SOLIDIFICAÇÃO DE LIGAS BINÁRIAS: EXISTÊNCIA DE SOLUÇÕES DE MODELOS DO TIPO CAMPO DE FASE

Gabriela del Valle Planas

Campinas
IMECC - UNICAMP
2002

SOLIDIFICAÇÃO DE LIGAS BINÁRIAS: EXISTÊNCIA DE SOLUÇÕES DE MODELOS DO TIPO CAMPO DE FASE

Este exemplar corresponde a redação final da tese devidamente corrigida e defendida por Gabriela del Valle Planas e aprovada pela Comissão Julgadora.

Campinas, 19 de Março de 2002

José Luiz Boldrini

Orientador: Prof. Dr. José Luiz Boldrini

Banca Examinadora:
Prof. Dr. José Luiz Boldrini
Prof. Dr. Milton da Costa Lopes Filho
Prof. Dr. Marcelo Martins Dos Santos
Prof. Dr. Gustavo Alberto Perla Menzala
Prof. Dr. Jaime Edilberto Muñoz Rivera

Tese apresentada ao Curso de Doutorado do Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas, como requisito parcial à obtenção do título de Doutora em Matemática Aplicada.

FICHA CATALOGRÁFICA ELABORADA PELA BIBLIOTECA DO IMECC DA UNICAMP

Planas, Gabriela del Valle

P692s Solidificação de ligas binárias: existência de soluções de modelos do tipo campo de fase / Gabriela del Valle Planas -- Campinas, [S.P. :s.n.], 2002.

Orientadór : José Luiz Boldrini

Tese (doutorado) - Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Solidificação. 2. Navier Stokes, Equações de. 3. Equações diferenciais parabólicas. I. Boldrini, José Luiz. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Tese de Doutorado defendida em 19 de março de 2002

e aprovada pela Banca Examinadora composta pelos Profs. Drs.

Prof (a). Dr (a). JOSÉ LUIZ BOLDRINI

Prof (a). Dr (a). MILTON DA COSTA LOPES FILHO

Prof (a). Dr (a). MARCELO MARTINS DOS SANTOS

Prof (a). Dr (a). GUSTAVO ALBERTO PERLA MENZALA

Prof (a). Dr (a). JAIME EDILBERTO MUÑOZ RIVERA

Agradecimentos

Ao Professor **José Luiz Boldrini**, pela orientação e dedicação durante a elaboração deste trabalho.

Aos professores e colegas do IMECC, pelo apoio e incentivo.

À Fapesp, pelo apoio financiero.

À minha familia

Resumo

Neste trabalho apresentamos resultados de existência de soluções para alguns modelos matemáticos do tipo campo de fase para a solidificação de ligas binárias. Inicialmente, consideramos um modelo composto por um sistema de equações diferenciais parciais altamente não lineares degenerado e parabólico, com três variáveis independentes: o campo de fase, a temperatura e a concentração. Depois incluímos termos convectivos para levar em consideração o fluxo nas regiões não sólidas. Estudamos alguns modelos desse tipo. A característica comum nesses modelos é que na equação da velocidade é utilizado um termo de penalização do tipo Carman-Kozeny para modelar o efeito *mushy*. Utilizamos técnicas de aproximação que envolvem regularização, o método de Faedo-Galerkin e o Teorema de Ponto Fixo de Leray-Schauder.

Abstract

In this work we present results of existence of solutions for some mathematical models of phase-field type for solidification of binary alloys. Firstly, we consider a model based on a highly non-linear degenerate parabolic system of partial differential equations, with three independent variables: phase-field, solute concentration and temperature. After that, we include convective terms in order to consider the flow in the non-solid regions. We study some models of this sort. All of them have the characteristic of modeling the mushy effect with a Carman-Kozeny penalization term added to the velocity equation. The proofs are based on an approximation technique which includes regularization, Faedo-Galerkin method and Leray-Schauder Fixed Point Theorem.

Índice

Capítulo 1

Introdução

Uma liga binária é um sistema composto de dois materiais A e B, no qual as moléculas de A e B formam uma união física criando um novo composto. O composto A é chamado solvente e o B soluto. Porém a liga não é uma nova substância química. Ligas podem ser metálicas ou misturas intermetálicas; por exemplo, Alumínio-Silício, Cobre-Níquel. As ligas são uma fonte principal de materiais novos e raros e por isso são muito importantes do ponto de vista tecnológico. A forma cristalina das ligas é, em geral, preparada por moldagem da substância derretida, i.e., por solidificação. Assim, os processos de solidificação/fusão têm muita relevância em metalurgia e na ciência dos materiais.

O fenômeno envolvido na solidificação de uma liga ocorre em duas escalas diferentes. O enfoque macroscópico, i.e., na escala do observador, envolve mudança de fase, transferência de massa e de calor e efeitos convectivos. Na escala microscópica são de interesse a estrutura cristalina e a morfologia da interface, a qual pode ter diversas formas, podendo ser planares, colunares, dendríticas ou simplesmente amorfas. A solidificação de ligas é um assunto de interesse tanto do ponto de vista teórico quanto da utilidade na prática.

Uma das características fundamentais dos problemas de mudança de fase é que as regiões correspondentes às diversas fases não são conhecidas *a priori*; por isso tais problemas são chamados "problemas de fronteira móvel"ou "problemas de fronteira livre".

Os problemas de mudança de fase têm sido extensivamente estudados desde que J. Stefan, no século XIX, formulou o problema de encontrar a distribução de temperatura durante a solidificação da água. A formulação clássica do problema de Stefan constitui a base para outros modelos mais complexos que levam o nome de problemas de tipo Stefan. Em tais modelos a hipótese fundamental é a de que as regiões de transição entre as fases são muito finas de tal modo que pode ser descrita por uma superfície regular, chamada de interface; a sua localização, inclusive, faz parte do problema. Estes modelos não incorporam de forma natural alguns efeitos tais como os causados pela tensão superficial, superresfriamento e nucleação. Mais detalhes sobre os problemas do tipo Stefan podem ser encontrados em Alexiades-Solomon [1] e Rubinstein [35].

Uma outra formulação para os problemas de mudança de fase é o método da entalpia, o qual pode ser interpretado como uma formulação fraca do problema de Stefan que incorpora a condição da interface. Nesta formulação não existe suposição sobre a interface. Assim, a região de transição sólido/líquido pode ser uma superfície regular ou pode ser interpretada microscopicamente como uma região intermediária entre a fase puramente líquida e puramente sólida, onde a fase líquida e sólida co-existem em certa proporção e são chamadas de regiões *mushy*. Porém, o método da entalpia tem a desvantagem de não incorporar também de forma natural alguns efeitos tais como superresfriamento.

Uma formulação alternativa para os problemas de mudança de fase são os modelos do tipo campo de fase (*phase-field models*). Estes são modelos contínuos que permitem que a interface tenha espessura e estrutura interna. Este postula a existência de uma função, chamada campo de fase, de tal forma que as interfaces são dadas por superfícies de nível adequadas desta função. Como referência histórica, lembramos que o primeiro modelo de campo de fase para transição sólido/líquido foi proposto por Langer [22] (veja também Caginalp [5]). Este método é particularmente adequado para a computação de situações realistas de interfaces de estrutura complicada, tais como os chamados crescimentos dendríticos (veja Caginalp-Socolovsky [7]).

A metodologia de campo de fase tem atingido nos últimos anos considerável importância na modelagem e simulação numérica dos processos de solidificação. Isto tem motivado vários artigos usando esta metodologia e propondo diferentes modelos matemáticos consistindo de sistemas de equações diferenciais parciais altamente não lineares. A análise matemática rigorosa torna-se em geral difícil, mas no caso de materiais puros alguns autores têm empreendido a tarefa. Por exemplo, veja [5, 18, 23, 29] onde tanto a existência como a unicidade de soluções foram estudadas para vários tipos de não linearidades. Para o caso de ligas binárias, vários modelos foram desenvolvidos. Um dos primeiros trabalhos nessa direção foi proposto por Wheeler et al.

[43], considerando o caso isotérmico. Warren-Boettinger [42] estenderam o modelo e Rappaz-Scheid [34] fizeram uma análise matemática sob hipóteses adequadas sobre as não linearidades. Caginalp et al. [5, 6] estenderam ainda mais o modelo incluindo as mudanças de temperatura. As equações do campo de fase e da concentração para este último modelo são derivadas de um funcional de energia livre; uma apropriada equação do calor é acrescentada, a qual para tomar conta da liberação do calor latente, foi modificada por um termo proporcional à derivada temporal do campo de fase. A análise matemática para o modelo descrito acima é o primeiro problema a ser estudado neste trabalho.

Outro aspecto importante é o de que, em diversas situações, o processo de solidificação se dá não apenas por condução de calor no meio físico, mas também pelo transporte convectivo, ou em outras palavras, deve-se que considerar a influência do movimento da parte fluida, o qual por sua vez é influenciado pelas variações de temperatura ou da concentração. Em muitos casos os fluxos ocorrem e têm importantes efeitos no processo de solidificação. Do ponto de vista matemático, a inclusão de tais termos convectivos torna o processo bastante mais não linear, trazendo maiores dificuldades, independentemente do tipo de formulação utilizada para modelar a mudança de fase. Exemplos destes estudos são os artigos [9, 10, 13, 14] que consideraram problemas do tipo Stefan para materiais puros. Os trabalhos [30, 37] abordaram o problema pelo método da entalpia. Voller et at. [39, 40], propuseram modelos usando a técnica da entalpia para os processos de mudança de fase com convecção/difusão incluindo no modelo as equações de Navier-Stokes modificadas por um certo termo que modela o fluxo na região *mushy*. Para obter expressões para este termo, a região *mushy* é modelada como um meio poroso. Um outro modelo foi proposto por Voss-Tsai [41]. No artigo de Blanc et al. [2] é feita uma análise matemática de um modelo estacionário para a solidificação de uma liga binária, usando o método da entalpia e um termo de penalização do tipo Carman-Kozeny, também sugerido por Voller, foi adicionado às equações de Navier-Stokes para modelar o efeito *mushy*. Outros autores têm proposto modelos usando a metodologia de campo de fase para os processos de solidificação de ligas binárias. Por exemplo, veja [3, 15], onde são propostos modelos usando argumentos da teoria da mistura. Eles também apresentam simulações numéricas para validar seus modelos.

Estamos interessados em modelos do tipo campo de fase para a solidificação de uma liga binária com convecção na fase não-sólida. Diferentemente dos modelos [3, 15], os modelos a serem considerados combinam as idéias de

Voller [39, 40] e Blanc et al. [2] para modelar a possibilidade do fluxo com as idéias de Caginalp et al [8] para o campo de fase e incluem as propriedades térmicas da liga binária. Assim, o segundo problema a ser considerado será uma formulação baseada nas idéias anteriores. O sistema de equações diferenciais parciais não lineares consistirá da equação do campo de fase, a equação do calor, a equação da concentração e as equações de Navier-Stokes modificadas por um termo de penalização do tipo Carman-Kozeny para tomar conta do efeito *mushy*, além de um termo do tipo Boussinesq para levar em consideração os efeitos de variações de temperatura e concentração no fluxo.

O terceiro problema a ser analizado será uma generalização em certos aspectos do problema anterior. Os termos convectivos serão incluídos em todas as equações, mas a dimensão espacial estará restrita a dois. Esta restrição surge pois a regularidade do campo de fase (continuidade), que é necessária para garantir que as regiões estejam bem definidas, depende da regularidade da velocidade. No caso bidimensional, tal regularidade é suficiente para obter a continuidade do campo de fase. No caso tridimensional, isto não parece possível. Portanto, para tratar o caso tridimensional, consideraremos uma variante das equações Navier-Stokes, sugerida por Ladyzenskaja ([21] p.193) e Lions ([24] p.207) para a descrição do movimento do fluido. Este será o quarto problema a ser estudado.

Para tratarmos os modelos, utilizamos técnicas de aproximação que envolvem uma regularização adequada do problema original. Analizamos estes problemas regularizados aplicando argumentos de ponto fixo, em particular, o Teorema de Ponto Fixo de Leray-Schauder (veja [16] p.189) e também o método de Faedo-Galerkin. Depois, por um processo de passagem ao limite nas equações regularizadas, obtemos soluções fracas dos problemas originais via argumentos de compacidade.

Este trabalho está organizado da seguinte forma: cada um dos quatro capítulos seguintes contém um artigo em inglês, cada um destes já foi aceito para publicação ou está sendo submetido a alguma revista internacional, precedido pelo correspondente resumo em lingua portuguesa. O Capítulo 2 está baseado no artigo "Weak Solutions of a Phase-Field Model for Phase Change of an Alloy with Thermal Properties", que já foi aceito e será publicado em *Mathematical Methods in the Applied Sciences.* O Capítulo 3 está baseado no artigo "Weak Solutions of a Phase-Field Model with Convection for Solidification of an Alloy", publicado como Relatório de Pesquisa do IMECC-UNICAMP (RP45/01) e está sendo submetido para publicação.

O Capítulo 4 está baseado no artigo "A Bidimensional Phase-Field Model with Convection for Change Phase of an Alloy" que em breve será submetido para publicação . O Capítulo 5 está baseado no trabalho "A Tridimensional Phase-Field Model with Convection for Change Phase of an Alloy", o qual será submetido para publicação brevemente. No Capítulo 6 apresentamos algumas conclusões gerais deste trabalho e por último as referências bibliográficas.

Capítulo 2

Soluções fracas de um modelo do tipo campo de fase para a mudança de fase de uma liga binária com propriedades térmicas

Resumo

A metodologia de campo de fase fornece uma descrição matemática para os problemas de fronteira livre associados a processos físicos de mudança de fase. Esta postula a existência de uma função, chamada campo de fase (*phase-field*), cujos valores identificam a fase num ponto particular no espaço e no tempo. O método é particularmente adequado para os casos onde ocorrem estruturas complexas de crescimento na mudança da fase.

O modelo matemático estudado neste trabalho descreve o processo de solidificação de uma liga binária com propriedades térmicas. O modelo é composto por um sistema de equações diferenciais parciais altamente não-linear, degenerado e parabólico com três variáveis independentes: o campo de fase, a concentração e a temperatura.

A existência de soluções fracas para o sistema é obtida introduzindo um problema regularizado, logo derivando estimativas *a priori* e aplicando argumentos de compacidade.

Weak Solutions of a Phase-Field Model for Phase Change of an Alloy with Thermal Properties

José Luiz Boldrini *
boldrini@ime.unicamp.br

Gabriela Planas †
gplanas@ime.unicamp.br

UNICAMP-IMECC
C.P. 6065
13083-970 Campinas, SP, Brazil

*Partially supported by CNPq, Brazil, grant 300513/87-9
†Supported by FAPESP, Brazil, grant 97/13932-4

Weak Solutions of a Phase-Field Model for Phase Change of an Alloy with Thermal Properties

MOS subject classification: 35K65, 80A22, 35K55, 82B26, 82C26

Abstract

The phase-field method provides a mathematical description for free-boundary problems associated to physical processes with phase transitions. It postulates the existence of a function, called the phase-field, whose value identifies the phase at a particular point in space and time. The method is particularly suitable for cases with complex growth structures occurring during phase transitions.

The mathematical model studied in this work describes the solidification process occurring in a binary alloy with temperature dependent properties. It is based on a highly nonlinear degenerate parabolic system of partial differential equations with three independent variables: phase-field, solute concentration and temperature.

Existence of weak solutions for this system is obtained via the introduction of a regularized problem, followed by the derivation of suitable estimates and the application of compactness arguments.

1 Introduction

We are interested in the rigorous mathematical analysis of a model for phase change processes occurring in binary alloys with thermal properties. Such a model, using a phase-field methodology, was proposed by Caginalp and Xie [3], and a detailed derivation of a family of models which includes the particular system of equations we will consider here was presented by Caginalp and Jones in [2]. This model can be expressed as the following coupled system of nonlinear partial differential equations:

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi-\phi^3) + \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \text{ in } \Omega\times(0,\infty), \quad (1)$$

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla\cdot[K_1(\phi)\nabla\theta] \quad \text{in } \Omega\times(0,\infty), \quad (2)$$

$$c_t = K_2\left(\Delta c + M\nabla\cdot[c(1-c)\nabla\phi]\right) \text{in}\Omega\times(0,\infty), \quad (3)$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega\times(0,\infty), \quad (4)$$

$$\phi(0) = \phi_0, \qquad \theta(0) = \theta_0, \qquad c(0) = c_0 \qquad \text{in } \Omega. \quad (5)$$

Here Ω is an open bounded domain of $I\!R^N$, $N = 2, 3$, with smooth boundary $\partial\Omega$. The order parameter (phase-field) ϕ is the state variable characterizing the different phases; the function θ represents the temperature; the concentration $c \in [0,1]$ denotes the fraction of one of the two materials in the mixture. The parameter $\alpha > 0$ is the relaxation scaling; the parameter β is given by $\beta = \epsilon[s]/3\sigma$, where $\epsilon > 0$ is a measure of the interface width, σ the surface tension and $[s]$ the entropy density difference between phases; $C_V > 0$ is the specific heat; the constant $l > 0$ the latent heat; θ_A , θ_B, are the respective melting temperatures of each of the two materials in the alloy; $K_2 > 0$ is the solute diffusivity; M is a constant related to the slopes of solidus and liquidus lines; $K_1 > 0$ denotes the thermal conductivity. Concerning this last physical parameter, throughout this paper we assume the conditions used by Laurençot in [11]:

(A) K_1 depends only on the order parameter ϕ, and it is a Lipschitz continuous function. Moreover, there exists $b > 0$ such that

$$0 \leq K_1(r) \leq b \qquad \text{for all } r \in I\!R.$$

We observe that one technical difficulty with the previous system is that, when K_1 vanishes, the equation (2) degenerates, losing its parabolic character.

We also remark that in [3], the concentration equation is written as

$$c_t = K_2 \nabla \cdot \left[c(1-c) \nabla \left(M\phi + \ln \frac{c}{1-c} \right) \right] \text{ in } Q,$$

which forces $c \in (0, 1)$. Equation (3) is more general, allowing c to assume the values 0 and 1. Moreover, note that for the pure materials, that is, when $c \equiv 0$ or $c \equiv 1$, the equations reduce to a standard phase field model for pure materials.

We should remark that in recent years the phase-field methodology has achieved considerable importance in the modelling and numerical simulation of a range of phase transitions and complex growth structures occurring during solidification. Phase-field models have been used to describe phase transitions of pure material due to thermal effects; as examples of papers where mathematical analyses of such models are performed, we single out [1, 8, 11, 16], where existence of solutions is investigated for various types of nonlinearities. We also remark that the phase-field equation has been derived in a thermodynamically consistent way by Penrose and Fife [17]. Several papers have been devoted to the mathematical analysis of the Penrose-Fife model; see for instance [4, 5] and references therein. We also remark that by linearization of the heat flux with respect to the temperature, from the model derived by Penrose and Fife it is possible to recover the phase-field equation employed by Caginalp [1].

Several phase-field models have also been developed for binary alloys. One of the first works in this direction was due to Wheeler *et al* [21] and was concerned with isothermal solidification. Warren and Boettinger [20] extended this model, while recently Rappaz and Scheid [18] investigated its well-posedness under suitable assumptions on the non-linearities. In such models, the phase-field and the concentration equations are derived from a free energy functional, and an appropriate balance equation for the temperature is then added to complete the model. As we stated, the model for binary alloys studied in this paper was developed by Caginalp and Xie [3], and it is similar to the previous ones, but includes thermal effects.

Standard notation will be used. We remark that $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. Also, for a given fixed $T > 0$, we denote $Q = \Omega \times (0, T)$.

The main result of this paper is the following.

Theorem 1 *Let be given functions satisfying:* $\phi_0 \in H^{1+\gamma}(\Omega)$ *with* $1/2 < \gamma \leq 1$ *and* $\dfrac{\partial \phi_0}{\partial n} = 0$ *a.e. on* $\partial\Omega$*;* $\theta_0 \in L^2(\Omega)$*;* $c_0 \in H^1(\Omega)$ *such that* $0 \leq c_0 \leq 1$ *a.e. in* Ω*. Then, under the assumption* **(A)***, there exist functions* (ϕ, θ, c, J) *satisfying, for any fixed* $T > 0$*,*

i) $\phi \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)), \quad \phi_t \in L^2(Q), \quad \phi(0) = \phi_0,$

ii) $\theta \in L^\infty(0,T; L^2(\Omega)), \quad \theta_t \in L^2(0,T; H^1(\Omega)'), \quad \theta(0) = \theta_0,$

iii) $c \in L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)), \quad c_t \in L^2(0,T; H^1(\Omega)'),$
$c(0) = c_0, \quad 0 \leq c \leq 1 \quad$ *a.e. in* Q*,*

iv) $J \in L^2(Q, I\!R^n), \quad J = \nabla(K_1(\phi)\theta) - \theta\nabla K_1(\phi),$

and such that

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \qquad a.e.\ in\ Q, \tag{6}$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad a.e.\ on\ \partial\Omega \times (0,T), \tag{7}$$

$$C_\nu \int_0^T \langle \theta_t, \eta \rangle dt + \frac{l}{2}\int_0^T \int_\Omega \phi_t \eta \, dxdt + \int_0^T \int_\Omega J \cdot \nabla\eta \, dxdt = 0, \tag{8}$$

$$\int_0^T \langle c_t, \eta \rangle dt + K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla \eta \, dxdt + K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi \cdot \nabla\eta \, dxdt = 0, \tag{9}$$

for any $\eta \in L^2(0,T; H^1(\Omega))$*.*

If, in addition, $K_1 \geq b_0$ *for some* $b_0 > 0$*, then* $\theta \in L^2(0,T; H^1(\Omega))$ *for each* $T > 0$*, and* $J = K_1(\phi)\nabla\theta$*.*

We remark that because the possibility of degeneracy of the parabolic character of the model, we were able neither to prove uniqueness nor to improve the global regularity of the constructed solution.

Finally, the outline of this paper is as follows. In Section 2, we study a family of regularized problems depending on a small accessory positive parameter; this study will be auxiliary in proving the main existence result by avoiding the above mentioned possibility of degeneracy of parabolicity in equation (3). Section 3 is devoted to proof of Theorem 1; it will be done with the help of suitable estimates derived for the regularized problem, together with compactness arguments.

2 A regularized problem

In this section we introduce a regularized problem related to (1)–(5); for it, we will prove a result of existence of solutions by using Leray-Schauder fixed point theorem in a form stated in ([6] p. 189).

Theorem (Leray-Schauder): *Consider a transformation $y = \mathcal{T}_\lambda(x)$ where x, y belong to a Banach space B and λ is a real parameter which varies in a bounded interval, say $0 \leq \lambda \leq 1$. Assume:*

(a) $\mathcal{T}_\lambda(x)$ is defined for all $x \in B$, $0 \leq \lambda \leq 1$,

(b) for any fixed λ, $\mathcal{T}_\lambda(x)$ is continuous in B,

(c) for x in bounded sets of B, $\mathcal{T}_\lambda(x)$ is uniformly continuous in λ,

(d) for any fixed $\lambda, \mathcal{T}_\lambda(x)$ is a compact transformation,

(e) there exists a (finite) constant M such that every possible solution x of $\mathcal{T}_\lambda(x) = x$ satisfies: $\|x\|_B \leq M$,

(f) the equation $\mathcal{T}_0(x) = x$ has a unique solution in B.

Under assumptions (a)-(f), there exists a solution of the equation $x - \mathcal{T}_1(x) = 0$.

Now, we recall certain results that will be helpful in the introduction of such regularized problem.

Recall that there is an extension operator $Ext(\cdot)$ taking any function w in the space

$$W_2^{2,1}(Q) = \left\{ w \in L^2(Q) \,/\, D_x w, D_x^2 w \in L^2(Q), w_t \in L^2(Q) \right\}$$

and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^{N+1})$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^{N+1})} \leq C \, \|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [15] pp.157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^{N+1})$ be a family of symmetric positive mollifier functions with compact support converging to the Dirac delta function, and denote by $*$ the convolution operation. Then, given a function $w \in W_q^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(I\!R^{N+1})$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

Now, we define K_1^δ for each $\delta \in (0,1)$ by

$$K_1^\delta(r) = K_1(r) + \delta$$

for all $r \in I\!R$. We infer from **(A)** that,

(B) K_1^δ is a Lipschitz continuous function and

$$0 < \delta \le K_1^\delta(r) \le b + 1 \qquad \text{for all } r \in I\!R.$$

Now, we are in position to define the following family of regularized problems. For $\delta \in (0, 1)$, we consider the system

$$\alpha\epsilon^2\phi_t^\delta - \epsilon^2\Delta\phi^\delta = \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) + \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q, \tag{10}$$

$$C_V\theta_t^\delta + \frac{l}{2}\phi_t^\delta = \nabla \cdot K_1^\delta(\rho_\delta(\phi^\delta))\nabla\theta^\delta \qquad \text{in } Q, \tag{11}$$

$$c_t^\delta - K_2\Delta c^\delta = K_2 M\nabla \cdot c^\delta(1 - c^\delta)\nabla\left(\rho_\delta(\phi^\delta)\right) \qquad \text{in } Q, \tag{12}$$

$$\frac{\partial\phi^\delta}{\partial n} = 0, \qquad \frac{\partial\theta^\delta}{\partial n} = 0, \qquad \frac{\partial c^\delta}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{13}$$

$$\phi^\delta(0) = \phi_0^\delta, \qquad \theta^\delta(0) = \theta_0^\delta, \qquad c^\delta(0) = c_0^\delta \qquad \text{in } \Omega. \tag{14}$$

We then have the following existence result.

Proposition 1 *For each $\delta \in (0, 1)$, let $(\phi_0^\delta, \theta_0^\delta, c_0^\delta) \in H^{1+\gamma}(\Omega) \times H^{1+\gamma}(\Omega) \times C^1(\bar{\Omega})$, $1/2 < \gamma \le 1$, satisfying the compatibility conditions $\frac{\partial\phi_0^\delta}{\partial n} = \frac{\partial\theta_0^\delta}{\partial n} = \frac{\partial c_0^\delta}{\partial n} = 0$ a.e. on $\partial\Omega$ and $0 \le c_0^\delta < 1$ in $\bar{\Omega}$. Under the assumption* **(B)**, *there exist functions $(\phi^\delta, \theta^\delta, c^\delta)$ satisfying, for any fixed $T > 0$,*

i) $\phi^\delta \in L^2(0, T; H^2(\Omega))$, $\quad \phi_t^\delta \in L^2(Q)$,

ii) $\theta^\delta \in L^2(0, T; H^2(\Omega))$, $\quad \theta_t^\delta \in L^2(Q)$,

iii) $c^\delta \in C^{2,1}(Q)$, $\quad 0 < c^\delta < 1$,

iv) $(\phi^\delta, \theta^\delta, c^\delta)$ *satisfies (10)-(14) almost everywhere.*

Proof: To ease the notation, in this proof we will omit the superscript δ of the variables $\phi^\delta, \theta^\delta, c^\delta$.

First of all, we consider the following family of operators, indexed by the parameter $0 \le \lambda \le 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^2(Q) \times L^2(Q) \times L^2(Q),$$

and defined as follows: given $(\hat{\phi}, \hat{\theta}, \hat{c}) \in B$, let $\mathcal{T}_\lambda(\hat{\phi}, \hat{\theta}, \hat{c}) = (\phi, \theta, c)$, where (ϕ, θ, c) is obtained by solving the problem

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q, \quad (15)$$

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla \cdot K_1^\delta(\rho_\delta(\phi))\nabla\theta \qquad \text{in } Q, \quad (16)$$

$$c_t - K_2\Delta c = K_2 M \nabla \cdot c(1-c)\nabla\left(\rho_\delta(\phi)\right) \qquad \text{in } Q, \quad (17)$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0,T), \quad (18)$$

$$\phi(0) = \phi_0^\delta, \qquad \theta(0) = \theta_0^\delta, \qquad c(0) = c_0^\delta \qquad \text{in } \Omega. \quad (19)$$

Before we prove that $\mathcal{T}_\lambda$ is well defined, we observe that clearly (ϕ, θ, c) is a solution of (10)-(14) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point by using the Leray-Schauder fixed point theorem stated at the beginning of this section ([6] p. 189).

To verify that $\mathcal{T}_\lambda$ is well defined, observe that since $\hat{\theta}$, $\hat{c} \in L^2(Q)$, we infer from Theorem 2.1 of [8] that there is a unique solution ϕ of equation (15) with $\phi \in W_2^{2,1}(Q)$ satisfying the first of the boundary conditions (18).

Now, since K_1^δ is a bounded Lipschitz continuous function and $\rho_\delta(\phi) \in C_0^\infty(\mathbb{R}^{N+1})$, we have that $K_1^\delta(\rho_\delta(\phi)) \in W_r^{1,1}(Q)$ for $1 \le r \le \infty$. Thus, since $\phi_t \in L^2(Q)$, we infer from L^p-theory of parabolic equations (see [10], Thm. 9.1 in Chapter IV, p. 341, and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ of equation (16) with $\theta \in W_2^{2,1}(Q)$.

We observe that equation (17) is a semilinear parabolic equation with smooth coefficients. Moreover, by looking at the right-hand side of this equation, written in form

$$c_t = K_2 \Delta c + K_2 M(1-2c)\nabla c \cdot \nabla \phi + K_2 M c(1-c)\Delta \phi,$$

we can see that it has the properties and growth conditions in order that the semigroup results about global existence, as stated in Henry [7], p.75, be applicable. Thus, we conclude that there is a unique global classical solution c.

In addition, note that equation (17) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principles together with conditions $0 < c_0^\delta < 1$ and $\dfrac{\partial c^\delta}{\partial n} = 0$, we can deduce that

$$0 < c(x,t) < 1, \qquad \forall\,(x,t) \in Q. \tag{20}$$

Therefore, for each $\lambda \in [0,1]$, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$, let $(\hat{\phi}_n, \hat{\theta}_n, \hat{c}_n) \in B$ strongly converging to $(\hat{\phi}, \hat{\theta}, \hat{c}) \in B$; for each n, let (ϕ_n, θ_n, c_n) the corresponding solution of the problem:

$$\alpha\epsilon^2 \phi_{nt} - \epsilon^2 \Delta\phi_n - \frac{1}{2}(\phi_n - \phi_n^3) = \lambda\beta\left(\hat{\theta}_n + (\theta_B - \theta_A)\hat{c}_n - \theta_B\right) \text{ in } Q, \tag{21}$$

$$C_V \theta_{nt} + \frac{l}{2}\phi_{nt} = \nabla \cdot K_1^\delta(\rho_\delta(\phi_n))\nabla\theta_n \qquad \text{in } Q, \tag{22}$$

$$c_{nt} - K_2 \Delta c_n = K_2 M \nabla \cdot c_n(1-c_n)\nabla\left(\rho_\delta(\phi_n)\right) \qquad \text{in } Q, \tag{23}$$

$$\frac{\partial \phi_n}{\partial n} = 0, \qquad \frac{\partial \theta_n}{\partial n} = 0, \qquad \frac{\partial c_n}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{24}$$

$$\phi_n(0) = \phi_0, \qquad \theta_n(0) = \theta_0, \qquad c_n(0) = c_0 \qquad \text{in } \Omega. \tag{25}$$

Next we show that the sequence (ϕ_n, θ_n, c_n) converges strongly to $(\phi, \theta, c) = \mathcal{T}_\lambda(\hat{\phi}, \hat{\theta}, \hat{c})$ in B. For that purpose, we will obtain estimates, uniformly with respect to n, for (ϕ_n, θ_n, c_n). We denote by C_i any positive constant independent of n.

We multiply (21) successively by ϕ_n, ϕ_{nt} and $-\Delta\phi_n$, and integrate over $\Omega \times (0, t)$. After integration by parts and the use the Hölder's and Young's inequalities, we obtain the following three estimates:

$$\begin{aligned}\frac{\alpha\epsilon^2}{2}\int_\Omega |\phi_n|^2 dx &+ \int_0^t \int_\Omega \left(\frac{\epsilon^2}{2}|\nabla\phi_n|^2 + \frac{1}{4}\phi_n^4\right) dxdt \\ &\leq C_1 + C_2 \int_0^t \int_\Omega \left(|\hat{\theta}_n|^2 + |\hat{c}_n|^2 + |\phi_n|^2\right) dxdt, \quad (26)\\ \frac{\alpha\epsilon^2}{2}\int_0^t \int_\Omega |\phi_{nt}|^2 dxdt &+ \int_\Omega \left(\frac{\epsilon}{2}|\nabla\phi_n|^2 + \frac{\phi_n^4}{8} - \frac{\phi_n^2}{4}\right) dx \\ &\leq C_1 + C_2 \int_0^t \int_\Omega \left(|\hat{\theta}_n|^2 + |\hat{c}_n|^2\right) dxdt, \quad (27)\\ \frac{\alpha\epsilon^2}{2}\int_\Omega |\nabla\phi_n|^2 dx &+ \frac{\epsilon^2}{2}\int_0^t \int_\Omega |\Delta\phi_n|^2 dxdt \\ &\leq C_1 + C_2 \int_0^t \int_\Omega \left(|\nabla\phi_n|^2 + |\hat{\theta}_n|^2 + |\hat{c}_n|^2\right) dxdt. \quad (28)\end{aligned}$$

By multiplying (27) by $\alpha\epsilon^2$ and adding the result to (26), we find

$$\begin{aligned}\int_\Omega \left(|\phi_n|^2\right. &+ \left.|\nabla\phi_n|^2 + \phi_n^4\right) dx \\ &\leq C_1 + C_2 \int_0^t \int_\Omega \left(|\hat{\theta}_n|^2 + |\hat{c}_n|^2 + |\phi_n|^2\right) dxdt. \quad (29)\end{aligned}$$

Since $\|\hat{\theta}_n\|_{L^2(Q)}$ and $\|\hat{c}_n\|_{L^2(Q)}$ are bounded independent of n, we infer from (29) and Gronwall's Lemma that

$$\|\phi_n\|_{L^\infty(0,T;H^1(\Omega))} \leq C_1. \quad (30)$$

Then, thanks to (26)-(28) we have

$$\|\phi_n\|_{L^2(0,T;H^2(\Omega))} + \|\phi_{nt}\|_{L^2(Q)} \leq C_1. \quad (31)$$

Now, by multiplying (22) by θ_n, one obtains in a similar way as above that

$$\int_\Omega |\theta_n|^2 dx + \int_0^t \int_\Omega |\nabla\theta_n|^2 dxdt \leq C_1 + C_2 \int_0^t \int_\Omega \left(|\phi_{nt}|^2 + |\theta_n|^2\right) dxdt. \quad (32)$$

Thus, with the help of (31) and Gronwall's Lemma, we infer that

$$\|\theta_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1. \quad (33)$$

Hence, it follows from (32) that

$$\|\theta_n\|_{L^2(0,T;H^1(\Omega))} \le C_2. \tag{34}$$

Now, we take scalar product in $L^2(\Omega)$ of (22) with $\eta \in H^1(\Omega)$. By integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\|\theta_{nt}\|_{H^1(\Omega)'} \le C_1 \left(\|\nabla\theta_n\|_{L^2(\Omega)} + \|\phi_{nt}\|_{L^2(\Omega)} \right).$$

Thus, we infer from (31) and (34) that

$$\|\theta_{nt}\|_{L^2(0,T;H^1(\Omega)')} \le C_1. \tag{35}$$

Next, by multiplying (23) by c_n, with the help of (20), as above we conclude that

$$\int_\Omega |c_n|^2 dx + \int_0^t \int_\Omega |\nabla c_n|^2 dxdt \le C_1 + C_2 \int_0^t \int_\Omega |\nabla \phi_n|^2 dxdt.$$

Hence, from (31) we have,

$$\|c_n\|_{L^2(0,T;H^1(\Omega))} + \|c_n\|_{L^\infty(0,T;L^2(\Omega))} \le C_1. \tag{36}$$

In order to get an estimate for c_{nt} in $L^2(0,T;H^1(\Omega)')$, we use arguments similar to the previous ones with equation (23) to obtain

$$\|c_{nt}\|_{L^2(0,T;H^1(\Omega)')} \le C_1. \tag{37}$$

We now infer from (30) and (31) that the sequence (ϕ_n) is bounded in

$$W_1 = \left\{ v \in L^2(0,T;H^2(\Omega)),\, v_t \in L^2(0,T;L^2(\Omega)) \right\}$$

and in

$$W_2 = \left\{ v \in L^\infty(0,T;H^1(\Omega)),\, v_t \in L^2(0,T;L^2(\Omega)). \right\},$$

We also infer from (33)-(36) that the sequences (θ_n) and (c_n) are bounded in

$$W_3 = \left\{ v \in L^2(0,T;H^1(\Omega)),\, v_t \in L^2(0,T;H^1(\Omega)') \right\}$$

and in

$$W_4 = \left\{ v \in L^\infty(0,T;L^2(\Omega)),\, v_t \in L^2(0,T;H^1(\Omega)') \right\}.$$

Since W_1 is compactly embedded in $L^2(0,T;H^1(\Omega))$, W_2 in $C([0,T];L^2(\Omega))$, W_3 in $L^2(0,T;L^2(\Omega))$ and W_4 in $C([0,T];H^1(\Omega)')$ ([19] Cor.4), it follows that there exist

$$
\begin{array}{lcl}
\phi & \in & L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \ \text{ with } \phi_t \in L^2(Q), \\
\theta & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \ \text{ with } \theta_t \in L^2(0,T;H^1(\Omega)'), \\
c & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \ \text{ with } c_t \in L^2(0,T;H^1(\Omega)'),
\end{array}
$$

and a subsequence of (ϕ_n, θ_n, c_n) (which we still denote by (ϕ_n, θ_n, c_n)), such that, as $n \to +\infty$,

$$
\begin{array}{lllll}
\phi_n & \to & \phi & \text{in} & L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega)) \ \text{ strongly}, \\
\phi_n & \rightharpoonup & \phi & \text{in} & L^2(0,T;H^2(\Omega)) \ \text{ weakly}, \\
\theta_n & \to & \theta & \text{in} & L^2(Q) \cap C([0,T];H^1(\Omega)') \ \text{ strongly}, \\
\theta_n & \rightharpoonup & \theta & \text{in} & L^2(0,T;H^1(\Omega)) \ \text{ weakly}, \\
c_n & \to & c & \text{in} & L^2(Q) \cap C([0,T];H^1(\Omega)') \ \text{ strongly}, \\
c_n & \rightharpoonup & c & \text{in} & L^2(0,T;H^1(\Omega)) \ \text{ weakly},
\end{array}
\tag{38}
$$

It now remains to pass to the limit as n tends to ∞ in (21)-(25). Since the embedding of $W_2^{2,1}(Q)$ into $L^9(Q)$ is compact ([12] pp.15), and (ϕ_n) is bounded in $W_2^{2,1}(Q)$, we infer that ϕ_n^3 converges to ϕ^3 in $L^2(Q)$. We then pass to the limit in (21) and get

$$
\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \qquad \text{a.e. in } Q.
$$

Since K_1^δ is bounded Lipschitz continuous function and $\rho_\delta(\phi_n)$ converges to $\rho_\delta(\phi)$ in $L^2(Q)$, we have that $K_1^\delta(\rho_\delta(\phi_n))$ converges to $K_1^\delta(\rho_\delta(\phi))$ in $L^p(Q)$ for any $p \in [1,\infty)$. This fact and (38) yield the weak convergence of $K_1^\delta(\rho_\delta(\phi_n))\nabla\theta_n$ to $K_1^\delta(\rho_\delta(\phi))\nabla\theta$ in $L^{3/2}(Q)$. Now, by multiplying (22) by $\eta \in L^2(0,T;H^1(\Omega))$ and integrating over $\Omega \times (0,T)$, after integration by parts, we obtain

$$
C_V\int_0^T\int_\Omega \theta_{nt}\eta\,dxdt + \frac{l}{2}\int_0^T\int_\Omega \phi_{nt}\eta\,dxdt + \int_0^T\int_\Omega K_1^\delta(\rho_\delta(\phi_n))\nabla\theta_n\cdot\nabla\eta\,dxdt = 0.
$$

Then, we may pass to the limit and find that

$$
C_V\int_0^T\langle\theta_t,\eta\rangle dt + \frac{l}{2}\int_0^T\int_\Omega \phi_t\eta\,dxdt + \int_0^T\int_\Omega K_1^\delta(\rho_\delta(\phi))\nabla\theta\cdot\nabla\eta\,dxdt = 0
$$

holds for any $\eta \in L^2(0,T;H^1(\Omega))$. This implies that

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla\cdot K_1^\delta(\rho_\delta(\phi))\nabla\theta \qquad \text{in } \mathcal{D}'(Q). \tag{39}$$

Now, by noting that $\phi_t \in L^2(Q)$ and using the L^p-theory of parabolic equations, it is easy to conclude that (39) holds almost everywhere in Q.

It remains to pass to the limit in (23). We infer from (38) that $\nabla\rho_\delta(\phi_n)$ converges to $\nabla\rho_\delta(\phi)$ in $L^2(Q)$ and since $\|c_n\|_{L^\infty(Q)}$ is bounded, it follows that $c_n(1-c_n)$ converges to $c(1-c)$ in $L^p(Q)$ for any $p\in[1,\infty)$. Similarly, we may pass to the limit in (23) to obtain

$$c_t - K_2\Delta c = K_2M\nabla\cdot c(1-c)\nabla\left(\rho_\delta(\phi)\right) \qquad \text{in } Q.$$

Therefore $\mathcal{T}_\lambda$ is continuous for all $0\le\lambda\le 1$. At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1\times W_3\times W_3$, and the embedding of this space in B is compact. We conclude that $\mathcal{T}_\lambda$ is a compact operator for each $\lambda\in[0,1]$.

To prove that for $(\hat\phi,\hat\theta,\hat c)$ in a bounded set of B, $\mathcal{T}_\lambda$ is uniformly continuous with respect to λ, let $0\le\lambda_1,\lambda_2\le 1$ and $(\phi_i,\theta_i,c_i)\,(i=1,2)$ be the corresponding solutions of (15)-(19). We observe that $\phi=\phi_1-\phi_2$, $\theta=\theta_1-\theta_2$ and $c=c_1-c_2$ satisfy the following problem:

$$\begin{aligned}\alpha\epsilon^2\phi_t-\epsilon^2\Delta\phi &= \frac{1}{2}\phi(1-(\phi_1^2+\phi_1\phi_2+\phi_2^2))\\ &+ (\lambda_1-\lambda_2)\beta\left(\hat\theta+(\theta_B-\theta_A)\hat c-\theta_B\right) \qquad \text{in } Q,\end{aligned} \tag{40}$$

$$\begin{aligned}C_V\theta_t+\frac{l}{2}\phi_t &= \nabla\cdot K_1^\delta(\rho_\delta(\phi_1))\nabla\theta\\ &+ \nabla\cdot\left[K_1^\delta(\rho_\delta(\phi_1))-K_1^\delta(\rho_\delta(\phi_2))\right]\nabla\theta_2 \qquad \text{in } Q,\end{aligned} \tag{41}$$

$$\begin{aligned}c_t-K_2\Delta c &= K_2M\nabla\cdot\left[c_1(1-c_1)\left(\nabla\rho_\delta(\phi_1)-\nabla\rho_\delta(\phi_2)\right)\right]\\ &+ K_2M\nabla\cdot\left[c(1-(c_1+c_2))\nabla\rho_\delta(\phi_2)\right] \qquad \text{in } Q,\end{aligned} \tag{42}$$

$$\frac{\partial\phi}{\partial n}=0,\qquad \frac{\partial\theta}{\partial n}=0,\qquad \frac{\partial c}{\partial n}=0 \qquad \text{on } \partial\Omega\times(0,T), \tag{43}$$

$$\phi(0)=0,\qquad \theta(0)=0,\qquad c(0)=0 \qquad \text{in } \Omega. \tag{44}$$

We remark that $d:=\phi_1^2+\phi_1\phi_2+\phi_2^2\ge 0$. Now, multiply equation (40) by ϕ and integrate over Q; after integration by parts and the use of Hölder's and Young's inequalities we, obtain

$$\begin{aligned}\int_\Omega |\phi|^2 dx + \int_0^t \int_\Omega |\nabla\phi|^2 dxdt &\leq C_1 \int_0^t \int_\Omega |\phi|^2 dxdt \\ &+ C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_\Omega \left(|\hat{\theta}|^2 + |\hat{c}|^2 \right) dxdt.\end{aligned}$$

By applying Gronwall's Lemma, we arrive at

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;H^1(\Omega))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{45}$$

Now, multiply (40) by ϕ_t and use Hölder's inequality to obtain

$$\begin{aligned}\alpha\epsilon^2 \int_0^t \int_\Omega |\phi_t|^2 dxdt &+ \frac{\epsilon^2}{2} \int_\Omega |\nabla\phi|^2 dx \\ &\leq C_1 \int_0^t \int_\Omega |\phi|^2 dxdt + \frac{\alpha\epsilon^2}{2} \int_0^t \int_\Omega |\phi_t|^2 dxdt \\ &+ C_2 \left(\int_0^t \int_\Omega |\phi|^{10/3} dxdt \right)^{3/5} \left(\int_0^t \int_\Omega |d|^5 dxdt \right)^{2/5} \\ &+ C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_\Omega \left(|\hat{\theta}|^2 + |\hat{c}|^2 \right) dxdt.\end{aligned}$$

Since $W_2^{2,1}(Q) \hookrightarrow L^{10}(Q)$, the following interpolation inequality holds

$$\|\phi\|^2_{L^{10/3}(Q)} \leq \eta \, \|\phi\|^2_{W_2^{2,1}(Q)} + \tilde{C} \, \|\phi\|^2_{L^2(Q)} \text{ for all } \eta > 0.$$

Moreover, since $\|d\|_{L^5(Q)} \leq C$, with C depending on $\|\phi_1\|_{L^{10}(Q)}$ and $\|\phi_2\|_{L^{10}(Q)}$, by rearranging the terms in the last inequality, we obtain

$$\begin{aligned}\int_0^t \int_\Omega |\phi_t|^2 dxdt &+ \int_\Omega |\nabla\phi|^2 dx \\ &\leq C_1 \int_0^t \int_\Omega |\phi|^2 dxdt + C_2 \, \eta \, \|\phi\|^2_{W_2^{2,1}(Q)} \\ &+ C_3 |\lambda_1 - \lambda_2|^2 \int_0^t \int_\Omega \left(|\hat{\theta}|^2 + |\hat{c}|^2 \right) dxdt.\end{aligned} \tag{46}$$

By multiplying (40) by $-\Delta\phi$, we infer in a similar way that

$$\begin{aligned}\int_\Omega |\nabla\phi|^2 dx &+ \int_0^t \int_\Omega |\Delta\phi|^2 dxdt \\ &\leq C_1 \int_0^t \int_\Omega \left(|\phi|^2 + |\nabla\phi|^2 \right) dxdt + C_2 \, \eta \, \|\phi\|^2_{W_2^{2,1}(Q)} \\ &+ C_3 |\lambda_1 - \lambda_2|^2 \int_0^t \int_\Omega \left(|\hat{\theta}|^2 + |\hat{c}|^2 \right) dxdt.\end{aligned} \tag{47}$$

By taking $\eta > 0$ small enough and considering (45) we conclude from (46) and (47) that

$$\|\phi\|^2_{W_2^{2,1}(Q)} + \|\phi\|^2_{L^\infty(0,T;H^1(\Omega))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{48}$$

By multiplying (41) by θ, integrating over Ω and using Hölder's inequality and **(B)**, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\theta|^2 dx + \delta \int_\Omega |\nabla\theta|^2 dxdt \;&\leq\; C_1 \int_\Omega \left(|\phi_t|^2 + |\theta|^2 + |\phi|\,|\nabla\theta_2|\,|\nabla\theta|\right) dxdt \\
&\leq\; C_1 \int_\Omega \left(|\phi_t|^2 + |\theta|^2\right) dxdt + \frac{\delta}{2}\int_\Omega |\nabla\theta|^2 \\
&+\; C_3 \|\phi\|^2_{L^\infty(0,T;H^1(\Omega))} \|\theta_2\|^2_{H^2(\Omega)}.
\end{aligned}$$

Therefore, integration with respect to t and the use of Gronwall's Lemma and (48) lead to the estimate

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{49}$$

Now, we multiply (42) by c and integrate over $\Omega \times (0,t)$. By integration by parts, using Hölder's and Young's inequalities, (20) and the known estimates for c_1 and c_2, we obtain

$$\begin{aligned}
\int_\Omega |c|^2 dx + \int_0^t \int_\Omega |\nabla c|^2 dxdt \;&\leq\; C_1 \int_0^t \int_\Omega \Big(|\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)|^2 \\
&+\; \|\nabla\rho_\delta(\phi_2)\|^2_{L^\infty(Q)} |c|^2\Big) dxdt.
\end{aligned}$$

The last term in this inequality can be estimated as follows:

$$\begin{aligned}
\|\nabla\rho_\delta(\phi_2)\|_{L^\infty(Q)} &= \|\nabla(\rho_\delta * Ext(\phi_2))\|_{L^\infty(Q)} = \|\rho_\delta * \nabla Ext(\phi_2))\|_{L^\infty(Q)} \\
&\leq \|\rho_\delta\|_{L^2(Q)} \|\nabla Ext(\phi_2))\|_{L^2(Q)} \leq C \|\nabla\phi_2\|_{L^2(Q)},
\end{aligned}$$

where we used the definition of ρ_δ and Young's inequality, as well as properties of the convolution and of the extension operator Ext and the fact that $\delta > 0$ is fixed.

From the last two results, we obtain

$$\int_\Omega |c|^2 dx + \int_0^t \int_\Omega |\nabla c|^2 dxdt \leq C_1 \int_0^t \int_\Omega \left(|\nabla\phi|^2 + |c|^2\right) dxdt.$$

By applying Gronwall's Lemma and using (48), we arrive at

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{50}$$

Therefore, it follows from (48)-(50) that $\mathcal{T}_\lambda$ is uniformly continuous with respect to λ on bounded sets of B.

Now we have to estimate the set of all fixed points of $\mathcal{T}_\lambda$, let $(\phi, \theta, c) \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) = \lambda\beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \text{ in } Q, \quad (51)$$

$$C_V\theta_t + \frac{l}{2}\phi_t = \nabla \cdot [K_1^\delta(\rho_\delta(\phi))\nabla\theta] \qquad \text{in } Q, \quad (52)$$

$$c_t - K_2\Delta c = K_2 M\nabla \cdot [c(1-c)\nabla(\rho_\delta(\phi))] \qquad \text{in } Q, \quad (53)$$

$$\frac{\partial\phi}{\partial n} = 0, \qquad \frac{\partial\theta}{\partial n} = 0, \qquad \frac{\partial c}{\partial n} = 0 \qquad \text{on } \partial\Omega \times (0,T), \quad (54)$$

$$\phi(0) = \phi_0^\delta, \qquad \theta(0) = \theta_0^\delta, \qquad c(0) = c_0^\delta \qquad \text{in } \Omega. \quad (55)$$

For this, we multiply the first equation (51) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over Ω. After integration by parts, using Hölder's and Young's inequalities, we obtain, respectively

$$\begin{aligned} \frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_\Omega |\phi|^2 dx &+ \int_\Omega \left(\epsilon^2|\nabla\phi|^2 + \frac{1}{4}\phi^4\right) dx \\ &\leq C_1 + C_2\int_\Omega \left(|\theta|^2 + |c|^2 + |\phi|^2\right) dx, \end{aligned} \quad (56)$$

$$\begin{aligned} \frac{\alpha\epsilon^2}{2}\int_\Omega |\phi_t|^2 dx &+ \frac{d}{dt}\int_\Omega \left(\frac{\epsilon^2}{2}|\nabla\phi|^2 + \frac{1}{8}\phi^4 - \frac{1}{4}|\phi|^2\right) dx \\ &\leq C_1 + C_2\int_\Omega \left(|\theta|^2 + |c|^2\right) dx, \end{aligned} \quad (57)$$

$$\begin{aligned} \frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_\Omega |\nabla\phi|^2 dx &+ \int_\Omega \frac{\epsilon^2}{2}|\Delta\phi|^2 dx \\ &\leq C_1 + C_2\int_\Omega \left(|\theta|^2 + |c|^2 + |\nabla\phi|^2\right) dx. \end{aligned} \quad (58)$$

By multiplying (52) by θ and (53) by c, arguments similar to the previous ones lead to the following estimates

$$\frac{d}{dt}\int_\Omega \frac{C_\nu}{2}|\theta|^2 dx + \delta\int_\Omega |\nabla\theta|^2 dx \leq \frac{\alpha^2\epsilon^4}{4}\int_\Omega |\phi_t|^2 dx + C_1\int_\Omega |\theta|^2 dx, \quad (59)$$

$$\frac{d}{dt}\int_\Omega |c|^2 dx + K_2\int_\Omega |\nabla c|^2 dx \;\;\leq\;\; C_1\int_\Omega |\nabla\phi|^2 dx, \tag{60}$$

where (20) was used to obtain the last inequality.

Now, multiply (57) by $\alpha\epsilon^2$ and add the result to (56), (58)- (60), to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{\alpha\epsilon^2}{4}|\phi|^2 + \left(\frac{\alpha\epsilon^2}{2}+\frac{\alpha\epsilon^4}{2}\right)|\nabla\phi|^2 + \frac{\alpha\epsilon^2}{8}\phi^4 + \frac{C_\nu}{2}|\theta|^2 + |c|^2\right) dx \\
&+ \int_\Omega \left(\epsilon^2|\nabla\phi|^2 + \frac{1}{4}\phi^4 + \frac{\alpha^2\epsilon^4}{4}|\phi_t|^2 + \frac{\epsilon^2}{2}|\Delta\phi|^2 + \delta|\nabla\theta|^2 + K_2|\nabla c|^2\right) dx \\
&\qquad\qquad \leq C_1 + C_2\int_\Omega \left(|\theta|^2 + |c|^2 + |\phi|^2 + |\nabla\phi|^2\right) dx.
\end{aligned} \tag{61}$$

Hence, the integration of (61) with respect t and the use of Gronwall's Lemma give us

$$||\phi||_{L^\infty(0,T;H^1(\Omega))} + ||\theta||_{L^\infty(0,T;L^2(\Omega))} + ||c||_{L^\infty(0,T;L^2(\Omega))} \leq C_1,$$

where C_1 is independent of λ and δ.

Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda \in [0, 1]$.

Finally, to verify the last assumption, observe that the equation $x - \mathcal{T}_0(x) = 0$ is equivalent to say that problem (15)-(19) for $\lambda = 0$ has a unique solution. This is concluded reasoning exactly as in the beginning of this proof, when we proved that $\mathcal{T}_\lambda$ was well defined.

Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $(\phi, \theta, c) \in B \cap W_2^{2,1}(Q) \times W_2^{2,1}(Q) \times C^{2,1}(Q)$ of the operator $\mathcal{T}_1$, i.e., $(\phi, \theta, c) = \mathcal{T}_1(\phi, \theta, c)$. This corresponds to a solution of problem (10)-(14) and the proof of Proposition 1 is thus complete. ∎

3 Proof of Theorem 1

To prove Theorem 1, we start by taking the initial condition in the previous regularized problem as follows.

For a sequence $\delta \to 0+$, we choose $\phi_0^\delta = \phi_0$ and pick two corresponding sequences $\theta_0^\delta \in H^{1+\gamma}(\Omega)$, γ as in Proposition 1, and $c_0^\delta \in C^1(\bar{\Omega})$ such that

$0 < c_0^\delta < 1$, $\dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ a.e. on $\partial\Omega$, and moreover $\theta_0^\delta \to \theta_0$ in $L^2(\Omega)$, $c_0^\delta \to c_0$ in $H^1(\Omega)$.

From Proposition 1, we know that there exist a sequence $(\phi^\delta, \theta^\delta, c^\delta)$ of corresponding solutions of problem (10)-(14). For such solutions, we will derive bounds, uniform with respect to δ; then, we will use compactness arguments to pass to the limit and establish the desired result.

Lemma 1 *There exists a constant C_1 such that, for any $\delta \in (0,1)$*

$$\begin{aligned} \|\phi^\delta\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \|\phi_t^\delta\|_{L^2(Q)} &\leq C_1 &(62)\\ \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))} + \int_0^T \int_\Omega K_1^\delta(\rho_\delta(\phi^\delta))|\nabla\theta^\delta|^2 dxdt &\leq C_1 &(63)\\ \|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} &\leq C_1. &(64)\end{aligned}$$

Proof: From inequality (61), it follows estimates (62), (64) and also

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1. \qquad (65)$$

By multiplying (11) by θ^δ and integrating over Q, we obtain

$$\int_\Omega |\theta^\delta|^2 dx + \int_0^T \int_\Omega K_1^\delta(\rho_\delta(\phi^\delta))|\nabla\theta^\delta|^2 dxdt \leq C_1 \int_0^T \int_\Omega \left(|\phi_t^\delta|^2 + |\theta^\delta|^2\right) dxdt.$$

In view of (62) and (65), this gives estimate (63). ∎

Lemma 2 *There exists a constant C_1 such that, for any $\delta \in (0,1)$*

$$\begin{aligned} \|\theta_t^\delta\|_{L^2(0,T;H^1(\Omega)')} &\leq C_1 &(66)\\ \|c_t^\delta\|_{L^2(0,T;H^1(\Omega)')} &\leq C_1. &(67)\end{aligned}$$

Proof: We take the scalar product in $L^2(\Omega)$ of (11) with $\eta \in H^1(\Omega)$. By using Hölder's inequality and **(B)**, we find

$$C_V \|\theta_t^\delta\|_{H^1(\Omega)'} \leq \left((b+1)\int_\Omega K_1^\delta(\rho_\delta(\phi^\delta))|\nabla\theta^\delta|^2 dx\right)^{1/2} + \frac{l}{2}\|\phi_t^\delta\|_{L^2(\Omega)}.$$

Then, (66) follows from (62)-(63). Estimate (67) can be similarly obtained by using (62) and (64). ∎

From Lemma 1 and Lemma 2, by using Aubin-Lions Lemma (see for instance Simon [19]) and the fact that $H^{2-\overline{\gamma}}(\Omega))$ is compactly immersed in $H^2(\Omega))$ for $0 < \overline{\gamma} \leq 1/2$, we conclude that there exist

$$
\begin{array}{lcl}
\phi & \in & L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega)) \ \text{ with } \phi_t \in L^2(Q), \\
\theta & \in & L^\infty(0,T;L^2(\Omega)) \ \text{ with } \theta_t \in L^2(0,T;H^1(\Omega)'), \\
c & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \ \text{ with } c_t \in L^2(0,T;H^1(\Omega)'), \\
J & \in & L^2(Q)
\end{array}
$$

and a subsequence, that to ease the notation we still denote $(\phi^\delta, \theta^\delta, c^\delta)$, satisfying as $\delta \to 0+$ that

$$
\begin{array}{lcll}
\phi^\delta & \to & \phi & \text{strongly in } L^2(0,T;H^{2-\overline{\gamma}}(\Omega)) \cap C([0,T];L^2(\Omega)), \\
 & & & 0 < \overline{\gamma} \leq 1/2, \\
\phi_t^\delta & \rightharpoonup & \phi_t & \text{weakly in } L^2(Q), \\
\theta^\delta & \to & \theta & \text{strongly in } C([0,T];H^1(\Omega)'), \\
\theta^\delta & \rightharpoonup & \theta & \text{weakly in } L^2(Q), \\
c^\delta & \to & c & \text{strongly in } L^2(Q) \cap C([0,T];H^1(\Omega)'), \\
c^\delta & \rightharpoonup & c & \text{weakly in } L^2(0,T;H^1(\Omega)), \\
K_1^\delta(\rho_\delta(\phi^\delta))\nabla\theta^\delta & \rightharpoonup & J & \text{weakly in } L^2(Q).
\end{array}
\tag{68}
$$

It now remains to identify J in terms of ϕ and θ and pass to the limit as δ approaches zero in (10)-(14).

It follows from (68) that we may pass to the limit in (10) and find that (6) holds almost everywhere.

We proceed similarly as in Laurençot [11]. Since K_1 is a Lipschitz continuous function, the Nemitsky operator associated to K_1 is continuous and bounded from $H^1(\Omega)$ in itself (see the statement of these results for instance in Kavian [9] Thm 16.7; the corresponding proofs can be found in Marcus and Mizel [14], Thm 1, and [13], Thm. 1). This and the fact that $\rho_\delta(\phi^\delta)$ converges to ϕ in $L^2(0,T;H^1(\Omega))$ imply that

$$K_1(\rho_\delta(\phi^\delta)) \to K_1(\phi) \text{ strongly in } L^2(0,T;H^1(\Omega)). \tag{69}$$

From (68)-(69), we conclude that

$$
\begin{array}{rcll}
\theta^\delta\nabla K_1(\rho_\delta(\phi^\delta)) & \rightharpoonup & \theta\nabla K_1(\phi) & \text{weakly in } \quad L^1(Q), \\
K_1(\rho_\delta(\phi^\delta))\theta^\delta & \rightharpoonup & K_1(\phi)\theta^\delta & \text{weakly in } \quad L^1(Q).
\end{array}
\tag{70}
$$

Also, since $K_1(\rho_\delta(\phi^\delta)) \in L^\infty(0,T;H^1(\Omega))$ and $\theta^\delta \in L^2(0,T;H^1(\Omega))$, we have

$$K_1(\rho_\delta(\phi^\delta))\theta^\delta \in L^2(0,T;W^{1,p}(\Omega)) \quad \text{for } p = \min\left\{2, \frac{N}{N-1}\right\}$$

and

$$\nabla \left(K_1(\rho_\delta(\phi^\delta))\theta^\delta \right) = K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta + \theta^\delta \nabla K_1(\rho_\delta(\phi^\delta)).$$

It then follows from (70) that

$$K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta \rightharpoonup \nabla\left(K_1(\phi)\theta\right) - \theta\nabla K_1(\phi) \text{ in } \mathcal{D}'(Q). \tag{71}$$

Since K_1 is nonnegative, the definition of K_1^δ and (63) yield that $\|\delta^{\frac{1}{2}}\nabla\theta^\delta\|_{L^2(Q)} \leq C$, with C independent of δ. Thus,

$$\delta\nabla\theta^\delta \to 0 \text{ in } L^2(Q). \tag{72}$$

From (68), (71) and (72), we conclude that

$$J = \nabla\left(K_1(\phi)\theta\right) - \theta\nabla K_1(\phi).$$

Moreover, we may pass to the limit in a weak sense in (11) and obtain (8).

In order to pass to the limit in (12), we take scalar product in $L^2(\Omega)$ of it with $\eta \in L^2(0,T;H^1(\Omega))$, to obtain

$$\begin{aligned}\int_0^T \int_\Omega c_t^\delta \eta\, dxdt \;&+\; K_2\int_0^T \int_\Omega \nabla c^\delta \cdot \nabla\eta\, dxdt \\ &+\; K_2 M\int_0^T \int_\Omega c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)) \cdot \nabla\eta\, dxdt = 0.\end{aligned}$$

Then, from (68), we have that

$$\int_0^T \langle c_t, \eta\rangle dxdt + K_2\int_0^T\int_\Omega \nabla c\cdot\nabla\eta\, dxdt + K_2 M\int_0^T\int_\Omega c(1-c)\nabla\phi\cdot\nabla\eta\, dxdt = 0$$

holds for any $\eta \in L^2(0,T;H^1(\Omega))$.

Moreover, since $0 < c^\delta < 1$ and c^δ converges to c in $L^2(Q)$, we have that $0 \leq c \leq 1$ a.e. in Q.

Finally, it follows from (68) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$.

The proof of Theorem 1 is then complete. ∎

Remarks

1. From the L^p-theory of parabolic equations, it is easy to conclude that $c \in W^{2,1}_{3/2}(Q)$, and therefore the equation for c holds almost everywhere.

2. The result stated in Theorem 1 still holds, exactly with the same proof, for an initial condition ϕ_0 in any functional space including $H^1(\Omega)$ and for which it makes sense to require that $\dfrac{\partial \phi_0}{\partial n} = 0$ a.e. on $\partial\Omega$. Moreover, a weaker version of the same theorem holds, with a natural weaker formulation of (6)-(7), for initial conditions ϕ_0 just in $H^1(\Omega)$. For the proof, it is enough to adapt that of Section 3 by taking a sequence $\phi_0^\delta \in H^{1+\gamma}(\Omega)$, such that $\dfrac{\partial \phi_0^\delta}{\partial n} = 0$ a.e. on $\partial\Omega$ and ϕ_0^δ converges to ϕ_0 in $H^1(\Omega)$ as $\delta \to 0+$.

References

[1] Caginalp, G., 'An analysis of a phase field model of a free boundary', *Arch. Rational Mech. Anal.*, **92**, 205-245 (1986).

[2] Caginalp, G. and Jones, J., 'A derivation and analysis of phase-field models of thermal alloys', *Annals of Phy.*, **237**, 66-107 (1995).

[3] Caginalp, G. and Xie, W., 'Phase-field and sharp-interfase alloys models', *Phys. Rev. E*, **48**(3), 1897-1909 (1993).

[4] Colli, P., Laurençot, Ph., 'Weak solution to the Penrose-Fife phase field model for a class of admissible heat flux laws', *Physica D* **111**, 311-334 (1998).

[5] Colli, P., Sprekels, J.,'Weak solution to some Penrose-Fife phase-field systems with temperature-dependent memory', *J. Diff. Eq.* **142**(1), 54-77 (1998).

[6] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[7] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math, Vol 840, Springer-Verlag, 1981.

[8] Hoffman, K-H. and Jiang, L., 'Optimal control of a phase field model for solidification', *Numer. Funct. Anal. and Optim.*, **13**, 11-27 (1992).

[9] Kavian, O., *Introduction à la Théorie des Points Critiques*, Mathématiques et Applications 13, Springer-Verlag, 1993.

[10] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[11] Laurençot, Ph., 'Weak solutions to a phase-field model with non-constant thermal conductivity', *Quart. Appl. Math.*, **15**(4), 739-760 (1997).

[12] Lions, J.L., *Control of Distributed Singular Systems*, Gauther-Villars, 1985.

[13] Marcus, M. and Mizel, V.J., 'Complete Characterization of Functions which act, via Superposition, on Sobolev Spaces', *Trans. American Math. Soc.*, **251**(July), 187-218 (1979).

[14] Marcus, M. and Mizel, V.J., 'Every Superposition Operator Mapping One Sobolev Space into Another Is Continuous', *J. Func. Anal*, **33**(2), 217-229 (1979).

[15] Mikhailov, V.P., *Partial Differential Equations*, Mir, 1978.

[16] Moroşanu, C. and Motreanu, D., 'A generalized phase-field system', *J. Math. Anal. Appl.*, **237**, 515-540 (1999).

[17] Penrose,O., Fife, P.C., 'Thermodynamically consistent model of phase-field type for the kinetics of phase transitions', *Physica D* **43**, 44-62 (1990).

[18] Rappaz, J. and Scheid, J.F., 'Existence of solutions to a Phase-field model for the isothermal solidification process of a binary alloy', *Math. Meth. Appl. Sci.*, **23**, 491-512 (2000).

[19] Simon, J., 'Compacts sets in the space $L^p(0, T, B)$', *Ann. Mat. Pura Appl.*, **146**, 65-96 (1987).

[20] Warren, J.A. and Boettinger, W.J., 'Prediction of dendritic growth and microsegregation patterns in a binary alloy using the phase-field method', *Acta Metall. Mater.*, **43**(2), 689-703 (1995).

[21] Wheeler, A.A., Boettinger, W.J. and McFadden, G.B., 'Phase-field model for isothermal phase transitions in binary alloys', *Phys. Rev. A*, **45**, 7424-7439 (1992).

Capítulo 3

Soluções fracas de um modelo do tipo campo de fase com convecção para a solidificação de uma liga binária

Resumo

A metodologia de campo de fase tem atingido nos últimos anos considerável importância na modelagem e simulação numérica dos processos de mudança de fase e de estruturas complexas de crescimento que ocorrem no processo de solidificação. Na tentativa de compreender os aspectos matemáticos de tal metodologia, considera-se neste trabalho um modelo de evolução para um processo de solidificação/fusão de uma liga binária com propriedades térmicas. O modelo inclui a possibilidade da ocorrência natural de convecção nas regiões não solidificadas e, assim, nos conduz a um sistema de equações diferenciais parciais altamente não linear, consistindo da equação do campo de fase, a equação do calor, a equação da concentração e as equações de Navier-Stokes modificadas por um termo de penalização do tipo Carman-Kozeny, o qual toma conta do efeito *mushy*.

É provada a existência de soluções fracas para o sistema. Primeiro o problema é aproximado e uma sequência de soluções aproximadas é obtida usando o Teorema de Ponto Fixo de Leray-Schauder. Então é mostrado que o limite desta sequência é uma solução fraca do problema usando argumentos de compacidade.

Weak Solutions of a Phase-Field Model with Convection for Solidification of an Alloy

Gabriela Planas *
gplanas@ime.unicamp.br

José Luiz Boldrini †
boldrini@ime.unicamp.br

UNICAMP-IMECC, CP 6065
13083-970 Campinas, SP, Brazil

*Supported by FAPESP, grant 97/13932-4
†Partially supported by CNPq, Brazil, grant 300513/87-9

Weak Solutions of a Phase-Field Model with Convection for Solidification of an Alloy

MSC Mathematics Subject Classification: 35K65, 76D05, 80A22, 35K55, 82B26, 35Q10, 76R99

Abstract

In recent years, the phase-field methodology has achieved considerable importance in modeling and numerically simulating a range of phase transitions and complex growth structures that occur during solidification processes. In attempt to understand the mathematical aspects of such methodology, in this article we consider a simplified model of this sort for a nonstationary process of solidification/melting of a binary alloy with thermal properties. The model includes the possibility of occurrence of natural convection in non-solidified regions and, therefore, leads to a free-boundary value problem for a highly non-linear system of partial differential equations consisting of a phase-field equation, a heat equation, a concentration equation and a modified Navier-Stokes equations by a penalization term of Carman-Kozeny type, which accounts for the mushy effects, and Boussinesq terms to take in consideration the effects of variations of temperature and concentration in the flow.

A proof of existence of weak solutions for the system is given. The problem is firstly approximated and a sequence of approximate solutions is obtained by Leray-Schauder's fixed point theorem. A solution of the original problem is then found by using compactness arguments.

1 Introduction

In recent years, the phase-field methodology, which is an alternative formulation both to the sharp-interface methodology (Stefan type approach) or to the enthalpy methodology, has achieved considerable importance in modeling and numerically simulating a range of phase transitions and complex growth structures that occur during solidification. This has spurred many articles using this approach and proposing several mathematical models consisting of highly nonlinear systems of partial differential equations.

Rigorous mathematical analysis is in general difficult, but for pure materials undergoing phase change, several authors have undertaking the task. Examples of this sort of analysis are [4, 14, 16, 20], where existence and uniqueness results are investigated for various types of non-linearities.

Several phase-field models have also been developed for binary alloys. One of the first works in this direction was due to Wheeler et al. [30] and was concerned with isothermal solidification. Warren and Boettinger [29] extended this model, while recently Rappaz and Scheid [22] investigated the well-posedness under suitable assumptions for the non-linearities. Caginalp et al. [6, 5] extended this kind of model by including temperatures changes. For such model, the governing equations for the phase-field and the concentration are derived from a free energy functional; then an appropriate balance equation for the temperature, accounting for the liberation of latent heat by addition of a term proportional to the time derivative of the phase-field, is added to complete the model. The existence of weak solutions for this model was recently studied in [3].

The previously mentioned phase-field models do not consider the possibility of flow of the non-solidified material. However, there are many cases where such flows do occur and are significant, having important effects on the outcome of the solidification process. From the mathematical point of view, the inclusion of such effects in the model brings another order of difficulty to the analysis, whatever the approach used for modeling phase change. For instance [7, 8, 9, 10, 21, 24] consider several mathematical aspects of the interplay between fluid motion and phase change for pure material; the first four of these papers used the Stefan approach, while the last two used the enthalpy approach.

Voller et al. [26, 27] proposed models using the enthalpy technique for a convection/diffusion phase change process by including in the model a modification of the Navier-Stokes equations by the inclusion of a certain term that takes in consideration the flow in mushy regions. Particular expressions for this term may be obtained by modeling the mushy region as porous medium. Another model of this type was proposed by Voss and Tsai [28]. In Blanc et al. [1] performed a rigorous mathematical analysis of a stationary model for the solidification process with convection of a binary alloy. The model in [1] used an enthalpy approach and, as suggested in Voller et al. [27], a Carman-Kozeny penalization term was added to the Navier-Stokes equations to model the flow in mushy regions. Other authors have proposed models using the phase-field method for solidification process of binary alloys in presence of convection. For instance, Beckermann et al. [2] and Diepers et al. [11] proposed models of this sort using arguments of mixture theory. They also presented numerical simulations to validate their models.

In this paper we are interested in the rigorous mathematical analysis of a phase-field type model for a non-stationary solidification process of a binary alloy, with the possibility of flow of the non-solid phase. Differently of models in [2] and [11], the model we consider here combines ideas of Voller et al. [27] and of Blanc et al. [1] to model the possibility of flow with those of Caginalp et al. [6] for the phase-field and the thermal properties of the alloy. Our system of equations will described in detail in the next section; here we just observe that our system includes the Navier-Stokes equations with a Carman-Kozeny type term as described above, and also a Boussinesq type term to take in consideration buoyancy forces due to thermal and concentration effects. Since these equations for the flow only hold in an a priori unknown non-solid region, the model corresponds to a free boundary value problem. Moreover, since the Carman-Kozeny term is dependent on the local solid fraction, this is assumed to be functionally related to the the phase-field.

Our objective is to present a result on existence of weak solutions for this mathematical model. The proof will be based on a regularization technique that combines ideas already used in [1] and [3]: an auxiliary positive parameter will be introduced in the equations in such way that the original free boundary value problem will be transformed in a more standard (penalized) problem. We say that this transformed problem is the regularized problem. By solving it, one hopes to recover a solution of the original problem as the parameter approaches zero. To accomplish such program, we will firstly solve the regularized problem by using the Faedo-Galerkin method, just in the modified Navier-Stokes equations, and the Leray-Schauder fixed point theorem. Then, by taking a sequence of values of the parameter approaching zero, we will have a sequence of approximate solutions. By obtaining suitable estimates for this sequence, we will then be able to take the limit along a subsequence and, by compactness arguments, to show that we have a solution of the original problem.

The paper is organized as follows. In Section 2 we describe the mathematical model and its variables; we fix the notation and describe the functional spaces to be used; we also state our technical hypotheses and main result. In Section 3 we introduce and analyze the regularized problem. Section 4 is dedicated to the proof of the existence of weak solutions of the original free boundary value problem.

2 The model and main result

Let $0 < T < +\infty$ and Ω be an open bounded domain in $I\!R^N$, $N = 2 \,\text{or}\, 3$, with smooth boundary $\partial\Omega$ (of class C^3 will be enough for our purposes). Being $Q = \Omega \times (0,T)$, we will consider the following system of equations:

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \quad \text{in } Q, \tag{1}$$

$$v_t - \nu\Delta v + \nabla p + v\cdot\nabla v + k(f_s(\phi))v = \mathcal{F}(c,\theta) \quad \text{in } Q_{ml}, \tag{2}$$

$$\text{div } v = 0 \quad \text{in } Q_{ml}, \tag{3}$$

$$v = 0 \quad \text{in } Q_s, \tag{4}$$

$$C_v\theta_t + C_v v\cdot\nabla\theta = \nabla\cdot[K_1(\phi)\nabla\theta] + \frac{l}{2}f_s(\phi)_t \quad \text{in } Q, \tag{5}$$

$$c_t + v.\nabla c = K_2\left(\Delta c + M\nabla\cdot[c(1-c)\nabla\phi]\right) \quad \text{in } Q, \tag{6}$$

$$\frac{\partial\phi}{\partial n} = \frac{\partial\theta}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \qquad v = 0 \quad \text{on } \partial Q_{ml}, \tag{7}$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad c(0) = c_0 \quad \text{in } \Omega, \quad v(0) = v_0 \quad \text{in } \Omega_{ml}(0), \tag{8}$$

Here, ϕ is the phase-field variable (sometimes called order parameter), which is the state variable characterizing the different phases; v is the velocity field; p is the associated hydrostatic pressure; $f_s \in [0,1]$ is the solid fraction; θ is the temperature; $c \in [0,1]$ is the concentration of the solute (i.e., the fraction of one of the two materials in the mixture).

We recall that the phase-field methodology in its simplest approach assumes the existence of two real numbers $\phi_s < \phi_l$ and a order parameter (phase-field) $\phi(x,t)$, depending on the spatial variable x and time t, such that if $\phi(x,t) \leq \phi_s$ then the material at point x at time t is in solid state; if $\phi_l \leq \phi(x,t)$ the material at point x at time t is in liquid state; if $\phi_s < \phi(x,t) < \phi_l$ then, at time t the point x is in the mushy region (a region of microscopic mixture of solid and liquid). This setting must be physically coherent with the concept of solid fraction, which we assume to be functionally dependent on the phase-field. This requires that $f_s(z)$ be a function such that $f_s(z) = 1$ for $z \leq \phi_s$, $f_s(z) = 0$ for $z \geq \phi_l$, and $0 < f_s(z) < 1$ for $\phi_s < z < \phi_l$. The required regularity assumptions on f_s will be described later on.

In the first of the previous equations (the phase-field equation), $\alpha > 0$ is the relaxation scaling; $\beta = \epsilon[s]/3\sigma$ where $\epsilon > 0$ is a measure of the interface width; σ the surface tension, and $[s]$ is the entropy density difference between phases. θ_A and θ_B are the melting temperatures of the two materials composing the binary alloy.

In the second of the previous equations, $\nu > 0$ is the viscosity, assumed to be constant. The penalization term $k(f_s)$ accounts for the mushy effect in the flow. The original Carman-Kozeny expression for it is $k(x) = C_0 x^2/(1-x)^3$; however, we will consider more general expressions for this term. The term $\mathcal{F}(c,\theta)$ is the buoyancy force, which by using Boussinesq approximation can be expressed as $\mathcal{F}(c,\theta) = \rho \mathrm{g}\,(c_1(\theta-\theta_r) + c_2(c-c_r)) + F$, where ρ is the mean value of the density (which for simplicity we will assume to be a positive constant); g is the acceleration of gravity (for simplicity also assumed to be constant); c_1 and c_2 are two constants; θ_r, c_r are respectively the reference temperature and concentration (again for simplicity of exposition, both will be assumed to be zero), and F is an external force field.

In the equation for the temperature, $C_\mathrm{v} > 0$ is the specific heat (constant); l is a positive constant associated to the latent heat. We also observe that this equation comes from the balance of the internal energy that in this case has the form $e = C_\mathrm{v}\theta + \dfrac{l}{2}(1-f_s)$, where $1 - f_s$ is the liquid fraction. The thermal conductivity $K_1 > 0$ is assumed to depend on the phase-field.

In the last equation, $K_2 > 0$ is the solute diffusivity and M is a constant related to the slopes of solidus and liquidus lines.

The domain Q is composed of three regions, Q_s, Q_m and Q_l. The first one corresponds to the fully solid region; the second one corresponds to the mushy region, while the third is fully liquid region. They are defined by

$$\begin{array}{rcl} Q_s & = & \{(x,t) \in Q \ : \ f_s(\phi(x,t)) = 1\}, \\ Q_m & = & \{(x,t) \in Q \ : \ 0 < f_s(\phi(x,t)) < 1\}, \\ Q_l & = & \{(x,t) \in Q \ : \ f_s(\phi(x,t)) = 0\}. \end{array} \tag{9}$$

Q_{ml} will refer to the non-solid region, i.e.,

$$Q_{ml} = Q_m \cup Q_l = \{(x,t) \in Q \ : \ 0 \leq f_s(\phi(x,t)) < 1\}. \tag{10}$$

We also define the subsets of Ω associated respectively to the solid and non-solid regions at time $t \in (0,T]$

$$\begin{array}{c} \Omega_s(t) = \{x \in \Omega \ : \ f_s(\phi(x,t)) = 1\}, \\ \Omega_{ml}(t) = \{x \in \Omega \ : \ 0 \leq f_s(\phi(x,t)) < 1\}. \end{array} \tag{11}$$

Observe that as we said above, all these previously described regions are a priori unknown, the model corresponds to a free boundary value problem.

Throughout this paper we will assume the following assumptions:

(H1) k is nondecreasing function of class $C^1[0,1)$ satisfying $k(0) = 0$ and $\lim\limits_{x \to 1^-} k(x) = +\infty$,

(H2) f_s depends only on the phase field and is a Lipschitz continuous function defined on $I\!R$ and satisfying $0 \leq f_s(r) \leq 1$ for $r \in I\!R$ with f_s' measurable,

(H3) K_1 depends only on the phase-field and is a Lipschitz continuous function defined on $I\!R$; moreover, there exist $a > 0$ and $b > 0$ such that

$$0 < a \leq K_1(r) \leq b \qquad \text{for all } r \in I\!R,$$

(H4) F is a given function in $L^2(Q)$.

We remark that the concentration equation as it is written in [6] (up to addition of a proper convection term) is the following:

$$c_t + v.\nabla c = K_2 \nabla \cdot \left[c(1-c) \nabla \left(M\phi + \ \ln \frac{c}{1-c} \right) \right] \ \text{ in } Q.$$

This form of the equation forces $c \in (0,1)$ and is equivalent to equation (6) in this case. Thus, (6) is more general than this last form since it allows c to assume the values 0 and 1, which are associated to regions of pure materials.

We use standard notation in this paper. We just briefly recall the following functional spaces associated to the Navier-Stokes equations. Let $G \subseteq I\!R^N$ be a non-void bounded open set; for $T > 0$, consider also $Q_G = G \times (0, T)$ Then,

$$\begin{array}{rcl} \mathcal{V}(G) & = & \left\{ w \in (C_0^\infty(G))^N, \ \mathrm{div}\ w = 0 \right\}, \\ H(G) & = & \text{closure of } \mathcal{V}(G) \text{ in } \left(L^2(G)\right)^N, \\ V(G) & = & \text{closure of } \mathcal{V}(G) \text{ in } \left(H_0^1(G)\right)^N, \\ H^{\tau,\tau/2}(\overline{Q}_G) & = & \text{Hölder continuous functions of exponent } \tau \text{ in } x \\ & & \text{and exponent } \tau/2 \text{ in } t, \\ W_q^{2,1}(Q_G) & = & \{ w \in L^q(Q_G) /\ D_x w, D_x^2 w \in L^q(Q_G), w_t \in L^q(Q_G) \}. \end{array}$$

When $G = \Omega$, we denote $H = H(\Omega)$, $V = V(\Omega)$. Properties of these functional spaces can be found for instance in [15, 25]. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. We also put $(\cdot, \cdot) = (\cdot, \cdot)_\Omega$ the inner product of $\left(L^2(\Omega)\right)^N$.

The main result of this paper is the following.

Theorem 1 *Let be given $T > 0$, $\Omega \subset I\!R^N$, $N = 2$, or 3, a bounded open domain of class C^3, and assume that* **(H1)-(H4)** *hold. Let also be given $(N+2)/2 < q \leq 2(N+2)/N$, $\phi_0 \in W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $1/2 < \gamma \leq 1$, satisfying the compatibility condition $\dfrac{\partial \phi_0}{\partial n} = 0$ on $\partial\Omega$, $v_0 \in H(\Omega_{ml}(0))$, $\theta_0 \in L^2(\Omega)$, and $c_0 \in L^2(\Omega)$ satisfying $0 \leq c_0 \leq 1$ a.e. in Ω. Then, there exist functions (ϕ, v, θ, c) satisfying:*

(i) $\phi \in W_q^{2,1}(Q)$, $\phi(0) = \phi_0$,

(ii) $v \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $v = 0$ *a.e. in* $\mathring{Q}_s$, $v(0) = v_0$ *in* $\Omega_{ml}(0)$, *where Q_s is defined by (9) and $\Omega_{ml}(0)$ by (11),*

(iii) $\theta \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, $\theta(0) = \theta_0$,

(iv) $c \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, $c(0) = c_0$, $0 \leq c \leq 1$ *a.e. in* Q

Moreover, they satisfy

$$\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \quad a.e.\ in\ Q, \tag{12}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad a.e.\ on\ \partial\Omega \times (0, T), \tag{13}$$

$$
\begin{aligned}
(v(t),\eta(t))_{\Omega_{ml}(t)} - \int_0^t (v,\eta_t)_{\Omega_{ml}(s)} ds + \nu \int_0^t (\nabla v, \nabla \eta)_{\Omega_{ml}(s)} ds \\
+ \int_0^t (v\cdot\nabla v, \eta)_{\Omega_{ml}(s)} ds + \int_0^t (k(f_s(\phi))v,\eta)_{\Omega_{ml}(s)} ds \\
= \int_0^t (\mathcal{F}(c,\theta),\eta)_{\Omega_{ml}(s)} ds + (v_0,\eta(0))_{\Omega_{ml}(0)},
\end{aligned} \tag{14}
$$

for $t \in (0,T]$ and any $\eta \in L^2(0,T;V(\Omega_{ml}(t)))$ with compact support contained in $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and $\eta_t \in L^2(0,T;V(\Omega_{ml}(t))')$ where Q_{ml} is defined by (10) and $\Omega_{ml}(t)$ by (11),

$$
\begin{aligned}
-C_v \int_\Omega \theta_0 \xi(0) dx - C_v \int_0^T \int_\Omega \theta \xi_t dx dt - C_v \int_0^T \int_\Omega v\theta\cdot\nabla\xi \, dx dt \\
+ \int_0^T \int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\xi \, dx dt = \frac{l}{2}\int_0^T \int_\Omega f_s(\phi)_t \xi \, dx dt
\end{aligned} \tag{15}
$$

for any $\xi \in L^4(0,T;H^1(\Omega))$ with $\xi_t \in L^2(Q)$ and $\xi(T) = 0$ in Ω,

$$
\begin{aligned}
-\int_0^T \int_\Omega c\zeta_t dx dt - \int_0^T \int_\Omega vc\cdot\nabla\zeta \, dx dt + K_2 \int_0^T \int_\Omega \nabla c\cdot\nabla\zeta \, dx dt \\
+K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi\cdot\nabla\zeta \, dx dt = \int_\Omega c_0\zeta(0) dx,
\end{aligned} \tag{16}
$$

for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(Q)$ and $\zeta(T) = 0$ in Ω.

Remarks:

1. The restriction $q > N + 2/2$ ensures the continuity of phase-field; in fact, in this case $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$, for $\tau = 2 - (N+2)/q$ ([15] p. 80). Therefore, the set Q_{ml} is open, and we have a suitable interpretation for the equations of velocity field. The restriction $q \leq 2(N+2)/N$ is consequence of the obtained regularity of the temperature. (iii) implies that $\theta \in L^{2(N+2)/N}(Q)$, and then, from the existence theorem for the phase-field equation given in ([14] Thm 2.1), we know that $\phi \in W^{2,1}_{2(N+2)/N}(Q)$.

2. We observe that the phase-field models without convection studied in [3] or [16] allow the thermal conductivity K_1 to vanish. In the presence of convection, we were not able to prove the existence of global weak solutions in this degenerate case; thus, we had to assume the more restrictive assumption **(H3)**. It is possible, however, to prove the existence of a slightly different local weak solution of (1)-(8) in the degenerate case. This will be done elsewhere.

3 A regularized problem

In this section we introduce an auxiliary regularized problem by performing suitable modifications of the original equations. The first objective of these modifications is to introduce coefficients ensuring enough regularity for the arguments to be used. The second objective, as in Blanc [1], is to change the modified Navier-Stokes equations in such way that it holds in the whole domain instead of holding just in an a priori unknown region.

The proof of existence of solutions for such regularized problem will be done by using Faedo-Galerkin method, with the help of the Leray-Schauder Fixed Point Theorem as stated in ([12], p. 189):

Theorem (Leray-Schauder): *Consider a transformation $y = \mathcal{T}_\lambda(x)$ where x, y belong to a Banach space B and λ is a real parameter which varies in a bounded interval, say $0 \le \lambda \le 1$. Assume:*

(a) $\mathcal{T}_\lambda(x)$ is defined for all $x \in B$, $0 \le \lambda \le 1$,

(b) for any fixed λ, $\mathcal{T}_\lambda(x)$ is continuous in B,

(c) for x in bounded sets of B, $\mathcal{T}_\lambda(x)$ is uniformly continuous in λ,

(d) for any fixed $\lambda, \mathcal{T}_\lambda(x)$ is a compact transformation,

(e) there exists a (finite) constant M such that every possible solution x of $\mathcal{T}_\lambda(x) = x$ satisfies: $\|x\|_B \le M$,

(f) the equation $\mathcal{T}_0(x) = x$ has a unique solution in B.

Under assumptions (a)-(f), there exists a solution of the equation $x - \mathcal{T}_1(x) = 0$.

Now, we recall certain results that will be helpful in the introduction of such regularized problem.

Recall that there is an extension operator $Ext(\cdot)$ taking any function w in the space $W_2^{2,1}(Q)$ and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^{N+1})$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^{N+1})} \le C\,\|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [19] p. 157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^{N+1})$ be a family of symmetric positive mollifier functions with compact support converging to the Dirac delta function (we can take the support of ρ_δ contained in the ball of radius δ), and denote by $*$ the convolution operation. Then, given a function $w \in W_2^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(\mathbb{R}^{N+1})$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

This sort of regularization will be used with the phase-field variable. We will also need a regularization for the velocity, and for it we proceed as follows.

Given $v \in L^2(0,T;V)$, first we extend it as zero in $\mathbb{R}^{N+1}\backslash Q$. Then, as in [19] p. 157, by using reflection and cutting-off, we extend the resulting function to another one defined on $\mathbb{R}^{N+1}$ and with compact support. Without the danger of confusion, we again denote such extension operator by $Ext(v)$. Then, being $\delta > 0$, ρ_δ and $*$ as above, operating on each component, we can again define a regularization $\rho_\delta(v) \in C_0^\infty(\mathbb{R}^{N+1})$ of v by

$$\rho_\delta(v) = \rho_\delta * Ext(v).$$

Besides having properties of control of Sobolev norms in terms of the corresponding norms of the original function (exactly as above), such extension has the property described below.

For $0 < \delta \leq 1$, define firstly the following family of uniformly bounded open sets

$$\Omega^\delta = \{x \in \mathbb{R}^N : d(x,\Omega) < \delta\}. \tag{17}$$

We also define the associated space-time cylinder

$$Q^\delta = \Omega^\delta \times (0,T). \tag{18}$$

Obviously, for any $0 < \delta_1 < \delta_2$, we have $\Omega \subset \Omega^{\delta_1} \subset \Omega^{\delta_2}$, $Q \subset Q^{\delta_1} \subset Q^{\delta_2}$. Also, by using properties of convolution, we conclude that $\rho_\delta(v)|_{\partial\Omega^\delta} = 0$. In particular, for $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$, we conclude that $\rho_\delta(v) \in L^\infty(0,T;H(\Omega^\delta)) \cap L^2(0,T;V(\Omega^\delta))$.

Moreover, since Ω is of class C^3, there exists $\delta(\Omega) > 0$ such that for $0 < \delta \leq \delta(\Omega)$, we conclude that Ω^δ is of class C^2 and such that the C^2 norms of the maps defining $\partial\Omega^\delta$ are uniformly estimated with respect to δ in terms of the C^3 norms of the maps defining $\partial\Omega$.

Since we will be working with the sets Ω^δ, the main objective of this last remark is to ensure that the constants associated to Sobolev immersions and interpolations inequalities, involving just up to second order derivatives and used with Ω^δ, are uniformly bounded for $0 < \delta \leq \delta(\Omega)$. This will be very important to guarantee that certain estimates will be independent of δ.

Finally, let f_s^δ be any regularization of f_s.

Now, we are in position to define the regularized problem. Let $\delta(\Omega)$ be as described after (17); for each $\delta \in (0, \delta(\Omega)]$, we consider the system

$$\alpha\epsilon^2\phi_t^\delta - \epsilon^2\Delta\phi^\delta - \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) = \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q^\delta, \quad (19)$$

$$\begin{aligned} \frac{d}{dt}(v^\delta, u) \;&+\; \nu(\nabla v^\delta, \nabla u) + (v^\delta \cdot \nabla v^\delta, u) + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, u) \\ &= \; (\mathcal{F}(c^\delta, \theta^\delta), u) \quad \text{ for all } u \in V,\; t \in (0, T), \end{aligned} \quad (20)$$

$$C_v\theta_t^\delta + C_v\rho_\delta(v^\delta)\cdot\nabla\theta^\delta = \nabla\cdot\left(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\right) + \frac{l}{2}f_s^\delta(\phi^\delta)_t \quad \text{in } Q^\delta, \quad (21)$$

$$c_t^\delta - K_2\Delta c^\delta + \rho_\delta(v^\delta)\cdot\nabla c^\delta = K_2M\nabla\cdot\left(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\right) \quad \text{in } Q^\delta, \quad (22)$$

$$\frac{\partial\phi^\delta}{\partial n} = 0, \qquad \frac{\partial\theta^\delta}{\partial n} = 0, \qquad \frac{\partial c^\delta}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0, T), \quad (23)$$

$$v^\delta(0) = v_0^\delta \text{ in } \Omega, \quad \phi^\delta(0) = \phi_0^\delta, \quad \theta^\delta(0) = \theta_0^\delta, \quad c^\delta(0) = c_0^\delta \text{ in } \Omega^\delta. \quad (24)$$

Concerning this system we will prove the following existence result.

Proposition 1 *Let $T > 0$, $\delta(\Omega) > 0$ be as described following (17), and $1/2 < \gamma \le 1$. For each $\delta \in (0, \delta(\Omega)]$, consider $\phi_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta \in H$, $\theta_0^\delta \in H^{1+\gamma}(\Omega^\delta)$ and $c_0^\delta \in C^1(\overline{\Omega^\delta})$ satisfying the compatibility conditions $\dfrac{\partial\phi_0^\delta}{\partial n} = \dfrac{\partial\theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$ and $0 \le c_0^\delta \le 1$ in $\overline{\Omega^\delta}$. Assume also that* **(H1)-(H4)** *hold. Then, there exist a solution $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ of (19)-(24) satisfying*

i) $\phi^\delta \in L^2(0, T; H^2(\Omega^\delta))$, $\phi_t^\delta \in L^2(Q^\delta)$,

ii) $v^\delta \in L^2(0, T; V) \cap L^\infty(0, T; H)$, $v_t^\delta \in L^2(0, T; V')$,

iii) $\theta^\delta \in L^2(0, T; H^2(\Omega^\delta))$, $\theta_t^\delta \in L^2(Q^\delta)$,

iv) $c^\delta \in C^{2,1}(Q^\delta)$, $\quad 0 \le c^\delta \le 1$.

The proof of this proposition will depend on an another existence result for other approximate problem, obtained from (19)-(24) by discretizing just the modified Navier-Stokes equations using Faedo-Galekin method. By solving this approximate problem, we will recover the solution of the regularized problem as the discretization dimension m increases to $+\infty$.

For this purpose, first we introduce the spaces V_s,

$$V_s = \text{ the closure of } \mathcal{V}(\Omega) \text{ in } (H^s(\Omega))^N, \ s \geq 1,$$

endowed with the usual Hilbert scalar product

$$((u,v))_s = \sum_{i=1}^{N} (u_i, v_i)_{H^s(\Omega)}.$$

We also consider the spectral problem:

$$((u,v))_s = \lambda(u,v) \quad \text{ for all } v \in V_s \text{ and } s = \frac{N}{2},$$

which admits a sequence of solutions w_j corresponding to the sequence of eigenvalues $\lambda_j > 0$.

With the help of these eigenfunctions, we define the following approximate problem of order m: find $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$, with

$$v^\delta_m(t) = \sum_{j=1}^{m} g^\delta_{jm}(t) w_j \in V_m = \operatorname{span}\{w_1, \ldots, w_m\},$$

such that

$$\alpha\epsilon^2 \phi^\delta_{mt} - \epsilon^2 \Delta\phi^\delta_m - \frac{1}{2}(\phi^\delta_m - ({\phi^\delta_m}^3) = \beta\left(\theta^\delta_m + (\theta_B - \theta_A)c^\delta_m - \theta_B\right) \text{ in } Q^\delta, \quad (25)$$

$$\begin{aligned} \frac{d}{dt}(v^\delta_m, w_j) \ &+ \ \nu(\nabla v^\delta_m, \nabla w_j) + (v^\delta_m \cdot \nabla v^\delta_m, w_j) + (k(f^\delta_s(\phi^\delta_m) - \delta)v^\delta_m, w_j) \\ &= \ (\mathcal{F}(c^\delta_m, \theta^\delta_m), w_j) \quad 1 \leq j \leq m, \ t \in (0,T), \end{aligned} \quad (26)$$

$$C_v \theta^\delta_{mt} + C_v \rho_\delta(v^\delta_m) \cdot \nabla\theta^\delta_m = \nabla \cdot \left(K_1(\rho_\delta(\phi^\delta_m))\nabla\theta^\delta_m\right) + \frac{l}{2} f^\delta_s(\phi^\delta_m)_t \quad \text{in } Q^\delta, \quad (27)$$

$$c^\delta_{mt} - K_2 \Delta c^\delta_m + \rho_\delta(v^\delta_m) \cdot \nabla c^\delta_m = K_2 M \nabla \cdot \left(c^\delta_m(1 - c^\delta_m)\nabla\rho_\delta(\phi^\delta_m)\right) \quad \text{in } Q^\delta, \quad (28)$$

$$\frac{\partial\phi^\delta_m}{\partial n} = 0, \quad \frac{\partial\theta^\delta_m}{\partial n} = 0, \quad \frac{\partial c^\delta_m}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \quad (29)$$

$$v^\delta_m(0) = v^\delta_{0m} \text{ in } \Omega, \ \phi^\delta_m(0) = \phi^\delta_{0m}, \ \theta^\delta_m(0) = \theta^\delta_{0m}, \ c^\delta_m(0) = c^\delta_{0m} \text{ in } \Omega^\delta. \quad (30)$$

We then have the following existence result.

Proposition 2 *Let $T > 0$, $\delta(\Omega)$ be as described after (17), and $1/2 < \gamma \leq 1$. Fix $\delta \in (0, \delta(\Omega)]$ and $m \in I\!N$; let $\phi^\delta_{0m} \in H^{1+\gamma}(\Omega^\delta)$, $v^\delta_{0m} \in V_m$, $\theta^\delta_{0m} \in H^{1+\gamma}(\Omega^\delta)$ and $c^\delta_{0m} \in C^1(\overline{\Omega^\delta})$ satisfying the compatibility conditions $\dfrac{\partial \phi^\delta_{0m}}{\partial n} = \dfrac{\partial \theta^\delta_{0m}}{\partial n} = \dfrac{\partial c^\delta_{0m}}{\partial n} = 0$ on $\partial\Omega^\delta$ and $0 < c^\delta_{0m} < 1$ in $\overline{\Omega^\delta}$. Assume also that* **(H1)**-**(H4)** *hold. Then, there exist a solution $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$ satisfying (25)-(30) and*

i) $\phi^\delta_m \in L^2(0,T; H^2(\Omega^\delta))$, $\phi^\delta_{mt} \in L^2(Q^\delta)$,

ii) $v^\delta_m \in C^1([0,T]; V_m)$,

iii) $\theta^\delta_m \in L^2(0,T; H^2(\Omega^\delta))$, $\theta^\delta_{mt} \in L^2(Q^\delta)$,

iv) $c^\delta_m \in C^{2,1}(Q^\delta)$, $0 < c^\delta_m < 1$.

Proof: For simplicity of notation, in this proof we shall omit the index δ used in ϕ^δ_m, v^δ_m, θ^δ_m, c^δ_m.

We consider the family of operators, for $0 \leq \lambda \leq 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^2(Q^\delta) \times L^2(0,T;H) \times L^2(Q^\delta) \times L^2(Q^\delta),$$

which maps $(\hat\phi_m, \hat v_m, \hat\theta_m, \hat c_m) \in B$ into $(\phi_m, v_m, \theta_m, c_m)$, with $v_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \in V_m$, obtained by solving the problem

$$\alpha\epsilon^2 \phi_{mt} - \epsilon^2 \Delta\phi_m - \frac{1}{2}(\phi_m - \phi_m^3) = \lambda\beta\left(\hat\theta_m + (\theta_B - \theta_A)\hat c_m - \theta_B\right) \text{ in } Q^\delta, \quad (31)$$

$$\begin{aligned} \frac{d}{dt}(v_m, w_j) &+ \nu(\nabla v_m, \nabla w_j) + (v_m \cdot \nabla v_m, w_j) + (k(f^\delta_s(\phi_m) - \delta)v_m, w_j) \\ &= \lambda(\mathcal{F}(\hat c_m, \hat\theta_m), w_j) \quad 1 \leq j \leq m,\ t \in (0,T), \end{aligned} \quad (32)$$

$$C_v \theta_{mt} + C_v \rho_\delta(v_m) \cdot \nabla\theta_m = \nabla \cdot (K_1(\rho_\delta(\phi_m))\nabla\theta_m) + \frac{l}{2} f^\delta_s(\phi_m)_t \quad \text{in } Q^\delta, \quad (33)$$

$$c_{mt} - K_2\Delta c_m + \rho_\delta(v_m) \cdot \nabla c_m = K_2 M \nabla \cdot (c_m(1 - c_m)\nabla\rho_\delta(\phi_m)) \quad \text{in } Q^\delta, \quad (34)$$

$$\frac{\partial \phi_m}{\partial n} = \frac{\partial \theta_m}{\partial n} = \frac{\partial c_m}{\partial n} = 0 \qquad \text{on } \partial\Omega^\delta \times (0,T), \tag{35}$$

$$v_m(0) = v_{0m}^\delta \text{ in } \Omega,\ \phi_m(0) = \phi_{0m}^\delta,\ \theta_m(0) = \theta_{0m}^\delta,\ c_m(0) = c_{0m}^\delta \text{ in } \Omega^\delta. \tag{36}$$

Clearly $(\phi_m, v_m, \theta_m, c_m)$ is a solution of (25)-(30) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point using the Leray-Schauder Fixed Point Theorem.

To begin with, observe that since $\hat{\theta}_m$, $\hat{c}_m \in L^2(Q^\delta)$ we infer from Theorem 2.1 [14] that there is a unique solution ϕ_m of equation (31) with $\phi_m \in W_2^{2,1}(Q^\delta)$.

Now, (32) is a nonlinear system of ordinary differential equations for the functions $g_{1m}, \ldots, g_{mm}$. This problem has an unique maximal solution defined on same interval $[0, t_m)$ and $v_m \in C^1([0, t_m); V_m)$. The *a priori* estimates we shall prove later will show in particular that $t_m = T$.

Observe that since K_1 is a bounded Lipschitz continuous function and $\rho_\delta(\phi_m) \in C^\infty(I\!R^{N+1})$, we have that $K_1(\rho_\delta(\phi_m)) \in W_r^{1,1}(Q^\delta)$, $1 \leq r \leq \infty$, and since $\rho_\delta(v_m) \in L^{N+2}(Q^\delta)$ and $f_s^\delta(\phi_m)_t = f_s^{\delta'}(\phi_m)\phi_{mt} \in L^2(Q^\delta)$, we infer from L^p-theory of parabolic equations ([15], Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ_m of equation (33) with $\theta_m \in W_2^{2,1}(Q^\delta)$.

We observe that equation (34) is a semi-linear parabolic equation with smooth coefficients and growth conditions on the nonlinear forcing terms as the ones required for a semigroup result on global existence result given in [13], p., 75. Thus, there is a unique classical global solution c_m. In addition, note that equation (34) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principle together with the conditions $0 < c_{0m}^\delta < 1$ and $\frac{c_m}{\partial n} = 0$ on $\partial\Omega^\delta$, we can deduce that

$$0 < c_m(x,t) < 1, \qquad \forall\,(x,t) \in Q^\delta. \tag{37}$$

Therefore, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove the continuity of $\mathcal{T}_\lambda$, let $(\hat{\phi}_m^k, \hat{v}_m^k, \hat{\theta}_m^k, \hat{c}_m^k)$, $k \in I\!N$ be a sequence in B strongly converging to $(\hat{\phi}_m, \hat{v}_m, \hat{\theta}_m, \hat{c}_m) \in B$ and for each k, let $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$, the solution of the problem:

$$\alpha\epsilon^2\phi_{mt}^k - \epsilon^2\Delta\phi_m^k - \frac{1}{2}(\phi_m^k - (\phi_m^k)^3) = \lambda\beta\left(\hat{\theta}_m^k + (\theta_B - \theta_A)\hat{c}_m^k - \theta_B\right) \text{ in } Q^\delta, \tag{38}$$

$$
\begin{aligned}
v_m^k(t) &= \sum_{j=1}^{m} g_{jm}^k(t) w_j \in V_m, \\
\frac{d}{dt}(v_m^k, w_j) &+ \nu(\nabla v_m^k, \nabla w_j) + (v_m^k \cdot \nabla v_m^k, w_j) + (k(f_s^\delta(\phi_m^k) - \delta) v_m^k, w_j) \\
&= \lambda(\mathcal{F}(\hat{c}_m^k, \hat{\theta}_m^k), w_j), \ 1 \le j \le m, \ t \in (0, T), \qquad (39)
\end{aligned}
$$

$$
C_v \theta_{mt}^k + C_v \rho_\delta(v_m^k) \cdot \nabla \theta_m^k = \nabla \cdot \left(K_1(\rho_\delta(\phi_m^k)) \nabla \theta_m^k \right) + \frac{l}{2} f_s^\delta(\phi_m^k)_t \text{ in } Q^\delta, \quad (40)
$$

$$
c_{mt}^k - K_2 \Delta c_m^k + \rho_\delta(v_m^k) \cdot \nabla c_m^k = K_2 M \nabla \cdot \left(c_m^k (1 - c_m^k) \nabla \rho_\delta(\phi_m^k) \right) \text{ in } Q^\delta, \quad (41)
$$

$$
\frac{\partial \phi_m^k}{\partial n} = \frac{\partial \theta_m^k}{\partial n} = \frac{\partial c_m^k}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0, T), \qquad (42)
$$

$$
v_m^k(0) = v_{0m}^\delta \text{ in } \Omega, \ \phi_m^k(0) = \phi_{0m}^\delta, \ \theta_m^k(0) = \theta_{0m}^\delta, \ c_m^k(0) = c_{0m}^\delta \text{in } \Omega^\delta. \qquad (43)
$$

We show that the sequence $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$ converges strongly in B to $(\phi_m, v_m, \theta_m, c_m) = \mathcal{T}_\lambda(\hat{\phi}_m, \hat{v}_m \hat{\theta}_m, \hat{c}_m)$. For that purpose, we will obtain estimates to $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$ independent of k. As usual, we will denote by C_i, with a proper indexes i, positive constants independent of k.

We multiply (38) by ϕ_m^k, ϕ_{mt}^k and $-\Delta \phi_m^k$, we integrate over $\Omega^\delta \times (0, t)$ and by parts, and we use the Hölder's and Young's inequalities to obtain the following three estimates:

$$
\begin{aligned}
&\frac{\alpha \epsilon^2}{2} \int_{\Omega^\delta} |\phi_m^k|^2 dx + \int_0^t \int_{\Omega^\delta} \left(\epsilon^2 |\nabla \phi_m^k|^2 + \frac{1}{4} (\phi_m^k)^4 \right) dx dt \\
&\qquad \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 + |\phi_m^k|^2 \right) dx dt, \qquad (44) \\
&\frac{\alpha \epsilon^2}{2} \int_0^t \int_{\Omega^\delta} |\phi_{mt}^k|^2 dx dt + \int_{\Omega^\delta} \left(\frac{\epsilon}{2} |\nabla \phi_m^k|^2 + \frac{(\phi_m^k)^4}{8} - \frac{(\phi_m^k)^2}{4} \right) dx \\
&\qquad \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 \right) dx dt, \qquad (45) \\
&\frac{\alpha \epsilon^2}{2} \int_{\Omega^\delta} |\nabla \phi_m^k|^2 dx + \frac{\epsilon^2}{2} \int_0^t \int_{\Omega^\delta} |\Delta \phi_m^k|^2 dx dt \\
&\qquad \le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\nabla \phi_m^k|^2 + |\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 \right) dx dt. \qquad (46)
\end{aligned}
$$

Multiplying (45) by $\alpha \epsilon^2$ and adding the result to (44), we find

$$
\begin{aligned}
\int_{\Omega^\delta} |\phi_m^k|^2 + |\nabla \phi_m^k|^2 &+ (\phi_m^k)^4 dx \\
&\le C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 + |\phi_m^k|^2 \right) dx dt. \qquad (47)
\end{aligned}
$$

Since $\|\hat{\theta}_m^k\|_{L^2(Q^\delta)}$ and $\|\hat{c}_m^k\|_{L^2(Q^\delta)}$ are bounded independent of k, we infer from (47) and Gronwall's inequality that

$$\|\phi_m^k\|_{L^\infty(0,T;H^1(\Omega^\delta))} \leq C_1. \tag{48}$$

Then, thanks to (44)-(46) we have

$$\|\phi_m^k\|_{L^2(0,T;H^2(\Omega^\delta))} + \|\phi_{mt}^k\|_{L^2(Q^\delta)} \leq C_1. \tag{49}$$

We multiply (39) by $g_{jm}^k(t)$ and add these equations for $j = 1, \ldots, m$. Using that $(u \cdot \nabla v, v) = 0$, $u \in V$, $v \in (H^1(\Omega))^N$ we get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |v_m^k|^2 dx \quad &+ \quad \int_\Omega \left(\nu|\nabla v_m^k|^2 + k(f_s^\delta(\phi_m^k) - \delta)|v_m^k|^2\right) dx \\
&\leq \quad C_1 \int_\Omega \left(|F|^2 + |\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2 + |v_m^k|^2\right) dx \\
&\leq \quad C_1 \int_\Omega \left(|F|^2 + |v_m^k|^2\right) dx + C_1 \int_{\Omega^\delta} \left(|\hat{\theta}_m^k|^2 + |\hat{c}_m^k|^2\right) dx.
\end{aligned}$$

By using Gronwall's inequality, we obtain

$$\|v_m^k\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C_1. \tag{50}$$

Let now P_m be the projector of H on the space V_m. Note that P_m is a V_s-orthogonal projector on V_m and thus $\|P_m\|_{\mathcal{L}(V_s,V_s)} \leq 1$. Therefore, from equation (39), we infer that

$$\begin{aligned}
\|v_{mt}^k\|_{V_s'} \quad &\leq \quad C_1 \Big(\|v_m^k\|_V + \|v_m^k\|^2_{L^{\frac{2N}{N-1}}(\Omega)} + \|F\|_{L^2(\Omega)} \\
&+ \quad \|\hat{\theta}_m^k\|_{L^2(\Omega^\delta)} + \|\hat{c}_m^k\|_{L^2(\Omega^\delta)} \Big).
\end{aligned}$$

Then, by using (50) and interpolation ([17] p.73), we obtain

$$\|v_{mt}^k\|_{L^2(0,T;V_s')} \leq C_1. \tag{51}$$

Now, by multiplying (40) by θ_m^k, one obtains similarly that

$$\int_{\Omega^\delta} |\theta_m^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla\theta_m^k|^2 dx dt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\phi_{mt}^k|^2 + |\theta_m^k|^2\right) dx dt, \tag{52}$$

and we infer from (49) and Gronwall's inequality that

$$\|\theta_m^k\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{53}$$

Hence, it follows from (52) that

$$\|\theta_m^k\|_{L^2(0,T;H^1(\Omega^\delta))} \leq C_1. \tag{54}$$

Now we take the scalar product of (40) with $\eta \in H^1(\Omega^\delta)$ and integrate by parts using Hölder's and Young's inequalities to obtain

$$\|\theta_{mt}^k\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla\theta_m^k\|_{L^2(\Omega^\delta)} + \|v_m^k\|_{L^4(\Omega)}\|\theta_m^k\|_{L^4(\Omega^\delta)} + \|\phi_{mt}^k\|_{L^2(\Omega^\delta)} \right)$$

and we infer from (49),(50) and (54) that

$$\|\theta_{mt}^k\|_{L^{4/3}(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{55}$$

Next, multiplying (41) by c_m^k and reasoning as before with the help of (37), we conclude that

$$\int_{\Omega^\delta} |c_m^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c_m^k|^2 dxdt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} |\nabla \phi_m^k|^2 dxdt.$$

Hence, from (49), we obtain

$$\|c_m^k\|_{L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{56}$$

In order to get an estimate for (c_{mt}^k) in $L^2(0,T;H^1(\Omega^\delta)')$, we go back to equation (41) and proceed similarly as before to obtain

$$\|c_{mt}^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{57}$$

We now infer from (48)-(57) that the sequence (ϕ_m^k) is uniformly bounded with respect to k in

$$W_1 = \left\{ w \in L^2(0,T;H^2(\Omega^\delta)),\ w_t \in L^2(0,T;L^2(\Omega^\delta)) \right\}$$

and in

$$W_2 = \left\{ w \in L^\infty(0,T;H^1(\Omega^\delta)),\ w_t \in L^2(0,T;L^2(\Omega^\delta)) \right\};$$

the sequence (v_m^k) is bounded in

$$W_3 = \left\{ w \in L^2(0,T;V),\ w_t \in L^2(0,T;V_s') \right\}$$

and in

$$W_4 = \left\{ w \in L^\infty(0,T;H),\ w_t \in L^2(0,T;V_s') \right\};$$

the sequence (θ_m^k) is bounded in

$$W_5 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\ w_t \in L^{4/3}(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_6 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\ w_t \in L^{4/3}(0,T;H^1(\Omega^\delta)') \right\};$$

and the sequence (c_m^k) is bounded in

$$W_7 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_8 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}.$$

Now we observe that W_1 is compactly embedded into $L^2(0,T;H^1(\Omega^\delta))$, and the same holds for W_2 into $C([0,T];L^2(\Omega^\delta))$; for W_3 into $L^2(Q)$; W_5 and W_7 into $L^2(Q^\delta)$; with W_4 into $C([0,T];V_s')$, and with W_6 and W_8 into $C([0,T];H^1(\Omega^\delta)')$ ([23] Cor.4).

It follows that there exist $(\phi_m,\ v_m,\ \theta_m,\ c_m)$ satisfying:

$$\begin{array}{lcl}
\phi_m & \in & L^2(0,T;H^2(\Omega^\delta)) \cap L^\infty(0,T;H^1(\Omega^\delta)), \text{ with } \phi_{mt} \in L^2(Q^\delta), \\
v_m & \in & L^2(0,T;V) \cap L^\infty(0,T;H), \text{ with } v_{mt} \in L^2(0,T;V_s'), \\
\theta_m & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)), \text{ with } \theta_{mt} \in L^{4/3}(0,T;H^1(\Omega^\delta)'), \\
c_m & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)), \text{ with } c_{mt} \in L^2(0,T;H^1(\Omega^\delta)'),
\end{array}$$

and a subsequence of $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$, which for simplicity of notation we keep denoting $(\phi_m^k, v_m^k, \theta_m^k, c_m^k)$, such that as $k \to +\infty$ we have

$$\begin{array}{lcll}
\phi_m^k & \to & \phi_m & \text{strongly in } L^2(0,T;H^1(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)), \\
\phi_m^k & \rightharpoonup & \phi_m & \text{weakly in } L^2(0,T;H^2(\Omega^\delta)), \\
v_m^k & \to & v_m & \text{strongly in } L^2(Q) \cap C([0,T];V_s'), \\
v_m^k & \rightharpoonup & v_m & \text{weakly in } L^2(0,T;V), \\
\theta_m^k & \to & \theta_m & \text{strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\
\theta_m^k & \rightharpoonup & \theta_m & \text{weakly in } L^2(0,T;H^1(\Omega^\delta)), \\
c_m^k & \to & c_m & \text{strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\
c_m^k & \rightharpoonup & c_m & \text{weakly in } L^2(0,T;H^1(\Omega^\delta)).
\end{array} \tag{58}$$

It now remains to pass to the limit as k tends to $+\infty$ in (38)-(43).

Since the embedding of $W_2^{2,1}(Q^\delta)$ into $L^9(Q^\delta)$ is compact ([18] p.15), and (ϕ_m^k) is bounded in $W_2^{2,1}(Q^\delta)$, we infer that $(\phi_m^k)^3$ converges to ϕ_m^3 in $L^2(Q^\delta)$. We then pass to the limit as k tends to $+\infty$ in (38) and get

$$\alpha\epsilon^2\phi_{mt} - \epsilon^2\Delta\phi_m - \frac{1}{2}(\phi_m - \phi_m^3) = \lambda\beta\left(\hat{\theta}_m + (\theta_B - \theta_A)\hat{c}_m - \theta_B\right) \text{ a.e. in } Q^\delta.$$

Now we observe that for fixed $\delta > 0$, $k(f_s^\delta(\cdot) - \delta)$ is a bounded Lipschitz continuous function from $\mathbb{R}$ in $\mathbb{R}$; therefore, $k(f_s^\delta(\phi_m^k) - \delta)$ converges to $k(f_s^\delta(\phi_m) - \delta)$ in $L^p(Q)$ for any $1 \le p < +\infty$. Since the passing to the limit of the other terms of (39) can be done in standard ways, we get

$$\begin{aligned}\frac{d}{dt}(v_m, w_j) + \nu(\nabla v_m, \nabla w_j) \quad &+ (v_m \cdot \nabla v_m, w_j) + (k(f_s^\delta(\phi_m) - \delta)v_m, w_j)\\ &= \lambda(\mathcal{F}(\hat{c}_m, \hat{\theta}_m), w_j) \quad 1 \le j \le m,\ t \in (0,T).\end{aligned}$$

Also, since V_m is a closed subspace, we have that $v_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j \in V_m$.

Since $K_1(\rho_\delta)$ and $f_s^{\delta'}$ are bounded Lipschitz continuous functions and ϕ_m^k converges to ϕ_m in $L^2(Q^\delta)$, we have that $K_1(\rho_\delta(\phi_m^k))$ converges to $K_1(\rho_\delta(\phi_m))$ and $f_s^{\delta'}(\phi_m^k)$ converges to $f_s^{\delta'}(\phi_m)$ in $L^p(Q^\delta)$ for any $p \in [1, \infty)$. These facts and (58) yield the weak convergence of $K_1(\rho_\delta(\phi_m^k))\nabla\theta_m^k$ to $K_1(\rho_\delta(\phi_m))\nabla\theta_m$ and $f_s^{\delta'}(\phi_m^k)\phi_{mt}^k$ to $f_s^{\delta'}(\phi_m)\phi_{mt}$ in $L^{3/2}(Q^\delta)$. Also, since v_m^k converges to v_m in $L^2(Q)$ we have that $\rho_\delta(v_m^k)$ converges to $\rho_\delta(v_m)$ in $L^2(Q^\delta)$. Now, multiplying (40) by $\eta \in \mathcal{D}(Q^\delta)$, integrating over $\Omega^\delta \times (0,T)$ and by parts, we obtain

$$\begin{aligned}\int_0^T \int_{\Omega^\delta} C_{\mathrm{v}}\left(\theta_{mt}^k + \rho_\delta(v_m^k)\cdot\nabla\theta_m^k\right)\eta \quad &+ \quad K_1(\rho_\delta(\phi_m^k))\nabla\theta_m^k \cdot \nabla\eta\, dxdt\\ &= \quad \int_0^T \int_{\Omega^\delta} \frac{l}{2} f_s^{\delta'}(\phi_m^k)\phi_{mt}^k\eta\, dxdt.\end{aligned}$$

Then, we may pass to the limit and find that

$$C_{\mathrm{v}}\theta_{mt} + C_{\mathrm{v}}\rho_\delta(v_m)\cdot\nabla\theta_m = \nabla\cdot(K_1(\rho_\delta(\phi_m))\nabla\theta_m) + \frac{l}{2}f_s^{\delta'}(\phi_m)\phi_{mt} \text{ in } \mathcal{D}'(Q^\delta). \tag{59}$$

Now, by using the L^p-theory of parabolic equations, we conclude that (59) holds almost everywhere in Q^δ.

It remains to pass to the limit in (41). We infer from (58) that $\nabla\rho_\delta(\phi_m^k)$ converges to $\nabla\rho_\delta(\phi_m)$ in $L^2(Q^\delta)$. Also, since $\|c_m^k\|_{L^\infty(Q^\delta)}$ is bounded, it follows that $c_m^k(1-c_m^k)$ converges to $c_m(1-c_m)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. Thus, we may pass to the limit in (41) to obtain

$$c_{mt} - K_2\Delta c_m + \rho_\delta(v_m)\cdot\nabla c_m = K_2 M\nabla\cdot(c_m(1-c_m)\nabla\rho_\delta(\phi_m)) \text{ in } Q^\delta.$$

Therefore, $\mathcal{T}_\lambda$ is continuous for each $0 \le \lambda \le 1$.

At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1\times W_3\times W_5\times W_7$, and the embedding of this space in B is compact. We conclude that $\mathcal{T}_\lambda$ is a compact operator.

To prove that for $(\hat{\phi}_m, \hat{v}_m, \hat{\theta}_m, \hat{c}_m)$ in a bounded set of B, $\mathcal{T}_\lambda$ is uniformly continuous with respect to λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and $(\phi_{mi}, v_{mi}, \theta_{mi}, c_{mi})$, $(i = 1,2)$ the corresponding solutions of (31)-(36). We observe that $\phi_m = \phi_{m1} - \phi_{m2}$, $v_m = v_{m1} - v_{m2}$ $\left(v_m(t) = \sum_{j=1}^{m} g_{jm}(t) w_j \in V_m\right)$, $\theta_m = \theta_{m1} - \theta_{m2}$ and $c_m = c_{m1} - c_{m2}$ satisfy the following problem:

$$\begin{aligned}\alpha\epsilon^2\phi_{mt} - \epsilon^2\Delta\phi_m = \ & \frac{1}{2}\phi_m\left(1 - (\phi_{m1}^2 + \phi_{m1}\phi_{m2} + \phi_{m2}^2)\right) \\ & +(\lambda_1-\lambda_2)\beta\left(\hat{\theta}_m + (\theta_B - \theta_A)\hat{c}_m - \theta_B\right) \quad \text{in } Q^\delta,\end{aligned} \tag{60}$$

$$\begin{aligned}&\frac{d}{dt}(v_m, w_j) + \nu(\nabla v_m, \nabla w_j) + (v_m\cdot\nabla v_{m1}, w_j) - (v_{m2}\cdot\nabla v_m, w_j) \\ &\quad +(k(f_s^\delta(\phi_{m1})-\delta)v_m, w_j) + \left(\left[k(f_s^\delta(\phi_{m1})-\delta) - k(f_s^\delta(\phi_{m2})-\delta)\right]v_{m2}, w_j\right) \\ &\quad = (\lambda_1-\lambda_2)(\mathcal{F}(\hat{c}_m, \hat{\theta}_m), w_j), \quad 1 \le j \le m,\end{aligned} \tag{61}$$

$$\begin{aligned}C_v\theta_{mt} - \nabla\cdot(K_1(\rho_\delta(\phi_{m1}))\nabla\theta_m) &- \nabla\cdot[K_1(\rho_\delta(\phi_{m1})) - K_1(\rho_\delta(\phi_{m2}))]\nabla\theta_{m2} \\ &+ C_v\rho_\delta(v_m)\cdot\nabla\theta_{m1} + C_v\rho_\delta(v_{m2})\cdot\nabla\theta_m \\ &= \frac{l}{2}f_s^{\delta\prime}(\phi_{m1})\phi_{mt} + \frac{l}{2}\left[f_s^{\delta\prime}(\phi_{m1}) - f_s^{\delta\prime}(\phi_{m2})\right]\phi_{m2t} \ \text{ in } Q^\delta,\end{aligned} \tag{62}$$

$$\begin{aligned}c_{mt} - K_2\Delta c_m &= K_2M\nabla\cdot(c_{m1}(1-c_{m1})\left[\nabla\rho_\delta(\phi_{m1}) - \nabla\rho_\delta(\phi_{m2})\right]) \\ &+ \rho_\delta(v_m)\cdot\nabla c_{m1} + \rho_\delta(v_{m2})\cdot\nabla c_m \\ &+ K_2M\nabla\cdot(c_m(1-(c_{m1}+c_{m2}))\nabla\rho_\delta(\phi_{m2})) \ \text{ in } Q^\delta,\end{aligned} \tag{63}$$

$$\frac{\partial\phi_m}{\partial n} = \frac{\partial\theta_m}{\partial n} = \frac{\partial c_m}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta\times(0,T), \tag{64}$$

$$v_m(0) = 0 \text{ in } \Omega, \quad \phi_m(0) = 0, \quad \theta_m(0) = 0, \quad c_m(0) = 0 \text{ in } \Omega^\delta. \tag{65}$$

We remark that $d := {\phi_{m}}_1^2 + \phi_{m1}\phi_{m2} + {\phi_{m}}_2^2 = (\phi_{m1}/\sqrt{2} + \phi_{m2}/\sqrt{2})^2 + {\phi_{m}}_1^2/2 + {\phi_{m}}_2^2/2 \geq 0$. Now, by multiplying equation (60) by ϕ_m, integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\begin{aligned}\int_{\Omega^\delta} |\phi_m|^2 dx \;\; &+ \;\; \int_0^t \int_{\Omega^\delta} |\nabla \phi_m|^2 dx dt \leq C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 (1-d) dx dt \\ &+ \;\; C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2 \right) dx dt.\end{aligned}$$

Applying Gronwall's inequality, we get

$$\|\phi_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} + \|\phi_m\|^2_{L^2(0,T;H^1(\Omega^\delta))} \leq C_1 \, |\lambda_1 - \lambda_2|^2. \tag{66}$$

Now, by multiplying (60) by ϕ_{m_t} and using Hölder's inequality, we conclude

$$\begin{aligned}\alpha\epsilon^2 \int_0^t \int_{\Omega^\delta} |\phi_{m_t}|^2 dx dt \;\; &+ \;\; \frac{\epsilon^2}{2} \int_{\Omega^\delta} |\nabla \phi_m|^2 dx \\ &\leq \;\; C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 dx dt + \frac{\alpha\epsilon^2}{2} \int_0^t \int_{\Omega^\delta} |\phi_{m_t}|^2 dx dt \\ &+ \;\; C_2 \left(\int_0^t \int_{\Omega^\delta} |\phi_m|^{10/3} dx dt \right)^{3/5} \left(\int_0^t \int_{\Omega^\delta} |d|^5 dx dt \right)^{2/5} \\ &+ \;\; C_2 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2 \right) dx dt.\end{aligned}$$

Since $W_2^{2,1}(Q^\delta) \hookrightarrow L^{10}(Q^\delta)$, the following interpolation inequality holds

$$\|\phi_m\|^2_{L^{10/3}(Q^\delta)} \leq \eta \, \|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} + \tilde{C} \, \|\phi_m\|^2_{L^2(Q^\delta)} \text{ for all } \eta > 0,$$

and since $\|d\|_{L^5(Q^\delta)} \leq C$, depending on $\|\phi_{m1}\|_{L^{10}(Q^\delta)}$ and $\|\phi_{m2}\|_{L^{10}(Q^\delta)}$, rearranging the different terms, we obtain

$$\begin{aligned}\int_0^t \int_{\Omega^\delta} |\phi_{m_t}|^2 dx dt \;\; &+ \;\; \int_{\Omega^\delta} |\nabla \phi_m|^2 dx \\ &\leq \;\; C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 dx dt + C_2 \, \eta \, \|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} \\ &+ \;\; C_3 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2 \right) dx dt.\end{aligned} \tag{67}$$

Multiplying (60) by $-\Delta\phi_m$, and proceeding similarly as before, we infer that

$$\begin{aligned}\int_{\Omega^\delta} |\nabla \phi_m|^2 dx \;\; &+ \;\; \int_0^t \int_{\Omega^\delta} |\Delta \phi_m|^2 dx dt \\ &\leq \;\; C_1 \int_0^t \int_{\Omega^\delta} |\phi_m|^2 + |\nabla \phi_m|^2 dx dt + C_2 \, \eta \, \|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} \\ &+ \;\; C_3 |\lambda_1 - \lambda_2|^2 \int_0^t \int_{\Omega^\delta} \left(|\hat{\theta}_m|^2 + |\hat{c}_m|^2 \right) dx dt.\end{aligned} \tag{68}$$

Taking $\eta > 0$ small enough and considering (66), we conclude from (67) and (68) that

$$\|\phi_m\|^2_{W_2^{2,1}(Q^\delta)} + \|\phi_m\|^2_{L^\infty(0,T;H^1(\Omega^\delta))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{69}$$

Multiplying (61) by $g_{jm}(t)$ and adding these equations for $j = 1, \cdots, m$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |v_m|^2 dx &+ \int_\Omega \nu|\nabla v_m|^2 + k(f_s^\delta(\phi_{m1}) - \delta)|v_m|^2 dx \\
&\leq \int_\Omega |k(f_s^\delta(\phi_{m1}) - \delta) - k(f_s^\delta(\phi_{m2}) - \delta)||v_{m2}||v_m|dx \\
&\quad + \int_\Omega |v_m||\nabla v_{m1}||v_m|dx + |\lambda_1 - \lambda_2|\int_\Omega \mathcal{F}(\hat{c}_m, \hat{\theta}_m)v_m dx \\
&\leq C_1\left(\|\phi_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))}\|v_{m2}\|^2_V + \|v_{m1}\|^2_{L^\infty(0,T;V_s)}\|v_m\|^2_{L^2(\Omega)}\right) \\
&\quad + \frac{\nu}{2}\int_\Omega |\nabla v_m|^2 dx + C_3\int_\Omega |v_m|^2 dx \\
&\quad + C_2|\lambda_1 - \lambda_2|^2\left(\int_\Omega |F|^2 dx + \int_{\Omega^\delta} |\hat{\theta}_m|^2 + |\hat{c}_m|^2 dx\right).
\end{aligned}$$

By integrating this last inequality with respect to t and using our previous estimates and Gronwall's inequality, we obtain

$$\|v_m\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{70}$$

Multiplying (62) by θ_m, integrating over Ω^δ using Hölder's inequality and that K_1 and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^\delta} |\theta_m|^2 dx + a\int_{\Omega^\delta} |\nabla\theta_m|^2 dx \\
&\quad \leq C_1\int_{\Omega^\delta} |\rho_\delta(\phi_m)||\nabla\theta_{m2}||\nabla\theta_m| + |\rho_\delta(v_m)||\nabla\theta_{m1}||\theta_m|dx \\
&\qquad + C_2\int_{\Omega^\delta} |\phi_{mt}||\theta_m| + |\phi_m||\phi_{m2t}|\,|\theta_m|dx \\
&\quad \leq C_1\|\phi_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))}\|\nabla\theta_{m2}\|^2_{L^2(\Omega^\delta)} + \frac{a}{2}\int_{\Omega^\delta} |\nabla\theta_m|^2 dx \\
&\qquad + C_2\|v_m\|^2_{L^\infty(0,T;H)}\|\nabla\theta_{m1}\|^2_{L^2(\Omega^\delta)} \\
&\qquad + C_4\|\phi_m\|^2_{L^\infty(0,T;H^1(\Omega^\delta))}\|\phi_{m2t}\|^2_{L^2(\Omega^\delta)} + C_3\int_{\Omega^\delta}\left(|\phi_{mt}|^2 + |\theta_m|^2\right)dx.
\end{aligned}$$

Integration with respect to t and the use of Gronwall's Lemma and (69)-(70) lead to the estimate

$$\|\theta_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1\,|\lambda_1 - \lambda_2|^2. \tag{71}$$

We multiply (63) by c_m, integrate over $\Omega^\delta \times (0,t)$ and by parts, and we use Hölder's and Young's inequalities and (37) to obtain

$$
\begin{aligned}
\int_{\Omega^\delta} |c_m|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c_m|^2 dx dt \\
&\le C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla \rho_\delta(\phi_{m1}) - \nabla \rho_\delta(\phi_{m2})|^2 + |\rho_\delta(v_m)|^2 + |c_m|^2 \right) dx dt \\
&\le C_1 \int_0^t \int_{\Omega^\delta} \left(|\nabla \phi_m|^2 + |c_m|^2 \right) dx dt + C_1 \int_0^t \int_\Omega |v_m|^2 dx dt.
\end{aligned}
$$

Applying Gronwall's inequality and using (69)-(70) we arrive at

$$
\|c_m\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1 \, |\lambda_1 - \lambda_2|^2. \tag{72}
$$

Therefore, it follows from (69)-(72) that $\mathcal{T}_\lambda$ is uniformly continuous with respect to λ on bounded sets of B.

To estimate the set of all fixed points of $\mathcal{T}_\lambda$, let $(\phi_m, v_m, \theta_m, c_m) \in B$ be such any given fixed point, i.e., it is a solution of the problem

$$
\alpha\epsilon^2 \phi_{mt} - \epsilon^2 \Delta \phi_m - \frac{1}{2}(\phi_m - \phi_m^3) = \lambda \beta \left(\theta_m + (\theta_B - \theta_A) c_m - \theta_B \right) \text{ in } Q^\delta, \tag{73}
$$

$$
\begin{aligned}
v_m(t) &= \sum_{j=1}^m g_{jm}(t) w_j \in V_m = \operatorname{span}\{w_1, \ldots, w_m\}, \\
\frac{d}{dt}(v_m, w_j) &+ \nu(\nabla v_m, \nabla w_j) + (v_m \cdot \nabla v_m, w_j) + (k(f_s^\delta(\phi_m) - \delta) v_m, w_j) \\
&= \lambda(\mathcal{F}(c_m, \theta_m), w_j) \quad 1 \le j \le m, \ t \in (0,T),
\end{aligned} \tag{74}
$$

$$
C_v \theta_{mt} + C_v \rho_\delta(v_m) \cdot \nabla \theta_m = \nabla \cdot (K_1(\rho_\delta(\phi_m)) \nabla \theta_m) + \frac{l}{2} f_s^\delta(\phi_m)_t \text{ in } Q^\delta, \tag{75}
$$

$$
c_{mt} - K_2 \Delta c_m + \rho_\delta(v_m) \cdot \nabla c_m = K_2 M \nabla \cdot \left(c_m (1 - c_m) \nabla \left(\rho_\delta(\phi_m) \right) \right) \text{ in } Q^\delta, \tag{76}
$$

$$
\frac{\partial \phi_m}{\partial n} = \frac{\partial \theta_m}{\partial n} = \frac{\partial c_m}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{77}
$$

$$
v_m(0) = v_{0m}^\delta \text{ in } \Omega, \ \phi_m(0) = \phi_{0m}^\delta, \ \theta_m(0) = \theta_{0m}^\delta, \ c_m(0) = c_{0m}^\delta \text{ in } \Omega^\delta. \tag{78}
$$

Multiplying the first equation (73) by ϕ_m, ϕ_{mt} and $-\Delta\phi_m$, respectively, integrating over Ω^δ and by parts, using Hölder's and Young's inequalities, we obtain

$$
\frac{\alpha\epsilon^2}{2} \frac{d}{dt} \int_{\Omega^\delta} |\phi_m|^2 dx \quad + \quad \int_{\Omega^\delta} \left(\epsilon^2 |\nabla \phi_m|^2 + \frac{1}{4} \phi_m^4 \right) dx
$$

$$
\leq C_1 + C_2 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 + |\phi_m|^2 \right) dx, \quad (79)
$$

$$
\begin{aligned}
\frac{\alpha\epsilon^2}{2} \int_{\Omega^\delta} |\phi_{mt}|^2 dx &+ \frac{d}{dt} \int_{\Omega^\delta} \left(\frac{\epsilon^2}{2} |\nabla \phi_m|^2 + \frac{1}{8}\phi_m^4 - \frac{1}{4}|\phi_m|^2 \right) dx \\
&\leq C_1 + C_2 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 \right) dx, \quad (80) \\
\frac{\alpha\epsilon^2}{2} \frac{d}{dt} \int_{\Omega^\delta} |\nabla \phi_m|^2 dx &+ \int_{\Omega^\delta} \frac{\epsilon^2}{2} |\Delta \phi_m|^2 dx \\
&\leq C_1 + C_2 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 + |\nabla \phi_m|^2 \right) dx. \quad (81)
\end{aligned}
$$

Now, for each $j = 1, \ldots, m$, we multiply (74) by $g_{jm}(t)$ and add the resulting equations to obtain

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_\Omega |v_m|^2 dx &+ \int_\Omega \left(\nu |\nabla v_m|^2 + k(f_s^\delta(\phi_m) - \delta)|v_m|^2 \right) dx \\
&\leq C_1 \int_\Omega |F|^2 + |\theta_m|^2 + |c_m|^2 + |v_m|^2 dx \quad (82) \\
&\leq C_1 \int_\Omega |F|^2 + |v_m|^2 dx + C_1 \int_{\Omega^\delta} |\theta_m|^2 + |c_m|^2 dx.
\end{aligned}
$$

By multiplying (75) by θ_m and (76) by c_m and proceeding similarly as above lead us to the following inequalities

$$
\frac{d}{dt} \int_{\Omega^\delta} \frac{C_v}{2} |\theta_m|^2 dx + a \int_{\Omega^\delta} |\nabla \theta_m|^2 dx \leq \frac{\alpha^2\epsilon^4}{4} \int_{\Omega^\delta} |\phi_{mt}|^2 dx + C_1 \int_{\Omega^\delta} |\theta_m|^2 dx, \quad (83)
$$

$$
\frac{d}{dt} \int_{\Omega^\delta} |c_m|^2 dx + K_2 \int_{\Omega^\delta} |\nabla c_m|^2 dx \leq C_1 \int_{\Omega^\delta} |\nabla \phi_m|^2 dx, \quad (84)
$$

where we used (37) to obtain the last inequality.

Now, by multiplying (80) by $\alpha\epsilon^2$ and adding the result to (79),(81)-(84), we obtain

$$
\begin{aligned}
&\frac{d}{dt} \int_{\Omega^\delta} \left(\frac{\alpha\epsilon^2}{4} |\phi_m|^2 + \left(\frac{\alpha\epsilon^2}{2} + \frac{\alpha\epsilon^4}{2} \right) |\nabla \phi_m|^2 + \frac{\alpha\epsilon^2}{8} \phi_m^4 + \frac{C_v}{2} |\theta_m|^2 + |c_m|^2 \right) dx \\
&\quad + \frac{d}{dt} \int_\Omega \frac{1}{2} |v_m|^2 dx + \int_\Omega \nu |\nabla v_m|^2 + k(f_s^\delta(\phi_m) - \delta)|v_m|^2 dx + \int_{\Omega^\delta} \left(\epsilon^2 |\nabla \phi_m|^2 \right. \\
&\quad \left. + \frac{1}{4}\phi_m^4 + \frac{\alpha^2\epsilon^4}{4} |\phi_{mt}|^2 + \frac{\epsilon^2}{2} |\Delta \phi_m|^2 + a|\nabla \theta_m|^2 + K_2 |\nabla c_m|^2 \right) dx \\
&\quad \leq C_1 + C_1 \int_{\Omega^\delta} \left(|\theta_m|^2 + |c_m|^2 + |\phi_m|^2 + |\nabla \phi_m|^2 \right) dx + C_1 \int_\Omega |v_m|^2 dx,
\end{aligned} \quad (85)
$$

where C_1 is independent of λ, m and δ.

Hence, integrating (85) with respect t and using Gronwall's inequality, we obtain

$$\begin{aligned}&\|\phi_m\|_{L^\infty(0,T;H^1(\Omega^\delta))} + \|v_m\|_{L^\infty(0,T;H)}\\ &+\|\theta_m\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|c_m\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1,\end{aligned}$$

where C_1 is independent of λ. Therefore, we have a bound for all fixed points of $\mathcal{T}_\lambda$ in B independent of λ.

Finally, proceeding exactly as we did to prove that $\mathcal{T}_\lambda$ is well defined, we conclude that for $\lambda = 0$, problem (31)-(36) has a unique solution.

Thus, we can apply Leray-Schauder theorem and conclude that there is a fixed point $(\phi_m, v_m, \theta_m, c_m) \in B \cap W_2^{2,1}(Q^\delta) \times C^1([0,T]; V_m) \times W_2^{2,1}(Q^\delta) \times C^{2,1}(Q^\delta)$ of the operator $\mathcal{T}_1$, that is, $(\phi_m, v_m, \theta_m, c_m) = \mathcal{T}_1(\phi_m, v_m, \theta_m, c_m)$. This is a solution of problem (25)-(30), and the proof of Proposition 2 is complete. ∎

We now proceed with the

Proof of Proposition 1: We choose $\phi_{0m}^\delta = \phi_0^\delta$, $\theta_{0m}^\delta = \theta_0^\delta$, $c_{0m}^\delta \in C^1(\bar{\Omega})$ with $0 < c_{0m}^\delta < 1$, and $v_{0m}^\delta \in V_m$ such that $c_{0m}^\delta \to c_0^\delta$ and $v_{0m}^\delta \to v_0^\delta$ in the norm of H as $m \to +\infty$. We then infer from Proposition 2 that, for each $\delta \in (0, \delta(\Omega)]$ and $m \in I\!N$, there exist functions $(\phi_m^\delta, v_m^\delta, \theta_m^\delta, c_m^\delta)$ satisfying the system (25)-(30). We will derive bounds, independent of m, for this solution and then pass to the limit in the approximate problem as m tends to $+\infty$ by using compactness arguments.

Lemma 1 *There exists a constant C_1 independent of $m \in I\!N$ such that*

$$\begin{aligned}\|\phi_m^\delta\|_{L^\infty(0,T;H^1(\Omega^\delta))\cap L^2(0,T;H^2(\Omega^\delta))} + \|\phi_{mt}^\delta\|_{L^2(Q^\delta)} &\leq C_1, &(86)\\ \|v_m^\delta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} &\leq C_1, &(87)\\ \|\theta_m^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} &\leq C_1, &(88)\\ \|c_m^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} &\leq C_1. &(89)\end{aligned}$$

Proof: It follows from the inequality (85). ∎

Lemma 2 *There exists a constant C_1 independent of $m \in I\!N$ such that*

$$\begin{aligned}\|v_{mt}^\delta\|_{L^2(0,T;V_s')} &\leq C_1, &(90)\\ \|\theta_{mt}^\delta\|_{L^{4/3}(0,T;H^1(\Omega^\delta)')} &\leq C_1, &(91)\\ \|c_{mt}^\delta\|_{L^2(0,T;H^1(\Omega^\delta)')} &\leq C_1. &(92)\end{aligned}$$

Proof: From the equation (26), we infer that

$$\begin{aligned}\|v^\delta_{mt}\|_{V_s'} \leq \;& C_1 \Big(\|v^\delta_m\|_V + \|v^\delta_m\|^2_{L^{\frac{2N}{N-1}}(\Omega)} \\ & + \|F\|_{L^2(\Omega)} + \|\theta^\delta_m\|_{L^2(\Omega^\delta)} + \|c^\delta_m\|_{L^2(\Omega^\delta)} \Big) .\end{aligned}$$

Then, by using (87)-(89) and interpolation ([17] p.73), we obtain (90). By taking the scalar product of (27) with $\eta \in H^1(\Omega)$ and using Hölder's inequality, we find

$$\|\theta^\delta_{mt}\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla \theta^\delta_m\|_{L^2(\Omega^\delta)} + \|\phi^\delta_{mt}\|_{L^2(\Omega^\delta)} + \|v^\delta_m\|_{L^4(\Omega)} \|\theta^\delta_m\|_{L^4(\Omega^\delta)} \right) .$$

Then, (91) follows from (86)-(88). (92) can be obtained similarly by using Lemma 1. ∎

We infer from Lemma 1 and 2 using the compact embedding ([23] Cor.4) that there exist

$$\begin{array}{lll} \phi^\delta & \in & L^2(0,T;H^2(\Omega^\delta)) \cap L^\infty(0,T;H^1(\Omega^\delta)) \text{ with } \phi^\delta_t \in L^2(Q), \\ v^\delta & \in & L^2(0,T;V) \cap L^\infty(0,T;H) \text{ with } v^\delta_t \in L^2(0,T;V_s'), \\ \theta^\delta & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } \theta^\delta_t \in L^{4/3}(0,T;H^1(\Omega^\delta)'), \\ c^\delta & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } c^\delta_t \in L^2(0,T;H^1(\Omega^\delta)'), \end{array}$$

and a subsequence of $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$, which we keep calling $(\phi^\delta_m, v^\delta_m, \theta^\delta_m, c^\delta_m)$ to ease the notation, such that, as $m \to +\infty$,

$$\begin{array}{lll} \phi^\delta_m & \to & \phi^\delta \quad \text{strongly in } L^2(0,T;H^{2-\overline{\gamma}}(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)), \\ & & \qquad 0 < \overline{\gamma} \leq 1/2 \\ \phi^\delta_{mt} & \rightharpoonup & \phi^\delta_t \quad \text{weakly in } L^2(Q^\delta), \\ v^\delta_m & \to & v^\delta \quad \text{stronly in } L^2(Q) \cap C([0,T];V_s'), \\ v^\delta_{mt} & \rightharpoonup & v^\delta_t \quad \text{weakly in } L^2(0,T;V_s'), \\ \theta^\delta_m & \to & \theta^\delta \quad \text{strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\ \theta^\delta_m & \rightharpoonup & \theta^\delta \quad \text{weakly in } L^2(0,T;H^1(\Omega^\delta)), \\ c^\delta_m & \to & c^\delta \quad \text{strongly in } L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)'), \\ c^\delta_m & \rightharpoonup & c^\delta \quad \text{weakly in } L^2(0,T;H^1(\Omega^\delta)). \end{array} \tag{93}$$

Thus, letting $m \to +\infty$ in (25), we get

$$\alpha\epsilon^2 \phi^\delta_t - \epsilon^2 \Delta \phi^\delta - \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) = \beta \left(\theta^\delta + (\theta_B - \theta_A) c^\delta - \theta_B \right) \text{ a.e. in } Q^\delta.$$

Since $k(f_s^\delta(\cdot)-\delta)$ is a bounded Lipschitz continuous function we have that $k(f_s^\delta(\phi_m^\delta)-\delta)$ converges to $k(f_s^\delta(\phi^\delta)-\delta)$ in $L^p(Q)$, for $p\in[1,\infty)$; then $k(f_s^\delta(\phi_m^\delta)-\delta)v_m^\delta$ converges to $k(f_s^\delta(\phi^\delta)-\delta)v^\delta$ in $L^{3/2}(Q)$ as m tends to $+\infty$. As usual ([17] p.76) we may pass to the limit in the other terms in (26) and get

$$\begin{aligned}\frac{d}{dt}(v^\delta,w_j)+\nu(\nabla v^\delta,\nabla w_j) \quad &+(v^\delta\cdot\nabla v^\delta,w_j)+(k(f_s^\delta(\phi^\delta)-\delta)v^\delta,w_j)\\ &=(\mathcal{F}(c^\delta,\theta^\delta),w_j) \quad \text{for all } j\in I\!N.\end{aligned}$$

We conclude that

$$\frac{d}{dt}(v^\delta,u)+\nu(\nabla v^\delta,\nabla u)+(v^\delta\cdot\nabla v^\delta,u)+(k(f_s^\delta(\phi^\delta)-\delta)v^\delta,u)=(\mathcal{F}(c^\delta,\theta^\delta),u),$$

for all $u\in V_s$, and then for all $u\in V$.

Since $K_1(\rho_\delta)$ and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions we have that $K_1(\rho_\delta(\phi_m^\delta))$ converges to $K_1(\rho_\delta(\phi^\delta))$ and $f_s^{\delta\prime}(\phi_m^\delta)$ to $f_s^{\delta\prime}(\phi^\delta)$ in $L^p(Q^\delta)$ for any $p\in[1,\infty)$ as m tends to $+\infty$. Also, since v_m^δ converges to v^δ in $L^2(Q)$ we have that $\rho_\delta(v_m^\delta)$ converges to $\rho_\delta(v^\delta)$ in $L^2(Q^\delta)$. Using these facts and (93) we pass to the limit in (27) and obtain

$$C_v\theta_t^\delta+C_v\rho_\delta(v^\delta)\cdot\nabla\theta^\delta=\nabla\cdot\Big(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\Big)+\frac{l}{2}f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta \text{ in } \mathcal{D}'(Q^\delta).$$

Applying L^p-theory of parabolic equations, we have that $\theta^\delta\in W_2^{2,1}(Q^\delta)$.

Similarly we pass to the limit in (28) and obtain

$$c_t^\delta-K_2\Delta c^\delta+\rho_\delta(v^\delta)\cdot\nabla c^\delta=K_2M\nabla\cdot\Big(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\Big) \text{ in } Q^\delta.$$

Observe that c^δ is a classical solution and satisfies $0\le c^\delta\le 1$. Finally, it follows from (93) that $\dfrac{\partial\phi^\delta}{\partial n}=\dfrac{\partial\theta^\delta}{\partial n}=\dfrac{\partial c^\delta}{\partial n}=0$, $\phi^\delta(0)=\phi_0^\delta$, $v^\delta(0)=v_0^\delta$, $\theta^\delta(0)=\theta_0^\delta$ and $c^\delta(0)=c_0^\delta$. Therefore, the proof of Proposition 1 is complete. ∎

4 Proof of Theorem 1

In this section we prove the existence Theorem 1. For $0<\delta\le\delta(\Omega)$ as in the statement of Theorem 1, we choose $\phi_0^\delta\in W^{2-2/q,q}(\Omega^\delta)\cap H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta\in H$,

$\theta_0^\delta \in H^{1+\gamma}(\Omega)$, $1/2 < \gamma \leq 1$, $c_0^\delta \in C^1(\overline{\Omega^\delta})$, satisfying $\dfrac{\partial \phi_0^\delta}{\partial n} = \dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$, $\|\theta_0^\delta\|_{L^2(\Omega^\delta)} \leq C$, and $0 \leq c_0^\delta \leq 1$ in $\overline{\Omega^\delta}$, $v_0^\delta \to v_0$ in the norm of $H(\Omega_{ml}(0))$, and such that the restrictions of these functions to Ω (recall that $\Omega \subset \Omega^\delta$) satisfy as $\delta \to 0+$ the following: $\phi_0^\delta \to \phi_0$ in the norm of $W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $\theta_0^\delta \to \theta_0$ in the norm of $L^2(\Omega)$, $c_0^\delta \to c_0$ in the norm of $L^2(\Omega)$.

We then infer from Proposition 1 that there exists $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ solution the regularized problem (19)-(24). We will derive bounds, independent of δ, for this solution and then use compactness arguments and passage to the limit procedure for δ tends to 0 to establish the desired existence result. They are stated in following in a sequence of lemmas; however, most of them are ease consequence of the previous estimates (those that are independent of δ) and the fact that $\Omega \subset \Omega^\delta$. We begin with the following:

Lemma 3 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$
\begin{aligned}
&\|\phi^\delta\|_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \|\phi_t^\delta\|_{L^2(Q)} \\
&\qquad \leq \|\phi^\delta\|_{L^\infty(0,T;H^1(\Omega^\delta))\cap L^2(0,T;H^2(\Omega^\delta))} + \|\phi_t^\delta\|_{L^2(Q^\delta)} \leq C_1, \qquad (94)
\end{aligned}
$$

$$
\|v^\delta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \int_0^T \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dx dt \leq C_1, \qquad (95)
$$

$$
\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1, \qquad (96)
$$

$$
\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|c^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1. \qquad (97)
$$

Proof: It follows from the inequality (85). ∎

Lemma 4 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$
\begin{aligned}
\|c_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\leq C_1, && (98)\\
\|\theta_t^\delta\|_{L^{4/3}(0,T;H_o^1(\Omega)')} &\leq C_1, && (99)\\
\|\phi^\delta\|_{W_q^{2,1}(Q)} &\leq C_1, \quad \textit{for any } 2 \leq q \leq 2(N+2)/N. && (100)
\end{aligned}
$$

Proof: Using that $0 \leq c^\delta \leq 1$ in Q, we infer from (22) that,

$$
\|c_t^\delta\|_{H_o^1(\Omega)'} \leq C_1 \left(\|\nabla c^\delta\|_{L^2(\Omega)} + \|v^\delta\|_{L^2(\Omega)} + \|\nabla \phi^\delta\|_{L^2(\Omega)} \right).
$$

Then, (98) follows from Lemma 3.

Now, we take the scalar product of (21) with $\eta \in H_o^1(\Omega)$, using Hölder's inequality and **(H3)** we find

$$C_v \|\theta_t^\delta\|_{H_o^1(\Omega)'} \leq C_1 \left(\|\nabla\theta^\delta\|_{L^2(\Omega)}^2 + \|\theta^\delta\|_{L^4(\Omega)} \|v^\delta\|_{L^4(\Omega)} + \|\phi_t^\delta\|_{L^2(\Omega)} \right).$$

Then, (99) follows from Lemma 3.

Now, from a result of Hoffman and Jiang ([14] Thm 2.1), we conclude that ϕ^δ satisfies the following inequality, for any $2 \leq q < \infty$,

$$\|\phi^\delta\|_{W_q^{2,1}(Q^\delta)} \leq C_1 \left(\|\theta^\delta\|_{L^q(Q^\delta)} + \|c^\delta\|_{L^q(Q^\delta)} + \|\phi_0^\delta\|_{W^{2-2/q,q}(\Omega^\delta)} + C_1 \right). \quad (101)$$

Then, (100) holds due to $\|c^\delta\|_{L^\infty(Q^\delta)}$ and by interpolation $\|\theta^\delta\|_{L^{2(N+2)/N}(\Omega^\delta)}$ are bounded independent of δ. ∎

Lemma 5 *There exist a constant C_1 and $\delta_0 \in (0, \delta(\Omega))$ such that, for any $\delta < \delta_0$,*

$$\|v_t^\delta\|_{L^{4/3}(t_1,t_2;V(U)')} \leq C_1 \quad (102)$$

where $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ and such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$.

Proof: Let $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ be such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. It is verified by means of (20) that for a.e. $t \in (t_1, t_2)$,

$$(v_t^\delta, u) = -\nu \int_U \nabla v^\delta \cdot \nabla u dx - \int_U v^\delta \cdot \nabla v^\delta u dx - \int_U k(f_s^\delta(\phi^\delta) - \delta) v^\delta u dx + \int_U \mathcal{F}(c^\delta, \theta^\delta) u dx \quad \text{for u } \in V(U).$$

In order to estimate $\|v_t^\delta\|_{V(U)'}$, we observe that the sequence (ϕ^δ) is bounded in $W_q^{2,1}(Q)$, for $2 \leq q \leq 2(N+2)/N$, in particular, for $q > (N+2)/2$ we have that $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - (N+2)/q$ ([15] p. 80). Due to theorem of Arzela-Ascoli, there exist ϕ and a subsequence of (ϕ^δ) (which we still denote by ϕ^δ), such that ϕ^δ converges uniformly to ϕ in $\bar{Q}$. Recall that $Q_{ml} = \{(x,t) \in Q : 0 \leq f_s(\phi(x,t)) < 1\}$ and $\Omega_{ml}(t) = \{x \in \Omega : 0 \leq f_s(\phi(x,t)) < 1\}$. Note that there is $\overline{\gamma} \in (0,1)$ such that for any $(x,t) \in [t_1, t_2] \times \bar{U}$, we have

$$f_s(\phi(x,t)) < 1 - \overline{\gamma}.$$

Due to the uniform convergence of f_s^δ towards f_s on any compact subset, there is an δ_0 such that for all $\delta \in (0, \delta_0)$ and for all $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s^\delta(\phi^\delta(x,t)) < 1 - \overline{\gamma}/2.$$

By assumption **(H1)** we infer that

$$k(f_s^\delta(\phi^\delta(x,t)) - \delta) < k(1 - \overline{\gamma}/2) \quad \text{for } (x,t) \in [t_1, t_2] \times \bar{U} \text{ and } \delta < \delta_0.$$

Thus,

$$\begin{aligned} \|v_t^\delta\|_{V(U)'} \;\; \leq \;\; & C_1\Big(\|v^\delta\|_V + \|v^\delta\|_{L^4(\Omega)}^2 + \|F\|_{L^2(\Omega)} + \|c^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^2(\Omega)} \\ & + \|k(f_s^\delta(\phi^\delta(x,t)) - \delta)\|_{L^\infty(U)}\|v^\delta\|_{L^2(\Omega)}\Big). \end{aligned}$$

Hence, (102) follows from Lemma 3. ∎

From (95), we conclude that the sequence (v^δ) is also uniformly bounded in $L^2(t_1, t_2; H^1(U))$. Then, by the compact embedding ([23] Cor. 4) , there exist v and a subsequence of (v^δ) (which we still denote by v^δ), such that

$$v^\delta \to v \quad \text{strongly in } L^2((t_1, t_2) \times U).$$

Observe that Q_{ml} is an open set and can be covered by a countable number of open sets $(t_i, t_{i+1}) \times U_i$ such that $U_i \subseteq \Omega_{ml}(t_i)$, then by means of a diagonal argument, we obtain

$$v^\delta \to v \quad \text{strongly in } L^2_{loc}(Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)). \tag{103}$$

Moreover, from (95) and the fact that $v^\delta \in L^2(0,T;V)$ we have that $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and

$$\begin{aligned} v^\delta \;\; &\rightharpoonup \;\; v \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \\ v^\delta \;\; &\overset{*}{\rightharpoonup} \;\; v \quad \text{weakly star in } L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{104}$$

Now, from Lemma 3 and Lemma 4, by using compact embedding ([23] Cor.4), we infer that there exist

$$\begin{aligned} \phi \;\; &\in \;\; W_q^{2,1}(Q) \text{ for } 2 \leq q \leq 2(N+2)/N, \\ \theta \;\; &\in \;\; L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \\ c \;\; &\in \;\; L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \end{aligned}$$

and a subsequence of $(\phi^\delta, \theta^\delta, c^\delta)$ (which we still denote by $(\phi^\delta, \theta^\delta, c^\delta)$) such that, as $\delta \rightarrow 0$,

$$\begin{array}{lcll}
\phi^\delta & \rightarrow & \phi & \text{uniformly in } Q, \\
\phi^\delta & \rightarrow & \phi & \text{strongly in } L^q(0,T;W^{2-\overline{\gamma},q}(\Omega)),\ 0<\overline{\gamma}<1/2, \\
\phi_t^\delta & \rightharpoonup & \phi_t & \text{weakly in } L^q(Q), \\
\theta^\delta & \rightarrow & \theta & \text{strongly in } L^2(Q)\cap C([0,T];H_o^1(\Omega)'), \\
\theta^\delta & \rightharpoonup & \theta & \text{weakly in } L^2(0,T;H^1(\Omega)), \\
c^\delta & \rightarrow & c & \text{strongly in } L^2(Q)\cap C([0,T];H_o^1(\Omega)'), \\
c^\delta & \rightharpoonup & c & \text{weakly in } L^2(0,T;H^1(\Omega)).
\end{array} \tag{105}$$

It now remains to pass to the limit as δ decreases to zero in (19)-(24).

It follows from (105) that we may pass to the limit in (19), and find that (12) holds almost everywhere.

Now, we take $u=\eta(t)$ in (20) where $\eta \in L^2(0,T;V(\Omega_{ml}(t)))$ with compact support contained in $Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$ and $\eta_t \in L^2(0,T;V(\Omega_{ml}(t))')$; after integration over $(0,t)$, we find

$$\begin{aligned}\int_0^t \Big((v_t^\delta,\eta)+(\nabla v^\delta,\nabla\eta)+(v^\delta\cdot\nabla v^\delta,\eta)+(k(f_s^\delta(\phi^\delta)-\delta)v^\delta,\eta)\Big)\,ds \\ = \int_0^t (\mathcal{F}(c^\delta,\theta^\delta),\eta)ds.\end{aligned} \tag{106}$$

Since supp $\eta \subseteq Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$ we have that supp $\eta(t)\subseteq\Omega_{ml}(t)$ a.e. $t\in[0,T]$. Moreover, we observe that

$$\int_0^t (v_t^\delta,\eta)ds = -\int_0^t (v^\delta,\eta_t)_{\Omega_{ml}(s)}ds + (v^\delta(t),\eta(t))_{\Omega_{ml}(t)} - (v_0^\delta,\eta(0))_{\Omega_{ml}(0)}.$$

Because of uniform convergence of f_s^δ to f_s on compact subsets, as well as the assumption **(H1)**, it follows that $k(f_s^\delta(\phi^\delta)-\delta)$ converges to $k(f_s(\phi))$ uniformly on compact subsets of $Q_{ml}\cup\Omega_{ml}(0)\cup\Omega_{ml}(T)$. These facts, together with (103)-(105), ensure that we may pass to the limit in (106) and get (14).

To check that $v=0$ a.e. in $\overset{o}{Q}_s$, take a compact set $K\subseteq\overset{o}{Q}_s$. Then there is an $\delta_K\in(0,1)$ such that

$$f_s^\delta(\phi^\delta(x,t)) = 1 \quad \text{in } K \text{ for } \delta<\delta_K,$$

hence, $k(f_s^\delta(\phi^\delta(x,t)-\delta) = k(1-\delta)$ in K for $\delta<\delta_K$. From (95) we infer that

$$k(1-\delta)\|v^\delta\|^2_{L^2(K)} \le C_1 \quad \text{for } \delta<\delta_K$$

where C_1 is independent of δ. As δ tends to 0, by assumption **(H1)**, $k(1-\delta)$ blows up and consequently $\|v^\delta\|_{L^2(K)}$ converges to 0. Therefore $v = 0$ a.e. in K. Since K is an arbitrary subset, we conclude that $v = 0$ a.e. in $\overset{\circ}{Q}_s$.

In order to pass to the limit in (21), we notice that given $\xi \in L^4(0,T;H^1(\Omega))$ with $\xi_t \in L^2(0,T;L^2(\Omega))$ satisfying $\xi(T) = 0$, we can consider an extension of ξ such that $\xi^\delta \in L^4(0,T;H^1(\Omega^\delta))$ with $\xi_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ satisfying $\xi^\delta(T) = 0$. Now, we take the scalar product of (21) with ξ^δ,

$$\begin{aligned}-C_{\mathrm{v}}\int_{\Omega^\delta}\theta_0^\delta\xi^\delta(0)dx - C_{\mathrm{v}}\int_0^T\int_{\Omega^\delta}\theta^\delta\xi_t^\delta dxdt - C_{\mathrm{v}}\int_0^T\int_{\Omega^\delta}\rho_\delta(v^\delta)\theta^\delta\cdot\nabla\xi^\delta dxdt \\ +\int_0^T\int_{\Omega^\delta}K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\cdot\nabla\xi^\delta dxdt = \frac{l}{2}\int_0^T\int_{\Omega^\delta}f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta\xi^\delta dxdt.\end{aligned} \tag{107}$$

Observe that, since K_1 is a bounded Lipschitz continuous function, $K_1(\rho_\delta(\phi^\delta))$ converges to $K_1(\phi)$ in $L^p(Q)$ for $p \in [1,\infty)$. We notice that since $\rho_\delta(v^\delta)$ converges weakly to v in $L^2(0,T;H^1(\Omega))$ and $\theta^\delta \to \theta$ strongly in $C([0,T];H_o^1(\Omega)')$ we have that $\rho_\delta(v^\delta)\theta^\delta$ converges to $v\theta$ in $\mathcal{D}'(Q)$. Observe also that $f_s^{\delta\prime} \to f_s'$ in $L^q(I\!R)$ for $2 \le q < \infty$, then from (105) we infer that $f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta$ converges weakly to $f_s'(\phi)\phi_t$ in $L^{q/2}(Q)$. Moreover, from Lemma 3 the integrals over $\Omega^\delta\backslash\Omega$ are bounded independent of δ and since $|\Omega^\delta\backslash\Omega| \to 0$ as $\delta \to 0$, we have that these integrals tend to zero as $\delta \to 0$. Therefore, we may pass to the limit in (107) and obtain

$$\begin{aligned}-C_{\mathrm{v}}\int_\Omega\theta_0\xi(0)dx - C_{\mathrm{v}}\int_0^T\int_\Omega\theta\xi_t dxdt - C_{\mathrm{v}}\int_0^T\int_\Omega v\,\theta\cdot\nabla\xi\,dxdt \\ +\int_0^T\int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\xi\,dxdt = \frac{l}{2}\int_0^T\int_\Omega f_s'(\phi)\phi_t\xi\,dxdt\end{aligned}$$

for all $\xi \in L^4(0,T;H^1(\Omega))$ with $\xi \in L^2(0,T;L^2(\Omega))$ and $\xi(T) = 0$.

It remains to pass to the limit in (22). For that purpose, we proceed in similar ways as before, taking the scalar product of it with $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ and $\zeta^\delta(T) = 0$,

$$\begin{aligned}-\int_0^T\int_{\Omega^\delta}c^\delta\zeta_t^\delta dxdt - \int_0^T\int_{\Omega^\delta}\rho_\delta(v^\delta)c^\delta\cdot\nabla\zeta^\delta dxdt + K_2\int_0^T\int_{\Omega^\delta}\nabla c^\delta\cdot\nabla\zeta^\delta dxdt \\ +K_2M\int_0^T\int_{\Omega^\delta}c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\cdot\nabla\zeta^\delta\,dxdt = \int_{\Omega^\delta}c_0^\delta\zeta^\delta(0)dx.\end{aligned}$$

Then, from (104),(105), and using the fact that sequence (c^δ) is bounded in

$L^\infty(Q)$, we may pass to the limit as $\delta \to 0$ to obtain

$$-\int_0^T \int_\Omega c\zeta_t dxdt - \int_0^T \int_\Omega vc \cdot \nabla\zeta \, dxdt + K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla \zeta \, dxdt$$
$$+K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi \cdot \nabla\zeta \, dxdt = \int_\Omega c_0 \zeta(0) dx,$$

which holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta \in L^2(0,T;L^2(\Omega))$ satisfying $\zeta(T) = 0$. Observe that since $0 \leq c^\delta \leq 1$ and c^δ converges to c in $L^2(Q)$ we have that $0 \leq c \leq 1$ a.e. in Q.

Finally, it follows from (105) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$. Furthermore, $v(0) = v_0$ in $\Omega_{ml}(0)$ because $v^\delta(0) \to v(0)$ in $V'(U)$ for any U such that $\bar{U} \subseteq \Omega_{ml}(0)$. The proof of Theorem 1 is then complete. ∎

References

[1] Blanc, Ph., Gasser, L., Rappaz,J., 'Existence for a stationary model of binary alloy solidification', *Math. Mod. and Num. Anal.* **29**(6),687-699 (1995).

[2] Beckerman, C., Diepers, H.-J., Steinbach, I., Karma, A., Tong, X., 'Modeling melt convection in phase-field simulations of solidification', *Journal of Computational Physics* **154**, 468-496 (1999).

[3] Boldrini, J.L., Planas, G., 'Weak solutions to a phase field model for an alloy with thermal properties', to be published in Mathematical Methods in Applied Science.

[4] Caginalp, G., 'An analysis of a phase field model of a free boundary', *Arch. Rat. Mech. Anal.*, **92**, 205-245 (1986).

[5] Caginalp, G. and Jones, J., 'A derivation and analysis of phase-field models of thermal alloys', *Annals of Phy.*, **237**, 66-107 (1995).

[6] Caginalp, G. and Xie, W., 'Phase-field and sharp-interfase alloys models', *Phys. Rev. E*, **48**(3), 1897-1909 (1993).

[7] Cannon, J.R., DiBenedetto,E., Knightly, G.H., 'The steady state Stefan Problem with convection', *Arch. Rat. Mech. Anal.* **73**, 79-97 (1980).

[8] Cannon, J.R., DiBenedetto,E., Knightly,G.H.,'The bidimensional Stefan Problem with convection: the time dependent case', *Comm. in Part. Diff. Eq.* **14**, 1549-1604 (1983).

[9] DiBenedetto, E., Friedman, A. 'Conduction-Convection Problems with Change of Phase', *J. of Diff. Eq.* **62**, 129-185 (1986).

[10] DiBenedetto, E., O'Leary, M. 'Three-Dimensional Conduction-Convection Problems with Change of Phase', *Arch. Rat. Mech. Anal.* **123**, 99-116 (1993).

[11] Diepers, H.-J., Beckermann, C., Steinbach, I., 'Simulation of convection and ripening in a binary alloy mush using the the phase-field method', *Acta mater.* **47** (13), 3663-3678 (1999).

[12] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[13] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math, Vol 840, Springer-Verlag, 1981.

[14] Hoffman, K-H. and Jiang, L., 'Optimal control of a phase field model for solidification', *Numer. Funct. Anal. and Optim.*, **13**, 11-27 (1992).

[15] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[16] Laurençot, Ph., 'Weak solutions to a phase-field model with non-constant thermal conductivity', *Quart. Appl. Math.*, **15**(4), 739-760 (1997).

[17] Lions, J.L., *Quelques méthodes de resolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, 1969.

[18] Lions, J.L., *Control of Distributed Singular Systems*, Gauthier-Villars, 1985.

[19] Mikhailov, V.P., *Partial Differential Equations*, Mir, 1978.

[20] Moroşanu, C. and Motreanu, D., 'A generalized phase-field system', *J. Math. Anal. Appl.*, **237**, 515-540 (1999).

[21] O'Leary, M., 'Analysis of the mushy region in conduction-convection problems with change of phase' , *Elect. J. Diff. Eqs.* **1997**(4), 1-14 (1997).

[22] Rappaz, J. and Scheid, J.F., 'Existence of solutions to a Phase-field model for the isothermal solidification process of a binary alloy', *Math. Meth. Appl. Sci.*, **23**, 491-512 (2000).

[23] Simon, J., 'Compacts sets in the space $L^p(0,T,B)$', *Ann. Mat. Pura Appl.*, **146**, 65-96 (1987).

[24] Soto Segura, H.P., 'Análise de um Modelo Matemático de Condução-Conveção do Tipo Entalpia para Solidificação', Ph.D. Thesis, IMECC-UNICAMP, Brazil (2000).

[25] Temam, R., *Navier-Stokes Equations*, AMS Chelsea Publishing, 2001.

[26] Voller,V.R., Cross,M., Markatos,N.C., 'An enthalpy method for convection/diffusion phase change', *Int. J. for Numer. Meth. in Eng.* **24**, 271-284 (1987).

[27] Voller, V.R., Prakash, C., 'A fixed grid numerical modeling methodology for convection-diffusion mushy region phase-change problems', *Int. J. Heat Mass Trans.*, **30** (8), 1709-1719 (1987).

[28] Voss, M.J., Tsai, H.L., 'Effects of the rate of latente heat release on fluid flow and solidification patterns during alloy solidification', *Int. J. Engng. Sci.*, **34** (6), 715-737 (1996).

[29] Warren, J.A. and Boettinger, W.J., 'Prediction of dendritic growth and microsegregation patterns in a binary alloy using the phase-field method', *Acta Metall. Mater.*, **43**(2), 689-703 (1995).

[30] Wheeler, A.A., Boettinger, W.J. and McFadden, G.B., 'Phase-field model for isothermal phase transitions in binary alloys', *Phys. Rev. A*, **45**, 7424-7439 (1992).

Capítulo 4

Um modelo bidimensional do tipo campo de fase com convecção para a mudança de fase de uma liga binária

Resumo

Neste trabalho analisamos um modelo bidimensional para um processo de evolução para a solidificação de uma liga binária com propriedades térmicas. O modelo inclui a possibilidade de fluxo nas regiões não-sólidas, as quais são desconhecidas *a priori*, e consiste de um sistema de equações diferenciais parciais altamente não-linear associado a um problema de fronteira livre. O sistema é composto pela equação do campo de fase, a equação do calor, a equação da concentração e as equações de Navier-Stokes modificadas por um termo de penalização do tipo Carman-Kozeny, o qual toma conta do efeito *mushy*, e um termo do tipo Boussinesq o qual considera os efeitos das variações de temperatura e de concentração. É provada a existência de soluções fracas para o sistema. O problema é aproximado e uma sequência de soluções aproximadas é obtida usando o Teorema de Ponto Fixo de Leray-Schauder. Uma solução é obtida usando argumentos de compacidade.

A Bidimensional Phase-Field Model with Convection for Change Phase of an Alloy

Gabriela Planas [*] gplanas@ime.unicamp.br

José Luiz Boldrini [†] boldrini@ime.unicamp.br

IMECC - UNICAMP

Caixa Postal 6065, CEP 13081-970, Campinas-SP, Brazil

*Supported by FAPESP, Brazil, grant 97/13932-4

†Partially supported by CNPq, Brazil, grant 300513/87-9

A Bidimensional Phase-Field Model with Convection for Change Phase of an Alloy

Abstract

The article analyzes a two-dimensional phase-field model for a non-stationary process of solidification of a binary alloy with thermal properties. The model allows the occurrence of fluid flow in non-solid regions, which are a priori unknown, and is thus associated to a free boundary value problem for a highly non-linear system of partial differential equations. These equations are the phase-field equation, the heat equation, the concentration equation and a modified Navier-Stokes equations obtained by the addition of a penalization term of Carman-Kozeny type, which accounts for the mushy effects, and also of a Boussinesq term to take in care of the effects of variations of temperature and concentration in the flow. A proof of existence of weak solutions for such system is given. The problem is firstly approximated and a sequence of approximate solutions is obtained by Leray-Schauder fixed point theorem. A solution is then found by using compactness argument.

MSC Mathematics Subject Classification: 35K65, 76D05, 80A22, 35K55, 82B26, 35Q10, 76R99

1 Introduction

Through the introduction of an extra variable to distinguish among physical phases, the phase-field methodology provides a continuum description of phase change processes. This method has proved itself to be a powerful tool for the study of situations with complex growth structures like dendrites, and recently phase-field models for solidification have been extended to include melt convection, bringing interesting new mathematical aspects to the methodology.

In an attempt to understand such mathematical aspects, we consider here a two-dimensional phase-field model for a non-stationary process of solidification with convection of a binary alloy with thermal properties. Our objective is to prove the existence of solutions of a mathematical model that combines ideas of Voller et al. [12, 13] and of Blanc et al. [1] for taking in consideration the possibility of flow, with those of Caginalp et al. [2] for the phase-field and the thermal properties of the alloy. The resulting system will be described in detail in the next section. Here, we just observe that, besides having a phase-field equation, a heat equation and a concentration equation, it also includes the Navier-Stokes equations modified by the addition of a Carman-Kozeny type term to take care of the flow in mushy regions and also by the addition a Boussinesq type term to take in consideration buoyancy forces due to thermal and concentration differences. Since these equations for the flow only hold in an a priori unknown non-solid region, the model corresponds to a free-boundary value problem. Moreover, since the Carman-Kozeny term is dependent on the local solid fraction, this is assumed to be functionally related to the phase-field.

The phase-field model with convection considered here includes advection terms in each of its equations. In a recent paper, [9], a simplified version of this model, which did not include the advection term in the phase-field equation, was analyzed. We should say that the inclusion of this term brings several new technical difficulties to an already hard problem. To overcome these difficulties is the purpose of this paper; for this, we had to adapt to our case the results presented in Hoffman and Jiang [5] concerning the phase-field equation. We also had to restrict the analysis to the two-dimensional situation. We will comment more about this point in the next section; here we just remark that this restriction in the dimension of the space is clearly of technical nature. We hope to remove it in the future.

Existence of solutions will be obtained by using a regularization technique similar to the one already used in [1] and [9]: with the help of an auxiliary parameter, we will transform the original free-boundary value problem into a more standard penalized one. This regularized problem will then be studied by using fixed point arguments, and then we pass to the limit to obtain a solution of the original problem.

The outline of this paper is as follows. In Section 2 we detail the model we consider; we also fix the notation and state our main result. In Section 3 we study an auxiliary phase-field problem. The description and the analysis of the regularized problem is done in Section 4. Section 5 is devoted to proof the main existence theorem.

2 The model and the main result

Consider $0 < T < +\infty$, a bounded open domain $\Omega \subset I\!R^2$ with smooth boundary $\partial\Omega$, and denote $Q = \Omega \times (0,T)$. Then, consider the following problem:

$$\begin{aligned} v_t - \nu\Delta v + \nabla p \quad &+ \quad v\cdot\nabla v + k(f_s(\phi))v = \mathcal{F}(c,\theta) \ \text{ in } Q_{ml}, && (1)\\ \operatorname{div} v = 0 \qquad & \text{in } Q_{ml}, && (2)\\ v = 0 \qquad & \text{in } Q_s, && (3)\end{aligned}$$

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi \quad &- \quad \epsilon^2\Delta\phi - \frac{1}{2}(\phi-\phi^3) \\ &= \quad \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \ \text{in}\, Q, && (4)\end{aligned}$$

$$\begin{aligned} C_v\theta_t + C_v v\cdot\nabla\theta \quad &= \quad \nabla\cdot K_1(\phi)\nabla\theta + \frac{l}{2}f_s(\phi)_t \quad \text{in } Q, && (5)\\ c_t + v\cdot\nabla c \quad &= \quad K_2\left(\Delta c + M\nabla\cdot c(1-c)\nabla\phi\right) \ \text{ in } Q, && (6)\end{aligned}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega\times(0,T), \quad v = 0 \text{ on } \partial Q_{ml}, \qquad (7)$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad c(0) = c_0 \text{ in } \Omega, \quad v(0) = v_0 \text{ in } \Omega_{ml}(0), \qquad (8)$$

In the previous equations, the order parameter (phase-field) ϕ is the state variable characterizing the different phases; v is the velocity field, and p is the associated hydrostatic pressure; $f_s \in [0,1]$ is the solid fraction; θ is the temperature; $c \in [0,1]$ is the concentration (the fraction of one of the two materials in the mixture.) The Carman-Kozeny type term $k(f_s)$ accounts for the mushy effect on the flow, and its usual form is $k(f_s) = C_0 {f_s}^2/(1-f_s)^3$. We do not restrict to this form and allow more general expressions. $\mathcal{F}(c,\theta)$ denotes the buoyancy forces, which by using Boussinesq approximation, we assume to be of form $\mathcal{F}(c,\theta) = \rho \mathbf{g}\,(c_1(\theta-\theta_r) + c_2(c-c_r)) + F$. Here, $\rho > 0$ is the mean value of the density (constant); $\mathbf{g}$ is the acceleration of gravity; c_1 and c_2 are two real constants; θ_r and c_r are respectively the reference temperature and concentration, which for simplicity of exposition will be assumed to be zero, and F is a given external force field. Also, $\alpha > 0$ is the relaxation scaling; $\beta = \epsilon[s]/3\sigma$, where $\epsilon > 0$ is a measure of the interface width; σ is the surface tension, and $[s]$ is the entropy density difference between phases; $\nu > 0$ is the viscosity; $C_v > 0$ is the specific heat; $l > 0$ the latent heat (constant); θ_A, θ_B are the melting temperatures of two materials composing the alloy; $K_2 > 0$ is the solute diffusivity, and M is a constant related to the slopes of solidus and liquidus lines. Finally, $K_1 > 0$ denotes the thermal conductivity which is assumed to depend on the phase-field.

The domain Q is composed of three regions, Q_s, Q_m and Q_l. The first region is fully solid, the second is mushy and the third is fully liquid . They are defined by

$$\begin{array}{lcll} Q_s & = & \{\ (x,t) \in Q \quad / & f_s(\phi(x,t)) = 1\ \}, \\ Q_m & = & \{\ (x,t) \in Q \quad / & 0 < f_s(\phi(x,t)) < 1\ \}, \\ Q_l & = & \{\ (x,t) \in Q \quad / & f_s(\phi(x,t)) = 0\ \}, \end{array} \tag{9}$$

and Q_{ml} will refer to the not-solid region, i.e.,

$$Q_{ml} = Q_m \cup Q_l = \{\ (x,t) \in Q \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{10}$$

At each time $t \in [0,T]$, $\Omega_{ml}(t)$ is defined by

$$\Omega_{ml}(t) = \{\ x \in \Omega \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{11}$$

In view of these regions are a priori unknown, the model is a free boundary problem.

Throughout this paper we assume the conditions,

(H1) k is a non decreasing function of class $C^1[0,1)$ satisfying $k(0) = 0$ and $\lim\limits_{x \to 1^-} k(x) = +\infty$,

(H2) f_s is a Lipschitz continuous function defined on $I\!R$ and satisfying $0 \le f_s(r) \le 1$ for $r \in I\!R$; f'_s is measurable,

(H3) K_1 is a Lipschitz continuous function defined on $I\!R$ such that there exist $a > 0$ and $b > 0$ for which

$$0 < a \leq K_1(r) \leq b \qquad \text{for all } r \in I\!R,$$

(H4) F is a given function in $L^2(Q)$.

Our purpose in this work is to show that problem (1)-(8) admits at least one solution in a sense to be made precise below.

Before that, we comment on the restriction on the spatial dimension. Since the modified Navier-Stokes equations only hold in the non-solid region Q_{ml}, this set must be open for these equations to be understood at least in the sense of distributions. This information is in particular implied by the continuity of phase-field ϕ which in turn depends on the smoothness of v. It turns out that only for the bidimensional case we are able to show enough regularity of v to yield the continuity of ϕ. As we wrote in the Introduction, such limitations are quite clearly of technical nature, and it is our hope to remove them in the future.

We use standard notation in this paper. We just briefly recall the following functional spaces associated to the Navier-Stokes equations. Let $G \subseteq I\!R^2$ be a non-void bounded open set; for $T > 0$, consider also $Q_G = G \times (0,T)$ Then,

$$\begin{array}{rcl} \mathcal{V}(G) & = & \left\{ w \in (C_0^\infty(G))^2, \text{ div } w = 0 \right\}, \\ H(G) & = & \text{closure of } \mathcal{V}(G) \text{ in } \left(L^2(G)\right)^2, \\ V(G) & = & \text{closure of } \mathcal{V}(G) \text{ in } \left(H_0^1(G)\right)^2, \\ H^{\tau,\tau/2}(\overline{Q}_G) & = & \text{Hölder continuous functions of exponent } \tau \text{ in } x \\ & & \text{and exponent } \tau/2 \text{ in } t, \\ W_q^{2,1}(Q_G) & = & \left\{ w \in L^q(Q_G) / \, D_x w, D_x^2 w \in L^q(Q_G), w_t \in L^q(Q_G) \right\}. \end{array}$$

When $G = \Omega$, we denote $H = H(\Omega)$, $V = V(\Omega)$. Properties of these functional spaces can be found for instance in [6, 11]. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. We also put $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ the inner product of $\left(L^2(\Omega)\right)^2$.

The main result of this paper is the following.

Theorem 1 *Let be $T > 0$, $\Omega \subseteq I\!R^2$ a bounded open domain of class C^3. Suppose that $v_0 \in H(\Omega_{ml}(0))$, $\phi_0 \in W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $2 < q < 4$, $1/2 < \gamma \leq 1$, satisfying the compatibility condition $\dfrac{\partial \phi_0}{\partial n} = 0$ on $\partial\Omega$, $\theta_0 \in L^2(\Omega)$*

and $c_0 \in L^2(\Omega)$ *satisfying* $0 \leq c_0 \leq 1$ *a.e. in* $\bar{\Omega}$. *Under the assumptions* **(H1)-(H4)**, *there exist functions* (v, ϕ, θ, c) *such that*

i) $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\quad v = 0$ *a.e. in* $\overset{\circ}{Q}_s$, $\quad v(0) = v_0$ *in* $\Omega_{ml}(0)$, *where* Q_s *is defined by (9) and* $\Omega_{ml}(0)$ *by (11),*

ii) $\phi \in W_q^{2,1}(Q)$, $\quad \phi(0) = \phi_0$,

iii) $\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\theta(0) = \theta_0$,

iv) $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $c(0) = c_0$, $0 \leq c \leq 1$ *a.e. in* Q,

and such that

$$\begin{aligned}
(v(t), \eta(t))_{\Omega_{ml}(t)} &- \int_0^t (v, \eta_t)_{\Omega_{ml}(s)} ds + \nu \int_0^t (\nabla v, \nabla \eta)_{\Omega_{ml}(s)} ds \\
&+ \int_0^t (v \cdot \nabla v, \eta)_{\Omega_{ml}(s)} ds + \int_0^t (k(f_s(\phi))v, \eta)_{\Omega_{ml}(s)} ds \quad (12) \\
&= \int_0^t (\mathcal{F}(c, \theta), \eta)_{\Omega_{ml}(s)} ds + (v_0, \eta(0))_{\Omega_{ml}(0)},
\end{aligned}$$

$t \in (0,T)$, *for any* $\eta \in L^2(0,T;V(\Omega_{ml}(t)))$ *with compact support contained in* $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ *and* $\eta_t \in L^2(0,T;V(\Omega_{ml}(t))')$ *where* Q_{ml} *is defined by (10) and* $\Omega_{ml}(t)$ *by (11),*

$$\alpha\epsilon^2 \phi_t + \alpha\epsilon^2 v \cdot \nabla\phi - \epsilon^2 \Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \quad \text{a.e. in } Q, \tag{13}$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{a.e. on } \partial\Omega \times (0,T), \tag{14}$$

$$\begin{aligned}
-C_v \int_0^T \int_\Omega \theta \zeta_t dxdt &- C_v \int_0^T \int_\Omega v\,\theta \cdot \nabla\zeta\, dxdt + \int_0^T \int_\Omega K_1(\phi) \nabla\theta \cdot \nabla\zeta\, dxdt \\
&= \frac{l}{2} \int_0^T \int_\Omega f_s(\phi)_t \zeta\, dxdt + C_v \int_\Omega \theta_0 \zeta(0) dx
\end{aligned} \tag{15}$$

$$\begin{aligned}
-\int_0^T \int_\Omega c\zeta_t dxdt - \int_0^T \int_\Omega v\, c \cdot \nabla\zeta\, dxdt &+ K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla\zeta\, dxdt \\
+K_2 M \int_0^T \int_\Omega c(1-c)\nabla\phi \cdot \nabla\zeta\, dxdt &= \int_\Omega c_0 \zeta(0) dx,
\end{aligned} \tag{16}$$

for any $\zeta \in L^2(0,T;H^1(\Omega))$ *with* $\zeta_t \in L^2(0,T;L^2(\Omega))$ *and* $\zeta(T) = 0$ *in* Ω.

Remark. The restriction $q > 2$ ensure the continuity of phase-field because $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - 4/q$ if $q > 2$ ([6] p. 80). Therefore the set Q_{ml} is open giving a meaningful interpretation to equation of velocity field. The restriction $q < 4$ comes from the regularity of velocity field. It will be clear in the next section.

3 An auxiliary problem

We consider the initial boundary value problem,

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{1}{2}(\phi - \phi^3) + g \quad \text{in } Q, & (17)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), & (18)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega, & (19) \end{aligned}$$

and prove the following result using a technique similar to the one already used in [5] to treat a phase-field equation without convective term.

Theorem 2 *Suppose that $g \in L^q(Q)$ with $2 \leq q < 4$, $v \in L^2(0,T;V)\cap L^\infty(0,T;H)$ and $\phi_0 \in W^{2-2/q,q}(\Omega)$ satisfying the compatibility conditions $\dfrac{\partial\phi_0}{\partial n} = 0$ on $\partial\Omega$. Then there exist a unique $\phi \in W_q^{2,1}(Q)$ solution of problem (17)-(19) for any $T > 0$, which satisfies the estimate*

$$\|\phi\|_{W_q^{2,1}(Q)} \leq C\left(\|\phi_0\|_{W^{2-2/q,q}(\Omega)} + \|g\|_{L^q(Q)} + \|\phi_0\|^3_{W^{2-2/q,q}(\Omega)} + \|g\|^3_{L^q(Q)}\right) \tag{20}$$

where C depends on $\|v\|_{L^4(Q)}$, on Ω and T.

Proof: In order to apply Leray-Schauder fixed point theorem ([3] p. 189) we consider the operator T_λ, $0 \leq \lambda \leq 1$, on the Banach space $B = L^6(Q)$, which maps $\hat{\phi} \in B$ into ϕ by solving the problem

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{\lambda}{2}(\hat{\phi} - \hat{\phi}^3) + \lambda g \quad \text{in } Q, & (21)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), & (22)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega. & (23) \end{aligned}$$

We define $G_\lambda = \dfrac{\lambda}{2}(\hat{\phi} - \hat{\phi}^3) + \lambda g$ and we observe that $G_\lambda \in L^2(Q)$. Since $v \in L^4(Q)$, we infer from L^p-theory of parabolic equations ([6], Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution ϕ of problem (21)-(23) with $\phi \in W_2^{2,1}(Q)$. Due to the embedding of $W_2^{2,1}(Q)$ into $L^p(Q)$, for any $p \in [1, \infty)$ ([7] p.15), the operator T_λ is well defined from B into B.

To prove continuity of T_λ, let $\hat{\phi}_n \in B$ strongly converging to $\hat{\phi} \in B$; for each n, let $\phi_n = T_\lambda(\hat{\phi}_n)$. We have that ϕ_n satisfies the following estimate ([6] p. 341)

$$\|\phi_n\|_{W_2^{2,1}(Q)} \leq C\left(\|\hat{\phi}_n\|_{L^2(Q)} + \|\hat{\phi}_n\|^3_{L^6(Q)} + \|g\|_{L^2(Q)} + \|\phi_0\|_{H^1(\Omega)}\right)$$

for some constant C independent of n. Since $W_2^{2,1}(Q)$ is compactly embedded in $L^2(0, T; W^{1,p}(\Omega))$ ([10] Cor.4) and in $L^p(Q)$, $p \in [1, \infty)$, it follows that there exist a subsequence of ϕ_n (which we still denote by ϕ_n) strongly converging to $\phi = T_\lambda(\hat{\phi})$ in B. Therefore T_λ is continuous for all $0 \leq \lambda \leq 1$. At the same time, T_λ is bounded in $W_2^{2,1}(Q)$, and the embedding of this space in B is compact. Thus, we conclude that T_λ is a compact operator for each $\lambda \in [0, 1]$.

To prove that for $\hat{\phi}$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \leq \lambda_1, \lambda_2 \leq 1$ and $\phi_i\,(i = 1, 2)$ be the corresponding solutions of (21)-(23). For $\phi = \phi_1 - \phi_2$ the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q)} \leq C|\lambda_1 - \lambda_2|\left(\|\hat{\phi}\|_{L^2(Q)} + \|\hat{\phi}\|^3_{L^6(Q)} + \|g\|_{L^2(Q)}\right)$$

where C is independent of λ_i. Therefore, T_λ is uniformly continuous in λ.

Now we have to estimate the set of all fixed points of T_λ, let $\phi \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v \cdot \nabla\phi - \epsilon^2\Delta\phi &= \frac{\lambda}{2}(\phi - \phi^3) + \lambda g \quad \text{in } Q, && (24)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega \times (0, T), && (25)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega. && (26) \end{aligned}$$

We multiply (24) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over $\Omega \times (0, t)$. After integration by parts and the use the Hölder's, Young's and interpolation inequalities, we obtain in the usual manner the following estimate

$$\begin{aligned}\int_\Omega \left(\phi^2 + |\nabla\phi|^2\right) dx + \|\phi\|^2_{W_2^{2,1}(Q)} \leq C\left(\|g\|^2_{L^2(Q)} + \|\phi_0\|^2_{H^1(\Omega)}\right)\\ +C\int_0^t \left(1+\|v\|^4_{L^4(\Omega)}\right)\left(\|\phi\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)}\right) dt\end{aligned} \tag{27}$$

where C is independent of λ. By applying Gronwall's Lemma we get

$$\|\phi\|_{L^6(Q)} \leq C\|\phi\|_{W_2^{2,1}(Q)} \leq C'$$

where C and C' are constants independent of λ. Therefore, all fixed points of T_λ in B are bounded independently of $\lambda \in [0,1]$.

Finally, for $\lambda = 0$, it is clear that problem (21)-(23) has a unique solution. Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $\phi \in B \cap W_2^{2,1}(Q)$ of the operator T_1, i.e., $\phi = T_1(\phi)$. This corresponds to a solution of problem (17)-(19). Observe that $W_2^{2,1}(Q)$ is embedded into $L^p(Q)$ for any $p \in [1,\infty)$, this implies that $G = \dfrac{1}{2}(\phi - \phi^3) + g \in L^q(Q)$ and further $\phi \in W_q^{2,1}(Q)$.

To prove estimate (20), observe that from L^p-theory of parabolic equations we have

$$\begin{aligned}\|\phi\|_{W_q^{2,1}(Q)} &\leq C\left(\|G\|_{L^q(Q)} + \|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\ &\leq C\left(\|g\|_{L^q(Q)} + \|\phi\|_{L^q(Q)} + \|\phi\|^3_{L^{3q}(Q)} + \|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\ &\leq C\left(\|g\|_{L^q(Q)} + \|\phi\|_{W_2^{2,1}(Q)} + \|\phi\|^3_{W_2^{2,1}(Q)} + \|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right).\end{aligned}$$

Using estimate (27) we deduce (20).

It remains to show uniqueness of the solution. Let us assume that ϕ_1 and ϕ_2 are two solutions of problem (17)-(19). Then the difference $\phi = \phi_1 - \phi_2$ satisfies the following initial boundary value problem

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{1}{2}\phi\left(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right) \quad \text{in } Q, \tag{28}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega\times(0,T), \tag{29}$$

$$\phi(0) = 0 \quad \text{in } \Omega, \tag{30}$$

We remark that $d := \phi_1^2 + \phi_1\phi_2 + \phi_2^2 \geq 0$. Multiplying (28) by ϕ and using the usual method of Gronwall's Lemma give us $\phi \equiv 0$. Therefore, the solution of problem (17)-(19) is unique and the proof of Theorem 2 is then complete. ∎

4 A regularized problem

In this section we introduce a regularized version of the original problem. As in [1] and [9], the idea is to modify the problem in such way that the Navier-Stokes equations will hold in the whole domain Ω instead of only in a apriori unknown region. For technical reason, we also introduce a suitable regularization of the coefficients of the equations. For this regularized problem, we prove an existence result by using Leray-Schauder Fixed Point Theorem ([3] p. 189).

For this, we need to recall certain results. We start by recalling that there is an extension operator $Ext(\cdot)$ taking any function w in the space $W_2^{2,1}(Q)$ and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^3)$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^3)} \leq C\,\|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [8] p.157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^3)$ be a family of symmetric positive mollifier functions converging to the Dirac delta function, and denote by $*$ the convolution operation. Then, given a function $w \in W_2^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(I\!R^3)$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

This sort of regularization will be used with the phase-field variable. We will also need a regularization for the velocity, and for it we proceed as follows.

Given $v \in L^2(0,T;V)$, first we extend it as zero in $I\!R^3 \backslash Q$. Then, as in [8] p. 157, by using reflection and cutting-off, we extend the resulting function to another one defined on $I\!R^3$ and with compact support. Without the danger of confusion, we again denote such extension operator by $Ext(v)$. Then, being $\delta > 0$, ρ_δ and $*$ as above, operating on each component, we can again define a regularization $\rho_\delta(v) \in C_0^\infty(I\!R^3)$ of v by

$$\rho_\delta(v) = \rho_\delta * Ext(v).$$

Besides having properties of control of Sobolev norms in terms of the corresponding norms of the original function (exactly as above), such extension has the property described below.

For $0 < \delta \leq 1$, define firstly the following family of uniformly bounded open sets

$$\Omega^\delta = \{x \in I\!R^2 : d(x,\Omega) < \delta\}. \tag{31}$$

We also define the associated space-time cylinder $Q^\delta = \Omega^\delta \times (0, T)$.

Obviously, for any $0 < \delta_1 < \delta_2$, we have $\Omega \subset \Omega^{\delta_1} \subset \Omega^{\delta_2}$, $Q \subset Q^{\delta_1} \subset Q^{\delta_2}$. Also, by using properties of convolution, we conclude that $\rho_\delta(v)|_{\partial\Omega^\delta} = 0$. In particular, for $v \in L^\infty(0,T;H) \cap L^2(0,T;V)$, we conclude that $\rho_\delta(v) \in L^\infty(0,T;H) \cap L^2(0,T;V(\Omega^\delta))$.

Moreover, since Ω is of class C^3, there exists $\delta(\Omega) > 0$ such that for $0 < \delta \leq \delta(\Omega)$, we conclude that Ω^δ is of class C^2 and such that the C^2 norms of the maps defining $\partial\Omega^\delta$ are uniformly estimated with respect to δ in terms of the C^3 norms of the maps defining $\partial\Omega$.

Since we will be working with the sets Ω^δ, the main objective of this last remark is to ensure that the constants associated to Sobolev immersions and interpolations inequalities, involving just up to second order derivatives and used with Ω^δ, are uniformly bounded for $0 < \delta \leq \delta(\Omega)$. This will be very important to guarantee that certain estimates will be independent of δ.

Finally, let f_s^δ be any regularization of f_s.

Now, we are in position to define the regularized problem. For $\delta \in (0, \delta(\Omega)]$, we consider the system

$$\begin{aligned}\frac{d}{dt}(v^\delta, u) \quad &+ \quad \nu(\nabla v^\delta, \nabla u) + (v^\delta \cdot \nabla v^\delta, u) + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, u) \\ &= \quad (\mathcal{F}(c^\delta, \theta^\delta), u) \text{ for all } u \in V,\ t \in (0,T), \qquad (32)\end{aligned}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t^\delta + \alpha\epsilon^2\rho_\delta(v^\delta)\cdot\nabla\phi^\delta \quad &- \quad \epsilon^2\Delta\phi^\delta - \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) \\ &= \quad \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q^\delta, \qquad (33)\end{aligned}$$

$$C_v\theta_t^\delta + C_v\rho_\delta(v^\delta)\cdot\nabla\theta^\delta = \nabla\cdot\left(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\right) + \frac{l}{2}f_s^\delta(\phi^\delta)_t \quad \text{in } Q^\delta, \qquad (34)$$

$$c_t^\delta - K_2\Delta c^\delta + \rho_\delta(v^\delta)\cdot\nabla c^\delta = K_2 M\nabla\cdot\left(c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\right) \quad \text{in } Q^\delta, \qquad (35)$$

$$\frac{\partial\phi^\delta}{\partial n} = 0, \quad \frac{\partial\theta^\delta}{\partial n} = 0, \quad \frac{\partial c^\delta}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0,T), \qquad (36)$$

$$v^\delta(0) = v_0^\delta \text{ in } \Omega, \quad \phi^\delta(0) = \phi_0^\delta, \quad \theta^\delta(0) = \theta_0^\delta, \quad c^\delta(0) = c_0^\delta \text{ in } \Omega^\delta. \qquad (37)$$

We then have the following existence result.

Proposition 1 *For each $\delta \in (0, \delta(\Omega)]$, let $v_0^\delta \in H$, $\phi_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $\theta_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $1/2 < \gamma \leq 1$, and $c_0^\delta \in C^1(\bar{\Omega}^\delta)$, $0 < c_0^\delta < 1$ in $\bar{\Omega}^\delta$ satisfying the* *compatibility conditions* $\dfrac{\partial \phi_0^\delta}{\partial n} = \dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ *on* $\partial\Omega^\delta$. *Assume that* **(H1)-(H4)** *hold. Then there exist functions* $(v^\delta, \phi^\delta, \theta^\delta, c^\delta)$ *which satisfy (32)-(37) for any* $T > 0$ *and*

i) $v^\delta \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad v_t^\delta \in L^2(0,T;V'),$

ii) $\phi^\delta \in L^2(0,T;H^2(\Omega^\delta)), \quad \phi_t^\delta \in L^2(Q^\delta),$

iii) $\theta^\delta \in L^2(0,T;H^2(\Omega^\delta)), \quad \theta_t^\delta \in L^2(Q^\delta),$

iv) $c^\delta \in C^{2,1}(Q^\delta), \quad 0 < c^\delta < 1.$

Proof: For simplicity we shall omit the superscript δ at v^δ, ϕ^δ, θ^δ, c^δ. First of all, we consider the following family of operators, indexed by the parameter $0 \le \lambda \le 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^2(0,T;H) \times L^2(Q^\delta) \times L^2(Q^\delta) \times L^2(Q^\delta),$$

and defined as follows: given $(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c}) \in B$, let $\mathcal{T}_\lambda(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c}) = (v, \phi, \theta, c)$, where (v, ϕ, θ, c) is obtained by solving the problem

$$\begin{aligned} \frac{d}{dt}(v,u) \ &+\ \nu(\nabla v, \nabla u) + (v \cdot \nabla v, u) = \lambda(\mathcal{F}(\hat{c}, \hat{\theta}), u) \\ &-\ \lambda(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u) \text{ for all } u \in V,\, t \in (0,T), \end{aligned} \tag{38}$$

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi \ &-\ \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) \\ &=\ \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta, \end{aligned} \tag{39}$$

$$C_{\rm v}\theta_t + C_{\rm v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{40}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M \nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \text{ in } Q^\delta, \tag{41}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{42}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{43}$$

We observe that clearly (v, ϕ, θ, c) is a solution of (32)-(37) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point by using the Leray-Schauder fixed point theorem ([3] p.189).

To verify that $\mathcal{T}_\lambda$ is well defined, observe that equation (38) is the classical Navier-Stokes equation and since $k(f_s^\delta(\hat{\phi})-\delta)\hat{v} \in L^2(Q)$, there exist a unique solution $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ ([11] p.198).

Since $\hat{\theta}$, $\hat{c} \in L^2(Q^\delta)$ and $\rho_\delta(v) \in L^4(Q^\delta)$ we infer from Theorem 2 that there is a unique solution ϕ of equation (39) with $\phi \in W_2^{2,1}(Q^\delta)$.

Since K_1 is a bounded Lipschitz continuous function and $\rho_\delta(\phi) \in C^\infty(Q^\delta)$, we have that $K_1(\rho_\delta(\phi)) \in W_r^{1,1}(Q^\delta)$, $1 \leq r \leq \infty$, and since $\rho_\delta(v) \in L^4(Q^\delta)$ and $f_s^\delta(\phi)_t = f_s^{\delta'}(\phi)\phi_t \in L^2(Q^\delta)$, we infer from L^p-theory of parabolic equations ([6], Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ of equation (40) with $\theta \in W_2^{2,1}(Q^\delta)$.

We observe that equation (41) is a semilinear parabolic equation with smooth coefficients and growth conditions on the non-linear forcing terms to apply semigroup results of Henry [4], p.75. Thus, there is a unique global classical solution c. In addition, note that equation (41) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principles together with conditions $0 < c_0^\delta < 1$ and $\dfrac{\partial c^\delta}{\partial n} = 0$, we can deduce that

$$0 < c(x,t) < 1, \qquad \forall\, (x,t) \in Q^\delta. \tag{44}$$

Therefore, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$ let $(\hat{v}^k, \hat{\phi}^k, \hat{\theta}^k, \hat{c}^k)$, $k \in I\!N$ be a sequence in B such that converges strongly in B to $(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c})$ and let $(v^k, \phi^k, \theta^k, c^k)$ the solution of the problem:

$$\begin{aligned} \frac{d}{dt}(v^k, u) \;&+\; \nu(\nabla v^k, \nabla u) + (v^k \cdot \nabla v^k, u) = \lambda(\mathcal{F}(\hat{c}^k, \hat{\theta}^k), u) \\ &-\; \lambda(k(f_s^\delta(\hat{\phi}^k) - \delta)\hat{v}^k, u) \text{ for all } u \in V,\; t \in (0,T), \end{aligned} \tag{45}$$

$$\begin{aligned} \alpha\epsilon^2\phi_t^k + \alpha\epsilon^2\rho_\delta(v^k)\cdot\nabla\phi^k \;&-\; \epsilon^2\Delta\phi^k - \frac{1}{2}(\phi^k - (\phi^k)^3) \\ &=\; \lambda\beta\left(\hat{\theta}^k + (\theta_B - \theta_A)\hat{c}^k - \theta_B\right) \text{ in } Q^\delta, \end{aligned} \tag{46}$$

$$C_v\theta_t^k + C_v\rho_\delta(v^k)\cdot\nabla\theta^k = \nabla\cdot\left(K_1(\rho_\delta(\phi^k))\nabla\theta^k\right) + \frac{l}{2}f_s^\delta(\phi^k)_t \text{ in } Q^\delta, \tag{47}$$

$$c_t^k - K_2\Delta c^k + \rho_\delta(v^k)\cdot\nabla c^k = K_2 M\nabla\cdot\left(c^k(1-c^k)\nabla\rho_\delta(\phi^k)\right) \text{ in } Q^\delta, \quad (48)$$

$$\frac{\partial\phi^k}{\partial n} = 0, \quad \frac{\partial\theta^k}{\partial n} = 0, \quad \frac{\partial c^k}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta\times(0,T), \quad (49)$$

$$v^k(0) = v_0^\delta \text{ in } \Omega, \quad \phi^k(0) = \phi_0^\delta, \quad \theta^k(0) = \theta_0^\delta, \quad c^k(0) = c_0^\delta \text{ in } \Omega^\delta. \quad (50)$$

We show that the sequence $(v^k,\phi^k,\theta^k,c^k)$ converges strongly in B to $(v,\phi,\theta,c) = \mathcal{T}_\lambda(\hat{v},\hat{\phi},\hat{\theta},\hat{c})$. For that purpose, we will obtain estimates to $(v^k,\phi^k,\theta^k,c^k)$ independent of k. We denote by C_i any positive constant independent of k.

We take $u = v^k$ in equation (45). Using Hölder's and Young's inequalities we obtain

$$\frac{d}{dt}\int_\Omega |v^k|^2 dx + \nu\int_\Omega |\nabla v^k|^2 dx \le C_1\int_\Omega \left(|F|^2 + |\hat{v}^k|^2 + |\hat{\theta}^k|^2 + |\hat{c}^k|^2 + |v^k|^2\right) dx.$$

Then, by the usual method of Gronwall's inequality, we get

$$\|v^k\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C_1. \quad (51)$$

From the equation (45) we infer that

$$\begin{aligned}\|v_t^k\|_{V'} \le C_1 &\left(\|v^k\|_V + \|v^k\|^2_{L^4(\Omega)} + \|F\|_{L^2(\Omega)}\right.\\ &\left.+\|\hat{v}^k\|_{L^2(\Omega)} + \|\hat{\theta}^k\|_{L^2(\Omega^\delta)} + \|\hat{c}^k\|_{L^2(\Omega^\delta)}\right),\end{aligned}$$

then, using (51) we obtain

$$\|v_t^k\|_{L^2(0,T;V')} \le C_1. \quad (52)$$

From estimate (20) we have that

$$\begin{aligned}\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C &\left(\|\phi_0\|_{H^1(\Omega^\delta)} + \|\hat{\theta}^k\|_{L^2(Q^\delta)} + \|\hat{c}^k\|_{L^2(Q^\delta)}\right.\\ &\left.+\|\phi_0\|^3_{H^1(\Omega^\delta)} + \|\hat{\theta}^k\|^3_{L^2(Q^\delta)} + \|\hat{c}^k\|^3_{L^2(Q^\delta)} + 1\right)\end{aligned}$$

where C depends on $\|\rho_\delta(v^k)\|_{L^4(Q^\delta)}$. Therefore, using (51) we conclude that

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C_1. \quad (53)$$

Now, multiplying (47) by θ^k one obtains

$$\int_{\Omega^\delta} |\theta^k|^2 dx + \int_0^t\int_{\Omega^\delta} |\nabla\theta^k|^2 dxdt \le C_1 + C_2\int_0^t\int_{\Omega^\delta}\left(|\phi_t^k|^2 + |\theta^k|^2\right) dxdt \quad (54)$$

and we infer from (53) and Gronwall's Lemma that

$$\|\theta^k\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1, \tag{55}$$

hence, it follows from (54) that

$$\|\theta^k\|_{L^2(0,T;H^1(\Omega^\delta))} \leq C_2. \tag{56}$$

We take scalar product of (47) with $\eta \in H^1(\Omega^\delta)$, integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\|\theta_t^k\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla\theta^k\|_{L^2(\Omega^\delta)} + +\|v^k\|_{L^4(\Omega)}\|\theta^k\|_{L^4(\Omega^\delta)} + \|\phi_t^k\|_{L^2(\Omega^\delta)} \right)$$

and we infer from (51),(53) and (56) that

$$\|\theta_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{57}$$

Next, multiplying (48) by c^k we conclude by analogous reasoning and using (44) that

$$\int_{\Omega^\delta} |c^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c^k|^2 dx dt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} |\nabla \phi^k|^2 dx dt,$$

hence, from (53) we have,

$$\|c^k\|_{L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{58}$$

In order to get an estimate for (c_t^k) in $L^2(0,T;H^1(\Omega^\delta)')$, we return to the equation (48) and use similar techniques, then

$$\|c_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{59}$$

We now infer from (51)-(59) that the sequence (v^k) is bounded (uniformly with respect to k) in

$$W_1 = \left\{ w \in L^2(0,T;V),\, w_t \in L^2(0,T;V') \right\}$$

and in

$$W_2 = \left\{ w \in L^\infty(0,T;H),\, w_t \in L^2(0,T;V') \right\},$$

the sequence (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$ and the sequences (θ^k) and (c^k) are bounded in

$$W_3 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\, w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_4 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}.$$

Since W_1 is compactly embedded in $L^2(Q)$, W_2 in $C([0,T];V')$, $W_2^{2,1}(Q^\delta)$ in $L^2(0,T;W^{1,p}(\Omega^\delta))$, $p \in [1,\infty)$, W_3 in $L^2(Q^\delta)$ and W_4 in $C([0,T];H^1(\Omega^\delta)')$ ([10] Cor.4), it follows that there exist

$$\begin{array}{rcl} v & \in & L^2(0,T;V) \cap L^\infty(0,T;H) \text{ with } v_t \in L^2(0,T;V'), \\ \phi & \in & L^2(0,T;H^2(\Omega^\delta)) \text{ with } \phi_t \in L^2(Q^\delta), \\ \theta & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } \theta_t \in L^2(0,T;H^1(\Omega^\delta)'), \\ c & \in & L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } c_t \in L^2(0,T;H^1(\Omega^\delta)'), \end{array}$$

and a subsequence of $(v^k, \phi^k, \theta^k, c^k)$ (which we still denote by $(v^k, \phi^k, \theta^k, c^k)$), such that, as $k \longrightarrow +\infty$,

$$\begin{array}{llll} v^k & \longrightarrow & v & \text{in} \quad L^2(Q) \cap C([0,T];V') \text{ strongly}, \\ v^k & \rightharpoonup & v & \text{in} \quad L^2(0,T;V) \text{ weakly}, \\ \phi^k & \longrightarrow & \phi & \text{in} \quad L^2(0,T;W^{1,p}(\Omega^\delta)) \cap C([0,T];L^2(\Omega^\delta)),\ p \in [1,\infty) \text{ strongly}, \\ \phi^k & \rightharpoonup & \phi & \text{in} \quad L^2(0,T;H^2(\Omega^\delta)) \text{ weakly}, \\ \theta^k & \longrightarrow & \theta & \text{in} \quad L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)') \text{ strongly}, \\ \theta^k & \rightharpoonup & \theta & \text{in} \quad L^2(0,T;H^1(\Omega^\delta)) \text{ weakly}, \\ c^k & \longrightarrow & c & \text{in} \quad L^2(Q^\delta) \cap C([0,T];H^1(\Omega^\delta)') \text{ strongly}, \\ c^k & \rightharpoonup & c & \text{in} \quad L^2(0,T;H^1(\Omega^\delta)) \text{ weakly}. \end{array} \tag{60}$$

It now remains to pass to the limit as k tends to $+\infty$ in (45)-(50).

We observe that $k(f_s^\delta(\cdot) - \delta)$ is bounded Lipschitz continuous function from $I\!R$ in $I\!R$ then $k(f_s^\delta(\hat{\phi}^k) - \delta)$ converges to $k(f_s^\delta(\hat{\phi}) - \delta)$ in $L^p(Q)$, for any $p \in [1,\infty)$. We then pass to the limit in standard ways as k tends to $+\infty$ in (45) and get

$$\begin{aligned} \frac{d}{dt}(v,u) \quad & +\nu(\nabla v, \nabla u) + (v \cdot \nabla v, u) = \lambda(\mathcal{F}(\hat{c},\hat{\theta}), u) \\ & -\lambda(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u) \text{ for all } u \in V,\ t \in (0,T). \end{aligned}$$

Since the embedding of $W_2^{2,1}(Q^\delta)$ into $L^p(Q^\delta)$ for any $p \in [1,\infty)$ is compact ([7] p.15), and (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$, we infer that $(\phi^k)^3$ converges to ϕ^3 in $L^{p/3}(Q^\delta)$. Also, since v^k converges to v in $L^2(Q)$ we have that $\rho_\delta(v^k)$

converges to $\rho_\delta(v)$ in $L^2(Q^\delta)$. We then pass to the limit as k tends to $+\infty$ in (46) and get

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi-\phi^3) = \lambda\beta\left(\hat\theta + (\theta_B-\theta_A)\hat c - \theta_B\right) \text{ in } Q^\delta.$$

Since $K_1(\rho_\delta)$ and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions and ϕ^k converges to ϕ in $L^p(Q^\delta)$, $p\in[1,\infty)$ we have that $K_1(\rho_\delta(\phi^k))$ converges to $K_1(\rho_\delta(\phi))$ and $f_s^{\delta\prime}(\phi^k)$ converges to $f_s^{\delta\prime}(\phi)$ in $L^p(Q^\delta)$ for any $p\in[1,\infty)$. These facts and (60) yield the weak convergence of $K_1(\rho_\delta(\phi^k))\nabla\theta^k$ to $K_1(\rho_\delta(\phi))\nabla\theta$ and $f_s^{\delta\prime}(\phi^k)\phi_t^k$ to $f_s^{\delta\prime}(\phi)\phi_t$ in $L^{3/2}(Q^\delta)$. Now, multiplying (47) by $\eta\in\mathcal{D}(Q^\delta)$, integrating over $\Omega^\delta\times(0,T)$ and by parts, we obtain

$$\begin{aligned}\int_0^T\int_{\Omega^\delta} C_{\mathrm{v}}\left(\theta_t^k + \rho_\delta(v^k)\cdot\nabla\theta^k\right)\eta + K_1(\rho_\delta(\phi^k))\nabla\theta^k\cdot\nabla\eta\,dxdt \\ = \int_0^T\int_{\Omega^\delta}\frac{l}{2}f_s^{\delta\prime}(\phi^k)\phi_t^k\eta\,dxdt,\end{aligned}$$

then we may pass to the limit and find that,

$$C_{\mathrm{v}}\theta_t + C_{\mathrm{v}}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^{\delta\prime}(\phi)\phi_t \quad \text{in } \mathcal{D}'(Q^\delta), \quad (61)$$

and using L^p-theory of parabolic equations we conclude that (61) holds almost everywhere in Q^δ.

It remains to pass to the limit in (48). We infer from (60) that $\nabla\rho_\delta(\phi^k)$ converges to $\nabla\rho_\delta(\phi)$ in $L^2(Q^\delta)$ and since $\|c^k\|_{L^\infty(Q^\delta)}$ is bounded, it follows that $c^k(1-c^k)$ converges to $c(1-c)$ in $L^p(Q^\delta)$ for any $p\in[1,\infty)$. Thus, we may pass to the limit in (48) to obtain

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2M\nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \qquad \text{in } Q^\delta.$$

Therefore $\mathcal{T}_\lambda$ is continuous for all $0\le\lambda\le 1$.

At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1\times W_2^{2,1}(Q^\delta)\times W_3\times W_3$ but, the embedding of this space in B is compact, then we conclude that $\mathcal{T}_\lambda$ is a compact operator.

To prove that for $(\hat v,\hat\phi,\hat\theta,\hat c)$ in a bounded set of B, $\mathcal{T}_\lambda$ is uniformly continuous in λ, let $0\le\lambda_1,\lambda_2\le 1$ and $(v_i,\phi_i,\theta_i,c_i)$ $(i=1,2)$ the corresponding solutions of (38)-(43). We observe that $v=v_1-v_2$, $\phi=\phi_1-\phi_2$, $\theta=\theta_1-\theta_2$

and $c = c_1 - c_2$, satisfy the following problem:

$$\begin{aligned}
\frac{d}{dt}(v,u) \quad &+ \quad \nu(\nabla v, \nabla u) + (v_1 \cdot \nabla v, u) - (v \cdot \nabla v_2, u) \\
&= \quad (\lambda_1 - \lambda_2)(\mathcal{F}(\hat{c},\hat{\theta}), u) + (\lambda_2 - \lambda_1)(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u), \quad (62) \\
&\quad\quad \text{for all } u \in V,\ t \in (0,T),
\end{aligned}$$

$$\begin{aligned}
\alpha\epsilon^2\phi_t \quad &- \quad \epsilon^2\Delta\phi + \alpha\epsilon^2\rho_\delta(v_1)\cdot\nabla\phi - \frac{1}{2}\phi\left(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right) \\
&= \quad \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi_2 + (\lambda_1 - \lambda_2)\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta, (63)
\end{aligned}$$

$$\begin{aligned}
C_v\theta_t \quad &- \quad \nabla\cdot(K_1(\rho_\delta(\phi_1))\nabla\theta) - \nabla\cdot\left[K_1(\rho_\delta(\phi_1)) - K_1(\rho_\delta(\phi_2))\right]\nabla\theta_2 \\
&+ \quad C_v\rho_\delta(v_1)\cdot\nabla\theta = C_v\rho_\delta(v)\cdot\nabla\theta_2 \\
&+ \quad \frac{l}{2}f_s^{\delta\prime}(\phi_1)\phi_t + \frac{l}{2}\left[f_s^{\delta\prime}(\phi_1) - f_s^{\delta\prime}(\phi_2)\right]\phi_{2t} \text{ in } Q^\delta, \qquad (64)
\end{aligned}$$

$$\begin{aligned}
c_t \quad &- \quad K_2\Delta c + \rho_\delta(v_1)\cdot\nabla c = K_2 M\nabla\cdot\left(c_1(1-c_1)\left[\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)\right]\right) \\
&+ \quad \rho_\delta(v)\cdot\nabla c_2 + K_2 M\nabla\cdot(c(1-(c_1+c_2))\nabla\rho_\delta(\phi_2)) \quad \text{in } Q^\delta, \qquad (65)
\end{aligned}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \qquad (66)$$

$$v(0) = 0 \text{ in } \Omega, \quad \phi(0) = 0, \quad \theta(0) = 0, \quad c(0) = 0 \text{ in } \Omega^\delta. \qquad (67)$$

Taking $u = v$ in equation (62), using Hölder's, Young's and interpolation inequalities we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx \quad &+ \quad \int_\Omega \nu|\nabla v|^2 dx \\
&\leq \quad \int_\Omega |v||\nabla v_2||v| dx \\
&+ \quad |\lambda_1 - \lambda_2|\int_\Omega \left(|\mathcal{F}(\hat{c},\hat{\theta})||v| + k(f_s^\delta(\hat{\phi}) - \delta)|\hat{v}||v|\right) dx \\
&\leq \quad C_1\|v_2\|_V^2\|v\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|v\|_V^2 + C_3\int_\Omega |v|^2 dx \\
&+ \quad C_2|\lambda_1 - \lambda_2|^2\left(\int_\Omega |F|^2 + |\hat{v}|^2 dx + \int_{\Omega^\delta} |\hat{\theta}|^2 + |\hat{c}|^2 dx\right).
\end{aligned}$$

Then, integration with respect t and Gronwall's Lemma give us

$$\|v\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}^2 \leq C_1\,|\lambda_1 - \lambda_2|^2. \qquad (68)$$

Applying L^p-theory of parabolic equations ([6] p. 341) to equation (63), the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C_1 \left(\|\rho_\delta(v)\cdot\nabla\phi_2\|_{L^2(Q^\delta)} + |\lambda_1-\lambda_2| \left(\|\hat{\theta}\|_{L^2(Q^\delta)} + \|\hat{c}\|_{L^2(Q^\delta)} + 1 \right) \right)$$

where C_1 depends on $\|\rho_\delta(v_1)\|_{L^4(Q^\delta)}$ and $\|\phi_1^2+\phi_1\phi_2+\phi_2^2\|_{L^{2+\eta}(Q^\delta)}$, $\eta > 0$, which are independent of λ_i. Therefore, using (68) we arrive at

$$\|\phi\|^2_{W_2^{2,1}(Q^\delta)} \le C_1\,|\lambda_1-\lambda_2|^2. \tag{69}$$

Multiplying (64) by θ, integrating over Ω^δ using Hölder's inequality and that K_1 and $f_s^{\delta'}$ are bounded Lipschitz continuous functions, we have

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega^\delta}|\theta|^2dx \;&+\; a\int_{\Omega^\delta}|\nabla\theta|^2dx \\
&\le\; C_1\int_{\Omega^\delta}|\rho_\delta(\phi)||\nabla\theta_2||\nabla\theta| + |\rho_\delta(v)||\nabla\theta_2||\theta|dx \\
&+\; C_2\int_{\Omega^\delta}|\phi_t||\theta| + |\phi||\phi_{2t}|\,|\theta|dx \\
&\le\; C_1\|\phi\|^2_{L^\infty(0,T;L^2(\Omega^\delta))}\|\nabla\theta_2\|_{L^2(\Omega^\delta)} \\
&+\; C_2\|v\|^2_{L^\infty(0,T;H)}\|\nabla\theta_2\|^2_{L^2(\Omega^\delta)} + C_3\int_{\Omega^\delta}\left(|\phi_t|^2+|\theta|^2\right)dx \\
&+\; C_4\|\phi\|^2_{L^\infty(0,T;H^1(\Omega^\delta))}\|\phi_{2t}\|^2_{L^2(\Omega^\delta)} + \frac{a}{2}\int_{\Omega^\delta}|\nabla\theta|^2dx.
\end{aligned}$$

Integration with respect to t and the use of Gronwall's Lemma and (68)-(69) lead to the estimate

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1\,|\lambda_1-\lambda_2|^2. \tag{70}$$

We multiply (65) by c, integrate over $\Omega^\delta\times(0,t)$ and by parts, and we use Hölder's and Young's inequalities and (44) to obtain

$$\begin{aligned}
\int_{\Omega^\delta}|c|^2dx \;&+\; \int_0^t\int_{\Omega^\delta}|\nabla c|^2dxdt \\
&\le\; C_1\int_0^t\int_{\Omega^\delta}\left(|\nabla\rho_\delta(\phi_1)-\nabla\rho_\delta(\phi_2)|^2 + |\rho_\delta(v)|^2 + |c|^2\right)dxdt \\
&\le\; C_1\int_0^t\int_{\Omega^\delta}\left(|\nabla\phi|^2+|c|^2\right)dxdt + C_1\int_0^t\int_\Omega|v|^2dxdt.
\end{aligned}$$

Applying Gronwall's Lemma and using (68)-(69) we arrive at

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1\,|\lambda_1-\lambda_2|^2. \tag{71}$$

Therefore, it follows from (68)-(71) that $\mathcal{T}_\lambda$ is uniformly continuous in λ.

To estimate the set of all fixed points of $\mathcal{T}_\lambda$ let $(v, \phi, \theta, c) \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\begin{aligned}\frac{d}{dt}(v,u) \quad &+ \quad \nu(\nabla v, \nabla u) + (v \cdot \nabla v, u) = \lambda(\mathcal{F}(c,\theta), u) \\ &- \quad \lambda(k(f_s^\delta(\phi) - \delta)v, u) \text{ for all } u \in V,\ t \in (0,T),\end{aligned} \tag{72}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi \quad &- \quad \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) \\ &= \quad \lambda\beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{73}$$

$$C_{\mathrm{v}}\theta_t + C_{\mathrm{v}}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{74}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot\left(c(1-c)\nabla\left(\rho_\delta(\phi)\right)\right) \text{ in } Q^\delta, \tag{75}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{76}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{77}$$

We take $u = v$ in equation (72). Then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dx \quad &+ \int_\Omega \left(\nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi) - \delta)|v|^2\right) dx \\ &\le C_1\int_\Omega |F|^2 + |\theta|^2 + |c|^2 + |v|^2 dx \\ &\le C_1\int_\Omega |F|^2 + |v|^2 dx + C_1\int_{\Omega^\delta} |\theta|^2 + |c|^2 dx.\end{aligned} \tag{78}$$

Multiplying equation (73) by ϕ, integrating over Ω^δ and by parts, using Hölder's and Young's inequalities we obtain,

$$\frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_{\Omega^\delta}|\phi|^2 dx + \int_{\Omega^\delta}\left(\epsilon^2|\nabla\phi|^2 + \frac{1}{2}\phi^4\right)dx \le C_1 + C_1\int_{\Omega^\delta}\left(|\theta|^2 + |c|^2 + |\phi|^2\right)dx. \tag{79}$$

By multiplying (74) by $e = C_{\mathrm{v}}\theta - \frac{l}{2}f_s^\delta(\phi)$ and (75) by c, arguments similar to the previous ones lead to the following estimates

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta}|e|^2 dx + \frac{C_{\mathrm{v}}a}{2}\int_{\Omega^\delta}|\nabla\theta|^2 dx \quad \le \quad C_2\int_{\Omega^\delta}|\nabla\phi|^2 dx + C_1\int_\Omega |v|^2 dx, \tag{80}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta}|c|^2 dx + \frac{K_2}{2}\int_{\Omega^\delta}|\nabla c|^2 dx \quad \le \quad C_2\int_{\Omega^\delta}|\nabla\phi|^2 dx, \tag{81}$$

where (44) was used to obtain the last inequality.

Now, multiplying (79) by A and adding the result to (78),(80)-(81), gives us

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}|v|^2 dx + \frac{d}{dt}\int_{\Omega^\delta}\left(\frac{A\alpha\epsilon^2}{4}|\phi|^2 + \frac{1}{2}|e|^2 + \frac{1}{2}|c|^2\right)dx \\
&\quad + \int_\Omega \left(\nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi)-\delta)|v|^2\right) \\
&\quad + \int_{\Omega^\delta}\left((A\epsilon^2 - 2C_2)|\nabla\phi|^2 + \frac{A}{2}\phi^4 + \frac{C_{\rm v}a}{2}|\nabla\theta|^2 + \frac{K_2}{2}|\nabla c|^2\right)dx \\
&\quad \le C_1 + C_1\int_\Omega |v|^2 dx + C_1\int_{\Omega^\delta}\left(|\phi|^2 + |\theta|^2 + |c|^2\right)dx \qquad (82)
\end{aligned}$$

where C_1 is independent of λ and δ, being $A \in I\!R$ an arbitrary parameter. Taking A large enough and using Gronwall's Lemma we obtain

$$\|v\|_{L^\infty(0,T;H)} + \|\phi\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|e\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|c\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1,$$

where C_1 is independent of λ. Since $\theta = \dfrac{1}{C_{\rm v}}\left(e + \dfrac{l}{2}f_s^\delta(\phi)\right)$ and $f_s^\delta(\phi)$ is bounded in $L^\infty(Q^\delta)$, we also have that $\|\theta\|_{L^\infty(0,T;L^2(\Omega^\delta))} \le C_1$. Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda \in [0,1]$.

Finally, for $\lambda = 0$, we can reason as in the proof that $\mathcal{T}_\lambda$ is well defined to conclude that the problem (38)-(43) has a unique solution. Therefore, we can apply Leray-Schauder's Theorem and so there is at least one fixed point $(v,\phi,\theta,c) \in B \cap \{L^2(0,T;V) \cap L^\infty(0,T;H)\} \times W_2^{2,1}(Q^\delta) \times W_2^{2,1}(Q^\delta) \times C^{2,1}(Q^\delta)$ of the operator $\mathcal{T}_1$, i.e. $(v,\phi,\theta,c) = \mathcal{T}_1(v,\phi,\theta,c)$. These functions are a solution of problem (32)-(37) and the proof of Proposition 1 is complete. ∎

5 Proof of Theorem 1

To prove Theorem 1, let $0 < \delta \le \delta(\Omega)$ be as in the statement of Theorem 1 and take $\phi_0^\delta \in W^{2-2/q,q}(\Omega^\delta) \cap H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta \in H$, $\theta_0^\delta \in H^{1+\gamma}(\Omega)$, $1/2 < \gamma \le 1$, $c_0^\delta \in C^1(\overline{\Omega^\delta})$, satisfying $\dfrac{\partial \phi_0^\delta}{\partial n} = \dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$, $\|\theta_0^\delta\|_{L^2(Q^\delta)} \le C$, $0 < c_0^\delta < 1$ in $\overline{\Omega^\delta}$, $v_0^\delta \to v_0$ in the norm of $H(\Omega_{ml}(0)$, and such that the restrictions of these functions to Ω (recall that $\Omega \subset \Omega^\delta$) satisfy as $\delta \to 0+$ the following: $\phi_0^\delta \to \phi_0$ in the norm of $W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $\theta_0^\delta \to \theta_0$ in the norm of $L^2(\Omega)$, $c_0^\delta \to c_0$ in the norm of $L^2(\Omega)$.

We then infer from Proposition 1 that there exists $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ solution the regularized problem (32)-(37). We will derive bounds, independent of δ, for this solution and then use compactness arguments and passage to the limit procedure for δ tends to 0 to establish the desired existence result. They are stated in following in a sequence of lemmas; however, most of them are ease consequence of the previous estimates (those that are independent of δ) and the fact that $\Omega \subset \Omega^\delta$. We begin with the following:

Lemma 1 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega)]$*

$$\|v^\delta\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} + \int_0^T \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dxdt \leq C_1, \tag{83}$$

$$\|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1, \tag{84}$$

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1, \tag{85}$$

$$\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|c^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1. \tag{86}$$

Proof: Observe that it follows from inequality (82). ∎

Lemma 2 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega)]$*

$$\begin{aligned} \|\phi^\delta\|_{W_q^{2,1}(Q)} &\leq C_1, \quad \textit{for any } 2 \leq q < 4, & (87)\\ \|\theta_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\leq C_1, & (88)\\ \|c_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\leq C_1, & (89)\end{aligned}$$

Proof: Note that (87) follows from estimate (20) of Theorem 2 and Lemma 1.

Next, we take the scalar product of (34) with $\eta \in H_o^1(\Omega)$, using Hölder's inequality and **(H3)** we find

$$C_v\|\theta_t^\delta\|_{H_o^1(\Omega)'} \leq C_1 \left(\|\nabla\theta^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^4(\Omega)}\|v^\delta\|_{L^4(\Omega)} + \|\phi_t^\delta\|_{L^2(\Omega)}\right).$$

Then, (88) follows from Lemma 1 and (87).

Using that $0 < c^\delta < 1$ in Q, we infer from (35) that,

$$\|c_t^\delta\|_{H_o^1(\Omega)'} \leq C_1 \left(\|\nabla c^\delta\|_{L^2(\Omega)} + \|v^\delta\|_{L^2(\Omega)} + \|\nabla\phi^\delta\|_{L^2(\Omega)}\right).$$

Then, (89) follows from Lemma 1. ∎

Lemma 3 *There exist a constant C_1 and $\delta_0 \in (0, \delta(\Omega)]$ such that, for any $\delta < \delta_0$,*

$$\|v_t^\delta\|_{L^2(t_1,t_2;V(U)')} \leq C_1 \tag{90}$$

where $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ and such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$.

Proof: Let $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ be such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. It is verified by means of (32) that for a.e. $t \in (t_1, t_2)$,

$$\begin{aligned}(v_t^\delta, u) = -\nu \int_U \nabla v^\delta \cdot \nabla u dx - \int_U v^\delta \cdot \nabla v^\delta u dx - \int_U k(f_s^\delta(\phi^\delta) - \delta) v^\delta u dx \\ + \int_U \mathcal{F}(c^\delta, \theta^\delta) u dx \quad \text{for } u \in V(U).\end{aligned}$$

In order to estimate $\|v_t^\delta\|_{V(U)'}$, we observe that the sequence (ϕ^δ) is bounded in $W_q^{2,1}(Q)$, for $2 \leq q < 4$, in particular, for $q > 2$ we have that $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - 4/q$ ([6] p.80). Consequently, because of Arzela-Ascoli's theorem, there exist ϕ and a subsequence of (ϕ^δ) (which we still denote by ϕ^δ), such that ϕ^δ converges uniformly to ϕ in $\bar{Q}$. Recall that $Q_{ml} = \{(x,t) \in Q \,/ 0 \leq f_s(\phi(x,t)) < 1\}$ and $\Omega_{ml}(t) = \{x \in \Omega \,/ 0 \leq f_s(\phi(x,t)) < 1\}$. Note that for a certain $\gamma \in (0,1)$ and for $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s(\phi(x,t)) < 1 - \gamma.$$

Due to the uniform convergence of f_s^δ towards f_s on any compact subset, there is an δ_0 such that for all $\delta \in (0, \delta_0)$ and for all $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s^\delta(\phi^\delta(x,t)) < 1 - \gamma/2.$$

By assumption **(H1)** we infer that

$$k(f_s^\delta(\phi^\delta(x,t)) - \delta) < k(1 - \gamma/2) \qquad \text{for } (x,t) \in [t_1, t_2] \times \bar{U} \text{ and } \delta < \delta_0.$$

Thus,

$$\begin{aligned}\|v_t^\delta\|_{V(U)'} \leq\; & C_1\Big(\|v^\delta\|_V + \|v^\delta\|^2_{L^4(\Omega)} + \|F\|_{L^2(\Omega)} + \|c^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^2(\Omega)} \\ & + \|k(f_s^\delta(\phi^\delta(x,t)) - \delta)\|_{L^\infty(U)} \|v^\delta\|_{L^2(\Omega)}\Big).\end{aligned}$$

Hence, (90) follows from Lemma 1. ∎

From (83) we conclude that the sequence (v^δ) is bounded in $L^2(t_1, t_2; H^1(U))$. Then, by the compact embedding ([10] Cor. 4), there exist v and a subsequence of (v^δ) (which we still denote by v^δ), such that

$$v^\delta \to v \quad \text{in } L^2((t_1, t_2) \times U) \text{ strongly.}$$

Observe that Q_{ml} is an open set and can be covered by a countable number of open sets $(t_i, t_{i+1}) \times U_i$ such that $U_i \subseteq \Omega_{ml}(t_i)$, then by means of a diagonal argument, we obtain

$$v^\delta \to v \quad \text{in } L^2_{loc}(Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)) \text{ strongly.} \tag{91}$$

Moreover, from (83) we have that $v \in L^2(0, T; V) \cap L^\infty(0, T; H)$ and

$$\begin{array}{lllll} v^\delta & \rightharpoonup & v & \text{in} & L^2(0, T; V) \text{ weakly,} \\ v^\delta & \overset{*}{\rightharpoonup} & v & \text{in} & L^\infty(0, T; H) \text{ weakly star.} \end{array} \tag{92}$$

We now infer from Lemma 1 and Lemma 2 using the compact embedding ([10] Cor.4) that there exist

$$\begin{array}{lll} \phi & \in & W^{2,1}_q(Q) \text{ for } 2 \leq q < 4, \\ \theta & \in & L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \\ c & \in & L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)), \end{array}$$

and a subsequence of $(\phi^\delta, \theta^\delta, c^\delta)$ (which we still denote by $(\phi^\delta, \theta^\delta, c^\delta)$) such that, as $\delta \to 0$,

$$\begin{array}{lllll} \phi^\delta & \to & \phi & & \text{uniformly in } Q, \\ \phi^\delta & \to & \phi & \text{in} & L^q(0, T; W^{1,q}(\Omega)) \text{ strongly,} \\ \phi^\delta_t & \rightharpoonup & \phi_t & \text{in} & L^q(Q) \text{ weakly,} \\ \theta^\delta & \to & \theta & \text{in} & L^2(Q) \cap C([0, T]; H^1_o(\Omega)') \text{ strongly,} \\ \theta^\delta & \rightharpoonup & \theta & \text{in} & L^2(0, T; H^1(\Omega)) \text{ weakly,} \\ c^\delta & \to & c & \text{in} & L^2(Q) \cap C([0, T]; H^1_o(\Omega)') \text{ strongly,} \\ c^\delta & \rightharpoonup & c & \text{in} & L^2(0, T; H^1(\Omega)) \text{ weakly.} \end{array} \tag{93}$$

It now remains pass to the limit as δ decreases to zero in (32)-(37).

Now, we take $u = \eta(t)$ in (32) where $\eta \in L^2(0, T; V(\Omega_{ml}(t)))$ with compact support contained in $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and $\eta_t \in L^2(0, T; V(\Omega_{ml}(t))')$; after integration over $(0, t)$, we find

$$\begin{aligned} \int_0^t \Big((v^\delta_t, \eta) + (\nabla v^\delta, \nabla \eta) + (v^\delta \cdot \nabla v^\delta, \eta) + (k(f^\delta_s(\phi^\delta) - \delta) v^\delta, \eta) \Big) \, ds \\ = \int_0^t (\mathcal{F}(c^\delta, \theta^\delta), \eta) ds. \end{aligned} \tag{94}$$

Since supp $\eta \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ we have that supp $\eta(t) \subseteq \Omega_{ml}(t)$ a.e. $t \in [0, T]$. Moreover, we observe that

$$\int_0^t (v_t^\delta, \eta) ds = -\int_0^t (v^\delta, \eta_t)_{\Omega_{ml}(s)} ds + (v^\delta(t), \eta(t))_{\Omega_{ml}(t)} - (v_0^\delta, \eta(0))_{\Omega_{ml}(0)}.$$

Because of uniform convergence of f_s^δ to f_s on compact subsets, as well as the assumption **(H1)**, it follows that $k(f_s^\delta(\phi^\delta) - \delta)$ converges to $k(f_s(\phi))$ uniformly on compact subsets of $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. These facts together with (91)-(93) ensure that we may pass to the limit in (94) and get (12).

To check that $v = 0$ a.e. in $\overset{\circ}{Q}_s$, take a compact set $K \subseteq \overset{\circ}{Q}_s$. Then there is an $\delta_K \in (0, 1)$ such that

$$f_s^\delta(\phi^\delta(x, t)) = 1 \quad \text{in } K \text{ for } \delta < \delta_K,$$

hence, $k(f_s^\delta(\phi^\delta(x, t) - \delta) = k(1 - \delta)$ in K for $\delta < \delta_K$. From (83) we infer that

$$k(1 - \delta) \|v^\delta\|_{L^2(K)}^2 \leq C_1 \quad \text{for } \delta < \delta_K$$

where C_1 is independent of δ. As δ tends to 0, by assumption **(H1)**, $k(1 - \delta)$ blows up and consequently $\|v^\delta\|_{L^2(K)}$ converges to 0. Therefore $v = 0$ a.e. in K. Since K is an arbitrary subset, we conclude that $v = 0$ a.e. in $\overset{\circ}{Q}_s$.

It follows from (92)-(93) that we may pass to the limit in (33), and find that (13) holds almost everywhere.

In order to pass to the limit in (34), we note that given $\zeta \in L^2(0, T; H^1(\Omega))$ with $\zeta_t \in L^2(0, T; L^2(\Omega))$ satisfying $\zeta(T) = 0$, we can consider an extension of ζ such that $\zeta^\delta \in L^2(0, T; H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0, T; L^2(\Omega^\delta))$ satisfying $\zeta^\delta(T) = 0$. Now, we take the scalar product of (34) with ζ^δ,

$$\begin{aligned} -C_{\mathrm{v}} \int_{\Omega^\delta} \theta_0^\delta \zeta^\delta(0) dx - C_{\mathrm{v}} \int_0^T \int_{\Omega^\delta} \theta^\delta \zeta_t^\delta dx dt \quad &- \quad C_{\mathrm{v}} \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta) \theta^\delta \cdot \nabla \zeta^\delta dx dt \\ + \int_0^T \int_{\Omega^\delta} K_1(\rho_\delta(\phi^\delta)) \nabla \theta^\delta \cdot \nabla \zeta^\delta dx dt \quad &= \quad \frac{l}{2} \int_0^T \int_{\Omega^\delta} f_s^{\delta\prime}(\phi^\delta) \phi_t^\delta \zeta^\delta dx dt. \end{aligned} \tag{95}$$

Observe that since $\rho_\delta(v^\delta)$ converges weakly to v in $L^2(0, T; H^1(\Omega))$ and $\theta^\delta \to \theta$ strongly in $C([0, T]; H_o^1(\Omega)')$ we have that $\rho_\delta(v^\delta)\theta^\delta$ converges to $v\theta$ in $\mathcal{D}'(Q)$. Observe also that $f_s^{\delta'} \to f_s'$ in $L^q(I\!R)$ for $2 \leq q < \infty$, then from (93) we infer that $f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta$ converges weakly to $f_s'(\phi)\phi_t$ in $L^{q/2}(Q)$. Moreover, from Lemma 1 the integrals over $\Omega^\delta \backslash \Omega$ are bounded independent of δ and since

$|\Omega^\delta\backslash\Omega| \to 0$ as $\delta \to 0$, we have that these integrals tend to zero as $\delta \to 0$. Therefore, we may pass to the limit in (95) and obtain

$$-C_v \int_0^T \int_\Omega \theta\zeta_t dxdt - C_v \int_0^T \int_\Omega v\,\theta \cdot \nabla\zeta\, dxdt + \int_0^T \int_\Omega K_1(\phi)\nabla\theta \cdot \nabla\zeta\, dxdt$$
$$= \frac{l}{2}\int_0^T \int_\Omega f_s(\phi)_t\zeta\, dxdt + C_v \int_\Omega \theta_0\zeta(0)dx$$

for $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$.

It remains to pass to the limit in (35). We proceed in similar ways as before, taking the scalar product of it with $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta^\delta_t \in L^2(0,T;L^2(\Omega^\delta))$ and $\zeta^\delta(T) = 0$,

$$-\int_0^T \int_{\Omega^\delta} c^\delta\zeta^\delta_t dxdt - \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta)c^\delta \cdot \nabla\zeta^\delta dxdt + K_2\int_0^T \int_{\Omega^\delta} \nabla c^\delta \cdot \nabla\zeta^\delta dxdt$$
$$+K_2M\int_0^T \int_{\Omega^\delta} c^\delta(1-c^\delta)\nabla\rho_\delta(\phi^\delta)\cdot\nabla\zeta^\delta dxdt = \int_{\Omega^\delta} c^\delta_0\zeta^\delta(0)dx,$$

then from (92),(93) and using that the sequence (c^δ) is bounded in $L^\infty(Q)$ we may pass to the limit as $\delta \to 0$ and obtain

$$-\int_0^T \int_\Omega c\zeta_t dxdt - \int_0^T \int_\Omega v\,c \cdot \nabla\zeta\, dxdt + K_2\int_0^T \int_\Omega \nabla c \cdot \nabla\zeta\, dxdt$$
$$+K_2M\int_0^T \int_\Omega c(1-c)\nabla\phi\cdot\nabla\zeta\, dxdt = \int_\Omega c_0\zeta(0)dx$$

holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$. Observe that since $0 < c^\delta < 1$ and c^δ converges to c in $L^2(Q)$ we have that $0 \le c \le 1$ a.e. in Q.

Finally, it follows from (93) that $\dfrac{\partial\phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$. Furthermore, $v(0) = v_0$ in $\Omega_{ml}(0)$ because $v^\delta(0) \to v(0)$ in $V'(U)$ for any U such that $\bar{U} \subseteq \Omega_{ml}(0)$. The proof of Theorem 1 is then complete. ∎

References

[1] Blanc, Ph., Gasser, L., Rappaz,J., 'Existence for a stationary model of binary alloy solidification', *Math. Mod. and Num. Anal.* **29**(6),687-699 (1995).

[2] Caginalp, G. and Xie, W., 'Phase-field and sharp-interfase alloys models', *Phys. Rev. E*, **48**(3), 1897-1909 (1993).

[3] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[4] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math, Vol 840, Springer-Verlag, 1981.

[5] Hoffman, K-H. and Jiang, L., 'Optimal control of a phase field model for solidification', *Numer. Funct. Anal. and Optim.*, **13**, 11-27 (1992).

[6] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[7] Lions, J.L., *Control of Distributed Singular Systems*, Gauthier-Villars, 1985.

[8] Mikhailov, V.P. *Partial Differential Equations*, Mir, 1978.

[9] Planas, G., Boldrini, J.L., 'Weak solutions to a phase-field model with convection for solidification of an alloy', UNICAMP-IMECC Preprint (2001).

[10] Simon, J., 'Compacts sets in the space $L^p(0,T,B)$', *Ann. Mat. Pura Appl.*, **146**, 65-96 (1987).

[11] Temam, R., *Navier-Stokes Equations*, AMS Chelsea Publishing, 2001.

[12] Voller,V.R., Cross,M., Markatos,N.C., 'An enthalpy method for convection/diffusion phase change', *Int. J. for Numer. Meth. in Eng.* **24**, 271-284 (1987).

[13] Voller, V.R., Prakash, C., 'A fixed grid numerical modeling methodology for convection-diffusion mushy region phase-change problems', *Int. J. Heat Mass Trans.*, **30** (8), 1709-1719 (1987).

Capítulo 5

Um modelo tridimensional do tipo campo de fase com convecção para a mudança de fase de uma liga binária

Resumo

Neste trabalho consideramos um modelo tridimensional do tipo campo de fase que incorpora propriedades físicas e térmicas de uma liga binária e a movimentação do fluido nas regiões não-solidificadas o qual ocorre no processo de solidificação. O modelo consiste de um sistema de equações diferenciais parciais altamente não linear, composto pela equação do campo de fase, a equação do calor, a equação da concentração e uma variante das equações de Navier-Stokes modificadas por um termo de penalização do tipo Carman-Kozeny, para levar em consideração o efeito *mushy*. É provada a existência de soluções fracas para o sistema. O problema é aproximado e uma sequência de soluções aproximadas é obtida usando o Teorema de Ponto Fixo de Leray-Schauder. Uma solução é obtida usando argumentos de compacidade.

A Tridimensional Phase-Field Model with Convection for Change Phase of an Alloy

José Luiz Boldrini *
boldrini@ime.unicamp.br

Gabriela Planas †
gplanas@ime.unicamp.br

IMECC - UNICAMP
Caixa Postal 6065, CEP 13081-970, Campinas-SP, Brazil

*Partially supported by CNPq, Brazil, grant 300513/87-9
†Supported by FAPESP, grant 97/13932-4

A Tridimensional Phase-Field Model with Convection for Change Phase of an Alloy

Abstract

We consider a tridimensional phase-field model for a solidification/melting non-stationary process, which incorporates the physics of binary alloys, thermal properties and fluid motion of non-solidified material. The model is a free-boundary value problem consisting of a non-linear parabolic system including a phase-field equation, a heat equation, a concentration equation and a variant of the Navier-Stokes equations modified by a penalization term of Carman-Kozeny type to model the flow in mushy regions and a Boussinesq type term to take in account the effects of the differences in temperature and concentration in the flow. A proof of existence of generalized solutions for the system is given. For this, the problem is firstly approximated and a sequence of approximate solutions is obtained by Leray-Schauder's fixed point theorem. A solution of the original problem is then found by using compactness arguments.

MSC Mathematics Subject Classification: 35K65, 76D05, 80A22, 35K55, 82B26, 35Q10, 76R99

1 Introduction

This paper is concerned with a non-isothermal phase-field model that accounts for both solidification/melting of a binary alloy and fluid motion. The present approach is based on ideas of Blanc et al.[1] and Voller et al. [15] to model the possibility of flow and those of Caginalp et al. [2] for the phase-field and the thermal properties of the alloy, and simpler versions were also considered in [11] and [12]. It is described as the following coupled system,

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v \cdot \nabla\phi - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta - c\theta_A - (1-c)\theta_B\right) \text{ in } Q, \quad (1)$$

$$C_v\theta_t + C_v v \cdot \nabla\theta = \nabla \cdot K_1(\phi)\nabla\theta + \frac{l}{2} f_s(\phi)_t \quad \text{in } Q, \quad (2)$$

$$c_t + v \cdot \nabla c = K_2\left(\Delta c + M\nabla \cdot c(1-c)\nabla\phi\right) \text{ in } Q, \quad (3)$$

$$v_t - \nu\Delta v + \nu_o A v + v \cdot \nabla v + \nabla p + k(f_s(\phi))v = \mathcal{F}(c,\theta) \text{ in } Q_{ml}, \quad (4)$$

$$\text{div } v = 0 \text{ in } Q_{ml}, \quad (5)$$

$$v = 0 \text{ in } Q_s, \quad (6)$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega \times (0,T), \quad v = 0 \text{ on } \partial Q_{ml}, \quad (7)$$

$$\phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad c(0) = c_0 \text{ in } \Omega, \quad v(0) = v_0 \quad \text{in } \Omega_{ml}(0), \quad (8)$$

where $Q = \Omega \times (0,T)$, $0 < T < +\infty$ and Ω is an open bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Here, ϕ is the phase-field which is the state variable characterizing the different phases; θ denotes the temperature, and $c \in [0,1]$ denotes the concentration, which is the fraction of one of the two materials in the mixture; v is the velocity field; p is the associated hydrostatic pressure; ν and ν_0 are positive constants corresponding to viscosities associated to the fluid material; $f_s \in [0,1]$ is the solid fraction.

The operator A is defined by

$$Av = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right), \qquad p \geq 3,$$

The penalization term $k(f_s)$ is the Carman-Kozeny type term and accounts for mushy effects in the flow; its usual expression is $k(f_s) = C_0{f_s}^2/(1-$

$f_s)^3$, but more general expressions will be allowed in this paper. $\mathcal{F}(c,\theta)$ is the buoyancy force, which by using Boussinesq approximation is given by $\mathcal{F}(c,\theta) = \rho \mathbf{g}\,(c_1(\theta-\theta_r)+c_2(c-c_r))+F$, where ρ is the mean value of the density; $\mathbf{g}$ is the acceleration of gravity; c_1 and c_2 are two real constants; θ_r, c_r are respectively the reference temperature and concentration, which for simplicity of exposition are assumed to be zero; F is an external force. The following physical parameter are assumed to be constant: $\alpha > 0$ the relaxation scaling; $\beta = \epsilon[s]/3\sigma$, where $\epsilon > 0$ is a measure of the interface width; σ is the surface tension, and $[s]$ is the entropy density difference between phases. $C_v > 0$ is the specific heat; $l > 0$ is associated to the latent heat; θ_A and θ_B are the respective melting temperatures of two materials composing the alloy; $K_2 > 0$ is the solute diffusivity, and M a constant related to the slopes of solidus and liquidus lines. Finally, $K_1 > 0$ denotes the thermal conductivity, which we will assume to depend on the phase-field.

We observe that equation (4) is associated to a modified form of the classical form of the Navier-Stokes equations as proposed by Ladyzenskaja in [6], in which the effective fluid viscosity depends on the gradient of the velocity.

The domain Q is composed of three regions: Q_s, Q_m and Q_l. The first region is fully solid; the second is mushy, and the third is fully liquid. They are defined by

$$\begin{array}{rcll} Q_s &=& \{\ (x,t)\in Q & / \quad f_s(\phi(x,t)) = 1\ \}, \\ Q_m &=& \{\ (x,t)\in Q & / \quad 0 < f_s(\phi(x,t)) < 1\ \}, \\ Q_l &=& \{\ (x,t)\in Q & / \quad f_s(\phi(x,t)) = 0\ \}, \end{array} \tag{9}$$

and Q_{ml} will refer to the not fully solid region, i.e.,

$$Q_{ml} = Q_m \cup Q_l = \{\ (x,t)\in Q \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{10}$$

At each time $t \in [0,T]$, $\Omega_{ml}(t)$ is defined by

$$\Omega_{ml}(t) = \{\ x \in \Omega \quad / \quad 0 \le f_s(\phi(x,t)) < 1\ \}. \tag{11}$$

In view of these regions are a priori unknown, the model is a free boundary problem.

Throughout this paper we assume the conditions,

(H1) f_s is a Lipschitz continuous function defined on $I\!R$ and satisfying $0 \le f_s(r) \le 1$ for all $r \in I\!R$; moreover f_s' is measurable,

(H2) k is a non decreasing function of class $C^1[0,1)$, satisfying $k(0)=0$, $\lim_{x\to 1^-} k(x) = +\infty$,

(H3) K_1 is a Lipschitz continuous function defined on $I\!R$; there exist $0 < a \leq b$ such that $0 < a \leq K_1(r) \leq b$ for all $r \in I\!R$,

(H4) F is a given function in $L^2(Q)$.

We use standard notation in this paper. We just briefly recall the following functional spaces associated to the Navier-Stokes equations. Let $G \subseteq I\!R^3$ be a non-void bounded open set; for $T > 0$, consider also $Q_G = G \times (0,T)$ Then,

$$\begin{aligned}
\mathcal{V}(G) &= \left\{ w \in \left(C_0^\infty(G)\right)^3,\ \operatorname{div} w = 0 \right\}, \\
H(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(L^2(G)\right)^3, \\
V^p(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(W_0^{1,p}(G)\right)^3, \\
V(G) &= \text{closure of } \mathcal{V}(G) \text{ in } \left(H_0^1(G)\right)^3, \\
H^{\tau,\tau/2}(\overline{Q}_G) &= \text{Hölder continuous functions of exponent } \tau \text{ in } x \\
& \text{ and exponent } \tau/2 \text{ in } t, \\
W_q^{2,1}(Q_G) &= \left\{ w \in L^q(Q_G) / \, D_x w, D_x^2 w \in L^q(Q_G), w_t \in L^q(Q_G) \right\}.
\end{aligned}$$

When $G = \Omega$, we denote $H = H(\Omega)$, $V = V(\Omega)$, $V^p = V^p(\Omega)$. Properties of these functional spaces can be found for instance in [7, 9, 14]. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^1(\Omega)$ and $H^1(\Omega)'$. We also put $(\cdot,\cdot) = (\cdot,\cdot)_\Omega$ the inner product of $\left(L^2(\Omega)\right)^3$.

Our purpose in this work is to show that problem (1)-(8) is solvable in a generalized sense to be made precise below.

The main result of this paper is the following.

Theorem 1 *Let be $T > 0$, $p \geq 3$, $5/2 < q \leq 10/3$, $\Omega \subseteq I\!R^3$ an open bounded domain de class C^3. Suppose that $v_0 \in H(\Omega_{ml}(0))$, $\phi_0 \in W^{2-2/q,q}(\Omega) \cap H^{1+\gamma}(\Omega)$, $1/2 < \gamma \leq 1$, $\theta_0 \in L^2(\Omega)$ and $c_0 \in L^2(\Omega)$, $0 \leq c_0 \leq 1$ a.e. in $\bar{\Omega}$, satisfying the compatibility conditions $\dfrac{\partial \phi_0}{\partial n} = 0$ on $\partial\Omega$. Under the assumptions* **(H1)-(H4)**, *there exist functions $(\phi, \theta, c, v, \chi)$ satisfying*

(i) $\phi \in W_q^{2,1}(Q)$, $\phi(0) = \phi_0$,

(ii) $\theta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $\theta(0) = \theta_0$,

(iii) $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, $c(0) = c_0$, $0 \leq c \leq 1$ *a.e. in* Q,

(iv) $v \in L^p(0,T;V^p) \cap L^\infty(0,T;H)$, $v = 0$ *a.e. in* $\mathring{Q}_s$, $v(0) = v_0$ *in* $\Omega_{ml}(0)$, *where* Q_s *is defined by (9) and* $\Omega_{ml}(0)$ *by (11),*

(v) $\chi \in L^{p'}(0,T;(V^p)')$

and such that for any $\zeta \in L^2(0,T;H^1(\Omega))$ *with* $\zeta_t \in L^2(0,T;L^2(\Omega))$ *and* $\zeta(T) = 0$ *in* Ω, *we have*

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi = \frac{1}{2}(\phi - \phi^3) + \beta\left(\theta + (\theta_B - \theta_A)c - \theta_B\right) \ \text{a.e. in } Q, \tag{12}$$

$$\frac{\partial\phi}{\partial n} = 0 \qquad \text{a.e. on } \partial\Omega\times(0,T), \tag{13}$$

$$\begin{aligned} -C_v\int_0^T\int_\Omega \theta\zeta_t dxdt - C_v\int_0^T\int_\Omega v\theta\cdot\nabla\zeta dxdt + \int_0^T\int_\Omega K_1(\phi)\nabla\theta\cdot\nabla\zeta\, dxdt \\ = \frac{l}{2}\int_0^T\int_\Omega f_s(\phi)_t\zeta\, dxdt + C_v\int_\Omega \theta_0\zeta(0)dx, \end{aligned} \tag{14}$$

$$\begin{aligned} -\int_0^T\int_\Omega c\zeta_t dxdt - \int_0^T\int_\Omega vc\cdot\nabla\zeta dxdt + K_2\int_0^T\int_\Omega \nabla c\cdot\nabla\zeta\, dxdt \\ +K_2M\int_0^T\int_\Omega c(1-c)\nabla\phi\cdot\nabla\zeta\, dxdt = \int_\Omega c_0\zeta(0)dx. \end{aligned} \tag{15}$$

Also, for any $t \in (0,T)$ *and* $\eta \in L^p(0,T;V^p)$ *with compact support contained in* $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ *and such that* $\eta_t \in L^p(0,T;(V^p)')$, *where* Q_{ml} *is defined by (10) and* $\Omega_{ml}(t)$ *by (11), there hold*

$$\begin{aligned} (v(t),\eta(t)) - \int_0^t (v,\eta_t)ds + \nu\int_0^t(\nabla v,\nabla\eta)ds + \nu_o\int_0^t(\chi,\eta)ds \\ + \int_0^t(v\cdot\nabla v,\eta)ds + \int_0^t(k(f_s(\phi))v,\eta)ds \\ = \int_0^t(\mathcal{F}(c,\theta),\eta)ds + (v_0,\eta(0)). \end{aligned} \tag{16}$$

Moreover, (ϕ,θ,c,v,χ) *is a generalized solution of (1)-(8), in the sense that under the following additional regularity and integrability assumptions:*

- *for a.e.* $t \in (0,T)$, *the boundary* $\partial\Omega_{ml}(t)$ *of* Ω_{ml} *in* Ω *has zero Lebesgue measure*
- *Suppose that either* $k(f_s(\phi)) \in L^{5/4}(0,T;L^{15/7}(\Omega_{ml}(t)))$, *when* $p = 3$, *or* $k(f_s(\phi)) \in L^s(0,T;L^1(\Omega_{ml}(t)))$, *with* $s = p/(p-2)$, *when* $p > 3$,

then $\chi = Av$ in the sense of distribution in Q_{ml}.

Remark: Interpreting the modified Navier-Stokes equations requires some topological information about the set occupied by the fluid. In fact, one should know that such a set is open to interpret the modified Navier-Stokes equations at least in the sense of distributions. This information is in particular implied by the continuity of phase-field, which in turn depends on the degree of smoothness of the other variables. In the two dimensional case, that is $\Omega \subseteq I\!R^2$, and when $\nu_o = 0$, an existence theorem for system (1)-(8) was obtained in [12]. The main feature of [12] is that the smoothness of v, θ and c suffice to yield the continuity of ϕ. In the three dimensional case, this does not appear possible. To stress this point, consider weak solutions of the classical Navier-Stokes equations with external force in $L^2(Q)$. It is well known that such solution satisfies $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ (see e.g. [14]). If $\Omega \subseteq I\!R^2$, this regularity implies that $|v| \in L^4(Q)$. This fact together with (12)-(13) suffices to prove that ϕ is continuous, and therefore the set Q_{ml} is open. In the three dimensional case, we just have that $v \in L^{10/3}(Q)$. This modest degree of integrability of velocity prevents us from proving that ϕ is continuous. When $\nu_o > 0$ and p is large enough, as it is the case of the present paper, it is possible to get more regularity of v and then the required continuity of ϕ. In fact, if $v \in L^p(0,T;V^p) \cap L^\infty(0,T;H)$, by interpolation ([8] p. 207), we conclude that $v \in L^{p5/3}(Q)$. Taking $p \geq 3$ is then enough to yield the continuity of ϕ (see Thm 2). In fact, the additional restriction $q > 5/2$ ensure the continuity of phase-field because in this case $W^{2,1}_q(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$, with $\tau = 2 - 5/q$ ([7] p. 80). Therefore the set Q_{ml} is open, giving a meaningful interpretation to the velocity equation. The restriction $q \leq 10/3$ is consequence of the obtained regularity of temperature. This will be clear in the next section.

The previous existence result will be obtained by using a regularization technique similar to the one already used in [1] and [11, 12]. The idea is to use a auxiliary parameter to transform the original free-boundary value problem in a penalized but more standard problem. This will be called the regularized problem and will be studied by using fixed point arguments. Then, by using compactness arguments as the auxiliary parameter goes to zero, we obtain a generalized solution of the original problem.

The outline of this paper is as follows. In Section 2, we study an auxiliary problem. Then, in Section 3, we study a regularized problem. Section 4 is devoted to the proof of the main existence theorem.

2 An auxiliary problem

We consider the initial boundary value problem,

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{1}{2}(\phi-\phi^3)+g \quad \text{in } Q, && (17)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), && (18)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega, && (19) \end{aligned}$$

and prove the following result using a technique similar to the one already used in [5] to treat a phase-field equation without convective term or in [12] to treat the two dimensional problem.

Theorem 2 *Let be $T>0$, $q\geq 2$, $p\geq 3$. Suppose that $g\in L^q(Q)$, $v\in L^p(0,T;V^p)\cap L^\infty(0,T;H)$ and $\phi_0\in W^{2-2/q,q}(\Omega)$ satisfying the compatibility condition $\dfrac{\partial\phi_0}{\partial n}=0$ on $\partial\Omega$. Then*

i) *If $2\leq q<5$, there exist a unique $\phi\in W_q^{2,1}(Q)$ solution of problem (17)-(19) , which satisfies the estimate*

$$\|\phi\|_{W_q^{2,1}(Q)} \leq C\left(\|\phi_0\|_{W^{2-2/q,q}(\Omega)} + \|g\|_{L^q(Q)} + \|\phi_0\|^3_{W^{2-2/q,q}(\Omega)} + \|g\|^3_{L^q(Q)}\right) \quad (20)$$

where C depends on $\|v\|_{L^5(Q)}$, on Ω and T,

ii) *If $q\geq 5$ and $p>3$, there exist a unique $\phi\in W_r^{2,1}(Q)$, $r=\min\{q,p5/3\}$ solution of problem (17)-(19), which satisfies the estimate (20) where C depends on $\|v\|_{L^{p5/3}(Q)}$, on Ω and T.*

Proof: In order to apply Leray-Schauder fixed point theorem ([3] p. 189) we consider the operator T_λ, $0\leq\lambda\leq 1$, on the Banach space $B=L^6(Q)$, which maps $\hat\phi\in B$ into ϕ by solving the problem

$$\begin{aligned} \alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{\lambda}{2}(\hat\phi-\hat\phi^3)+\lambda g \quad \text{in } Q, && (21)\\ \frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega\times(0,T), && (22)\\ \phi(0) &= \phi_0 \quad \text{in } \Omega. && (23) \end{aligned}$$

We define $G_\lambda = \frac{\lambda}{2}(\hat{\phi} - \hat{\phi}^3) + \lambda g$ and we observe that $G_\lambda \in L^2(Q)$. Since $v \in L^5(Q)$, we infer from L^p-theory of parabolic equations ([7] Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution ϕ of problem (21)-(23) with $\phi \in W_2^{2,1}(Q)$. Due to the embedding of $W_2^{2,1}(Q)$ into $L^{10}(Q)$ ([9] p.15), the operator T_λ is well defined from B into B.

To prove continuity of T_λ, let $\hat{\phi}_n \in B$ strongly converging to $\hat{\phi} \in B$; for each n, let $\phi_n = T_\lambda(\hat{\phi}_n)$. We have that ϕ_n satisfies the following estimate ([7] p. 341)

$$\|\phi_n\|_{W_2^{2,1}(Q)} \leq C \left(\|\hat{\phi}_n\|_{L^2(Q)} + \|\hat{\phi}_n\|^3_{L^6(Q)} + \|g\|_{L^2(Q)} + \|\phi_0\|_{H^1(\Omega)} \right)$$

for some constant C independent of n. Since $W_2^{2,1}(Q)$ is compactly embedded in $L^2(0,T;H^1(\Omega))$ ([13] Cor.4) and in $L^9(Q)$, it follows that there exist a subsequence of ϕ_n (which we still denote by ϕ_n) strongly converging to $\phi = T_\lambda(\hat{\phi})$ in B. Hence T_λ is continuous for all $0 \leq \lambda \leq 1$. At the same time, T_λ is bounded in $W_2^{2,1}(Q)$, and the embedding of this space in B is compact. Thus, we conclude that T_λ is a compact operator for each $\lambda \in [0,1]$.

To prove that for $\hat{\phi}$ in a bounded set of B, T_λ is uniformly continuous with respect to λ, let $0 \leq \lambda_1, \lambda_2 \leq 1$ and $\phi_i\,(i = 1,2)$ be the corresponding solutions of (21)-(23). For $\phi = \phi_1 - \phi_2$ the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q)} \leq C|\lambda_1 - \lambda_2| \left(\|\hat{\phi}\|_{L^2(Q)} + \|\hat{\phi}\|^3_{L^6(Q)} + \|g\|_{L^2(Q)} \right)$$

where C is independent of λ_i. Therefore, T_λ is uniformly continuous in λ.

Now we have to estimate the set of all fixed points of T_λ, let $\phi \in B$ be such a fixed point, i.e., it is a solution of the problem

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v \cdot \nabla\phi - \epsilon^2\Delta\phi = \frac{\lambda}{2}(\phi - \phi^3) + \lambda g \quad \text{in } Q, \tag{24}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{25}$$

$$\phi(0) = \phi_0 \quad \text{in } \Omega. \tag{26}$$

We multiply (24) successively by ϕ, ϕ_t and $-\Delta\phi$, and integrate over $\Omega \times (0,t)$. After integration by parts and the use the Hölder's, Young's and Gagliardo-Nirenberg's inequalities, we obtain in the usual manner the following estimate

$$\begin{aligned} \int_\Omega \left(\phi^2 + |\nabla\phi|^2\right) dx + \|\phi\|^2_{W_2^{2,1}(Q)} &\leq C \left(\|g\|^2_{L^2(Q)} + \|\phi_0\|^2_{H^1(\Omega)} \right) \\ &+ C \int_0^t \left(1 + \|v\|^5_{L^5(\Omega)}\right) \left(\|\phi\|^2_{L^2(\Omega)} + \|\nabla\phi\|^2_{L^2(\Omega)} \right) dt \end{aligned} \tag{27}$$

where C is independent of λ. By applying Gronwall's Lemma we get

$$\|\phi\|_{L^6(Q)} \leq C\|\phi\|_{W_2^{2,1}(Q)} \leq C'$$

where C and C' are constants independent of λ. Therefore, all fixed points of T_λ in B are bounded independently of $\lambda \in [0, 1]$.

Finally, for $\lambda = 0$, it is clear that problem (21)-(23) has a unique solution. Therefore, we can apply Leray-Schauder's fixed point theorem, and so there is at least one fixed point $\phi \in B \cap W_2^{2,1}(Q)$ of the operator T_1, i.e., $\phi = T_1(\phi)$. This corresponds to a solution of problem (17)-(19). Now we have to examine the regularity of ϕ. To prove **i)** we discuss the cases $2 \leq q \leq 3$ and $3 < q < 5$ separately.

If $2 \leq q \leq 3$, since $W_2^{2,1}(Q)$ is embedded into $L^9(Q)$, we have that $G = \dfrac{1}{2}(\phi - \phi^3) + g \in L^q(Q)$ and this implies $\phi \in W_q^{2,1}(Q)$. If $3 < q < 5$, we have that $G \in L^3(Q)$ and as a consequence $\phi \in W_3^{2,1}(Q)$. According to embedding ([9] p.15) we can conclude that $\phi \in L^\infty(Q)$ and consequently $\phi \in W_q^{2,1}(Q)$. To prove estimate (20) we restrict to the case $2 \leq q \leq 3$. The proof for $3 < q < 5$ is similar. Observe that from L^p-theory of parabolic equations we have

$$\begin{aligned}
\|\phi\|_{W_q^{2,1}(Q)} &\leq C\left(\|G\|_{L^q(Q)} + \|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\
&\leq C\left(\|g\|_{L^q(Q)} + \|\phi\|_{L^q(Q)} + \|\phi\|^3_{L^{3q}(Q)} + \|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right)\\
&\leq C\left(\|g\|_{L^q(Q)} + \|\phi\|_{W_2^{2,1}(Q)} + \|\phi\|^3_{W_2^{2,1}(Q)} + \|\phi_0\|_{W^{2-2/q,q}(\Omega)}\right).
\end{aligned}$$

Using estimate (27) we deduce (20).

If $q \geq 5$ and $p > 3$, we have that $G \in L^q(Q)$ and since $v \in L^{p5/3}(Q)$, from L^p-theory of parabolic equations we can conclude that $\phi \in W_r^{2,1}(Q)$ where $r = \min\{q, p5/3\}$. The estimative (20) is proved by analogous reasoning.

It remains to show uniqueness of the solution. Let us assume that ϕ_1 and ϕ_2 are two solutions of problem (17)-(19). Then the difference $\phi = \phi_1 - \phi_2$ satisfies the following initial boundary value problem

$$\begin{aligned}
\alpha\epsilon^2\phi_t + \alpha\epsilon^2 v\cdot\nabla\phi - \epsilon^2\Delta\phi &= \frac{1}{2}\phi\left(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right) \quad \text{in } Q, && (28)\\
\frac{\partial\phi}{\partial n} &= 0 \quad \text{on } \partial\Omega \times (0, T), && (29)\\
\phi(0) &= 0 \quad \text{in } \Omega, && (30)
\end{aligned}$$

We remark that $d := \phi_1^2 + \phi_1\phi_2 + \phi_2^2 \geq 0$. Multiplying (28) by ϕ and using the usual method of Gronwall's Lemma give us $\phi \equiv 0$. Therefore, the solution of problem (17)-(19) is unique and the proof of Theorem 2 is then complete. ∎

3 A regularized problem

In this section we turn to the full problem and introduce a regularized problem to lead with the modified Navier-Stokes equations in the whole domain instead of unknown regions, as well as with suitable regularity to the coefficients. We prove an existence result using Leray-Schauder Fixed Point Theorem ([3] p. 189).

Before doing so, we recall certain results that will be helpful in the introduction of such regularized problem.

Recall that there is an extension operator $Ext(\cdot)$ taking any function w in the space $W_2^{2,1}(Q)$ and extending it to a function $Ext(w) \in W_2^{2,1}(I\!R^4)$ with compact support satisfying

$$\|Ext(w)\|_{W_2^{2,1}(I\!R^4)} \leq C\,\|w\|_{W_2^{2,1}(Q)},$$

with C independent of w (see [10] p.157).

For $\delta \in (0,1)$, let $\rho_\delta \in C_0^\infty(I\!R^3)$ be a family of symmetric positive mollifier functions converging to the Dirac delta function, and denote by $*$ the convolution operation. Then, given a function $w \in W_2^{2,1}(Q)$, we define a regularization $\rho_\delta(w) \in C_0^\infty(I\!R^3)$ of w by

$$\rho_\delta(w) = \rho_\delta * Ext(w).$$

This sort of regularization will be used with the phase-field variable. We will also need a regularization for the velocity, and for it we proceed as follows.

Given $v \in L^2(0,T;V)$, first we extend it as zero in $I\!R^4 \backslash Q$. Then, as in [10] p. 157, by using reflection and cutting-off, we extend the resulting function to another one defined on $I\!R^4$ and with compact support. Without the danger of confusion, we again denote such extension operator by $Ext(v)$. Then, being $\delta > 0$, ρ_δ and $*$ as above, operating on each component, we can again define a regularization $\rho_\delta(v) \in C_0^\infty(I\!R^4)$ of v by

$$\rho_\delta(v) = \rho_\delta * Ext(v).$$

Besides having properties of control of Sobolev norms in terms of the corresponding norms of the original function (exactly as above), such extension has the property described below.

For $0 < \delta \leq 1$, define firstly the following family of uniformly bounded open sets

$$\Omega^\delta = \{x \in I\!R^3 : d(x, \Omega) < \delta\}. \tag{31}$$

We also define the associated space-time cylinder

$$Q^\delta = \Omega^\delta \times (0, T). \tag{32}$$

Obviously, for any $0 < \delta_1 < \delta_2$, we have $\Omega \subset \Omega^{\delta_1} \subset \Omega^{\delta_2}$, $Q \subset Q^{\delta_1} \subset Q^{\delta_2}$. Also, by using properties of convolution, we conclude that $\rho_\delta(v)|_{\partial\Omega^\delta} = 0$. In particular, for $v \in L^\infty(0, T; H) \cap L^2(0, T; V)$, we conclude that $\rho_\delta(v) \in L^\infty(0, T; H(\Omega^\delta)) \cap L^2(0, T; V(\Omega^\delta))$.

Moreover, since Ω is of class C^3, there exists $\delta(\Omega) > 0$ such that for $0 < \delta \leq \delta(\Omega)$, we conclude that Ω^δ is of class C^2 and such that the C^2 norms of the maps defining $\partial\Omega^\delta$ are uniformly estimated with respect to δ in terms of the C^3 norms of the maps defining $\partial\Omega$.

Since we will be working with the sets Ω^δ, the main objective of this last remark is to ensure that the constants associated to Sobolev immersions and interpolations inequalities, involving just up to second order derivatives and used with Ω^δ, are uniformly bounded for $0 < \delta \leq \delta(\Omega)$. This will be very important to guarantee that certain estimates will be independent of δ.

Finally, let f_s^δ be any regularization of f_s.

Now, we are in position to define the regularized problem. For $\delta \in (0, \delta(\Omega))$, we consider the system

$$\begin{aligned}(v_t^\delta, u) + \nu(\nabla v^\delta, \nabla u) + \nu_o(Av^\delta, u) + (v^\delta \cdot \nabla v^\delta, u) + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, u) \\ = (\mathcal{F}(c^\delta, \theta^\delta), u) \quad \forall\, u \in V^p, \text{ a.e. } t \in (0, T),\end{aligned} \tag{33}$$

$$\begin{aligned}&\alpha\epsilon^2\phi_t^\delta + \alpha\epsilon^2\rho_\delta(v^\delta) \cdot \nabla\phi^\delta - \epsilon^2\Delta\phi^\delta \\ &\quad = \frac{1}{2}(\phi^\delta - (\phi^\delta)^3) + \beta\left(\theta^\delta + (\theta_B - \theta_A)c^\delta - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{34}$$

$$C_v\theta_t^\delta + C_v\rho_\delta(v^\delta) \cdot \nabla\theta^\delta = \nabla \cdot \left(K_1(\rho_\delta(\phi^\delta))\nabla\theta^\delta\right) + \frac{l}{2}f_s^\delta(\phi^\delta)_t \text{ in } Q^\delta, \tag{35}$$

$$c_t^\delta - K_2\Delta c^\delta + \rho_\delta(v^\delta) \cdot \nabla c^\delta = K_2 M \nabla \cdot \left(c^\delta(1 - c^\delta)\nabla\rho_\delta(\phi^\delta)\right) \text{ in } Q^\delta, \tag{36}$$

$$\frac{\partial \phi^\delta}{\partial n} = 0, \quad \frac{\partial \theta^\delta}{\partial n} = 0, \quad \frac{\partial c^\delta}{\partial n} = 0 \quad \text{on } \partial\Omega^\delta \times (0,T), \tag{37}$$

$$v^\delta(0) = v_0^\delta \text{ in } \Omega, \quad \phi^\delta(0) = \phi_0^\delta, \quad \theta^\delta(0) = \theta_0^\delta, \quad c^\delta(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{38}$$

We then have the following existence result.

Proposition 1 *Let be $T > 0$, $p \geq 3$. For each $\delta \in (0, \delta(\Omega))$, let $v_0^\delta \in H$, $\phi_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $\theta_0^\delta \in H^{1+\gamma}(\Omega^\delta)$, $1/2 < \gamma \leq 1$ and $c_0^\delta \in C^1(\bar{\Omega}^\delta), 0 < c_0^\delta < 1$ in $\bar{\Omega}^\delta$, satisfying the compatibility conditions $\dfrac{\partial \phi_0^\delta}{\partial n} = \dfrac{\partial \theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$. Assume that* **(H1)-(H4)** *hold. Then there exist functions $(v^\delta, \phi^\delta, \theta^\delta, c^\delta)$ which satisfy (33)-(38) and*

i) $v^\delta \in L^p(0,T;V^p) \cap L^\infty(0,T;H)$, $\quad v_t^\delta \in L^{p'}(0,T;(V^p)')$,

ii) $\phi^\delta \in L^2(0,T;H^2(\Omega^\delta))$, $\quad \phi_t^\delta \in L^2(Q^\delta)$,

iii) $\theta^\delta \in L^2(0,T;H^2(\Omega^\delta))$, $\quad \theta_t^\delta \in L^2(Q^\delta)$,

iv) $c^\delta \in C^{2,1}(Q^\delta)$, $\quad 0 < c^\delta < 1$.

Remark: It is possible to obtain more regularity for ϕ^δ when the initial data are more regular. This will be done in the last section.

Proof: For simplicity we shall omit the superscript δ at v^δ, ϕ^δ, θ^δ, c^δ. First of all, we consider the following family of operators, indexed by the parameter $0 \leq \lambda \leq 1$,

$$\mathcal{T}_\lambda : B \to B,$$

where B is the Banach space

$$B = L^p(0,T;H) \times L^2(Q^\delta) \times L^2(Q^\delta) \times L^2(Q^\delta),$$

and defined as follows: given $(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c}) \in B$, let $\mathcal{T}_\lambda(\hat{v}, \hat{\phi}, \hat{\theta}, \hat{c}) = (v, \phi, \theta, c)$, where (v, ϕ, θ, c) is obtained by solving the problem

$$\begin{aligned}(v_t, u) &+ \nu(\nabla v, \nabla u) + \nu_0(Av, u) + (v \cdot \nabla v, u) \\ &= \lambda(\mathcal{F}(\hat{c}, \hat{\theta}), u) - \lambda(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u) \; \forall u \in V^p, \; t \in (0,T),\end{aligned} \tag{39}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v) \cdot \nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi - \phi^3) \\ = \lambda\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta.\end{aligned} \tag{40}$$

$$C_v\theta_t + C_v\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{41}$$

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \text{ in } Q^\delta, \tag{42}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega^\delta\times(0,T), \tag{43}$$

$$v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{44}$$

Clearly (v,ϕ,θ,c) is a solution of (33)-(38) if and only if it is a fixed point of the operator $\mathcal{T}_1$. In the following, we prove that $\mathcal{T}_1$ has at least one fixed point by using the Leray-Schauder fixed point theorem ([3] p. 189).

To verify that $\mathcal{T}_\lambda$ is well defined, observe that equation (39) is a variant of Navier-Stokes equation and since $k(f_s^\delta(\hat{\phi})-\delta)\hat{v} \in L^2(Q)$, there exist a unique solution $v \in L^p(0,T;V^p)\cap L^\infty(0,T;H)\cap L^{p5/3}(Q)$ ([8] p. 207).

Since $\hat{\theta}$, $\hat{c} \in L^2(Q^\delta)$ and $\rho_\delta(v) \in L^5(Q^\delta)$, we infer from Theorem 2 that there is a unique solution ϕ of equation (40) with $\phi \in W_2^{2,1}(Q^\delta)$.

Since K_1 is a bounded Lipschitz continuous function and $\rho_\delta(\phi) \in C^\infty(Q^\delta)$, we have that $K_1(\rho_\delta(\phi)) \in W_r^{1,1}(Q^\delta)$, $1 \le r \le \infty$, and since $\rho_\delta(v) \in L^5(Q^\delta)$ and $f_s^\delta(\phi)_t = f_s^{\delta\prime}(\phi)\phi_t \in L^2(Q^\delta)$, we infer from L^p-theory of parabolic equations ([7] Thm. 9.1 in Chapter IV, p. 341 and the remark at the end of Section 9 of the same chapter, p. 351) that there is a unique solution θ of equation (41) with $\theta \in W_2^{2,1}(Q^\delta)$.

We observe that equation (42) is a semilinear parabolic equation with smooth coefficients and growth conditions on the non-linear forcing terms to apply semigroup results of Henry [4] p.75. Thus, there is a unique global classical solution c.

In addition, note that equation (42) does not admit constant solutions, except $c \equiv 0$ and $c \equiv 1$. Thus, by using Maximum Principles together with conditions $0 < c_0^\delta < 1$ and $\dfrac{\partial c^\delta}{\partial n} = 0$, we can deduce that

$$0 < c(x,t) < 1, \qquad \forall\,(x,t) \in Q^\delta. \tag{45}$$

Therefore, the mapping $\mathcal{T}_\lambda$ is well defined from B into B.

To prove continuity of $\mathcal{T}_\lambda$ let $(\hat{v}^k,\hat{\phi}^k,\hat{\theta}^k,\hat{c}^k)$, $k \in I\!N$ be a sequence in B such that converges strongly in B to $(\hat{v},\hat{\phi},\hat{\theta},\hat{c})$ and let $(v^k,\phi^k,\theta^k,c^k)$ be the solution of the problem:

$$\begin{aligned}&(v_t^k,u) + \nu(\nabla v^k,\nabla u) + \nu_o(Av^k,u) + (v^k\cdot\nabla v^k,u)\\&\quad = \lambda(\mathcal{F}(\hat{c}^k,\hat{\theta}^k),u) - \lambda(k(f_s^\delta(\hat{\phi}^k)-\delta)\hat{v}^k,u)\;\forall\,u \in V^p,\; t\in(0,T),\end{aligned} \tag{46}$$

$$\begin{aligned}\alpha\epsilon^2\phi_t^k + \alpha\epsilon^2\rho_\delta(v^k)\cdot\nabla\phi^k - \epsilon^2\Delta\phi^k \\ = \frac{1}{2}(\phi^k - (\phi^k)^3) + \lambda\beta\left(\hat{\theta}^k + (\theta_B - \theta_A)\hat{c}^k - \theta_B\right) \text{ in } Q^\delta,\end{aligned} \tag{47}$$

$$C_v\theta_t^k + C_v\rho_\delta(v^k)\cdot\nabla\theta^k = \nabla\cdot\left(K_1(\rho_\delta(\phi^k))\nabla\theta^k\right) + \frac{l}{2}f_s^\delta(\phi^k)_t \text{ in } Q^\delta, \tag{48}$$

$$c_t^k - K_2\Delta c^k + \rho_\delta(v^k)\cdot\nabla c^k = K_2 M\nabla\cdot\left(c^k(1-c^k)\nabla\rho_\delta(\phi^k)\right) \text{ in } Q^\delta, \tag{49}$$

$$\frac{\partial\phi^k}{\partial n} = 0, \quad \frac{\partial\theta^k}{\partial n} = 0, \quad \frac{\partial c^k}{\partial n} = 0 \text{ on } \partial\Omega^\delta\times(0,T), \tag{50}$$

$$v^k(0) = v_0^\delta \text{ in } \Omega, \quad \phi^k(0) = \phi_0^\delta, \quad \theta^k(0) = \theta_0^\delta, \quad c^k(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{51}$$

We show that the sequence $(v^k,\phi^k,\theta^k,c^k)$ converges strongly in B to (v,ϕ,θ,c) $= \mathcal{T}_\lambda(\hat{v},\hat{\phi},\hat{\theta},\hat{c})$. For that purpose, we will obtain estimates to $(v^k,\phi^k,\theta^k,c^k)$ independent of k. We denote by C_i any positive constant independent of k.

We take $u = v^k$ in equation (46). Using Hölder's and Young's inequalities we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |v^k|^2 dx \quad &+ \nu_o\int_\Omega |\nabla v^k|^p dx + \nu\int_\Omega |\nabla v^k|^2 dx \\ &\le C_1\int_\Omega\left(|F|^2 + |\hat{v}^k|^2 + |\hat{\theta}^k|^2 + |\hat{c}^k|^2 + |v^k|^2\right)dx.\end{aligned}$$

Then, by the usual method of Gronwall's inequality, we get

$$\|v^k\|_{L^\infty(0,T;H)\cap L^p(0,T;V^p)} \le C_1. \tag{52}$$

Observe that operator A satisfies $\|Av\| \le C\|v\|_{V^p}^{p-1}$. Now, from the equation (46) we infer that

$$\begin{aligned}\|v_t^k\|_{(V^p)'} \le \quad & C_1\left(\|v^k\|_{V^p}^{p-1} + \|v^k\|_V + \|v^k\|_{L^{2p'}(\Omega)}^2 + \|F\|_{L^2(\Omega)}\right. \\ &\left. + \|\hat{v}^k\|_{L^2(\Omega)} + \|\hat{\theta}^k\|_{L^2(\Omega^\delta)} + \|\hat{c}^k\|_{L^2(\Omega^\delta)}\right),\end{aligned}$$

then, using (52) and since $2p' \le p5/3$, we obtain

$$\|v_t^k\|_{L^{p'}(0,T;(V^p)')} \le C_1. \tag{53}$$

From estimate (20) we have that

$$\begin{aligned}\|\phi\|_{W_2^{2,1}(Q^\delta)} \le \quad & C\left(\|\phi_0\|_{H^1(\Omega^\delta)} + \|\hat{\theta}^k\|_{L^2(Q^\delta)} + \|\hat{c}^k\|_{L^2(Q^\delta)} + \|\phi_0\|_{H^1(\Omega^\delta)}^3\right. \\ &\left. + \|\hat{\theta}^k\|_{L^2(Q^\delta)}^3 + \|\hat{c}^k\|_{L^2(Q^\delta)}^3 + 1\right)\end{aligned}$$

where C depends on $\|\rho_\delta(v^k)\|_{L^5(Q^\delta)}$. Therefore, using (52) we conclude that

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \leq C_1. \tag{54}$$

Now, multiplying (48) by θ^k one obtains

$$\int_{\Omega^\delta} |\theta^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla \theta^k|^2 dx dt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} \left(|\phi_t^k|^2 + |\theta^k|^2 \right) dx dt \tag{55}$$

and we infer from (54) and Gronwall's Lemma that

$$\|\theta^k\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1, \tag{56}$$

hence, it follows from (55) that

$$\|\theta^k\|_{L^2(0,T;H^1(\Omega^\delta))} \leq C_2. \tag{57}$$

We take scalar product of (48) with $\eta \in H^1(\Omega^\delta)$, integrating by parts and using Hölder's and Young's inequalities, we obtain

$$\|\theta_t^k\|_{H^1(\Omega^\delta)'} \leq C_1 \left(\|\nabla \theta^k\|_{L^2(\Omega^\delta)} + \|\rho_\delta\|_{L^\infty(Q^\delta)} \|v^k\|_{L^2(\Omega)} \|\theta^k\|_{L^2(\Omega^\delta)} + \|\phi_t^k\|_{L^2(\Omega^\delta)} \right)$$

and we infer from (52),(54) and (57) that

$$\|\theta_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{58}$$

Next, multiplying (49) by c^k we conclude by analogous reasoning and using (45) that

$$\int_{\Omega^\delta} |c^k|^2 dx + \int_0^t \int_{\Omega^\delta} |\nabla c^k|^2 dx dt \leq C_1 + C_2 \int_0^t \int_{\Omega^\delta} |\nabla \phi^k|^2 dx dt,$$

hence, from (54) we have,

$$\|c^k\|_{L^2(0,T;H^1(\Omega^\delta)) \cap L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1. \tag{59}$$

In order to get an estimate for (c_t^k) in $L^2(0,T;H^1(\Omega^\delta)')$, we return to the equation (49) and use similar techniques, then

$$\|c_t^k\|_{L^2(0,T;H^1(\Omega^\delta)')} \leq C_1. \tag{60}$$

We now infer from (52)-(60) that the sequence (v^k) is uniformly bounded with respect to k in

$$W_1 = \left\{ w \in L^p(0,T;V^p),\ w_t \in L^{p'}(0,T;(V^p)') \right\}$$

and in

$$W_2 = \left\{ w \in L^\infty(0,T;H),\ w_t \in L^{p'}(0,T;(V^p)') \right\},$$

the sequence (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$ and the sequences (θ^k) and (c^k) are bounded in

$$W_3 = \left\{ w \in L^2(0,T;H^1(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}$$

and in

$$W_4 = \left\{ w \in L^\infty(0,T;L^2(\Omega^\delta)),\ w_t \in L^2(0,T;H^1(\Omega^\delta)') \right\}.$$

Since W_1 is compactly embedded in $L^p(0,T;H)$, W_2 in $C([0,T];(V^p)')$, $W_2^{2,1}(Q^\delta)$ in $L^2(0,T;H^1(\Omega^\delta))$, W_3 in $L^2(Q^\delta)$ and W_4 in $C([0,T];H^1(\Omega^\delta)')$ ([13] Cor.4), it follows that there exist

$$\begin{array}{rcl}
v & \in & L^p(0,T;V^p)\cap L^\infty(0,T;H) \text{ with } v_t \in L^{p'}(0,T;(V^p)'),\\
\chi & \in & L^{p'}(0,T;(V^p)'),\\
\phi & \in & L^2(0,T;H^2(\Omega^\delta)) \text{ with } \phi_t \in L^2(Q^\delta),\\
\theta & \in & L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } \theta_t \in L^2(0,T;H^1(\Omega^\delta)'),\\
c & \in & L^2(0,T;H^1(\Omega^\delta))\cap L^\infty(0,T;L^2(\Omega^\delta)) \text{ with } c_t \in L^2(0,T;H^1(\Omega^\delta)'),
\end{array}$$

and a subsequence of $(v^k,\phi^k,\theta^k,c^k)$ (which we still denote by $(v^k,\phi^k,\theta^k,c^k)$), such that, as $k\to+\infty$,

$$\begin{array}{rclll}
v^k & \rightarrow & v & \text{in} & L^p(0,T;H)\cap C([0,T];(V^p)') \text{ strongly},\\
v^k & \rightharpoonup & v & \text{in} & L^p(0,T;V^p) \text{ weakly},\\
Av^k & \rightharpoonup & \chi & \text{in} & L^{p'}(0,T;(V^p)') \text{ weakly},\\
\phi^k & \rightarrow & \phi & \text{in} & L^2(0,T;H^1(\Omega^\delta))\cap C([0,T];L^2(\Omega^\delta)) \text{ strongly},\\
\phi^k & \rightharpoonup & \phi & \text{in} & L^2(0,T;H^2(\Omega^\delta)) \text{ weakly},\\
\theta^k & \rightarrow & \theta & \text{in} & L^2(Q^\delta)\cap C([0,T];H^1(\Omega^\delta)') \text{ strongly},\\
\theta^k & \rightharpoonup & \theta & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \text{ weakly},\\
c^k & \rightarrow & c & \text{in} & L^2(Q^\delta)\cap C([0,T];H^1(\Omega^\delta)') \text{ strongly},\\
c^k & \rightharpoonup & c & \text{in} & L^2(0,T;H^1(\Omega^\delta)) \text{ weakly}.
\end{array} \tag{61}$$

It now remains to pass to the limit as k tends to $+\infty$ in (46)-(51).

We observe that $k(f_s^\delta(\cdot)-\delta)$ is bounded Lipschitz continuous function from $I\!R$ in $I\!R$ then $k(f_s^\delta(\hat{\phi}^k)-\delta)$ converges to $k(f_s^\delta(\hat{\phi})-\delta)$ in $L^p(Q)$, for any $p\in[1,\infty)$. We then pass to the limit in the usual form as k tends to $+\infty$ in (46) and get

$$\frac{d}{dt}(v,u)+\nu_o(\chi,u)+\nu(\nabla v,\nabla u)+(v\cdot\nabla v,u) = \lambda(\mathcal{F}(\hat{c},\hat{\theta}),u)-\lambda(k(f_s^\delta(\hat{\phi})-\delta)\hat{v},u)$$

for all $u \in V^p$, $t \in (0,T)$. Using that the operator A is monotone we can conclude that $\chi = Av$.

Since the embedding of $W_2^{2,1}(Q^\delta)$ into $L^9(Q^\delta)$ is compact ([9] p.15), and (ϕ^k) is bounded in $W_2^{2,1}(Q^\delta)$, we infer that $(\phi^k)^3$ converges to ϕ^3 in $L^2(Q^\delta)$. Also, since v^k converges to v in $L^p(0,T;H)$ we have that $\rho_\delta(v^k)$ converges to $\rho_\delta(v)$ in $L^p(0,T;H(\Omega^\delta))$. We then pass to the limit as k tends to $+\infty$ in (47) and get

$$\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi-\phi^3) = \lambda\beta\left(\hat{\theta} + (\theta_B-\theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta.$$

Since $K_1(\rho_\delta)$ and $f_s^{\delta\prime}$ are bounded Lipschitz continuous functions and ϕ^k converges to ϕ in $L^9(Q^\delta)$ we have that $K_1(\rho_\delta(\phi^k))$ converges to $K_1(\rho_\delta(\phi))$ and $f_s^{\delta\prime}(\phi^k)$ converges to $f_s^{\delta\prime}(\phi)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. These facts and (61) yield the weak convergence of $K_1(\rho_\delta(\phi^k))\nabla\theta^k$ to $K_1(\rho_\delta(\phi))\nabla\theta$ and $f_s^{\delta\prime}(\phi^k)\phi_t^k$ to $f_s^{\delta\prime}(\phi)\phi_t$ in $L^{3/2}(Q^\delta)$. Now, multiplying (48) by $\eta \in \mathcal{D}(Q^\delta)$, integrating over $\Omega^\delta \times (0,T)$ and by parts, we obtain

$$\begin{aligned}\int_0^T \int_{\Omega^\delta} C_\mathrm{v}\left(\theta_t^k + \rho_\delta(v^k)\cdot\nabla\theta^k\right)\eta + K_1(\rho_\delta(\phi^k))\nabla\theta^k\cdot\nabla\eta \, dxdt \\ = \int_0^T \int_{\Omega^\delta} \frac{l}{2} f_s^{\delta\prime}(\phi^k)\phi_t^k\eta \, dxdt,\end{aligned}$$

then we may pass to the limit and find that,

$$C_\mathrm{v}\theta_t + C_\mathrm{v}\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^{\delta\prime}(\phi)\phi_t \quad \text{in } \mathcal{D}'(Q^\delta), \quad (62)$$

and using L^p-theory of parabolic equations we have that (62) holds almost everywhere in Q^δ.

It remains to pass to the limit in (49). We infer from (61) that $\nabla\rho_\delta(\phi^k)$ converges to $\nabla\rho_\delta(\phi)$ in $L^2(Q^\delta)$ and since $\|c^k\|_{L^\infty(Q^\delta)}$ is bounded, it follows that $c^k(1-c^k)$ converges to $c(1-c)$ in $L^p(Q^\delta)$ for any $p \in [1,\infty)$. Similarly, we may pass to the limit in (49) to obtain

$$c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2M\nabla\cdot(c(1-c)\nabla\rho_\delta(\phi)) \qquad \text{in } Q^\delta.$$

Therefore $\mathcal{T}_\lambda$ is continuous for all $0 \le \lambda \le 1$. At the same time, $\mathcal{T}_\lambda$ is bounded in $W_1 \times W_2^{2,1}(Q^\delta) \times W_3 \times W_3$ and the embedding of this space in B is compact; then we conclude that $\mathcal{T}_\lambda$ is a compact operator.

To prove that for $(\hat{v},\hat{\phi},\hat{\theta},\hat{c})$ in a bounded set of B, $\mathcal{T}_\lambda$ is uniformly continuous in λ, let $0 \le \lambda_1, \lambda_2 \le 1$ and $(v_i,\phi_i,\theta_i,c_i)$ $(i=1,2)$ be the corresponding solutions of (39)-(44). We observe that $v = v_1 - v_2$, $\phi = \phi_1 - \phi_2$, $\theta = \theta_1 - \theta_2$ and $c = c_1 - c_2$ satisfy the following problem:

$$\begin{gathered}(v_t, u) + \nu(\nabla v, \nabla u) + \nu_o(Av_1 - Av_2, u) + (v_1 \cdot \nabla v, u) - (v \cdot \nabla v_2, u) \\ = (\lambda_1 - \lambda_2)(\mathcal{F}(\hat{c}, \hat{\theta}), u) + (\lambda_2 - \lambda_1)(k(f_s^\delta(\hat{\phi}) - \delta)\hat{v}, u), \\ \text{for all } u \in V^p,\ t \in (0,T),\end{gathered} \tag{63}$$

$$\begin{gathered}\alpha\epsilon^2\phi_t - \epsilon^2\Delta\phi + \alpha\epsilon^2\rho_\delta(v_1)\cdot\nabla\phi - \frac{1}{2}\phi\left(1 - (\phi_1^2 + \phi_1\phi_2 + \phi_2^2)\right) \\ = \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi_2 + (\lambda_1 - \lambda_2)\beta\left(\hat{\theta} + (\theta_B - \theta_A)\hat{c} - \theta_B\right) \text{ in } Q^\delta,\end{gathered} \tag{64}$$

$$\begin{gathered}C_{\mathrm{v}}\theta_t - \nabla\cdot K_1(\rho_\delta(\phi_1))\nabla\theta - \nabla\cdot\left[K_1(\rho_\delta(\phi_1)) - K_1(\rho_\delta(\phi_2))\right]\nabla\theta_2 \\ + C_{\mathrm{v}}\rho_\delta(v_1)\cdot\nabla\theta = C_{\mathrm{v}}\rho_\delta(v)\cdot\nabla\theta_2 \\ + \frac{l}{2}f_s^{\delta\prime}(\phi_1)\phi_t + \frac{l}{2}\left[f_s^{\delta\prime}(\phi_1) - f_s^{\delta\prime}(\phi_2)\right]\phi_{2t} \text{ in } Q^\delta,\end{gathered} \tag{65}$$

$$\begin{gathered}c_t - K_2\Delta c + \rho_\delta(v_1)\cdot\nabla c = K_2M\nabla\cdot\left(c_1(1-c_1)\left[\nabla\rho_\delta(\phi_1) - \nabla\rho_\delta(\phi_2)\right]\right) \\ + \rho_\delta(v)\cdot\nabla c_2 + K_2M\nabla\cdot\left(c(1-(c_1+c_2))\nabla\rho_\delta(\phi_2)\right) \text{ in } Q^\delta,\end{gathered} \tag{66}$$

$$\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega^\delta \times (0,T), \tag{67}$$

$$v(0) = 0 \text{ in } \Omega, \quad \phi(0) = 0, \quad \theta(0) = 0, \quad c(0) = 0 \text{ in } \Omega^\delta. \tag{68}$$

Taking $u = v$ in equation (63), using Hölder's, Young's and interpolation inequalities and the monotonicity of operator A we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2dx \;&+\; \int_\Omega \nu|\nabla v|^2dx \le \int_\Omega |v||\nabla v_2||v|dx \\ &+\; |\lambda_1 - \lambda_2|\int_\Omega\left(|\mathcal{F}(\hat{c},\hat{\theta})||v| + k(f_s^\delta(\hat{\phi}) - \delta)|\hat{v}||v|\right)dx \\ &\le\; C_1\|v_2\|^s_{L^r(\Omega)}\|v\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|v\|^2_V \\ &+\; C_2|\lambda_1 - \lambda_2|^2\int_\Omega |F|^2 + |\hat{\theta}|^2 + |\hat{c}|^2 + |\hat{v}|^2dx + C_3\int_\Omega |v|^2dx.\end{aligned}$$

where $2/s+3/r = 1$ and $r > 3$. Observe that due to assumption $p \ge 3$ we have that $v \in L^s(0,T;L^r(\Omega))$. Then, integration with respect t and Gronwall's Lemma give us

$$\|v\|^2_{L^\infty(0,T;H)\cap L^2(0,T;V)} \le C_1\,|\lambda_1 - \lambda_2|^2. \tag{69}$$

Applying L^p-theory of parabolic equations ([7] p. 341) to equation (64), the following estimate holds

$$\|\phi\|_{W_2^{2,1}(Q^\delta)} \le C_1\left(\|\rho_\delta(v)\cdot\nabla\phi_2\|_{L^2(Q^\delta)} + |\lambda_1 - \lambda_2|\left(\|\hat{\theta}\|_{L^2(Q^\delta)} + \|\hat{c}\|_{L^2(Q^\delta)} + 1\right)\right)$$

where C_1 depends on $\|\rho_\delta(v_1)\|_{L^5(Q^\delta)}$ and $\|\phi_1^2+\phi_1\phi_2+\phi_2^2\|_{L^{5/2}(Q^\delta)}$, which are independent of λ_i. Therefore, using (69) we arrive at

$$\|\phi\|^2_{W_2^{2,1}(Q^\delta)} \leq C_1\,|\lambda_1-\lambda_2|^2. \tag{70}$$

Multiplying (65) by θ, integrating over Ω^δ using Hölder's inequality and that K_1 and $f_s^{\delta'}$ are bounded Lipschitz continuous functions, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^\delta}|\theta|^2dx + a\int_{\Omega^\delta}|\nabla\theta|^2dx\\
&\quad\leq C_1\int_{\Omega^\delta}|\rho_\delta(\phi)||\nabla\theta_2||\nabla\theta| + |\rho_\delta(v)||\nabla\theta_2||\theta| + |\phi_t||\theta| + |\phi||\phi_{2t}|\,|\theta|dx\\
&\quad\leq C_1\|\phi\|^2_{L^\infty(0,T;L^2(\Omega^\delta))}\|\nabla\theta_2\|_{L^2(\Omega^\delta)} + C_2\|v\|^2_{L^\infty(0,T;H)}\|\nabla\theta_2\|^2_{L^2(\Omega^\delta)}\\
&\quad+C_3\int_{\Omega^\delta}|\phi_t|^2+|\theta|^2dx + C_4\|\phi\|^2_{L^\infty(0,T;H^1(\Omega^\delta))}\|\phi_{2t}\|^2_{L^2(\Omega^\delta)} + \frac{a}{2}\int_{\Omega^\delta}|\nabla\theta|^2dx.
\end{aligned}$$

Integration with respect to t and the use of Gronwall's Lemma and (69)-(70) lead to the estimate

$$\|\theta\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1\,|\lambda_1-\lambda_2|^2. \tag{71}$$

We multiply (66) by c, integrate over $\Omega^\delta\times(0,t)$ and by parts, and we use Hölder's and Young's inequalities and (45) to obtain

$$\begin{aligned}
\int_{\Omega^\delta}|c|^2dx \;&+\; \int_0^t\int_{\Omega^\delta}|\nabla c|^2dxdt\\
&\leq\; C_1\int_0^t\int_{\Omega^\delta}\left(|\nabla\rho_\delta(\phi_1)-\nabla\rho_\delta(\phi_2)|^2+|\rho_\delta(v)|^2+|c|^2\right)dxdt\\
&\leq\; C_1\int_0^t\int_{\Omega^\delta}\left(|\nabla\phi|^2+|c|^2\right)dxdt + C_1\int_0^t\int_\Omega|v|^2dxdt.
\end{aligned}$$

Applying Gronwall's Lemma and using (69)-(70) we arrive at

$$\|c\|^2_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1\,|\lambda_1-\lambda_2|^2. \tag{72}$$

Therefore, it follows from (69)-(72) that $\mathcal{T}_\lambda$ is uniformly continuous in λ.

To estimate the set of all fixed points of $\mathcal{T}_\lambda$ let $(v,\phi,\theta,c)\in B$ be such a fixed point, i.e., it is a solution of the problem

$$\begin{aligned}
&(v_t,u)+\nu(\nabla v,\nabla u)+\nu_o(Av,u)+(v\cdot\nabla v,u)\\
&\quad=\lambda(\mathcal{F}(c,\theta),u)-\lambda(k(f_s^\delta(\phi)-\delta)v,u)\;\forall u\in V,\; t\in(0,T),
\end{aligned}\tag{73}$$

$$
\begin{aligned}
\alpha\epsilon^2\phi_t + \alpha\epsilon^2\rho_\delta(v)\cdot\nabla\phi - \epsilon^2\Delta\phi - \frac{1}{2}(\phi-\phi^3)& \\
= \lambda\beta\left(\theta + (\theta_B-\theta_A)c - \theta_B\right) \text{ in } Q^\delta&,
\end{aligned} \tag{74}
$$

$$
C_v\theta_t + C_v\rho_\delta(v)\cdot\nabla\theta = \nabla\cdot(K_1(\rho_\delta(\phi))\nabla\theta) + \frac{l}{2}f_s^\delta(\phi)_t \text{ in } Q^\delta, \tag{75}
$$

$$
c_t - K_2\Delta c + \rho_\delta(v)\cdot\nabla c = K_2 M\nabla\cdot\left(c(1-c)\nabla\left(\rho_\delta(\phi)\right)\right) \text{ in } Q^\delta, \tag{76}
$$

$$
\frac{\partial\phi}{\partial n} = 0, \quad \frac{\partial\theta}{\partial n} = 0, \quad \frac{\partial c}{\partial n} = 0 \text{ on } \partial\Omega^\delta\times(0,T), \tag{77}
$$

$$
v(0) = v_0^\delta \text{ in } \Omega, \quad \phi(0) = \phi_0^\delta, \quad \theta(0) = \theta_0^\delta, \quad c(0) = c_0^\delta \text{ in } \Omega^\delta. \tag{78}
$$

We take $u = v$ in equation (73). Then

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2dx \quad &+ \int_\Omega\left(\nu_o|\nabla v|^p + \nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi)-\delta)|v|^2\right)dx \\
&\leq C_1\int_\Omega |F|^2 + |\theta|^2 + |c|^2 + |v|^2dx \\
&\leq C_1\int_\Omega |F|^2 + |v|^2dx + C_1\int_{\Omega^\delta}|\theta|^2 + |c|^2dx.
\end{aligned} \tag{79}
$$

Multiplying equation (74) by ϕ, integrating over Ω^δ and by parts, using Hölder's and Young's inequalities we obtain,

$$
\begin{aligned}
\frac{\alpha\epsilon^2}{2}\frac{d}{dt}\int_{\Omega^\delta}|\phi|^2dx \quad &+ \quad \int_{\Omega^\delta}\left(\epsilon^2|\nabla\phi|^2 + \frac{1}{2}\phi^4\right)dx \\
&\leq \quad C_1 + C_1\int_{\Omega^\delta}\left(|\theta|^2 + |c|^2 + |\phi|^2\right)dx.
\end{aligned} \tag{80}
$$

By multiplying (75) by $e = C_v\theta - \frac{l}{2}f_s^\delta(\phi)$ and (76) by c, arguments similar to the previous ones lead to the following estimates

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta}|e|^2dx + \frac{C_v a}{2}\int_{\Omega^\delta}|\nabla\theta|^2dx \leq C_2\int_{\Omega^\delta}|\nabla\phi|^2dx + C_1\int_\Omega|v|^2dx, \tag{81}
$$

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^\delta}|c|^2dx + \frac{K_2}{2}\int_{\Omega^\delta}|\nabla c|^2dx \leq C_2\int_{\Omega^\delta}|\nabla\phi|^2dx, \tag{82}
$$

where (45) was used to obtain the last inequality.

Now, multiplying (80) by A and adding the result to (79),(81)-(82), gives us

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{1}{2}|v|^2 + \frac{d}{dt}\int_{\Omega^\delta}\left(\frac{A\alpha\epsilon^2}{4}|\phi|^2 + \frac{1}{2}|e|^2 + \frac{1}{2}|c|^2\right)dx \\
&\quad + \int_\Omega \left(\nu_o|\nabla v|^p + \nu|\nabla v|^2 + \lambda k(f_s^\delta(\phi)-\delta)|v|^2\right)dx \\
&\quad + \int_{\Omega^\delta}\left((A\epsilon^2 - 2C_2)|\nabla\phi|^2 + \frac{A}{2}\phi^4 + \frac{C_v a}{2}|\nabla\theta|^2 + \frac{K_2}{2}|\nabla c|^2\right)dx \\
&\qquad \leq C_1 + C_1\int_\Omega |v|^2 dx + C_1\int_{\Omega^\delta}\left(|\phi|^2 + |\theta|^2 + |c|^2\right)dx \qquad (83)
\end{aligned}$$

where C_1 is independent of λ and δ, being $A \in I\!R$ an arbitrary parameter. Taking A large enough and using Gronwall's Lemma to obtain

$$\|v\|_{L^\infty(0,T;H)} + \|\phi\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|e\|_{L^\infty(0,T;L^2(\Omega^\delta))} + \|c\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1,$$

where C_1 is independent of λ. Since $\theta = \dfrac{1}{C_v}\left(e + \dfrac{l}{2}f_s^\delta(\phi)\right)$ and $f_s^\delta(\phi)$ is bounded in $L^\infty(Q^\delta)$, we also have that $\|\theta\|_{L^\infty(0,T;L^2(\Omega^\delta))} \leq C_1$. Therefore, all fixed points of $\mathcal{T}_\lambda$ in B are bounded independently of $\lambda \in [0,1]$.

Finally, for $\lambda = 0$, we can reason as in the proof that $\mathcal{T}_\lambda$ is well defined to conclude that the problem (39)-(44) has a unique solution. Therefore, we can apply Leray-Schauder's Theorem and so there is at least one fixed point $(v,\phi,\theta,c) \in B\cap\{L^p(0,T;V^p)\cap L^\infty(0,T;H)\}\times W_2^{2,1}(Q^\delta)\times W_2^{2,1}(Q^\delta)\times C^{2,1}(Q^\delta)$ of the operator $\mathcal{T}_1$, i.e. $(v,\phi,\theta,c) = \mathcal{T}_1(v,\phi,\theta,c)$. These functions are a solution of problem (33)-(38) and the proof of Proposition 1 is complete. ∎

4 The proof of Theorem 1

To prove Theorem 1, we start by taking the initial condition in the previous regularized problem as follows. For $0 < \delta \leq \delta(\Omega)$ as in the statement of Theorem 1, we choose $\phi_0^\delta \in W^{2-2/q,q}(\Omega^\delta)\cap H^{1+\gamma}(\Omega^\delta)$, $v_0^\delta \in H$, $\theta_0^\delta \in H^{1+\gamma}(\Omega)$, $1/2 < \gamma \leq 1$, $c_0^\delta \in C^1(\overline{\Omega^\delta})$, satisfying $\dfrac{\partial\phi_0^\delta}{\partial n} = \dfrac{\partial\theta_0^\delta}{\partial n} = \dfrac{\partial c_0^\delta}{\partial n} = 0$ on $\partial\Omega^\delta$ and $0 < c_0^\delta < 1$ in $\overline{\Omega^\delta}$, $v_0^\delta \to v_0$ in the norm of $H(\Omega_{ml}(0))$, and such that the restrictions of these functions to Ω (recall that $\Omega \subset \Omega^\delta$) satisfy as $\delta \to 0+$ the following: $\phi_0^\delta \to \phi_0$ in the norm of $W^{2-2/q,q}(\Omega)\cap H^{1+\gamma}(\Omega)$, $\theta_0^\delta \to \theta_0$ in the norm of $L^2(\Omega)$, $c_0^\delta \to c_0$ in the norm of $L^2(\Omega)$.

We then infer from Proposition 1 that there exists $(\phi^\delta, v^\delta, \theta^\delta, c^\delta)$ solution the regularized problem (33)-(38).

In the following, we will derive bounds, independent of δ, for such solutions and then use compactness arguments to pass to the limit as δ approach 0 to establish the desired existence result. Such estimates will stated in following in a sequence of lemmas; however, most of them are ease consequence of the estimates obtained in the last section (those that are independent of δ) and the fact that $\Omega \subset \Omega^\delta$. We begin with the following:

Lemma 1 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$\|v^\delta\|_{L^\infty(0,T;H)\cap L^p(0,T;V^p)} + \int_0^T \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dxdt \leq C_1, \tag{84}$$

$$\|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|\phi^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1, \tag{85}$$

$$\|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|\theta^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1, \tag{86}$$

$$\|c^\delta\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))} \leq \|c^\delta\|_{L^\infty(0,T;L^2(\Omega^\delta))\cap L^2(0,T;H^1(\Omega^\delta))} \leq C_1. \tag{87}$$

Proof: The result follows from inequality (83). ∎

Lemma 2 *There exists a constant C_1 such that, for any $\delta \in (0, \delta(\Omega))$*

$$\begin{aligned} \|\phi^\delta\|_{W_q^{2,1}(Q)} &\leq C_1, \quad \text{for any } 2 \leq q \leq 10/3, &(88)\\ \|\theta_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\leq C_1, &(89)\\ \|c_t^\delta\|_{L^2(0,T;H_o^1(\Omega)')} &\leq C_1. &(90) \end{aligned}$$

Proof: Note that (88) follows from estimate (20) of Theorem 2 and Lemma 1.

Next, we take the scalar product of (35) with $\eta \in H_o^1(\Omega)$, using Hölder's inequality and **(H3)** we find

$$C_v\|\theta_t^\delta\|_{H_o^1(\Omega)'} \leq C_1 \left(\|\nabla\theta^\delta\|_{L^2(\Omega)} + \|\theta^\delta\|_{L^{10/3}(\Omega)}\|v^\delta\|_{L^5(\Omega)} + \|\phi_t^\delta\|_{L^2(\Omega)}\right).$$

Then, (89) follows from Lemma 1 and (88).

Using that $0 < c^\delta < 1$ in Q, we infer from (36) that,

$$\|c_t^\delta\|_{H_o^1(\Omega)'} \leq C_1 \left(\|\nabla c^\delta\|_{L^2(\Omega)} + \|v^\delta\|_{L^2(\Omega)} + \|\nabla\phi^\delta\|_{L^2(\Omega)}\right).$$

Then, (90) follows from Lemma 1. ∎

Lemma 3 *There exist a constant C_1 and $\delta_0 \in (0, \delta(\Omega))$ such that, for any $\delta < \delta_0$,*

$$\|v_t^\delta\|_{L^{p'}(t_1,t_2;V^p(U)')} \leq C_1 \tag{91}$$

where $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ and such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$.

Proof: Let $0 \leq t_1 < t_2 \leq T$, $U \subseteq \Omega_{ml}(t_1)$ be such that $[t_1, t_2] \times \bar{U} \subseteq Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. It is verified by means of (33) that for a.e. $t \in (t_1, t_2)$,

$$\begin{aligned}(v_t^\delta, u) &= -\nu_o \int_U Av^\delta\, u dx - \nu \int_U \nabla v^\delta \cdot \nabla u dx - \int_U v^\delta \cdot \nabla v^\delta u dx \\ &\quad - \int_U k(f_s^\delta(\phi^\delta) - \delta) v^\delta u dx + \int_U \mathcal{F}(c^\delta, \theta^\delta) u dx, \quad u \in V^p(U).\end{aligned}$$

In order to estimate $\|v_t^\delta\|_{V^p(U)'}$, we observe that the sequence (ϕ^δ) is bounded in $W_q^{2,1}(Q)$, for $2 \leq q < 5$, in particular, for $q > 5/2$ we have that $W_q^{2,1}(Q) \subseteq H^{\tau,\tau/2}(\bar{Q})$ where $\tau = 2 - 5/q$ ([7] p.80). Consequently, because of Arzela-Ascoli's theorem, there exist ϕ and a subsequence of (ϕ^δ) (which we still denote by ϕ^δ), such that ϕ^δ converges uniformly to ϕ in $\bar{Q}$. Recall that $Q_{ml} = \{(x,t) \in Q \,/\, 0 \leq f_s(\phi(x,t)) < 1\}$ and $\Omega_{ml}(t) = \{x \in \Omega \,/\, 0 \leq f_s(\phi(x,t)) < 1\}$. Note that for a certain $\overline{\gamma} \in (0,1)$ and for $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s(\phi(x,t)) < 1 - \overline{\gamma}.$$

Due to the uniform convergence of f_s^δ towards f_s on any compact subset, there is an δ_0 such that for all $\delta \in (0, \delta_0)$ and for all $(x,t) \in [t_1, t_2] \times \bar{U}$,

$$f_s^\delta(\phi^\delta(x,t)) < 1 - \overline{\gamma}/2.$$

By assumption **(H2)** we infer that

$$k(f_s^\delta(\phi^\delta(x,t)) - \delta) < k(1 - \overline{\gamma}/2) \qquad \text{for } (x,t) \in [t_1, t_2] \times \bar{U} \text{ and } \delta < \delta_0.$$

Thus,

$$\begin{aligned}\|v_t^\delta\|_{V^p(U)'} \leq\ & C_1\Big(\|v^\delta\|_{V^p}^{p-1} + \|v^\delta\|_V + \|v^\delta\|_{L^s(\Omega)}^2 + \|F\|_{L^2(\Omega)} + \|c^\delta\|_{L^2(\Omega)} \\ & + \|\theta^\delta\|_{L^2(\Omega)} + \|k(f_s^\delta(\phi^\delta(x,t)) - \delta)\|_{L^\infty(U)} \|v^\delta\|_{L^2(\Omega)}\Big),\end{aligned}$$

where $2/s + 1/p = 1$. Hence, (91) follows from Lemma 1. ∎

From (84), the sequence (v^δ) is also bounded in $L^p(t_1, t_2; W^{1,p}(U))$; then, by compact embedding ([13] Cor. 4), there exist v and a subsequence of (v^δ) (which we still denote v^δ), such that

$$v^\delta \to v \quad \text{strongly in } L^p((t_1, t_2) \times U).$$

Observe that Q_{ml} is an open set and can be covered by a countable number of open sets $(t_i, t_{i+1}) \times U_i$ such that $U_i \subseteq \Omega_{ml}(t_i)$, then by means of a diagonal argument, we obtain

$$v^\delta \to v \quad \text{strongly in } L^p_{loc}(Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)). \tag{92}$$

Moreover, from (84) we have that $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and

$$\begin{array}{lcll} v^\delta & \rightharpoonup & v & \text{weakly in} \quad L^2(0,T;V), \\ v^\delta & \overset{*}{\rightharpoonup} & v & \text{weakly * in} \quad L^\infty(0,T;H). \end{array} \tag{93}$$

Since Av^δ is bounded in $L^{p'}(0,T;(V^p)')$ there exists $\chi \in L^{p'}(0,T;(V^p)')$ such that

$$Av^\delta \rightharpoonup \chi \in L^{p'}(0,T;(V^p)') \text{ weakly.} \tag{94}$$

We now infer from Lemma 1 and Lemma 2, using compact embedding ([13] Cor.4), that there exist

$$\begin{array}{lcl} \phi & \in & W^{2,1}_q(Q) \text{ for } 2 \leq q \leq 10/3, \\ \theta & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \\ c & \in & L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)), \end{array}$$

and a subsequence of $(\phi^\delta, \theta^\delta, c^\delta)$ (which we still denote by $(\phi^\delta, \theta^\delta, c^\delta)$) such that, as $\delta \to 0$,

$$\begin{array}{lcll} \phi^\delta & \to & \phi & \text{uniformly in } Q, \\ \phi^\delta & \to & \phi & \text{strongly in } L^q(0,T;W^{1,q}(\Omega)), \\ \phi^\delta_t & \rightharpoonup & \phi_t & \text{weakly in } L^q(Q), \\ \theta^\delta & \to & \theta & \text{strongly in } L^2(Q) \cap C([0,T];H^1_o(\Omega)'), \\ \theta^\delta & \rightharpoonup & \theta & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ c^\delta & \to & c & \text{strongly in } L^2(Q) \cap C([0,T];H^1_o(\Omega)'), \\ c^\delta & \rightharpoonup & c & \text{weakly in } L^2(0,T;H^1(\Omega)). \end{array} \tag{95}$$

It now remains pass to the limit as δ decreases to zero in (33)-(38). We start with the velocity equation.

We take $u = \eta(t)$ in (33) where $\eta \in L^p(0,T;V^p)$ with compact support contained in $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and $\eta_t \in L^{p'}(0,T;V^p(\Omega_{ml}(t))')$; after integration over $(0,t)$, we find

$$\int_0^t \Big((v_t^\delta, \eta) + \nu(\nabla v^\delta, \nabla\eta) + \nu_o(Av^\delta, \eta) + (v^\delta \cdot \nabla v^\delta, \eta) \\ + (k(f_s^\delta(\phi^\delta) - \delta)v^\delta, \eta) \Big)\, ds = \int_0^t (\mathcal{F}(c^\delta, \theta^\delta), \eta) ds. \tag{96}$$

Moreover, we observe that

$$\int_0^t (v_t^\delta, \eta) ds = -\int_0^t (v^\delta, \eta_t) ds + (v^\delta(t), \eta(t)) - (v_0^\delta, \eta(0)).$$

Also, because of uniform convergence of f_s^δ to f_s on compact subsets, as well as the assumption **(H2)**, it follows that $k(f_s^\delta(\phi^\delta) - \delta)$ converges to $k(f_s(\phi))$ uniformly on compact subsets of $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$. These facts, together with (92)-(95), ensure that we can pass to the limit in (96) and get

$$(v(t), \eta(t)) - \int_0^t (v, \eta_t) ds + \nu \int_0^t (\nabla v, \nabla \eta) ds + \nu_o \int_0^t (\chi, \eta) ds \\ + \int_0^t (v \cdot \nabla v, \eta) ds + \int_0^t (k(f_s(\phi))v, \eta) ds = \int_0^t (\mathcal{F}(c, \theta), \eta) ds + (v_0, \eta(0)), \tag{97}$$

Since $v^\delta(0) \to v(0)$ in $(V^p(U))'$, for any U such that $\bar{U} \subseteq \Omega_{ml}(0)$, by using (97) it is easy to see that $v(0) = v_0$ in $\Omega_{ml}(0)$.

Now, we check that $v = 0$ a.e. in $\mathring{Q}_s$. For this, take a compact set $K \subseteq \mathring{Q}_s$. Then there is an $\delta_K \in (0, \delta(\Omega))$ such that

$$f_s^\delta(\phi^\delta(x,t)) = 1 \quad \text{in } K \text{ for } \delta < \delta_K.$$

Hence, $k(f_s^\delta(\phi^\delta(x,t) - \delta) = k(1-\delta)$ in K for $\delta < \delta_K$. From (84) we infer that

$$k(1-\delta)\|v^\delta\|^2_{L^2(K)} \le C_1 \quad \text{for } \delta < \delta_K,$$

where C_1 is independent of δ. Thus, as δ tends to 0, by assumption **(H2)**, $k(1-\delta)$ blows up and, consequently, $\|v^\delta\|_{L^2(K)}$ converges to 0. Therefore $v = 0$ a.e. in K, and since K is an arbitrary compact subset, we conclude that

$$v = 0 \quad \text{a.e. in } \mathring{Q}_s$$

Now, we proceed with the other equations.

It follows from (93)-(95) that we may pass to the limit in (34), and find that (12) holds almost everywhere.

In order to pass to the limit in (35), we note that given $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(0,T;L^2(\Omega))$ satisfying $\zeta(T) = 0$, we can consider an extension of ζ such that $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ satisfying $\zeta^\delta(T) = 0$. Now, we take the scalar product of (35) with ζ^δ,

$$\begin{aligned} -C_{\mathrm{v}} \int_{\Omega^\delta} \theta_0^\delta \zeta^\delta(0)dx - C_{\mathrm{v}} \int_0^T \int_{\Omega^\delta} \theta^\delta \zeta_t^\delta dxdt \quad &- \quad C_{\mathrm{v}} \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta)\theta^\delta \cdot \nabla \zeta^\delta dxdt \\ + \int_0^T \int_{\Omega^\delta} K_1(\rho_\delta(\phi^\delta)) \nabla \theta^\delta \cdot \nabla \zeta^\delta dxdt \quad &= \quad \frac{l}{2} \int_0^T \int_{\Omega^\delta} f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta \zeta^\delta dxdt. (98) \end{aligned}$$

Observe that since $\rho_\delta(v^\delta)$ converges weakly to v in $L^2(0,T;H^1(\Omega))$ and $\theta^\delta \to \theta$ strongly in $C([0,T];H_o^1(\Omega)')$ we have that $\rho_\delta(v^\delta)\theta^\delta$ converges to $v\theta$ in $\mathcal{D}'(Q)$. Observe that $f_s^{\delta'} \to f_s'$ in $L^q(I\!R)$ for $2 \leq q < \infty$, then from (95) we infer that $f_s^{\delta\prime}(\phi^\delta)\phi_t^\delta$ converges weakly to $f_s'(\phi)\phi_t$ in $L^{q/2}(Q)$. Moreover, from Lemma 1 the integrals over $\Omega^\delta \backslash \Omega$ are bounded independent of δ and since $|\Omega^\delta \backslash \Omega| \to 0$ as $\delta \to 0$, we have that these integrals tend to zero as $\delta \to 0$. Therefore, we may pass to the limit in (98) and obtain

$$\begin{aligned} -C_{\mathrm{v}} \int_0^T \int_\Omega \theta \zeta_t dxdt - C_{\mathrm{v}} \int_0^T \int_\Omega v\theta \cdot \nabla \zeta dxdt + \int_0^T \int_\Omega K_1(\phi) \nabla \theta \cdot \nabla \zeta \, dxdt \\ = \frac{l}{2} \int_0^T \int_\Omega f_s'(\phi)\phi_t \zeta \, dxdt + C_{\mathrm{v}} \int_\Omega \theta_0 \zeta(0) dx \end{aligned}$$

for all $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(0,T;L^2(\Omega))$ and $\zeta(T) = 0$.

It remains to pass to the limit in (36). We proceed in similar ways as before, taking the scalar product of it with $\zeta^\delta \in L^2(0,T;H^1(\Omega^\delta))$ with $\zeta_t^\delta \in L^2(0,T;L^2(\Omega^\delta))$ and $\zeta^\delta(T) = 0$,

$$\begin{aligned} -\int_0^T \int_{\Omega^\delta} c^\delta \zeta_t^\delta dxdt - \int_0^T \int_{\Omega^\delta} \rho_\delta(v^\delta) c^\delta \cdot \nabla \zeta^\delta dxdt + K_2 \int_0^T \int_{\Omega^\delta} \nabla c^\delta \cdot \nabla \zeta^\delta dxdt \\ + K_2 M \int_0^T \int_{\Omega^\delta} c^\delta(1 - c^\delta) \nabla \rho_\delta(\phi^\delta) \cdot \nabla \zeta^\delta dxdt = \int_{\Omega^\delta} c_0^\delta \zeta^\delta(0) dx, \end{aligned}$$

then from (93),(95) and using that the sequence (c^δ) is bounded in $L^\infty(Q)$ we may pass to the limit as $\delta \to 0$ and obtain

$$\begin{aligned} -\int_0^T \int_\Omega c\zeta_t dxdt - \int_0^T \int_\Omega vc \cdot \nabla \zeta dxdt + K_2 \int_0^T \int_\Omega \nabla c \cdot \nabla \zeta \, dxdt \\ + K_2 M \int_0^T \int_\Omega c(1-c) \nabla \phi \cdot \nabla \zeta \, dxdt = \int_\Omega c_0 \zeta(0) dx \end{aligned}$$

holds for any $\zeta \in L^2(0,T;H^1(\Omega))$ with $\zeta_t \in L^2(0,T;L^2(\Omega))$ and $\zeta(T)=0$. Observe that since $0 < c^\delta < 1$ and c^δ converges to c in $L^2(Q)$ we have that $0 \le c \le 1$ a.e. in Q.

Now, it follows from (95) that $\dfrac{\partial \phi}{\partial n} = 0$, $\phi(0) = \phi_0$, $\theta(0) = \theta_0$ and $c(0) = c_0$, and the first part of the proof of Theorem 1 is complete.

Under the additional regularity and integrability hypotheses stated in the second part of the statement of Theorem 1, in the following we will show that $\chi = Av$. We will use the monotonicity and the hemicontinuity of operator A ([9], Chap. 2.) by adapting an argument that is usual in the theory of monotone operators. For this, we take any $\psi \in L^p(0,T;V^p)$ such that $\operatorname{supp}\psi$ is contained in the closure of $\Omega_{ml}(0) \cup Q_m \cup \Omega_{ml}(T)$ and define

$$\begin{aligned} X_\delta^t &= \nu_o \int_0^t \left(Av^\delta - A\psi, v^\delta - \psi \right) ds + \frac{1}{2}\|v^\delta(t)\|^2_{L^2(\Omega)} \\ &\quad + \nu \int_0^t \|\nabla v^\delta\|^2_{L^2(\Omega)} ds + \int_0^t \int_\Omega k(f_s^\delta(\phi^\delta) - \delta)|v^\delta|^2 dx ds. \end{aligned} \tag{99}$$

Since A is monotone and $\Omega_{ml}(t) \subseteq \Omega$,

$$\begin{aligned} X_\delta^t \geq\ & \frac{1}{2}\|v^\delta(t)\|^2_{L^2(\Omega_{ml}(t))} + \nu \int_0^t \|\nabla v^\delta\|^2_{L^2(\Omega_{ml}(s))} ds \\ & + \int_0^t \|k^{1/2}(f_s^\delta(\phi^\delta) - \delta)|v^\delta|\|^2_{L^2(\Omega_{ml}(s))} ds. \end{aligned} \tag{100}$$

Observe that $v^\delta(t) \rightharpoonup v(t)$ weakly in H, $v^\delta \rightharpoonup v$ weakly in $L^p(Q)$; thus, thanks to (92), $v^\delta \to v$ a.e. in Q_{ml}. Note also that $k^{1/2}(f_s^\delta(\phi^\delta) - \delta) \to k^{1/2}(f_s(\phi))$ a.e. in Q_{ml}; hence

$$k^{1/2}(f_s^\delta(\phi^\delta) - \delta)v^\delta \to k^{1/2}(f_s(\phi))v \quad \text{ a.e. in } Q_{ml}$$

From (84) we have that $\int_0^t \|k^{1/2}(f_s^\delta(\phi^\delta) - \delta)|v^\delta|\|^2_{L^2(\Omega_{ml}(s))} ds$ is bounded. Therefore ([8] Lema1.3),

$$k^{1/2}(f_s^\delta(\phi^\delta) - \delta)v^\delta \rightharpoonup k^{1/2}(f_s(\phi))v \quad \text{weakly in } L^2(Q_{ml}).$$

Thus, we conclude from (100) that

$$\begin{aligned} \liminf_{\delta \to 0} X_\delta^t \geq\ & \frac{1}{2}\|v(t)\|^2_{L^2(\Omega_{ml}(t))} + \nu \int_0^t \|\nabla v\|^2_{L^2(\Omega_{ml}(s))} ds \\ & + \int_0^t \|k^{1/2}(f_s(\phi))|v|\|^2_{L^2(\Omega_{ml}(s))} ds. \end{aligned} \tag{101}$$

On the other hand, by using (33) with $u = v^\delta$, after integrating in $[0, t]$, we obtain an expression for $\nu_0 \int_0^t (Av^\delta, v^\delta) ds$ that substituted into (99), gives

$$X_\delta^t = \frac{1}{2}\|v_0^\delta\|^2_{L^2(\Omega)} + \int_0^t (\mathcal{F}(c^\delta, \theta^\delta), v^\delta) ds - \nu_o \int_0^t (Av^\delta, \psi) ds - \nu_o \int_0^t (A\psi, v^\delta - \psi) ds.$$

By letting $\delta \to 0$ in this last expression, we conclude that

$$X_\delta^t \to X^t = \frac{1}{2}\|v_0\|^2_{L^2(\Omega)} + \int_0^t (\mathcal{F}(c, \theta), v) ds - \nu_o \int_0^t (\chi, \psi) ds - \nu_o \int_0^t (A\psi, v - \psi) ds$$

Now, from the fact that $v = 0$ a.e. in $\mathring{Q}_s$ and our additional hypothesis that the measure of $\partial\Omega_{ml}$ is zero for a.e. $t \in (0, t)$, we can write X^t as

$$\begin{aligned} X^t = \; & \frac{1}{2}\|v_0\|^2_{L^2(\Omega_{ml}(0))} + \int_0^t (\mathcal{F}(c, \theta), v)_{\Omega_{ml}(s)} ds - \nu_o \int_0^t (\chi, \psi) ds \\ & - \nu_o \int_0^t (A\psi, v - \psi) ds. \end{aligned}$$

This and (101) imply that

$$\begin{aligned} & \frac{1}{2}\|v_0\|^2_{L^2(\Omega_{ml}(0))} + \int_0^t (\mathcal{F}(c, \theta), v)_{\Omega_{ml}(s)} ds - \nu_o \int_0^t (\chi, \psi) ds - \nu_o \int_0^t (A\psi, v - \psi) ds \\ & \geq \frac{1}{2}\|v(t)\|^2_{L^2(\Omega_{ml}(t))} + \nu \int_0^t \|\nabla v\|^2_{L^2(\Omega_{ml}(s))} ds + \int_0^t \|k^{1/2}(f_s(\phi))|v|\|^2_{L^2(\Omega_{ml}(s))} ds. \end{aligned}$$

Now, we recall that (97) holds for a.e. $t \in (0, T)$ and any $\eta \in L^p(0, T; V^p)$ with compact support contained in $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and such that $\eta_t \in L^{p'}(0, T; (V^p)')$. Thus, our previous estimates and our additional hypothesis on the integrability of $k(f_s(\phi))$ allow us to use density arguments to conclude that (97) holds for any $\eta \in L^p(0, T; V^p)$ with support contained in the closure of $Q_{ml} \cup \Omega_{ml}(0) \cup \Omega_{ml}(T)$ and such that $\eta_t \in L^{p'}(0, T; (V^p)')$. In particular, v has this properties, and we can take $\eta = v$ in (97) and integrate in time on the interval $[0, t]$ to find an energy identity that used with the last inequality furnishes

$$\nu_o \int_0^t (\chi - A\psi, v - \psi) ds \geq 0 \quad \text{a.e. } t.$$

Therefore, by standard arguments using the hemicontinuity of operator A ([8] Chp.2), we can conclude that $\chi = Av$, and the proof of Theorem 1 is complete. ∎

References

[1] Blanc, Ph., Gasser, L., Rappaz,J., 'Existence for a stationary model of binary alloy solidification', *Math. Mod. and Num. Anal.* **29**(6),687-699 (1995).

[2] Caginalp, G. and Xie, W., 'Phase-field and sharp-interfase alloys models', *Phys. Rev. E*, **48**(3), 1897-1909 (1993).

[3] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[4] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math, Vol 840, Springer-Verlag, 1981.

[5] Hoffman, K-H. and Jiang, L., 'Optimal control of a phase field model for solidification', *Numer. Funct. Anal. and Optim.*, **13**, 11-27 (1992).

[6] Ladyzenskaja, O.A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers Inc, 1969.

[7] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[8] Lions, J.L., *Quelques méthodes de resolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, 1969.

[9] Lions, J.L., *Control of Distributed Singular Systems*, Gauthier-Villars, 1985.

[10] Mikhailov, V.P. *Partial Differential Equations*, Mir, 1978.

[11] Planas, G., Boldrini, J.L., 'Weak solutions to a phase field model with convection for solidification of an alloy', Preprint (2001).

[12] Planas, G., Boldrini, J.L., 'A Bidimensional Phase-Field Model with Convection for Change Phase of an Alloy ', Preprint (2001).

[13] Simon, J., 'Compacts sets in the space $L^p(0,T,B)$', *Ann. Mat. Pura Appl.*, **146**, 65-96 (1987).

[14] Temam, R., *Navier-Stokes Equations*, AMS Chelsea Publishing, 2001.

[15] Voller,V.R., Cross,M., Markatos,N.C., 'An enthalpy method for convection/diffusion phase change', *Int. J. for Numer. Meth. in Eng.* **24**, 271-284 (1987).

Capítulo 6

Conclusões Gerais

Neste trabalho apresentamos resultados de existência de soluções para alguns modelos matemáticos do tipo campo de fase que tratam de problemas de solidificação de ligas binárias.

Observemos que no primeiro modelo (que não inclui convecção) a condutividade térmica, que depende do campo de fase, podia se anular. Ao introduzir os termos convectivos tivemos que fortalecer essa hipótese para obter um resultado de exitência global no tempo. Em particular, no segundo modelo, a dificuldade técnica surge quando precisamos obter mais regularidade do campo de fase para que as regiões estejam bem definidas. Tal regularidade depende do termo fonte da equação, que por sua vez depende da regularidade da temperatura e da concentração. Como a concentração pertence a $L^\infty(Q)$, a temperatura é a que controla a regularidade do campo de fase. A informação que se consegue obter sobre a temperatura ($L^\infty(0, T; L^2(\Omega))$) não é suficiente para garantir a continuidade do campo de fase. Torna-se então necessário aumentar tal regularidade para $L^p(Q)$ com $p > N + 2/2$. Uma alternativa possível seria a de tentar melhorar a regularidade da temperatura, tal vez apenas num intervalo de tempo pequeno, obtendo assim uma solução local. Esta análise está na fase inicial. Também poderia ser analizado a possibilidade de existência de soluções locais nos outros modelos estudados.

Uma outra proposta interessante seria a de fazer a análise matemática de um modelo de solidificação para ligas que usa a metodologia de campo de fase, inclui convecção, diferente dos modelos estudados e que foi proposto por Beckermann et al. [3] e Diepers [15]. Eles propoêm uma equação para a velocidade válida no dominio todo; o fato da velocidade ter que ser zero na região sólida está embutido na propria equação.

Bibliografia

[1] V. Alexiades and A.D. Solomon, *Mathematical Modeling of Melting and Freezing Processes*, Hemisphere Publishing Corporation, Washington, 1993.

[2] Blanc, Ph., Gasser, L., Rappaz,J., 'Existence for a stationary model of binary alloy solidification', *Math. Mod. and Num. Anal.* **29**(6),687-699 (1995).

[3] Beckermann, C., Diepers, H.-J., Steinbach, I., Karma, A., Tong, X., 'Modeling melt convection in phase-field simulations of solidification', *Journal of Computational Physics* **154**, 468-496 (1999).

[4] Boldrini, J.L., Planas, G., 'Weak solutions to a phase field model for an alloy with thermal properties', to be published in *Mathematical Methods in Applied Science*.

[5] Caginalp, G., 'An analysis of a phase field model of a free boundary', *Arch. Rational Mech. Anal.*, **92**, 205-245 (1986).

[6] Caginalp, G. and Jones, J., 'A derivation and analysis of phase-field models of thermal alloys', *Annals of Phy.*, **237**, 66-107 (1995).

[7] Caginalp, G. and Socolovsky, E.A., 'Computation of sharp phase boundaries by spreading the planar ans spherically symmtric cases', *J. Comp. Phys.*, **95**(1), 85-100 (1991).

[8] Caginalp, G. and Xie, W., 'Phase-field and sharp-interfase alloys models', *Phys. Rev. E*, **48**(3), 1897-1909 (1993).

[9] Cannon, J.R., DiBenedetto,E., Knightly, G.H., 'The steady state Stefan Problem with convection', *Arch. Rat. Mech. Anal.* **73**, 79-97 (1980).

[10] Cannon, J.R., DiBenedetto,E., Knightly,G.H.,'The bidimensional Stefan Problem with convection: the time dependent case', *Comm. in Part. Diff. Eq.* **14**, 1549-1604 (1983).

[11] Colli, P., Laurençot, Ph., 'Weak solution to the Penrose-Fife phase field model for a class of admissible heat flux laws', *Physica D* **111**, 311-334 (1998).

[12] Colli, P., Sprekels, J.,'Weak solution to some Penrose-Fife phase-field systems with temperature-dependent memory', *J. Diff. Eq.* **142**(1), 54-77 (1998).

[13] DiBenedetto, E., Friedman, A. 'Conduction-Convection Problems with Change of Phase', *J. of Diff. Eq.* **62**, 129-185 (1986).

[14] DiBenedetto, E., O'Leary, M. 'Three-Dimensional Conduction-Convection Problems with Change of Phase', *Arch. Rat. Mech. Anal.* **123**, 99-116 (1993).

[15] Diepers, H.-J., Beckermann, C., Steinbach, I., 'Simulation of convection and ripening in a binary alloy mush using the the phase-field method', *Acta mater.* **47** (13), 3663-3678 (1999).

[16] Friedman, A., *Partial Differential Equations of Parabolic Type*, Prentice-Hall, 1964.

[17] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Math, Vol 840, Springer-Verlag, 1981.

[18] Hoffman, K-H. and Jiang, L., 'Optimal control of a phase field model for solidification', *Numer. Funct. Anal. and Optim.*, **13**, 11-27 (1992).

[19] Kavian, O., *Introduction à la Théorie des Points Critiques*, Mathématiques et Applications 13, Springer-Verlag, 1993.

[20] Ladyzenskaja, O.A., Solonnikov, V.A. and Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, 1968.

[21] Ladyzenskaja, O.A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers Inc, 1969.

[22] J.S. Langer, *Models of pattern formation in first-order phase transitions*, in G. Grinstein and G. Mazenko (Eds.), Directions in Condensed Mather. Physics, World Scientific, Philadelphia, 1986.

[23] Laurençot, Ph., 'Weak solutions to a phase-field model with non-constant thermal conductivity', *Quart. Appl. Math.*, **15**(4), 739-760 (1997).

[24] Lions, J.L., *Quelques méthodes de resolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, 1969.

[25] Lions, J.L., *Control of Distributed Singular Systems*, Gauther-Villars, 1985.

[26] Marcus, M. and Mizel, V.J., 'Complete Characterization of Functions which act, via vuperposition, on Sobolev Spaces', *Trans. American Math. Soc.*, **251**(July), 187-218 (1979).

[27] Marcus, M. and Mizel, V.J., 'Every Superposition Operator Mapping One Sobolev Space into Another Is Continuous', *J. Func. Anal*, **33**(2), 217-229 (1979).

[28] Mikhailov, V.P., *Partial Differential Equations*, Mir, 1978.

[29] Moroşanu, C. and Motreanu, D., 'A generalized phase-field system', *J. Math. Anal. Appl.*, **237**, 515-540 (1999).

[30] O'Leary, M., 'Analysis of the mushy region in conduction-convection problems with change of phase' , *Elect. J. Diff. Eqs.* **1997**(4), 1-14 (1997).

[31] Penrose,O., Fife, P.C., 'Thermodynamically consistent model of phase-field type for the kinetics of phase transitions', *Physica D* **43**, 44-62 (1990).

[32] Planas, G., Boldrini, J.L., 'Weak solutions to a phase field model with convection for solidification of an alloy', Preprint (2001).

[33] Planas, G., Boldrini, J.L., 'A Bidimensional Phase-Field Model with Convection for Change Phase of an Alloy', Preprint (2001).

[34] Rappaz, J. and Scheid, J.F., 'Existence of solutions to a Phase-field model for the isothermal solidification process of a binary alloy', *Math. Meth. Appl. Sci.*, **23**, 491-512 (2000).

[35] L. Rubinstein, *The Stefan Problem*, AMS. Transl, 27, Amer. Math. Society, Providence, 1971.

[36] Simon, J., 'Compacts sets in the space $L^p(0,T,B)$', *Ann. Mat. Pura Appl.*, **146**, 65-96 (1987).

[37] Soto Segura, H.P., 'Análise de um Modelo Matemático de Condução-Conveção do Tipo Entalpia para Solidificação', Ph.D. Thesis, IMECC-UNICAMP, Brazil (2000).

[38] Temam, R., *Navier-Stokes Equations*, AMS Chelsea Publishing, 2001.

[39] Voller,V.R., Cross,M., Markatos,N.C., 'An enthalpy method for convection/diffusion phase change', *Int. J. for Numer. Meth. in Eng.* **24**, 271-284 (1987).

[40] Voller, V.R., Prakash, C., 'A fixed grid numerical modeling methodology for convection-diffusion mushy region phase-change problems', *Int. J. Heat Mass Trans.*, **30** (8), 1709-1719 (1987).

[41] Voss, M.J., Tsai, H.L., 'Effects of the rate of latente heat release on fluid flow and solidification patterns during alloy solidification', *Int. J. Engng. Sci.*, **34** (6), 715-737 (1996).

[42] Warren, J.A. and Boettinger, W.J., 'Prediction of dendritic growth and microsegregation patterns in a binary alloy using the phase-field method', *Acta Metall. Mater.*, **43**(2), 689-703 (1995).

[43] Wheeler, A.A., Boettinger, W.J. and McFadden, G.B., 'Phase-field model for isothermal phase transitions in binary alloys', *Phys. Rev. A*, **45**, 7424-7439 (1992).