], que nos diz que

Teorema 2. *Sejam $t, x^1, \ldots, x^{n-1}$ coordenadas de $\mathbb{R}^n$. Considere um sistema de m equações diferenciais para m funções desconhecidas $\phi_1, \ldots, \phi_m$ em $\mathbb{R}^n$, tendo a forma*

$$\frac{\partial^2\phi_i}{\partial t^2} = F_i(t, x^a; \phi_j; \partial\phi_j/\partial t; \partial\phi_j/\partial x^a; \partial^2\phi_j/\partial t\partial x^a; \partial^2\phi_j/\partial x^a\partial x^b), \tag{2.41}$$

onde cada F_i é uma função analítica de suas variáveis. Sejam $f_i(x^\alpha)$ e $g_i(x^\alpha)$ funções analíticas. Então existe uma vizinhança aberta $\mathcal{O}$ da hipersuperfície $t = t_0$ tal que em $\mathcal{O}$ existe uma única solução analítica da equação (2.41) tal que $\phi_i(t_0, x^\alpha) = f_i(x^\alpha)$ e $\frac{\partial\phi_i}{\partial t}(t_0, x^\alpha) = g_i(x^\alpha)$.

Note que a componente temporal da equação (2.40) não possui segunda derivada temporal. De fato, em notação vetorial temos

$$\nabla^2 A_0 - \vec{\nabla}\cdot(\partial\vec{A}/\partial t) = 0, \tag{2.42}$$

isto é,

$$\vec{\nabla} \cdot \vec{E} = 0, \tag{2.43}$$

onde o campo elétrico $\vec{E}$ é definido por

$$\vec{E} = \vec{\nabla} A_0 - \partial \vec{A}/\partial t. \tag{2.44}$$

As equações de Maxwell restantes contém segundas derivadas temporais, nos permitindo resolver $\partial^2 A_\mu/\partial t^2$ para $\mu = 1, 2, 3$ da maneira requerida pelo teorema de Cauchy-Kowalewski. As equações de Maxwell são de fato um conjunto de três equações para as componentes espaciais de A_μ, juntamente com um vínculo de valor inicial (2.43). Escolhendo arbitrariamente A_0 no espaço-tempo, obtemos uma única solução $A_\mu(x^\alpha)$. Vemos assim que o vínculo (2.43) diminui o número de graus de liberdade na eletrodinâmica de quatro para três e nos dá a liberdade de escolha de um gauge específico através de A_0. Isto é de se esperar, já que sabemos que dois potenciais vetores A_μ e A'_μ que diferem um do outro por um gradiente $\partial_\mu \chi$ são soluções do mesmo problema físico.

A teoria da relatividade geral é invariante por mudanças arbitrária de coordenadas do espaço-tempo de $x^\alpha \to x^{\alpha'}$, então é de se esperar que algo parecido com o que foi descrito acima para o caso da eletrodinâmica aconteça. De fato, utilizando as técnicas da seção anterior, podemos encontrar imediatamente quatro das dez equações de Einstein. São elas

$$G_{\alpha\beta} n^\alpha n^\beta = \frac{1}{2} {}^{(3)}R - \frac{1}{2}(K^{ab}K_{ab} - K^2) = 8\pi\rho \tag{2.45}$$

$$G_{\alpha\beta} e^\alpha_a n^\beta = K^b{}_{a|b} - K_{,a} = 8\pi j_a, \tag{2.46}$$

onde ρ e j_a são a densidade de energia e a corrente de momento, respectivamente. Note as duas equações acima não possuem segundas derivadas temporais e funcionam como vínculos de valor inicial. As quantidades iniciais relevantes são uma hipersuperfície Σ do tipo espaço, escolhida arbitrariamente, a geometria desta hipersuperfície, representada pela métrica induzida h_{ab}, e sua primeira derivada temporal, representada pela curvatura extrínsica K_{ab}. Note que a primeira e a segunda forma fundamental caracterizam completamente a hipersuperfície Σ, as demais componentes da métrica são arbitrárias e representam exatamente a liberdade de gauge da teoria. Note também que h_{ab} e K_{ab} contabilizam seis graus de liberdade, exatamente o número de equações de Einstein que não foram explicitadas acima e que dão origem à dinâmica da teoria.

2.6 Divisão do espaço-tempo em espaço + tempo

A idéia do formalismo ADM é dividir o espaço-tempo em uma família de um parâmetro de hipersuperfícies do tipo espaço (ver figura abaixo). A partir disto, dada as 3-geometrias das duas faces de um sanduíche de espaço-tempo, ajusta-se a 4-geometria entre as faces de modo a extremizar a ação de Einstein-Hilbert

$$S = \int d^4x(-g)^{1/2}R, \tag{2.47}$$

onde $g = \det(g_{\alpha\beta})$ é o determinante da métrica e R é o escalar de curvatura (escalar de Ricci).

O sanduíche mais simples é o de comprimento infinitesimal, no qual utilizamos coordenadas adaptadas de forma que as duas faces do sanduíche são dadas pelas equações $t =$contante e $t + dt =$constante.

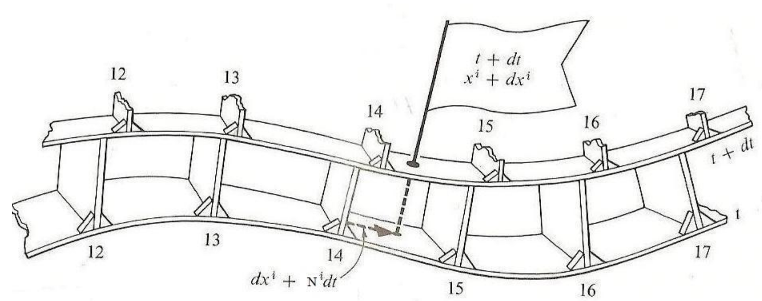

Figura 2.6-1. Divisão do espaço-tempo em espaço + tempo. Figura retirada da referência [16].

Sejam Σ_t e Σ_{t+dt} tais hipersuperfícies com métricas $g(t, x^i)$ e $g(t + dt, x^i)$, respectivamente. Denotemos por t^μ o vetor fluxo de tempo t ilustrado na figura 2.6-2.

Os pontos (t, x^i) em Σ_t e $(t + dt, x^i)$ em Σ_{t+dt} são ligados pelo vetor t^μ, não necessariamente perpendicular à hipersuperfície base. Ao ponto da superfície Σ_{t+dt} que é ligado perpendicularmente à (t, x^i) denotamos $(t+dt, x^i - N^i dt)$, onde $N^i(t, x^j)$ é chamada função de deslocamento, isto é,

$$x^i_{\text{sup}\perp}(x^m) = x^i - N^i(t, x^j)dt. \tag{2.48}$$

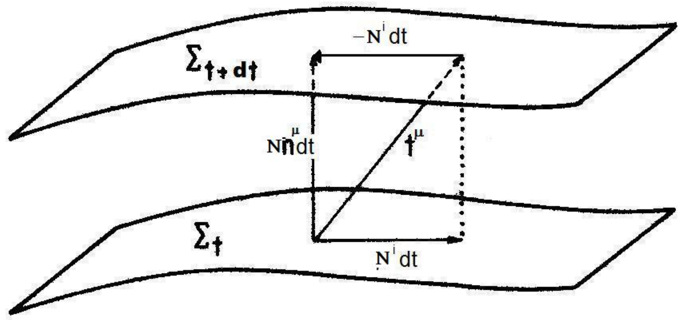

Figura 2.6-2. Fluxo de tempo t^μ de Σ_t para Σ_{t+dt}. Figura retirada da referência [15].

O comprimento da componente de t^μ que liga (t, x^i) à $(t + dt, x^i - N^i dt)$ é igual ao tempo próprio entre as duas hipersuperfícies e denotamos por $d\tau = N(t, x^i)dt$, com $N(t, x^i)$ chamada função lapso. O ponto de Σ_t que é ligado perpendicularmente ao ponto $(t + dt, x + dx^i)$ é dado por $(t, x^i + dx^i + N^i dt)$.

O teorema de Pitágoras em sua forma 4-dimensional nos diz que o elemento de linha ds^2 entre os pontos (t, x^i) e $(t + dt, x^i + dx^i)$ é dado por

$$ds^2 = -(\text{tempo próprio entre as hipersuperfícies})^2 + (\text{distância própria na 3-geometria base})^2. \tag{2.49}$$

Portanto, com o auxílio da figura 2.6-1 vemos que a métrica relatada acima assume a forma

$$\begin{aligned} ds^2 &= -(Ndt)^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\ &= (N_i N^i - N^2)dt^2 + h_{ij}dx^i dx^j + 2N_i dx^i dt, \end{aligned} \tag{2.50}$$

onde $N_i = h_{ij}N^j$.

As matrizes representando a métrica e sua inversa são dadas por

$$(g_{\mu\nu}) = \begin{pmatrix} N_k N^k - N^2 & N_k^T \\ Ni & h_{ij} \end{pmatrix}, \tag{2.51}$$

onde N_i e N_i^T na matriz acima representam o vetor coluna e o vetor linha, respectivamente, formados pelas componentes covariantes da função deslocamento e

$$(g^{\mu\nu}) = \begin{pmatrix} -N^{-2} & N^{-2}(N^j)^T \\ N^{-2}N^i & h^{ij} - N^{-2}N^i(N^j)^T \end{pmatrix}, \tag{2.52}$$

onde N^i e $(N^i)^T$ representam o vetor coluna e o vetor linha, respectivamente, formados pelas componentes contravariantes da função deslocamento.

O cálculo do elemento de volume é um pouco mais complicado e por isso fazemos em detalhes a seguir:

$$\begin{aligned} det(g_{\mu\nu}) &= det\begin{pmatrix} N_iN^i - N^2 & N_i^T \\ N_i & h_{ij} \end{pmatrix} = det\begin{pmatrix} N_iN^i & N_i^T \\ N_i & h_{ij} \end{pmatrix} + det\begin{pmatrix} -N^2 & 0 \\ Ni & h_{ij} \end{pmatrix} \\ &= det\begin{pmatrix} (N^i)^T h_{ij} N^j & (N^i)^T h_{ij} \\ h_{ij}N^j & h_{ij} \end{pmatrix} - N^2 h \\ &= \underbrace{det\left[\begin{pmatrix} (N^i)^T h_{ij} & 0 \\ h_{ij} & 0 \end{pmatrix}\begin{pmatrix} N^j & I_{3x3} \\ 0 & 0 \end{pmatrix}\right]}_{=0} - N^2 h \\ &= -N^2 h, \end{aligned} \tag{2.53}$$

onde na primeira linha foi usado que o determinante é linear em suas linhas e na segunda linha usamos que as componentes covariantes da função deslocamento são obtidas através das componentes contravariantes por uma multiplicação à esquerda pela matriz da 3-métrica h_{ij} da hipersuperfície base.

Portanto, o elemento de volume tem a forma

$$\sqrt{-g}dx^0dx^1dx^2dx^3 = N\sqrt{h}dtdx^1dx^2dx^3. \tag{2.54}$$

Como a hipersuperfície Σ_t é uma hipersuperfície de tempo constante, ela pode ser descrita pela restrição $\Phi(x^\alpha) = 0$, onde $\Phi = t -$ constante. O vetor normal à essa superfície é dado por (ver eq. 2.3)

$$n_\alpha = (-N, 0, 0, 0). \tag{2.55}$$

A forma contravariante de n^α é dada por

$$n^\mu = g^{\mu\nu}n_\nu = N^{-1}(1, -N^i). \tag{2.56}$$

Veja que $n_\mu n^\mu = -1$ de forma que o vetor normal n^α é unitário.

Então um conector ligando perpendicularmente duas superfícies infinitesimalmente próximas tem componentes

$$n^\mu d\tau = n^\mu N dt = (dt, -N^i dt). \tag{2.57}$$

O tensor de curvatura extrínsica K_{ab} é dado por

$$\begin{aligned}
K_{ab} &= n_{\alpha;\beta} e_a^\alpha e_b^\beta \\
&= n_{a;b} \\
&= -\Gamma^\gamma_{ab} n_\gamma \\
&= N\Gamma^0_{ab} \\
&= N(g^{00(4)}\Gamma_{0ik} + g^{0p(4)}\Gamma_{pik}) \\
&= \frac{1}{2N}\left[-N_{i,k} - N_{k,i} + h_{ik,0} - 2N^p\Gamma_{pik}\right] \\
&= \frac{1}{2N}\left[h_{ik,0} - N_{i|k} - N_{k|i}\right].
\end{aligned} \tag{2.58}$$

Sabemos através da equação acima calcular as componentes da curvatura extrínsica de Σ em termos da 3-métrica h_{ij} e das funções lapso, N, e deslocamento, N^i.

2.7 Ação para a relatividade geral

Por fim chegamos que a densidade Lagrangeana de Einstein-Hilbert é expressa por

$$\begin{aligned}
\sqrt{-g}^{(4)}R &= \sqrt{-g}\{^{(3)}R - K^2 + K_{ab}K^{ab} + V^\alpha_{\ ;\alpha}\} \\
&= \sqrt{-g}[^{(3)}R - K^2 + K_{ab}K^{ab}] + (\sqrt{-g}V^\alpha)_{,\alpha},
\end{aligned} \tag{2.59}$$

onde

$$V^\alpha = -2(n^\alpha{}_{;\beta} n^\beta - n^\alpha n^\beta{}_{;\beta}). \tag{2.60}$$

É possível mostrar [4, 17, 18] que

$$V^\alpha{}_{,\alpha} = 2(h^{1/2}K)_{,0} - 2\left(h^{1/2}KN^i - h^{1/2}h^{ij}\alpha_{,j}\right)_{,i}. \tag{2.61}$$

Temos assim que a ação para a relatividade geral é dada por[3]

$$S = \int_{\mathcal{V}} d^4x \left[N h^{1/2} (K_{ab}K^{ab} - K^2 + {}^{(3)}R) - 2(h^{1/2}K)_{,0} + 2(h^{1/2}KN^i - h^{1/2}h^{ij}N_{,j})_{,i} \right], \tag{2.62}$$

onde $\mathcal{V}$ é um sanduiche de duas hipersuperfícies do tipo expaço que se extendem por todo o infinito e são envolvidas por um tres-cilindro no infinito.

O último termo da integral acima não contribui por ser um termo de divergência total. O penúltimo termo contribui da forma

$$-2 \int_{\dot{\mathcal{V}}} \sqrt{h} K \mathrm{d}^3 x. \tag{2.63}$$

Este termo pode ser removido adicionando a ação de Einstein-Hilbert um termo de superfície de forma que nossa a nova ação é dada por [18]

$$S = \int_{\mathcal{V}} {}^{(4)}R\sqrt{-g} d^4x + 2 \int_{\dot{\mathcal{V}}} K\sqrt{h} d^3x. \tag{2.64}$$

É possível mostrar (ver [13]) que a variação da ação acima nos leva à equação de Einstein.

De qualquer forma, a ação que consideraremos daqui para frente é dada por

$$S = \int N\sqrt{h}({}^{(3)}R + K_{ab}K^{ab} - K^2)\mathrm{d}^4x \tag{2.65}$$

[3]Há aqui um abuso de notação quando consideramos o elemento de volume de toda a variedade por $\sqrt{-g}d^4x$, já que geralmente a variedade não pode ser coberta por uma única carta.

Capítulo 3

Quântização de Sistemas com Vínculos

3.1 Sistemas Hamiltonianos com Vínculos

Na seção anterior encontramos (equação (2.59)) uma expressão para a Lagrangeana da relatividade geral em função dos novos conceitos introduzidos a partir da separação $3+1$ do espaço-tempo. Como Dirac relatou na referência [12], se quisermos uma teoria que seja relativística, basta exigirmos que a Lagrangeana seja um escalar. No nosso caso, como partimos de uma Lagrangeana claramente invariante ($\mathcal{L} = (-g)^{1/2(4)}R$), podemos garantir que a teoria obtida através do formalismo Hamiltoniano é relativística. Mesmo que não se mostre manifestadamente covariante.

A teoria da relatividade geral é covariante com relação à uma mudança arbitrária de coordenadas. Esta invariância, assim como a invariância de gauge na eletrodinâmica de Maxwell, dá origem ao aparecimento de vínculos. Estes vínculos, como veremos mais tarde, dão origem à mudanças arbitrárias nas variáveis canônicas sem alterar o estado físico do sistema. A seguir vamos descrever como trabalhamos com o formalismo Hamiltoniano quando nosso sistema posssui vínculos. Vamos estudar o caso de um sistema com número finito de graus de liberdade por questão de simplicidade. A extensão para campos será imediata.

Como dito anteriormente é sempre bom iniciarmos o estudo de um sistema através de uma Lagrangeana, definindo-se a ação

$$S = \int L\mathrm{d}t. \tag{3.1}$$

Com N graus de liberdade temos N coordenadas generalizadas q^n, $n = 1, \ldots, N$ e N velocidades $\dot{q^n}$. Aqui vemos a importância que a variável tempo tem no formalismo. Isto é estranho para uma teoria relativística e obviamento os resultados que encontraremos não serão covariantes. Mas lembre-se que partindo de uma Lagrangeana invariante, nossa teoria é verdadeiramente relativística.

A partir das equações de Euler-Lagrange, encontramos as funcões $q^n(t)$ que extremizam a ação (3.1)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^n}\right) - \frac{\partial L}{\partial q^n} = 0 \quad n = 1, \ldots, N. \tag{3.2}$$

Definimos então o momento p_n conjugado à q^n por

$$p_n = \frac{\partial L}{\partial \dot{q}^n}. \tag{3.3}$$

No nosso caso, não vamos exigir que os momentos $p's$ sejam funções independentes da velocidade, como é feito nas teorias dinâmicas usuais. Podem existir então relações conectando as variáveis canônicas, isto é, podem existir vínculos

$$\phi_m(q's, p's) = 0 \quad m = 1, \ldots, M. \tag{3.4}$$

Obviamente não podemos ter mais vínculos do que graus de liberdade pois se todos os vínculos fossem independentes não teríamos solução.

Estes vínculos (equação (3.4)) são chamados vínculos primários. Eles provém direto da Lagrangeana, não das equações de movimento, como será o caso dos vínculos secundários que veremos mais pra frente.

Definimos agora a quantidade H, que chamaremos Hamiltoniana, da forma:

$$H = \sum_{n=1}^{N} p_n \dot{q}^n - L. \tag{3.5}$$

Vamos a partir de agora esquecer o símbolo de soma e usar a convensão que índices repetidos se somam.

Variando-se H em relação à q^n, $\dot{q}^n$ e p_n e utilizando o fato que $p_n = \frac{\partial L}{\partial \dot{q}^n}$, temos

$$\delta H = \dot{q}^n \delta p_n + p_n \delta \dot{q}^n - \frac{\partial L}{\partial q^n}\delta q^n - \frac{\partial L}{\partial \dot{q}^n}\delta \dot{q}^n = \dot{q}^n \delta p_n - \frac{\partial L}{\partial q^n}\delta q^n, \tag{3.6}$$

logo a Hamiltoniana H é uma função dos $q's$ e $p's$ apenas, isto é, $H \equiv H(q's, p's)$.

Enunciaremos agora um teorema clássico do cálculo variacional que será importante a seguir (para demonstração ver referência [19]).

Teorema 3. *Se $\lambda_n \delta q^n + \mu^n \delta p_n = 0$ para variações δq^n e δp_n que satisfazem os vínculos $\phi_m = 0$, $m = 1, \ldots, M$, então*

$$\begin{aligned} \lambda_n &= u^m \frac{\partial \phi_m}{\partial q^n}, \\ \mu^n &= u^m \frac{\partial \phi_m}{\partial p_n} \end{aligned} \tag{3.7}$$

para algumas funções u^m.

Temos agora que

$$(p_n \dot{q}^n - L) - H = 0, \tag{3.8}$$

logo

$$\delta q^n \left(\underbrace{-\frac{\partial L}{\partial q^n}}_{=\dot{p}_n} - \frac{\partial H}{\partial q^n} \right) + \delta p_n \left(\dot{q}^n - \frac{\partial H}{\partial p_n} \right) = 0 \tag{3.9}$$

Portanto, pelo teorema acima enunciado temos

$$\begin{cases} \dot{q}^n = \dfrac{\partial H}{\partial p_n} + u^m \dfrac{\partial \phi_m}{\partial p_n} \\ \dot{p}_n = -\dfrac{\partial H}{\partial q^n} - u^m \dfrac{\partial \phi_m}{\partial q^n}. \end{cases} \tag{3.10}$$

onde os u^m's são coeficientes desconhecidos.

Definindo-se o parênteses de Poisson como

$$\{f, g\} = \frac{\partial f}{\partial q^n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q^n}, \tag{3.11}$$

temos, se f é um observável,

$$\dot{f} = \frac{\partial f}{\partial q^n}\dot{q}^n + \frac{\partial f}{\partial p_n}\dot{p}_n = \{f, H\} + u^m\{f, \phi_n\}. \tag{3.12}$$

O parênteses de Poisson é apenas definido para funções de p e q. Vamos expandir sua definição para funcões independentes de p e q, como u^m, da seguinte maneira

$$\dot{f} = \{f, H + u^m\phi_m\} = \{f, H\} + \{f, u^m\phi_m\} = \{f, H\} + \{f, u^m\}\underbrace{\phi_m}_{=0} + u^m\{f, \phi_m\}. \tag{3.13}$$

Como observado por Dirac [12], não devemos utilizar os vínculos antes de colocá-los nos parênteses de Poisson. Para nos lembrar disto, Dirac utiliza a seguinte notação:

$$\phi_m \approx 0. \tag{3.14}$$

Temos assim a equação temporal para o observável f da forma

$$\dot{f} \approx \{f, H_T\}, \tag{3.15}$$

onde

$$H_T = H + u^m\phi_m. \tag{3.16}$$

Note que, como $\phi_n = 0$, devemos ter $\dot{\phi}_n = 0$, logo, pela equação (3.15), temos as equações de consistência

$$\{\phi_n, H\} + u^m\{\phi_n, \phi_m\} \approx 0. \tag{3.17}$$

A partir da equação acima podemos destacar as três possibilidades de ocorrência nesta equação. Ela pode levar à uma inconsistência do tipo $0 \neq 0$, o que nos diz que as equações de movimento (e portanto a Lagrangeana) são inconsistentes. Ela pode se reduzir à $0 = 0$, o que não nos diz nada, todos os vínculos já foram dados. Ela pode nos dar uma equação independente dos u's do tipo

$$\chi(q's, p's) = 0, \tag{3.18}$$

o que nos dá outra restrição e portanto mais um vínculo, este agora chamado de vínculo secundário. Ela pode nos dar condições sobre os u's.

Se outros vínculos (vínculos secundários) surgirem, eles nos levam à outras equações de consistência

$$\{\chi, H\} + u^m\{\chi, \phi_m\} \approx 0. \tag{3.19}$$

Devemos analisar estas equações como no caso anterior e proceder desta maneira até que todas os vínculos sejam encontrados. Aí denominamos todos os vínculos (primários ou secundários) por

$$\phi_j \approx 0, \quad j = 1, \ldots, M, \ldots J. \tag{3.20}$$

À cada vínculo corresponde uma equação

$$\{\phi_j, H\} + u^m\{\phi_j, \phi_m\} \approx 0, \tag{3.21}$$

com $m = 1, \ldots, M$ e $j = 1, \ldots J$.

Denominemos a solução deste sistema por

$$u^m = U^m(p, q). \tag{3.22}$$

Esta solução não é única. Qualquer solução $U^m + V^m$, onde V^m é solução de

$$V^m\{\phi_j, \phi_m\} = 0 \tag{3.23}$$

é tambem solução de (3.21). A solução geral deste problema é então dada por

$$u^m = U_m + v^a V_a^m, \tag{3.24}$$

onde os coeficientes v^a, $a = 1, \ldots A$, são arbitrários (Note aqui que estamos com o clássico problema de álgebra linear que diz que as soluções de um sistema de equações são, uma solução particular mais qualquer solução do sistema homogêneo).

Portanto a Hamiltoniana total é dada por

$$
\begin{aligned}
H_T &= H + U^m\phi_m + v^a V_a^m \phi_m \\
&= H + U^m\phi_m + v^a\phi'_a,
\end{aligned} \tag{3.25}
$$

ainda valendo a equação de movimento

$$\dot{f} = \{f, H_T\}. \tag{3.26}$$

As funções arbitrárias v^a nos dizem que nem todos os $q's$ e $p's$ estão em ligação única com os estados físicos. Em outras palavras, apesar de um estado físico ser unicamente determinado pelos $p's$ e $q's$, o contrário não é verdade. Existe mais do que um conjunto de variáveis canônicas descrevendo o mesmo sistema físico.

Vamos definir agora o conceito de vínculos de primeira classe que, como veremos mais tarde são os geradores infinitesimais de transformações de gauge. Uma função $F(q's, p's)$ é dita de primeira classe se $\forall\, j = 1, \ldots, J$ temos

$$\{F, \phi_j\} \approx 0. \tag{3.27}$$

Um teorema simples de ser demonstrado, porém muito importante, nos diz que o parênteses de Poisson de duas funções de primeira classe é ainda uma função de primeira classe.

Note agora que $H' = H + U^m\phi_m$ na equação (3.25) é de primeira classe, já que pelas próprias equações de consistência

$$\{\phi_j, H + U^m\phi_m\} \approx 0. \tag{3.28}$$

Além disso as funções $\phi'_a s$ são de primeira classe pois

$$\{\phi_j, \phi'_a\} = \{\phi_j, V_a^m\phi_m\} \approx V_a^m\{\phi_j, \phi_m\} = 0. \tag{3.29}$$

pela equação (3.23).

Temos assim que a Hamiltoniana total é a soma de uma Hamiltoniana de primeira classe mais uma combinação linear de vínculos primários de primeira classe.

Vamos agora tomar valores iniciais de $p's$ e $q's$ num tempo $t = 0$ e analisar uma variável dinâmica g num tempo $t = \delta t$:

$$\begin{aligned} g(\delta t) &= g_0 + \dot{g}\delta t \\ &= g_0 + \{g, H_T\}\delta t \\ &= g_0 + \delta t\left(\{g, H'\} + v^a\{g, \phi_a\}\right). \end{aligned} \tag{3.30}$$

A diferença de valores de g correspondentes a duas escolhas distintas dos coeficientes arbitrários v^a e $\tilde{v}^a$ é dada por

$$\delta g(\delta t) = \delta v^a\{g, \phi_a\}, \tag{3.31}$$

com $\delta v^a = \delta t(v^a - \tilde{v}^a)$. Dizemos assim que os vínculos primários de primeira classe geram transformações de gauge. Mas a transformação acima não é a única que deixa o sistema físico inalterado. Em particular temos dois resultados que serão apresentados a seguir sem demonstração (ver [19]):

1. O parênteses de Poisson de dois vínculos primários de primeira classe, isto é, $\{\phi_a, \phi_{a'}\}$, geram transformações de gauge.

2. o parênteses de Poisson de qualquer vínculo primário de primeira classe com a Hamiltoniana H', isto é, $\{\phi_a, H'\}$, geram transformações de gauge.

Estes resultados indicam que vínculos secundários de primeira classe também geram transformações de gauge. O parênteses acima também são de primeira classe, mas nada indica que são combinações lineares de vínculos primários apenas. De fato Dirac conjecturou se todos os vínculos secundários de primeira classe são geradores de transformações de gauge. Isto não é necessariamente verdade para um caso completamente geral, mas é verdade para todos os sistemas físicos conhecidos.

Denotando então todos os vínculos de primeira classe por γ_a, definimos a Hamiltoniana extendida por

$$H_E = H' + w^a\gamma_a. \tag{3.32}$$

Esta é a Hamiltoniana mais geral para um sistema com vínculos.

Como primeira ilustração dos métodos discutidos nesta seção, vamos tomar um exemplo artificial e por isso muito simples de ser analizado. Seja um sistema com dois graus de liberdade x e y com Lagrangeana

$$L = \frac{1}{2}\dot{x}^2. \tag{3.33}$$

Imediatamente reconhecemos um vínculo primário

$$p_y = \frac{\partial L}{\partial \dot{y}} = 0. \tag{3.34}$$

O momento canônico relacionado a variável x é dada por $p_x = \dot{x}$. Logo

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{1}{2}\dot{x}^2. \tag{3.35}$$

Como $\{p_y, H\} = 0$, a equação de consistência nos resulta em $0 = 0$ e portanto só temos um vínculo, que é logicamente de primeira classe. A Hamiltoniana total do sistema é dada por

$$H_T = H + vp_y. \tag{3.36}$$

A diferença entre os valores da variável dinâmica y num intervalo de tempo δt com diferentes escolhas de v é dada por (ver equação (3.31))

$$\delta y = \delta v\{y, p_y\} = \delta v. \tag{3.37}$$

Vemos portanto que a liberdade de gauge do nosso sistema se mostra através de uma invariância com relação ao eixo y.

Como segundo exemplo vamos estudar uma teoria clássica qualquer, invariante por reparametrização temporal (exemplo encontrado na referência [20]). Seja $L(q, \frac{dq}{dt}, t)$ a Lagrangeana do sistema e

$$S = \int L(q, \frac{dq}{dt}, t)dt \tag{3.38}$$

sua ação. Como a teoria é invariante por reparametrização temporal temos que a ação é também dada por

$$S = \int L(q, \frac{dq}{d\sigma}\frac{d\sigma}{dt}, t)\frac{dt}{d\sigma}d\sigma = \int L(q, \frac{\dot{q}}{\dot{t}}, t)\dot{t}d\sigma = \int \tilde{L}(q, \dot{q}, t, \dot{t})d\sigma, \tag{3.39}$$

onde $\dot{}$ denota a derivada em relação ao parâmetro σ. Note que $\tilde{L}(q, \dot{q}, t, \dot{t}) = \dot{t}L(q, \frac{dq}{dt}, t)$. O momento conjugado à q^i na teoria reparametrizada é dado por

$$\pi_i = \frac{\partial \tilde{L}}{\partial \dot{q}^i} = \dot{t}\frac{\partial L}{\partial(\frac{dq^j}{dt})}\frac{\partial(\frac{dq^j}{dt})}{\partial \dot{q}^i} = \dot{t}p_j\frac{\delta_i^j}{\dot{t}} = p_i, \tag{3.40}$$

onde p_i é o momento à q^i, isto é,

$$p_i = \frac{\partial L}{\partial(\frac{dq^i}{dt})}. \tag{3.41}$$

O momento conjugado à t é dado por

$$\begin{aligned} \pi_0 = \frac{\partial \tilde{L}}{\partial \dot{t}} &= \frac{\partial}{\partial \dot{t}}(\dot{t}L) = L + \dot{t}\frac{\partial L}{\partial(\frac{dq^j}{dt})}\frac{\partial(\frac{dq^j}{dt})}{\partial \dot{t}} \\ &= L - \frac{1}{\dot{t}}p_j\dot{q}^j = L - p_j\frac{dq^j}{dt} = -H(q^i, p_i, t), \end{aligned} \tag{3.42}$$

logo temos o vínculo

$$\phi_0 = \pi_0 + H(q^i, \pi_i, t) = 0. \tag{3.43}$$

Vejamos a hamiltoniana do sistema reparametrizado:

$$\begin{aligned} \tilde{H}(q, \pi) &= \pi_\mu q^\mu - \tilde{L} = \pi_0\dot{t} + \pi_i q^i - \dot{t}L = \dot{t}\left[\pi_0 + p_i\frac{dq^i}{dt} - L\right] \\ &= \dot{t}(\pi_0 + H) = 0 \end{aligned} \tag{3.44}$$

pela equação (3.43).

Portanto numa teoria invariante por reparametrização temporal, a Hamiltoniana se anula[1]. A Hamiltoniana total do sistema é dada por

$$H_T = \tilde{H} + \lambda\phi_0 = \lambda\phi_0. \tag{3.45}$$

[1]Obviamente a teoria da relatividade geral é invariante por reparametrizações temporais. Logo a Hamiltoniana da TRG também se anula.

O parênteses de Poisson para duas funções A e B é dado por

$$\{A, B\} = \frac{\partial A}{\partial t}\frac{\partial B}{\partial \pi_0} - \frac{\partial A}{\partial \pi_0}\frac{\partial B}{\partial t} + \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial \pi_i} - \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial q^i}, \tag{3.46}$$

Portanto temos as equações

$$\begin{aligned} \dot{t} &= \{t, H_T\} = \{t, \lambda(\pi_0 + H)\} = \lambda \\ \dot{q^i} &= \{q^i, \lambda(\pi_0 + H)\} = \lambda\{q^i, H\} = \lambda\frac{\partial H}{\partial p_i} \\ \dot{\pi}_0 &= \{\pi_0, \lambda(\pi_0 + H)\} = \lambda\{\pi_0, H\} = -\lambda\frac{\partial H}{\partial t} \\ \dot{\pi}_i &= \{\pi_i, \lambda(\pi_0 + H)\} = \lambda\{\pi_i, H\} = -\lambda\frac{\partial H}{\partial q^i}. \end{aligned} \tag{3.47}$$

Pela primeira das equações acima vemos que $\dot{t}$ é igual a um multiplicador de Lagrange, que como vimos anteriormente é totalmente arbitrário. Isto nos diz que o tempo é irrelevante para teoria, o que já era de se esperar já que a teoria é invariante por reparametrização temporal.

Para analisar a tranformação de gauge que o vínculo ϕ_0 induz, tomemos

$$\delta t = \epsilon\{t, \phi_0\} = \epsilon, \tag{3.48}$$

logo ϕ é o gerador de transformações infinitesimal no tempo que não alteram o estado físico.

3.2 Quantização

Vamos agora estudar o problema da quantização de sistemas com vínculos. Para a quantização, as variáveis dinâmicas q e p se tornam operadores satisfazendo a relação de comutação

$$[q, p] = i\hbar, \tag{3.49}$$

cada função das variáveis dinâmicas é promovida à condição de operador e os parênteses de Poisson são substituídos por comutadores de operadores.

Cada vínculo impões condições na função de onda da forma

$$\phi_j \psi = 0. \tag{3.50}$$

Devemos ver agora se essas condições são consistentes, para isto note que um outro vínculo qualquer $\phi_{j'}$ impões outra condição para o estado ψ

$$\phi_{j'} \psi = 0. \tag{3.51}$$

Aplicando $\phi_{j'}$ em (3.50) temos

$$\phi_{j'} \phi_j \psi = 0. \tag{3.52}$$

Aplicando ϕ_j em (3.51) temos

$$\phi_j \phi_{j'} \psi = 0. \tag{3.53}$$

Subtraindo as duas equações acima temos

$$[\phi_j, \phi_{j'}] \psi = 0. \tag{3.54}$$

Esta condição é necessária para consistência. Agora, não queremos que nenhuma condição nova sobre ψ apareça, logo requeremos que $[\phi_j, \phi_{j'}]$ seja uma combinação linear dos vínculos ϕ_m, isto é,

$$[\phi_j, \phi_{j'}] = c_{jj'}^{j''} \phi_{j''}. \tag{3.55}$$

Se isto for verdade, então (3.54) é uma simples consequência de (3.50).

Na teoria clássica, sabemos que como os $\phi's$ são de primeira classe, o parênteses de Poisson de dois vínculos é fracamente nulo, isto é, é uma combinação de todos os $\phi's$. Quando vamos para a mecânica quântica, algo similar acontece, porém não necessariamente os $c's$ estarão todos à esquerda. Isto é um problema, já que os $c's$ dependerão das coordenadas canônicas e portanto não comutarão com os $\phi's$ em geral. Devemos portanto tentar arranjar os fatores nos $\phi's$ de forma que os coeficientes em (C.24) fiquem realmente todos a esquerda. Se conseguirmos isto, as condições suplementares serão consistentes.

A equação que dará a dinâmica do sistema quântico é a equação de Schödinger

$$i\hbar\frac{d\psi}{dt} = H\psi. \tag{3.56}$$

As equações de vínculo devem ser válidas para qualquer instante de tempo. Seja então ψ um estado que satisfaz as equações de vínculo num instante de tempo t, isto é,

$$\phi_j\psi(t) = 0. \tag{3.57}$$

Temos assim, num instante de tempo $t + dt$,

$$\phi_j\psi(t+dt) = \phi_j\psi(t) - \frac{i}{\hbar}\phi_j H dt\psi(t) \Longrightarrow \phi_j H\psi = 0. \tag{3.58}$$

Obviamente

$$H\phi_j\psi = 0, \tag{3.59}$$

logo

$$[\phi_j, H]\psi = 0. \tag{3.60}$$

A fim de que não tenhamos novas condições suplementared devemos ter

$$[\phi_j, H] = b_{jj'}\phi_{j'}. \tag{3.61}$$

Novamente temos que arranjar as coisas de forma que os coeficientes $b_{jj'}$ fiquem todos à esquerda.

Capítulo 4

Equação de Wheller-Dewitt

4.1 Formulação Hamiltoniana da Relatividade Geral

Após a divisão do espaço-tempo em espaço+tempo, a ação que encontramos foi (equação (4.1))

$$S = \int N\sqrt{h}({}^{(3)}R + K_{ab}K^{ab} - K^2)\mathrm{d}^4x \tag{4.1}$$

Desta forma a Lagrangeana para a relatividade geral é dada por

$$L = \int \mathcal{L}\mathrm{d}^3x = \int \left[K_{ij}K^{ij} - (K^i_i)^2 + {}^{(3)}R\right] Nh^{1/2}\mathrm{d}^3x. \tag{4.2}$$

Se introduzirmos a supermétrica definida por

$$G^{ijkl} = h^{1/2}\left[\frac{1}{2}(h^{ik}h^{jl} + h^{il}h^{jk}) - h^{ij}h^{kl}\right], \tag{4.3}$$

temos

$$G^{ijkl}K_{ij}K_{kl} = h^{1/2}\left[K_{ij}K^{ij} - (K^i_i)^2\right]. \tag{4.4}$$

Portanto a Lagrangeana (4.2) pode ser escrita como

$$L = \int N(G^{ijkl}K_{ij}K_{kl} + h^{1/2(3)}R)\mathrm{d}^3x. \tag{4.5}$$

Os momentos π e π^i associados à N e N_i são dados por

$$\pi = \frac{\partial \mathcal{L}}{\partial N_{,0}} = 0 \tag{4.6}$$

$$\pi^i = \frac{\partial \mathcal{L}}{\partial N_{i,0}} = 0 \tag{4.7}$$

já que claramente a Lagrangeana não depende das velocidades de N e N^i. Note que as equações acima representam os vínculos primários da teoria (ver (3.4)). Já o momento canônico π^{ij} associado à h_{ij} é dado por [21]

$$\begin{aligned}\pi^{ij} &= \frac{\partial \mathcal{L}}{\partial h_{ij,0}} = 2NG^{lkmn}K_{lk}\frac{\partial K_{mn}}{\partial h_{ij,0}} \\ &= 2N(\frac{1}{2N}G^{klmn}K_{kl}\delta^i_k\delta^j_l) \\ &= G^{ijkl}K_{kl} \\ &= \gamma^{1/2}(K^{ij} - h^{ij}K) \\ &= G^{ijkl}K_{kl},\end{aligned} \tag{4.8}$$

onde na passagem para a segunda linha foi utilizada a equação (2.58).

A supermétrica também possui uma inversa dada por

$$G_{ijkl} = \frac{1}{2}h^{-1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}), \tag{4.9}$$

que satisfaz

$$G_{ijkl}G^{ijmn} = \frac{1}{2}(\delta^m_k\delta^n_l + \delta^m_l\delta^n_k) \equiv \delta_{kl}{}^{mn}. \tag{4.10}$$

Assim podemos inverter a equação (4.8) para obter

$$K_{ij} = G_{ijkl}\pi^{kl}. \tag{4.11}$$

Uma variedade 6-dimensional M, constituida de pontos $\{h_{ij}\}$, pode ser definida e uma métrica G_{ijkl} pode ser introduzida nesta variedade. Pares de índices podem ser mapeados em índices simples de acordo com a regra

$$
\begin{gathered}
h_{11} = h^1, \quad h_{22} = h^2, \quad h_{33} = h^3, \\
h_{23} = 2^{-1/2} h^4, \quad h_{31} = h^5, \quad h_{12} = h^6, \\
\delta_{ij}{}^{kl} \to \delta^{\Gamma}_{\Delta}, \quad \Gamma, \Delta = 1 \ldots 6, \quad \text{etc.}
\end{gathered}
\tag{4.12}
$$

A assinatura desta métrica é hiperbólica $- + + + + +$ [4].

A Hamiltoniana para a relatividade geral é dada por (ver (3.5))

$$
\begin{aligned}
H &= \int (\pi N_{,0} + \pi^i N_{i,0} + \pi^{ij} h_{ij,0}) \mathrm{d}^3x - L \\
&= \int (\pi N_{,0} + \pi^i N_{i,0} + \pi^{ij} h_{ij,0}) - h^{1/2} N \left[K_{ij} K^{ij} - K^2 + {}^{(3)}R \right] \mathrm{d}^3x.
\end{aligned}
\tag{4.13}
$$

Pela equação (2.58) temos

$$
h_{ij,0} = N_{i|j} + N_{j|i} + 2NK_{ij}, \tag{4.14}
$$

logo

$$
\pi^{ij} h_{ij,0} = 2\pi^{ij} N_{i|j} + 2N G_{ijkl} \pi^{ij} \pi^{kl}. \tag{4.15}
$$

Note que

$$
\begin{aligned}
G^{ijkl} K_{ij} K_{kl} &= \underbrace{G_{ijkl} G^{ijmn}}_{\frac{1}{2}(\delta^m_k \delta^n_l + \delta^m_l \delta^n_k)} G^{klop} K_{mn} K_{op} \\
&= G_{ijkl} \pi^{ij} \pi^{kl}.
\end{aligned}
\tag{4.16}
$$

Até agora temos

$$
H = \int \mathrm{d}^3x \left[N\mathcal{H} + 2\pi^{ij} N_{i|j} \right], \tag{4.17}
$$

onde

$$
\mathcal{H} = \frac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}) \pi^{ij} \pi^{kl} - h^{1/2(3)} R. \tag{4.18}
$$

Utilizando o teorema de Gauss no último termo da equação obtemos

$$H = \int \mathrm{d}^3x \left[N\mathcal{H} + N_i\chi^i \right], \tag{4.19}$$

com $\chi^i = -2{\pi^{ij}}_{|j} = -2{\pi^{ij}}_{,j} - h^{il}(2h_{jl,k} - h_{jk,l})\pi^{jk}$.

Pela equação (3.16) temos

$$H_T = \int \mathrm{d}^3x \left[N\mathcal{H} + N_i\chi^i + \lambda\pi + \lambda_i\pi^i \right]. \tag{4.20}$$

Os parênteses de Poisson para as variáveis canônicas $N, \pi, N_i, \pi^i, \gamma_{ij}, \pi^{ij}$ são dados por

$$\{N(t,\vec{x}), \pi(t,\vec{x}')\} = \delta(\vec{x} - \vec{x}') \tag{4.21}$$

$$\{N_i(t,\vec{x}), \pi^j(t,\vec{x}')\} = \delta_i^j\delta(\vec{x} - \vec{x}') \tag{4.22}$$

$$\{h_{ij}(t,\vec{x}), \pi^{kl}(t,\vec{x}')\} = {\delta_{ij}}^{kl}\delta(\vec{x} - \vec{x}'), \tag{4.23}$$

todos os outros comutadores se anulam e

$${\delta_{ij}}^{kl} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k). \tag{4.24}$$

Pelas equações de consistência (3.17) e pelas equações (4.6) e (4.7), temos

$$0 \approx \dot{\pi}(t,\vec{x}) = \{\pi(t,\vec{x}), H\} = \int \mathrm{d}^3z [\underbrace{\{\pi(t,\vec{x}), N(t,\vec{z})\}}_{=-\delta(\vec{x}-\vec{z})} \mathcal{H}(t,\vec{z})] = -\mathcal{H}(t,\vec{x}) \approx 0 \tag{4.25}$$

$$0 \approx \dot{\pi^i} = \{\pi^i, H\} = \int \mathrm{d}^3z \underbrace{\{\pi^i(t,\vec{x}), N_j(t,\vec{z})\}}_{=-\delta^i_j\delta(\vec{x}-\vec{z})} \chi^j(t,\vec{z}) = -\chi^i(t,\vec{x}) \approx 0. \tag{4.26}$$

Encontramos portanto os vínculos secundários

$$\begin{cases} \mathcal{H} \approx 0 \\ \chi^i \approx 0. \end{cases} \tag{4.27}$$

Para o vínculo $\chi^i = 0$, abaixaremos o índice i da forma

$$\chi_j = h_{ij}\chi^i = 0. \tag{4.28}$$

É facil notar que os vínculos primários π e π^i comutam entre si e comutam com os vínculos secundários $\mathcal{H}$ e χ_i. São portanto vínculos de primeira classe. Porém não é tão simples mostrar que os vínculos secundários também são de primeira classe. Sua álgebra é dada por $[4, 22]$

$$\begin{aligned} \{\mathcal{H}(t,\vec{x}), \mathcal{H}(t,\vec{x}')\} &= \chi^i(t,\vec{x})\delta^3_{,i}(\vec{x}-\vec{x}') - \chi^i(t,\vec{x}')\delta^3_{,i}(\vec{x}-\vec{x}'), \\ \{\chi_i(t,\vec{x}), \mathcal{H}(t,\vec{x}')\} &= \mathcal{H}(t,\vec{x})\delta^3_{,i}(\vec{x}-\vec{x}'), \\ \{\chi_i(t,\vec{x}), \chi_j(t,\vec{x}')\} &= \chi_i(t,\vec{x})\delta^3_{,j}(\vec{x}-\vec{x}') - \chi_j(t,\vec{x}')\delta^3_{,i}(\vec{x}-\vec{x}')). \end{aligned} \tag{4.29}$$

Como as relações acima são fracamente nulas (≈ 0), vemos que os vínculos secundários também são vínculos de primeira classe. Todos os vínculos de primeira classe, primários ou secundários, aparecem na expressão para a Hamiltoniana (equação (4.20)). Esta é portanto a Hamiltoniana mais geral para a relatividade geral.

As expressões seguintes

$$\begin{aligned} 0 \approx \dot{N} = \{N(t,\vec{x}), H_T\} = \lambda(t,\vec{x}), \\ 0 \approx \dot{N}_i = \{N(t,\vec{x}), H_T\} = \lambda_i(t,\vec{x}), \end{aligned} \tag{4.30}$$

nos mostram que os multiplicadores de Lagrange λ e λ_i que aparecem na expressão (4.20) nada mais são do que as derivadas temporais de N e N_i. Como λ e λ_i são arbitrários, vemos que N e N_i também são, o que nos mostra que as funções N e N_i são irrelevantes para a teoria, isto é, a teoria não depende de como a hipersuperfície Σ é puxada com relação ao tempo tanto na direção ortogonal quanto na direção paralela à Σ.

Vamos agora ver qual o significado dos vínculos secundários. Segundo a conjectura de Dirac, todos os vínculos de primeira classe são geradores de transformações de gauge. Vejamos que[1] (ver (3.31))

$$\begin{aligned}
\delta h_{ij}(x) &= \{h_{ij}(x), \int d^3x'\chi_k(x')\delta\xi^k(x')\} = \int d^3x'\delta\xi^k(x')\{h_{ij}(x),\chi_k(x')\}\\
&= \int d^3x'\delta\xi^k(x')\{[-2h_{mk}(x')\underbrace{\{h_{ij}(x),\pi^{mn}{}_{,n}(x')\}}_{\delta_{ij}{}^{mn}\delta^3_{n'}(x-x')}]-\\
&\quad [(2h_{mk,n}(x')-h_{mn,k}(x'))\underbrace{\{h_{ij}(x),\pi^{mn}(x')\}}_{\delta_{ij}{}^{mn}\delta^3(x-x')}]\}\\
&= \int d^3x'\delta\xi^k(x')\{[-h_{ik}(x')\delta^3_{j'}(x-x')-h_{jk}(x')\delta^3_{i'}(x-x')]-\\
&\quad [h_{ik,j}(x')\delta^3(x-x')+h_{jk,i}(x-x')\delta^3(x-x')-h_{ij,k}\delta^3(x-x')]\}\\
&= (h_{ik}(x)\delta\xi^k(x))_j+(h_{jk}(x)\delta\xi^k(x))_i-h_{ik,j}(x)\delta\xi^k(x)-h_{jk,i}(x)\delta\xi^k(x)+h_{ij,k}(x)\delta^k(x)\\
&= h_{ij,k}(x)\delta\xi^k(x)+h_{ik}(x)\delta\xi^k_{,j}(x)+h_{jk}(x)\delta\xi^k_{,i}(x)=\mathcal{L}_{\delta\xi^k}(h_{ij}),
\end{aligned} \tag{4.31}$$

onde $\mathcal{L}_v$ denota a derivada de Lie na direção do vetor v. Vemos portanto que o vínculo χ_i, também chamado de super-momento, é o gerador de transformações espaciais de coordenadas [23]. Sob uma transformação infinitesimal de coordenadas, a transformação da métrica é dada por $\mathcal{L}_{\delta\xi^k}(h_{ij})$. Analogamente, pode-se mostrar que

$$\delta h_{ij}(x) = \{h_{ij}(x), \int d^3x'\mu(x')\mathcal{H}(x')\} = \mu(x)\mathcal{L}_{\mathbf{n}}(h_{ij}), \tag{4.32}$$

o que nos mostra que $\mathcal{H}$ é o gerador de reparametrização temporal e nos dá a dinâmica do sistema.

4.2 Quantização

A arena na qual a dinâmica clássica toma lugar é o super-espaço, o espaço de todas as 3-métricas (h_{ij}). Não entraremos em detalhe no estudo do super-espaço, mas vale dizer que este é um espaço de dimensão infinita com um número finito de coordenadas (h_{ij}) em cada ponto $\vec{x}$ da hipersuperfície Σ [5]. A supermétrica G_{ijkl} providencia uma métrica

[1] O cálculo a seguir é laborioso porém importante para ilustrar como se trabalha com os parênteses de Poisson.

neste espaço. Esta métrica tem a propriedade de ter assinatura hiperbólica em cada ponto $\vec{x}$ da hipersuperfície. A assinatura da supermétrica independe da assinatura do espaço-tempo [4].

No processo de quantização canônica, o estado quântico na representação da métrica é representado por um funcional $\Psi(g_{\mu\nu})$ no super-espaço.

Para este vetor de estado, temos as condições

$$\begin{cases} \pi\Psi = 0, \\ \pi^i\Psi = 0, \\ \mathcal{H}\Psi = 0, \\ \chi_i\Psi = 0. \end{cases} \tag{4.33}$$

Os momentos conjugados às coordenadas canônicas N, N_i, h_{ij} se tornam operadores diferenciais funcionais[2]

$$\pi = -i\frac{\delta}{\delta N}, \qquad \pi^i = -i\frac{\delta}{\delta N_i}, \qquad \pi^{ij} = -i\frac{\delta}{\delta h_{ij}}. \tag{4.34}$$

Os vínculos primários nos dizem que o funcional Ψ depende apenas de h_{ij}. Denotaremos isto indicando Ψ por $\Psi[h]$. Vejamos que condição o vínculo $\chi^i = -2\pi^{ij}{}_{|j}$ impõe sobre o estado Ψ. Temos

$$2i\left(\frac{\delta\Psi[\gamma]}{\delta h_{ij}}\right)_{|j} = 0. \tag{4.35}$$

Esta expressão nos diz que a função de onda Ψ é a mesma para toda configuração (h_{ij}) que se relacionam por uma transformação de coordenadas na hipersuperfície Σ, isto é, o estado Ψ depende apenas da 3-geometria ${}^{(3)}\mathcal{G}$ da hipersuperfície. Para ver isso, consideremos uma mudança de coordenadas pelo difeomorfismo $x^i \to x^i - \xi^i$. A métrica após este difeormorfismo é dada por $h_{ij} + \xi_{(j|i)}$ [16]. Temos assim

$$\Psi[h_{ij} + \xi_{(i|j)}] = \Psi[h_{ij}] + \int d^3x\xi_{(i|j)}\frac{\delta\Psi}{\delta h_{ij}}. \tag{4.36}$$

[2]A variação de um funcional pode sempre ser escrita da forma $\delta S = \int \frac{\delta S}{\delta q(t)}\delta q(t)dt + O(\delta q^2)$. A expressão $\delta S/\delta q(t)$ é chamada derivada functional de $S[q]$ com respeito a $q(t)$.

Integrando por partes, temos

$$\delta\Psi = -\int d^3x \xi_i (\delta\Psi/\delta h_{ij})_{|j} = \frac{1}{2i}\int d^3x \xi_i \chi^i \Psi = 0, \tag{4.37}$$

mostrando que a função de onda é invariante por mudanças de coordenadas na hipersuperfície Σ. Para enfatizar que Ψ depende apenas da 3-geometria da hipersuperfície, substituimos $\Psi[h]$ por $\Psi[{}^{(3)}\mathcal{G}]$.

Nos resta agora a equação para o vínculo $\mathcal{H}$, que é a única condição que pode ditar a dinâmica do sistema. Ela é dada por

$$\left(G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + h^{1/2 (3)} R \right) \Psi[{}^{(3)}\mathcal{G}] = 0 \tag{4.38}$$

e é chamada equação de Wheller-deWitt.

4.3 Modelos de Minisuperespaço

O espaço de configurações na qual a cosmologia quântica lida tem dimensão infinita: os infinitos valores da métrica em todo o espaço-tempo, logo o formalismo completo da cosmologia quântica é extremamente complicado. Mas mesmo em cosmologia clássica, algumas simplificações são feitas devido ao universo parecer homogêneo e isotrópico em grandes escalas. Num contexto cosmológico então, parece natural considerarmos inicialmente modelos homogêneos e isotrópicos, nos atendo a um número finito de graus de liberdade. Tais modelos são chamados modelos de minisuperespaço. Lidamos assim com um problema de mecânica quântica ao invés de uma teoria de campos.

Num modelo geral de minisuperespaço, a função lapso é imposta ser homogênea $N = N(t)$ e a função deslocamento é nula $N_i = 0$. Além disso a 3-métrica h_{ij} é suposta homogênea e é descrita por funções do tempo $q^\alpha(t)$, onde $\alpha = 0, 1, 2, \ldots, n-1$. Dessa forma temos [5]

$$ds^2 = -N(t)dt^2 + h_{ij}(q^\alpha(t))dx^i dx^j. \tag{4.39}$$

A ação para a relatividade geral foi encontrada como sendo

$$S = \int \left[K_{ij}K^{ij} - (K^i_i)^2 + {}^{(3)}R\right] \alpha h^{1/2} dt d^3x,$$

com

$$K_{ij} = \frac{1}{2N}\left[\frac{\partial h_{ij}}{\partial t} - N_{i|j} - Nj|i\right].$$

Como estamos lidando com o caso $N_i = 0$ e $h_{ij} = h_{ij}(q^\alpha(t))$, temos

$$K_{ij} = \frac{1}{2N}\left[\frac{\partial h_{ij}}{\partial q^\alpha}\dot{q}^\alpha\right]. \tag{4.40}$$

Logo a ação pode ser escrita de forma geral como[3]

$$S[q^\alpha(t), N(t)] = \int_0^1 dt N\left[\frac{1}{2N^2} f_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta - U(q)\right] \equiv \int L dt. \tag{4.41}$$

Aqui, $f_{\alpha\beta}$ é uma versão reduzida da supermétrica G_{ijkl} e o intervalo de integração $(0,1)$ pode ser obtido deslocando-se o tempo e reescalando a função lapso $N(t)$.

O Hamiltoniano é obtido da maneira usual. Primeiro definimos os momentos canônicos

$$p_\lambda = \frac{\partial L}{\partial \dot{q}^\lambda} = f_{\lambda\beta}\frac{\dot{q}^\beta}{N}, \tag{4.42}$$

logo

$$H_c = p_\alpha\dot{q}^\alpha - L = N\left[\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q)\right] \equiv NH. \tag{4.43}$$

A forma Hamiltoniana para a ação é dada por

$$S = \int_0^1 [p_\alpha\dot{q}^\alpha - NH], \tag{4.44}$$

onde vemos que N é o multiplicador de Lagrange do vinculo Hamiltoniano

$$H(q^\alpha, p_\alpha) = \frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q) = 0. \tag{4.45}$$

[3]A ação a seguir foi renormalizada de modo a incorporar a divergência espacial $\int d^3x h^{1/2}$.

Para o processo de quantização, introduzimos uma função de onda independente do tempo $\Psi(q^\alpha)$ e impomos que ela é aniquilada pelo operador correspondente ao vínculo clássico (4.45). Isto nos leva à equação de Wheeler-de Witt

$$H(q^\alpha, -i\frac{\partial}{\partial q^\alpha})\Psi(q^\alpha) = 0. \tag{4.46}$$

Tomemos como exemplo a métrica de Robertson-Walker

$$ds^2 = -N(t)^2 dt + a(t)^2 d\Omega_3^2(k), \tag{4.47}$$

onde $d\Omega_3^2(k)$ é a metrica da seção espacial com curvatura constante $k = -1, 0, +1$. Temos neste caso

$$\begin{aligned}
&\int dtd^3xNh^{1/2}[K_{ij}K^{ij} - (K_i^i)^2 + {}^{(3)}R] \\
&= \int dtd^3xN\frac{a(t)^3 \sin\theta r^2}{\sqrt{1-kr^2}}\left[\frac{3}{N^2}\left(\frac{\dot{a}}{a}\right)^2 - \left(-\frac{3}{N}\frac{\dot{a}}{a}\right)^2 + 6\frac{k}{a^2}\right] \\
&= \int d^3x\frac{\sin\theta r^2}{\sqrt{1-kr^2}}\int dtN\left[-\frac{6}{N^2}\dot{a}^2 a + 6ka\right] \\
&= \int d^3x\frac{\sin\theta r^2}{\sqrt{1-kr^2}}\int dtN\left[\frac{1}{2N^2}(-12a)\dot{a}^2 + 6ka\right].
\end{aligned} \tag{4.48}$$

Portanto $f_{11} = -12a$ e utilizando a equação (4.45), temos

$$H(q^\alpha, p^\alpha) = -\frac{1}{24a}p_a^2 - 6ka = 0, \tag{4.49}$$

o que nos leva à equação quântica

$$\frac{1}{24a}\frac{\partial^2\Psi}{\partial a^2} - 6ka\Psi = 0. \tag{4.50}$$

Capítulo 5

Interpretação de de Broglie-Bohm da Mecânica Quântica

5.1 A interpretação causal da mecânica quântica

A interpretação de Copenhaguen da mecânica quântica é a mais aceita e até agora não existe um único experimento que ponha em dúvida sua validade. Ela é baseada na suposição que um sistema físico é completamente especificado por uma função de onda que determina somente as probabilidades de determinados resultados serem medidos. Neste processo de medida acontece o que chamamos de o colapso da função de onda, onde apenas um possível estado do sistema é obtido e se medirmos imediatamente em seguida obteremos o mesmo resultado. Tal colapso não pode ser descrito por uma evolução unitária [23], sendo necessária assim a presença de um mundo clássico fora do sistema físico, a medida sendo feita neste domínio clássico. O universo é um sistema físico, podemos portanto nos perguntar qual o seu comportamento segundo as leis da mecânica quântica. Porém obviamente não há um domínio clássico fora do universo e portanto a interpretação probabilística de Copenhaguen não pode ser aplicada à este sistema. Além disso, nesta interpretação, o papel dos observadores é decisivo, já que apenas um processo de medida pode dar sentido à teoria quântica, pois não é possível acompanhar passo a passo a evolução do sistema sem provocar o colapso da função de onda. Mas quem são os observadores de todo o universo?

Graças ao que foi dito no parágrafo anterior, iniciaremos o estudo da interpretação de de Broglie-Bohm da mecânica quântica [10]. Diferentemente da interpretação de Copenhaguen, esta interpretação permite a descrição de estados precisos do sistema individual, que evoluem segundo leis parecidas com as equações clássicas de movimento e têm sentido mesmo sem a presença decisiva de um observador.

Para iniciarmmos o estudo da interpretação causal das mecânica quântica, comecemos analisando a equação de Schrödinger

$$i\hbar\frac{\partial\psi}{\partial t} = -\left(\frac{\hbar^2}{2m}\right)\nabla^2\psi + V(\vec{x})\psi. \tag{5.1}$$

Como $\psi(\vec{x}, t) \in \mathbb{C}$, pode ser expresso como

$$\psi = Re^{iS/\hbar}, \tag{5.2}$$

onde R e S são funções reais.

Substituindo (5.2) em (5.1) e separando as partes real e imaginária da equação resultante temos

$$\begin{aligned} \frac{\partial R}{\partial t} &= -\frac{1}{2m}[R\nabla^2 S + 2\nabla R \cdot \nabla S] \\ \frac{\partial S}{\partial t} &= -\left[\frac{(\nabla S)^2}{2m} + V(\vec{x}) - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}\right]. \end{aligned} \tag{5.3}$$

A seguir vamos interpretar $R(\vec{x})^2$ como uma densidade de probabilidade, é conveniente portanto introduzir $P(t, \vec{x}) = R(t, \vec{x})^2$. Temos assim

$$\begin{aligned} &\frac{\partial P}{\partial t} + \nabla \cdot \left(P\frac{\nabla S}{m}\right) = 0 \\ &\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\vec{x}) - \frac{\hbar^2}{4m}\left[\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right] = 0. \end{aligned} \tag{5.4}$$

Note na equação acima que se $\hbar \to 0$ temos que S é solução da equação de Hamilton-Jacobi. Interpretamos portanto $\nabla S/m$ como o campo de velocidades de um ensemble de partículas e fazemos a identificação $\vec{p} = \nabla S$. Desta forma, a primeira equação em (5.4) pode ser reescrita como

$$\frac{\partial P}{\partial t} + \nabla \cdot (P\vec{v}) = 0. \tag{5.5}$$

A equação acima é uma equação de conservação. Para que os resultados na nova interpretação sejam os mesmos que os da interpretação usual, tomamos P como sendo a densidade de probabilidade para partículas no nosso ensemble.

Quando $\hbar \neq 0$, assumimos que cada partícula no nosso ensemble é afetada não apenas pelo potencial usual $V(\vec{x})$, mas também pelo potencial quântico

$$U(\vec{x}) = \frac{-\hbar^2}{4m}\left[\frac{\nabla^2 P}{P} - \frac{1}{2}\frac{(\nabla P)^2}{P^2}\right] = 0 = -\frac{-\hbar^2}{2m}\frac{\nabla^2 R}{R} \tag{5.6}$$

e satisfaz a equação de Newton modificada

$$m\frac{d^2\vec{x}}{dt^2} = -\nabla\{V(\vec{x}) + U(\vec{x})\}. \tag{5.7}$$

Listamos a seguir os postulados da interpretação de de Broglie-Bohm [25]:

- Um sistema quântico é descrito por uma partícula e um campo ψ que obedece a equação de Schrödinger;
- As tragetórias $\vec{x}(t)$ das partículas independe de observação (note que na interpretação de Copenhaguen não podemos falar em trajetória de uma partícula quântica);
- O momento da partícula é dado por $\vec{p} = \nabla S$;
- Para um ensemble de partículas no mesmo estado quântico ψ, a densidade de probabilidade é P.

Na interpretação de Copenhaguen toda a informação do sistema quântico isolado está guardada em $P = R^2$ e a fase S não é importante. Já na interpretação causal, S nos dirá a trajetória da partícula ao integrarmos a relação $\vec{v} = \nabla S/m$. Além disso, na interpretação de Copenhaguen, a probablididade $|\psi|^2$ é inerente da matéria. A melhor informação possível que podemos encontrar sobre um sistema quântico é uma densidade de probabilidade. Na interpretação causal, a probabilidade aparece porque não podemos controlar precisamente a posição e a velocidade iniciais da partícula devido à interferência do aparato de medida. Mesmo em regiões onde ψ é desprezível, o potencial quântico pode ser muito grande. Este potencial depende da forma de ψ e não do seu valor absoluto. Isto trás à tona a característica não local e contextual do potencial quântico. A questão de porque não vemos, num processo de medida, uma superposição de funções de onda que não se interceptam é respondida vendo-se que a partícula entrará em uma região (que depende de suas condições iniciais) e será influenciada pelo potencial quântico que corresponde a este ramo particular, que é o único não desprezivel na regão onde a partícula realmente está [23].

5.2 A relação entre as propriedades de partículas e operadores quânticos

Considere um operador Hermitiano $\hat{A}$ que é uma função dos operadores posição $\hat{\vec{x}}$ e momento $\hat{\vec{p}}$: $\hat{A} = \hat{A}(\hat{\vec{x}}, \hat{\vec{p}})$. O valor esperado quântico deste operador é dado por

$$\left\langle \hat{A} \right\rangle = \frac{\left\langle \psi \left| \hat{A} \right| \psi \right\rangle}{\langle \psi | \psi \rangle}. \tag{5.8}$$

Na representação $\{|\vec{x}\rangle\}$ temos

$$\left\langle \hat{A} \right\rangle = \frac{\int d^3x \int d^3x' \, \langle \psi | \vec{x} \rangle \left\langle \vec{x} \left| \hat{A} \right| \vec{x}' \right\rangle \langle \vec{x}' | \psi \rangle}{\int d^3x \psi^*(\vec{x}, t) \psi(\vec{x}, t)}. \tag{5.9}$$

Definindo [24]

$$\left(\hat{A}\psi \right)(x, t) = \int d^3x' \left\langle \vec{x} \left| \hat{A} \right| \vec{x}' \right\rangle \psi(\vec{x}') \tag{5.10}$$

e utilizando o fato que $\hat{A}$ é Hermitiano, o que faz com que apenas a parte real da equação (5.9) importe, temos

$$\left\langle \hat{A} \right\rangle = \frac{\int d^3x \mathrm{Re} \left[\psi^*(\vec{x}, t) \left(\hat{A}\psi \right)(x, t) \right]}{\int d^3x \psi^*(\vec{x}, t) \psi(\vec{x}, t)}. \tag{5.11}$$

É então razoável chamar a seguinte expressão de “valor esperado local” do operador $\hat{A}$

no estado $|\psi\rangle$ na representação $|a\rangle$:

$$\langle A(\vec{x},t)\rangle_L = \mathrm{Re}\left[\frac{\psi^*(\vec{x},t)\left(\hat{A}\psi\right)\psi(\vec{x},t)}{\psi^*(\vec{x},t)\psi(\vec{x},t)}\right]. \tag{5.12}$$

Esta é uma quantidade que reúne informações sobre o operador e a função de estado. Vamos ver sua importância através de três simples exemplos.

1) *Operador Posição.* O operador posição é dado por $\hat{A} = \hat{\vec{x}}$. Logo $\langle \vec{x}\,|\vec{x}|\,\vec{x}'\rangle = \vec{x}\delta(\vec{x}-\vec{x}')$ e portanto

$$\langle \vec{x}\rangle_L = \psi^*\vec{x}\psi/(\psi^*\psi) = \vec{x}. \tag{5.13}$$

O valor esperado local de $\vec{x}$ é a própria posição da partícula.

2) *Operador Momento.* No caso em que $\hat{A} = \hat{\vec{p}}$ temos $\langle \vec{x}\,|\vec{p}|\,\vec{x}'\rangle = -i\hbar\nabla\delta(\vec{x}-\vec{x}')$. Logo

$$\begin{aligned}\langle \vec{p}\rangle_L &= \mathrm{Re}\left[\psi^*(-i\hbar\nabla)\psi\right]/\left|\psi\right|^2\\ &= (\frac{\hbar}{2i\left|\psi\right|^2}\left[\psi^*\nabla\psi - (\nabla\psi^*)\psi\right]\\ &= \nabla S,\end{aligned} \tag{5.14}$$

onde $\psi = Re^{iS}$. Note que o valor esperado do operador momento é o próprio momento da partícula.

3) *Operador Energia.* Por fim consideremos o operador energia. Neste caso $\hat{H} = \vec{p}^2/2m + V$ e temos então que o valor esperado local de $\hat{H}$ é dado por

$$\begin{aligned}\left\langle \hat{H}\right\rangle_L &= \mathrm{Re}\left[\psi^*(-i\hbar^2\nabla^2\psi/2m) + V\right]/\psi^*\psi\\ &= (\nabla S)^2/2m - \hbar^2\nabla^2 R/2mR + V\\ &= \vec{p}^2/2m + Q + V.\end{aligned} \tag{5.15}$$

O valor esperado local da energia nada mais é do que a energia total da partícula.

5.3 A aplicação das interpretação causal para a cosmologia quântica

Na interpretação de de Brogli-Bohm, o objeto fundamental da gravitação quântica, a geometria das hipersuperfícies do tipo espaço, é suposta existir independentemente de um observador ou de uma medida, juntamente com seu momento conjugado, a curvatura extrínsica das hipersuperfícies [8]. Sua evolução, ditada por um parâmetro do tipo tempo, é diferente da equação clássica devido à presença de um potencial quântico que aparece naturalmente na equação de Wheeler-deWitt.

Fazendo a decomposição $\Psi = Re^{iS/\hbar}$ e substituindo na equação de Wheeler-deWitt, obtemos

$$\begin{gathered} G_{ijkl}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{kl}} - h^{1/2(3)}R(h_{ij}) + h^{1/2}Q(hij) = 0, \\ G_{ijkl}\frac{\delta}{\delta h_{ij}}\left(R^2\frac{\delta S}{\delta h_{ij}}\right), \end{gathered} \tag{5.16}$$

onde Q é o potencial quântico dado por

$$Q = -\frac{1}{R}G_{ijkl}\frac{\delta^2 R}{\delta h_{ij}\delta h_{kl}}. \tag{5.17}$$

Note que a equação (5.16) é como a equação de Hamilton-Jacobi para a relatividade geral [16], exceto pela presença do potencial quântico Q. O limite clássico é quando o potencial quântico Q se torna desprezível em comparação com o termo clássico

$$V = -h^{1/2(3)}R(h_{ij}). \tag{5.18}$$

Tudo se assemelha com o caso de uma partícula relativística exceto pelo fato que a segunda equação em (5.16) não pode ser interpretada como uma equação de conservação de probabilidade devido a natureza hiperbólica da supermétrica G_{ijkl}.

Para o caso de modelos de minisuperespaços, a equação (5.16) se torna

$$\frac{1}{2}f_{\alpha\beta}(q^\alpha)\frac{\partial S}{\partial q_\alpha}\frac{\partial S}{\partial q_\beta} + V(q^\alpha) + Q(q^\alpha) = 0, \tag{5.19}$$

com

$$Q(q^{\alpha}) = -\frac{1}{R}\frac{\partial^2 R}{\partial q_{\alpha}\partial q_{\beta}} \tag{5.20}$$

e $f_{\alpha\beta}$ e $V(q^{\alpha})$ particularizações de G_{ijkl} e $h^{-1/2(3)}R$, respectivamente. Temos também a equação para o momento canônico

$$p_{\alpha} = \frac{\partial S}{\partial q^{\alpha}} = f_{\alpha\beta}\frac{1}{N}\frac{\partial q^{\alpha}}{\partial t}. \tag{5.21}$$

Note que uma reparametrização temporal $\tau(t)$ deixa o momento invariante já que

$$\tilde{p}_{\alpha}(\tau) = f_{\alpha\beta}\frac{1}{\tilde{N}(\tau)}\frac{\partial q^{\beta}}{\partial \tau} = f_{\alpha\beta}\frac{1}{N(t)(\partial t/\partial \tau)}\frac{\partial q^{\beta}}{\partial t}\frac{\partial t}{\partial \tau} = p_{\alpha}. \tag{5.22}$$

Logo, mesmo a um nível quântico, diferentes escolhas de $N(t)$ resultam na mesma geometria do espaço-tempo para uma solução não clássica dada $q^{\alpha}(t)$.

Capítulo 6

Fluidos perfeitos

Neste capítulo faremos um resumo do estudo de fluidos perfeitos em relatividade geral realizado por Bernard F. Schutz entre 1970 e 1971. Em dois artigos [6, 7], Schutz propõe uma representação do campo de velocidades por 6 parâmetros, cada um com uma equação de evolução e quase todos com um significado claro. A partir daí ele estuda um princípio variacional, descoberto por Selinger e Whitham [26] para o caso Newtoniano, e desenvolve a teoria Hamiltoniana para fluidos perfeitos em relatividade geral.

6.1 Fluidos Newtonianos

Potenciais velocidades já são conhecidos desde a hidrodinâmica Newtoniana. O campo de velocidades para um fluido irrotacional pode ser derivado de um potencial escalar ϕ a partir da representação $\mathbf{v} = \nabla\phi$. Em 1959, Clebsch [27] mostrou que qualquer movimento Newtoniano pode ser representado por três potenciais da forma

$$\mathbf{v} = \nabla\phi + \alpha\nabla\beta. \tag{6.1}$$

Estes três potenciais não têm significado individualmente e não existem equações individuais para cada um dos potenciais.

Selinger e Whitham procederam com uma descrição Euleriana[1] do fluido. Nesta des-

[1] Na descrição Euleriana o observador é fixo e o fluido flui neste referencial. Ao contrário, na descrição

crição as equações de movimento são

$$\rho\left(\frac{\partial x_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}, \tag{6.2}$$

juntamente com as equações para a densidade ρ e a entropia S[2]

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho v_j)=0, \tag{6.3}$$

$$\frac{\partial S}{\partial t}+v_j\frac{\partial S}{\partial x_j}=0, \tag{6.4}$$

e com a equação de estado

$$p=p(\rho,S). \tag{6.5}$$

O princípio de Hamilton neste caso é dado por

$$\delta\iint\left\{\frac{1}{2}\rho\mathbf{v}^2-\rho E(\rho,S)\right\}d\mathbf{x}dt=0, \tag{6.6}$$

onde E é a energia interna por unidade de massa. Pela primeira lei da termodinâmica

$$\omega_Q=dU+pdV, \tag{6.7}$$

onde ω_Q é a 1-forma de calor, logo

$$TdS=\omega_q\equiv\frac{\omega_Q}{m}=dU/m+pdV/m=dE+pd(1/\rho). \tag{6.8}$$

Portanto E se relaciona com as demais variáveis termodinâmicas da forma

$$dE=TdS-pd\left(\frac{1}{\rho}\right). \tag{6.9}$$

Se variarmos a ação (6.6) com relação a v_i obtemos $v_i=0$. É necessário então restrições nas variações a fim de obter resultados não triviais. É razoável variar $\frac{1}{2}\rho v_i^2-\rho E$ sujeito às condições (6.3) e (6.4). Isto é feito introduzindo multiplicadores de Lagrange ϕ e η e

Lagrangeana o observador se move com o fluido.

[2]Para um fluxo adiabático a entropia se conserva ao longo do movimento.

considerando a variação

$$\iint \left\{ \frac{1}{2}\rho\mathbf{v}^2 - \rho E(\rho, S) + \phi \left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) \right] + \eta \left[\frac{\partial(\rho S)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j S) \right] \right\} d\mathbf{x}dt. \quad (6.10)$$

Agora, a variação com respeito à v_i nos dá

$$\delta v_i : \qquad v_i = \frac{\partial\phi}{\partial x_i} + S\frac{\partial\eta}{\partial x_i}, \quad (6.11)$$

isto é, $\mathbf{v} = \nabla\phi + S\nabla\eta$.

Note que se $S = const.$, então $\mathbf{v} = \nabla(\phi + S\eta)$ e só fluidos irrotacionais são permitidos nesta representação. Baseado no trabalho de Lin [28], Selinger e Whitham contornam este problema adicionando dois novos potenciais α e β de forma que

$$\mathbf{v} = \nabla\phi + S\nabla\eta + \alpha\nabla\beta \quad (6.12)$$

e o princípio de Hamilton assume a forma

$$\begin{aligned} \iint \left\{ \frac{1}{2}\rho\mathbf{v}^2 - \rho E(\rho, S) \right. & + \phi \left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) \right] + \eta \left[\frac{\partial(\rho S)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j S) \right] \\ & \left. + \beta \left[\frac{\partial(\rho\alpha)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j \alpha) \right] \right\} d\mathbf{x}dt. \end{aligned} \quad (6.13)$$

As variações da integral acima com respeito a v_i e ρ nos dão

$$\delta v_i : \qquad v_i = \frac{\partial\phi}{\partial x_i} + S\frac{\partial\eta}{\partial x_i} + \alpha\frac{\partial\beta}{\partial x_i}, \quad (6.14)$$

$$\delta\rho : \qquad \frac{1}{2}\mathbf{v}^2 - (\rho E)_\rho = \left(\frac{\partial\phi}{\partial t} + v_j\frac{\partial\phi}{\partial x_j} \right) + S\left(\frac{\partial\eta}{\partial t} + v_j\frac{\partial\eta}{\partial x_j} \right) + \alpha\left(\frac{\partial\beta}{\partial t} + v_j\frac{\partial\beta}{\partial x_j} \right). \quad (6.15)$$

Se adotarmos a seguinte notação

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}, \quad (6.16)$$

temos que a Lagrangeana do Fluido é dada por

$$L = \frac{1}{2}\rho\mathbf{v}^2 - \rho E - \rho\frac{D\phi}{Dt} - \rho S\frac{D\eta}{Dt} - \rho\alpha\frac{D\beta}{Dt}. \tag{6.17}$$

Pela relação termodinâmica (6.8) temos

$$\begin{cases} (\rho E)_\rho = E + \rho E_\rho = E + p/\rho = H, \\ \quad E_S = T, \end{cases} \tag{6.18}$$

onde $H(\rho, S)$ e $T(\rho, S)$ denotam a entalpia e a temperatura, respectivamente.

Pelas equações (6.15) e (6.18) temos

$$\frac{1}{2}\mathbf{v}^2 - \frac{D\phi}{Dt} - S\frac{D\eta}{Dt} - \alpha\frac{D\beta}{Dt} = H, \tag{6.19}$$

logo

$$L = \rho H - \rho E = p. \tag{6.20}$$

Note que resultado surpreendente obtivemos acima. A densidade lagrangeana para o fluido é dada somente pela pressão p.

6.2 Teoria Relativística para um fluido perfeito

Consideremos um fluido perfeito composto por bárions[3]. Definimos a massa de repouso de uma mostra de matéria contendo N bárions por $m_H N$, onde m_H é a massa de um hidrogênio em seu estado fundamental. A diferença entre a energia total do sistema e $m_H N$ é chamada energia interna. Definimos por ρ_0 a densidade de massa de repouso e por $\Pi \equiv U/(m_H N)$ a energia interna específica, ambas medidas num referencial inercial em repouso no fluido. Então a densidade total de energia é dada por

$$\rho = E/V = (U + m_H N)/V = \frac{m_H N}{V}\left(U/m_H N + 1\right) = \rho_0(1 + \Pi). \tag{6.21}$$

[3]Bárions porque eles podem sofrer transmutação, a massa de repouso pode não se conservar, mas o número bariônico N se conserva.

Assumimos uma equação de estado da forma $p = p(\rho_0, \Pi)$. Pela primeira lei da termodinâmica

$$\omega_Q = dU + pdV \Rightarrow \delta q \equiv \delta Q/m_H N = d\Pi + pd(1/\rho_0), \tag{6.22}$$

logo, pelo teorema de Pfaff[4] existem duas funções $S(\rho_0, \Pi)$ e $T(\rho_0, \Pi)$ (a entropia específica e a temperatura, respectivamente), tal que

$$d\Pi + pd(1/\rho_0) = TdS. \tag{6.23}$$

Se definirmos agora a quantidade

$$\mu = (\rho + p)/\rho_0 = 1 + \Pi + p/\rho_0, \tag{6.24}$$

podemos usar $d\mu$ para eliminar $d\Pi$ em (6.23), obtendo

$$d\mu - \rho_0^{-1} dp = TdS \Rightarrow dp = \rho_0 d\mu - \rho_0 TdS. \tag{6.25}$$

Consideramos agora uma equação de estado da forma $p = p(\mu, S)$.

Para o caso relativístico, Schutz propõe uma representação do movimento do fluido com seis potenciais (um a mais que no caso não-relativístico, já que estamos trabalhando com uma dimensão a mais) da forma

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}). \tag{6.26}$$

A normalização do quadrivetor velocidade $U^\nu U_\nu = -1$ implica que

$$\mu^2 = -g^{\mu\nu}(\phi_{,\mu} + \alpha\beta_{,\mu} + \theta S_{,\mu})(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}). \tag{6.27}$$

Seguindo o exemplo da ação de Selinger e Whitham, mas agora acoplando o fluido à gravidades temos a ação

$$I = \int (R + 16\pi p)(-g)^{1/2} d^4x. \tag{6.28}$$

[4] Ver apêndice A da referência [6].

Vamos primeiramente variar esta ação com relação à métrica $g^{\mu\nu}$. variemos o primeiro termo de (6.28).

$$\delta[(-g)^{1/2}R] = (-g)^{1/2}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R]\delta g^{\mu\nu} \tag{6.29}$$

Variemos agora o segundo termo de (6.28)

$$\delta[(-g)^{1/2}p] = \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}p\delta g^{\mu\nu} + \sqrt{-g}\rho_0\frac{\delta\mu}{\delta g^{\mu\nu}}\delta g^{\mu\nu}, \tag{6.30}$$

onde utilizamos a equação (6.25).

Pela equação (6.27), temos

$$2\mu\frac{\delta\mu}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = -(\phi_{,\mu} + \alpha\beta_{,\mu} + \theta S_{,\mu})(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu})\delta g^{\mu\nu} = -\mu^2 U_\mu U_\nu \delta g^{\mu\nu}, \tag{6.31}$$

logo

$$\begin{aligned}\delta[(-g)^{1/2}p] &= -\frac{\sqrt{-g}}{2}g_{\mu\nu}p\delta g^{\mu\nu} - \frac{\sqrt{-g}}{2}\mu\rho_0 U_\mu U_\nu \delta g^{\mu\nu} \\ &= -\frac{\sqrt{-g}}{2}g_{\mu\nu}p\delta g^{\mu\nu} - \frac{\sqrt{-g}}{2}(\rho + p)U_\mu U_\nu \delta g^{\mu\nu}.\end{aligned} \tag{6.32}$$

Temos assim que a variação da ação (6.28) com relação a $g^{\mu\nu}$ nos dá a equação de Einstein

$$\delta g^{\mu\nu}: \qquad G_{\mu\nu} - 8\pi[(\rho + p)U_\mu U_\nu + p g_{\mu\nu}] = 0. \tag{6.33}$$

variando agora a ação I com relação à ϕ, temos

$$\delta I = \delta\int 16\pi p(-g)^{1/2}d^4x = 16\pi\int (-g)^{1/2}\rho_0\delta\mu d^4x. \tag{6.34}$$

Pela expressão (6.27) vemos que

$$2\mu\frac{\delta\mu}{\delta\phi_{,\lambda}}\delta\phi_{,\lambda} = -2g^{\lambda\nu}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu})\delta\phi_{,\lambda} = -2\mu U^\lambda\delta\phi_{,\lambda}, \tag{6.35}$$

logo

$$\delta I = -16\pi\int \rho_0 U^\lambda\delta\phi_{,\lambda}(-g)^{1/2}d^4x = 16\pi\int(\rho_0 U^\lambda)_{;\lambda}(-g)^{1/2}\delta\phi d^4x. \tag{6.36}$$

Portanto a variação da ação com relação à ϕ nos dá a equação de conservação do número bariônico

$$(\rho_0 U^\nu)_{;\nu} = 0. \tag{6.37}$$

Procedendo desta maneira para os próximos casos e sumarizando os resultados temos

$$\delta g^{\mu\nu}: \qquad G_{\mu\nu} = 8\pi[(\rho+p)U_\mu U_\nu + p g_{\mu\nu}]; \tag{6.38}$$

$$\delta\phi \qquad (\rho_0 U^\nu)_{;\nu} = 0; \tag{6.39}$$

$$\delta\theta: \qquad U^\nu S_{,\nu} = 0; \tag{6.40}$$

$$\delta S: \qquad U^\nu \theta_{,\nu} = T; \tag{6.41}$$

$$\delta\alpha: \qquad U^\nu \beta_{,\nu} = 0; \tag{6.42}$$

$$\delta\beta: \qquad U^\nu \alpha_{,\nu} = 0. \tag{6.43}$$

Com a ajuda das equações acima ainda podemos deduzir a equação

$$U^\nu \phi_{,\nu} = -\mu. \tag{6.44}$$

Note que cada um dos potenciais têm sua própria equação de evolução.

6.3 Teoria Hamiltoniana para fluido relavístico perfeito

A densidade Lagrangeana para um fluido relativístico (utilizando a notação ADM) é dada por

$$\mathcal{L} = p\sqrt{-g} = pNh^{1/2}. \tag{6.45}$$

Podemos agora calcular os momentos canônicos assiciados aos potenciais $\phi, \alpha, \theta, \beta$ e S com a ajuda da relação

$$\delta p = \rho_0 \delta\mu - \rho_0 T \delta S. \tag{6.46}$$

Comecemos com p^ϕ:

$$p^\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial p}{\partial \dot{\phi}} N h^{1/2} = \rho_o \frac{\partial \mu}{\partial \dot{\phi}} N h^{1/2}, \tag{6.47}$$

utilizando a relação

$$\mu^2 = -g^{\sigma\nu}(\phi_{,\sigma} + \alpha\beta_{,\sigma} + \theta S_{,\sigma})(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}),$$

temos

$$2\mu\frac{\partial\mu}{\partial\dot{\phi}} = -2g^{\sigma\nu}\delta^0_\sigma(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_\nu) = -2g^{0\nu}\mu U_\nu = -2\mu U^0. \tag{6.48}$$

Portanto

$$p^\phi = -\rho_0 U^0 N h^{1/2}. \tag{6.49}$$

Procedendo analogamente para os outros potenciais e sumarizando os resultados temos

$$\begin{aligned} p^\phi &= -\rho_0 U^0 N h^{1/2}, \\ p^\alpha &= 0, \\ p^\theta &= 0, \\ p^\beta &= \alpha p^\phi, \\ p^S &= \theta p^\phi. \end{aligned} \tag{6.50}$$

Note que temos quatro vínculos:

$$\begin{aligned} \varphi_1 &= p^\alpha = 0, \\ \varphi_2 &= p^\theta = 0, \\ \varphi_3 &= p^\beta - \alpha p^\phi = 0, \\ \varphi_4 &= p^S - \theta p^\phi = 0. \end{aligned} \tag{6.51}$$

Schutz observa que nenhum destes vínculos é de primeira classe, logo nenhum deles é gerador de alguma transformação de gauge.

É de se estranhar que tenhamos apenas um momento independente. Esperaríamos pelo menos três (um para cada componente espacial da velocidade). A razão matemática para isso é que de todas as equações de campo, somente a equação

$$(\rho_0 U^\nu)_{;\nu} = 0$$

é de segunda ordem. Esta equação é obtida variando a ação com relação à ϕ e p^ϕ é o único momento independente.

A densidade Hamiltoniana é definida da forma tradicional

$$
\begin{aligned}
\mathcal{H} = \sum_a p^a \dot{q}_a - L = p^\phi(\dot{\phi} + \alpha\dot{\beta} + \theta\dot{S}) - pNh^{1/2} & \\
= -\rho_0 U^0 N h^{1/2}(\dot{\phi} + \alpha\dot{\beta} + \theta\dot{S}) - pNh & \\
= -\rho_0 \mu U^0 U_0 N h^{1/2} - pNh^{1/2} & \\
= -Nh^{1/2}[(\rho + p)U^0 U_0 + p] & \\
= T^0_0 N h^{1/2}. &
\end{aligned} \tag{6.52}
$$

Para ilustrar o que foi discutido neste capítulo, vamos complementar o exemplo do capítulo 4 onde estudamos a quantização canônica da métrica de Robertson-Walker

$$ds^2 = -N(t)^2 dt + a(t)^2 d\Omega_3^2(k). \tag{6.53}$$

Este exemplo foi tirado da referência [29].

As relações termodinâmicas básicas da seção anterior para um fluido perfeito com equação de estado $p = w\rho$ tomam a forma

$$\rho = \rho_0[1 + \Pi],\ \mu = 1 + \Pi + p/\rho_0,\ TdS = d\Pi + pd(1/\rho_0) = (1 + \Pi)d[\ln(1 + \Pi) - w \ln \rho_0]. \tag{6.54}$$

Portanto, a menos de um fator $T = 1 + \Pi$, $S = \ln(1 + \Pi) - k \ln \rho_0$. Logo a densidade de partículas e a densidade de energia são, respectivamente,

$$\rho_0 = \frac{\mu^{1/w}}{(1 + w)^{1/w}} \exp(-S/w),\ \rho = \frac{\mu^{1+1/w}}{(1 + w)^{1+1/w}} \exp(-S/w). \tag{6.55}$$

A equação de estado é dada então por

$$p = \frac{w\mu^{1+1/w}}{(1 + w)^{1+1/w}} \exp(-S/w). \tag{6.56}$$

A quadri-velocidade do fluido é representada na forma

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \theta S_{,\nu}), \tag{6.57}$$

já que α e β descreve movimentos de vórtice, que são nulos no espaço tempo de Robertson-Walker devido a sua simetria. Alem disso possui uma única componente neste modelo

$$U_\nu = (N, 0, 0, 0). \tag{6.58}$$

Portanto

$$\mu = (\dot{\phi} + \theta\dot{S})/N. \tag{6.59}$$

Logo a Hamiltoniana do fluido é dada por

$$\mathcal{H}_f = p^\phi(\dot{\phi} + \theta\dot{S}) - pNh^{1/2} = p^\phi N\mu - w\frac{\mu^{1+1/w}}{(1+w)^{1+1/w}}e^{-S/w}Nh^{1/2}. \tag{6.60}$$

Lembre-se que $p^\phi = -\rho_0 U^0 N h^{1/2} = \rho_0 h^{1/2}$, logo

$$p^\phi = h^{1/2}\frac{\mu^{1/w}}{(1+w)^{1/w}}e^{-S/w} \Rightarrow \mu = (p^\phi)^w(1+w)e^S h^{-w/2}. \tag{6.61}$$

Temos assim

$$\begin{aligned}\mathcal{H}_f &= N(p^\phi)^{w+1}(1+w)e^S h^{-w/2} - wN\frac{[(p^\phi)^w(1+w)e^S h^{-w/2}]^{1+1/w}}{(1+w)^{1+1/w}}e^{-S/w}h^{1/2} \\ &= N(p^\phi)^{1+w}e^S\gamma^{-w/2} = N(p^\phi)^{1+w}e^S a^{-3w}.\end{aligned} \tag{6.62}$$

A Hamiltonana geral é então dada por (ver (4.49))

$$\mathcal{H} = -\frac{1}{24a}p_a^2 - 6ka + p_\phi^{1+w}a^{-3w}e^S. \tag{6.63}$$

Fazemos agora uma transformação canônica da forma [30]

$$T = -p_S e^{-S}p_\phi^{-(w+1)},\ p_T = p_\phi^{(w+1)}e^S,\ \bar{\phi} = \phi - (w+1)\frac{p_S}{p_\phi},\ \bar{p}_\phi = p_\phi, \tag{6.64}$$

e obtemos a nova Hamiltoniana

$$\bar{\mathcal{H}} = -\frac{1}{24a}p_a^2 - 6ka + p_T a^{-3w}. \tag{6.65}$$

Para mostrar que as tranformações (6.64) são canônicas, basta mostrar que elas levam às equações

$$\dot{T} = \frac{\partial \bar{\mathcal{H}}}{\partial p_T} = a^{-3w};\ \dot{p}_T = -\frac{\partial \bar{\mathcal{H}}}{\partial T} = 0;\ \dot{\bar{\phi}} = \frac{\partial \bar{\mathcal{H}}}{\partial \bar{p}_\phi} = 0;\ \bar{p}_\phi = -\frac{\partial \bar{\mathcal{H}}}{\partial \bar{\phi}} = 0. \tag{6.66}$$

Apenas para ilustração vejamos o caso da variável T:

$$\dot{T} = -\dot{p}_s e^{-S} p_\phi^{-(1+w)} + p_S \dot{S} e^{-S} p_\phi^{-(1+w)} + (1+w) p_S e^{-S} p_\phi^{-(2+w)} \dot{p}_\phi = a^{-3w} + 0 + 0 = a^{-3w}, \tag{6.67}$$

já que

$$\dot{p}_S = -\frac{\partial \mathcal{H}}{\partial S} = p_\phi^{(1+w)} a^{-3w} e^S;\ \dot{S} = \frac{\partial \mathcal{H}}{\partial p_S} = 0;\ \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0. \tag{6.68}$$

Tendo em mãos a Hamiltoniana (6.65) seguimos com o processo de quantização e obtemos (exigindo que a Hamiltoniana aniquile a função de onda)

$$\frac{\partial^2 \Psi}{\partial a^2} - 144 k a^2 \Psi + i 24 a^{1-3w} \frac{\partial \Psi}{\partial t} = 0, \tag{6.69}$$

com $t = -T$.

Note que a equação acima é uma equação parecida com a equação de Schrödinger com a variável dinâmica $t = -T$ fazendo o papel do tempo.

Capítulo 7

Singularidades Quânticas em Espaços-Tempos Estáticos

Neste capítulo seguiremos um caminho diferente dos capítulos anteriores e utilizaremos a teoria quântica de campos em espaços curvos a fim de criar um critério para decidir se um espaço-tempo estático classicamente singular, se mantém singular quando testados por partículas quânticas satisfazendo as equações de Klein-Gordon e de Dirac.

Para isto, primeiramente desenvolveremos o ferramental matemático baseado na teoria de operadores lineares em espaços de Hilbert, necessário para entender a definição física de singularidades quânticas. Em seguida introduziremos o conceito de singularidades quânticas em espaços-tempos estáticos, que provém naturalmente da definição de singularidades clássicas como indicadas por geodésicas incompletas ou caminhos incompletos de aceleração limitada.

7.1 Ferramental Matemático

Um espaço de Hilbert sobre o corpo dos complexos $\mathbb{C}$ é um espaço vetorial munido com um produto interno

$$\begin{aligned} \langle\ ,\ \rangle : H \times H &\longrightarrow \mathbb{C} \\ (\varphi, \psi) &\longmapsto \langle \varphi, \psi \rangle\,, \end{aligned} \tag{7.1}$$

anti-linear na primeita entrada e linear na segunda. Além disso, este espaço é completo na norma $\|\cdot\|$, definida pelo produto interno por $\|\psi\| = \langle\psi,\psi\rangle^{1/2}$, completo no sentido que toda sequência de Cauchy converge.

Um operador linear num espaço de Hilbert H é um operador satisfazendo a seguinte propriedade:

$$T(\alpha\psi + \varphi) = \alpha T\psi + T\varphi \qquad \forall\, \psi, \varphi \in H;\ \alpha \in \mathbb{C}. \tag{7.2}$$

Porém, um detalhe crucial, nem sempre lembrados em livros de mecânica quântica, é que um operador não está completamente definido até que seu domínio seja especificado. Operadores com diferentes domínios devem ser considerados distintos. A definição formal é esta: um operador linear $T : D(T) \to H$ é uma aplicação linear de um subconjunto $D(T) \subseteq H$, chamado domínio de T, num subconjunto $R(T) \subseteq H$, chamado imagem de T. O domínio e a imagem de T são subespaço vetoriais de H [34].

Um operador $T : D(T) \to H$ é chamado limitado se existir um número $c > 0$ tal que $\forall\, \psi \in D(T)$ temos

$$\|T\psi\| \leq c\,\|\psi\|\,. \tag{7.3}$$

Pelo teorema de representação de Riesz-Fréchet [34,35], sabemos que para um operador limitado $T : H \to H$, o operador adjunto de Hilbert $T^* : H \to H$ definido pela igualdade

$$\langle T^*\varphi, \psi\rangle = \langle\varphi, T\psi\rangle \qquad \forall\, \psi, \varphi \in H \tag{7.4}$$

existe e é limitado.

Entretanto, é um fato que em mecânica quântica muitos operadores são não limitados, como os operadores de posição e de momento. Para operadores não limitados nós não temos um teorema como o de Riesz-Fréchet, mas temos uma definição mais fraca de operador adjunto de Hilbert.

Definição 1 (Operador adjunto de Hilbert)**.** *Seja $T : D(T) \to H$ um operador linear densamente definido num espaço de Hilbert H. O operador adjunto de Hilbert é definido como se segue: o domínio $D(T^*)$ de T^* consiste de todos os $\varphi \in H$ para os quais existe $\varphi^* \in H$ tal que, para todo $\psi \in D(T)$,*

$$\langle\varphi^*, \psi\rangle = \langle\varphi, T\psi\rangle\,. \tag{7.5}$$

Para cada $\varphi \in D(T)$, a ação de T^* é dado por

$$T^* \varphi = \varphi^*. \tag{7.6}$$

No teorema acima, um operador é dito densamente definido num espaço de Hilbert H se $D(T)$ é denso em H ($\overline{D(T)} = H$, onde $\overline{D(T)}$ denota o fecho de $D(T)$). Esta é uma condição necessária para que T^* seja uma aplicação, i.e., para que T^* associe um único φ^* para cada $\varphi \in D(T^*)$ [34].

Um operador $T : D(T) \to H$ é chamado auto-adjunto se $T = T^*$. Note que os domínios devem ser iguais, i.e., $D(T) = D(T^*)$.

Um operador auto-adjunto é também simétrico, i.e., ele satisfaz

$$\langle \varphi, T\psi \rangle = \langle T\varphi, \psi \rangle \tag{7.7}$$

$\forall\, \varphi, \psi \in D(T)$.

O conceito de operador linear fechado será importante no que se segue.

Definição 2 (Operador linear fechado)**.** *Seja $T : D(T) \to H$ um operador linear num espaço de Hilbert H. T é dito fechado se*

$$\begin{cases} \psi_n \to \psi & [\psi_n \in D(T)] \\ T\psi_n \to \varphi \end{cases} \tag{7.8}$$

implica $\psi \in D(T)$ e $T\psi = \varphi$, onde ψ_n é uma sequência de elementos no domínio $D(T)$.

Um operador T é **fechável** se ele possui uma extensão fechada. Todo operador fechável possui uma extensão fechada mínima, chamada **fecho**, que será denotada por $\overline{T}$.

Vamos agora introduzir o conceito de grupo contínuo unitário de um parâmetro. Este será o caso para o operador de evolução em mecânica quântica.

Definição 3. *Um operador função de um parâmetro $U(t)$ é chamado grupo unitário contínuo de um parâmetro se*

a) Para cada $t \in \mathbb{R}$, $U(t)$ é um operador unitário e $U(t+s) = U(t)U(s)\, \forall\, t, s \in \mathbb{R}$.

b) Se $\psi \in H$ e $t \to t_0$ então $U(t)\psi \to U(t_0)\psi$.

Existe um teorema devido a M. Stone que relaciona $U(t)$ com a expenencial de um operador auto-adjunto [36].

Teorema 4 (Teorema de Stone)**.** *Seja $U(t)$ um grupo unitário de um parâmetro num espaço de Hilbert H. Então existe um operador auto-adjunto A em H tal que $U(t) = e^{itA}$.*

Em mecânica quântica não relativística, a equação que governa o sistema físico é a equação de Schrödinger

$$\mathcal{H}\left|\psi\right\rangle = -i\frac{\partial}{\partial t}\left|\psi\right\rangle, \tag{7.9}$$

onde $\mathcal{H}$ é o operador Hamiltoniano do sistema (suponha que $\mathcal{H}$ é independente do tempo).

Se o operador $\mathcal{H}$ é auto adjunto, o estado do sistema num instante t é dado por

$$\left|\psi(t)\right\rangle = U(t)\left|\psi(0)\right\rangle = e^{-i\mathcal{H}t/\hbar}\left|\psi(0)\right\rangle \tag{7.10}$$

Motivações físicas dão origem a uma expressão para o operador Hamiltoniano. Geralmente é um operador com derivadas parciais num espaço L^2 apropriado. Em princípio, o domínio do operador não é especificado e é simples encontrar um domínio onde o operador Hamiltoniano é um operador simétrico bem definido (auto-adjunto ou não). Então, se o fecho $\overline{\mathcal{H}}$ é auto-adjunto nós podemos usá-lo. Caso contrário, se o fecho de $\mathcal{H}$ não é auto-adjunto, devemos procurar quantas extensões auto-adjuntas $\mathcal{H}$ possui (possivelmente nenhuma).

Para responder esta pergunta duas definições serão necessárias [36].

Definição 4 (Operadores essencialmente auto-adjuntos)**.** *Um operador linear simétrico T é chamado essencialmente auto-adjunto se seu fecho $\overline{T}$ é auto-adjunto.*

Pode ser mostrado que se T é essencialmente auto-adjunto, possui uma única extensão auto-adjunta.

Definição 5. *Seja T um operador simétrico. Seja*

$$\begin{aligned} \mathcal{K}_+ &= Ker(i - T^*) \\ \mathcal{K}_- &= Ker(i + T^*). \end{aligned} \tag{7.11}$$

$\mathcal{K}_+$ e $\mathcal{K}_-$ são chamados **subspaços deficientes** *de T. Os números n_+ e n_-, dados por $n_+(T) = dim[\mathcal{K}_+]$, $n_-(T) = dim[\mathcal{K}_-]$ são chamados* **índices deficientes** *de T.*

O seguinte teorema nos dará um critério para decidir se um operador é essencialmente auto-adjunto, e um método para encontrar suas extensões auto-adjuntas [37].

Teorema 5 (Critério para operadores auto-adjuntos)**.** *Seja T um operador simétrico com índices deficientes n_+ e n_-. Então,*

(a) Se $n_+ = n_- = 0$, T é essencialmente auto-adjunto.

(b) Se $n_+ = n_- = n \geq 1$, T possui um número infinito de extensões auto-adjuntas parametrizadas por uma matriz $n \times n$.

(c) Se $n_+ \neq n_-$, T não possui extensões auto-adjuntas.

Então, a fim de determinar se um operador simétrico T é essencialmente auto-adjunto nós devemos resolver o par de equações

$$(T^* \mp i)\psi = 0 \tag{7.12}$$

e contar o número de soluções independentes em H, i.e., a dimensão de $\mathrm{Ker}(T^* \mp i)$. O teorema $X.2$ da Ref. [37] nos diz que as extensões auto-adjutas do operador são representadas pela família de um parâmetro de domínios extendidos do operador T dada por

$$D^U = \{\psi = \phi + \phi^+ + U\phi^- : \phi \in D(T)\}, \tag{7.13}$$

onde

$$T^*\phi^\pm = \pm i\phi^\pm, \tag{7.14}$$

$\phi^\pm \in H$, e U é uma isometria de $\ker(T^* - i)$ para $\ker(T^* + i)$.

Para exemplificar tudo o que dissemos nesta seção, vamos estudar o operador momento dado por

Exemplo 1 (Operador momento)**.** *Seja $H = L^2(0,1)$ e T definido por*

$$\begin{aligned} D(T) &= C_0^\infty(0,1) \\ Tf &= -if'. \end{aligned} \tag{7.15}$$

Primeiramente, é fácil checar que T é um operador simétrico, já que $\forall\, f, g \in D(T)$

$$\langle g, Tf\rangle = \int_0^1 \overline{g}(-if') = -i \underbrace{gf|_0^1}_{=0} + i\int_0^1 \overline{g'}f = \int_0^1 \overline{-ig'}f = \langle Tg, f\rangle\,, \tag{7.16}$$

e, como $C_0^\infty(0,1)$ é denso em $L^2(0,1)$, T é densamente definido em $L^2(0,1)$.

A fim de encontrar o operador adjunto de Hilbert, devemos procurar por todos os pares $(g, g^*) \in D(T^*) \times L^2(0,1)$ tais que $\forall\, f \in D(T)$

$$\langle g, Tf\rangle = \langle g^*, f\rangle\,. \tag{7.17}$$

Então

$$\begin{aligned}\langle g^*, f\rangle &= \langle g, Tf\rangle = \langle g, -if'\rangle = -i\,\langle g, f'\rangle = \\ &= -i\int_a^b \overline{g}f'dx = -i\overline{g}f|_0^1 + i\int_a^b \overline{g'}f = \int_a^b \overline{-ig'}f dx = \langle -ig', f\rangle\,,\end{aligned} \tag{7.18}$$

logo $T^*g = -ig'$. Mas, se $g_1 \in L^2(0,1)$ e $g_1^* = -ig_1' \in L^2$, então (lembre-se que f tem suporte compacto em $(0,1)$)

$$\langle g_1^*, f\rangle = \langle -ig_1', f\rangle = i\,\langle g_1', f\rangle = -i\,\langle g_1, f'\rangle = \langle g_1, -if'\rangle = \langle g_1, Tf\rangle\,. \tag{7.19}$$

Logo nenhuma condição de contorno em g é necessária. Consequentemente T^* é dado por

$$\begin{aligned}D(T^*) &= \left\{g \in L^2(0,1) : g' \in L^2(0,1)\right\} \\ T^*g &= -ig'.\end{aligned} \tag{7.20}$$

O operador momento dado pela Eq.(7.15) não é auto-adjunto. Isto acontece porque o domínio escolhido é tão pequeno, i.e., as restrições sob as funções são tão fortes que permitem que o domínio de T^* seja extremamente grande.

Vamos utilizar o teorema 5 para encontrar quantas extensões auto-adjuntas T possui.

$$(-ig' - \mp ig) = 0 \Rightarrow' g' = \mp g \Rightarrow g = e^{\mp x}. \tag{7.21}$$

Portanto, existe uma solução em $L^2(0,1)$ para cada equação em (7.12) e as extensões auto-adjuntas de T são parametrizadas pelo parâmetro θ.

Agora, as isometrias de $\ker(T^* - i)$ *para* $\ker(T^* + i)$ *são dadas por* $U : e^{-x} \to e^{i\gamma - 1} e^x$, $\gamma \in \mathbb{R}$*, pois*

$$||e^{i\gamma-1}e^x|| = \sqrt{\int_0^1 |e^{i\gamma-1}e^x|^2 dx} = \frac{1}{\sqrt{2}e}\sqrt{e^2-1} = ||e^{-x}||. \tag{7.22}$$

Os domínios extendidos de T *são dados por*

$$D^\gamma = \left\{ f + e^{-x} + e^{i\gamma-1}e^x : f \in D(T) \right\}. \tag{7.23}$$

Logo, para uma função $\psi \in D^\gamma$ *temos*

$$\begin{aligned} \psi(0) &= f(0) + 1 + e^{i\gamma-1} = 1 + e^{i\gamma-1} \\ \psi(1) &= f(1) + e^{-1} + e^{i\gamma} = e^{-1} + e^{i\gamma}. \end{aligned} \tag{7.24}$$

portanto

$$\psi(1) = \frac{e^{-1} + e^{i\gamma}}{1 + e^{i\gamma-1}}\psi(0) \tag{7.25}$$

e, como $\left|\frac{e^{-1}+e^{i\gamma}}{1+e^{i\gamma-1}}\right| = 1$*, temos*

$$\psi(1) = e^{i\theta}\psi(0), \qquad \theta \in \mathbb{R}. \tag{7.26}$$

As extensões auto-adjuntas de T *são dadas por*

$$\begin{aligned} D(T_\theta) &= \left\{ \psi(x) \in L^2(0,1) : \psi' \in L^2(0,1), \psi(1) = e^{i\theta}\psi(0) \right\} \\ T_\theta \psi &= -i\psi'. \end{aligned} \tag{7.27}$$

Esta condição de contorno nos assegura conservação de probabilidade.

Neste exemplo vimos que relaxando as condições nas funções pertencentes a $D(T)$*, podemos obter um operador auto-adjunto.*

7.2 Singularidades Quânticas

Nesta seção utilizaremos a teoria quântica de campos em espaços curvos e, analogamente ao caso clássico, seguiremos as idéias de Horowitz e Marolf [11] e diremos que um espaço-tempo é singular em mecânica quântica, se a evolução de um pacote de onda representando um estado de uma partícula, não é unicamente determinado pelo pacote de onda inicial. Neste caso, o futuro da partícula quântica é também imprevisível, a menos que uma nova informação seja adicionada, uma condição de contorno nos pontos singulares para sermos mais precisos. Devido à crença geral que o princíio da censura cósmica é valido, estudaremos apenas singularidades nuas, i.e., singularidades que não estão escondidas por um horizonte de eventos. Um caso muito interessante onde singularidades nuas surgem é devido a defeitos topológicos, que são caracterizados por um tensor de curvatura nulo em todo lugar, exceto numa subvariedade, onde é proporcional a uma função delta de Dirac.

Seguiremos as idéias introduzidas por Wald [41], Horowitz e Marolf [11]. Assim como em relatividade geral, onde uma informação extra deve ser adicionada nos pontos singulares, já que perdemos o poder de prever o futuro de uma partícula seguindo uma linha de mundo incompleta, definimos um espaço-tempo como singular em mecânica quântica se a evolução de um pacote de onda qualquer não é completamente determinada pelo pacote inicial numa superfície de Cauchy.

Seja $(M, g_{\mu\nu})$ um espaço-tempo estático com um campo vetorial de Killing do tipo-tempo ξ^μ e t o parâmetro de Killing. A equação de Klein-Gordon neste espaço-tempo

$$\Box\Psi = M^2\Psi \tag{7.28}$$

pode ser dividida numa parte temporal e uma espçial,

$$\frac{\partial^2\Psi}{\partial t^2} = -A\Psi = VD^i(VD_i\Psi) + M^2V^2\Psi, \tag{7.29}$$

onde $V^2 = \xi^\mu\xi_\mu$ e D_i é a derivada covariante na fatia espacial estática Σ do espaço-tempo (lembre-se que os pontos singulares não fazem parte do espaço-tempo).

Podemos ver A como um operador no espaço de Hilbert H das funções quadrado-integráveis em Σ (seguindo as referências [11, 41–43]). Porém, encontrar o domínio do operador A, $D(A)$, é uma tarefa muito mais difícil e geralmente nenhuma informação é dada. Então, um domínio mínimo é tomado (um núcleo), onde o operador pode ser definido e que nao inclui as singularidades do espaço-tempo. Um conjunto apropriado é $C_0^\infty(\Sigma)$, o conjunto da funções suaves de suporte compacto em Σ. Mas o domínio escolhido é tão pequeno, i.e., as restrições sobre as funções são tão fortes, que o domínio do operador adjunto de Hilbert A^* é extremamente grande e é composto de todas as funçõe ψ in $L^2(\Sigma, V^{-1}d\mu)$ tais que $A\psi \in L^2$. Então, A não é auto-adjunto. Portanto, enfrentamos o problema de encontrar extensões auto-adjuntas de A e descobrir se A possui apenas uma ou várias extensões auto-adjuntas. No espaço de Hilbert $L^2(\Sigma, V^{-1}d\mu)$, onde $d\mu$ é o elemento prório de volume em Σ, não é difícil mostrar (integrando por partes) que o operador $(A, C_0^\infty(\Sigma))$ é simétrico positivo definido. Então, extensões auto-adjuntas sempre existem [36] (ao menos a extensão de Friedrichs[1]).

Se A possui uma única extensão auto-adjunta (o fecho $\overline{A}$ de A), então A é essencialmente auto-adjunto e, como estamos preocupados com uma descrição de uma partícula, não uma teoria de campos, as soluções com frequências positivas satisfazem

$$i\frac{\partial \Psi}{\partial t} = (\overline{A})^{1/2}\Psi, \tag{7.30}$$

e a evolução de um pacote de onda é unicamente determinado pelo pacote inicial

$$\Psi(t, \mathbf{x}) = e^{-it(\overline{A})^{1/2}}\Psi(0, \mathbf{x}). \tag{7.31}$$

Dizemos que o espaço-tempo é quanticamente não singular.

Agora, se A possui infinitas extensões auto-adjunta A_α, onde α é um parâmetro real, devemos escolher uma a fim de evoluir o pacote de onda. Qualquer solução da forma

$$\Psi(t, \mathbf{x}) = e^{-it(A_\alpha)^{1/2}}\Psi(0, \mathbf{x}), \tag{7.32}$$

é uma boa solução e uma informação extra deve ser dada para nos dizer qual escolha devemos fazer. O espaço-tempo é dito quanticamente singular e vemos claramente a relação com o caso clássico.

[1] Ver Teorema X.23 da referência [37].

Capítulo 8

Conclusões

Ao longo deste trabalho nos aprofundamos na cosmologia quântica do universo de FRW. Como pode ser visto no apêndice A, através da quantização das componentes do tensor de curvatura na base tetrada, quantizamos o fluido perfeito que preenche o universo e constatamos a violação das condições de energia, explicando de maneira mais fundamental o porque que as singularidades clássicas são excluídas no modelo quântico. Demos também evidências de que, por mais que a expressão para o fator de escala do universo dependa da escolha do pacote de ondas do universo, a não singularidade do universo quântico e a expressão para $a(t)$ no limite clássico nao dependem da escolha da função de onda do universo. Mostramos que um universo quântico representado por um pacote de ondas não normalizável é singular na interpretação de de Broglie-Bohm e encontramos um pacote de ondas que reproduz a expressão clássica para o fator de escala.

Já no apêndice B, generalizamos o modelo já conhecido de cosmologia quântica para um universo homogêneo e isotrópico n-dimensional. Neste novo modelo constatamos que o universo quântico n-dimensional é não singular e que a expressão para o fator de escala, tanto na visão de vários mundos quanto na visão de de Broglie-Bohm da mecânica quântica, tende para a clássica quanto $t \to \infty$. Além disso, constatamos que para a matéria de radiação quantizada, a expressão p/ρ tende para a expressão clássica $1/(n-1)$ somente para o caso $n = 4$. De certa forma, o universo quadri-dimensional é privilegiado perante os de outras dimensões.

Mostramos que importantes resultados obtidos em cosmologia quântica podem ser também obtidos em cosmologia clássica desde que o universo seja preenchido com uma matéria exótica, que pode ser um fluido que satisfaz uma equação de estado implícita ou dois campos escalares, um deles do tipo fantasma. Resultados que podem ser vistos no apêndice C.

Fizemos uma revisão das ferramentas matemáticas envolvidas no conceito de singularidades quânticas em espaços-tempos estáticos no apêndice D e estudamos um caso importante de singularidades quânticas ao redor de um monopólo global no apêndice E, constatando que este espaço-tempo permanece singular com a introdução de partículas quânticas e encontrando todas as condições de contorno que podem ser escolhidas a fim de tornar o problema bem posto, no sentido que as condições iniciais são suficientes para determinar a evolução temporal do pacote de ondas.

Constatamos também, no apêndice F, que as flutuações quânticas do vácuo implicam numa forte focalização do cone de luz, o que faz com que vizinhanças próximas nao estão mais em contato causal. Este fenômeno, chamado silêncio assintótico quando ocorre em cosmologia, pode explicar vários quebra-cabeças da gravitação quântica, incluindo evidências de redução dimensional em pequenas escalas.

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Apêndice A

Singularidades Quânticas no Universo FRW revisitadas

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Quantum singularities in FRW universe revisited

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The components of the Riemann tensor in the tetrad basis are quantized and, through the Einstein equation, we find the local expectation value in the ontological interpretation of quantum mechanics of the energy density and pressure of a perfect fluid with equation of state $p = \frac{1}{3}\rho$ in the flat Friedmann-Robertson-Walker quantum cosmological model. The quantum behavior of the equation of state and energy conditions are then studied and it is shown that these latter are violated since the singularity is removed with the introduction

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of quantum cosmology, but in the classical limit both the equation of state and the energy conditions behave as in the classical model. We also calculate the expectation value of the scale factor for several wave packets in the many-worlds interpretation in order to show the independence of the nonsingular character of the quantum cosmological model with respect to the wave packet representing the wave function of the Universe. It is also shown that, with the introduction of non normalizable wave packets, solutions of the Wheeler-DeWitt equation, the singular character of the scale factor, can be recovered in the ontological interpretation.

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A.1 Introduction

One of the main problems of modern cosmology is the presence of singularities in cosmological models. Classical singularities in general relativity are indicated by incomplete geodesics or incomplete paths of bounded acceleration [1]. There are three types of singularities [2,3]: the quasi regular singularity, where no observer sees any physical quantities diverging even if its world line reaches the singularity (for example the singularity in the spacetime of a cosmic string); the scalar curvature singularity, where every observer near the singularity sees physical quantities diverging [for example the singularity in the Schwarchild spacetime or the big-bang singularity in Friedmann-Robertson-Walker (FRW) cosmology]; the non scalar curvature singularity, where there are some curves in which the observers experience unbounded tidal forces (whimper cosmologies are a good example). It was shown that under very reasonable conditions (the energy conditions) singularities are always present in cosmological models [1]. Since general relativity cannot escape this burden, we hope that a complete theory of quantum gravity will overcome this situation, teaching us how to deal with such spacetime near the singularities or excluding the singularities at all. While such a theory does not exist, there are many attempts to incorporate quantum mechanics into general relativity. One of the first attempts to do this was quantum cosmology [4, 5]. One of the major problem of quantum cosmology is the absence of a natural time variable, since the Wheeler-DeWitt equation is a second order functional differential equation in the superspace and there is no first order functional differential to play the role of time. The introduction of matter fields may overcome this situation, since the evolution of a dynamical parameter of the matter field can recover the notion of time. With the introduction of a perfect fluid in Schutz's formalism [6, 7], we can recover a Schrödinger-like equation in the minisuperspace in which the momentum associated with the dynamical degree of freedom of the fluid appears linearly in the equation.

Another problem of quantum cosmology is the interpretation of the wave function of the Universe. Since the Universe includes everything, the probabilistic interpretation becomes impossible, since it assumes there is a fundamental process of measure outside the quantum world, in a classical domain [8]. Two of the most common alternative interpretations to the wave function in quantum cosmology are the *many-worlds* [9] and the *de Broglie-Bohm* [10] interpretation of quantum mechanics. In the former, all the possibilities in the splitting are actually realized, and there is one observer in each branch with the knowledge of the correspondent eigenvalue, and these branches do not interfere. Every time an experiment is performed, every one of the outcomes are obtained, each in a different world. In the later, a trajectory in the phase space for each dynamical variable is supposed to exist independently of any measure. In a measurement, this trajectory enters in one of the branches, depending on the initial condition, which is unknown.

In this paper we use the machinery of quantum cosmology and the de Broglie-Bohm interpretation of quantum mechanics in order to study the energy conditions in the quantum FRW universe with a perfect fluid in the radiation dominated era. We hope that the energy conditions will be violated at the quantum domain since it was shown that the big-bang singularity in the FRW universe is removed with the introduction of quantum mechanics [11]. We also study the behavior of the scale factor in the many-worlds interpretation for several wave packets to see if the exclusion of the big-bang singularity is not particular to the simple wave packets found in the literature so far.

In what follows we proceed in the following manner: In Sec. A.2, we present, for easy reference, a brief summary of quantum cosmology in the flat FRW universe in the radiation dominated era. In Sec. A.3, we quantize the components of the Riemann tensor in the tetrad basis and study the energy conditions and the equation of state for the quantum cosmological model of the Universe in order to show the consistency of the model. In Sec. A.4, we find the expectation value of the scale factor for several wave packets in the many-worlds interpretation to show the independence of the results found previously in the literature with respect to the particular wave packet. In Sec. A.5, we show that with the introduction of nonnormalizable wave packets the classical singular behavior can be recovered. Finally, in section A.6, we discuss the main results presented in this work.

A.2 Wheeler-DeWitt equation for the flat FRW Universe with a perfect fluid

The action for general relativity is given by

$$S_G = \int_{\mathcal{M}} d^4x\sqrt{-g}R + 2\int_{\partial\mathcal{M}} d^3x\sqrt{h}h_{ab}K^{ab}, \tag{A.1}$$

where h_{ab} is the induced metric over the boundary $\partial\mathcal{M}$ of the four-dimensional manifold $\mathcal{M}$ and K^{ab} is the extrinsic curvature of the hypersurface $\partial\mathcal{M}$, that is, the curvature of $\partial\mathcal{M}$ with respect to $\mathcal{M}$.

Schutz [7] showed that for a perfect fluid with four velocity expressed in terms of five potentials

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \alpha\beta_{,\nu} + \theta S_{,\nu}), \tag{A.2}$$

where μ is the specific enthalpy, and respecting the normalization condition

$$U_\nu U^\nu = -1, \tag{A.3}$$

the action is given by

$$S_f = \int_{\mathcal{M}} d^4x\sqrt{-g}p, \tag{A.4}$$

where p is the fluid pressure, which is linked to the energy density by the equation of state, $p = \alpha\rho$.

The super-Hamiltonian for the total action,

$$S = S_G + S_f, \tag{A.5}$$

for the flat FRW universe,

$$ds^2 = -N(t)^2 dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \tag{A.6}$$

with matter represented by perfect fluid with equation of state $p = \frac{1}{3}\rho$ is given by [12,13]

$$\mathcal{H} = \frac{p_a^2}{24} - p_T, \tag{A.7}$$

where $p_a = -12\frac{da}{dT}a/N$ is the momentum conjugated to the scale factor $a(t)$ and p_T is the momentum conjugated to the dynamical degree of freedom of the fluid. In fact, the super-Hamiltonian (C.3) is a constraint of the theory. By following the Dirac approach [14] for quantization of Hamiltonian systems with constraints, imposing

$$p_a = -i\frac{\partial}{\partial a}; \qquad p_T = -i\frac{\partial}{\partial T} \tag{A.8}$$

and demanding that the super-Hamiltonian constraint annihilate the wave function we find [13]

$$\frac{\partial^2\Psi}{\partial a^2} + 24i\frac{\partial\Psi}{\partial t} = 0, \tag{A.9}$$

where $t = -T$ is the time coordinate in the conformal-time gauge $N = a$.

The internal product between two wave functions is defined by

$$\langle\Psi|\Phi\rangle = \int_0^\infty \Psi(a,t)^*\Phi(a,t)da, \tag{A.10}$$

so that the condition for self-adjointness of the operator $\hat{H} = -\frac{\partial^2}{\partial a^2}$ is given by [15]

$$\Psi'(0,t) = \beta\Psi(0,t), \quad \beta \in \mathbb{R}. \tag{A.11}$$

A.3 Energy conditions

Equation (C.5) is analogous to the Schrödinger equation for a free particle. Its Green function in the usual $L^2(-\infty,\infty)$ space is given by [15]

$$G(a,a',t) = \left(\frac{6}{\pi i t}\right)^{1/2} \exp\left[\frac{6i}{t}(a-a')^2\right]. \tag{A.12}$$

With the propagator above we can evolve any given initial wave packet for the universe. We choose a normalized wave packet satisfying the boundary condition $\Psi'(0,t)=0$ ($\beta = 0$),

$$\Psi(a,0) = \left(\frac{8\sigma}{\pi}\right)^{1/4} e^{-\sigma a^2}. \tag{A.13}$$

The Green function for the chosen boundary condition is given by

$$G^{\beta=0}(a,a',t) = G(a,a',t) + G(a,-a',t), \tag{A.14}$$

so that

$$\Psi(a,t) = \int_0^\infty G^{\beta=0}(a,a',t)\Psi(a',0)da' = \int_{-\infty}^\infty G(a,a',t)\Psi(a',0)da' \tag{A.15a}$$

$$= \left(\frac{8\sigma}{\pi}\right)^{1/4} \left(\frac{6}{\sigma t - 6i}\right)^{1/2} \exp\left(\frac{6i\sigma a^2}{\sigma t - 6i}\right). \tag{A.15b}$$

In the first line of Eq. (A.15a) we used the fact that $\Psi(a',0)$ is an even function of a' so that the first integral above is mathematically equivalent to the second one.

The analysis of this section will be made using the above wave packet, which satisfies the Neumann boundary condition $\Psi'(0,t)=0$. If we chose a wave packet satisfying the Dirichlet boundary condition $\Psi(0,t)=0$, the result is the same, since in both cases the quantum Universe is nonsingular (the wave packet satisfying the Dirichlet boundary condition is more complicated [15]). In fact, the expectation value of the scale factor in the many-worlds interpretation for the wave packet satisfying the Dirichlet boundary condition is twice the value of that satisfying the Neumann boundary condition [15]. The Bohmian trajectory of the scale factor is the same on both cases. No matter what choice we make, the Hamiltonian operator is self-adjoint so that the evolution of the wave packet is unitary.

In order to use the de Broglie-Bohm interpretation, we need to write the wave function of the Universe in its polar form

$$\Psi = \Theta e^{iS}, \tag{A.16}$$

where Θ and S are real functions. In our case,

$$\begin{aligned} S &= \frac{6\sigma^2 t a^2}{\sigma^2 t^2 + 36} + f_0(t), \\ \Theta &= g_0(t) \exp\left(-\frac{36\sigma a^2}{\sigma^2 t^2 + 36}\right), \end{aligned} \tag{A.17}$$

where $f_0(t)$ and $g_0(t)$ will not matter in what comes.

Now we can calculate the Bohmian trajectories for the scale factor $a(t)$ by the use of the equation [10, 16]

$$p_a = \frac{\partial S}{\partial a}. \tag{A.18}$$

So we have

$$12\dot{a} = \frac{12\sigma^2 t a}{\sigma^2 t^2 + 36} \Rightarrow a(t) = a_0\sqrt{36 + \sigma^2 t^2}, \tag{A.19}$$

where a_0 is an integration constant.

For the metric (A.6) we have as a basis for the cotangent space the following 1-forms

$$\omega^{\hat{t}} = N(t)dt; \quad \omega^{\hat{r}} = a(t)dr; \quad \omega^{\hat{\theta}} = a(t)rd\theta; \quad \omega^{\hat{\phi}} = a(t)r\sin\theta d\phi. \tag{A.20}$$

In this basis the metric can be written as

$$ds^2 = -(\omega^{\hat{t}})^2 + (\omega^{\hat{r}})^2 + (\omega^{\hat{\theta}})^2 + (\omega^{\hat{\phi}})^2 = \eta_{\hat{a}\hat{b}}\omega^{\hat{a}}\omega^{\hat{b}}. \tag{A.21}$$

There are only two independent components of the curvature tensor in the tetrad basis. They are

$$\begin{aligned} R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{t}\hat{\phi}} = -\frac{\ddot{a}}{a^3} + \frac{\dot{a}^2}{a^4}, \\ R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} &= R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{\dot{a}^2}{a^4}. \end{aligned} \tag{A.22}$$

By the relation $p_a = 12\dot{a}$, we have

$$\begin{aligned} R_{\hat{t}\hat{r}\hat{t}\hat{r}} &= -\frac{\dot{p}_a}{12a^3} + \frac{p_a^2}{144a^4}, \\ R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} &= \frac{p_a^2}{144a^4}. \end{aligned} \tag{A.23}$$

Here we promote the components of the curvature tensor to the condition of quantum operators. Then we can take the local expectation value of each component through the relation [16]

$$\langle R_{\hat{a}\hat{b}\hat{c}\hat{d}} \rangle_L = \mathrm{Re}\left(\frac{\Psi^* \hat{R}_{\hat{a}\hat{b}\hat{c}\hat{d}} \Psi}{\Psi^* \Psi} \right). \tag{A.24}$$

Equation (A.23) mixes the coordinate a with its respective momentum p_a. So we face a factor ordering problem since a and p_a do not commute. Here, following the choice of Ref. [11], we chose the Weyl ordering [17]. Weyl ordering is a kind of symmetrization procedure that take all possible orders of the a's and the p_a's and then divides the result by the number of terms in the final expression. In the case of an operator of the form $f(a)p_a^2$, Weyl ordering corresponds to the following expression

$$\left(f(a)\hat{p}_a^2\right)_W = \frac{1}{4}\left[f(a)\hat{p}_a^2 + 2\hat{p}_a f(a)\hat{p}_a + \hat{p}_a^2 f(a) \right]. \tag{A.25}$$

Using the relations

$$\begin{aligned} \left[a^{-4}, \hat{p}_a\right] &= -4ia^{-5}, \\ \left[a^{-4}, \hat{p}_a^2\right] &= -8ia^{-5}p_a + 20a^{-6}, \end{aligned} \tag{A.26}$$

we have

$$\begin{aligned} \hat{p}_a a^{-4} &= a^{-4}\hat{p}_a + 4ia^{-5}, \\ \hat{p}_a^2 a^{-4} &= a^{-4}\hat{p}_a^2 + 8ia^{-5}\hat{p}_a - 20a^{-6}. \end{aligned} \tag{A.27}$$

Then, after Weyl ordering operation, the operator $a^{-4}\hat{p}_a$ becomes

$$\left(\frac{\hat{p}_a^2}{a^4}\right)_W = a^{-4}\hat{p}_a^2 + 4ia^{-5}\hat{p}_a - 5a^{-6}. \tag{A.28}$$

The components of the curvature tensor can be written as

$$\begin{aligned} \left(\hat{R}_{\hat{t}\hat{r}\hat{t}\hat{r}}\right)_W &= -\frac{\dot{p}_a}{12a^3} + \frac{1}{144}\left(a^{-4}\hat{p}_a^2 + 4ia^{-5}\hat{p}_a - 5a^{-6}\right), \\ \left(\hat{R}_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}\right)_W &= \frac{1}{144}\left(a^{-4}\hat{p}_a^2 + 4ia^{-5}\hat{p}_a - 5a^{-6}\right). \end{aligned} \tag{A.29}$$

Now the local expectation value of the components of the Riemann tensor are obtained by the relation

$$\left\langle \hat{R}_{\hat{a}\hat{b}\hat{c}\hat{d}} \right\rangle_L = \mathrm{Re}\left[\frac{\Psi^*\left(\hat{R}_{\hat{a}\hat{b}\hat{c}\hat{d}}\right)_W \Psi}{\Psi^*\Psi}\right]. \tag{A.30}$$

We have

$$\Psi^*\left(\frac{\hat{p}_a^2}{a^4}\right)_W \Psi = \Theta e^{-iS}\left[a^{-4}\hat{p}_a^2 + 4ia^{-5}\hat{p}_a - 5a^{-6}\right]\Theta e^{iS}. \tag{A.31}$$

But

$$\begin{aligned} \left(\Theta e^{-iS}\right) a^{-4}\hat{p}_a^2\left(\Theta e^{iS}\right) &= -\left(\Theta e^{-iS}\right) a^{-4}\frac{\partial^2}{\partial a^2}\left(\Theta e^{iS}\right) \\ &= -\Theta a^{-4}\left[\frac{\partial^2\Theta}{\partial a^2} + 2i\frac{\partial\Theta}{\partial a}\frac{\partial S}{\partial a} + i\Theta\frac{\partial^2 S}{\partial a^2} - \Theta\left(\frac{\partial S}{\partial a}\right)^2\right] \end{aligned} \tag{A.32}$$

and

$$\left(\Theta e^{-iS}\right) 4ia^{-5}\hat{p}_a\left(\Theta e^{iS}\right) = \left(\Theta e^{-iS}\right) 4a^{-5}\frac{\partial}{\partial a}\left(\Theta e^{iS}\right) = 4\Theta a^{-5}\left(\frac{\partial\Theta}{\partial a} + i\Theta\frac{\partial S}{\partial a}\right). \tag{A.33}$$

Therefore

$$\left\langle \frac{\hat{p}_a^2}{a^4} \right\rangle_L = a^{-4}\left(\frac{\partial S}{\partial a}\right)^2 - \frac{a^{-4}}{\Theta}\frac{\partial^2\Theta}{\partial a^2} + \frac{4a^{-5}}{\Theta}\frac{\partial\Theta}{\partial a} - 5a^{-6}. \tag{A.34}$$

The quantum mechanical potential [10] in the de Broglie-Bohm interpretation is obtained through the equation

$$Q = -\frac{1}{24\Theta}\frac{\partial^2\Theta}{\partial a^2} = \frac{3\sigma}{\sigma^2t^2 + 36} - \frac{216\sigma^2a^2}{(\sigma^2t^2 + 36)^2}. \tag{A.35}$$

The time rate of the momentum p_a has the following value [16]

$$\dot{p}_a = -\frac{\partial}{\partial a}(V+Q), \tag{A.36}$$

where V is the classical potential. In our case $V=0$ so that

$$\dot{p}_a = \frac{432\sigma^2 a}{(\sigma^2 t^2+36)^2}. \tag{A.37}$$

From Eqs. (A.17) and (A.34), we have

$$\begin{aligned}\left\langle \frac{\hat{p}_a^2}{a^4}\right\rangle_L &= \frac{1}{a(t)^4}\left(\frac{12\sigma^2 t a(t)}{36+\sigma^2t^2}\right)^2 - \frac{1}{a(t)^4}\left[\frac{-72\sigma}{36+\sigma^2t^2}+\left(\frac{72\sigma a(t)}{36+\sigma^2t^2}\right)^2\right]\\ &+ \frac{4}{a(t)^5}\left(\frac{-72\sigma a(t)}{36+\sigma^2t^2}\right) - \frac{5}{a(t)^6}\\ &= \frac{-5+72\sigma a_0^2[-3+2\sigma(-36+t^2\sigma^2)a_0^2]}{(36+t^2\sigma^2)^3 a_0^6}.\end{aligned} \tag{A.38}$$

Substituting Eqs. (A.37) and (A.38) into Eq. (A.23) we have

$$\begin{aligned}\left\langle \hat{R}_{\hat{t}\hat{r}\hat{t}\hat{r}}\right\rangle_L &= \frac{-5+72\sigma a_0^2[-3+2\sigma(-72+\sigma^2t^2)a_0^2]}{144(36+\sigma^2t^2)^3 a_0^6}\\ \left\langle \hat{R}_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}\right\rangle_L &= \frac{-5+72\sigma a_0^2[-3+2\sigma(-36+\sigma^2t^2)a_0^2]}{144(36+\sigma^2t^2)^3 a_0^6}.\end{aligned} \tag{A.39}$$

The graphics of $\left\langle \hat{R}_{\hat{t}\hat{r}\hat{t}\hat{r}}\right\rangle_L$ and $\left\langle \hat{R}_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}\right\rangle_L$ are shown in Fig. A.3-1. Note that they are perfectly regular for all times t, indicating the absence of a big-bang singularity in the quantum cosmological model. If they were classical quantities, every curvature scalar could be constructed from them. Of course this is not true in our case (for example if we would like to quantize the Kretschmann scalar, we would face a factor ordering problem involving a term like p_a^4/a^8, see Ref. [11]), but it indicates that all the curvature scalars are perfectly regular.

The Ricci scalar is related to the independent components of the curvature tensor by the equation

$$R = -6R_{\hat{t}\hat{r}\hat{t}\hat{r}} + 6R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = \frac{\dot{p}_a}{2a^3} = \frac{216\sigma^2}{a^2(\sigma^2t^2+36)^2}. \tag{A.40}$$

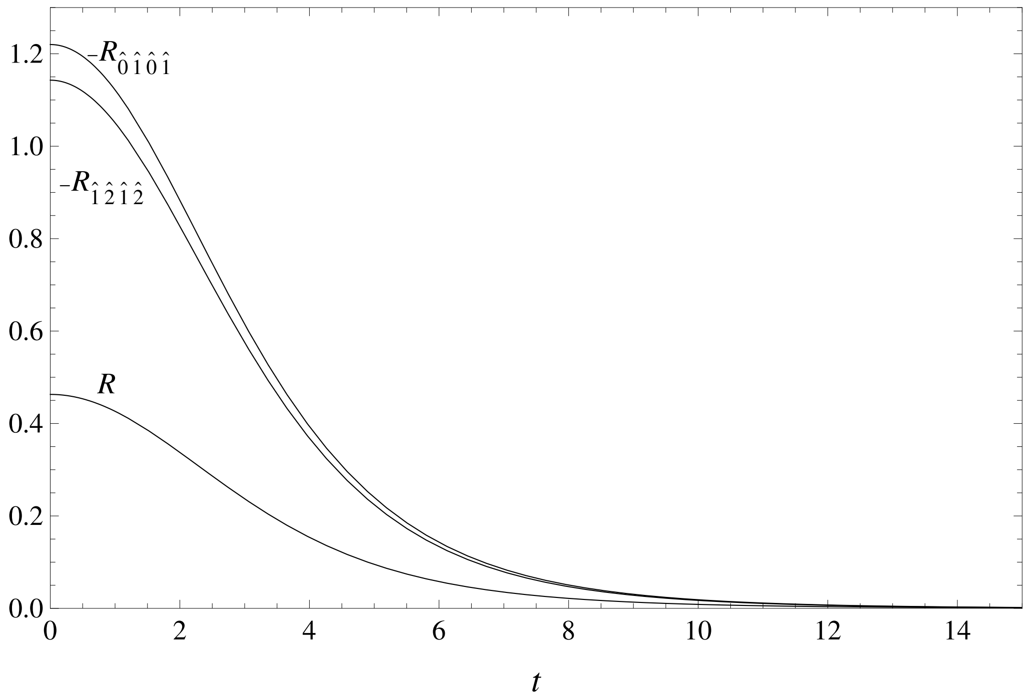

Figura A.3-1. The local expectation value of the non-null curvature components in the tetrad basis and of the Ricci scalar. They are perfect for all values of t. Here we have chosen $\sigma = 1$ and $a_0 = 0.1$.

Its graphic, Fig. A.3-1, is perfectly regular like in Ref. [11], giving further evidence of the nonsingular character of the quantum cosmological model.

In the tetrad basis, the tensor $T^{\mu}{}_{\nu}$ is in the diagonal form

$$T^{\hat{\mu}}{}_{\hat{\nu}} = \mathrm{diag}(-\rho, p, p, p). \tag{A.41}$$

Using the Einstein equation

$$T^{\mu}{}_{\nu} = \frac{1}{8\pi}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R\right) \tag{A.42}$$

we have

$$\begin{aligned} T^{\hat{0}}_{\hat{0}} &= -\rho = -\frac{3}{8\pi} R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}, \\ T^{\hat{i}}_{\hat{i}} &= p = \frac{1}{8\pi}\left(2R_{\hat{t}\hat{r}\hat{t}\hat{r}} - R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}\right). \end{aligned} \tag{A.43}$$

Now we can plot the graphics of $\langle\hat{\rho}\rangle_L$ and $\langle\hat{p}\rangle_L$ (Fig. A.3-2). At the quantum level both are negative and turn positive as quantum effects become negligible. They are also regular for all time t, including the beginning of times $t = 0$.

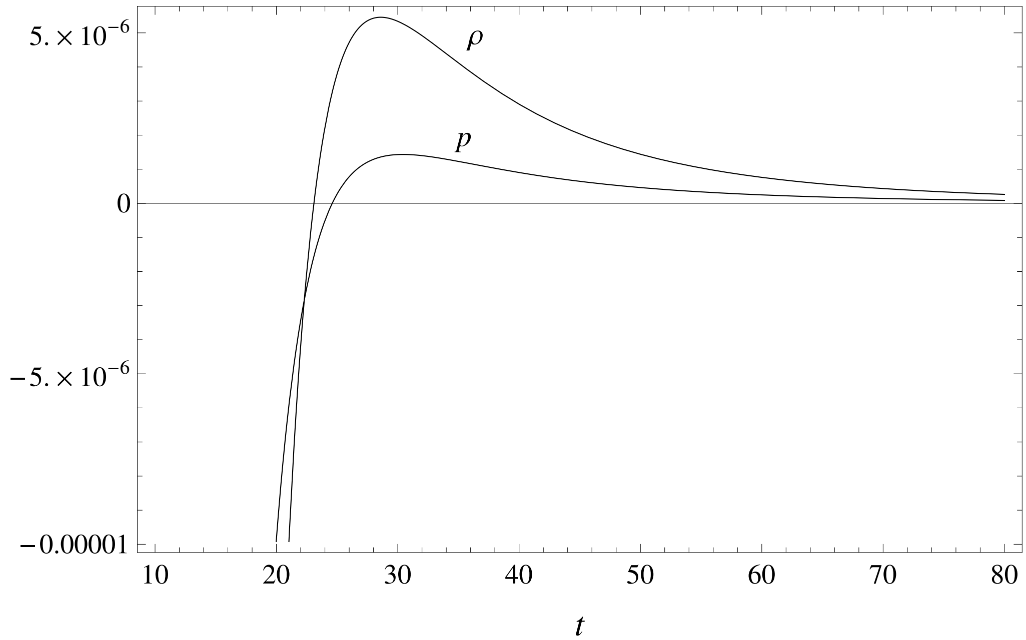

Figura A.3-2. The local expectation value of the energy density and pressure for $\sigma = 1$ and $a_0 = 0.1$.

We can also study $\langle\hat{p}\rangle_L / \langle\hat{\rho}\rangle_L$ in order to see if the equation of state is respected during the evolution of the Universe. In Fig. A.3-3 we note that in the classical limit $\langle\hat{p}\rangle_L / \langle\hat{\rho}\rangle_L \to 1/3$ as expected. This shows the consistency of the model, since we expect that as t becomes large, classical aspects begin to appear.

We now turn to analyze the breakdown in the energy conditions. There are three types of physical reasonable energy conditions to be considered [1]. The *weak energy condition* states that $T_{\mu\nu}W^{\mu}W^{\nu} \geq 0$ for any timelike vector W^{μ}. For an energy-momentum tensor expressed in the form (A.41), this will be true if and only if $\rho \geq 0$ and $\rho + p \geq 0$. The *dominant energy condition* says that for every timelike vector W^{μ}, $T_{\mu\nu}W^{\mu}W^{\nu} \geq 0$ and $T^{\mu\nu}W_{\nu}$ is a nonspacelike vector. This holds if $\rho \geq 0$ and $\rho \geq |p|$. Finally the most common energy condition, called *strong energy condition*, states that $R_{\mu\nu}W^{\mu}W^{\nu} \geq 0$ for every timelike vector W^{μ}. According to the Einstein equation, this is equivalent to saying that the energy-momentum tensor satisfies $T_{\mu\nu}W^{\mu}W^{\nu} \geq \frac{1}{2}W^{\mu}W^{\mu}T$ and this will be true for the energy-momentum tensor (A.41) if $\rho + p \geq 0$ and $\rho + 3p \geq 0$. These conditions, along with some simple restrictions on the spacetime manifold [**?**], such as certain reasonable initial conditions (the existence of trapped surfaces or the existence

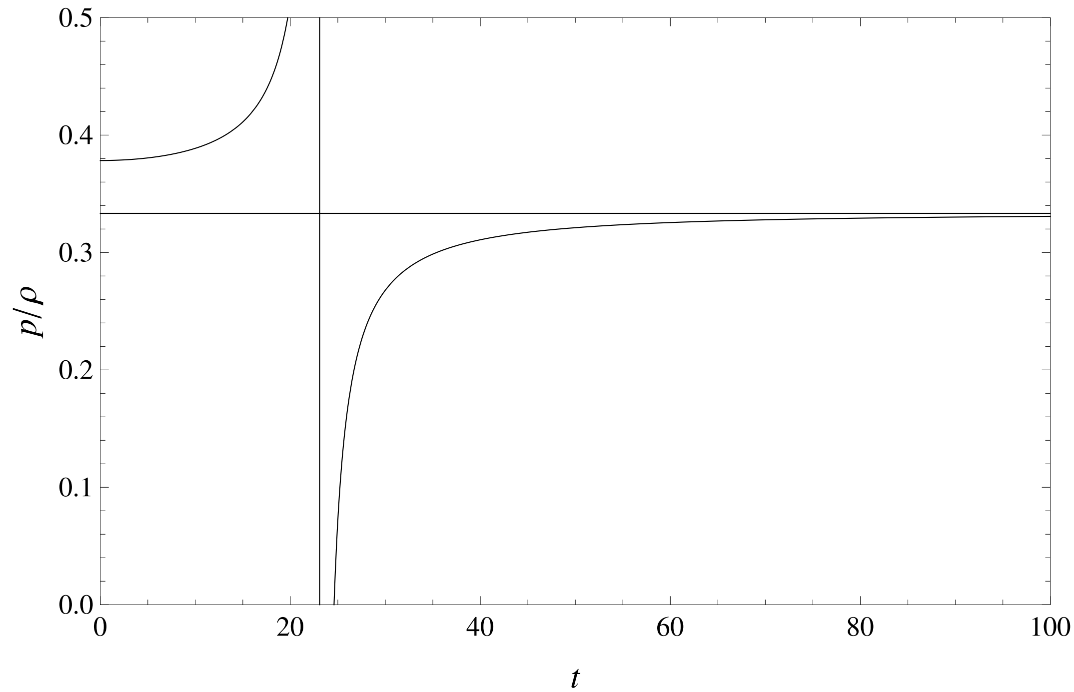

Figura A.3-3. The local expectation value of p/ρ for $\sigma = 1$ and $a_0 = 0.1$. The function explodes in t between 20 and 25 because ρ becomes zero at this point.

of a spacelike hypersurface) and restrictions on the causal structure (the existence of a Cauchy surface or the absence of closed timelike curves), give rise to the Hawking-Penrose singularity theorems. They state that under these conditions singularities must occur in the spacetime.

In Fig. A.3-4, we can see the local expectation value of the various relations between ρ and p implied by the energy conditions. We note that they are all violated at the quantum level as we would expect, since the singularity has been removed, and becomes valid in the classical limit, showing the consistency of the model.

Now we will show that these results are inherent of the quantum model, not of a particular choice of the wave packet. For this we will do the same analysis using a different wave packet satisfying the Dirichlet boundary condition, i.e., $\Psi(0,t) = 0$ [$\beta = \infty$ in Eq. (C.7)].

The Green function for the Dirichlet boundary condition is given by

$$G^{\beta=\infty}(a,a',t) = G(a,a',t) - G(a,a',t). \tag{A.44}$$

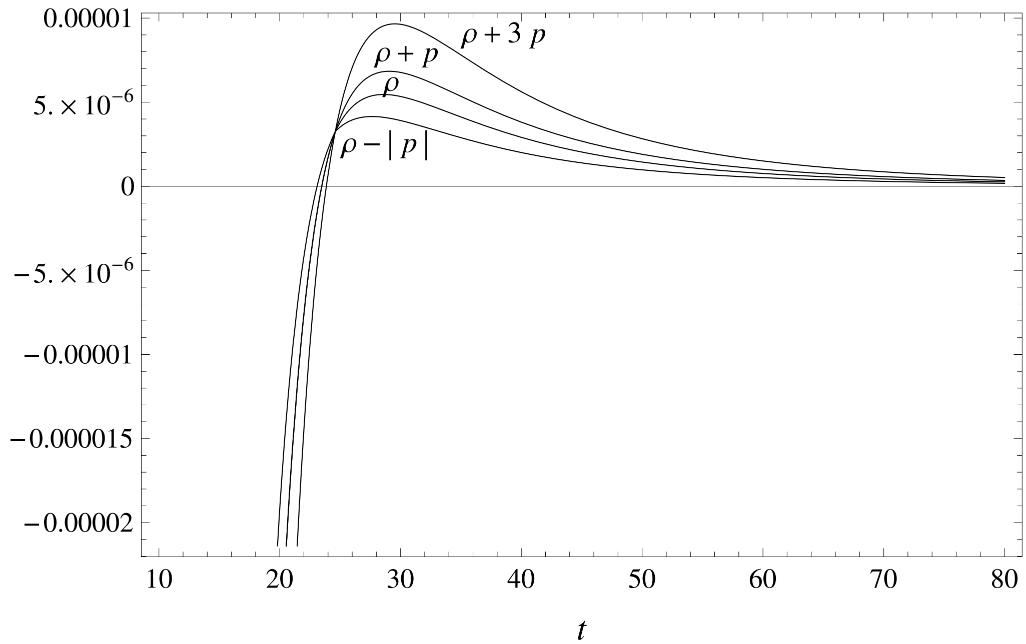

Figura A.3-4. The energy conditions for $\sigma = 1$ and $a_0 = 0.1$. We see that every one of them are violated at the quantum level.

If we start with a normalized wave packet

$$\Psi(a,0) = \left(\frac{128\sigma^3}{\pi}\right)^{1/4} a e^{-\sigma a^2} \tag{A.45}$$

we have

$$\begin{aligned}\Psi(a,t) &= \int_0^\infty G^{\beta=\infty}(a,a't)\Psi(a',0)da' = \int_{-\infty}^\infty G(a,a't)\Psi(a',0)da' \\ &= \left(\frac{128\sigma^3}{\pi}\right)\frac{(216i)^{1/2}}{(\sigma t - 6i)^{3/2}} a \exp\left(\frac{6i\sigma a^2}{\sigma t - 6i}\right).\end{aligned} \tag{A.46}$$

The polar decomposition of Ψ has the form

$$\Psi = \Theta e^{iS},$$

where

$$\begin{aligned}S &= \frac{6\sigma^2 t a^2}{\sigma^2 t^2 + 36} + f_0(t), \\ \Theta &= g_0(t) a \exp\left(\frac{-36\sigma a^2}{\sigma^2 t^2 + 36}\right).\end{aligned} \tag{A.47}$$

The Bohmian trajectory for the scale factor is the same as the previous case, i.e.,

$$a(t) = a_0\sqrt{36+\sigma^2t^2}.$$

The local expectation value for the quantity $\hat{p}_a^2/a^4$ is now given by

$$\begin{aligned}\left\langle\frac{\hat{p}_a^2}{a^4}\right\rangle_L =&\frac{1}{a(t)^4}\left(\frac{12\sigma^2ta(t)}{36+\sigma^2t^2}\right)^2-\frac{1}{a(t)^4}\frac{1}{a(t)}\left[-\frac{144\sigma a(t)}{26+\sigma^2t^2}+\left(1-\frac{72\sigma a(t)^2}{\sigma^2t^2+36}\right)\left(-\frac{72\sigma a(t)}{\sigma^2t^2+36}\right)\right]\\ &+\frac{4}{a(t)^5}\frac{1}{a(t)}\left(1-\frac{72\sigma a(t)^2}{36+\sigma^2t^2}\right)-\frac{5}{a(t)^6}=\frac{-1+72\sigma a_0^2[-1+2\sigma(-36+t^2\sigma^2)a_0^2]}{(36+t^2\sigma^2)^3a_0^6}\end{aligned} \tag{A.48}$$

The local expectation values of the components of the curvature tensor in the tetrad basis are given by

$$\begin{aligned}\left\langle\hat{R}_{\hat{t}\hat{r}\hat{t}\hat{r}}\right\rangle_L &= \frac{-1+72\sigma a_0^2[-1+2\sigma(-72+\sigma^2t^2)a_0^2]}{144(36+\sigma^2t^2)^3a_0^6}\\ \left\langle\hat{R}_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}\right\rangle_L &= \frac{-1+72\sigma a_0^2[-1+2\sigma(-36+\sigma^2t^2)a_0^2]}{144(36+\sigma^2t^2)^3a_0^6}.\end{aligned} \tag{A.49}$$

In what follows we will show the graphics of $\left\langle\hat{R}_{\hat{t}\hat{r}\hat{t}\hat{r}}\right\rangle_L$, $\left\langle\hat{R}_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}\right\rangle_L$, and $\langle R\rangle_L$ in Fig. A.3-5 and the relations between ρ and p implied by the energy conditions in Fig. A.3-6. As we can see, the graphics in this case are very similar to those showed previously.

The other graphics representing ρ, p, and p/ρ, and we did not show here are very similar to those showed previously too.

A.4 Independence of the nonsingular character of the quantum cosmological model with respect to the wave packet

In this section we calculate the expectation value of the scale factor in the many-worlds interpretation of quantum mechanics for several wave packets representing the wave function of the universe. We want to speculate as to whether the nonsingular character of the

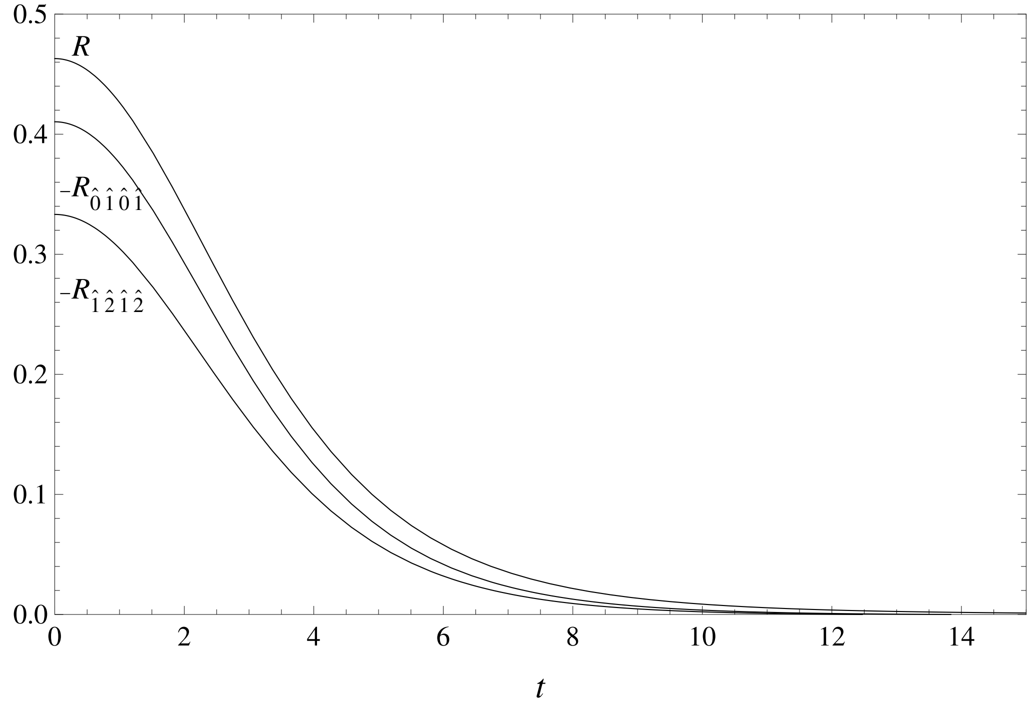

Figura A.3-5. The local expectation value of the non-null curvature components in the tetrad basis and of the Ricci scalar for a wave packet satisfying the Dirichlet boundary condition for $\sigma = 1$ and $a_0 = 0.1$.

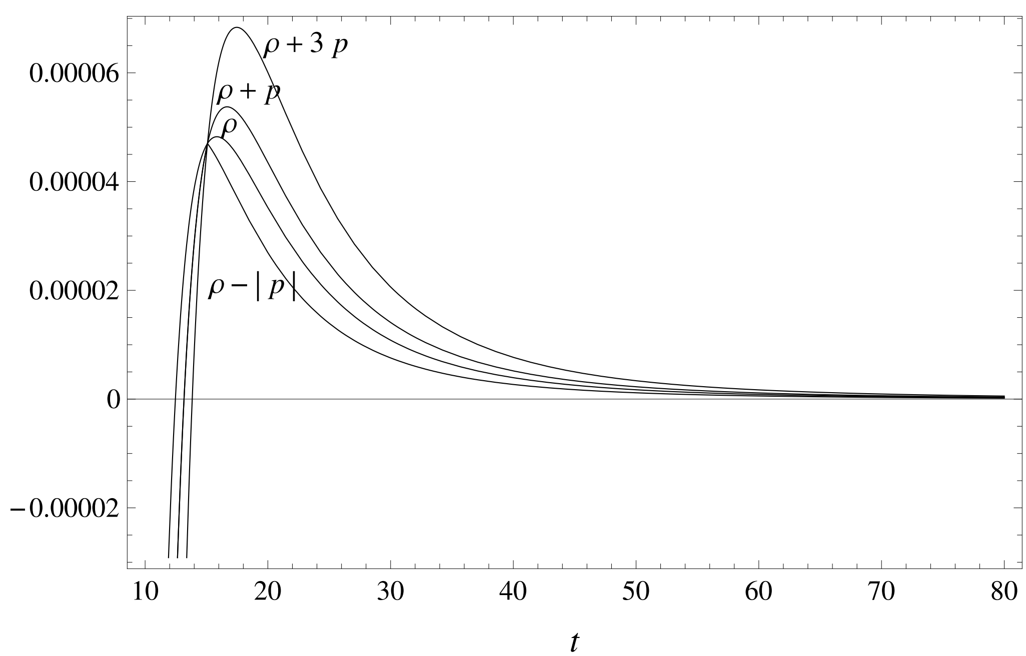

Figura A.3-6. The energy conditions for a wave packet satisfying $\Psi(a, 0) = 0$ for $\sigma = 1$ and $a_0 = 0.1$.

quantum Universe is associated with the simple wave packets found in the literature so far.

The wave packets we will use in this section will be obtained in two different ways. The first by using the Green function [Eq. (A.12)] with the boundary condition $\Psi'(0,t)=0$. The second is through the eigenfunctions [13]

$$\Psi_E(a,t) = e^{-iEt}\sqrt{a}J_{1/2}\left(\frac{\sqrt{96E}}{2}a\right) \tag{A.50}$$

of Eq. (C.5). The superposition

$$\Psi(a,t) = \int_0^\infty A(E)\Psi_E(a,t)dE \tag{A.51}$$

implies the boundary condition $\Psi(0,t)=0$.

The first wave packet we use is given in Eq. (A.15b). The expectation value of the scale factor for such a wave function was calculated in Ref. [15] and is given by

$$< a(t) >= \frac{1}{12}\sqrt{\frac{2}{\pi\sigma}}\sqrt{\sigma^2t^2+36}. \tag{A.52}$$

Now, we will take the initial wave packets of the form

$$\Psi(a,0) = a^n e^{-\gamma a^2} \qquad \gamma > 0, \tag{A.53}$$

n even. Then, we have

$$\Psi(a,t) = \left(\frac{6}{\pi i t}\right)^{1/2}\int_{-\infty}^{\infty} e^{\frac{6i}{t}(a-a')^2}a'^n e^{-\gamma a'^2}da'. \tag{A.54}$$

This integral can be performed with the help of Ref. [20] and gives

$$\Psi(a,t) = \alpha_n(\gamma,t)\exp\left(-\frac{36a^2\gamma t^2}{\gamma^2t^4+36t^2}\right)\exp\left[i\left(-\frac{216a^2t}{\gamma^2t^4+36t^2}+\frac{6a^2}{t}\right)\right]H_n\left(\frac{6a}{(\gamma t^2-6i)^{1/2}}\right), \tag{A.55}$$

where

$$\alpha_n(\gamma,t) = 6^{1/2}t^{n/2}(i)^{-(n+1)/2}2^{-n}(\gamma t-6i)^{-(n+1)/2}. \tag{A.56}$$

and $H_n(x)$ is the Hermite polynomial of order n. Let us consider the case $n = 2$ and $n = 4$. First, let us take $\Psi(a, 0) = a^2 e^{-\gamma a^2}$. In this case,

$$\Psi(a,t) = \alpha_2(\gamma,t)e^{-\frac{\xi}{2}a^2}e^{i\nu}(\eta a^2 - 2), \tag{A.57}$$

$$\begin{aligned} \xi &= -\frac{72\gamma t^2}{\gamma^2 t^4 + 36t^2}, \\ \nu &= -\frac{216a^2 t}{\gamma^2 t^4 + 36t^2} + \frac{6a^2}{t}, \\ \eta &= \frac{144}{\gamma t^2 - 6it}. \end{aligned} \tag{A.58}$$

The expectation value of the scale factor a for this wave packet is given by

$$\begin{aligned} \langle a \rangle (t) &= \frac{\int_0^\infty \Psi(a,t)^* a \Psi(a,t) da}{\int_0^\infty \Psi(a,t)^* \Psi(a,t) da} = \frac{\int_0^\infty |\alpha(\gamma,t)|\, e^{-\xi a^2} \left[|\eta|^2 a^5 - 2\mathrm{Re}(\eta) a^3 + 4a\right] da}{\int_0^\infty |\alpha(\gamma,t)|\, e^{-\xi a^2} \left[|\eta|^2 a^4 - 2\mathrm{Re}(\eta) a^2 + 4\right] da} \\ &= \frac{\sqrt{\frac{\pi}{2}}(72 + t^2\gamma^2)}{9\sqrt{\gamma(36 + t^2\gamma^2)}}. \end{aligned} \tag{A.59}$$

We note that the expectation value of the scale factor is nonzero for all times, showing that the Universe is nonsingular. Now let us take $\Psi(a, 0) = a^4 e^{-\gamma a^2}$. In this case,

$$\Psi(a,t) = \alpha_4(\gamma,t)e^{-\frac{\xi}{2}a^2}e^{i\nu}(\eta a^4 - \varsigma a^2 + 12), \tag{A.60}$$

where

$$\begin{aligned} \eta &= \frac{20736}{(\gamma t^2 - 6it)^2}, \\ \varsigma &= \frac{1728}{\gamma t^2 - 6it}. \end{aligned} \tag{A.61}$$

Then we have

$$\langle a \rangle (t) = \frac{4\sqrt{\frac{2}{\pi}}(3456 + 144t^2\gamma^2 + t^4\gamma^4)}{35\sqrt{\gamma}(36 + t^2\gamma^2)^{3/2}}. \tag{A.62}$$

Again the expectation value of the scale factor is nonzero for all values of t. Note that on both cases the asymptotic behavior of the scale factor is $\langle a \rangle (t) \propto t$, like its classical analogue, since we are working with the gauge choice $N = a$ so that the cosmological time τ (in the asymptotic behavior) is given by

$$d\tau = tdt \Rightarrow \tau \propto t^2 \Rightarrow \langle a \rangle (\tau) \propto \tau^{1/2}. \tag{A.63}$$

Let us now take more elaborate wave packets. Consider $A(E) = e^{-E}$ in Eq. (A.51). The integral

$$\Psi(a,t) = \sqrt{a}\int_0^\infty e^{-(1+it)E} J_{1/2}(\frac{\sqrt{96E}}{2}a)dE \tag{A.64}$$

can be solved analytically [20], and we get

$$\Psi(a,t) = \frac{\sqrt{96}}{8}\sqrt{\frac{\pi}{(\gamma+it)^3}}a^{3/2}e^{\frac{3a^2it}{\gamma^2+t^2}}e^{-\frac{3\gamma a^2}{\gamma^2+t^2}}\left[I_{-1/4}\left(\frac{3a^2}{\gamma+it}\right) - I_{3/4}\left(\frac{3a^2}{\gamma+it}\right)\right]. \tag{A.65}$$

We were unable to solve analytically the expectation value of the scale factor for this case, but we solved it numerically. The result is shown in Fig. A.4-7.

If we now define $r = \frac{\sqrt{96}}{2}$ in Eq. (A.50) we will have, instead of Eq. (A.51),

$$\Psi(a,t) = \frac{\sqrt{a}}{12}\int_0^\infty re^{\frac{-ir^2t}{24}} J_{1/2}(ra)A(r)dr. \tag{A.66}$$

Consider $A(r) = e^{-\gamma r^2} I_{1/2}(\beta r)$. Then, the integral (A.66) can be solved [20] and results in

$$\begin{aligned}\Psi(a,t) =& \frac{\sqrt{a}}{24\left(\gamma+\frac{it}{24}\right)}\exp\left(-i\frac{\beta^2 t}{96\left(\gamma^2+\frac{t^2}{24^2}\right)}\right)\exp\left(i\frac{a^2 t}{96\left(\gamma^2+\frac{t^2}{24^2}\right)}\right)\times \\ &\times\exp\left(\frac{\beta^2\gamma}{4\left(\gamma^2+\frac{t^2}{24^2}\right)}\right)\exp\left(-\frac{\gamma a^2}{4\left(\gamma^2+\frac{t^2}{24^2}\right)}\right)J_{1/2}\left(\frac{\beta a}{2(\gamma+\frac{it}{24})}\right).\end{aligned} \tag{A.67}$$

This wave packet respects the boundary condition $\Psi(0,t) = 0$, and the initial wave packet is given by

$$\Psi(a,0) = \frac{\sqrt{a}}{24\gamma}e^{\frac{\beta^2\gamma}{4\gamma^2}}e^{-\frac{\gamma a^2}{4\gamma^2}}J_{1/2}\left(\frac{\beta a}{2\gamma}\right). \tag{A.68}$$

We could not find either an analytic expression for the expectation value of the scale factor, but we have shown the numerical result in Fig. A.4-7.

Figure A.4-7 shows the expectation value of the scale factor for (1) $\Psi(a,0) = e^{-a^2}$, (2) $\Psi(a,0) = a^2 e^{-a^2}$, (3) $\Psi(a,0) = a^4 e^{-a^2}$, (4) $A(E) = e^{-E}$, and (5) $A(r) = e^{-r^2} I_{1/2}(r)$. We see that they have qualitatively the same behavior, every one of them growing linearly proportional to t as $t \to \infty$. The main point here is that asymptotically the graphics behave as straight lines which pass trough the origin. In principle the proportionality constants need not be the same. In every case, the singularity has been excluded, indicating that this is a property of the introduction quantum cosmology, not of a particular wave packet.

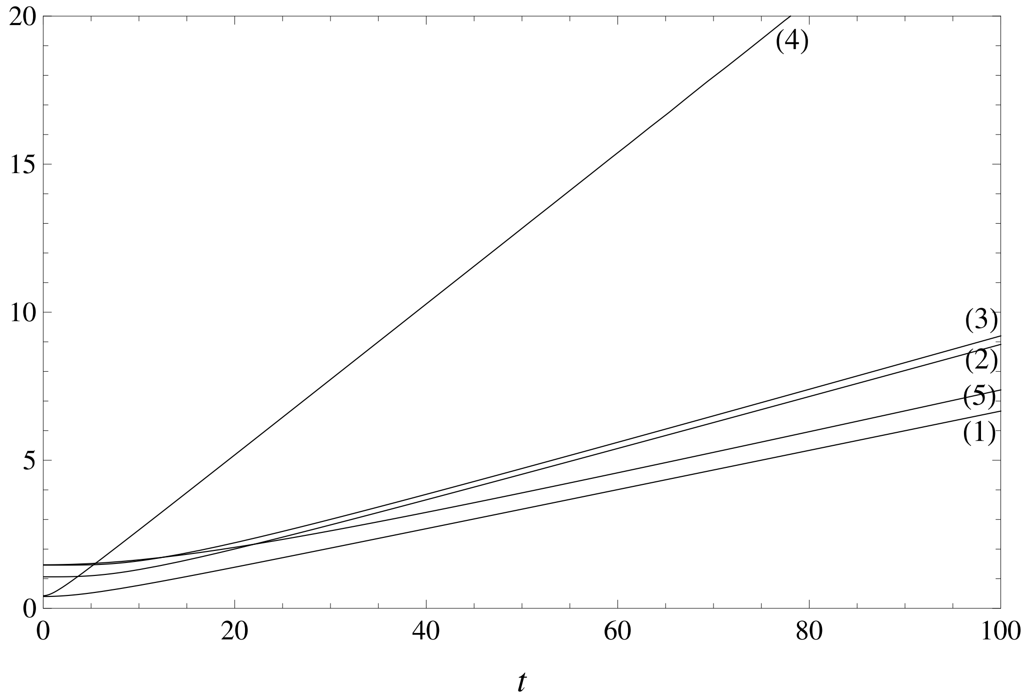

Figura A.4-7. The expectation value of the scale factor for (1) $\Psi(a,0) = e^{-a^2}$, (2) $\Psi(a,0) = a^2 e^{-a^2}$, (3) $\Psi(a,0) = a^4 e^{-a^2}$, (4) $A(E) = e^{-E}$ and (5) $A(r) = e^{-r^2} I_{1/2}(r)$. We see that they have qualitatively the same behavior, i.e., asymptotically they behave as straight lines which pass trough the origin

.

A.5 Recovering the Big Bang by the introduction of a nonnormalizable wave packet

It has been argued that a nonnormalizable wave function is unavoidable in quantum cosmology [18]. With such a wave packet, we cannot find the expectation value of the scale factor, but we can still find the trajectory in the de Broglie-Bohm interpretation. The role of the wave function in the ontological interpretation of quantum mechanics is to provide a wave guidance [Eq. (A.18)] for the dynamical variables, and this clearly does not require normalization [21]. Square integrability of the wave function is necessary only when probability takes place. So it is valid to work with nonnormalizable wave functions if we are looking for the trajectory of a dynamical variable. Now we will show that the singular character of the scale factor can be recovered with the introduction of a nonnormalizable solution of Eq. (C.5) in the de Broglie-Bohm interpretation of quantum mechanics.

First, let us assume that the wave function satisfies the boundary condition $\Psi(0,t)=0$. This boundary condition was defended by DeWitt [4] because it keeps wave packets away from the singularity. Now we take $A(E)=J_{1/2}(\beta\sqrt{E})$ in Eq. (A.51), and we have

$$\Psi(a,t)=\sqrt{a}\int_0^{\infty}e^{-iEt}J_{1/2}\left(\frac{\sqrt{96}a}{2}\sqrt{E}\right)J_{1/2}(\beta\sqrt{E})dE. \tag{A.69}$$

The above integral can be analytically solved with the help of Ref. [20]. The result is

$$\Psi(a,t)=\sqrt{a}\frac{1}{t}J_{1/2}\left(\frac{\sqrt{96}a\beta}{4t}\right)\exp\left[i\left(\frac{24a^2+\beta^2}{4t}-\frac{3\pi}{4}\right)\right]. \tag{A.70}$$

It is already separated in the polar form Θe^{iS}, so we have

$$S=\frac{24a^2+\beta^2}{4t}-\frac{3\pi}{4}. \tag{A.71}$$

By Eq. (A.18), we have

$$12\dot{a}=\frac{12a}{t}\Rightarrow a(t)=a_0t. \tag{A.72}$$

Note that the above Bohmian trajectory of the scale factor recovers the classical behavior $a(\tau) \propto \tau^{1/2}$ for the radiation dominated era, where τ is the cosmological time.

Now, following the quantization procedure discussed by Tipler in reference [18], we require that, like the classical case in which all solutions pass through the singularity $a = 0$ in $t = 0$, all quantum universes do the same. Then, we pick $\Psi(a, 0) = \delta(a)$ as the initial wave packet. It is easy to show that for such an initial wave packet, the DeWitt boundary condition gives rise to a null wave function for the Universe. So we take the boundary condition used by Tipler in [18] $\Psi'(0, t) = 0$. Then, we have

$$\Psi(a,t) = \int_{-\infty}^{\infty} G(a, a', t)\delta(a')da' = G(a, 0, t) = \left(\frac{6}{\pi i t}\right)^{1/2} \exp\left(\frac{6i}{t}a^2\right). \tag{A.73}$$

This is what Tipler called the Green function of the Universe. Note that in this case we have

$$S = \frac{6a^2}{t}, \tag{A.74}$$

so that

$$12\dot{a} = \frac{12a}{t} \Rightarrow a(t) = a_0 t \tag{A.75}$$

like in the previous case.

It is worth noting that, as observed by Lemos in [15], these boundary conditions are somewhat arbitrary for the wave packets studied in this section. This is because, since these functions are not square-integrable, they do not satisfy the condition for the Hermiticity of the operator $\hat{H} = -\frac{\partial^2}{\partial a^2}$

$$\left(\Psi^* \frac{d\Psi}{da} - \frac{d\Psi^*}{da}\Psi\right)(\infty) = \left(\Psi^* \frac{d\Psi}{da} - \frac{d\Psi^*}{da}\Psi\right)(0), \tag{A.76}$$

so they are not in the domain of functions where $\hat{H}$ is Hermitian. So Eq. (C.7) is not a condition for the self-adjointness of the operator $\hat{H}$ in this case.

A.6 conclusions

By the quantization of the components of the curvature tensor in the tetrad basis and with the use of a simple wave packet for the Universe, we were able to take the local expectation value of these components in the de Broglie-Bohm interpretation of quantum mechanics. We then showed graphically that these quantities are perfectly regular for all times, giving one further evidence that the Universe is nonsingular with the introduction of quantum cosmology, like in Ref. [11]. We also took the local expectation value of the energy density ρ and the pressure p of the fluid and showed that the energy conditions are violated in the quantum era as expected, since under the validity of these conditions the Universe is undoubtedly singular. We also found the consistency of the theory by showing that the classical equation of state relating ρ and p is recovered in the classical limit and that the classical energy conditions are again observed when quantum mechanics become unimportant.

We also find the expectation value of the scale factor in the many-worlds interpretation for several wave packets representing the wave function of the Universe. We showed graphically that for every wave packet the qualitative behavior of the scale factor is the same, every one of them tending to the classical expression $a(t) \propto t$ $(a(\tau) \propto \tau^{1/2})$. As we said before, it has been argued that nonnormalizable wave packets are unavoidable to quantum cosmology, so we took two nonnormalizable wave packets representing wave functions for the Universe and showed that for these wave packets, the Bohmian trajectory of the scale factor is singular like its classical analogue, so we could speculate that if the big-bang singularity really existed, the wave function of the Universe is in fact nonnormalizable.

Acknowledgements

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Apêndice B

Cosmologia Quântica FLRW n-dimensional

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n-dimensional FLRW Quantum Cosmology

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We introduce the formalism of quantum cosmology in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe of arbitrary dimension filled with a perfect fluid with $p = \alpha\rho$ equation of state. First we show that the Schutz formalism, developed in four dimensions, can be extended to a n-dimensional universe. We compute the quantum representant of the scale factor $a(t)$, in the Many-Worlds, as well as, in the de Broglie-Bohm interpretation of quantum mechanics. We show that the singularities, which are still present in

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the n-dimensional generalization of FLRW universe, are excluded with the introduction of quantum theory. We quantize, via the de Broglie-Bohm interpretation of quantum mechanics, the components of the Riemann curvature tensor in a tetrad basis in a n-dimensional FLRW universe filled with radiation ($p = \frac{1}{n-1}\rho$). We show that the quantized version of the Ricci scalar are perfectly regular for all time t. We also study the behavior of the energy density and pressure and show that the ratio $\langle p \rangle_L / \langle \rho \rangle_L$ tends to the classical value $1/(n-1)$ only for $n = 4$, showing that $n = 4$ is somewhat privileged among the other dimensions. Besides that, as $n \to \infty$, $\langle p \rangle_L / \langle \rho \rangle_L \to 1$.

PACS numbers: 04.50.-h, 04.60.-m, 98.80.Qc

B.1 Introduction

Recently there has been a current interest in the study of models that involve extra dimensions. The first works on this subject date back to the 1920s with Kaluza and his assumption that spacetime is five-dimensional in order to unify gravity with the electromagnetism. The inclusion of the electromagnetic potential within a geometrical picture as components of the metric tensor along the fifth dimension, leads to the Einstein-Maxwell equations, which follows from the vacuum Einstein equations in five dimensions. The nonobservability of the extra dimension was considered by Klein, who showed that this would explained if the extra dimension is compatified, having a circular topology and a small enough scale (of the order of the Planck length). This tiny radius of curvature implies that the extra dimension would remain hidden to all low energy physics considerations [1]. Over the years, the Kaluza-Klein idea was generalized to higher dimensions in order to unify non-Abelian gauge fields with gravity [2, 3] and now it is believed that spacetime could have much more than four dimensions, such as in the 10-dimensional superstring [4] theory and in the 11-dimensional supergravity [5].

One of the main problems with the 4-dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) universe filled with a perfect fluid is the presence of the big-bang singularity. Such a singularity is a scalar curvature singularity, where every observer near the singularity sees physical quantities diverging [6, 7]. It was shown [8] that under very reasonable assumption, singularities are always present in cosmology. Near the singularity, the laws of physics break-down and we hope that a full quantum theory of gravity will overcome this situation. While such a theory does not exist, there are many attempts to incorporate quantum mechanics into general relativity. One of the first attempts to do this was quantum cosmology [9]. The introduction of quantum cosmology in the minisuperspace of FLRW universe has been shown successfully in the sense that in the Many-Worlds and in the de Broglie-Bohm interpretation of quantum mechanics, the singularity are removed [10–12].

In the n-dimensional FLRW universe, the big-bang singularity is still present. So in this paper we follow the idea that our physical four-dimensional Universe is actually embedded in a higher-dimensional spacetime and generalize the concept of quantum cosmology to a flat, homogeneous, and isotropic n-dimensional FLRW universe in order to see if the singularities are excluded with the introduction of quantum mechanics. The generalization of classical inflationary cosmological solutions for a n-dimensional FLRW universe has already been made [13] and it was shown that the derived solutions possess properties similar to those ones of the corresponding solutions of the standard four-dimensional FLRW cosmology. Here we show that, like in the 4-dimensional FLRW quantum cosmology, the singularity in the n-dimensional universe is removed with the introduction of quantum cosmology and that the expectation value of the scale factor in the Many-Worlds interpretation, and the trajectory of the scale factor in the de Broglie-Bohm interpretation, tends to the classical behavior as $t \to \infty$. All of our results reduces to the well known results for $n = 4$. We also study the local expectation value of the energy density and pressure of the fluid for a universe filled with radiation and show that the ratio $\langle p \rangle_L \,/\, \langle \rho \rangle_L$ tends to the classical limit as $t \to \infty$ only for the $n = 4$ case, fact that single out four dimensions.

In this paper we proceed as follows: in Sec. II we generalize Schutz formalism [14, 15] to n-dimensional fluids and solve the Einstein equations via Hamiltonian formalism to find the scale factor for the n-dimensional FLRW universe filled with a perfect fluid with equation of state $p = \alpha\rho$. We compare this scale factor with the one found via Einstein equations. In Sec. III we solve the Wheeler-deWitt equation of the universe, find the expectation value and the trajectory of the scale factor and compare with the classical value. In Sec. IV we quantize the independent components of the Riemann curvature tensor in a tetrad basis in a radiation-dominated universe and show that the Ricci scalar is perfectly regular for all time t, illustrating the exclusion of the classical singularity with the introduction of quantum theory. We also quantize the energy density and pressure of the fluid and study the behavior of the ratio p/ρ with relation to time. Finally, in Sec. V, we discuss the main results presented in this work.

B.2 Classical formalism

B.2.1 Einstein equations

Let us find the scale factor in the n-dimensional flat FLRW universe filled with a perfect fluid through the Einstein equations. A flat n-dimensional FLRW universe has the metric

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \qquad (i, j = 1 \dots n-1), \tag{B.1}$$

where δ_{ij} is the Kronecker delta symbol and x^i are the comoving coordinates of the universe. In this coordinate set, the n-velocity of the fluid which fills the universe is given by

$$U^\mu = (1, 0, \dots, 0)\,, \tag{B.2}$$

so that the energy momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{B.3}$$

takes the form

$$T_{\mu\nu} = \mathrm{diag}(\rho, a^2 p, \dots, a^2 p). \tag{B.4}$$

The Einstein equations for an n-dimensional spacetime are given by [13]

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa_n T_{\alpha\beta}, \tag{B.5}$$

with κ_n representing the multidimensional gravitational constant. The independent Einstein equations are

$$G_{00} = \kappa_n T_{00} \Rightarrow \frac{(n-1)(n-2)}{2}\left(\frac{\dot{a}}{a}\right)^2 = \kappa_n \rho, \tag{B.6a}$$

$$G_{ii} = \kappa_n T_{ii} \Rightarrow -\frac{(n-2)(n-3)}{2}\dot{a}^2 - (n-2)\ddot{a}a = \kappa_n a^2 p \qquad (i = 1, \ldots, n-1). \tag{B.6b}$$

These are the Friedmann equations for a flat n-dimensional FLRW universe filled with a perfect fluid. Note that there are only two independent equation due to isotropy and homogeneity. By introducing an equation of state for the fluid $p = \alpha\rho$, we can replace Eq. (B.6a) into Eq. (B.6b) obtaining

$$\frac{1}{2}\left[\alpha(n-1) + (n-3)\right]\frac{\dot{a}}{a} + \frac{\ddot{a}}{\dot{a}} = 0. \tag{B.7}$$

The above equation can be integrated,

$$\ln a^{\frac{\alpha(n-1)+(n-3)}{2}}\dot{a} = \ln \mathrm{const}. \tag{B.8}$$

Therefore

$$\dot{a} \propto a^{-\frac{\alpha(n-1)+(n-3)}{2}} \Rightarrow a \propto t^{\frac{2}{(n-1)(1+\alpha)}}. \tag{B.9}$$

This is the scale factor for a flat n-dimensional FLRW universe filled with a perfect fluid with equation of state $p = \alpha\rho$. The big-bang singularity is present no matter what dimension we take. Note that for $n = 4$ and $\alpha = 1/3$ (a four-dimensional flat radiation-dominated universe) we have $a \propto t^{1/2}$, and for $\alpha = 0$ (matter-dominated universe) we have $a \propto t^{2/3}$ as required.

B.2.2 Hamiltonian formalism

Now we will develop the Hamiltonian formalism for the n-dimensional FLRW universe filled with a perfect fluid with equation of state $p = \alpha\rho$. We will generalize Schutz formalism [14, 15] to n-dimensions and check its validity by calculating the scale factor via a variational principle and comparing with the scale factor calculated via Friedmann equations in the previous subsection.

The action for general relativity is given by

$$S_{GR} = \int_{\mathcal{M}} d^n x \sqrt{-g} R + 2 \int_{\partial \mathcal{M}} d^{n-1} x \sqrt{h} h_{ab} K^{ab}, \tag{B.10}$$

where h_{ab} is the induced metric over the boundary $\partial \mathcal{M}$ of the four-dimensional manifold $\mathcal{M}$ and K^{ab} is the extrinsic curvature of the hypersurface $\partial \mathcal{M}$, that is, the curvature of $\partial \mathcal{M}$ with respect to $\mathcal{M}$.

In a four-dimensional spacetime, Schutz showed that the four-velocity of a perfect fluid is expressed in terms of five potentials [14]

$$U_\nu = \mu^{-1}(\phi_{,\nu} + \alpha \beta_{,\nu} + \theta S_{,\nu}), \tag{B.11}$$

where μ is the specific enthalpy and S is the specific entropy. The potentials α and β are connected with rotation [14] so they are not present in the FLRW universe due to its symmetry. The potentials ϕ and θ have no clear physical meaning and the four-velocity satisfies the normalization condition

$$U_\nu U^\nu = -1. \tag{B.12}$$

Schutz also showed that the action for the fluid is given by

$$S_f = \int_{\mathcal{M}} d^4 x \sqrt{-g} p, \tag{B.13}$$

where p is the pressure of the fluid and is linked with the density by the equation of state $p = \alpha \rho$.

By thermodynamical considerations, Lapchinskii and Rubakov [16] found that the expression for the pressure is given in terms of the potentials by

$$p = \frac{\alpha}{(\alpha+1)^{1/\alpha+1}} (\dot{\phi} + \theta \dot{S})^{1/\alpha+1} \exp\left(-\frac{S}{\alpha}\right), \tag{B.14}$$

so that the action for the four-dimensional flat FLRW universe with metric

$$ds^2 = -N(t)^2 dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{B.15}$$

where $N(t)$ is the lapse function, is given by [10] (in units where $16\pi G = 1$)

$$S = \int dt \left[-6\frac{\dot{a}^2 a}{N} + N^{-1/\alpha} a^3 \frac{\alpha}{(\alpha+1)^{1/\alpha+1}} (\dot{\phi} + \theta\dot{S})^{1/\alpha+1} \exp\left(-\frac{S}{\alpha}\right)\right]. \tag{B.16}$$

In an n-dimensional spacetime, more potentials should be added in Eq. (B.11). We suppose that these new potentials, just like α and β in Eq. (B.11), are related with generalized rotations and are absent of the n-dimensional FLRW universe. Therefore the action for the flat n-dimensional FLRW universe filled with a perfect fluid with equation of state $p = \alpha\rho$ is given by

$$\begin{aligned} S &= \int dtL \\ &= \int dt \Big[-(n-1)(n-2)\frac{\dot{a}^2 a^{n-3}}{N} + N^{-1/\alpha} a^{n-1} \frac{\alpha}{(\alpha+1)^{1/\alpha+1}} \times \\ &(\dot{\phi} + \theta\dot{S})^{1/\alpha+1} \exp\left(-\frac{S}{\alpha}\right)\Big]. \end{aligned} \tag{B.17}$$

The momenta associated with a, ϕ and S are given by

$$p_a = \frac{\partial L}{\partial \dot{a}} = -2(n-1)(n-2)\frac{\dot{a} a^{n-3}}{N}, \tag{B.18a}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = N^{-1/\alpha} a^{n-1} \frac{(\dot{\phi} + \theta\dot{S})^{1/\alpha}}{(1+\alpha)^{1/\alpha}} e^{-S/\alpha}, \tag{B.18b}$$

$$p_S = \frac{\partial L}{\partial \dot{S}} = \theta p_\phi. \tag{B.18c}$$

The Hamiltonian of the system is then given by

$$H = p_a\dot{a} + p_\phi(\dot{\phi} + \theta\dot{S}) - L = N\left[-\frac{p_a^2}{4(n-1)(n-2)a^{n-3}} + a^{(n-1)\alpha} e^S p_\phi^{\alpha+1}\right]. \tag{B.19}$$

After a canonical transformation of the form [10, 16]

$$T = -p_S e^{-S} p_\phi^{-(\alpha+1)}, \quad p_T = p_\phi^{\alpha+1}, \quad \overline{\phi} = \phi - (\alpha+1)\frac{p_S}{p_\phi}, \quad \overline{p}_\phi = p_\phi, \tag{B.20}$$

and, since the lapse function N plays the role of a Lagrange multiplier in the formalism, we have that the super-Hamiltonian of the system reads

$$\mathcal{H} = -\frac{p_a^2}{4(n-1)(n-2)a^{n-3}} + \frac{p_T}{a^{(n-1)\alpha}} \approx 0, \tag{B.21}$$

where p_T is a new canonical variable conjugated to dynamical degree of the fluid. Having the Hamiltonian of the system we can extract the equations of motion. They are [17]

$$\begin{aligned}
\dot{a} &= \frac{\partial(N\mathcal{H})}{\partial p_a} = -\frac{Np_a}{2(n-1)(n-2)a^{n-3}}, \\
\dot{p}_a &= -\frac{\partial(N\mathcal{H})}{\partial a} = -\frac{N(n-3)}{4(n-1)(n-2)}\frac{p_a^2}{a^{n-2}} + \frac{\alpha(n-1)Np_T}{a^{(n-1)\alpha}+1}, \\
\dot{T} &= \frac{\partial(N\mathcal{H})}{\partial p_T} = \frac{N}{a^{(n-1)\alpha}}, \quad \dot{p}_T = -\frac{\partial(N\mathcal{H})}{\partial T} = 0.
\end{aligned} \tag{B.22}$$

By the super-Hamiltonian constraint we have

$$p_a = -2\sqrt{(n-1)(n-2)p_T}a^{-[(n-1)\alpha-(n-3)]/2}, \tag{B.23}$$

where we chose the negative sign in order to have an expanding universe [17]. Therefore, in the cosmic gauge $N = 1$ the equation for $a(t)$ reads

$$\dot{a} = a^{-[(n-1)\alpha+(n-3)]/2}\sqrt{\frac{p_T}{(n-1)(n-2)}}, \tag{B.24}$$

whose solution is

$$a^{(n-1)(1+\alpha)/2} = a_0^{(n-1)(1+\alpha)/2} + \sqrt{\frac{p_T}{4(n-1)(n-2)}}(n-1)(1+\alpha)(t-t_0), \tag{B.25}$$

where a_0 is the present value of the scale factor. Hence

$$a(t) = \left[a_0^{(n-1)(1+\alpha)/2} + \sqrt{\frac{p_T}{4(n-1)(n-2)}}(n-1)(1+\alpha)(t-t_0)\right]^{\frac{2}{(n-1)(1+\alpha)}}, \tag{B.26}$$

and by a suitable redefinition of time we have

$$a(t) \propto t^{\frac{2}{(n-1)(1+\alpha)}} \tag{B.27}$$

just like Eq. (B.9), fact that validates our assumptions.

In Fig. B.2-1 we show the behavior of the scale factor for $n = 4, 10, 20$ and 40 in the radiation-dominated era $\alpha = 1/(n-1)$. We see that as $n \to \infty$ the universe tends to a static universe.

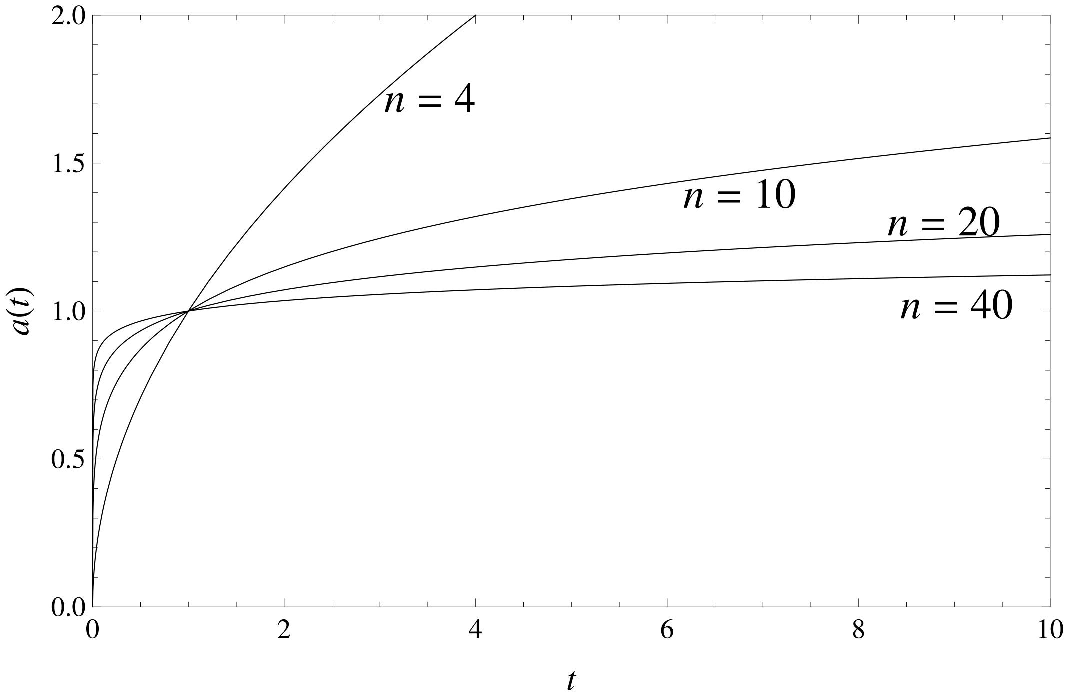

Figura B.2-1. The scale factor for $n = 4, 10, 20$ and 40 (for $\alpha = 1/(n-1)$). We note that the big-bang is still present.

B.3 Wheeler-deWitt equation for the flat n-dimensional FLRW universe with a perfect fluid

By following the Dirac approach [18] for quantization of Hamiltonian systems with constraints, making the substitutions

$$p_a \to -i\frac{\partial}{\partial a}, \quad p_T = -i\frac{\partial}{\partial T} \tag{B.28}$$

and demanding

$$\mathcal{H}\Psi = 0, \tag{B.29}$$

where Ψ is the wave function of the universe, we have

$$\frac{a^{(n-1)\alpha-(n-3)}}{4(n-1)(n-2)}\frac{\partial^2\Psi}{\partial a^2}+i\frac{\partial\Psi}{\partial t}=0, \tag{B.30}$$

with $t=-T$ being the time coordinate in the conformal gauge $N=a^{(n-1)\alpha}$ [see Eq. (B.22)].

This equation has the form of a Schrödinger equation $i\frac{\partial\Psi}{\partial t}=\hat{H}\Psi$ with

$$\hat{H}=-\frac{a^{(n-1)\alpha-(n-3)}}{4(n-1)(n-2)}\frac{\partial^2}{\partial a^2}. \tag{B.31}$$

In order to ensure self-adjointness of the operator $\hat{H}$ above, the inner product between two state equations is defined by [11]

$$\langle\Phi,\Psi\rangle=\int_0^\infty a^{(n-3)-(n-1)\alpha}\Phi^*\Psi da. \tag{B.32}$$

Moreover, boundary conditions near $a=0$ on the wave function must be imposed. The most common ones are

$$\Psi(0,t)=0 \quad \text{(Dirichlet boundary condition)}, \tag{B.33a}$$

$$\left.\frac{\partial\Psi}{\partial a}\right|_{a=0}=0 \quad \text{(Neumann boundary condition)}. \tag{B.33b}$$

In the course of the paper we will choose for simplicity the Dirichlet boundary condition. Let us now solve Eq. (B.30). By a separation of variables of the form $\Psi(a,t)=e^{-iEt}\psi(a)$ we have

$$\psi''(a)+4E(n-1)(n-2)a^{[(n-3)-(n-1)\alpha]}\psi(a)=0. \tag{B.34}$$

The solution of the above equation, respecting the Dirichlet boundary condition, is given by [19]

$$\psi_E(a)=a^{1/2}J_{\frac{1}{(n-1)(1-\alpha)}}\left(\frac{4\sqrt{E}}{(1-\alpha)}\sqrt{\frac{n-2}{n-1}}a^{\frac{(n-1)(1-\alpha)}{2}}\right), \tag{B.35}$$

where $J_\nu(x)$ is the Bessel function of order ν. Note that this solution is valid only for $\alpha\neq 1$. For $\alpha=1$, solution of Eq. (B.34) is given by $a^{\frac{1\pm\sqrt{1-16E(n-1)(n-2)}}{2}}$, which diverge either at $a=0$ or $a=\infty$ (note that E is a variable to be integrated in order to find wave packets representing the wave function of the universe).

From these stationary states we can find finite norm wave packets by superposing states of the form [10]

$$\Psi(a,t) = a^{1/2}\int_0^\infty dEA(E)e^{-iEt}J_{\frac{1}{(n-1)(1-\alpha)}}\left(\frac{4\sqrt{E}}{(1-\alpha)}\sqrt{\frac{n-2}{n-1}}a^{\frac{(n-1)(1-\alpha)}{2}}\right). \tag{B.36}$$

Defining $u=\sqrt{E}$ we have

$$\Psi(a,t) = 2a^{1/2}\int_0^\infty duA(u)ue^{-itu^2}J_{\frac{1}{(n-1)(1-\alpha)}}\left(\frac{4u}{(1-\alpha)}\sqrt{\frac{n-2}{n-1}}a^{\frac{(n-1)(1-\alpha)}{2}}\right). \tag{B.37}$$

Let us take a quasi-Gaussian superposition factor $A(u) = u^{\frac{1}{(n-1)(1-\alpha)}}e^{-\gamma u^2}/2$. We have

$$\Psi(a,t) = 2a^{1/2}\int_0^\infty duu^{\frac{1}{(n-1)(1-\alpha)}+1}e^{-(\gamma+it)u^2}J_{\frac{1}{(n-1)(1-\alpha)}}\left(\frac{4u}{(1-\alpha)}\sqrt{\frac{n-2}{n-1}}a^{\frac{(n-1)(1-\alpha)}{2}}\right). \tag{B.38}$$

Now we use the following relation [20]

$$\int_0^\infty x^{\nu+1}e^{-\alpha u^2}J_\nu(\beta x)dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left(-\frac{\beta^2}{4\alpha}\right) \tag{B.39}$$

to obtain

$$\begin{aligned}\Psi(a,t) &= \frac{\left[\frac{4}{(1-\alpha)}\sqrt{\frac{n-2}{n-1}}\right]^{\frac{1}{(n-1)(1-\alpha)}}}{[2(\gamma+it)]^{\frac{1+(n-1)(1-\alpha)}{(n-1)(1-\alpha)}}}a\exp\left[-\frac{4}{(1-\alpha)^2(\gamma+it)}\left(\frac{n-2}{n-1}\right)a^{(n-1)(1-\alpha)}\right]\\ &= \frac{\text{const.}}{[2(\gamma+it)]^{\frac{(n-1)(1-\alpha)+1}{(n-1)(1-\alpha)}}}a\exp\left[-\frac{4\gamma}{(1-\alpha)^2}\left(\frac{n-2}{n-1}\right)\frac{a^{(n-1)(1-\alpha)}}{\gamma^2+t^2}\right]\times\\ &\quad\exp\left[\frac{4it}{(1-\alpha)^2}\left(\frac{n-2}{n-1}\right)\frac{a^{(n-1)(1-\alpha)}}{\gamma^2+t^2}\right].\end{aligned} \tag{B.40}$$

By adopting the many-worlds interpretation of quantum mechanics, we can calculate the expectation value of the scale factor

$$\langle a\rangle(t) = \frac{\langle\Psi\,|a|\,\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\int_0^\infty a^{(n-3)-(n-1)\alpha}\Psi^*(a,t)a\Psi(a,t)da}{\int_0^\infty a^{(n-3)-(n-1)\alpha}\Psi^*(a,t)\Psi(a,t)da}. \tag{B.41}$$

Therefore

$$\begin{aligned}\langle a\rangle(t) \propto &\frac{1}{(\gamma^2+t^2)^{\frac{1+(n-1)(1-\alpha)}{(n-1)(1-\alpha)}}}\int_0^\infty a^{(n-3)-(n-1)\alpha}a^3\times \\ &\exp\left[-\frac{8\gamma}{(1-\alpha)^2}\left(\frac{n-2}{n-1}\right)\frac{a^{(n-1)(1-\alpha)}}{\gamma^2+t^2}\right].\end{aligned} \tag{B.42}$$

The identity [20]

$$\int_0^\infty x^{\nu-1}e^{-\mu x^p} = \frac{1}{p}\mu^{-\frac{\nu}{p}}\Gamma\left(\frac{\nu}{p}\right) \tag{B.43}$$

gives us

$$\langle a\rangle(t) \propto \frac{1}{(\gamma^2+t^2)^{\frac{1+(n-1)(1-\alpha)}{(n-1)(1-\alpha)}}}\frac{1}{(\gamma^2+t^2)^{\frac{-(n+1)+(n-1)\alpha}{(n-1)(1-\alpha)}}} = (\gamma^2+t^2)^{\frac{1}{(n-1)(1-\alpha)}}. \tag{B.44}$$

Note that $\langle a\rangle(t) \neq 0$ for all t so that the big-bang singularity is eliminated with the introduction of quantum cosmology. Note also that $\langle a\rangle(t)$ goes asymptotically as $\langle a\rangle(t) \propto t^{\frac{2}{(n-1)(1-\alpha)}}$. Let us see that in the cosmic time $\langle a\rangle(t)$ mimics Eq. (B.9) in the classical limit $t \to \infty$. To do this we remember that we are working in the conformal gauge $N = a^{(n-1)\alpha}$. Therefore,

$$N(t)dt = d\tau, \tag{B.45}$$

where τ is the cosmic time. Hence

$$a^{(n-1)\alpha}dt = d\tau \Rightarrow t^{\frac{2(n-1)\alpha}{(n-1)(1-\alpha)}}dt = d\tau \Rightarrow t^{\frac{1+\alpha}{1-\alpha}} \propto \tau \Rightarrow t \propto \tau^{\frac{1-\alpha}{1+\alpha}} \tag{B.46}$$

and

$$\langle a\rangle(t) \to t^{\frac{2}{(n-1)(1-\alpha)}} \propto \tau^{\frac{2}{(n-1)(1+\alpha)}}. \tag{B.47}$$

We see that in the classical limit, where quantum effects become negligible, the quantum scale factor reproduces the classical one as expected.

We can also calculate the scale factor through the de Broglie-Bohm interpretation of quantum mechanics. In order to use the de Broglie-Bohm interpretation, we need to write the wave function of the universe in its polar form

$$\Psi = \Theta e^{iS}. \tag{B.48}$$

In our case

$$\begin{aligned} \Theta &= g(t) a \exp\left[-\frac{4\gamma}{(1-\alpha)^2}\left(\frac{n-2}{n-1}\right)\frac{a^{(n-1)(1-\alpha)}}{\gamma^2+t^2}\right], \\ S &= \frac{4t}{(1-\alpha)^2}\left(\frac{n-2}{n-1}\right)\frac{a^{(n-1)(1-\alpha)}}{\gamma^2+t^2} + f(t), \end{aligned} \tag{B.49}$$

where $g(t)$ and $f(t)$ are complicate functions that will not be necessary in what follows. Now we can calculate the Bohmian trajectories for the scale factor $a(t)$ by the use of equation [21, 22]

$$p_a = \frac{\partial S}{\partial a}. \tag{B.50}$$

Therefore,

$$\begin{aligned} 2(n-1)(n-2)\dot{a}\frac{a^{n-3}}{a^{(n-1)\alpha}} &= \frac{4t}{(1-\alpha)^2}\left(\frac{n-2}{n-1}\right)(n-1)(1-\alpha)\frac{a^{(n-1)(1-\alpha)-1}}{\gamma^2+t^2} \\ \Rightarrow \dot{a} &= \frac{2t}{(n-1)(1-\alpha)}\frac{a}{\gamma^2+t^2}, \end{aligned} \tag{B.51}$$

and the solution of the above equation is

$$a(t) = a_0(\gamma^2+t^2)^{\frac{1}{(n-1)(1-\alpha)}}, \tag{B.52}$$

where a_0 is a constant of integration. We see that both visions of quantum mechanics give us to the same behavior of the scale factor.

B.4 Examples

In this section we will consider the n-dimensional FLRW universe in the radiation-dominated era. By quantizing the components of the curvature tensor in the tetrad basis via the de Broglie-Bohm interpretation of quantum mechanics, we will show that the quantized Ricci scalar is perfect regular for all times t, illustrating the exclusion of the big-bang singularity with the introduction of quantum cosmology. We will also find the expectation value of the energy density and pressure and study the behavior of $\langle p \rangle / \langle \rho \rangle$. Like we did in a previous paper [23] we will show that in four-dimensions $\langle p \rangle / \langle \rho \rangle \to 1/3$ in the classical limit. Unexpectedly, in higher dimensions this is not the case, i.e., the ratio $\langle p \rangle / \langle \rho \rangle$ does not tend to the classical value, indicating a distinguish behavior for $n = 4$.

Let us first consider the energy momentum tensor of a perfect fluid in n-dimensions

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}. \tag{B.53}$$

In order to represent a radiation fluid we must have $T = T^\mu{}_\mu = 0$. Therefore

$$T = T^\mu{}_\mu = -\rho + (n-1)p = 0. \tag{B.54}$$

Thus, $p = \frac{1}{n-1}\rho$ for radiation.

The non-null components of the Riemman tensor in the natural tetrad basis in n-dimensions are given by

$$R^{\hat{i}}{}_{\hat{0}\hat{i}\hat{0}} = -\frac{\ddot{a}}{a^3} + \frac{\dot{a}^2}{a^4} \qquad (i = 1, \ldots, n-1), \tag{B.55a}$$

$$R^{\hat{i}}{}_{\hat{j}\hat{i}\hat{j}} = \frac{\dot{a}^2}{a^4} \qquad (i, j = 1, \ldots, n-1; i \neq j). \tag{B.55b}$$

The momentum associated with a in n-dimensions in the conformal gauge $N = a$ is given by [see Eq. (B.18a)]

$$p_a = 2(n-1)(n-2)\dot{a}a^{n-4}. \tag{B.56}$$

Therefore the components of the Riemann tensor can be related to a and p_a by

$$R^{\hat{i}}{}_{\hat{0}\hat{i}\hat{0}} = -\frac{\dot{p}_a}{2(n-1)(n-2)a^{n-1}} + (n-3)\frac{p_a^2}{4(n-1)^2(n-2)^2a^{2n-4}} \qquad (i = 1, \ldots, n-1), \tag{B.57a}$$

$$R^{\hat{i}}{}_{\hat{j}\hat{i}\hat{j}} = \frac{p_a^2}{4(n-1)^2(n-2)^2a^{2n-4}} \qquad (i, j = 1, \ldots, n-1; i \neq j). \tag{B.57b}$$

Now we promote the components of the curvature tensor to the condition of quantum operators. Then we can take the local expectation value of each component through the relation [12, 22, 23]

$$\langle R^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} \rangle_L = \mathrm{Re}\left(\frac{\Psi^* \hat{R}^{\hat{a}}{}_{\hat{b}\hat{c}\hat{d}} \Psi}{\Psi^* \Psi} \right) \tag{B.58}$$

Since a and p_a do not commute, we must choose an ordering factor in order to calculate the local expectation values of Eqs. (B.57a) and (B.57b). We choose the Weyl ordering [24] which is a kind of symmetrization procedure that takes all possible orders of the $a's$ and the $p's$ and then divides the result by the number of terms in the final expression. In our case we have p_a^2/a^{2n-4}. For the general case $f(a)p_a^2$, Weyl ordering gives us

$$(f(a)\hat{p}_a^2)_W = \frac{1}{4}\left[f(a)\hat{p}_a^2 + 2\hat{p}_a f(a)\hat{p}_a + \hat{p}_a^2 f(a) \right]. \tag{B.59}$$

With the aid of the following commutators,

$$\begin{aligned} [a^m, \hat{p}_a] &= ima^{m-1} \\ \left[a^m, \hat{p}_a^2\right] &= 2ima^{m-1}\hat{p}_a + m(m-1)a^{m-2}, \end{aligned} \tag{B.60}$$

we can put powers of p_a at the right of powers of a in Eq. (B.59), obtaining

$$\left(\frac{\hat{p}_a^2}{a^{2n-4}} \right)_W = a^{2n-4}\hat{p}_a^2 + i(2n-4)a^{2n-3}\hat{p}_a - \frac{(2n-4)(2n-3)}{4a^{2n-2}}. \tag{B.61}$$

Therefore the local expectation value of the operator $\hat{p}_a^2/a^{2n-4}$ is given by

$$\begin{aligned} \left\langle \frac{p_a^2}{a^{2n-4}} \right\rangle_L =& \mathrm{Re}\left[\frac{\Psi^* \left(\hat{p}_a^2/a^{2n-4}\right)_W \Psi}{\Psi^*\Psi} \right] = -\frac{1}{a^{2n-4}\Theta}\frac{\partial^2\Theta}{\partial a^2} + \frac{1}{a^{2n-4}}\left(\frac{\partial S}{\partial a}\right)^2 \\ &+ \frac{2n-4}{a^{2n-3}\Theta}\frac{\partial\Theta}{\partial a} - \frac{(2n-4)(2n-3)}{4a^{2n-2}}, \end{aligned} \tag{B.62}$$

where $\Psi = \Theta e^{iS}$.

The time rate of the momentum $\dot{p}_a$ can be found through the equation [22]

$$\dot{p}_a = -\frac{\partial Q}{\partial a}, \tag{B.63}$$

where Q is the quantum mechanical potential [21] [see Eq. (B.30)]

$$Q = -\frac{1}{4(n-1)(n-2)a^{n-4}}\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial a^2}. \tag{B.64}$$

Substituting Eq. (B.49) with $\alpha = 1/(n-1)$ into the above equation we have

$$Q = \frac{n-1}{n-2}\frac{\gamma}{\gamma^2+t^2} - \frac{4\gamma^2}{(\gamma^2+t^2)^2}\frac{n-1}{n-2}a^{n-2} \tag{B.65}$$

which implies

$$\dot{p}_a = \frac{4\gamma^2(n-1)}{(\gamma^2+t^2)^2}a^{n-3}. \tag{B.66}$$

Substituting Eq. (B.49) with $\alpha = 1/(n-1)$ into Eq. (B.62) we have

$$\begin{aligned}\left\langle \frac{p_a^2}{a^{2n-4}} \right\rangle_L = &-\frac{1}{a^{2n-3}}\left\{-\frac{4\gamma(n-1)^2}{\gamma^2+t^2}a^{n-3} + \left[\frac{4\gamma(n-1)}{\gamma^2+t^2}\right]^2 a^{2n-5}\right\} \\ &+ \frac{1}{a^{2n-4}}\left[\frac{4t(n-1)}{\gamma^2+t^2}a^{n-3}\right]^2 + \frac{2n-4}{a^{2n-2}}\left[1-\frac{4\gamma(n-1)}{\gamma^2+t^2}a^{n-2}\right] \\ &- \frac{(2n-4)(2n-3)}{4a^{2n-2}}.\end{aligned} \tag{B.67}$$

Therefore

$$\begin{aligned}\left\langle R^{\hat{i}}{}_{\hat{0}\hat{i}\hat{0}} \right\rangle_L &= -\frac{2\gamma^2}{(\gamma^2+t^2)^2}\frac{1}{n-2}\frac{1}{a^2} + \frac{n-3}{4(n-1)^2(n-2)^2}\left\langle \frac{p_a^2}{a^{2n-4}} \right\rangle_L \\ \left\langle R^{\hat{i}}{}_{\hat{j}\hat{i}\hat{j}} \right\rangle_L &= \frac{1}{4(n-1)^2(n-2)^2}\left\langle \frac{p_a^2}{a^{2n-4}} \right\rangle_L.\end{aligned} \tag{B.68}$$

The Ricci scalar is given in terms of the components of the curvature tensor in the tetrad basis by

$$\langle R \rangle_L = -2(n-1)\left\langle R^{\hat{i}}{}_{\hat{0}\hat{i}\hat{0}} \right\rangle_L + (n-1)(n-2)\left\langle R^{\hat{i}}{}_{\hat{j}\hat{i}\hat{j}} \right\rangle_L. \tag{B.69}$$

Let us now present some graphics relating the dimension of the FLRW universe. We start with the graphic of the scale factor in the de Broglie-Bohm interpretation of quantum mechanics, Fig. B.4-2 (remember that the scale factor in the many-worlds and in the de Broglie-Bohm interpretation differs only by an overall constant factor). For $\alpha = \frac{1}{n-1}$ the scale factor behaves as [see Eq. (B.52)]

$$a(t) = a_0(\gamma^2 + t^2)^{\frac{1}{n-2}} \tag{B.70}$$

in the conformal gauge. Note that for $n \to \infty$ the scale factor approaches the constant $a(t) = a_0$ and that the expansion is slower as n grows up.

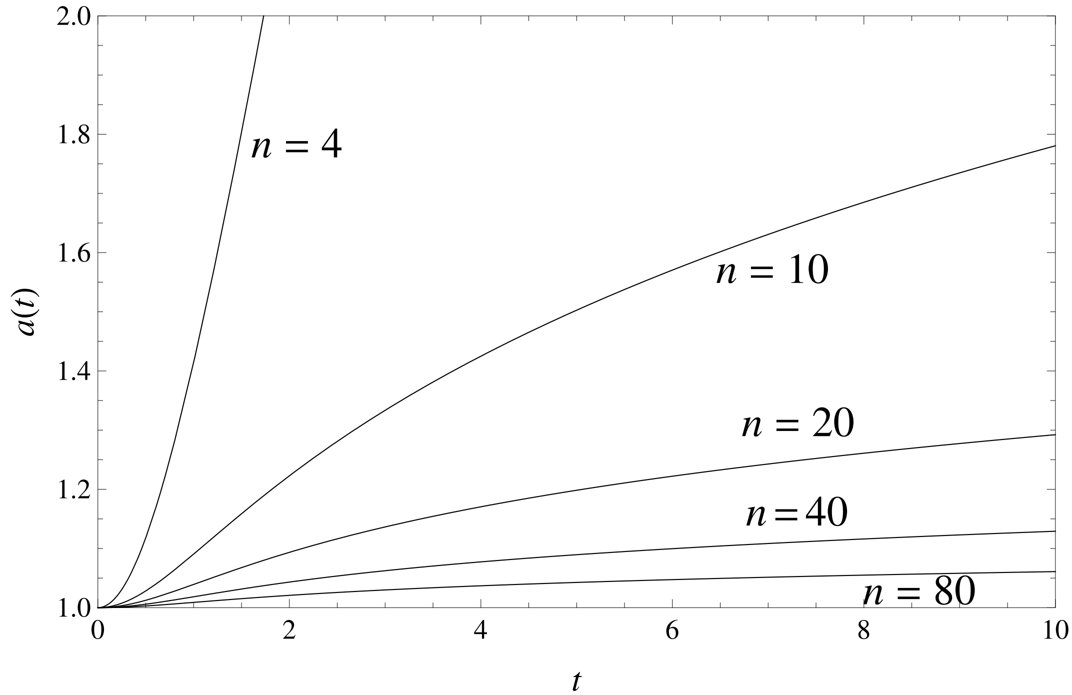

Figura B.4-2. The Bohmian trajectory of the scale factor for $n = 4, 10, 20, 40$ and 80, for $a_0 = \gamma = 1$. The expansion of the universe slower as n grows up and as $n \to \infty$ we approach a static universe.

The graphic for the local expectation value of the Ricci scalar, which is perfectly regular for all n, is shown in Fig. B.4-3.

With the components of the Riemann curvature tensor in the tetrad basis we can find the energy density and pressure of the fluid filling the universe through the Einstein equations

$$T^{\alpha}{}_{\beta} = \frac{1}{k_n}\left(R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R\right) = \mathrm{diag}(-\rho, p, \ldots, p). \tag{B.71}$$

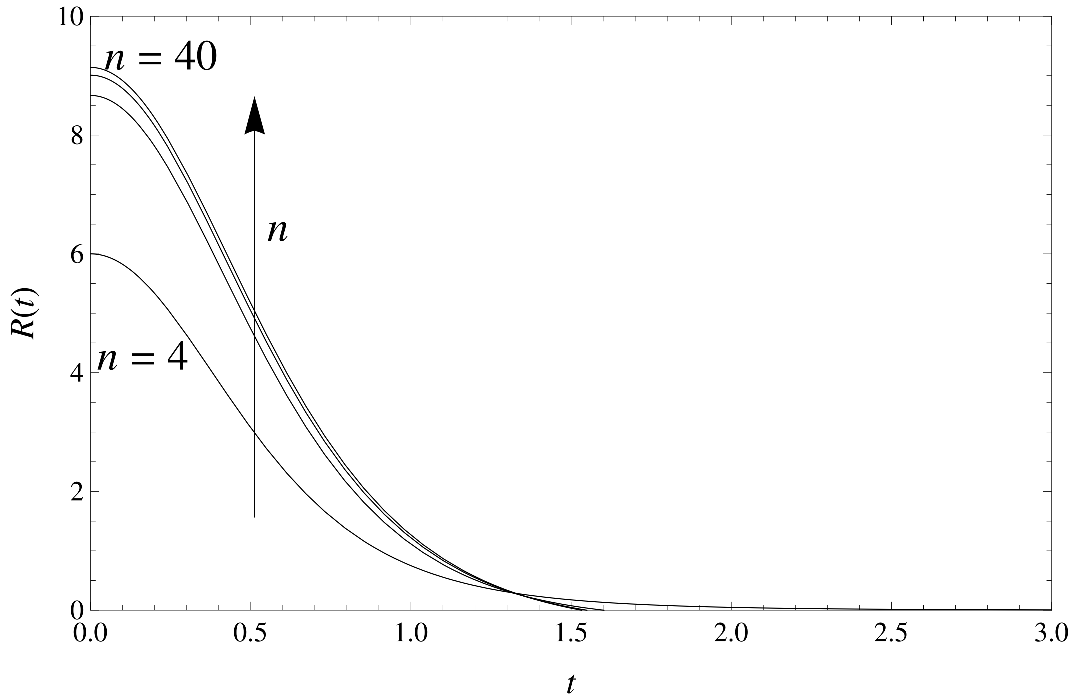

Figura B.4-3. The local expectation value of the Ricci scalar $R(t)$ for $n = 4, 10, 20, 40$ ($a_0 = \gamma = 1$). They are perfectly regular for all values of t indicating that the singularity is excluded by the introduction of quantum cosmology.

Therefore

$$\begin{aligned} \rho &= \frac{1}{k_n}\left[\frac{(n-1)(n-2)}{2}R^{\hat{i}}{}_{\hat{j}\hat{i}\hat{j}}\right] \\ p &= \frac{1}{k_n}\left[(n-2)R^{\hat{i}}{}_{\hat{0}\hat{i}\hat{0}} - \frac{(n-2)(n-3)}{2}R^{\hat{i}}{}_{\hat{j}\hat{i}\hat{j}}\right], \end{aligned} \tag{B.72}$$

and we can plot the ratio $\langle p\rangle / \langle\rho\rangle$ as shown in Fig. B.4-4. Wee see that in the classical limit, where quantum effects become negligible, $\langle p\rangle / \langle\rho\rangle \to 1/3$ in four-dimensions as expected. For other dimensions $\langle p\rangle / \langle\rho\rangle$ does not tend to the classical limit and we can also see that as $n \to \infty$, $\langle p\rangle / \langle\rho\rangle \to 1$. In this sense four-dimensions are privileged among the others. The introduction of quantum cosmology in four-dimensions reproduces the classical behavior of the fluid.

We can also study the behavior of the quantity $\kappa_n[\langle\rho\rangle_L + (n-1)\langle p\rangle_L]$, which comes from the strong energy condition. We see in Fig. B.4-5 that the strong energy condition, which demands that $\rho + (n-1)p \geq 0$, is violated for small t. This is expected since the singularities have been removed. But note that for large t the quantity $\rho + (n-1)p$ is greater than zero, returning to the classical behavior.

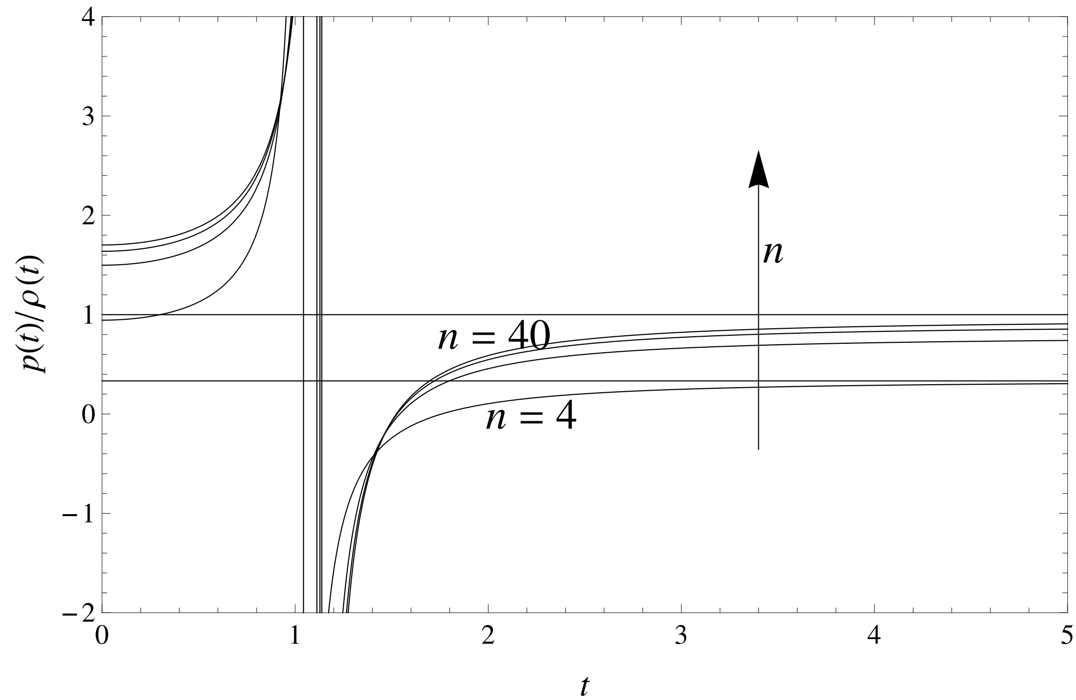

Figura B.4-4. The ratio $\langle p\rangle_L / \langle\rho\rangle_L$ for $n = 4, 10, 20{,}40$ and $a_0 = \gamma = 1$. In four dimensions the classical behavior of the fluid is recovered while $n \to \infty$ $\langle p\rangle_L / \langle\rho\rangle_L \to 1$.

B.5 conclusions

The time dependence of the scale factor in and n-dimensional FLRW universe with a perfect fluid was found by two different ways. By solving Einstein equations and via Hamiltonian formalism, where we extended Schutz formalism to dimensions other than four. By comparing both results we could test the validity of this extrapolation. We then found that the big-bang singularity is still present in the n-dimensional FLRW universe. By solving the Wheeler-deWitt equation for the universe, we obtained by the superposition principle a finite norm wave packet for the universe and we could find the expectation value, in the many-worlds interpretation, and the trajectory in de Broglie-Bohm interpretation, of the scale factor and discovered that the big-bang singularity is removed with the introduction of quantum cosmology.

By quantizing the independent components of the curvature tensor in the tetrad basis, we showed graphically that the local expectation value of the Ricci scalar is well behaved for all time, giving one more indication that the big-bang singularity has been removed. We also studied the quantum behavior of the ratio p/ρ and showed that this ratio tends to

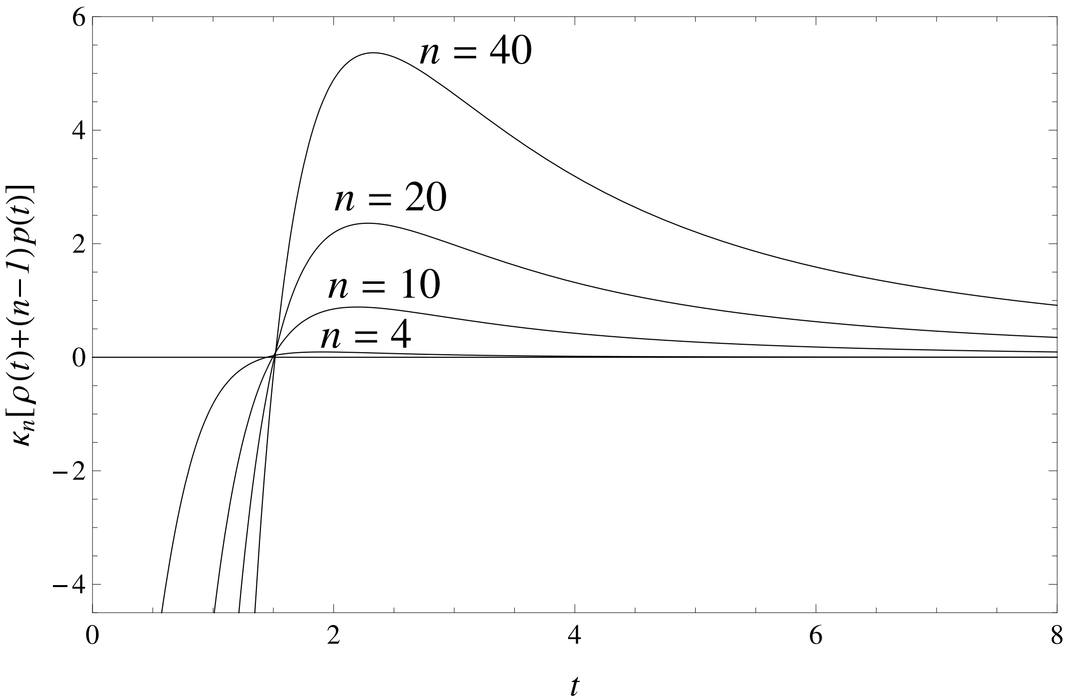

Figura B.4-5. The graphic of $\kappa_n[\langle\rho\rangle_L + (n-1)\langle p\rangle_L]$ for $n = 4, 10, 20$ and 40 . We see that the strong energy condition is violated when t is small and returns to which is classically expected as t gets bigger.

the classical limit when $t \to \infty$ only in four-dimensions. This indicates that, in a certain way, four dimensions are single out from the view point of the quantum cosmology since we recover classical quantities as the quantum behavior becomes negligible. We also showed that for any dimension the strong energy condition is violated as expected since we do not have singularities.

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Apêndice C

Modelando a Evolução Quântica do Universo Através de Matéria Clássica

ref.: arXiv:1107.5802 [gr-qc].

Modeling the quantum evolution of the universe through classical matter

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We show that the quantum behavior of the radiation-dominated flat Friedmann-Robertson-Walker (FRW) universe can be reproduced by classical cosmology given that the universe is filled with an exotic matter. In the case of a fluid, we find an implicit equation of state (EOS) which asymptotically approaches $p = \rho/3$, showing consistency with the classical model. In the case of two non-interacting scalar fields, where one of them is of the phantom type, we find their potential energy. These scalar fields behaves asymptotically as radiation.

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C.1 Introduction

Classical singularities are always present in cosmological models [1]. This is a burden of general relativity and there is a general agreement that a full quantum theory of gravitation will solve this problem, either indicating how to deal with the singularities or excluding them at all. There are several pieces of evidences suggesting that quantum mechanics will be able to exclude the classical singularities on cosmological models. It was shown in [2,3] that the quantum evolution of the FRW universe filled with a perfect fluid with EOS $p = w\rho$ is non-singular and matches the classical evolution when $t \to \infty$. In this context we have shown that the energy conditions for matter are quantum mechanically violated [4], a necessary requirement for the exclusion of singularities.

Here we consider another possibility, a classical exotic matter which, through Einstein equations, reproduces the quantum behavior of the universe. For simplicity, we will consider the quantum flat FRW universe filled with radiation. We then show that in the classical limit, the matter content which fills the universe tends to become radiation with EOS $p = \rho/3$. This indicates that it is possible to get important results from quantum cosmology in the classical scenario and still recover the main features of classical cosmology as time flows. We will consider two kinds of matter. A perfect fluid, for which we will find an implicit EOS and two non-interacting scalar fields, one of them of the phantom type, for which we will be looking for their potential energy.

Non-linear equations of state have already been considered in previous works. For instance, in [5] the general quadratic EOS $p = p_0 + \alpha\rho + \beta\rho^2$ was studied and it was shown that with the right choice of the parameters p_0, α and β this exotic fluid can model dark matter. A more general EOS was studied in [6] and again it was shown that cosmic acceleration can be modeled by exotic fluids. An example of implicit EOS can be found in [7]. By considering an implicit EOS depending also on the Hubble parameter $H = \dot{a}/a$, a universe with one big bang and one big rip singularity was found and it was shown that the phantom divide $w = -1$ is crossed thanks to the effect of the inhomogeneous term H in the EOS.

A universe filled with multiple (canonical and/or phantom) scalar fields was studied in [8]. There, they reconstructed the entire evolution of the universe, including early inflation and late-time acceleration, using these fields.

In this paper we search for an exotic matter in a non-singular universe. Since the energy conditions are violated due to the exclusion of singularities, cosmic acceleration comes as a bonus. However, the expression for the scale factor is algebraic in time, so this cosmic acceleration is not enough to generate inflation.

C.2 Quantum representative of the scale factor

The action for general relativity with a perfect fluid in Schutz formalism [9, 10] is given by (in units where $16\pi G = 1$)

$$\begin{aligned} S_G = &\int_{\mathcal{M}} d^4x\sqrt{-g}R + 2\int_{\partial\mathcal{M}} d^3x\sqrt{h}h_{ab}K^{ab} \\ &+ \int_{\mathcal{M}} d^4x\sqrt{-g}p, \end{aligned} \tag{C.1}$$

where h_{ab} is the induced metric over the boundary $\partial\mathcal{M}$ of the four-dimensional manifold $\mathcal{M}$, K^{ab} is the second fundamental form of the hypersurface $\partial\mathcal{M}$ and p is the fluid pressure, which is linked to the energy density by the EOS $p = w\rho$.

The super-Hamiltonian for this action in a flat FRW universe given by the metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{C.2}$$

and filled with radiation, is given by

$$\mathcal{H} = \frac{p_a^2}{24} - p_T \approx 0, \tag{C.3}$$

where $p_a = -12\frac{da}{dT}a/N$ is the momentum conjugated to the scale factor $a(t)$ and p_T is the momentum conjugated to the dynamical degree of freedom of the fluid [11]. As we can see in Eq. (C.3), this super-Hamiltonian is a constraint. Following Dirac algorithm [12] for quantization of constrained Hamiltonian systems, we make the substitutions

$$p_a \to -i\frac{\partial}{\partial a}; \quad p_T \to -i\frac{\partial}{\partial T} \tag{C.4}$$

and demand that the super-Hamiltonian operator annihilate the wave function of the universe. The result is

$$\frac{\partial^2 \Psi}{\partial a^2} + 24i\frac{\partial \Psi}{\partial t} = 0, \tag{C.5}$$

where $t = -T$ is the time coordinate in the conformal-time gauge $N(t) = a(t)$ [2]. This is the so called Wheeler-DeWitt equation of the universe [13]. Note that in this case we do not have operator ambiguities.

The scale factor is defined on the half-line $(0, \infty)$, therefore with the usual scalar product

$$\langle \Psi | \Phi \rangle = \int_0^\infty \Psi(a,t)^* \Phi(a,t) da, \tag{C.6}$$

a boundary condition in $a = 0$ is necessary in order to ensure self-adjointness of Eq. (C.5). Lemos found in [2] that all possible boundary conditions are given by

$$\Psi'(0,t) = \beta \Psi(0,t), \quad \beta \in \mathbb{R}. \tag{C.7}$$

For the sake of simplicity, let us consider only two possibilities $\beta = 0$ and $\beta = \infty$, corresponding to Neumann and Dirichlet boundary conditions, respectively. Eq. (C.5) can be easily solved by finding its propagator [2] or separating variables [3]. Then the expected value of the scale factor can be readily computed from the wave function $\Psi(a,t)$ through the formula

$$\langle a \rangle (t) = \frac{\langle \Psi | a | \Psi \rangle}{\langle \Psi | \Psi \rangle}. \tag{C.8}$$

The result is [2, 3]

$$a(t) = a_0 \sqrt{\gamma^2 + t^2}, \tag{C.9}$$

where a_0 and γ are positive constants, γ being related to the initial wave packet. Note that when $t \to \infty$ we have that $a(t) \sim t$. In the cosmic gauge, τ is given by $a(t)dt = d\tau$. Therefore $a(\tau) \sim \tau^{1/2}$, as expected from the classical model [14].

C.3 Exotic Equation of state

The four-velocity of a perfect fluid in the comoving coordinates in the conformal gauge $N(t) = a(t)$ is given by

$$U_\mu = (a(t), 0, 0, 0), \tag{C.10}$$

in such a way that the energy-momentum tensor of the fluid

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{C.11}$$

is given by

$$T_{\mu\nu} = a^2(t)\mathrm{diag}(\rho, p, p, p). \tag{C.12}$$

Einstein field equations (in units where $16\pi G = 1$) become

$$\begin{cases} G_{00} & = 3\frac{\dot{a}^2(t)}{a^2(t)} = \frac{1}{2}a^2(t)\rho \\ G_{ii} & = \frac{\dot{a}^2(t)}{a^2(t)} - 2\frac{\ddot{a}(t)}{a(t)} = \frac{1}{2}a^2(t)p, \qquad i = 1, 2, 3. \end{cases} \tag{C.13}$$

Substituting Eq. (C.9) into the above equations leads to

$$\rho = \frac{6t^2}{a_0^2(\gamma^2 + t^2)^3}, \tag{C.14a}$$

$$p = \frac{\rho}{3} - \frac{4\gamma^2}{a_0^2(\gamma^2 + t^2)^3} = \frac{2}{a_0^2}\frac{t^2 - 2\gamma^2}{(\gamma^2 + t^2)^3}. \tag{C.14b}$$

Prior to find the EOS of this fluid, let us study the expressions for ρ and p as functions of time.

In Fig. C.3-1-a) we see that $\rho(t)$ does not start with a singularity. This is expected since the universe with the scale factor (C.9) is non-singular. Then $\rho(t)$ reaches a maximum $\rho_{\max} = 8/(9a_0^2\gamma^4)$ at $t = \gamma/\sqrt{2}$, thereafter it starts to decay to 0 as $t \to \infty$ (as the universe expands). Note that $\rho(t)$ is not a one-to-one function, so if we wish to invert t as a function of ρ, we must do this in two branches (more on this later). We know that the energy conditions, along with some reasonable restrictions on the causal structure of the spacetime, lead to singularities in cosmological models. Therefore, for a non-singular universe, we expect the violation of the energy conditions. This can be verified in Fig.

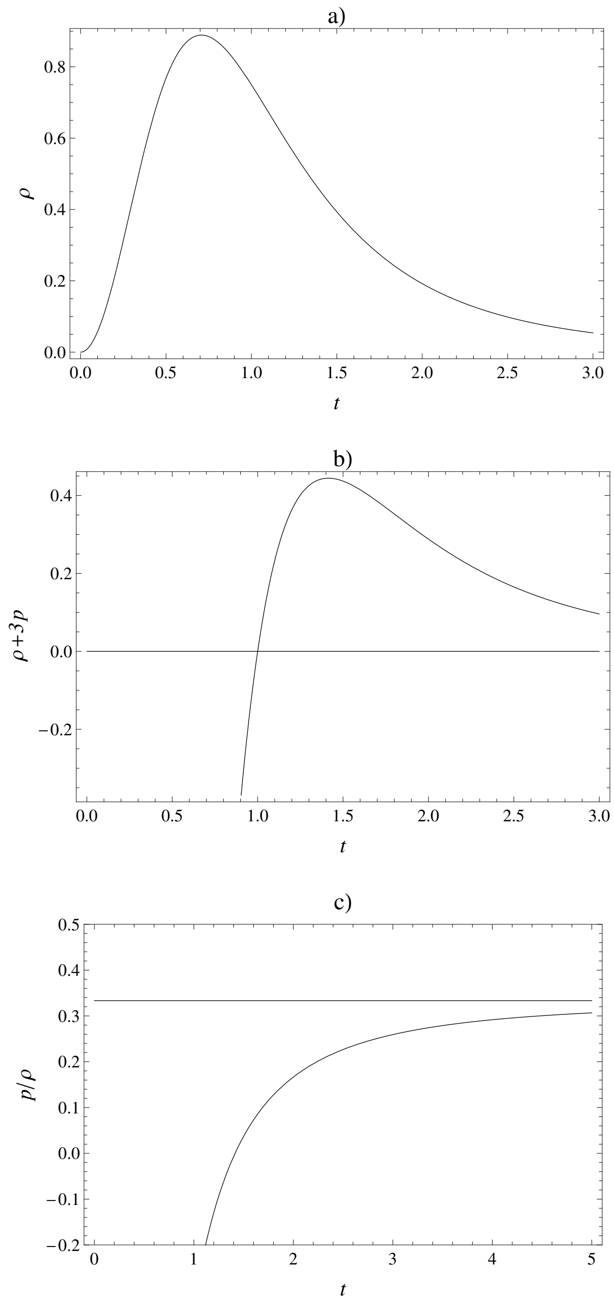

Figura C.3-1. *a*) Letting $a_0 = \gamma = 1$, we see that the curve of $\rho(t)$ starts at 0, it has a maximum $\rho_{\max} = 9/8$ at $t = 1/\sqrt{2}$ and it decays to 0 as $t \to \infty$. *b*) Also, looking to the graphic of $\rho(t) + 3p(t)$ for the same values of a_0 and γ, we clearly see that the strong energy condition is violated for small values of t. *c*). Finally, we consider the graphic of $p(t)/\rho(t)$ for $a_0 = \gamma = 1$. In the classical limit $(t \to \infty)$, $p/\rho \to 1/3$.

C.3-1-b), where we analyzed the strong energy condition, which states that $\rho + 3p \geq 0$. For small t, we have $\rho + 3p < 0$, but we recover the classical inequality as time flows. Finally, we see in Fig. C.3-1-c) that the classical expression $p = \rho/3$ is recovered as $t \to \infty$. This is not a surprise since $a(t) \sim t$ in this limit.

We can now find the exotic EOS of the fluid which will reproduce the quantum behavior of the universe. By Eqs. (C.14a) and (C.14b) we have

$$\begin{cases} \frac{a_0^2}{6}\rho = \frac{t^2}{(\gamma^2+t^2)^3} \\ \frac{a_0^2}{4}\left(\frac{\rho}{3} - p\right) = \frac{\gamma^2}{(\gamma^2+t^2)^3}. \end{cases} \tag{C.15}$$

Adding both equations leads to

$$\frac{a_0^2}{4}(\rho - p) = \frac{1}{(\gamma^2 + t^2)^2}. \tag{C.16}$$

Eq. (C.14b) asserts that

$$\frac{1}{(\gamma^2 + t^2)^2} = \frac{a_0^{4/3}}{4^{2/3}\gamma^{4/3}}\left(\frac{\rho}{3} - p\right)^{2/3}, \tag{C.17}$$

so that

$$\frac{a_0^2}{4}(\rho - p) = \frac{a_0^{4/3}}{4^{2/3}\gamma^{4/3}}\left(\frac{\rho}{3} - p\right)^{2/3}. \tag{C.18}$$

Therefore, we have the implicit EOS $f(\rho, p) = 0$, where

$$f(\rho, p) = \left(\frac{\rho}{3} - p\right)^2 - \frac{a_0^2\gamma^4}{4}(\rho - p)^3. \tag{C.19}$$

Eq. (C.19) can be used to express p as a function of ρ. Doing this inversion we come out with three branches, as can be seen in Fig. C.3-2. Two of these branches (denoted by dashed lines) are smoothly connected and they represent the physical "trajectory" in the diagram ρ-p. The third branch (the dotted line) is spurious. In fact, in Eq. (C.14a) we can invert t^2 as a function of ρ for $0 < \rho < \rho_{\max}$. We find three different expressions for t^2 (since we are working with a third order polynomial in t^2), two positives and one negative. The negative one is non-physical since it corresponds to an imaginary time. Because $\rho(t)$ is not a one-to-one function, it must be inverted in two steps. Therefore, one of the positive expressions for t^2 corresponds to the phase where ρ grows up to $\rho_{\max}$

while the other one corresponds to the phase where ρ decays from ρ_{max} to 0. Finally, replacing the expressions of t^2 in Eq. (C.14b) we find the three branches of $p(\rho)$, the one corresponding to the imaginary time being spurious.

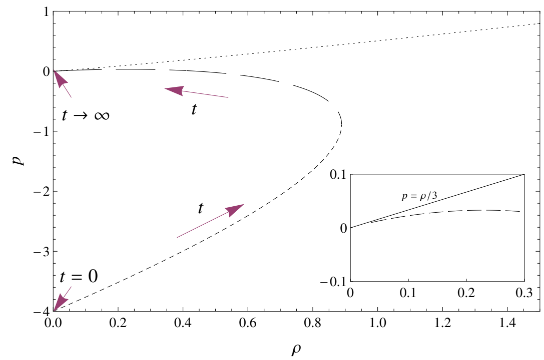

Figura C.3-2. The three branches of $p = p(\rho)$ obtained from Eq. (C.19). Two of them (denoted by dashed lines) are connected smoothly and represent the physical trajectory of p as a function of ρ as time flows (see the arrows representing the direction of time). The other one is spurious. Inset we see that as $t \to \infty$, we recover the classical EOS $p = \rho/3$.

Now we will show that the classical EOS $p = \rho/3$ is recovered when $t \to \infty$, or similarly, $\rho \to 0$ and $p \to 0$ (see Fig. C.3-2). To do this, we expand $f(\rho, p)$ in Taylor series up to second order. The result is

$$f(\rho, p) = 2\left(\frac{\rho}{3} - p\right)^2 + O(3). \tag{C.20}$$

When $\rho \to 0$ and $p \to 0$ we have $f(\rho, p) = 0 \Rightarrow p = \rho/3$. This shows that when $t \to \infty$ the classical limit is recovered.

C.4 Scalar fields

We can try to reproduce the quantum behavior of the universe by adding a scalar field instead of a perfect fluid. We choose to add a phantom and a standard scalar field. As we will see in a moment, with these two non-interacting fields we can fit the initial cosmic acceleration and still recover the classical behavior of the universe as $t \to \infty$.

The Lagrangian for these two fields is given by

$$\begin{aligned} L = & -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{\rm st}\partial_\nu\phi_{\rm st} - V(\phi_{\rm st}) \\ & + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi_{\rm ph}\partial_\nu\phi_{\rm ph} - U(\phi_{\rm ph}), \end{aligned} \tag{C.21}$$

where $\phi_{\rm st}$ corresponds to the standard scalar field while $\phi_{\rm ph}$ corresponds to the phantom one.

The energy density and the pressure for these fields can be expressed as

$$\begin{aligned} \rho &= \frac{\dot{\phi}_{\rm st}^2}{2a^2} - \frac{\dot{\phi}_{\rm ph}^2}{2a^2} + V(\phi_{\rm st}) + U(\phi_{\rm ph}) \\ p &= \frac{\dot{\phi}_{\rm st}^2}{2a^2} - \frac{\dot{\phi}_{\rm ph}^2}{2a^2} - V(\phi_{\rm st}) - U(\phi_{\rm ph}). \end{aligned} \tag{C.22}$$

By adding and subtracting the above equations we have

$$\begin{aligned} \rho + p &= \frac{\dot{\phi}_{\rm st}^2}{a^2} - \frac{\dot{\phi}_{\rm ph}^2}{a^2}, \\ \rho - p &= 2V(\phi_{\rm st}) + 2U(\phi_{\rm ph}). \end{aligned} \tag{C.23}$$

Using Eqs. (C.14a) and (C.14b) leads to

$$\dot{\phi}_{\rm st}^2 - \dot{\phi}_{\rm ph}^2 = 4\left[\frac{2t^2-\gamma^2}{(\gamma^2+t^2)^2}\right], \tag{C.24a}$$

$$V(\phi_{\rm st}) + U(\phi_{\rm ph}) = \frac{2}{a_0^2(\gamma^2+t^2)^2}. \tag{C.24b}$$

We demand the standard scalar field and the phantom one to take care of the positive and negative parts of the right side of Eq. (C.24), respectively. In this way

$$\begin{aligned} \dot{\phi}_{\rm st}^2 &= \frac{8t^2}{(\gamma^2+t^2)^2} \Rightarrow \phi_{\rm st}(t) = \phi_{0\rm st} + \sqrt{2}\ln\left(1 + t^2/\gamma^2\right) \\ \dot{\phi}_{\rm ph}^2 &= \frac{4\gamma^2}{(\gamma^2+t^2)^2} \Rightarrow \phi_{\rm ph}(t) = \phi_{0\rm ph} + 2\arctan\left(t/\gamma\right). \end{aligned} \tag{C.25}$$

Since

$$\gamma^2 + t^2 = \gamma^2 \exp\left(\frac{\phi_{\rm st} - \phi_{0\rm st}}{\sqrt{2}}\right), \tag{C.26}$$

it is easy to show that [by using Eq. (C.24b)]

$$V(\phi_{\rm st}) + U(\phi_{\rm ph}) = \frac{2}{a_0^2\gamma^4} \exp\left[-\sqrt{2}(\phi_{\rm st} - \phi_{0\rm st})\right]. \tag{C.27}$$

Equivalently, we have

$$t = \gamma \tan\left(\frac{\phi_{\rm ph} - \phi_{0\rm ph}}{2}\right), \tag{C.28}$$

so that

$$V(\phi_{\rm st}) + U(\phi_{\rm ph}) = \frac{2}{a_0^2\gamma^4\left[1 + \tan^2\left(\frac{\phi_{\rm ph} - \phi_{0\rm ph}}{2}\right)\right]^2}. \tag{C.29}$$

Obviously the right hand sides of Eqs. (C.27) and (C.29) are equal. Therefore, both equations are satisfied if

$$\begin{aligned} V(\phi_{\rm st}) &= \alpha \frac{2}{a_0^2\gamma^4} \exp\left[-\sqrt{2}(\phi_{\rm st} - \phi_{0\rm st})\right], \\ U(\phi_{\rm ph}) &= (1-\alpha) \frac{2}{a_0^2\gamma^4\left[1 + \tan^2\left(\frac{\phi_{\rm ph} - \phi_{0\rm ph}}{2}\right)\right]^2}, \end{aligned} \tag{C.30}$$

where $0 \le \alpha \le 1$.

It is also simple to check that in this case $p = \rho/3$ as $t \to \infty$.

C.5 Final Remarks

We have found an exotic classical fluid which reproduces the quantum behavior of the universe. This fluid obeys an implicit equation of state. When we use this equation to find p as a function of ρ we find three different branches. Two of them correspond to the physical "trajectory" in the ρ-p diagram. The third one is spurious. Moreover, this equation of state approaches $p = \rho/3$ as $t \to \infty$ (or ρ and p goes to zero as the universe expands). We can also reproduce the quantum behavior of the universe with the insertion of two non-interacting scalar fields, one of them being usual and the other of the phantom type. We are free to choose the fraction of potential energy for each field. In particular, we can set one of them to be free. In either case, we showed that it is possible to get important results from quantum cosmology and still recover the classical limit with the use of classical matter.

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Apêndice D

Singularidades Quânticas ao Redor de um Monopólo Global

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Quantum Singularities Around a Global Monopole

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The behavior of a massive scalar particle on the spacetime surrounding a monopole is studied from a quantum mechanical point of view. All the boundary conditions necessary to turn the spatial portion of the wave operator self-adjoint are found and their importance to the quantum interpretation of singularities is emphasized.

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D.1 Introduction

Classical singularities in general relativity are indicated by incomplete geodesics or incomplete paths of bounded acceleration [1]. They are classified into three basic types : A singular point p in the spacetime is a quasiregular singularity if no observer sees any physical quantities diverging even if its world line reaches the singularity. A singular point p is called a scalar curvature singularity if every observer that approaches the singularity sees physical quantities such as tidal forces and energy density diverging. Finally, in a nonscalar curvature singularity, there are some curves in which the observers experience unbounded tidal forces [2, 3]. Because the spacetime is by definition differentiable, points representing singularities must be excluded from our manifold. The geodesic incompleteness leads to the lack of predictability of the future of a classical test particle which reaches the singularity.

It is this lack of predictability that links classical and quantum singularity. Analogous to the classical case, we say that a spacetime is quantum mechanically singular if the evolution of a wave function representing a one particle state is not uniquely determined by the initial state. That is to say that we need a boundary condition near the singularity in order to obtain the time evolution of the wave packet [4]. An example of a classical singularity which becomes nonsingular with the introduction of quantum mechanics is the hydrogen atom. Solving the Schrödinger equation for the Coulomb potential equation and imposing square-integrability of the solutions is enough to obtain a complete set of solutions which span $L^2(\mathbb{R}^3)$. Then the evolution of the initial wave packet is uniquely determined. There are others examples of classically singular spacetimes that become nonsingular in view of quantum mechanics [4, 5]. But, unfortunately, there are much more examples of spacetimes which remain singular [2, 6–8].

In this paper we will study the spacetime of a global monopole from a quantum mechanical point of view. We believe that this will be the first time that the ideas of Horowitz and Marolf [4] are applied to a non vacuum solution of Einstein equations, since the metric of a global monopole due to Barriola and Vilenkin [9] (deficit of a solid angle) represents a solution of the Einstein field equations with spherical symmetry with matter that extends to infinity (cloud of cosmic strings with spherical symmetry [10]). This point will be discussed in the next section.

This paper is organized as follows: In Sec. D.2 we briefly review the spacetime of a global monopole. In Sec. D.3 we explore the definition of quantum singularities and present the criterion that will decide whether the spacetime is quantum mechanically singular or not. In Sec. D.4 we use the methods described in Sec. D.3 to the case of the global monopole. Finally in Sec. D.5 we discuss the implications of the results of Sec. D.4.

D.2 The Metric of a Global Monopole

One of the predictions of the grand unification theories is the arising of topological defects. They are produced during the phase transitions in the early universe and their existence is a very attractive scenario for large scale structure formation; see, for instance, [11, 12]. The most simple example of a topological defect is the cosmic string; see, for instance, [13]. This defect appears in the breaking of a $U(1)$ symmetry group and has the metric,

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\varphi^2 + dz^2. \tag{D.1}$$

The factor α^2 in the metric (D.1) introduces a deficit angle $\Delta = 2\pi(1-\alpha)$ on a spatial section $z =$ const. This metric is characterized by a null Riemann-Christoffel curvature tensor everywhere except on a line (z axis), where it is proportional to a Dirac delta function; see [14]. The energy-momentum tensor associated with (D.1) is

$$T_t^t = T_z^z = \frac{(1-\alpha)}{4\alpha\rho}\delta(\rho), \tag{D.2}$$

i. e., for strings we have the equation of state: energy density equal to tension.

Barriola and Vilenkin [9] considered a monopole as associated with a triplet of scalar fields ϕ^a $(a = 1, 2, 3)$ given by

$$\phi^a = \eta f(r) x^a / r, \tag{D.3}$$

with $x^a x^a = r^2$ and η is the spontaneous symmetry breaking scale.

Considering the most general static metric with spherical symmetry,

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{D.4}$$

and that outside the monopole the function f takes the value 1, they obtain the solution of the Einstein equations,

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{D.5}$$

where $\alpha^2 = (1 - 8\pi G\eta^2)$.

Note that on the hypersurface given by $\theta = \frac{\pi}{2}$, the spacetime is conical [13] and geodesics on this surface behaves as geodesics on a cone with angular deficit $\Delta = 2\pi(1-\alpha)$. This fact can mislead us to think that this is an empty flat spacetime with a topological defect. But this is not the case, as we will see in the following paragraphs.

In a previous work [10], one of the authors found the metric (D.5) following a completely different approach, the search of the metric associated with a cloud of cosmic strings with spherical symmetry. This spacetime is not isometric to Minkowski spacetime (as is the spacetime surrounding a cosmic string [13]) since we have a non zero tetradical component of the curvature tensor,

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1-\alpha^2}{\alpha^2 r^2}, \tag{D.6}$$

nonvanishing everywhere, opposed to the curvature in the spacetime of a cosmic string, which is proportional to a Dirac delta function with support on the string [14]. The spacetime around a monopole has a scalar curvature singularity, since all observers who approach the singularity will see physical quantities, such as tidal forces, diverging [2]. Note that the Ricci scalar is twice the value of $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$.

Despite the fact that the curvature tends to zero when $r \to \infty$, the spacetime of a global monopole is not asymptotically flat. The energy-momentum tensor $T_{\mu\nu}$ has only the components $T_t^t = T_r^r = \eta^2/r^2$ that are nonvanishing everywhere. A similar situation due to the existence of closed timelike curves was discussed in [15]. In the present case it arises due to the slow falling off of T_t^t that has as a consequence the divergence of $M(r) = \int_0^r T_t^t(r) r^2 \mathrm{d}r$ as $r \to \infty$.

The Newtonian potential $\Phi = GM(r)/r$ defined in Ref. [9] is constant and since the T_t^t component of the energy-momentum tensor is given by $T_t^t = \eta^2/r^2$, we have $M(r)$ proportional to the distance r. But, in view of the Poisson equation, such a constant potential is inconsistent with a nonvanishing density, as noted by Raychoudhuri [16].

Since $T_t^t \propto 1/r^2$ we have that the correct expression for the potential is $\Phi \sim \ln r$ with gravitational intensity $\propto 1/r$. So there is in fact a Newtonian gravitational force on the matter around the monopole [16].

It is clear that we can not interpret the metric (D.5) as representing an isolated massive object introduced into a previously flat universe. So it must be regarded as a symmetric cloud of cosmic string, with all the string forming the cloud intersect in a single point $r = 0$.

D.3 Quantum Singularities

Classical singularities can be interpreted via quantum mechanics by using the definition of Horowitz and Marolf, who considered a classically singular spacetime as quantum mechanically nonsingular when the evolution of a general state is uniquely determined for all time [4]; in other words, that the spatial portion of the wave operator is self-adjoint.

The wave operator for a massive scalar field is given by

$$\frac{\partial^2 \Psi}{\partial t^2} = -A\Psi, \tag{D.7}$$

where $A = -VD^i(VD_i) + V^2M^2$ and $V = -\xi_\mu \xi^\mu$, with ξ^μ being a timelike Killing vector field and D_i the spatial covariant derivative on a static slice Σ. Let us choose the domain of A to be $\mathcal{D}(A) = C_0^\infty(\Sigma)$ in order to avoid the singular points. In this way A is a symmetric positive definite operator, but this domain is so small, so restrictive that the adjoint of A, i.e., A^* is defined on a much larger domain $\mathcal{D}(A^*) = \{\psi \in L^2 : A\psi \in L^2\}$ and A is not self-adjoint. In order to transform the operator A into self-adjoint one we must extend its domain until the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ are equal. If the extended operator $(\overline{A}, \mathcal{D}(\overline{A}))$ is unique [3] then A is said to be essentially self-adjoint and the evolution of a quantum test particle obeying (D.7) is given by

$$\psi(t) = exp(-it\overline{A}^{1/2})\psi(0). \tag{D.8}$$

The spacetime is said to be quantum mechanically nonsingular. Otherwise there is one

[3] In this case the closure of A. See Refs. [17] and [18].

specific evolution for each self-adjoint extension A_E

$$\psi_E(t) = exp(-itA_E^{1/2})\psi(0) \tag{D.9}$$

and the spacetime is quantum mechanically singular. The criterion used to determine if an operator is essentially self-adjoint comes from a theorem by von Neumann [17], which says that the self-adjoint extensions of an operator A are in one-to-one correspondence with the isometries from $Ker(A^* - i)$ to $Ker(A^* + i)$. We solve the equations

$$A^*\psi \mp i\psi = 0 \tag{D.10}$$

and count the number of linear independent solutions in L^2. If there is no solution for the above equations, then $\dim(Ker(A^*) \mp i) = 0$, and if there are no isometries from $Ker(A^* - i)$ to $Ker(A^* + i)$, the operator is essentially self-adjoint. Otherwise, if there is one solution for each equation (D.10), there is a one-parameter family of isometries and therefore a one-parameter family of self-adjoint extensions and so on.

D.4 Quantum singularities on the global monopole background

From the metric (D.5) and the identity

$$\Box\Psi = g^{-1/2}\partial_\mu \left[g^{1/2}g^{\mu\nu}\partial_\nu\right]\Psi \tag{D.11}$$

we have that the Klein-Gordon equation reads

$$\begin{aligned}\frac{\partial^2\Psi}{\partial t^2} =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{\alpha^2 r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right)\\ &+ \frac{1}{\alpha^2 r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} - M^2\Psi.\end{aligned} \tag{D.12}$$

From (D.7) we find

$$\begin{aligned} -A =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\alpha^2 r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \\ &+ \frac{1}{\alpha^2 r^2 \sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - M^2\Psi \end{aligned} \tag{D.13}$$

and the equation to be solved is

$$\begin{aligned} (A^* \mp i)\psi =& \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{\alpha^2 r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) \\ &+ \frac{1}{\alpha^2 r^2 \sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + (\pm i - M^2)\psi. \end{aligned} \tag{D.14}$$

We shall closely follow Sec. III of Ref. [6], where an illustrative example of a flat spacetime with a point removed (texture) is studied.

By separating variables, $\psi = R(r)Y_l^m(\theta,\varphi)$, we get the radial equation

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[(\pm i - M^2) - \frac{l(l+1)}{\alpha^2 r^2}\right]R(r). \tag{D.15}$$

Let us first consider the case $r = \infty$. The above equation takes the form

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + (\pm i - M^2)R(r) = 0, \tag{D.16}$$

whose solution is

$$R(r) = \frac{1}{r}[C_1 e^{\beta r} + C_2 e^{-\beta r}], \tag{D.17}$$

where

$$\beta = \frac{1}{\sqrt{2}}\left[(\sqrt{1+M^4} + M^2)^{1/2} \mp i(\sqrt{1+M^4} - M^2)^{1/2}\right]. \tag{D.18}$$

Obviously, solution (D.17) is square-integrable near infinity if and only if $C_1 = 0$. Then the asymptotic behavior of $R(r)$ is given by $R(r) \sim \frac{1}{r}e^{-\beta r}$.

Near $r = 0$, Eq. (D.15) reduces to

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} - \frac{l(l+1)}{\alpha^2 r^2}R(r) = 0, \tag{D.19}$$

whose solution is $R(r) \sim r^{\gamma}$, where

$$\gamma = \frac{-1 \pm \sqrt{1 + 4\frac{l(l+1)}{\alpha^2}}}{2}. \tag{D.20}$$

For $\gamma = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\frac{l(l+1)}{\alpha^2}}$ the solution $R(r) \sim r^{\gamma}$ is square-integrable near $r = 0$, that is,

$$\int_0^{\text{constant}} |r^{\gamma}|^2 r^2 dr < \infty. \tag{D.21}$$

And for $\gamma = -\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{l(l+1)}{\alpha^2}}$ we have

$$\int_0^{\text{contant}} |r^{\gamma}|^2 r^2 dr = \int_0^{\text{contant}} r^{1-\sqrt{1+4\frac{l(l+1)}{\alpha^2}}} dr. \tag{D.22}$$

Therefore in order for r^{γ} be square integrable we have that

$$1 - \sqrt{1 + 4\frac{l(l+1)}{\alpha^2}} > -1 \tag{D.23}$$

and

$$l(l+1) < \frac{3}{4}\alpha^2. \tag{D.24}$$

This condition is satisfied only if $l = 0$. In fact, the mode $R_0(r)$ does not belong to $L^2(\mathbb{R}^3)$ because $\nabla^2(1/r) = 4\pi\delta^3(r)$. But, we have a physical singularity at $r = 0$ so $r = 0 \notin \Sigma$ and $R_0(r)$ is an allowed mode. Then near origin we have

$$R_0(r) = \tilde{C}_1 + \tilde{C}_2 r^{-1} \tag{D.25}$$

and we can adjust the constants in Eq. (D.25) to meet the asymptotical behavior $R_0(r) \sim e^{-\beta r}$ [4]. There is one solution for each sign in Eq. (D.10), so there is a one-parameter family of self-adjoint extensions of A.

In order to better understand this result, let us solve exactly equation (D.12) using a separation of variables of the form

$$\Psi(t, r, \theta, \varphi) = e^{-i\omega t} R(r) Y_l^m(\theta, \varphi). \tag{D.26}$$

[4]We say that $r = \infty$ is in the point-limit case and $r = 0$ is in the circle-limit case. When this happens, we can assure that there is one solution of Eq. (D.10). See Refs. [17] and [18].

For the radial equation we have

$$r^2R''(r) + 2rR'(r) + [(k^2r^2 - l(l+1)/\alpha^2]R(r) = 0, \tag{D.27}$$

where $k^2 = \omega^2 - M^2$. By doing $u \equiv kr$ we find

$$u^2R''(u) + 2uR'(u) + [u^2 - l(l+1)/\alpha^2]R(u) = 0. \tag{D.28}$$

And defining $Z(u)$ by

$$R(u) = \frac{Z(u)}{u^{1/2}}, \tag{D.29}$$

we get

$$Z''(u) + \frac{1}{u}Z'(u) + \left\{1 - \frac{1}{4u^2}\left[1 + 4\frac{l(l+1)}{\alpha^2}\right]\right\}Z(u) = 0 \tag{D.30}$$

which is the Bessel equation of order δ_l,

$$\delta_l = \frac{1}{2}\sqrt{1 + 4\frac{l(l+1)}{\alpha^2}}. \tag{D.31}$$

Therefore the general solution of Eq. (D.27) is

$$R_{\omega,l,m}(r) = A\frac{J_{\delta l}(kr)}{\sqrt{kr}} + B\frac{N_{\delta l}(kr)}{\sqrt{kr}}. \tag{D.32}$$

Let us analyze the square-integrability near $r = 0$ of each one of the functions appearing in the above equation.

The Bessel functions are always square-integrable near the origin, i.e.,

$$\int_0^{\text{constant}} \left|\frac{J_{\delta_l}(kr)}{\sqrt{kr}}\right|^2 r^2 dr < \infty. \tag{D.33}$$

The behavior of the Newmann functions near the origin is given by $N_\nu(x) \propto \left(\frac{2}{x}\right)^\nu$. Therefore,

$$\int_0^{\text{constant}} \left|\frac{N_{\delta_l}(kr)}{\sqrt{kr}}\right|^2 r^2 dr \sim \int_0^{\text{constant}} r^{-2\delta_l+1} \tag{D.34}$$

so that

$$-2\delta_l + 1 > -1 \Rightarrow \delta_l < 1 \Rightarrow \sqrt{1 + 4\frac{l(l+1)}{\alpha^2}} < 2 \Rightarrow l = 0 \tag{D.35}$$

in order to be square integrable.

Therefore, for $l \neq 0$ square integrability suffices to determine uniquely solution (D.32), while for $l = 0$ we need an extra boundary condition. To find this extra boundary condition, let us define $G(r) = rR_0(r)$. Then Eq. (D.27) with $l = 0$ becomes

$$\frac{d^2G(r)}{dr^2} + k^2G(r) = 0. \tag{D.36}$$

The boundary condition for the above equation is simple (see Refs. [6] and [18]) and it is given by

$$G(0) - aG'(0) = 0. \tag{D.37}$$

Therefore $G(r)$ we have

$$G(r) \propto \begin{cases} \cos kr + \dfrac{1}{ak}\sin kr & a \neq 0 \\ \cos kr & a = 0 \end{cases} \tag{D.38}$$

The general solution of Eq. (D.12) is

$$\Psi_a = \int \mathrm{d}\omega e^{-i\omega t} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C(\omega, l, m) R_{\omega,l,m}(r) Y_l^m(\theta, \varphi), \tag{D.39}$$

with

$$R_{\omega,l,m} = \frac{J_{\delta_l}(kr)}{\sqrt{kr}} \quad l \neq 0 \tag{D.40}$$

and

$$R_{\omega,0,0}(r) = \begin{cases} \dfrac{\cos kr}{r} + \dfrac{1}{ak}\dfrac{\sin kr}{r} & a \neq 0 \\ \dfrac{\cos kr}{r} & a = 0. \end{cases} \tag{D.41}$$

D.5 Discussion

In the previous section we found [Eq. (D.41)] a one-parameter family of solutions of Eq. (D.12), each one corresponding to a determined value of $a \in \mathbb{R}$. The theory does not tell us how to pick up one determined solution, or even if there exists such a distinguished one. Any solution is as good as the others, so the spacetime of a global monopole (or of a cloud of strings with spherical symmetry to be more precise) remains singular in the view of quantum mechanics. The future of a given initial wave packet obeying the Klein-Gordon equation is uncertain, as well as it is uncertain of the future of a classical particle which reaches the classical singularity in $r = 0$.

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Apêndice E

Singularidades Quânticas em espaços-tempos estáticos

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Quantum singularities in static spacetimes

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We review the mathematical framework necessary to understand the physical content of quantum singularities in static spacetimes. We present many examples of classical singular spacetimes and study their singularities by using wave packets satisfying Klein-Gordon and Dirac equations. We show that in many cases the classical singularities are excluded when tested by quantum particles but unfortunately there are other cases where the singularities remain from the quantum mechanical point of view. When it is possible we also find,

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for spacetimes where quantum mechanics does not exclude the singularities, the boundary conditions necessary to turn the spatial portion of the wave operator into self-adjoint and emphasize their importance to the interpretation of quantum singularities.

keywords: quantum singularities, von Newmann theorem, self-adjointness

E.1 Introduction

Classical singularities in general relativity are indicated by incomplete geodesics or incomplete paths of bounded acceleration [1]. There are three types of singularities [2, 3]: the quasi regular singularity, where no observer sees any physical quantities diverging even if its world line reaches the singularity (for example, the singularity in the spacetime of a cosmic string); the scalar curvature singularity, where every observer near the singularity sees physical quantities diverging (for example, the singularity in the Schwarzchild spacetime or the big bang singularity in FRW cosmology); the non scalar curvature singularity, where there are some curves in which the observers experience unbounded tidal forces (whimper cosmologies are a good example). In general relativity, singularities are not part of the spacetime since the spacetime is differentiable by definition, so the future of a test particle which reaches a singular point at a finite proper time is unpredictable unless some extra information is imposed. Moreover, in the classical singular points the laws of physics are no longer valid since they are formulated having a smooth classical spacetime background.

In order to try to avoid this conclusion it was conjectured the cosmic censorship hypothesis [4], which says basically that nature abhors naked singularities. Therefore, every singularity which appears as a solution of Einstein equation must be hidden by an event horizon. This hypothesis assures us that no observer at infinity or at least careful enough not to fall into a black hole will see any effect caused by the singularity. As Hawking said [5], this hypothesis is selfish because it ignores the question of what happens to an observer who does fall through an event horizon and, despite the fact that it has not been proved, many authors suspect that in a fully classical context this conjecture is correct. But recently, some examples of violation of the cosmic censorship hypothesis have been found, even when the spacetime is filled with reasonable matter, for example when gravity is coupled to a scalar field with potential $V(\phi)$ satisfying the positive energy theorem [6].

Under very reasonable conditions (the energy conditions) which state that gravity is always attractive, singularities are inevitable to general relativity. Since general relativity can not escape this burden, we hope that a complete theory of quantum gravity will overcome this situation, teaching us how to deal with such spacetime near the singularities or excluding the singularities at all. While such a theory does not exist, there are many attempts to incorporate quantum mechanics into general relativity.

In this paper we use quantum field theory in curved spacetimes and, analogous to the classical case, we follow the ideas of Horowitz and Marolf [7] and say that a spacetime is quantum mechanically singular if the evolution of a wave packet representing an one particle state is not uniquely determined by the initial wave packet. In this case, the future of the quantum particle is also unpredictable unless we state some extra information, a boundary condition near the singularity to be more precise. Because of the general belief that the cosmic censorship is valid, we study here only naked singularities, i.e., singularities which are not hidden by an event horizon. A very interesting case where naked singularities appears is due to a topological defect, which are characterized by a null curvature tensor everywhere, except on a submanifold, where it is proportional to a Dirac delta function. This case will also be studied here.

This paper is organized as follows: in Sec. 2 we present a brief summary of the theory of unbounded operators in Hilbert spaces, giving emphasis to a theorem due to von Neumann which will provide a method to decide if the spacetime is quantum mechanically singular or regular. In Sec. 3 we introduce the concept of quantum singularities. In Sec. 4 we show several examples of the quantum behavior of test particles in classical singular spacetimes found in the literature. Finally, in Sec. 5 we discuss the examples presented in the previous section.

E.2 Mathematical Framework

A Hilbert space over the complex field $\mathbb{C}$ is a vector space provided with an inner product

$$\begin{aligned} \langle\ ,\ \rangle : H \times H &\longrightarrow \mathbb{C} \\ (\varphi, \psi) &\longmapsto \langle \varphi, \psi \rangle\,, \end{aligned} \tag{E.1}$$

which is anti linear in the first input and linear in the second one. Moreover, this space is complete in the norm $\|\cdot\|$, defined by the inner product by $\|\psi\| = \langle \psi, \psi \rangle^{1/2}$, complete in the sense that every Cauchy sequence converges.

A linear operator in a Hilbert space H is an operator satisfying the following property:

$$T(\alpha\psi + \varphi) = \alpha T\psi + T\varphi \qquad \forall\, \psi, \varphi \in H;\ \alpha \in \mathbb{C}. \tag{E.2}$$

Nevertheless, a crucial detail, which sometimes quantum mechanical textbooks do not mention, is that an operator is not currently defined if its domain is not specified. Operators with different domains must be considered distinct. The formal definition is: a linear operator $T : D(T) \to H$ is a linear map from a subset $D(T) \subseteq H$, which is called domain of T, into a subset $R(T) \subseteq H$, which is called range of T. The domain and the range of T are vector subspaces of H [8].

An operator $T : D(T) \to H$ is said to be bounded if there exists a number $c > 0$ such that $\forall\, \psi \in D(T)$ we have

$$\|T\psi\| \leq c\, \|\psi\|\,. \tag{E.3}$$

By the representation theorem of Riesz-Fréchet [8, 9] we know that for a bounded operator $T : H \to H$, the Hilbert adjoint operator $T^* : H \to H$ defined by the equality

$$\langle T^*\varphi, \psi \rangle = \langle \varphi, T\psi \rangle \qquad \forall\, \psi, \varphi \in H \tag{E.4}$$

exists and it is bounded.

However, it is a fact that many operators we work in quantum mechanics are unbounded, like the position and the momentum operators. For unbounded operators we do not have such a theorem like the Riesz-Fréchet theorem, but we have a weaker definition of

Hilbert adjoint operator.

Definição 6 (Hilbert adjoint operator)**.** *Let $T : D(T) \to H$ be a linear operator densely defined in a Hilbert space H. The Hilbert adjoint operator is defined as follows: the domain $D(T^*)$ of T^* consists of all $\varphi \in H$ for which there is $\varphi^* \in H$ such that, for all $\psi \in D(T)$,*

$$\langle \varphi^*, \psi \rangle = \langle \varphi, T\psi \rangle \,. \tag{E.5}$$

For each $\varphi \in D(T)$, the action of T^* is given by

$$T^*\varphi = \varphi^*. \tag{E.6}$$

In the theorem above, an operator is said to be densely defined in the Hilbert space H if $D(T)$ is dense in H ($\overline{D(T)} = H$, where $\overline{D(T)}$ denotes the closure of $D(T)$). It is a necessary condition for T^* to be a map, i.e., for T^* associate a unique φ^* to each $\varphi \in D(T^*)$ [8].

An operator $T : D(T) \to H$ is called self adjoint if $T = T^*$. Note that the domains must be the same, i.e., $D(T) = D(T^*)$.

A self-adjoint operator is also symmetric, i.e., it satisfies

$$\langle \varphi, T\psi \rangle = \langle T\varphi, \psi \rangle \tag{E.7}$$

$\forall\, \varphi, \psi \in D(T)$.

The concept of closed linear operators will be important in what follows.

Definição 7 (Closed linear operator)**.** *Let $T : D(T) \to H$ be a linear operator on a Hilbert space H. T is said to be closed if*

$$\begin{cases} \psi_n \to \psi & [\psi_n \in D(T)] \\ T\psi_n \to \varphi & \end{cases} \tag{E.8}$$

implies $\psi \in D(T)$ and $T\psi = \varphi$.

An operator T is **closable** if it has a closed extension. Every closed operator has a smallest closed extension, called its **closure**, which we denote by $\overline{T}$.

Let us now introduce the concept of continuous one-parameter unitary group. This will be the case for the evolution operator in quantum mechanics.

Definição 8. *A one-parameter function operator $U(t)$ is called a continuous one-parameter unitary group if*

a) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)\,\forall\, t, s \in \mathbb{R}$.

b) If $\psi \in H$ and $t \to t_0$ then $U(t)\psi \to U(t_0)\psi$.

There is a theorem due to M. Stone which relates $U(t)$ with the exponential of a self-adjoint operator [10].

Teorema 6 (Stone's theorem)**.** *Let $U(t)$ be a one parameter continuous unitary group in a Hilbert space H. Then exists a self-adjoint operator A in H such that $U(t) = e^{itA}$.*

In non-relativistic quantum mechanics, the equation which governs the physical system is the Schrödinger equation

$$\mathcal{H}\,|\psi\rangle = -i\frac{\partial}{\partial t}\,|\psi\rangle\,, \tag{E.9}$$

where $\mathcal{H}$ is the Hamiltonian operator of the system (suppose $\mathcal{H}$ is time independent).

If the operator $\mathcal{H}$ is self-adjoint, the state of the system at instant t can be found by

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle = e^{-i\mathcal{H}t/\hbar}\,|\psi(0)\rangle \tag{E.10}$$

Physical motivations give rise to an expression to the system Hamiltonian operator. Usually it is an operator with partial derivatives in an appropriate L^2 space. At first, the domain of the operator is not specified and it is simple to find a domain where the Hamiltonian operator is a well-defined symmetric operator (self-adjoint or not). Then, if the closure of $\overline{\mathcal{H}}$ is self-adjoint we can use it. Otherwise, if the closure of $\mathcal{H}$ is not self-adjoint we seek to figure out how many self-adjoint extensions does it has (possibly none).

To answer this question another two definitions will be necessary [10].

Definição 9 (Essentially self-adjoint operators)**.** *A linear symmetric operator T is called essentially self-adjoint if its closure $\overline{T}$ is self-adjoint.*

It can be shown that if T is essentially self-adjoint it has a unique self-adjoint extension.

Definição 10. *Let T be a symmetric operator. Let*

$$\begin{aligned} \mathcal{K}_+ &= Ker(i - T^*) \\ \mathcal{K}_- &= Ker(i + T^*). \end{aligned} \tag{E.11}$$

$\mathcal{K}_+$ and $\mathcal{K}_-$ are called the **deficiency subspaces** *of T. The pair of numbers n_+,n_-, given by $n_+(T) = dim[\mathcal{K}_+]$, $n_-(T) = dim[\mathcal{K}_-]$ are called the* **deficiency indices** *of T.*

The following theorem will give us a criterion to decide if the operator is essentially self-adjoint or not, and a method to find its self-adjoint extensions [11].

Teorema 7 (Criterion for essentially self-adjoint operators)**.** *Let T be a symmetric operator with deficiency indices n_+ and n_-. Then,*

(a) If $n_+ = n_- = 0$, T is essentially self-adjoint.

(b) If $n_+ = n_- \geq 1$, T has an infinite number of self-adjoint extensions parametrized by a $n \times n$ matrix.

(c) If $n_+ \neq n_-$, T does not have self-adjoint extensions.

So, in order to determine if a symmetric operator T is essentially self-adjoint we must solve the pair of equations

$$(T^* \mp i)\psi = 0 \tag{E.12}$$

and count the number of independent solutions in H, i.e., the dimension of Ker$(T^* \mp i)$. Theorem $X.2$ of Ref. [11] states that the self-adjoint extensions of the operator are represented by the one-parameter family of the extended domains of operator T given by

$$D^U = \{\psi = \phi + \phi^+ + U\phi^- : \phi \in D(T)\}, \tag{E.13}$$

where

$$T^*\phi^{\pm} = \pm i\phi^{\pm}, \tag{E.14}$$

$\phi^{\pm} \in H$, and U is an isometry from $\ker(T^* - i)$ to $\ker(T^* + i)$.

As an example of everything we said before let us study the momentum operator given by

Exemplo 2 (Momentum operator)**.** *Let $H = L^2(0,1)$ and T defined by*

$$\begin{aligned} D(T) &= C_0^\infty(0,1) \\ Tf &= -if'. \end{aligned} \tag{E.15}$$

First of all, it is easy to check that T is a symmetric operator since $\forall\, f, g \in D(T)$

$$\langle g, Tf\rangle = \int_0^1 \overline{g}(-if') = -i \underbrace{gf|_0^1}_{=0} + i\int_0^1 \overline{g'}f = \int_0^1 \overline{-ig'}f = \langle Tg, f\rangle\,, \tag{E.16}$$

and, since $C_0^\infty(0,1)$ is dense in $L^2(0,1)$, T is densely defined in $L^2(0,1)$.

In order to find the Hilbert adjoint operator we must look for all pairs $(g, g^*) \in D(T^*)\times L^2(0,1)$ such that $\forall\, f \in D(T)$

$$\langle g, Tf\rangle = \langle g^*, f\rangle\,. \tag{E.17}$$

So

$$\begin{aligned} \langle g^*, f\rangle &= \langle g, Tf\rangle = \langle g, -if'\rangle = -i\,\langle g, f'\rangle = \\ &= -i\int_a^b \overline{g}f'dx = -i\overline{g}f|_0^1 + i\int_a^b \overline{g'}f = \int_a^b \overline{-ig'}fdx = \langle -ig', f\rangle\,, \end{aligned} \tag{E.18}$$

therefore $T^*g = -ig'$. But now, if $g_1 \in L^2(0,1)$ and $g_1^* = -ig_1' \in L^2$, then (remember f has compact support in $(0,1)$)

$$\langle g_1^*, f\rangle = \langle -ig_1', f\rangle = i\,\langle g_1', f\rangle = -i\,\langle g_1, f'\rangle = \langle g_1, -if'\rangle = \langle g_1, Tf\rangle\,. \tag{E.19}$$

Hence no boundary condition on g is necessary. Consequently T^* is given by

$$\begin{aligned} D(T^*) &= \left\{g \in L^2(0,1) : g' \in L^2(0,1)\right\} \\ T^*g &= -ig'. \end{aligned} \tag{E.20}$$

The momentum operator given by Eq.(E.15) is not self-adjoint. This happens because the chosen domain is so small, i.e., the restrictions on functions are so strong that allows the domain of T^* to be extremely large.

Let us use Theorem 7 to find how many self-adjoint extensions does T have.

$$(-ig' - \mp ig) = 0 \Rightarrow' g' = \mp g \Rightarrow g = e^{\mp x}. \tag{E.21}$$

Therefore, there is one solution in $L^2(0,1)$ to each equation in (E.12) and the self-adjoint extensions of T are parametrized by the parameter θ.

Now, the isometries from from $\ker(T^*-i)$ to $\ker(T^*+i)$ are given by $U : e^{-x} \to e^{i\gamma-1}e^x$, $\gamma \in \mathbb{R}$, because

$$||e^{i\gamma-1}e^x|| = \sqrt{\int_0^1 |e^{i\gamma-1}e^x|^2 dx} = \frac{1}{\sqrt{2}e}\sqrt{e^2-1} = ||e^{-x}||. \tag{E.22}$$

The extended domains of T are given by

$$D^\gamma = \left\{ f + e^{-x} + e^{i\gamma-1}e^x : f \in D(T) \right\}. \tag{E.23}$$

Therefore, for a function $\psi \in D^\gamma$ we have

$$\begin{aligned} \psi(0) &= f(0) + 1 + e^{i\gamma-1} = 1 + e^{i\gamma-1} \\ \psi(1) &= f(1) + e^{-1} + e^{i\gamma} = e^{-1} + e^{i\gamma}. \end{aligned} \tag{E.24}$$

Hence

$$\psi(1) = \frac{e^{-1} + e^{i\gamma}}{1 + e^{i\gamma-1}}\psi(0) \tag{E.25}$$

and, since $\left|\frac{e^{-1}+e^{i\gamma}}{1+e^{i\gamma-1}}\right| = 1$, we have

$$\psi(1) = e^{i\theta}\psi(0), \qquad \theta \in \mathbb{R}. \tag{E.26}$$

The self-adjoint extensions of T are given by

$$\begin{aligned} D(T_\theta) &= \left\{ \psi(x) \in L^2(0,1) : \psi' \in L^2(0,1), \psi(1) = e^{i\theta}\psi(0) \right\} \\ T_\theta \psi &= -i\psi'. \end{aligned} \tag{E.27}$$

This boundary condition assures conservation of probability.

In this example we saw that by relaxing the condition on functions belonging to $D(T)$

we could obtain a self-adjoint operator.

E.3 Quantum singularities

In this section we follow the ideas introduced by Wald [12], Horowitz and Marolf [7]. Just like in general relativity, where and extra information must be added at the singular points since we lose the aptness to predict the future of a particle following an incomplete world line, we define a spacetime as quantum mechanically singular if the time evolution of any wave packet is not completely determined by the initial wave data on a Cauchy surface.

Let $(M, g_{\mu\nu})$ be a static spacetime with a timelike Killing vector field ξ^μ and t be the Killing parameter. Klein-Gordon equation on this spacetime

$$\Box\Psi = M^2\Psi \tag{E.28}$$

can be splitted into a temporal and a spatial part,

$$\frac{\partial^2\Psi}{\partial t^2} = -A\Psi = VD^i(VD_i\Psi) + M^2V^2\Psi, \tag{E.29}$$

where $V^2 = \xi^\mu\xi_\mu$ and D_i is the spatial covariant derivative on a static spatial slice Σ of the spacetime (remember that the singular points are not part of the spacetime).

We may view A as an operator on the Hilbert space $\mathcal{H}$ of the square-integrable functions on Σ (following references 7, 12, 13 and 14). We chose this because we will consider an one-particle description of the field, which is mathematically equivalent to solve the first-order pseudo-differential equation

$$i\frac{\partial\Phi}{\partial t} = \sqrt{A}\Psi. \tag{E.30}$$

We then take the physical axiom [15] which says that the Hilbert space of possible quantum states of a single particle, in the theory governed by the relativistic wave equation (E.29), is the space of functions of the form

$$\Psi(t,x) = \int \frac{d\sigma(j)}{\sqrt{2\omega_j}} \tilde{\phi}_+(j)\psi_j(x)e^{-i\omega_j t}, \tag{E.31}$$

where $\tilde{\phi}_+ \in L^2_\sigma$ and $\{\psi_j\}$ is a complete set of eingeinfunctions of the operator A in Eq. (E.29). So, given $\tilde{\phi}_+ \in L^2_\sigma$ we can define

$$\Psi_{NW}(t,x) \equiv \int \tilde{\phi}_+(j)\psi_j(x)e^{-i\omega_j t} d\sigma(j), \tag{E.32}$$

where NW means Newton-Wigner. Note that $\Psi_{NW} \neq \Psi$, but both represent the same state vector and solve Eq. (E.30). We therefore have

$$\|\Phi_{NW}\|^2_\mu = \int |\Psi_{NW}(t,x)|^2 \, d\mu = \left\|\tilde{\phi}_+\right\|^2_\sigma, \tag{E.33}$$

which is independent of time (here $d\mu$ is the proper element on Σ). So it makes sense to intepret $|\Psi_{NW}(t,x)|^2$ as the probability density for observations of x at time t. This justify our choice to take our Hilbert space as being L^2.

But of course the interpretation of $|\Psi_{NW}(t,x)|^2$ as the probability density has some problems as pointed out by Fulling [15]. In particular, a particle localized in a compact set at time t has nonzero probability of being detected at any given point at an instant later. This problem is solved by taking the negative frequence solutions into account. It makes clear that, despite the fact that our choice of the Hilbert space seems appropriate for an one-particle description, it is by no means as unique as in ordinary quantum mechanics. In fact, Ishibashi and Hosoya [16] used the Sobolev space as the natural Hilbert space of the wave functions. The physical idea behind this choice is that they could prepare an initial data only with finite energy. They showed that in some cases, operators which are not essentially self-adjoint in L^2 are essentially self-adjoint if we use the Sobolev space instead.

To find the domain of the operator A, $D(A)$, is a more difficult task and generally no information is provided. So, a minimum domain is taken (a core), where the operator can be defined and which does not enclose the spacetime singularities. An appropriate set is $C_0^\infty(\Sigma)$, the set of the smooth function of compact support on Σ. But the chosen domain is so small, i.e., the restrictions on functions are so strong, that the domain of the Hilbert adjoint operator A^* is extremely large and is composed of all functions ψ in $L^2(\Sigma, V^{-1}d\mu)$ such that $A\psi \in L^2$. Then, A is not self-adjoint. Hence, we are faced with the problem to find self-adjoint extensions of A and to discover if it has only one or many of such extensions. On the Hilbert space $L^2(\Sigma, V^{-1}d\mu)$, where $d\mu$ is the proper element on Σ, it is not difficult to show (integrating by parts) that the operator $(A, C_0^\infty(\Sigma))$ is a positive symmetric operator. Then, self-adjoint extensions always exist [10] (at least the Friedrichs extension).

If A has only one self-adjoint extension (the closure $\overline{A}$ of A), then A is essentially self-adjoint and, since we are worried with a one particle description, not a field theory, the positive frequency solution satisfy

$$i\frac{\partial \Psi}{\partial t} = (\overline{A})^{1/2}\Psi, \tag{E.34}$$

and the evolution of a wave packet is uniquely determined by the initial data

$$\Psi(t, \mathbf{x}) = e^{-it(\overline{A})^{1/2}}\Psi(0, \mathbf{x}). \tag{E.35}$$

We say that the spacetime is quantum mechanically non-singular.

Now, if A has many self-adjoint extensions A_α, where α is a real parameter, we must choose one in order to evolve the wave packet. Any solution of the form

$$\Psi(t, \mathbf{x}) = e^{-it(A_\alpha)^{1/2}}\Psi(0, \mathbf{x}), \tag{E.36}$$

is a good solution and an extra information must be given to tell us which one has to be chosen. The spacetime is said quantum mechanically singular and we can clearly see the resemblance to the classical case.

E.4 Study of wave packets in classically singular spacetimes

The first example of a classical singular theory, which becomes nonsingular in the view of quantum mechanics, is the nonrelativistic hydrogen atom. The imposition of quadratic integrability of the solutions of Schrödinger equation are sufficient to provide a complete set of eingenfunctions. Given an initial wave packet, its time evolution is uniquely determined.

Another example, now of a singularity which remains singular when tested by quantum mechanics, is the nonrelativistic particle trapped in a 1-dimensional box. A boundary condition is necessary on both edges (it is usually taken $\psi(0) = \psi(1) = 0$) in order to evolve uniquely the wave packet.

Let us now study in a complete way some classical spacetimes already found in the literature.

E.4.1 Global Monopole

The metric around a global monopole [17] is given by

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{E.37}$$

where $\alpha^2 = (1 - 8\pi G\eta^2)$ and η is the spontaneous symmetry breaking scale. It represents a symmetric cloud of cosmic strings, with all the strings forming the cloud intersecting at a single point $r = 0$ [18].

Klein-Gordon equation in this spacetime is given by [19]

$$\frac{\partial^2\Psi}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{\alpha^2 r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\alpha^2 r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} - M^2\Psi. \tag{E.38}$$

By separating variables $\psi = R(r)Y_l^m(\theta,\varphi)$, Eq. (E.12) reads

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[(\pm i - M^2) - \frac{l(l+1)}{\alpha^2 r^2}\right]R(r). \tag{E.39}$$

The solution of the above equation near infinity is given by

$$R(r) = \frac{1}{r}[C_1 e^{\beta r} + C_2 e^{-\beta r}], \tag{E.40}$$

where
$$\beta = \frac{1}{\sqrt{2}}\left[(\sqrt{1+M^4}+M^2)^{1/2} \mp i(\sqrt{1+M^4}-M^2)^{1/2}\right]. \tag{E.41}$$

This solution is square integrable only if $C_1 = 0$. So the asymptotic behavior of $R(r)$ is given by $R(r) \sim \frac{1}{r}e^{-\beta r}$.

Near $r = 0$, solution of Eq. (E.39) is given by r^γ, with

$$\gamma = \frac{-1 \pm \sqrt{1+4\frac{l(l+1)}{\alpha^2}}}{2}. \tag{E.42}$$

For $\gamma = -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\frac{l(l+1)}{\alpha^2}}$ the solution $R(r) \sim r^\gamma$ is square-integrable near $r = 0$. For $\gamma = -\frac{1}{2} - \frac{1}{2}\sqrt{1+\frac{l(l+1)}{\alpha^2}}$, r^γ is square integrable only if $l = 0$. Therefore near origin we have
$$R_0(r) = \tilde{C}_1 + \tilde{C}_2 r^{-1} \tag{E.43}$$

and we can adjust the constants in Eq. (E.43) to meet the asymptotic behavior $R_0(r) \sim \frac{1}{r}e^{-\beta r}$ [20]. There is one solution for each sign in Eq. (E.12), so there is a one-parameter family of self-adjoint extensions of A. The spacetime is quantum mechanically singular.

The positive frequency solutions of Eq. (E.38) are given by

$$\Psi(t,r,\theta,\varphi) = e^{-i\omega t}R_{\omega,l,m}Y_l^m(\theta,\varphi), \tag{E.44}$$

where
$$R_{\omega,l,m}(r) = A\frac{J_{\delta l}(kr)}{\sqrt{kr}} + B\frac{N_{\delta l}(kr)}{\sqrt{kr}} \tag{E.45}$$

and
$$\delta_l = \frac{1}{2}\sqrt{1+4\frac{l(l+1)}{\alpha^2}}. \tag{E.46}$$

The functions J_{δ_l} are always square-integrable near the origin and N_{δ_l} is square integrable only if $l = 0$. For this value of l, the boundary condition on the function $R(r)$ is given by [16, 19]
$$R_{\omega,0,0}(r) = \begin{cases} \dfrac{\cos kr}{r} + \dfrac{1}{ak}\dfrac{\sin kr}{r} & a \neq 0 \\ \dfrac{\cos kr}{r} & a = 0. \end{cases} \tag{E.47}$$

so the general solution of Eq. (E.38) is given by

$$\Psi_a = \int \mathrm{d}\omega e^{-i\omega t} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C(\omega, l, m) R_{\omega,l,m}(r) Y_l^m(\theta, \varphi), \tag{E.48}$$

with

$$R_{\omega,l,m} = \frac{J_{\delta_l}(kr)}{\sqrt{kr}} \quad l \neq 0. \tag{E.49}$$

In order to choose one solution of Eq. (E.38) we must fix one value of the parameter a. This choice is somewhat arbitrary since there is nothing in the theory which indicates the right choice.

E.4.2 Cosmic string

Klein-Gordon equation in the spacetime of a cosmic string with metric

$$ds^2 = -dt^2 + dr^2 + \beta^2 r^2 d\phi^2 + dz^2 \tag{E.50}$$

is given by [2]

$$-\Phi_{,tt} + \Phi_{,rr} + \frac{1}{r}\Phi_{,r} + \frac{1}{\beta^2 r^2}\Phi_{,\phi\phi} + \Phi_{,zz} = M^2\Phi. \tag{E.51}$$

Equation (E.12) now reads $(\nabla^2 \pm i)\Phi = 0$ where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\beta^2 r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \tag{E.52}$$

After separating variables in the form $\Phi = e^{im\phi}e^{ikz}R(r)$ we have

$$R'' + \frac{1}{r}R' + \left[(\pm i - k^2) - \frac{m^2}{\beta^2 r^2}\right] R = 0. \tag{E.53}$$

Near infinity the asymptotic solution is

$$R(r) = \frac{1}{\sqrt{r}}\left(C_1 e^{\alpha r} + C_2 e^{-\alpha r}\right) \tag{E.54}$$

where

$$\alpha = \frac{1}{\sqrt{2}}\left[(\sqrt{1+k^4}+k^2)^{1/2} \mp i\left(\sqrt{1+k^4}-k^2\right)^{1/2}\right]. \tag{E.55}$$

This solution is square integrable only if $C_1 = 0$. Near $r = 0$ the solution of equation (E.53) is given by $R(r) \sim r^{\pm|m/\beta|}$. Both solutions are square integrable only if $|m/\beta| < 1$ [2], i. e., $m = 0$. Therefore there is a solution of equation (E.12) and the spacetime is quantum mechanically singular.

For free spin-1/2 particles we have the Dirac equation

$$\left\{i\gamma^{(0)}\partial_t + i\gamma^{(r)}\left[\partial_r - \frac{1}{2r}\left(\frac{1-\beta}{\beta}\right)\right] + \frac{i}{\beta r}\gamma^{(\theta)}\left(\partial_\theta + \partial_t\right) + i\gamma^{(3)}\partial_z - m\right\}\Psi = 0, \tag{E.56}$$

where $\gamma^{(r)} = \cos\theta\gamma^{(1)} + \sin\theta\gamma^{(2)}$ and $\gamma^{(\theta)} = -\sin\theta\gamma^{(1)} + \cos\theta\gamma^{(2)}$ and $\gamma^{(\mu)}$ are given in terms of the Pauli matrices by

$$\gamma^{(0)} = \begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}, \gamma^{(1)} = \begin{pmatrix}i\sigma^2 & 0\\ 0 & -i\sigma^2\end{pmatrix}, \gamma^{(2)} = \begin{pmatrix}-i\sigma^1 & 0\\ 0 & i\sigma^1\end{pmatrix}, \gamma^{(3)} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{E.57}$$

Solution of equation (E.56) is given by [21, 22]

$$\Psi(t,r,\phi,z) = \begin{pmatrix}\sqrt{E+m}R_1(r)\\ i\sqrt{E+m}R_2(r)e^{i\phi}\end{pmatrix} e^{-iEt+im\phi+ikz} \tag{E.58}$$

where

$$R_j'' + \frac{1}{r}R_j' + \left[\left(E^2-m^2\right) - \frac{(\nu+j-1)^2}{r^2}\right]R_j = 0 \qquad (j=1,2) \tag{E.59}$$

with $\nu = \frac{m+1/2}{\beta} - \frac{1}{2}$.

We must analyse the following equation

$$R_j'' + \frac{1}{r}R_j' + \left[\left(E^2-m^2\right) - \frac{(\nu+j-1)^2}{r^2} \pm i\right]R_j = 0. \tag{E.60}$$

Both solutions of the above equation are square integrable near $r = 0$ if, and only if, $|\nu + j - 1| < 1$ [2]. Therefore the range of modes for which there is a quantum singularity is given by

$$-\frac{3}{2} < \frac{m+1/2}{\beta} < \frac{3}{2}. \tag{E.61}$$

The spacetime is quantum mechanically singular when tested by Dirac particles too.

E.4.3 BTZ spacetime

The metric for the spinless BTZ spacetime [23] is given by

$$ds^2 = -V(r)^2 dt^2 + V(r)^{-2} dr^2 + r^2 d\theta^2, \tag{E.62}$$

with the usual ranges of cylindrical coordinates and $V(r)$ is given by

$$V(r)^2 = -m + \frac{r^2}{l^2}, \tag{E.63}$$

where m is the mass parameter.

For $-1 < m \leq 0$, there appears a continuous sequence of naked singularities at the origin. Since we are worried with naked singularities, we will work with these values of m.

As $r \to \infty$ the metric (E.62) takes the form

$$ds^2 \approx -\left(\frac{r^2}{l^2}\right) dt^2 + \left(\frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\theta^2. \tag{E.64}$$

so that after separating variables in the form $\psi = R(r)e^{in\theta}$ equation (E.12) takes the form (for large r) [24]

$$R_n'' + \frac{3}{r} R_n' = 0, \tag{E.65}$$

whose solution is

$$R_n(r) = C_{1n} + C_{2n} r^{-2}, \tag{E.66}$$

where C_{1n} and C_{2n} are arbitrary constants. It is clear that $R(r) \in L^2$ if, and only if $C_1 n = 0$. To analyse the case $r \to 0$ we note that, after a redefinition of the coordinates

$(t \to \alpha t, r \to \alpha^{-1} r)$ [24] the metric (E.62) takes the form

$$ds^2 \approx -dt^2 + dr^2 + \alpha^2 r^2 d\theta^2. \tag{E.67}$$

This represents a conical singularity. For this metric, the radial portion of equation (E.12) is given by

$$R_n'' + \frac{1}{r} R_n' + \left[\pm i - \frac{n^2}{\alpha^2 r^2} \right] R_n = 0. \tag{E.68}$$

The general solution of the above equation is

$$R_n(r) = A_n J_{|n/\alpha|}(kr) + B_n N_{|n/\alpha|r}, \tag{E.69}$$

with $k = \sqrt{i}$. Bessel function is always square integrable near the origin but $N_{|n/\alpha|}(r)$ is square integrable only for $n = 0$. Then, for $n = 0$ there is a solution of equation (E.12) in L^2. The spacetime is quantum mechanically singular.

Because near $r = 0$ the spacetime is similar to a conic spacetime, we can use the results of Ref. [25] where the boundary conditions necessary to turn self-adjoint the spatial portion of the wave operator in a conic spacetime where studied. They are

$$\begin{array}{ll} \lim_{r\to 0} \left\{ \left[\ln(qr/2) + \gamma \right] r R_0'(r) - R_0 \right\} = 0, & q \in (0, \mu], \\ \lim_{r\to 0} r R_0'(r) = 0, & q = 0, \end{array} \tag{E.70}$$

where μ is the particle mass and γ is Euler-Mascheroni constant. Note that if we are working with massless particles $\mu = 0$ we must choose the boundary condition

$$\lim_{r \to 0} r R_0'(r) = 0 \tag{E.71}$$

so that the spacetime is nonsingular when tested by massless particles.

E.4.4 Spherical Spacetimes

Consider the spherical spacetimes with metric [26]

$$ds^2 = -dt^2 + dr^2 + r^{2p}(d\theta^2 + \sin^2 \theta d\phi^2) \tag{E.72}$$

After separating variables in the form $\psi \sim f(r)Y(\text{angles})$ equation (E.12) becomes

$$f'' + \frac{2p}{r}f' + \left[\pm i - \frac{c}{r^{2p}}\right] f = 0 \tag{E.73}$$

where c is a constant related with the angular variables. Let us concentrate in solutions with spherical symmetry, i.e., $c = 0$.

Near $r = 0$ equation (E.73) reads

$$f'' + \frac{2p}{r}f' = 0. \tag{E.74}$$

One solution goes like a constant while the other goes like r^{1-2p}. A constant is clearly square integrable near the origin, but r^{1-2p} is square integrable only if $p < 3/2$. Therefore the spacetime is quantum mechanically singular for $p < 3/2$ and quantum mechanically regular for $p \geq 3/2$ when tested by spherical symmetric test particles.

E.5 Concluding Remarks

A review of quantum singularities in static spacetimes following the ideas of Horowitz and Marolf was presented. A brief summary of the theory of unbounded operators in Hilbert spaces was given and the main results about essentially self-adjoint operators was established. We also studied quantum test particles in many classically singular spacetimes and found that there it is easier to find spacetimes where the introduction of quantum mechanics does not exclude the singularities. These examples were found in the literature and in particular we saw a curious result in the BTZ spacetime. There, the singularity remains when tested by massive particles but is excluded when tested by massless ones. Despite the fact that the theory of Horowitz and Marolf does not exclude the singularities in all classically singular spacetimes, it gives us evidence that a full quantum theory of gravity may remove the singular points of the classical theory.

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Apêndice F

Flutuações de Vácuo e a Estrutura do Espaço-tempo em Pequena Escala

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Vacuum Fluctuations and the Small Scale Structure of Spacetime

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We show that vacuum fluctuations of the stress-energy tensor in two-dimensional dilaton gravity lead to a sharp focusing of light cones near the

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Planck scale, effectively breaking space up into a large number of causally disconnected regions. This phenomenon, called “asymptotic silence” when it occurs in cosmology, might help explain several puzzling features of quantum gravity, including evidence of spontaneous dimensional reduction at short distances. While our analysis focuses on a simplified two-dimensional model, we argue that the qualitative features should still be present in four dimensions.

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F.1 Introduction

A fundamental goal of quantum gravity is to understand the causal structure of spacetime—the behavior of light cones—at very small scales. One recent proposal [1,2], based on the strong coupling approximation to the Wheeler-DeWitt equation [3,4], is that quantum effects may lead to strong focusing near the Planck scale, essentially collapsing light cones and breaking space up into a large number of very small, causally disconnected regions. Such a phenomenon occurs in classical cosmology near a spacelike singularity, where it has been called “asymptotic silence” [5]; it can be viewed as a kind of anti-Newtonian limit, in which the effective speed of light drops to zero. Near the Planck scale, asymptotic silence could explain several puzzling features of quantum gravity, most notably the apparent dimensional reduction to two dimensions [1,2,6] that occurs in lattice models [7], in some renormalization group analyses [8], and elsewhere [9,10]

In cosmology, the strong focusing of null geodesics comes from the presence of a singularity. For quantum gravity, a different source is required. Null geodesics are governed by the Raychaudhuri equation [11,12]. For a congruence of null geodesics—a pencil of light—with affinely parametrized null normals ℓ^a, this equation tells us that

$$\ell^a\nabla_a\theta = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - 8\pi T_{ab}\ell^a\ell^b, \tag{F.1}$$

where the expansion

$$\theta = \frac{1}{A}\ell^a\nabla_a A \tag{F.2}$$

is the fractional rate of change of area of a cross section of the pencil, σ_{ab} is its shear, and ω_{ab} is its vorticity. The stress-energy tensor T_{ab} appears by virtue of the Einstein field equations, which relate it to the Ricci tensor. (We use natural units, $\hbar = c = G_N = 1$.) In particular, vacuum fluctuations with positive energy focus null geodesics, decreasing the expansion, while fluctuations with negative energy defocus them.

It has recently been shown that for many forms of matter, most vacuum fluctuations have negative energy [13]. These fluctuations have a strict lower bound, however, while the rarer positive fluctuations are unbounded. A key question is then which of these dominate the behavior of light cones near the Planck scale.

In this paper, we answer this question in the simplified context of two-dimensional dilaton gravity, a model obtained by dimensionally reducing general relativity to one space and one time dimension. We show that the positive energy fluctuations win, and cause a collapse of light cones in a time on the order of 15 Planck times. The reduction to two dimensions is, for the moment, required for exact calculations; it is only in this setting that the spectrum of vacuum fluctuations is fully understood. But we show that the qualitative features leading to strong focusing are also present in the full four-dimensional theory, strongly suggesting that our results should apply to full general relativity.

F.2 Dilaton gravity and the Raychaudhuri equation

Our starting point is two-dimensional dilaton gravity, a theory that can be described by the action [14, 15]

$$I = \int d^2x\sqrt{|g|}\left[\frac{1}{16\pi}\varphi R + V[\varphi] + \varphi L_m\right], \tag{F.3}$$

where φ is a scalar field, the dilaton, with a potential $V[\varphi]$ whose details will be unimportant. This action can be obtained from standard general relativity by dimensional reduction, assuming either spherical or planar symmetry. A conformal redefinition of fields is needed to bring the action into this form; the dilaton coupling to the matter Lagrangian L_m is fixed provided the two-dimensional matter action is conformally invariant. From now on, we assume such an invariance.

In two dimensions, the Raychaudhuri equation (F.1) does not quite make sense, since there are no transverse directions in which to define the area or the expansion. Its generalization to dilaton gravity is simple, however. For any model of dilaton gravity obtained by dimensional reduction, the dilaton has a direct physical interpretation as the transverse area in the "missing" dimensions. We can therefore define a generalized expansion

$$\bar{\theta} = \frac{1}{\varphi}\ell^a \nabla_a \varphi = \frac{d}{d\lambda}\ln\varphi, \tag{F.4}$$

where λ is the affine parameter. It is then an easy consequence of the dilaton gravity field equations that

$$\frac{d\bar{\theta}}{d\lambda} = -\bar{\theta}^2 - 8\pi T_{ab}\ell^a\ell^b = -\bar{\theta}^2 - 16\pi T_L, \tag{F.5}$$

where T_L is the left-moving component of the stress-energy tensor.

We next need the vacuum fluctuations of the stress-energy tensor. For conformally invariant matter in two dimensions, Fewster, Ford, and Roman have found these exactly [13]. A conformally invariant field is characterized by a central charge c; a massless scalar, for instance, has $c = 1$. To obtain a finite value for the stress-energy tensor, one must smear it over a small interval; Fewster et al. use a Gaussian smearing function with width τ. The probability distribution for the quantity T_L in the Minkowski vacuum is then given by a shifted Gamma distribution

$$Pr(T_L = \omega) = \vartheta(\omega + \omega_0)\frac{(\pi\tau^2)^\alpha(\omega+\omega_0)^{\alpha-1}}{\Gamma(\alpha)}e^{-\pi\tau^2(\omega+\omega_0)} \tag{F.6}$$

where

$$\omega_0 = \frac{c}{24\pi\tau^2}, \quad \alpha = \frac{c}{24}, \tag{F.7}$$

and ϑ is the Heaviside step function. As noted earlier, this distribution is peaked at negative values of the energy; for $c = 1$, the probability of a positive fluctuation is only .16. There is, however, a long positive tail, as there must be in order that the average $\langle T_L\rangle$ be zero.

Before proceeding with the calculation, it is useful to look at the behavior of the

Raychaudhuri equation (F.5) with a constant source $T_L = \omega$. It is easy to see that

$$\bar{\theta}(\lambda) = \begin{cases} -\sqrt{16\pi\omega}\tan\sqrt{16\pi\omega}(\lambda-\lambda_0) \\ \qquad \text{if } \omega > 0 \\ \sqrt{|16\pi\omega|}\tanh\sqrt{|16\pi\omega|}(\lambda-\lambda_0) \\ \qquad \text{if } \omega < 0 \text{ and } |\bar{\theta}(0)| < \sqrt{|16\pi\omega|} \\ \pm\sqrt{|16\pi\omega|} \\ \qquad \text{if } \omega < 0 \text{ and } \bar{\theta}(0) = \pm\sqrt{|16\pi\omega|} \\ \sqrt{|16\pi\omega|}\coth\sqrt{|16\pi\omega|}(\lambda-\lambda_0) \\ \qquad \text{if } \omega \leq 0 \text{ and } |\bar{\theta}(0)| > \sqrt{|16\pi\omega|} \end{cases} \tag{F.8}$$

where the integration constant λ_0 is determined from the initial value of $\bar{\theta}$.

As noted earlier, positive energy fluctuations can quickly drive the expansion to $-\infty$. If $\bar{\theta}$ is initially negative, even small negative energy fluctuations—those on the coth branch of (F.8)—cannot overcome the nonlinearities that also drive the expansion to $-\infty$. Larger negative energy fluctuations tend to defocus null geodesics, increasing the expansion, but even these have a limited effect. The lower bound $-\omega_0$ for energy in the distribution (F.6) determines a maximum asymptotic value

$$\bar{\theta}_+ = \sqrt{16\pi\omega_0} = \sqrt{2c/3\tau^2}, \tag{F.9}$$

and if the expansion starts below this value, it will asymptote to at most $\bar{\theta}_+$.

Moreover, the same bound on negative energy fluctuations means that the positive contribution to the right-hand side of (F.5) cannot be larger than $16\pi\omega_0 = \bar{\theta}_+{}^2$. The expansion thus has a critical negative value. Any fluctuation that brings $\bar{\theta}$ to a value lower than

$$\bar{\theta}_{crit} = -\sqrt{\frac{2c}{3\tau^2}} = -\bar{\theta}_+ \tag{F.10}$$

is irreversible: once the expansion becomes this negative, it will necessarily continue to decrease. This gives a qualitative answer to our central question—in the long run, the positive energy fluctuations will always win, and the expansion will diverge to $-\infty$. Note the crucial role of the nonlinear term in (F.5); a similar problem was considered in [16], but only in the approximation that $\bar{\theta}$ was small enough that this term could be neglected.

To estimate the time to this "collapse" of the light cones, let us assume that the initial negative energy fluctuations push $\bar{\theta}$ to near its peak value of $\bar{\theta}_+$. We can then ask for the probability of a positive energy fluctuation large enough to drive $\bar{\theta}$ to $\bar{\theta}_{crit}$ within a characteristic time τ. This can be determined from (F.6); for a massless scalar field ($c = 1$), it is $\rho \approx .065$. If we now treat these fluctuations as a Poisson process—i.e., independent events occurring randomly with probability $\rho\Delta t/\tau$ in any interval Δt—and ignore all smaller fluctuations, the time to "collapse" will be given by an exponential distribution $(\rho/\tau)e^{-\rho t/\tau}$, with a mean of approximately 15.4τ.

This is, of course, an oversimplified picture of the combined effect of many vacuum fluctuations. To obtain a more precise result, we next turn to a numerical analysis of the Raychaudhuri equation.

F.3 Quantitative results

Our analysis so far has been semiclassical: we consider quantum fluctuations of matter, but ignore purely quantum gravitational effects. We can, in principle, incorporate weak quantum fluctuations of the metric into the stress-energy tensor in (F.5), but there is no reason to trust our methods at scales at or below the Planck scale. We therefore choose the width τ of the Gaussian smearing function to be the Planck length. The overall system (F.5)–(F.6) is invariant under a simultaneous rescaling of τ and the affine parameter λ, so this choice is not critical; given any cutoff τ, our results can be interpreted as giving the focusing time in units of τ. For our matter field, we choose a massless scalar, that is, a conformal field with central charge $c = 1$.

We proceed as follows:

1. We choose an initial condition $\bar{\theta}_0 = 0$, and select a random value of the vacuum fluctuation ω, with a probability given by the distribution (F.6).
2. We evolve forward one Planck time using (F.8), to determine a new value $\bar{\theta}_1$.
3. Using $\bar{\theta}_1$ as an initial condition and choosing a new value of ω, we evolve another Planck time to determine $\bar{\theta}_2$. We repeat the process, keeping track of the number of

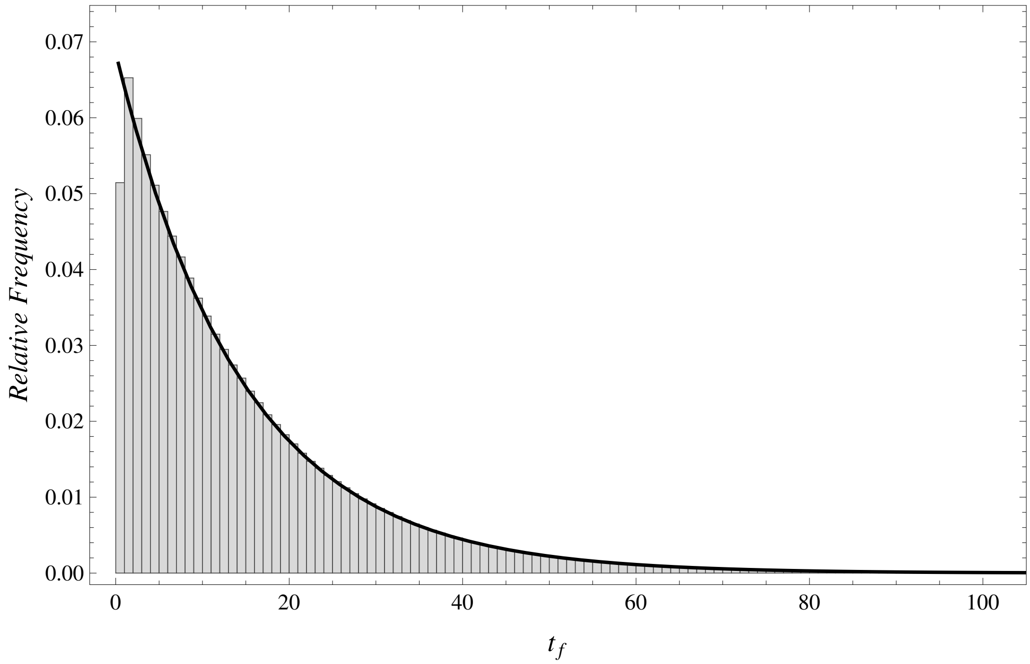

Figura F.3-1. Probability of the expansion diverging to $-\infty$ as a function of Planck time steps. The solid line is the exponential distribution (F.11).

iterations, until we land on a negative branch of (F.8) with the expansion diverging to $-\infty$ during the step.

Using Mathematica [17], we have performed ten million runs of this simulation. Figure F.3-1 shows the probability density of light cone "collapse" as a function of time in Planck units. The mean time to collapse is $14.73\,t_P$, where t_P is the Planck time; the standard deviation is $14.53\,t_P$. These are perhaps large enough to justify our neglect of strong quantum gravitational effects, but small enough to probe the causal structure of spacetime in a physically very interesting region.

We see from the figure that the probability for "collapse" at the first step is approximately .051. This provides a useful check, since this probability can be obtained directly from the distribution (F.6): it is just the probability that the argument of the tangent in the collapsing branch of (F.8) is at least $\pi/2$ when $\lambda = 1$, $\lambda_0 = 0$. The exact result matches our simulation.

As shown in Figure F.3-1, after the first few steps the distribution can be fit very accurately to an exponential distribution:

$$Pr(t_f = n) = \rho e^{-\rho n} \qquad \text{with } \rho \approx .0686. \tag{F.11}$$

This is surprisingly close to our approximation at the end of the preceding section. Moreover, while the details of individual runs vary widely, almost all show $\bar{\theta}$ rising quickly to near its maximum value of $\bar{\theta}_+$, confirming the starting assumption of our approximation.

Two-dimensional dilaton gravity thus appears to exhibit short distance asymptotic silence, with vacuum fluctuations causing a rapid convergence of null cones.

F.4 Generalizations and implications

Our computations have been restricted to two-dimensional dilaton gravity, primarily because this is the only setting in which the probability distribution (F.6) is known. We have also restricted ourselves to conformally invariant matter, for the same reason. But the qualitative features of our results are largely independent of such details, and we expect them to carry over to the full four-dimensional theory. In particular,

1. The vacuum fluctuations of the stress-energy tensor in four dimensions are also expected to have a strict lower bound and an infinitely long positive tail [13]. If anything, the positive tail seems to fall off more slowly than in two dimensions. Moreover, while our detailed results used a Gaussian test function to smear the stress-energy tensor, these features hold for any sufficiently compact test function [18].

2. If the stress-energy tensor is bounded below, the expansion in four dimensions has bounds exactly analogous to those of (F.8), even taking the same functional form [19].

3. Adding real matter would change the quantitative details of our results. But as long as that matter satisfies the null energy condition, it can only lead to further focusing.

We thus expect something akin to short distance asymptotic silence in four-dimensional general relativity. We should, however, add three caveats. First, we have treated our sequence of vacuum fluctuations as if they were statistically independent. This is not quite right: the results of [13] imply that the (Gaussian smeared) stress-energy tensor T_L at coordinate u is weakly anticorrelated with the same object at $u+\tau$. The correlation drops off sharply with distance, and should not affect our qualitative results. It may be possible, though, to take this effect into account to produce a more accurate quantitative picture. Work on this question is in progress.

Second, the vacuum fluctuations found in [13] are fluctuations of the Minkowski vacuum (or, by conformal invariance, of any conformally equivalent vacuum). While any curved spacetime is approximately flat at short enough distances, this is not enough to determine a unique vacuum, and we do not know how sensitive our results are to this choice.

Third, we have by necessity neglected purely quantum gravitational effects, which could also compete with the vacuum fluctuations of matter. This is not independent of the problem of choosing a vacuum; for instance, it is known that if one chooses the Unruh vacuum near a black hole horizon, the renormalized value of the shear term $\sigma_{ab}\sigma^{ab}$ in (F.1) is negative [20].

If a proper handling of these caveats does not drastically change our conclusions, though, we have learned something very interesting about the small scale structure of spacetime. The strong focusing of null cones means that "nearby" neighborhoods of space are no longer in causal contact. This sort of breakup of the causal structure has been studied in cosmology [5], where it leads to BKL behavior [21]: each small neighborhood spends most of its time as an anisotropically expanding Kasner space with essentially random expansion axes and speeds, but periodically undergoes a chaotic "bounce" to a new Kasner space with different axes and speeds. It was argued in [1, 2] that such a local Kasner behavior could explain the apparent spontaneous dimensional reduction of spacetime near the Planck scale, while preserving Lorentz invariance at large scales. Whether or not this proves to be the case, the short distance collapse of light cones suggests both a new picture of spacetime and a new set of approximations for short distances.

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