BRIAN CALLANDER

Lefschetz Fibrations

Fibrações de Lefschetz

CAMPINAS
2013

UNIVERSIDADE ESTADUAL DE CAMPINAS

INSTITUTO DE MATEMÁTICA, ESTATÍSTICA
E COMPUTAÇÃO CIENTÍFICA

BRIAN CALLANDER

Lefschetz Fibrations

Orientadora: Profa. Dra. Elizabeth Terezinha Gasparim

Fibrações de Lefschetz

Dissertação de Mestrado apresentada ao Instituto de
Matemática, Estatística e Computação Científica da Unicamp para obtenção
do título de Mestre em Matemática

Master's Dissertation presented to the Institute of
Mathematics, Statistics and Scientific Computing of Unicamp to obtain the
title of Master in Mathematics

ESTE EXEMPLAR CORRESPONDE À VERSÃO FINAL DA DISSERTAÇÃO
DEFENDIDA PELO ALUNO BRIAN CALLANDER,
E ORIENTADA PELA PROFA. DRA ELIZABETH TEREZINHA GASPARIM

Assinatura do Orientador

Elizabeth T. Gasparim

CAMPINAS
2013

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

C131L

Callander, Brian, 1986-
Lefschetz fibrations / Brian Callander. – Campinas, SP : [s.n.], 2013.

Orientador: Elizabeth Terezinha Gasparim.
Dissertação (mestrado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Leftschetz, Fibrações de. 2. Singularidades (Matemática). 3. Variedades simpléticas. 4. Floer, Homologia de. 5. Hodge, Teoria de. I. Gasparim, Elizabeth Terezinha,1963-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em outro idioma: Fibrações de Lefschetz
Palavras-chave em inglês:
Lefschetz fibrations
Singularities (Mathematics)
Symplectic manifolds
Floer homology
Hodge theory
Área de concentração: Matemática
Titulação: Mestre em Matemática
Banca examinadora:
Elizabeth Terezinha Gasparim [Orientador]
Paolo Piccione
Luiz Antonio Barrera San Martin
Data de defesa: 20-06-2013
Programa de Pós-Graduação: Matemática

Dissertação de Mestrado defendida em 20 de junho de 2013 e aprovada

Pela Banca Examinadora composta pelos Profs. Drs.

Elizabeth T. Gasparim

Prof.(a). Dr(a). ELIZABETH TEREZINHA GASPARIM

Paolo Piccione

Prof.(a). Dr(a). PAOLO PICCIONE

Luiz A. B. San Martin

Prof.(a). Dr(a). LUIZ ANTONIO BARRERA SAN MARTIN

Agradecimentos

Eu gostaria de expressar minha profunda gratidão a minha orientadora Dr. Elizabeth Gasparim, que está sempre disposta a compartilhar seu imenso conhecimento e criatividade com os outros. Em particular, eu me sinto tanto matematicamente como pessoalmente enriquecido pelas inúmeras horas de discussão matemática, conselhos, e apoio que ela deu, que não só fez esta dissertação possível, mas também me apresentou a cultura matemática mais ampla além da universidade própria.

Eu também gostaria de agradecer a CAPES e CNPq, cujo apoio financeiro tornou possível para me fazer meu mestrado na UNICAMP.

Acknowledgements

I would like to express my deep gratitude towards my supervisor Dr. Elizabeth Gasparim, who is always willing to share her immense knowledge and creativity with others. In particular, I feel both mathematically and personally enriched by the countless hours of mathematical discussion, advice, and support she has generously given, which have not only made this dissertation possible but have also introduced me to the wider mathematical culture beyond the university itself.

I would also like to thank CAPES and CNPq, whose financial support has made it possible for me to pursue my master's degree at UNICAMP.

RESUMO. O propósito desta tese é estudar fibrações de Lefschetz simpléticas, nas quais os ciclos evanescentes são subvariedades Lagrangianas das fibras. Para a descrição da teoria de interseção dos ciclos evanescentes utilizamos cohomologia de Floer Lagrangiana, cujo conceito revemos nesta tese. Apresentamos três exemplos principais e de caráteres distintos:

(1) twists de Dehn generalizados,
(2) o "espelho" da reta projetiva, e
(3) uma fibração numa órbita adjunta de $\mathfrak{sl}(3,\mathbb{C})$.

O terceiro destes exemplos é original e utiliza um teorema recente de Gasparim-Grama-San Martin.

ABSTRACT. The objective of this thesis is to study symplectic Lefschetz fibrations, in which the vanishing cycles are Lagrangian submanifolds of the fibres. In order to describe the intersection theory of vanishing cycles we use Lagrangian intersection Floer cohomology, which we review. We present three main examples of distinct characters:

(1) generalised Dehn twists,
(2) the "mirror" of the projective line, and
(3) a fibration on an adjoint orbit of $\mathfrak{sl}(3,\mathbb{C})$.

The third of these examples is original and uses a recent theorem of Gasparim-Grama-San Martin.

Contents

Agradecimentos vi
Acknowledgements vi
List of Figures ix
1. Introduction 1
2. Symplectic Manifolds 2
3. Topological Lefschetz Fibrations 5
4. Symplectic Lefschetz Fibrations 11
5. Generalised Dehn Twists 12
6. Vanishing Cycles 14
7. Floer Cohomology 15
8. Examples of Lefschetz Fibrations 20
Appendix A. Computations 23
A.1. Mathematica 24
A.2. Macaulay 2 26
References 30

List of Figures

1 Dehn Twists and deformations 18
2 Thimbles for the mirror of $\mathbb{P}^1$ 23

1. Introduction

The objective of this thesis is to study symplectic Lefschetz fibrations, in which the vanishing cycles are Lagrangian submanifolds of the fibres. In order to describe the intersection theory of vanishing cycles we use Lagrangian intersection Floer cohomology, which we review. We present three main examples of distinct characters:

(1) generalised Dehn twists,
(2) the "mirror" of the projective line, and
(3) a fibration on an adjoint orbit of $\mathfrak{sl}(3,\mathbb{C})$.

The third of these examples is original and uses a recent theorem of Gasparim-Grama-San Martin.

One method of creating a Lefschetz fibration is by blowing up a Lefschetz pencil at its base locus. Lefschetz pencils have become of great interest due to a theorem of Donaldson that every manifold of dimension 4 that admits an (integral) symplectic form also admits a Lefschetz pencil. A theorem of Gompf shows the converse: that any 4 dimensional manifold admitting a Lefschetz pencil also admits a symplectic form. This is one example of a link between algebraic geometry and symplectic geometry. In fact, a much deeper connection between these two areas is predicted by Kontsevich's homological mirror symmetry conjecture as an equivalence of the category of derived categories of coherent sheaves of one manifold with the Fukaya category of Lagrangian submanifolds of its "mirror". One context for studying the Fukaya category is via the category of vanishing cycles of Lefschetz fibrations.

Section 2 gives an introduction to the basic results of symplectic geometry we require. In Sections 3 and 4 we introduce topological and symplectic Lefschetz fibrations, proving in particular that there do not exist any Lefschetz fibrations with total space $\mathbb{CP}^n$ (with base space $\mathbb{CP}^1$). We also present a number of examples of Lefschetz pencils and illustrate how to construct Lefschetz fibrations from them. In Section 5 we present the first of our three main examples: that of the generalised Dehn twist. Section 6 gives an account of Picard-Lefschetz theory and shows how Lefschetz fibrations are in some sense the complex analogue of Morse functions; indeed, their behaviour near the singularities determines the topology of the total space and regular fibres. This Morse-like flavour is also reflected in the description of Floer homology we give in Section 7. Indeed, Floer homology is considered the infinite dimensional analogue of Morse homology. In Section 8 we finish with our two remaining main examples: the first is original and uses a recent result by Gasparim-Grama-San Martin; the second is the "mirror" of the projective line.

2. Symplectic Manifolds

2.1. **Theorem.** *Let Ω be an antisymmetric form on a vector space V. Then there exists a basis $\mathscr{B} = \{u_1,\dots,u_k,e_1,\dots,e_n,f_1,\dots,f_n\}$ with respect to which Ω has the matrix*

$$\begin{pmatrix} 0_k & 0 & 0 \\ 0 & 0 & \mathrm{id}_n \\ 0 & -\mathrm{id}_n & 0 \end{pmatrix} .= \sum_{i=1}^{n} e^i \wedge f^i, \qquad (1)$$

where $\{e^1,\dots,e^n,f^1,\dots,f^n\}$ is the dual basis to $\mathscr{B}$. We shall call $\mathscr{B}$ the standard basis *of an antisymmetric form.*

2.2. **Corollary.** *If V admits a non-degenerate antisymmetric form, then V has even dimension.*

2.3. Definition. We say that an antisymmetric form Ω on V is a *symplectic form* if the map $\Omega\colon V \to V^*, v \mapsto \Omega(v,-)$ is bijective. Any vector space which admits a symplectic form is called a *symplectic space*. We denote such a space by (V,Ω).

A linear isomorphism $\varphi\colon (V,\Omega) \to (V',\Omega')$ between symplectic spaces is *symplectomorphism* if it preserves the symplectic form, i.e. $\varphi^*\Omega' = \Omega$.

2.4. *Remark.* Since the matrix of a symplectic form must be non-singular, it is easily seen from Theorem 2.1 that any symplectic space is even dimensional.

2.5. Definition. A subspace L of a symplectic space V is called *Lagrangian* if $\Omega_{|L} = 0$ and $\dim L = \frac{1}{2}\dim V$.

2.6. Definition. A *symplectic form* ω on a manifold M is a closed 2-form which is symplectic on every tangent space. A *symplectic manifold* is a smooth manifold M with a symplectic form ω. We write (M,ω) to denote such a manifold.

2.7. *Example.* The simplest example of a symplectic manifold is $(\mathbb{R}^{2n},\omega_0)$, where

$$\omega_0 := \sum_1^n dx_i \wedge dy_i$$

for coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$.

2.8. **Theorem.** *Any complex (Riemann) manifold (M,J,g) admits the symplectic structure defined by $\omega(X,Y) = g(X,JY)$.*

2.9. *Example.* Continuing Example 2.7, we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and pullback the symplectic structure to make $\mathbb{C}^n$ symplectic too. Indeed, in complex coordinates we get the following expression for the symplectic form:

$$\omega_0 = \frac{i}{2}\sum_1^n dz_i \wedge d\overline{z}_i.$$

Moreover, ω_0 is compatible with the usual complex structure J and hermitian metric g in the sense that $\omega_0(X,Y) = g(X,JY)$, making $\mathbb{C}^n$ a Kähler manifold.

2.10. *Example.* Complex projective space $\mathbb{P}^n := \mathbb{CP}^n$ also admits a symplectic structure. With the usual complex structure on $\mathbb{P}^n$, we can use the Fubini-Study metric g to define a symplectic form via $\omega(X,Y) := g(X,JY)$.

The Fubini-Study metric can be defined via the $\mathbb{S}^1$-bundle

$$p\colon\ \mathbb{S}^{2n+1} \to \mathbb{P}^n$$
$$z \mapsto [z]$$

by identifying the horizontal tangent space $\ker p_*$ with the tangent space of $\mathbb{P}^n$. This allows the restriction of the round metric $g_\circ$ to the horizontal tangent space to descend to a metric g on $\mathbb{P}^n$, i.e. $g_\circ(X,Y) = g(p_*X, p_*Y)$ for any horizontal vectors X,Y.

To show that this defines a bona fide metric, we must show that the $\mathbb{S}^1$-action

(1) acts on $\mathbb{S}^{2n+1}$ by isometries,
(2) satisfies $p \circ g = p$ for all $g \in \mathbb{S}^1$, and
(3) acts transitively on each fibre.

Item 1 is clear since $g_\circ(\lambda X, \lambda Y) = \lambda\overline{\lambda} g_\circ(X,Y) = g_\circ(X,Y)$ for $\lambda \in \mathbb{S}^1 \subset \mathbb{C}^1$. The defining equivalence relation on $\mathbb{C}^{2n+1}$ guarantees that Item 2 is satisfied. The fibre over $[z_0,\ldots,z_n]$ is given by $\{\lambda(z_0,\ldots,z_n) \mid \lambda \in \mathbb{C}^1\} = \{\lambda(z_0,\ldots,z_n) \mid \lambda \in \mathbb{S}^1\}$, the orbit of $(z_0,\ldots,z_n) \in \mathbb{S}^1$, showing that Item 3 is indeed satisfied.

2.11. **Lemma.** *Let M^{2m} be a manifold with a closed 2-form ω. Then ω is a symplectic form if and only if ω^m is a volume form on M.*

Proof. By Theorem 2.1, a 2-form ω can be written locally as

$$\omega = \sum_1^n e^i \wedge f^i,$$

for some $n \leq m$. If ω is non-degenerate, then $n = m$ and the m-th exterior product is given by

$$\omega^{\wedge m} = m! \bigwedge_{i=1}^m e^i \wedge f^i \neq 0$$

where we used the fact that $(e^i \wedge f^i) \wedge (e^j \wedge f^j) = (e^j \wedge f^j) \wedge (e^i \wedge f^i)$. Therefore, $\omega^{\wedge m}$ is a volume form.

Conversely, if ω is degenerate, then $n < m$ and the m-th exterior product is zero and thus $\omega^{\wedge m}$ is not a volume form. $\square$

2.12. *Remark.* Since the existence of a volume form is equivalent to having an orientation, we now know that any non-orientable manifold, such as the Möbius strip, does not admit a symplectic structure.

2.13. DEFINITION. We say that a smooth map $\varphi\colon (M,\omega) \to (M',\omega')$ of manifolds is a *symplectomorphism* if it preserves the symplectic form, i.e. $\varphi^*\omega' = \omega$.

2.14. **Theorem** (Darboux)**.** *Let (M,ω) be a symplectic manifold of dimension $2n$ and $p \in M$. Then there exists a local coordinate system $(U, x_1, \dots, x_n, y_1, \dots, y_n)$ such that*

$$\omega = \sum_1^n dx_i \wedge dy_i$$

on U. Such a coordinate system is called a Darboux system.

2.15. **Proposition.** *The sphere $\mathbb{S}^n$ admits a symplectic structure only for $n =$ 2.*

Proof. We can immediately exclude all odd dimensional spheres by Remark 2.4.

Let $n = 2k$ for $k > 1$ and suppose for a contradiction that ω is a symplectic form on $\mathbb{S}^{2k}$. If ω^k were exact, say $d\alpha = \omega^k$, then Stokes' theorem would give

$$0 \neq \int_{\mathbb{S}^k} \omega^k = \int_{\partial \mathbb{S}^k} \alpha = 0,$$

a contradiction. Thus, ω^k is not exact and defines a non-zero element $0 \neq [\omega^k] \in H^n(\mathbb{S}^{2k}, \mathbb{R})$. This implies that $0 \neq [\omega] \in H^2(\mathbb{S}^{2k}, \mathbb{R})$, which can be seen by considering the cohomological product or by direct calculation: $\omega^k = (d\alpha)^k = d(\alpha \wedge (d\alpha)^{n-1})$. But

$$H^2_{dR}\left(\mathbb{S}^{2k}; \mathbb{R}\right) = \begin{cases} \mathbb{R} & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases},$$

which is a contradiction if $k > 1$.

The symplectic structure on $\mathbb{S}^2$ is given by pulling back the symplectic structure of $\mathbb{P}^1$ via a smooth identification of $\mathbb{S}^2$ with $\mathbb{P}^1$. □

2.16. **Theorem.** *Given any manifold X, the associated cotangent manifold T^*X is a symplectic manifold.*

Proof. Let $p = (x, \xi) \in T^*X$ and define the *tautological* 1*-form* $\alpha_p := (\pi^*\xi)_p$, where $\pi \colon T^*X \to X$ is the natural projection. Now define the *canonical symplectic* 2*-form* $\omega := -d\alpha$.

To see that this does indeed define a symplectic form, we write it in a local system of coordinates $\left(\pi^{-1}(U), x_1, \dots, x_n, \xi_1, \dots, \xi_n\right)$. First, $\alpha_p\left(\frac{\partial}{\partial x_i}\right) = \xi_x\left(\frac{\partial}{\partial x_i}\right) = \xi_i$, so that

$$\alpha = \sum_{i=1}^n \xi_i dx_i \tag{2}$$

in $\pi^{-1}(U)$. It now follows that

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i, \tag{3}$$

which is ω_0 from Example 2.7. □

2.17. DEFINITION. Let (M,ω) be a symplectic manifold. We say that a submanifold $i \colon L \to M$ is a *Lagrangian submanifold* if $i^*\omega \equiv 0$ and $\dim L = \frac{1}{2}\dim M$.

2.18. **Theorem.** *Let μ be a* 1*-form on a manifold X. Then*

$$X_\mu := \left\{ (x, \mu_x) \,\middle|\, x \in X, \mu_x \in T_x^* X \right\}$$

is a Lagrangian submanifold of T^*X if and only if μ is a closed form.

Proof. First of all, X_μ is an embedded submanifold of T^*X since it is the image of the section μ. Let $s_\mu : X \to T^*X$ be this embedding, i.e. the 1-form μ considered as a function. Then

$$\begin{aligned} \left(s_\mu^* \alpha\right)_x &= \left(ds_\mu\right)_x^* \alpha_p \\ &= \left(ds_\mu\right)_x^* (d\pi)_p^* \mu_x \\ &= (\pi s)_x^* \mu_x \\ &= \mu_x. \end{aligned} \tag{4}$$

Now let $\tau : X \to X_\mu$ be the diffeomorphism making the diagram commute

$$\begin{tikzcd} X \arrow[r, "s_\mu"] \arrow[d, "\tau"'] & T^*X \\ X_\mu \arrow[ur, "i"'] & \end{tikzcd}$$

where $i : X_\mu \to T^*X$ is the inclusion. By definition, X_μ is a Lagrangian submanifold if and only if $i^* d\alpha = 0$, which occurs if and only if $\tau^* i^* d\alpha = 0$ since τ is a diffeomorphism. Now, $\tau^* i^* d\alpha = (i\tau)^* d\alpha = s_\mu^* d\alpha = d(s_\mu^* \alpha) = d\mu$, where the last identity follows by eq. (4). □

2.19. **Theorem.** *Let Z be a submanifold of X. Then the conormal bundle N^*Z is a Lagrangian submanifold of T^*X.*

Proof. For the inclusion $i : N^*Z \to T^*X$, we need to show that $i^*\omega = 0$. We do this by first showing that $i^*\alpha = 0$. To that end, let $(U, x_1, \ldots, x_n)$ be a local coordinate system around $x \in Z$ adapted to Z, i.e. $x_{k+1}, \ldots, x_n = 0$ on Z. Thus, $N^*Z \cap T^*U$ is defined by $x_{k+1} = \cdots = x_n = 0 = \xi_1 = \cdots = \xi_k$. Combining this with eq. (2), we see that $\alpha = \sum_{i>k} \xi_i dx_i$, and so

$$\begin{aligned} i^* \alpha_p &= \alpha|_{T_p(N^*Z)} \\ &= \sum_{i>k} \xi_i dx_i|_{\operatorname{span}\left\{ \frac{\partial}{\partial x_j} \,\middle|\, j \le k \right\}} \\ &= 0. \end{aligned}$$

□

3. Topological Lefschetz Fibrations

3.1. Definition. A *holomorphic Morse function* on a manifold X is a holomorphic function $f : M \to \mathbb{P}^1$ which has only non-degenerate critical points.

3.2. Definition. Let X be a complex manifold of dimension n and $f\colon X \to \mathbb{P}^1$ a surjective smooth fibration. We say that f is a *topological Lefschetz fibration* if

(1) there are finitely many critical points $p_1, \ldots, p_k$, and $f(p_i) \neq f(p_j)$ for $i \neq j$;
(2) any pair of regular fibres is homeomorphic;
(3) for each critical point p, there are complex neighbourhoods $p \in U \subset X$, $f(p) \in V \subset \mathbb{P}^1$ on which $f_{|U}$ is represented by the holomorphic Morse function
$$f_{|U}(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2,$$
and such that $\operatorname{crit} f \cap U = \{p\}$; and
(4) the restriction $f_{\mathrm{reg}} := f|_{X - \bigcup X_i}$ to the complement of the singular fibres X_i is a locally trivial fibre bundle.

3.3. *Remark.* There are many examples in the literature where a Lefschetz fibration is defined as above but with a target of $\mathbb{C}$ instead of $\mathbb{P}^1$. We will also refer to such fibrations as Lefschetz fibrations but will assume the target space to be $\mathbb{P}^1$ unless explicitly stated otherwise.

3.4. *Example.* The holomorphic Morse function
$$\begin{aligned} m\colon \mathbb{C}^{n+1} &\to \mathbb{C} \\ (z_0, \ldots, z_n) &\mapsto \sum_1^n z_j^2 \end{aligned}$$
may not be a Lefschetz fibration under our definition, but it does give us a flavour of the local behaviour in virtue of Item 3. We show here that m is a fibration satisfying Items 1, 2 and 4 of Definition 3.2.

(Item 1) There is precisely one critical point at 0.

(Item 2) The map
$$\begin{aligned} \varphi_\lambda\colon m^{-1}(\lambda) &\to m^{-1}(1) \\ z &\mapsto \frac{z}{\sqrt{\lambda}} \end{aligned}$$
is a homeomorphism for any $\lambda \in \mathbb{C} - \{0\}$.

(Item 4) Define $\ell_\theta := \{re^{2\pi i\theta} \mid r \in \mathbb{R}_{\geq 0}\}$ for $\theta \in [0, 2\pi]$. Then the local triviality condition is satisfied for $U_{\ell_\theta} := \mathbb{C} - \ell_\theta$. Indeed, the function
$$\begin{aligned} \varphi\colon m^{-1}(U_{\ell_\theta}) &\to U_{\ell_\theta} \times m^{-1}(1), \\ z &\mapsto \left(m(z), \varphi_{m(z)}(z)\right), \end{aligned}$$
is a diffeomorphism and the following diagram commutes.

$$\begin{tikzcd} m^{-1}(U_{\ell_\theta}) \arrow[r, "\varphi"] \arrow[d, "m"'] & U_{\ell_\theta} \times m^{-1}(1) \arrow[dl, "\mathrm{proj}"] \\ U_{\ell_\theta} & \end{tikzcd}$$

3.5. DEFINITION. Let (X, ω) be a symplectic manifold of dimension $2n$, $A \subset X$ a codimension 4 submanifold, and $f : X - A \to \mathbb{P}^1$ a smooth map. We say that f is a *topological Lefschetz pencil* if

(1) there are finitely many critical points, $p_1, \ldots, p_k$, and $f(p_i) \neq f(p_j)$ for $i \neq j$;
(2) for each $a \in A$, there is a compatible local system of complex coordinates U in which
$$f : (z_1, \ldots, z_n) \mapsto \frac{z_1}{z_2} \in \mathbb{P}^1$$
and where $A \cap U = \{z \in U \mid z_1(z) = z_2(z) = 0\}$; and
(3) f is represented by the holomorphic Morse function
$$f(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$$
in some compatible local system of complex coordinates around each critical point p_i.

We denote a topological Lefschetz pencil by (X, A, f).

3.6. **Theorem.** [Don99] *Let ω be a symplectic form on a manifold M of real dimension 4. If $[\omega] \in H^2(M, \mathbb{R})$ is the reduction of an integral class h, then M admits a Lefschetz pencil.*

3.7. **Theorem.** [GS91] *If a 4-manifold admits a Lefschetz pencil, then it admits a symplectic structure.*

3.8. **Theorem.** [GS91] *The blow up of a Lefschetz pencil on a complex surface at the base locus is a Lefschetz fibration.*

3.9. *Example.* We present here an explicit example of blowing up a Lefschetz pencil at the base locus. Let $p_0 := x_0, p_1 := x_1$ be two homogeneous polynomials on $\mathbb{C}^3$. The base locus is then given by $B = \{[0,0,1]\}$, and the function

$$\begin{aligned} \pi : \mathbb{P}^2 - B &\to \mathbb{P}^1 \\ x = [x_0, x_1, x_2] &\mapsto [p_1(x), -p_0(x)] = [x_1, -x_0] \end{aligned}$$

is a Lefschetz pencil. The fibres are given by

$$\begin{aligned} \pi^{-1}([s_0, s_1]) &= \{[-s_1, s_0, s_2] \mid s_2 \in \mathbb{C}\} \\ &= V(s_0 p_0 + s_1 p_1) - B \\ &= \mathbb{P}^1 - \infty. \end{aligned}$$

To construct a Lefschetz fibration from this pencil, we need to blow up π at the base locus. Writing $\mathbb{P}^2 = U^0 \cup U^1 \cup U^2$ as the union of the standard open sets $U^i := \{[x_0, x_1, x_2] \mid x_i \neq 0\}$, we see that $B \subset U^2 \cong \mathbb{C}^2$. Blowing up U^2 at B (which we can think of as $\mathbb{C}^2$ at $(0,0)$) gives

$$\tilde{U}^2_B = \{([y_0, y_1, y_2], [t_0, t_1]) \mid t_0 y_0 = -t_1 y_1\}.$$

We obtain the desired blow-up by glueing this onto the remaining standard open sets, i.e.

$$\tilde{\mathbb{P}}^2_B = \big(U^0 \cup U^1 \cup \tilde{U}^2_B\big)/\sim,$$

where $[x_0, x_1, 1] \sim ([y_0, y_1, 1], [t_0, t_1])$ if and only if $[x_0, x_1, 1] = [y_0, y_1, 1]$ and $(x_0, x_1) \neq (0,0)$. The Lefschetz fibration is then given by

$$\begin{aligned} \tilde{\pi} \colon \tilde{\mathbb{P}}^2_B &\to \mathbb{P}^1 \\ [x_0, x_1, x_2] &\mapsto \pi(x) && \text{if } (x_0, x_1) \neq (0,0) \\ ([y_0, y_1, y_2], [t_0, t_1]) &\mapsto [t_0, t_1] && \text{if } y_2 \neq 0. \end{aligned}$$

This is well-defined on the overlap since, for $[x_0, x_1, 1] \sim ([y_0, y_1, y_2], [t_0, t_1])$, $\tilde{\pi}([y_0, y_1, 1], [t_0, t_1]) = [t_0, t_1] = [y_1, -y_0] = [x_1, -x_0] = \pi([x_0, x_1, 1])$.

The fibres of this Lefschetz fibration are homeomorphic to $\mathbb{P}^1$ since $\tilde{\pi}^{-1}([s_0, s_1]) = \{[-s_1, s_0, s_2] \mid s_2 \in \mathbb{C}\} \cup \{([0,0,1], [s_0, s_1])\} \cong \mathbb{P}^1$.

We now wish to prove the two non-existence results of Theorems 3.17 and 3.22, but shall first require a number of lemmas.

3.10. **Lemma.** [Spa66, Theorem 2.8.12] *Let $p \colon E \to B$ be a fibration, $b \in B$, and R a ring. Then there is a natural action of $\pi_1(B, b)$ on $H_*(F_b, R)$.*

3.11. Definition. A fibration is said to be *orientable* if the action of Lemma 3.10 is the trivial action.

3.12. **Lemma.** *Any fibration $p \colon E \to B$ which has a simply connected base is orientable.*

3.13. **Lemma.** [Spa66, Theorem 3.1.1] *Let $p \colon E \to B$ be a fibration that is orientable over a field K, has fibre F, and has path-connected base B. Then the Euler characteristic with coefficients in the field K is multiplicative; that is, it satisfies*

$$\chi(E) = \chi(F) \cdot \chi(B).$$

3.14. **Lemma.** *For coefficients in a ring R,*

$$H_i(\mathbb{P}^n, R) = \begin{cases} R & \text{if } i = 2k < 2n \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Consider $\mathbb{P}^n$ as the quotient of $\mathbb{S}^n$ via the action of $\mathbb{S}^1$ and define the function

$$\begin{aligned} f \colon \mathbb{P}^n &\to \mathbb{R} \\ [z_0, \ldots, z_n] &\mapsto \sum_i i \|z_i\|^2. \end{aligned}$$

This is well-defined since we are only considering two points to be equivalent up to scalar multiples in $\mathbb{S}^1$.

We show that f is a Morse function with critical points $p_i = [e_i]$ of index $2i$ (where e_i denotes the standard basis vector in $\mathbb{C}^{n+1}$). Using the relation $\sum \|z_i\|^2 = 1$ for a point $z \in \mathbb{P}^n$, we can express f on the canonical open set $U_j = \left\{[z_0, \dots, z_n] \in \mathbb{P}^n \,\middle|\, z_j \neq 0\right\}$ as

$$j + \sum_i (i-j)\left(x_i^2 + y_i^2\right),$$

where $z_i = x_i + \sqrt{-1} y_i$. It is clear that there is precisely one critical point at the origin of U_j; that is, at $p_j := [e_j]$. Moreover, this critical point is non-degenerate since the Hessian

$$Hf = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad A := 2 \begin{pmatrix} -j & & & & & & \\ & -j+1 & & & & & \\ & & \ddots & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & -j+n \end{pmatrix}$$

has determinant $(n-j)!(j!)(-1)^j 2^{2n} \neq 0$. The index of p_j is given by the number of negative eigenvalues of Hf, which can clearly be seen on the above matrix to be $2j$.

Morse functions furnish the manifold with the homotopy type of a CW-complex with an i-cell for every critical point of index i. In our case, we have exactly one cell in every even dimension. Therefore, the CW chain complex is

$$\cdots \to 0 \to R \to 0 \to R \ ,$$

the homology of which is as stated. Homotopically equivalent manifolds have the same homology and it is well-known that the cellular homology groups are isomorphic to the singular homology groups. □

3.15. **Corollary.** $\chi(\mathbb{P}^n) = n+1$.

3.16. *Remark.* Although Corollary 3.15 can be proved directly from the definition of the Euler characteristic as the alternating sum of the Betti numbers, it is also known that

$$\chi = \sum (-1)^\lambda C_\lambda,$$

where C_λ is the number of critical points of index λ (with respect to any Morse function). Our proof of Lemma 3.14 then gives the same Euler characteristic by summing the $n+1$ critical points p_i.

We are now ready to prove our non-existence results.

3.17. **Theorem.** *There are no continuous fibrations $f : \mathbb{P}^{2k} \to \mathbb{P}^1$ for any k.*

Proof. Suppose there exists a fibration $f : \mathbb{P}^n \to \mathbb{P}^1$ for some $n = 2k$. By Corollary 3.15, $\chi(\mathbb{P}^n) = 2k+1$, which is odd, and $\chi\left(\mathbb{P}^1\right) = 2$. However, by Lemma 3.13, $2k+1 = \chi(\mathbb{P}^n) = \chi(F)\cdot\chi\left(\mathbb{P}^1\right) = 2\cdot\chi(F)$, which is a contradiction. □

3.18. **Lemma.** $H^*(\mathbb{P}^n,\mathbb{Z}) = \bigoplus_k H^k(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}[x]/\langle x^{n+1}\rangle$ *for some* $x \in H^2(\mathbb{P}^n,\mathbb{Z})$.

3.19. **Proposition.** *Let* $n \geq 2$ *and* $f : \mathbb{P}^n \to \mathbb{P}^1$ *be* C^1*. Then* f *has no regular points.*

Proof. The function f induces a homomorfism of graded rings $f^* : H^*\left(\mathbb{P}^1,\mathbb{Z}\right) \to H^*(\mathbb{P}^n,\mathbb{Z})$. Let ω be the volume form of $\mathbb{P}^1$. This represents a class $[\omega] \in H^2\left(\mathbb{P}^1,\mathbb{Z}\right)$. As $H^4\left(\mathbb{P}^1,\mathbb{Z}\right) = 0$, we have that $0 = [\omega^2]$. Therefore, $(f^*[\omega])^2 = 0$. By Lemma 3.18, $f^*[\omega] = 0$ since $2 \leq n$. This implies that $f^*[\omega]$ is a coboundary; that is, there exists a 1-form α such that $f^*\omega = d\alpha$. But $H^1(\mathbb{P}^n,\mathbb{Z}) = 0$, and thus $\alpha = dg$ for some 0-form g. It follows that $f^*\omega = 0$. For a regular point $z \in \mathbb{P}^n$, the non-degeneracy of ω_z prevents df_z being surjective, which contradicts that assumption that z is regular. Therefore, there are no regular points. □

3.20. **Corollary.** *There are no submersions* $f : \mathbb{P}^n \to \mathbb{P}^1$ *for* $n > 1$*. In particular, there do not exist any Lefschetz fibrations* $f : \mathbb{P}^n \to \mathbb{P}^1$ *for* $n > 1$.

3.21. *Example.* There are non-constant differentiable functions $f : \mathbb{P}^n \to \mathbb{P}^1$. Indeed, the function

$$\begin{aligned} f : \mathbb{P}^n &\to \mathbb{P}^1 \\ z = [z_0,\ldots,z_n] &\mapsto \left[\|\mathfrak{Re}\,z\|^2, \|\mathfrak{Im}\,z\|^2\right]. \end{aligned}$$

is well-defined and differentiable sinde the entries are homogenious polynomials of equal degree. It is clear that it is not constant.

3.22. **Theorem.** *Any* C^1 *map* $f : \mathbb{P}^n \to \mathbb{P}^1$ *has no regular values. There are no differentiable non-constant maps* $f : \mathbb{P}^n \to \mathbb{P}^1$*. In particular, there are no smooth fibrations* $f : \mathbb{P}^n \to \mathbb{P}^1$.

Proof. Suppose for a contradiction that there exists such a function f. This map induces a map of cohomology groups $f^* : H^*(\mathbb{P}^1,\mathbb{Z}) \to H^*(\mathbb{P}^n,\mathbb{Z})$. By Lemma 3.14, both groups are isomorphic to $\mathbb{Z}$ so we can choose a generator $[\omega] \in H^2(\mathbb{P}^1,\mathbb{Z})$, the class of the symplectic form for example. Since $[\omega]^2 \in H^4(\mathbb{P}^1,\mathbb{Z}) = 0$ and $H^4(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$, we know that $f^2 = 0$. But this forces f to be constant by the non-degeneracy of ω, contradicting the assumption that f is non-constant. □

3.23. **Corollary.** *There are no topological Lefschetz fibrations* $f : \mathbb{P}^n \to \mathbb{P}^1$ *for any* n.

3.24. *Remark.* It is known that any symplectic 4-manifold admits a Lefschetz pencil [Don99], which can in turn be blown up to a Lefschetz fibration. The space $\mathbb{P}^2$ is symplectic but Theorem 3.17 tells us that we cannot hope to have a Lefschetz fibration without blowing up.

4. Symplectic Lefschetz Fibrations

4.1. Definition. Let X be a complex manifold and ω a symplectic form making (X,ω) into a symplectic manifold. We say that a topological Lefschetz fibration is a *symplectic Lefschetz fibration* if

(1) the smooth part of any fibre is a symplectic submanifold of (X,ω); and
(2) for each critical point p_i, the form ω_{p_i} is non-degenerate on the tangent cone of X_i at p_i.

4.2. **Theorem.** [Smi06] *Let $p: E \to B$ be a holomorphic fibration where E is a quasi-projective variety and a Kähler manifold. Then any fibre of the restriction $p: p^{-1}(B_0) \to B_0$ to a submanifold $B_0 \to B$ is a symplectic submanifold of E.*

4.3. *Example.* Our local model m from Example 3.4 is a symplectic fibration satisfying Items 1 and 2 of Definition 4.1.

(Item 1) The regular fibre $m^{-1}(1)$ is symplectic by Theorem 4.2. We can even give an explicit description of it as the cotangent bundle of the sphere (equipped with the canonical symplectic form)

$$T^* \mathbb{S}^n = \left\{ (q,p) \in \mathbb{R}^n \times \mathbb{R}^n \;\middle|\; \|q\| = 1, \langle q,p \rangle = 0 \right\} \tag{5}$$

via the symplectomorphism

$$\begin{aligned} \psi : m^{-1}(1) &\to T^* \mathbb{S}^n \\ z &\mapsto \left(\frac{\mathfrak{Re}\, z}{|\mathfrak{Re}\, z|}, -|\mathfrak{Re}\, z|\, \mathfrak{Im}\, z \right). \end{aligned}$$

Note that fixing a point on the sphere in the image does *not* fix $\mathfrak{Re}\, z$.

The smooth part of the critical fibre is given by

$$m^{-1}(0) - \{0\} = \left\{ z \in \mathbb{C}^{n+1} \;\middle|\; |\mathfrak{Re}\, z|^2 - |\mathfrak{Im}\, z|^2 = 0 = \langle \mathfrak{Re}\, z, \mathfrak{Im}\, z \rangle \right\} - \{0\}.$$

We can see that it is a symplectic submanifold by considering the restriction $M := m_{|\mathbb{C}^{n+1} \setminus \{0\}} : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}$. Then $M^{-1}(0) = m^{-1}(0) \setminus \{0\}$, which is a symplectic submanifold by Theorem 4.2.

(Item 2) The affine variety $m^{-1}(0)$ is defined by precisely one homogeneous equation and is, therefore, a cone. It follows that the tangent cone at 0 is the variety itself. We have just seen that the restriction of ω to this variety is a symplectic form and, thus, non-degenerate.

4.4. Definition. A topological Lefschetz Fibration $f: X \to \mathbb{P}^1$ is said to be *orientable* if X is orientable and the local system of coordinates in Item 3 can be chosen to be compatible with the orientations of X and $\mathbb{P}^1$.

4.5. **Proposition.** [GS91, Theorem 10.2.18] *A topological Lefschetz fibration $f: X \to \mathbb{P}^1$ of curves of genus $g \geq 2$ on a 4-manifold X is orientable if and only if X admits a symplectic structure.*

4.6. *Example.* Let $p_0 := x_0^3 + x_1^3$ and $p_1 := x_0^3 + x_2^3$. In the base locus B, $x_0x_1x_2 \neq 0$ and so we can assume, without loss of generality, that $x_2 = 1$. Therefore,

$$B = \left\{ [-e^{2k\pi i/3}, e^{2k\pi i/9}, 1] \,\middle|\, k = 0, \ldots, 8 \right\}.$$

The corresponding Lefschetz pencil is given by

$$\begin{aligned} \pi \colon \mathbb{P}^2 - B &\to \mathbb{P}^1 \\ [x_0, x_1, x_2] &\mapsto [p_1(x), -p_0(x)] \\ &\qquad = [x_0^3 + x_2^3, -x_0^3 - x_3^3] \end{aligned}$$

and the fibres are $\pi^{-1}([s_0, s_1]) = V(s_0p_0 + s_1p_1) - B$. By the degree-genus formula, the fibres have genus 1. Thus, there are cycles. To show that there are vanishing cycles, the fundamental theorem of Picard-Lefschetz theory Theorem 6.7 tells us that it is sufficient to show that

$$f := \frac{x_0^3 + x_2^3}{x_0^3 + x_1^3}$$

has singularities. Indeed,

$$\begin{aligned} \frac{\partial f}{\partial x_0} &= \frac{(x_0^3 + x_1^3)3x_0^2 - (x_0^2 + x_2^3)3x_0^2}{(x_0^3 + x_1^3)^2} = \frac{(x_1^3 - x_2^3)3x_0^2}{(x_0^3 + x_1^3)^2} \\ \frac{\partial f}{\partial x_1} &= \frac{-(x_0^3 + x_2^3)3x_1^2}{(x_0^3 + x_1^3)^2} \\ \frac{\partial f}{\partial x_2} &= \frac{3x_2^2}{x_0^3 + x_1^3}. \end{aligned}$$

Solving for 0 shows us that $[1,0,0]$ and $[0,1,0]$ are critical points.

5. Generalised Dehn Twists

We use the representation of the cotangent bundle of the sphere given in Example 4.3 and define

$$T_\varepsilon^* \mathbb{S}^2 := \left\{ (q, p) \in T^* \mathbb{S}^2 \,\middle|\, \|p\| < \varepsilon \right\}$$

for $\varepsilon > 0$. We can define the Hamiltonian function

$$\begin{aligned} \mu \colon T^* \mathbb{S}^2 &\to \mathbb{R} \\ (q, p) &\mapsto \|p\|, \end{aligned}$$

which is smooth on the complement of $\mathbb{S}^2$. It is well known that the integral curves for $\mu^2/2$ are the geodesics. It can then be shown that the Hamiltonian flow, φ_t^μ, transports a cotangent vector p along the unit speed geodesic through p. It cannot be extended smoothly to the total space for all t, but φ_π^μ does extend smoothly to the total space as the antipodal map on $\mathbb{S}^2$.

The flow of μ gives us a smooth circle action on $T^*\mathbb{S}^2 \setminus \mathbb{S}^2$ given by $e^{it} \cdot (q,p) = \varphi_t^{\mu}(q,p)$. This is well-defined since the geodesics on $\mathbb{S}^2$ are closed and of period 2π. This action can be written explicitly as

$$e^{it} \cdot (q,p) = \left(\cos(t)q + \sin(t)\frac{p}{\|p\|}, \cos(t)p - \sin(t)q\|p\|\right).$$

Note that $\varphi_\pi^\mu(x) = (-1)\cdot x = -x$, which we call the *antipodal map* and denote by A.

We now want to modify φ_t^μ in such a way as to obtain a function with compact support. Given a function $r \in C^\infty(\mathbb{R},\mathbb{R})$, we have

$$\varphi_t^{r(\mu)}(x) = e^{itr'(\mu(x))} \cdot x.$$

Suppose further that

$$r(t) = 0 \text{ for } t \geq \frac{\varepsilon}{2} \quad \text{ and } \quad r(-t) = r(t) - t \text{ for all } t.$$

Then $R(t) := r(t) - t/2$ is even, and thus there exists a smooth function $\rho \in C^\infty(\mathbb{R})$ such that $R(t) = \rho(t^2)$. It then follows that $R(\mu)$ is smooth on $T^*\mathbb{S}^2$ and induces a Hamiltonian flow on all of $T^*\mathbb{S}^2$. But

$$\begin{aligned}\varphi_{2\pi}^{R(\mu)} &= e^{i2\pi r'(\mu) - i\pi} \\ &= e^{i2\pi r'(\mu)} e^{-i\pi} = e^{-i\pi} e^{i2\pi r'(\mu)} \\ &= \tau A = A\tau\end{aligned}$$

showing that τ commutes with A. It also follows that τ is a symplectic automorphism since both A and $\varphi_{2\pi}^{R(\mu)}$ are.

5.1. **Lemma.** [Sei97, Lemma 2.1]. *Let $r \in C^\infty(\mathbb{R},\mathbb{R})$ be a function such that*

$$r(t) = 0 \qquad \text{for } t \geq \frac{\varepsilon}{2}, \tag{6}$$

$$r(t) = r(-t) + t \qquad \text{for all } t. \tag{7}$$

Then

(1) *the map*

$$\begin{aligned}\tau \colon T^*\mathbb{S}^2 &\mapsto T^*\mathbb{S}^2 \\ x &\mapsto \varphi_{2\pi}^{r(\mu)}(x) = e^{i2\pi r'(\mu)} \cdot x\end{aligned}$$

is a symplectic automorphism supported inside $T_\varepsilon^*\mathbb{S}^2$,

(2) *$\tau \circ A = A \circ \tau$,*

(3) *any two choices of r are isotopic in* $\mathrm{Aut}^c(T_\varepsilon^*\mathbb{S}^2)$.

5.2. **Lemma.** *Let (M,ω) be a compact symplectic 4-manifold and $V \subset M$ an embedded Lagrangian 2-sphere. Then there is an $\varepsilon > 0$ and a symplectic embedding $i \colon T_\varepsilon^*\mathbb{S}^2 \to M$ with $i(\mathbb{S}^2) = V$.*

Moreover, for any two such embeddings i, i' as above such that $i^{-1}i'|_{\mathbb{S}^2} \in$ $\mathrm{Diff}(\mathbb{S}^2)$ *is diffeotopic to the identity, there is a $\delta < \varepsilon$ such that $i|_{T_\delta^*\mathbb{S}^2}$ can be deformed into $i'|_{T_\delta^*\mathbb{S}^2}$ within the space of symplectic embeddings which map $\mathbb{S}^2$ to V.*

5.3. **Proposition.** *Choose $\varepsilon > 0$, an embedding i as in Lemma 5.2, and a function r as in Lemma 5.1. Then*

$$\tau_V(x) = \begin{cases} i\tau i^{-1} & \text{if } x \in \operatorname{im}(i) \\ x & \text{if } x \notin \operatorname{im}(i) \end{cases} \tag{8}$$

defines a symplectic automorphism of M. Moreover, τ_V maps V to itself but reverses its orientation. It is independent of r and i up to symplectic isotopy.

5.4. Definition. The function τ_V from Proposition 5.3 is called the *generalised Dehn twist along V*.

6. Vanishing Cycles

6.1. Definition. Let $f : X \to \mathbb{P}^1$ be a Lefschetz fibration. Throughout this section we denote the set of regular points by X_{reg}, the set of regular values by $\mathbb{P}^1_{\text{reg}}$, and fix a regular fibre X_ρ.

Given a loop $\gamma : I \to \mathbb{P}^1_{\text{reg}}$, we can pull back the fibre bundle f_{reg} to get the fibre bundle $\gamma^* f_{\text{reg}}$. We can assume has total space $X_\rho \times I$ since any fibre bundle over a contractible space is trivial.

$$\begin{tikzcd} X_\rho \times I \arrow[r, "\Gamma"] \arrow[d, "\gamma^* f_{\text{reg}}"'] & X_{\text{reg}} \arrow[d, "f_{\text{reg}}"] \\ I \arrow[r, "\gamma"'] & \mathbb{P}^1_{\text{reg}} \end{tikzcd}$$

The map $\Gamma(-,0) = \text{id}$ but $\Gamma_1 := \Gamma(-,1)$ gives us what shall be called the *geometric monodromy of γ*. This induces a map of homology groups

$$(\Gamma_1)_* \in \operatorname{Aut}\left(H_*\left(X_\rho\right)\right),$$

called the *algebraic monodromy of γ*. Since this map depends only on the homotopy class of γ, we can define the *monodromy of f* as the action

$$\pi_1\left(\mathbb{P}^1_{\text{reg}}\right) \to \operatorname{Aut}\left(H_*(X_\rho)\right).$$

We seek to understand this action in terms of the behaviour of f near its critical points.

Choose a regular value $\rho \in \mathbb{P}^1$ and for each critical value $q_i \in \mathbb{P}^1$ a disk D_i small enough so that they are pairwise disjoint. Let a_i be arcs such that $a_i(0) = \rho$ and $a_i(1) \in \partial D_i$, and let b_i be the arcs traversing ∂D_i in the anticlockwise direction, starting at $b_i(0) = a_i(1)$. Then we can define the loops $\gamma_i := a_i * b_i * \overline{a}_i$, where $*$ denotes concatenation of paths and $\overline{a}_i(t) := a_i(1-t)$.

6.2. **Lemma.** [ABKP00] *Denote the critical points of the Lefschetz fibration by q_i for $i = 1, \ldots, k$ and define the paths γ_i as above. Then*

$$\pi_1\left(\mathbb{P}^1 \setminus \{q_i \mid i = 1, \ldots, k\}\right) = \left\langle \gamma_1, \ldots, \gamma_k \,\middle|\, \gamma_1 \cdots \gamma_k = 1 \right\rangle.$$

The preceeding lemma shows that we need only consider the action of the generators γ_i on the homology groups to understand the monodromy action. The full expression of the monodromy action is given in terms of the vanishing cycles, which we now define.

6.3. **Lemma.** [ABKP00] *There are retractions $r_i \colon X_\rho \to X_i$ for each singular fibre X_i.*

6.4. DEFINITION. A *geometric vanishing cycle* is a Lagrangian sphere $s_i \in X_\rho$ such that $r_i(s_i)$ is a point. Now let $X_D := f^{-1}(D)$, where D is a disk in $\mathbb{P}^1$ containing all of the critical values. Then an *algebraic vanishing cycle* is understood as an element $\delta \in \ker\left(\iota_* \colon H_{n-1}(X_\rho) \to H_{n-1}(X_D)\right)$, where ι is the inclusion map $\iota \colon X_\rho \to X_D$. We denote the set of all algebraic vanishing cycles by $\mathbb{V}_{X_\rho}$.

6.5. *Example.* We look again at the local model given by the symplectic fibration m from Example 4.3. The fact that $m^{-1}(t) \cap \mathbb{R}^{n+1} = \mathbb{S}^n_t$ shrinks to a point as $t \to 0$ shows that they are our vanishing cycles. Note that they correspond to the Lagrangian spheres given by the zero sections of of $T^*\mathbb{S}^n$ under our identification.

6.6. *Example.* Unfortunately, Example 3.9 does not contain any singularities or vanishing cycles. Indeed, any polynomial on $\mathbb{P}^2$ of degree 1 or 2 defines a curve C homeomorphic to $\mathbb{P}^1$ and, thus, has no homology $H_1\left(\mathbb{P}^1, \mathbb{Z}\right)$. Therefore, there are no vanishing cycles.

6.7. **Theorem** (Fundamental theorem of Picard-Lefschetz theory)**.** *Let $f \colon X \to \mathbb{P}^1$ be a Lefschetz fibration for X a projective complex manifold of complex dimension n. Then for each loop γ_i around the critical value $f(p_i)$, there corresponds a vanishing cycle $\delta_i \in H_{n-1}(X_\rho)$. Moreover, the monodromy action of f can be written explicitly in terms of the generators γ_i as*

$$(\gamma_i)_*(c) = c + (-1)^{\frac{n(n+1)}{2}} \langle c, \delta_i \rangle \, \delta_i,$$

where the bracket $\langle \cdot, \cdot \rangle$ is the Kronecker pairing. Moreover, the monodromy action on $H_k(X_\rho)$ is trivial for $k \neq n-1$.

6.8. **Theorem** (Hard Lefschetz Theorem)**.** *With $\mathbb{I}_{X_\rho} := \left\{ c \in H_{n-1}(X_\rho) \,\middle|\, \gamma_i(c) = c \ \forall i \right\}$ and $\mathbb{V}_{X_\rho}$ the set of vanishing cycles, we have*

$$H_{n-1}(X_\rho) = \mathbb{I}_{X_\rho} \oplus \mathbb{V}_{X_\rho}.$$

7. FLOER COHOMOLOGY

Floer cohomology was originally defined by Floer to prove the Arnold Conjecture. One form of this conjecture is the content of Theorem 7.2 Item 1. Floer homology is defined analogously to Morse homology, the details of which can be found in [Sal99, Section 1.3].

For Morse homology, we start with a compact Riemannian manifold (M, g) and a Morse function $h \colon M \to \mathbb{R}$ (also called a height function). To each critical point $p \in \mathfrak{crit}\, h$, we can associate an integer λ_p, called the *index*, given by the dimension of largest negative definite subspace of the Hessian of h. We then consider the flow of $-\nabla f$. The flow lines must converge to critical points and, in fact, flow from one critical point to another of smaller index. This allows us to define the Morse chain complex:

$$\begin{aligned} CM_\lambda &:= \mathfrak{crit}_\lambda h \\ \partial_\lambda : CM_\lambda &\to CM_{\lambda-1} \\ p &\to \sum_{q \in \mathfrak{crit}_{\lambda-1} h} n(p,q) q, \end{aligned}$$

where $n(p,q)$ is the number of flow lines connecting p,q, counting with orientation. It can be shown that the homology of this complex is isomorphic to the singular homology $H_* M$. This also implies that the Morse homology is independent of the metric g and the height function h.

Now we give a brief summary of Lagrangian intersection Floer cohomology as given in [FOOO09]. To define the Floer cohomology $HF_*(L_0, L_1)$ of M with respect to two (transversal) Lagrangian submanifolds L_0, L_1 of M, we take a slightly different approach to that of Morse homology. Consider the universal cover $\tilde{P}$ of the space of paths

$$P := \{x : [0,1] \to M \mid x(i) \in L_i, i = 0,1\}$$

joining one Lagrangian to the other. We want to integrate the (symplectic) area delineated by x in the following way. A homotopy map for x is a function $\hat{x} : [0,1] \times [0,1] \to M$ such that

- $\hat{x}(i,s) \in L_i$,
- $\hat{x}(t,0) = y \in L_0 \cap L_1$ for fixed y, and
- $\hat{x}(t,1) = x$.

Set H_x to be the set of such homotopy maps. Then we can write $\tilde{P}$ as

$$\tilde{P} = \{(x,\hat{x}) \mid x \in P,\ \hat{x} \in H_x\}$$

and define the action

$$\begin{aligned} A : \tilde{P} &\to \mathbb{R} \\ (x,\hat{x}) &\mapsto \iint_{[0,1]^2} \hat{x}^* \omega \end{aligned}$$

as the symplectic area between the two Lagrangians delineated by x. The action A is the analogue of the height function h in Morse homology.

The problem now is that the index of a critical point can be infinite since we are on an infinite dimensional manifold. To overcome this obstacle, we instead use the Maslov index and define $\mathfrak{crit}_\lambda A$ to be the set of critical points with Maslov index λ. We do not wish to enter into the details of the Maslov index, which can be found in [PT08, Chapter 5.4].

To define the Floer chain complex, we now need the analogue of the flow lines connecting two critical points. To that end, choose an almost complex structure J on M. The pseudo-holomorphic strips $u\colon \mathbb{R}\times[0,1]\to M$ satisfying $u(\mathbb{R}\times\{0\})\subset L_0$, $u(\mathbb{R}\times\{1\})\subset L_1$ and the system of equations

$$\begin{aligned} 0 &= \frac{\partial u}{\partial s} + J(t,u)\frac{\partial u}{\partial t} \\ p &= \lim_{s\to-\infty} u(s,t) \\ q &= \lim_{t\to\infty} u(s,t) \end{aligned}$$

are said to connect p and q. We set $n(p,q)$ to be the number pseudo-holomorphic strips connecting p and q.

Now we have all the tools in place to define the Floer chain complex:

$$\begin{aligned} CF_\mu(L_0,L_1) &:= \mathfrak{crit}_\mu A \\ \partial_\mu\colon CF_\mu(L_0,L_1) &\to CF_{\mu-1}(L_0,L_1) \\ p &\to \sum_{q\in\mathfrak{crit}_{\mu-1}A} n(p,q)q. \end{aligned}$$

Strictly speaking, for $\partial^2=0$ to hold, we require the extra conditions on M given in the statement of Theorem 7.2.

7.1. DEFINITION (Condition T). Let L_0 and L_1 be Lagrangian submanifolds. They are said to satisfy Condition T if one of the following holds:

(1) the images of $H_1(L_0;\mathbb{Z})$ and $H_1(L_1;\mathbb{Z})$ in $H_1(M;\mathbb{Z})$ are finite.
(2) L_0 is homotopic to L_1 in M.

As the following theorem shows, the Floer cohomology groups behave well with respect to the symplectic structure. In particular, Item 2 shows that it is invariant under Hamiltonian symplectomorphisms.

7.2. **Theorem.** [Flo88] *Let L_0 and L_1 be Lagrangian submanifolds satisfying Condition T and such that $\pi_2(M,L_i)=0$. Then there exists a $\mathbb{Z}$-graded Floer cohomology group $HF^*(L_0,L_1)$ with $\mathbb{Z}_2$ coefficients satisfying the following:*

(1) *If L_0 is transversal to L_1, then*

$$\sum_k \operatorname{rank} HF^k(L_0,L_1) \geq |L_0\cap L_1|, \tag{9}$$

$$\sum_k (-1)^k \operatorname{rank} HF^k(L_0,L_1) = L_0\cdot L_1, \tag{10}$$

where $|L_0\cap L_1|$ is the order of the set $L_0\cap L_1$, and $L_0\cdot L_1$ is the intersection number;

(2) *If $\varphi\colon M\to M$ is a Hamiltonian symplectomorphism, then*

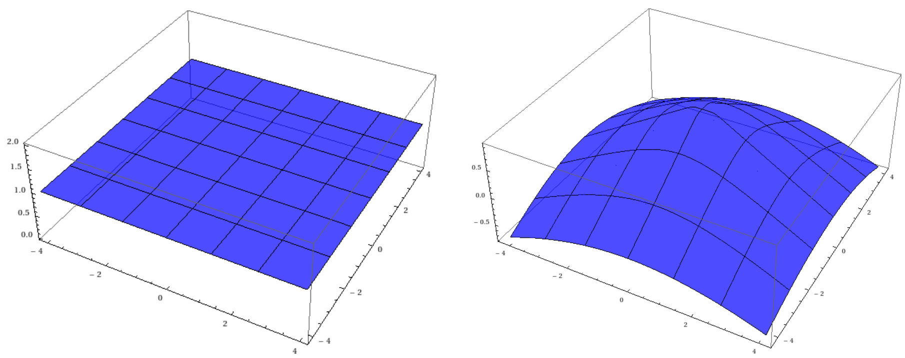
$$HF^*(\varphi(L_0),\varphi(L_1)) \cong HF^*(L_0,L_1);$$

(3) *$HF^*(L_i,L_i)\cong H^*(L_i;\mathbb{Z}_2)$.*

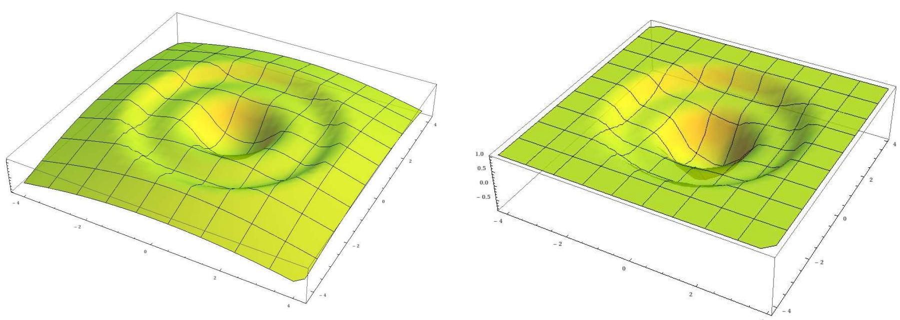
We can define the Floer cohomology with respect to any two Lagrangian submanifolds simply by deforming one of them by a Hamiltonian flow until they are transversal. However, care must be taken as to which entry we deform, as Lemma 7.3 shows.

7.3. **Lemma.** *Let F_0 be a fibre of the bundle $T^* \mathbb{S}^n$, $F_1 := \tau(F_0)$ the Dehn twist of the fibre, and Z the zero section. Then*

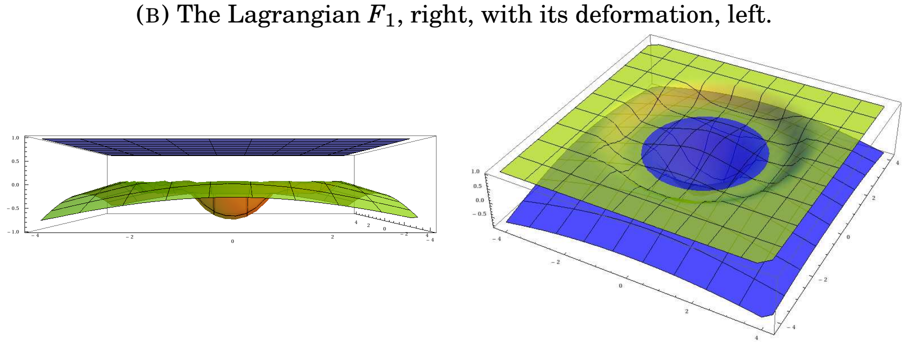
(1) $HF^{n-1}(F_0, F_1) \neq 0$*; but*



(A) The Lagrangian F_0, left, with its deformation, right.



(B) The Lagrangian F_1, right, with its deformation, left.



(C) The Lagrangian F_0 together with the deformation of F_1, left, and F_1 with the deformation of F_0, right. The intersections are clearly $\emptyset$ and $\mathbb{S}^1$, respectively.

FIGURE 1. The Lagrangians $F_0 := T^*_{(0,0,1)} S^2$ and $F_1 := \tau(F_0)$ and their deformations with respect to the Hamiltonian flow. The function $r(t) = \frac{-\sqrt{t^2+\varphi(t)}+t}{2}$ was used to define the Dehn twist (see Lemma 5.1), where φ is the bump function only taking non-zero values $\varphi(t) = \exp\left(\frac{3^2}{t^2-3^2}\right)$ for $|t| < 3$.

(2) $HF^*(F_1, F_0) = 0$.

Proof. As Seidel comments in [Sei04, Lemma 2], the Floer cohomology groups of a pair of non-compact Lagrangian submanifolds are given by deforming the first by the Hamiltonian flow of $\mu(q,p) = \|p\|$ until they are transversal and then calculating the singular cohomology groups of the intersection.

(Item 1) We must deform F_0 along the Hamiltonian flow and find the intersection of this with F_1. To that end, fix $q \in \mathbb{S}^n$. Then we can write

$$F_1 = \left\{ \left(\cos(2\pi r'(|p|))q + \sin(2\pi r'(|p|))\frac{p}{|p|}, \cos(2\pi r'(|p|))p - \sin(2\pi r'(|p|))|p|q \right) \right\},$$
$$\varphi_\delta F_0 = \left\{ \left(\cos(\delta)q + \sin(\delta)\frac{p}{|p|}, \cos(\delta)p - \sin(\delta)|p|q \right) \right\}.$$

These sets intersect if, for a sufficiently small δ, there exist p_1, p_2 solving the equations

$$0 = \left(\cos(2\pi r'(|p_1|)) - \cos(\delta)\right) q + \sin(2\pi r'(|p_1|))\frac{p_1}{|p_1|} - \sin(\delta)\frac{p_2}{|p_2|} \tag{11}$$
$$0 = \left(\sin(2\pi r'(|p_1|))|p_1| - \sin(\delta)|p_2|\right) q - \cos(2\pi r'(|p_1|))p_1 + \cos(\delta)p_2 \tag{12}$$

The orthogonality conditions on q, p_1, and p_2 allow us to reduce this problem to that of finding a solution $t = |p|$ with $p = p_1 = p_2$ to the equation

$$0 = \cos(2\pi r'(t)) - \cos(\delta). \tag{13}$$

Differentiating eqs. (6) and (7), we obtain

$$r'(t) = 0 \qquad \text{for } t \geq \frac{\varepsilon}{2}, \tag{14}$$
$$r'(t) = -r'(-t) + 1 \qquad \text{for all } t. \tag{15}$$

It follows that $r'(0) = \frac{1}{2}$ and $r'\left(\frac{\varepsilon}{2}\right) = 0$, so the intermediate value theorem guarantees the existence of some $\frac{\varepsilon}{2} > t > 0$ such that eq. (13) is satisfied. Choose any $p = p_1 = p_2$ with $|p| = t$ to obtain a solution to eqs. (11) and (12). The set of all such possible solutions p with $|p| = t$ is given by the sphere $\mathbb{S}^{n-1}_t \subset F_0$ of radius t inside the fibre. Since $t = 0$ is not a solution to eq. (13), the intersection $F_1 \cap \varphi_\delta F_0$ has the homotopy type of a disjoint union of (possibly uncountably many) $(n-1)$-spheres.

(Item 2) We must deform F_1 along the Hamiltonian flow and find the intersection of this with F_0. To that end, fix $q \in \mathbb{S}^n$ and set $\rho = 2\pi r'(|p|)$. Then the points of $\varphi_\delta F_1$ are of the form

$$\Bigg(\cos(\delta)\left[\cos(\rho)q+\sin(\rho)\frac{p}{|p|}\right]+\sin(\delta)\left[\frac{\cos(\rho)p-\sin(\rho)|p|q}{|\cos(\rho)p-\sin(\rho)|p|q|}\right],$$

$$\cos(\delta)\left[\cos(\rho)p-\sin(\rho)|p|q\right]-\sin(\delta)|\cos(\rho)p-\sin(\rho)|p|q|\left[\cos(\rho)q+\sin(\rho)\frac{p}{|p|}\right]\Bigg)$$

$$=\Bigg(\left[\cos(\delta)\cos(\rho)-\frac{\sin(\delta)\sin(\rho)|p|}{|\cos(\rho)p-\sin(\rho)|p|q|}\right]q + \left[\frac{\cos(\delta)\sin(\rho)}{|p|}+\frac{\sin(\delta)\cos(\rho)}{|\cos(\rho)p-\sin(\rho)|p|q|}\right]p\,,$$

$$-\left[\cos(\delta)\sin(\rho)|p|+\sin(\delta)\cos(\rho)|\cos(\rho)p-\sin(\rho)|p|q|\right]q + \left[\cos(\delta)\cos(\rho)-\frac{\sin(\delta)\sin(\rho)|\cos(\rho)p-\sin(\rho)|p|q|}{|p|}\right]p\Bigg).$$

Supposing there exists a solution in $\varphi_\delta F_1\cap F_0$, the p-coefficient of the first component and the q-coefficient of the second must both be zero, again by using the orthogonality conditions. This implies

$$0=\frac{\cos(\delta)\sin(\rho)}{|p|}+\frac{\sin(\delta)\cos(\rho)}{\cos(\rho)p-\sin(\rho)|p|q},$$
$$0=\cos(\delta)\sin(\rho)|p|+\sin(\delta)\cos(\rho)|\cos(\rho)p-\sin(\rho)|p|q|,$$

which we can reduce to

$$|\cos(\rho)|+|\sin(\rho)|=1.$$

Since $\cos^2(\rho)+\sin^2(\rho)=1$, we have $\cos(\rho)\sin(\rho)=0$. By inspecting the expression for the points in $\varphi_\delta F_1$, we see that this is not possible. Hence, $\varphi_\delta F_1\cap F_0=\emptyset$. □

7.4. *Remark.* In the proof of Lemma 7.3 Item 1, if we could show that the set of solutions t to eq. (13) is connected, we would be able to deduce that $HF^*(F_0,F_1)\cong H^*\left(\mathbb{S}^{n-1},\mathbb{C}\right)$. Figure 1 certainly suggests that this is the case for $n=2$. Indeed, Seidel claims that the solution is unique in the general case but does not provide a proof.

8. Examples of Lefschetz Fibrations

We present two examples of Lefschetz fibrations: one from Lie theory; the other from mirror symmetry. The former is constructed on the adjoint orbit of a regular element $H_0\in\mathfrak{sl}(3\mathbb{C})$ using Theorem 8.1. The second is well-known as the mirror of $\mathbb{P}^1$, where the Lefschetz fibration is given by the "superpotential". Both are functions into $\mathbb{C}$ instead of $\mathbb{P}^1$.

8.1. **Theorem.** [GGM] *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathscr{W}$ its Weyl group, $\mathfrak{h}$ a Cartan subalgebra, G a Lie group with Lie algebra $\mathfrak{g}$, and denote the adjoint orbit of an element $X\in\mathfrak{g}$ by $\mathscr{O}(X)$. Then for any $H_0\in\mathfrak{h}$ and $H\in\mathfrak{h}_{\mathbb{R}}$ with H a regular element, the function $f_H:\mathscr{O}(H_0)\to\mathbb{C}$ defined by*

$$f_H(x)=\langle H,x\rangle \qquad x\in\mathscr{O}(H_0)$$

is a Lefschetz fibration in the following sense.

(1) *There are finitely many singularities. This number is given by* $|\mathscr{W}|/|\mathscr{W}_{H_0}|$.
(2) *The singularities are non-degenerate.*
(3) *The regular fibres are diffeomorphic.*
(4) *There exists a symplectic form* ω *on* $\mathscr{O}(H_0)$*, with respect to which the regular fibres are symplectic submanifolds*
(5) *The singular fibres contain affine subspaces, which are each symplectic with respect to* ω*.*

8.2. *Example.* We write a general element of $\mathfrak{sl}(3,\mathbb{C})$ as

$$A = \begin{pmatrix} x_1 & y_1 & y_2 \\ z_1 & x_2 & y_3 \\ z_2 & z_3 & -(x_1+x_2) \end{pmatrix}.$$

Choosing

$$H_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then $\mathscr{O}(H_0)$ is the set of matrices in $\mathfrak{sl}(3,\mathbb{C})$ with minimal polynomial $p(x) := (x-2\,\mathrm{id})(x+\mathrm{id})$. We can also obtain $\mathscr{O}(H_0)$ as the affine variety in $\mathfrak{sl}(3,\mathbb{C}) \cong \mathbb{C}^8$ cut out by the ideal $\langle a_{ij} \,|\, i,j = 1,2,3 \rangle$, where the a_{ij} are the polynomial entries of the matrix $p(A) = (A-2\,\mathrm{id})(A+\mathrm{id})$.

Choosing

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

yields the Lefschetz fibration f_H

$$\begin{aligned} f_H \colon \mathscr{O}(H_0) &\to \mathbb{C} \\ A &\mapsto x_1 - x_2. \end{aligned}$$

Moreover, the Weyl group $\mathscr{W}$ acts via conjugation by permutation matrices. Therefore, f_H has 3 singularities: the 3 diagonal matrices with diagonal entries $2,-1,-1$. The critical values are 3, 0, and -3.

The regular fibre $X_1 := f_H^{-1}(1)$ is cut out as an affine variety in $\mathfrak{sl}(3,\mathbb{C}) \simeq \mathbb{C}^8$ by the ideal

$$I = \langle x_1 - x_2 - 1, a_{ij} \,|\, i,j = 1,2,3 \rangle.$$

Projectivising gives us a compactification $\overline{X}_1 \in \mathbb{P}^8$. Let $\overline{a}_{ij}$ be the homogenisation of a_{ij}, or, equivalently, the polynomial entries of $(A-2t\,\mathrm{id})(A+t\,\mathrm{id})$. Then the ideal corresponding to $\overline{X}_1$ is

$$J = \langle x_1 - x_2 - t, \overline{a}_{ij} \,|\, i,j = 1,2,3 \rangle.$$

Using the Macaulay2 algorithms described in appendix A.2 to calculate the Hodge cohomology groups, we obtain the following Hodge diamond for $\overline{X}_1$:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We can also projectivise the orbit $X := \mathcal{O}(H_0)$. For example, with $K := \langle \overline{a}_{ij} \mid i,j = 1,2,3 \rangle$, we can choose the projectivisation $\overline{X}$ of X cut out by K in $\mathbb{P}^8$. Again using the Macaulay2 algorithms described in appendix A.2, we obtain the following Hodge diamond for $\overline{X}$:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 2 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 3 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Mirror of $\mathbb{P}^1$. Now we turn to our last example and study the variety $\mathbb{C}^*$ with the potential

$$\begin{aligned} W\colon \mathbb{C}^* &\to \mathbb{C} \\ z &\mapsto z + \frac{1}{z} \end{aligned}$$

and the symplectic form

$$\omega := \sqrt{-1}\frac{dz \wedge d\overline{z}}{z\overline{z}}.$$

This is known as the mirror of $\mathbb{P}^1$ in the sense of Kontsevich's homological mirror symmetry conjecture, as is shown in [AKO08]. The interesting part for us will be the vanishing cycles determined by the potential W.

The critical points of W are $\{z \mid 1 - z^{-2} = 0\} = \{\pm 1\}$. The images of the critical points are pairwise distinct, with $W(1) = 2 = -W(-1)$. The Hessian is $2z^{-3}$, which is never zero, so the critical points are non-degenerate.

The fibres of W are given by

$$W^{-1}(\lambda) = \{z \mid z^2 - \lambda z + 1 = 0\},$$

which has 2 points unless $z(z-\lambda)+1$ has a double root. So the critical values are the solutions to $\lambda^2 = 4$, which we denote by

$$\begin{aligned} \lambda_0 &= 2 \\ \lambda_1 &= -2. \end{aligned} \tag{16}$$

Furthermore, since the fibres are all finite, the map W must be proper. Thus, by the Ehresmann fibration theorem, the restriction of W to $W^{-1}(\mathbb{C}\setminus\{\pm 2\})$ is a locally trivial fibre bundle.

Following the notation of [AKO08], we denote our regular reference fibre by $\Sigma_0 := W^{-1}(0) = \{\iota, -\iota\}$, where $\iota := \sqrt{-1}$. To find the vanishing cycles, we look for the spheres in Σ_0 that contract to a point under parallel transport. Each of the arcs

$$\begin{aligned}\gamma_0(t) &= 2t\\ \gamma_1(t) &= -2t,\end{aligned} \tag{17}$$

has two possible lifts to $\mathbb{C}^*$

$$\begin{aligned}\tilde{\gamma}_0(t) &= t \pm \sqrt{t^2-1}\\ \tilde{\gamma}_1(t) &= -t \pm \sqrt{t^2-1}\end{aligned}$$

since $\tilde{\gamma}_j(t) \in W^{-1}(\pm 2t)$ must satisfy the equation $z^2 + (-1)^j 2tz + 1 = 0$. Thus, there are two vanishing cycles, both of which coincide with Σ_0 as sets Σ_0. We distinguish them by their thimbles D_0, D_1 traced out under parallel transport, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, as shown in Figure 2.

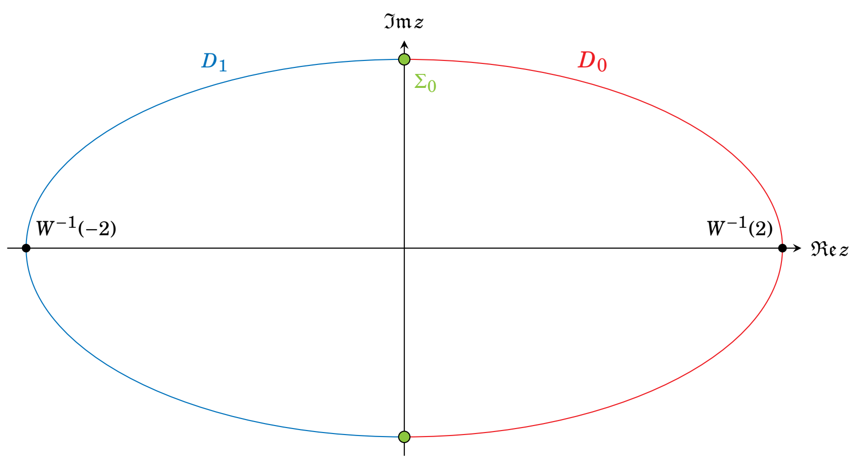


FIGURE 2. The thimbles for W on X.

We must also check that there are no cycles vanishing to infinity. But there are only two solutions of $x^2 - \lambda x + 1 = 0$ as $\lambda \to \infty$: one near zero and the other near λ. Thus, there are no more vanishing cycles.

Now consider the Floer chain complex $CP^*(L_0, L_1)$. Since they are each generated by the thimbles, it follows that $CF^*(L_0, L_1)$ is a free module of rank 2. Since Σ_0 is discrete, the pseudo-holomorphic strips (see Section 7) must be constant. Hence, the differentials vanish identically.

APPENDIX A. COMPUTATIONS

A.1. **Mathematica.** We use Mathematica to obtain a concrete visualisation of a fibre of $T^* \mathbb{S}^n$, it's generalised Dehn twist, and their deformations under a Hamiltonian deformation. The general theory is described in Lemma 7.3 and Section 5, and the results are presented in Figure 1.

We fix a point $u := (0,0,1) \in \mathbb{S}^2$.

```
u = {0, 0, 1};
```

The flow used in our example is given by the circle action which rotates the vector $q \in \mathbb{S}^2$ in the direction $p \in T_q \mathbb{S}^2$ (along the geodesic) by an angle t.

```
flow = Compile[    (* The definition of the flow *)
  {
    {t, _Real},    (* Distance along geodesic *)
    {q, _Real, 1}, (* Point on the Sphere *)
    {p, _Real, 1}  (* Point on the cotangent fibre at q *)
  },
  If[
    Norm[p] == 0,
    Transpose[ {q, p } ],
    RotationMatrix[t, { q, p } ].Transpose[ {q, p }]
  ]
];
```

The algorithm `RotationMatrix` requires $p \neq 0$ to be well-defined, in which case the flow should be the identity map.

The construction of the Dehn twist depends on a function r satisfying eqs. (6) and (7). Here we use the function $r(t) = \frac{-\sqrt{t^2+B(t)}}{2} + \frac{t}{2}$, where B is the bump function only taking non-zero values $B(t) = \exp\left(\frac{3^2}{t^2-3^2}\right)$ for $|t| < 3$.

```
b = 1/3;
B[x_] := Piecewise[
  {
    {Exp[-1/(1 - ((b*x)^2))], Abs[b*x] < 1} ,
    {0, Abs[b*x] >= 1}
  }
];
R[t_] := -Sqrt[t^2 + B[t]]/2;
r[t_] := R[t] + t/2;
```

The Dehn twist of the fibre $T_u \mathbb{S}^2$ is then the function that rotates $u + p$ by the angle $2\pi r'(\|p\|)$ in the direction of $p \in T_u \mathbb{S}^2$.

```
dehn = Compile[
  {
    {p, _Real, 1}
  },
  flow[ 2*\[Pi]*r'[Norm[p]], u, p ]
];
```

We specify the points of the fibre `F0` $:= T_u\,\mathbb{S}^2$ by:

```
F0 = Flatten[ (* The points on the fibre F_0 *)
  Table[
    {p1, p2, 1},
    {p1, -\[Pi] - 1, \[Pi] + 1, 0.1},
    {p2, -\[Pi] - 1, \[Pi] + 1, 0.1}
  ],
  1
];
```

and the points of the Dehn twist `F1` of `F0` by:

```
F1 = Flatten[ (* The Dehn twist of F_0 *)
  Table[
    Total[
      dehn[ {p1, p2, 0} ]
    ],
    {p1, -\[Pi] - 1, \[Pi] + 1, 0.3},
    {p2, -\[Pi] - 1, \[Pi] + 1, 0.3}
  ],
  1
];
```

We fix the parameter t of the deformation φ_t with

```
def = 0.3;
```

and specify the points of the deformation `DefF0` of `F0` by:

```
DefF0 = Flatten[ (* The deformation of F_0 *)
  Table[
    Total[
      flow[ def, u, {p1, p2, 0} ]
    ],
    {p1, -\[Pi] - 1, \[Pi] + 1, 0.1},
    {p2, -\[Pi] - 1, \[Pi] + 1, 0.1}
  ],
  1
];
```

and also the points of the deformation `DefF1` of `F1` by:

```
DefF1 = Flatten[ (* The deformation of F_1 *)
  Table[
    Total[
      flow[
        def,
        dehn[ {p1, p2, 0} ][[1]],
        dehn[ {p1, p2, 0} ][[2]]
      ]
    ],
```

```
    {p1, -\[Pi] - 1, \[Pi] + 1, 0.1},
    {p2, -\[Pi] - 1, \[Pi] + 1, 0.1}
  ],
  1
];
```

Plotting these points with

```
ListPlot3D[
  {F1, DefF0},
  MaxPlotPoints -> 30,
  PlotRange -> All,
  Mesh -> 8,
  BoxRatios -> Automatic,
  PlotStyle -> {
    Directive[Opacity[0.7], Yellow],
    Directive[Opacity[0.7], Blue]
  },
  Boxed -> False,
  Axes -> False
]

ListPlot3D[
  {DefF1, F0},
  MaxPlotPoints -> 30,
  PlotRange -> All,
  Mesh -> 8,
  BoxRatios -> Automatic,
  PlotStyle -> {
    Directive[Opacity[0.7], Yellow],
    Directive[Opacity[0.7], Blue]
  },
  Boxed -> False,
  Axes -> False]
```

then gives the diagrams of Figure 1.

A.2. **Macaulay 2.** We use Macaulay to calculate the Hodge diamonds of the projectivisations of the regular fibre and of the orbit from Section 8. We must first define our base field k and ring of polynomials R. Note that the prime number 32749 is the largest prime that Macaulay2 can work with.

```
i1 : k = ZZ/32749;

i2 : R = k[x_1, x_2, y_1..y_3, z_1..z_3, t];
```

The variable t will be used for projectivisation. A general element $A \in \mathfrak{sl}(3,\mathbb{C})$ has the form

```
i3 : A = matrix{
```

```
    {x_1, y_1, y_2},
    {z_1, x_2, y_3},
    {z_2, z_3, -x_1 - x_2}
   };

              3       3
o3 : Matrix R  <--- R
```

and we will also make use of the identity matrix

```
i4 : Id = id_(k^3);

              3       3
o4 : Matrix k  <--- k
```

In our example, the adjoint orbit $\mathcal{O}(H_0)$ consists of all the matrices with the minimal polynomial $(X+\mathrm{id})(X-2*\mathrm{id})$, so we are interested in the variety cut out by the equation `minPoly`:

```
i6 : minPoly = (A + Id)*(A - 2*Id);

              3       3
o6 : Matrix R  <--- R

i7 : I = ideal minPoly;

o7 : Ideal of R
```

To obtain a projectivisation of X, we first homogenize I:

```
i8 : Iproj = homogenize(I,t);

o8 : Ideal of R

i9 : Xproj = Proj(R/Iproj);
```

One checks with the command `dim Xproj` that $\dim\overline{X}=4$. To check that $\overline{X}$ is non-singular, we use:

```
i10 : codim singularLocus Iproj

o10 = 9
```

Since the codimension of the singularities is 9, our projective variety $\overline{X}$ must be non-singular. Now we calculate $h^{i,j}$ for $(i,j)=(0,0)$, $(1,0)$, $(2,0)$, $(3,0)$, $(4,0)$, $(1,1)$, $(2,1)$, $(3,1)$, and $(2,2)$ with the commands:

```
i11 : hh^(0,0) Xproj

o11 = 1

i12 : hh^(1,0) Xproj
```

```
o12 = 0

i13 : hh^(2,0) Xproj

o13 = 0

i14 : hh^(3,0) Xproj

o14 = 0

i15 : hh^(4,0) Xproj

o15 = 0

i16 : hh^(1,1) Xproj

o16 = 2

i17 : hh^(2,1) Xproj

o17 = 0

i18 : hh^(3,1) Xproj

o18 = 0

i19 : hh^(2,2) Xproj

o19 = 3
```

Since $\overline{X}$ is non-singular, the other entries of the Hodge diamond are given by the classical symmetries.

To define the regular and critical fibres, we will also need the potential, which in our case is given by:

```
i20 : potential = x_1 - x_2;
```

Since 1 is a regular value, we define the regular fibre X_1 as the variety in $\mathfrak{sl}(3,\mathbb{C}) \cong \mathbb{C}^8$ corresponding to the ideal J:

```
i21 : J = ideal(minPoly) + ideal(potential-1);

o21 : Ideal of R
```

We then homogenise J to obtain a projectivisation $\overline{X}_1$ of the regular fibre X_1:

```
i22 : Jproj = homogenize(J,t);

o22 : Ideal of R
```

```
i23 : X1proj = Proj(R/Jproj);
```

One checks with the command `dim X1proj` that $\dim \overline{X}_1 = 3$. We use the command

```
i24 : codim singularLocus Jproj

o24 = 9
```

to test for singularity. Since this codimension is 9, we see that $\overline{X}_1$ is indeed non-singular. Now we calculate $h^{i,j}$ for $(i,j) = (0,0)$, $(1,0)$, $(2,0)$, $(3,0)$, $(2,1)$, and $(3,1)$ with the commands:

```
i25 : hh^(0,0) X1proj

o25 = 1

i26 : hh^(1,0) X1proj

o26 = 0

i27 : hh^(2,0) X1proj

o27 = 0

i28 : hh^(3,0) X1proj

o28 = 0

i29 : hh^(1,1) X1proj

o29 = 2

i30 : hh^(2,1) X1proj

o30 = 0
```

Since $\overline{X}_1$ is non-singular, the other entries of the Hodge diamond are obtained via the classical symmetries.

REFERENCES

[ABKP00] J. Amorós, F. Bogomolov, L. Katzarkov, and T. Pantev, *Symplectic Lefschetz fibrations with arbitrary fundamental groups*, J. Differential Geom. **54** (2000), no. 3, 489–545.

[AKO08] D. Auroux, L. Katzarkov, and D. Orlov, *Mirror symmetry for weighted projective planes and their noncommutative deformations*, Ann. of Math. (2) **167** (2008), no. 3, 867–943.

[Don99] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Differential Geometry **53** (1999), no. 2, 205–236.

[Flo88] A. Floer, *Morse theory for Lagrangian intersections*, J. Differential Geometry **28** (1988), no. 3, 513–547.

[FOOO09] K. Fukaya, Y. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer homology: Anomaly and obstruction*, American Mathematical Society, 2009.

[GGM] E. Gasparim, L. Grama, and L. A. B. San Martin, *Lefschetz fibrations on adjoint orbits*, Preprint.

[GS91] R. Gompf and A. Stipsicz, 4-*manifolds and Kirby calculus*, American Mathematical Society, 1991.

[PT08] P. Piccione and D. V. Tausk, *A students' guide to symplectic spaces, Grassmannians and Maslov index*, Publicações matemáticas do IMPA, 2008.

[Sal99] D. Salamon, *Lectures on Floer homology*, Symplectic Geometry and Topology (Y. Eliashberg and L. Traynor, eds.), IAS/Park City Mathematics Series, vol. 7, American Mathematical Society, 1999, pp. 143–229.

[Sei97] P. Seidel, *Floer homology and the symplectic isotopy problem*, Ph.D. thesis, University of Oxford, 1997.

[Sei04] ———, *Exact Lagrangian submanifolds of T^*S^n and the graded Kronecker quiver*, Different Faces of Geometry, Int. Math. Ser. (N. Y.), vol. 3, Kluwer/Plenum, 2004, pp. 349–364.

[Smi06] I. Smith, *Monodromy, vanishing cycles, knots and the adjoint quotient*, Floer Homology, Gauge Theory, and Low-dimensional Topology, Clay Math. Proc., vol. 5, Amer. Math. Soc., 2006, pp. 277–292.

[Spa66] E. H. Spanier, *Algebraic topology*, McGraw Hill Book Co., 1966.