Dahisy Valadão de Souza Lima

Dynamical Spectral Sequences for Morse-Novikov and Morse-Bott Complexes

Sequências Espectrais Dinâmicas para Complexos de Morse-Novikov e Morse-Bott

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DAHISY VALADÃO DE SOUZA LIMA

DYNAMICAL SPECTRAL SEQUENCES FOR MORSE-NOVIKOV AND MORSE-BOTT COMPLEXES

SEQUÊNCIAS ESPECTRAIS DINÂMICAS PARA COMPLEXOS DE MORSE-NOVIKOV E MORSE-BOTT

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in mathematics.

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em matemática.

Orientadora: Ketty Abaroa de Rezende

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Ketty A de Rezende

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Ketty A. de Rezende

Prof(a). Dr(a). KETTY ABAROA DE REZENDE

Prof(a). Dr(a). MARCO ANTONIO TEIXEIRA

Prof(a). Dr(a). OZIRIDE MANZOLI NETO

Prof(a). Dr(a). REGILENE DELAZARI DOS SANTOS OLIVEIRA

Maria do Carmo Carbinatto

Prof(a). Dr(a). MARIA DO CARMO CARBINATTO

Abstract

The main theme in this thesis is the study of gradient flows associated to a vector field $-\nabla f$ on closed manifolds, where f is either a Morse function, a circle-valued Morse function or a Morse-Bott function. In order to obtain dynamical information, we make use of algebraic and topological tools such as spectral sequences and connection matrices.

In the Morse context, consider a chain complex (C, Δ) generated by the critical points of f, where Δ counts the number of flow lines between consecutive critical points with signs. A spectral sequence (E^r, d^r) analysis is used to obtain results on global continuation of flows on surfaces. A link is established between the differentials on the r-th page of (E^r, d^r) and cancellation of critical points.

In the circle-valued Morse case $f : M \to S^1$, a sweeping algorithm for the Novikov chain complex $(\mathcal{N}, \Delta)$ associated to f and generated by the critical points of f is defined over the ring $\mathbb{Z}((t))$. This algorithm produces at each stage Novikov matrices. We prove that the last Novikov matrix has polynomial entries which is quite surprising since the matrices in the intermediary stages may have infinite series entries. We also present results showing that the modules and differentials of the spectral sequence associated to $(\mathcal{N}, \Delta)$ can be retrieved through the sweeping algorithm.

For gradient flows associated to Morse-Bott functions, the singularities form critical manifolds. We use the Conley index theory for the critical manifolds in order to characterize the set of connection matrices for Morse-Bott flows. Results are obtained on the effects on the set of connection matrices caused by a change in the partial ordering and Morse decomposition of isolated invariant sets.

Resumo

O tema principal desta tese é o estudo de fluxos gradientes associados a campos vetoriais $-\nabla f$ em variedades fechadas, onde f é uma função do tipo Morse, Morse circular e Morse-Bott. Para obter informações dinâmicas em cada caso, utilizamos ferramentas algébricas e topológicas, tais como sequências espectrais e matrizes de conexão.

No contexto de Morse, consideramos um complexo de cadeias (C, Δ) gerado pelos pontos críticos de f onde Δ conta (com sinal) o número de linhas do fluxo entre dois pontos críticos consecutivos. Uma análise via sequências espectrais (E^r, d^r) é feita para se obter resultados de continuação global em superfícies. Nós relacionamos as diferenciais da r-ésima página de (E^r, d^r) com cancelamentos dinâmicos entre pontos críticos.

No caso de função de Morse circular $f : M \to S^1$, o método da varredura para um complexo de Novikov $(\mathcal{N}, \Delta)$ associado f e gerado pelos pontos críticos de f é definido sobre o anel $\mathbb{Z}((t))$. Este método produz a cada etapa matrizes de Novikov. Provamos que a matriz final produzida pelo método da varredura tem entradas polinomiais, o que é surpreendente, já que as matrizes intermediárias podem ter séries infinitas como entradas. Apresentamos resultados que mostram que os módulos e diferenciais de uma sequência espectral associada a $(\mathcal{N}, \Delta)$ podem ser recuperados através do método da varredura.

Para fluxos gradientes associados a funções de Morse-Bott, as singularidades formam variedades críticas. Usamos a teoria do índice de Conley para obter uma caracterização do conjunto de matrizes de conexão para fluxos Morse-Bott. Obtemos resultados sobre o efeito no conjunto de matrizes de conexão causado por mudanças na ordem parcial e na decomposição de Morse de um conjunto invariante isolado.

Contents

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Introduction

Algebraic-topological tools have been widely used in dynamical systems in order to determine structural properties which remain invariant under small perturbations, as in Conley index theory [12, 38].

The basic idea of the Conley index theory is the notion of Morse decomposition which provides a way to decompose an invariant set inside a flow into smaller components by using appropriate attractor-repeller pairs. In this way, if one can understand the smallest invariant sets of the flow then one can investigate some more complex invariant sets such as the ones consisting of attractor-repeller pairs given by a pair of invariant sets of the first type together with all the flow lines joining them. After that, one can repeat this procedure to study the next class of complex invariant sets by taking into account "longer" and "longer" flow lines.

Given a Morse decomposition of an isolated invariant set, the Conley index provides a topological description of the local dynamics around the Morse sets. The connection matrices introduced by Franzosa [20, 21] are algebraic-topological tools which enable us to study the connections between Morse sets. Roughly speaking, a connection matrix for a Morse decomposition is a matrix which has, as entries, maps between the homology Conley indices of Morse sets. Connection matrices encode some information about the structure of the invariant set considered. In fact, non null entries in a connection matrix detects existence of connections between Morse sets. The most important property of this tool is the invariance under continuation, see [22].

In the case of negative gradient flows generated by a Morse function f on a finite dimensional closed manifold, the critical points of f and connecting orbits between them determine a Morse chain complex (C, Δ) [3, 39, 42] whose differential Δ is a special case of a connection matrix and plays an important role in the study of the dynamics associated with this chain complex. For instance, it was proved in [13, 30] that a spectral sequence of a filtered Morse chain complex can be retrieved from its connection matrix. The main idea behind this result is a sweeping algorithm which generates a collection of connection and transition matrices, Δ^r and T^r respectively. The matrix Δ^r contains the necessary information to recover the module E^r and differential d^r of the spectral sequence.

The sweeping algorithm singles out important nonzero entries in Δ^r, which we refer to as primary pivots and change-of-basis pivots, of the r-th diagonal of Δ^r in order to define a matrix Δ^{r+1}. At each step, Δ^{r+1} is a change of basis of Δ^r. Hence, all Δ^r represent in some sense the initial connection matrix, that is, they all represent the same linear transformation. As r increases, the $\mathbb{Z}$-modules E^r_p of the spectral sequence change generators. In [13], these algebraic changes of the generators of the $\mathbb{Z}$-modules of the spectral sequence are connected to a particular family of changes of basis over $\mathbb{Q}$ of the connection matrix Δ.

The Zig-Zag Theorem in [13] states that, whenever $\Delta_{p-r+1,p+1}$ corresponds to a non-zero differential $d^r_p : E^r_p \to E^r_{p-r}$, there exists a path of connecting orbits joining the critical points generating E^0_p and E^0_{p-r} . Inspired by this particular case of algebraic-dynamical correspondence, the next natural, albeit difficult, step is to find out how much of the algebraic information in the spectral sequence can be interpreted dynamically. As one "turns the pages" of the spectral sequence, i.e. considers progressively the modules E^r, one observes algebraic cancellations within the E^r's. What are the dynamical meaning of these algebraic cancellations? Does the sweeping algorithm provide a continuation of the flow?

The spirit of this thesis is to investigate the correspondence between the algebra coded in the spectral sequence through the sweeping algorithm and the dynamics. In a first instance, one considers a Morse function on two dimensional manifolds and the associated Morse chain complex. We also apply this type of investigation to gradient flows generated by Morse-Bott function, where critical manifolds are admissible; and to gradient flows generated by circle-valued Morse functions $f : M \to S^1$ and the corresponding Novikov chain complex. We also present a connection matrix approach to Morse-Bott flows.

The work reported here is a compilation of several papers [4, 5, 25, 26, 27] where we investigate the dynamical properties provided by the sweeping algorithm for Morse chain complexes in [4, 5] and for Novikov complexes in [27]. In [25, 26], we apply the Conley index theory for Morse-Bott flows on compact manifolds.

This thesis is organized in five chapters as follows. Chapter 1 is dedicated to the necessary background material: Conley index theory, spectral sequences and Lyapunov graphs.

In Chapter 2, connection matrix theory and a spectral sequence analysis of a filtered Morse chain complex (C, Δ) are used to study global continuation results for flows on surfaces. The novelty herein is a global dynamical cancellation theorem inferred from the differentials of the spectral sequence (E^r, d^r). The local version of this theorem relates differentials d^r of the r-th page E^r to Smale's Theorem on cancellation of critical points.

In Chapter 3, a Spectral Sequence Sweeping Algorithm (SSSA) is established for a two dimensional Novikov complex over $\mathbb{Z}((t))$ associated to a circle-valued Morse function. We prove that the SSSA is well defined and in the process we obtain a characterization result for the family of Novikov matrices that it produces. We also prove that the final matrix produced by the SSSA over $\mathbb{Z}((t))$ has only polynomial entries which is quite surprising, mainly because the intermediate matrices in the process may have infinite series entries. We present results that retrieve the modules and differentials of the spectral sequence (E^r, d^r) from the SSSA computed for a Novikov chain complex.

In Chapter 4, we introduce the generalized Morse-Bott inequalities for compact manifolds with possibly non-empty boundary. It encompasses the classical Morse-Bott inequalities for closed manifolds. We make use of these inequalities and the Conley index to establish a continuation theorem for Morse-Bott graphs.

In Chapter 5, a connection matrix theory approach is presented for Morse-Bott flows φ on smooth closed n-manifolds by characterizing the set of connection matrices in terms of Morse-Smale perturbations. Further results are obtained on the effect on the set of connection matrices $\mathcal{CM}(S)$ caused by changes in the partial orderings and in the Morse decompositions of an isolated invariant set S.

Chapter 1

Background

The aim of this first chapter is to present some background on topological dynamical systems and algebraic topology tools which will be used throughout this thesis.

The basic ideas of Conley index theory are presented in the first section. In Section 1.2, one gives a brief introduction to spectral sequences and to the Spectral Sequence Sweeping Algorithm which recovers a spectral sequence associated to a chain complex. Section 1.3 is dedicated to graphs, more specifically, abstract Lyapunov graphs. The background material used in this chapter can be found in [6, 13, 21, 30, 41].

1.1 Conley Index Theory

In this section, the Conley index theory is addressed. This theory has several applications in the study of the dynamics of a system, including the existence of periodic orbits in Hamiltonian systems, proof of chaotic behaviour in dynamical systems and bifurcation theory. The connection matrix is a central concept in this theory and enables one to investigate and prove the existence of heteroclinic connections between isolated invariant sets.

Conley index theory generalizes Morse theory, which in essence describes the dynamical structure of a closed manifold through the non-degenerate critical points of a gradient vector field. The Morse index is not well defined for more general invariant sets, while the Conley index is well defined for any isolated invariant set.

We assume that the reader is familiar with the basic ideas in Conley index theory, hence only a brief introduction to homotopy and homology Conley indices, Morse decompositions, homology index braids and connection matrices will be presented. References on this section are [12, 20, 21, 22, 39].

1.1.1 Conley Index of an Isolated Invariant Set

Let X be a Hausdorff topological space and ϕ_t a continuous flow on X, i.e., a continuous map $\phi : X \times \mathbb{R} \to X$, $(x,t) \mapsto \phi_t$, which satisfies $\phi_0 = \mathrm{id}$ and $\phi_s \circ \phi_t = \phi_{s+t}$. A subset $S \subset X$ is called an *invariant set* under ϕ if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. Given a subset $N \subset X$, let

$$\mathrm{Inv}_\phi(N) = \{x \in N \mid \phi_t(x) \in N, \forall t \in \mathbb{R}\},$$

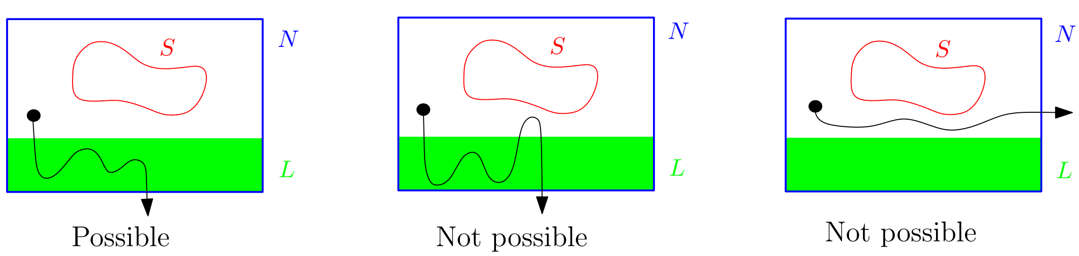
that is, $\mathrm{Inv}_\phi(N)$ is the maximal invariant subset in N. A subset $S \subset X$ is called an *isolated invariant set* with respect to the flow ϕ_t if there exists a compact set $N \subset X$ such that $S = \mathrm{Inv}_\phi(N) \subset \mathrm{int}(N)$. In this case, N is called an *isolating neighbourhood* for S. A particular case of an isolating neighbourhood N is an *isolating block*, where the exiting set of the flow $N^- = \{x \in N \mid \phi_{[0,t)}(x) \nsubseteq N, \forall t > 0\}$ is closed[1].

Given an isolated invariant set S, an *index pair* for S in X is a pair of compact sets (N, L) such that $L \subset N$ and

(1) $\overline{N \backslash L}$ is an isolating neighborhood of S in X, i.e., $S = \mathrm{Inv}_\phi(\overline{N \backslash L}) \subset \mathrm{int}(N \backslash L)$;

(2) L is positively invariant relative to N, i.e., if $x \in L$ and $\phi(x, [0,T]) \subset N$ then $\phi(x, [0,T]) \subset L$;

(3) and L is the exit set of the flow in N, i.e., if $x \in N$ and $\phi(x, [0,\infty)) \nsubseteq N$ then there exists $T > 0$ such that $\phi(x, [0,T]) \subset N$ and $\phi(x,T) \in L$.



Conley proved in [12] that, given an isolated invariant set, there exists an index pair. Of course, the index pair is not uniquely determined. However, given two index pairs (N, L)

[1] We define $N^+ = \{x \in N \mid \phi'_{[0,t)}(x) \nsubseteq N, \forall t > 0\}$, where ϕ' is the reverse flow of ϕ, as the entering set of the flow ϕ. We assume it to be closed as well.

and $(\overline{N}, \overline{L})$ for S, the pointed spaces N/L and $\overline{N}/\overline{L}$, obtained by collapsing the exit sets L and $\overline{L}$, respectively, to a point, have the same homotopy type.

The *homotopy Conley index* $I(S, \phi)$ of S is defined as the homotopy type of the pointed space N/L and the *homology Conley index* $CH(S)$ of S is defined as the reduced homology of N/L, where (N, L) is an index pair for S. Denote h_* the rank of the homology Conley index $CH(S)$.

Figure 1.1 represents a flow on $\mathbb{R}^2$ containing a saddle-saddle connection. The set S consisting of the two saddles and the connection between them is an isolated invariant set, hence the Conley index is well defined for S. Considering the index pair (N, L) illustrated in Figure 1.1, one has that the homotopy Conley index of S is the wedge sum of two pointed one-spheres, i.e. $I(S) = \sum^1 \vee \sum^1$.

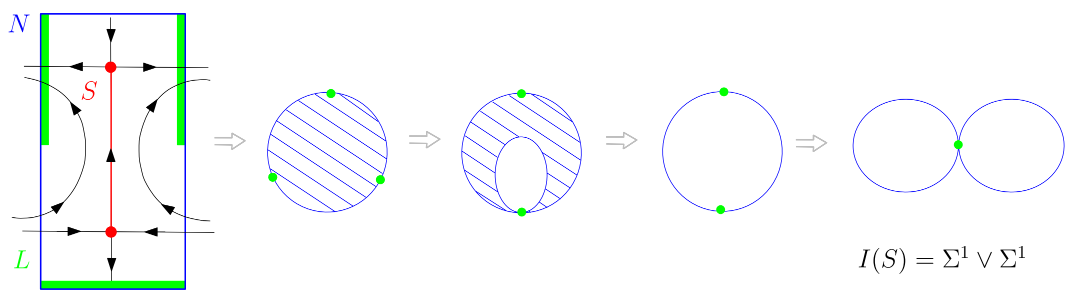


Figure 1.1: Homotopy type of the space N/L.

An index pair (N, L) is called regular if the inclusion map $L \subset N$ is a cofibration. In this case, it follows that $H(N, L) \cong H(N/L)$. An index pair can always be modified to a regular index pair. Since for some algebraic techniques it is more convenient to work with the pair (N, L) than with the pointed space N/L, we assume from now on that the index pairs are regular.

As previously stated, the Conley index generalizes the Morse index, see Section 2.1 for the definition. The Conley index is the homotopy type of a pointed space and is well defined for all isolated invariant sets. In the case of non degenerate critical points, these two notions are related as follows: if x is a singularity with Morse index k then the Conley index of x is the homotopy type of the k-sphere, i.e. $I(x) = \sum^k$. Figure 1.2 illustrates this relation for singularities in $\mathbb{R}^3$.

For more details on Conley index theory see [12, 38].

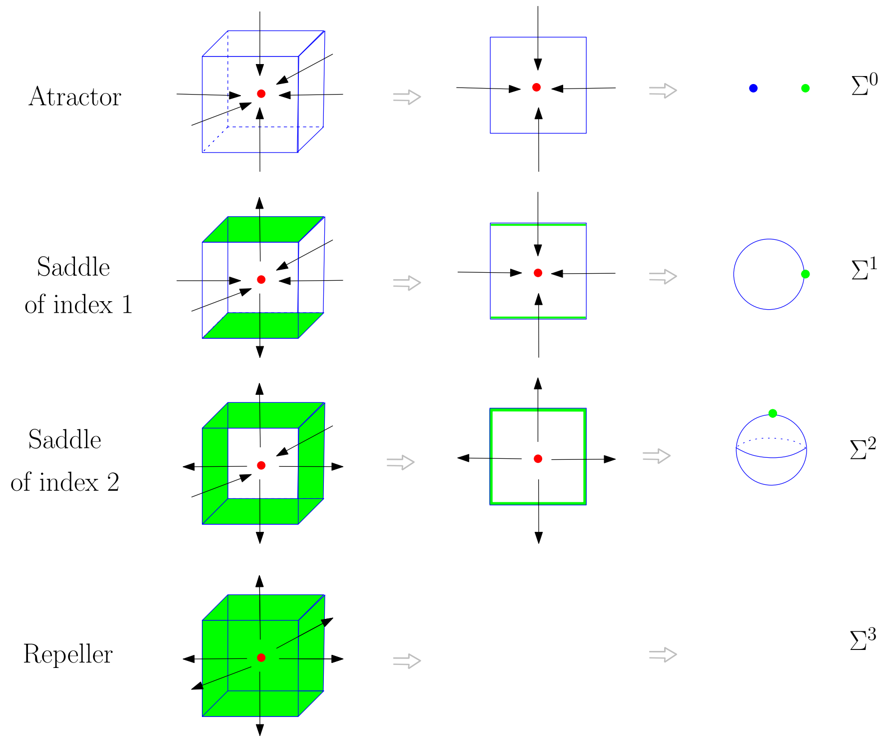


Figure 1.2: Index pairs and Conley indices for singularities in $\mathbb{R}^3$.

1.1.2 Connection Matrix Theory

In this subsection, we present some definitions and results on connection matrices, Morse decompositions and partial orders. Further details can be found in [20] and [21], for example.

Let P be a finite set. A partial order on P is a transitive relation $<$ on the elements of P for which $\pi < \pi$ never holds, for all $\pi \in P$. The pair $(P, <)$ is called a partially ordered set.

An *interval* in $(P, <)$ is a subset $I \subset P$, such that, if $\pi, \pi' \in I$ and $\pi < \pi'' < \pi'$, where $\pi'' \in P$, then $\pi'' \in I$. The set of all intervals in $(P, <)$ is denoted by $\mathcal{I}(P, <)$. Two elements π, π' of P are said to be *adjacent* if $\{\pi, \pi'\} \in \mathcal{I}(P, <)$.

An ordered collection $(I_1, \cdots, I_n)$ of intervals in $(P, <)$ is called an *adjacent n-tuple of intervals* if $\cup_{j=1}^{n} I_j \in \mathcal{I}(P, <)$ and if $\pi \in I_j$ and $\pi' \in I_k$, with $k < j$, implies $\pi \not< \pi'$. The set of all adjacent n-tuple of intervals is denoted by $\mathcal{I}_n(P, <)$. If $(I, J) \in \mathcal{I}_2(P, <)$ then $I \cup J$ is denoted by IJ. An n-tuple $(I_1, \ldots, I_n)$ is called a *decomposition* of an interval I if $(I_1, \ldots, I_n) \in \mathcal{I}_n(P, <)$ and $\cup_{j=1}^{n} I_j = I$.

Let Γ be a Hausdorff topological space with a continuous flow and S an isolated invariant set in Γ. A *$<$-ordered Morse decomposition* of S is a collection $\mathcal{D}(S) = \{M_\pi\}_{\pi \in P}$ of mutually disjoint compact invariant subsets of S for which the following property holds: if $\gamma \in S$ does not belong to any M_π with $\pi \in P$, then there must exist $\pi, \pi' \in P$ such that $\pi' < \pi$ and $\gamma \in C(M_\pi, M_{\pi'})$, where $C(M_\pi, M_{\pi'})$ is the set of orbits connecting M_π to $M_{\pi'}$, i.e.,

$$C(M_\pi, M_{\pi'}) = \{x \in S \mid \omega^* \subset M_\pi \text{ and } \omega \subset M_{\pi'}\}.$$

Note that the partial order $<$ on P induces a partial order on $\mathcal{D}(S)$, which is also denoted by $<$ and called an *admissible ordering* of $\mathcal{D}(S)$. The flow defines an admissible ordering $<_F$ of $\mathcal{D}(S)$, defined as follows: $M_\pi <_F M_{\pi'}$ if and only if there exists a sequence $\pi = \pi_0, \pi_1, \ldots, \pi_{n-1}, \pi_n = \pi'$ of elements of P such that $C(M_{\pi_j}, M_{\pi_{j-1}}) \neq \emptyset$, for all $j = 1, \ldots n$. Every admissible ordering of $\mathcal{D}(S)$ is an extension of $<_F$, in other words, all other admissible orders are obtained by adding relations to $<_F$.

For each interval I of $(P, <)$, one can associate the set

$$M_I = \left(\bigcup_{\pi \in I} M_\pi\right) \cup \left(\bigcup_{\pi, \pi \in I} C(M_{\pi'}, M_\pi)\right),$$

which is called a *Morse set* of the admissible ordering $<$. Franzosa proves in [20] that if $(I, J) \in \mathcal{I}_2(P, <)$ then (M_I, M_J) is an attractor-repeller pair in M_{IJ} and that there is a long exact sequence

$$\cdots \longrightarrow CH(M_I) \xrightarrow{i_*} CH(M_{IJ}) \xrightarrow{p_*} CH(M_J) \xrightarrow{\partial_*} CH(M_I) \longrightarrow \cdots \tag{1.1}$$

associated to the pair (M_I, M_J), where $CH(M_I)$ denotes de homology index of M_I. The collection of the homology index $CH(M_I)$, for all $I \in \mathcal{I}(P, <)$, and the maps i_*, p_*, ∂_*, for all pair $(I, J) \in \mathcal{I}_2(P, <)$, is a graded module braid over $<$. This graded module braid is denoted by $\mathcal{H}(<)$ and is called the *homology index braid* of the admissible ordering $<$ of $\mathcal{D}(S)$. Moreover, $\mathcal{H}(<)$ is chain complex generated. See [20] and [21] for more details.

Now, let $C = \{C\Delta(\pi)\}_{\pi \in P}$ be a collection of free chain complexes with trivial boundary operator, where $C\Delta(\pi) = CH(M_\pi)$, for all $\pi \in P$, the homology index of M_π with coefficients in G. A map $\Delta : C\Delta(P) \to C\Delta(P)$ can be viewed as a matrix

$$\Delta = \left(\quad \Delta(\pi', \pi) \quad \right)_{\pi, \pi' \in P}$$

where each entry $\Delta(\pi', \pi)$ is a map of degree -1 from $CH(M_{\pi'})$ to $CH(M_\pi)$. One says that

(a) Δ is strictly upper triangular if $\Delta(\pi', \pi) \neq 0$ implies $\pi < \pi'$;

(b) Δ is a boundary map if each map $\Delta(\pi', \pi)$ is of degree -1 and $\Delta \circ \Delta = 0$;

Let Δ be strictly upper triangular boundary map. Given $(I, J) \in \mathcal{I}_2(P, <)$, let $\Delta(J, I)$ be the map defined by the matrix

$$\Delta(J, I) = \begin{pmatrix} & & \\ & \Delta(\pi', \pi) & \\ & & \end{pmatrix}_{\pi \in I,\ \pi' \in J}$$

and denote the map $\Delta(I, I)$ by $\Delta(I)$. For each $I \in \mathcal{I}(P, <)$, considering the module $C\Delta(I) = \oplus_{\pi \in I} CH(M_\pi)$, $(C\Delta(I), \Delta(I))$ is a chain complex. For each pair of adjacent intervals (I, J), there is a short exact sequence associated to it:

$$0 \longrightarrow C\Delta(I) \overset{i(I,IJ)}{\longrightarrow} C\Delta(IJ) \overset{p(IJ,J)}{\longrightarrow} C\Delta(J) \longrightarrow 0.$$

Passing to homology, one has the long exact sequence

$$\cdots \longrightarrow H\Delta(I) \overset{i_*(I,IJ)}{\longrightarrow} H\Delta(IJ) \overset{p_*(IJ,J)}{\longrightarrow} H\Delta(J) \overset{\Delta_*(J,I)}{\longrightarrow} H\Delta(I) \longrightarrow \cdots ,$$

where $H\Delta(K)$ denotes the homology of the chain complex $(C\Delta(K), \Delta(K))$, the maps $i_*(I, IJ)$, $p_*(IJ, J)$ are induced by the inclusion $i(I, IJ)$ and projection $p(IJ, J)$ maps, respectively. The map $\Delta_*(J, I)$ is induced by $\Delta(J, I)$ as follows: $\Delta_*(J, I)[a] = [\Delta(J, I)a]$. The collection of $H\Delta(I)$, for all $I \in \mathcal{I}(P, <)$, and the maps $i_*(I, IJ)$, $p_*(IJ, J)$, $\Delta_*(J, I)$, for each $(I, J) \in \mathcal{I}_2(P, <)$, is a graded module braid denoted by $\mathcal{H}\Delta$.

A strictly upper triangular boundary map $\Delta : C\Delta(P) \to C\Delta(P)$ is called a *connection matrix* of $\mathcal{H}(<)$ if and only if the graded module braid $\mathcal{H}\Delta$ generated by Δ is isomorphic to $\mathcal{H}(<)$, that is, if there is a collection of isomorphism $\{\theta(I) : H\Delta(I) \to CH(M_I) | I \in \mathcal{I}(P, <)\}$, such that, the following diagram commutes for all pair $(I, J) \in \mathcal{I}_2(P, <)$:

$$\begin{tikzcd}
\cdots \arrow[r, "\Delta_*(J,I)"] & H\Delta(I) \arrow[r, "i_*(I,IJ)"] \arrow[d, "\theta(I)"] & H\Delta(IJ) \arrow[r, "p_*(IJ,J)"] \arrow[d, "\theta(IJ)"] & H\Delta(J) \arrow[r, "\Delta_*(J,I)"] \arrow[d, "\theta(J)"] & H\Delta(I) \arrow[r, "i_*"] \arrow[d, "\theta(I)"] & \cdots \\
\cdots \arrow[r, "\partial(J,I)"] & CH(M_I) \arrow[r, "i_*"] & CH(M_{IJ}) \arrow[r, "p_*"] & CH(M_J) \arrow[r, "\partial(J,I)"] & CH(M_I) \arrow[r, "i_*"] & \cdots
\end{tikzcd}$$

If $<$ is the flow ordering, then Δ is said to be a connection matrix of the Morse decomposition $\mathcal{D}(S)$.

The set $\mathcal{CM}(<)$ of connection matrices of $\mathcal{H}(<)$ is non empty, as Franzosa proved in [21]. Moreover, if $<_1$ and $<_2$ are admissible orderings of $\mathcal{D}(S)$ such that $<_2$ is an extension of $<_1$, then $\mathcal{CM}(<_1) \subset \mathcal{CM}(<_2)$. In particularly, $\mathcal{CM}(<_F) \subset \mathcal{CM}(<)$, for all admissible ordering $<$ of $\mathcal{D}(S)$.

The set $\mathcal{CM}(<)$ provides some dynamical information about the structure of an invariant set S. A well known fact is that if $\Delta \in \mathcal{CM}(<_F)$, π and π' are adjacent in the flow ordering and $\Delta(\pi', \pi) \neq 0$ then $C(M_{\pi'}, M_\pi) \neq \emptyset$.

Note that algebraic properties of Δ put restrictions on the maps $\partial(\pi, \pi')$. Δ can be used to prove the existence of connecting orbits between Morse sets. Moreover, this theory can also be applied to the study of parameterized families of flows, according to the following two approaches: first by studying the stability of connection matrices under perturbations, whenever some stable connecting orbits are identified; and secondly by studying the changes in connection matrices under perturbation, whenever bifurcations are detected, see [22] and [24].

Example 1.1. Consider the flow illustrated in Figure 1.3, where the isolated invariant set S consists of singularities a, b and c and the connections between them. Let $M_1 = \{a\}$, $M_2 = \{b\}$ and $M_3 = \{c\}$. $\mathcal{D}(S) = \{M_1, M_2, M_3\}$ is a $<$-ordered Morse decomposition of S, where $(P, <)$ is the ordered set $P = \{1, 2, 3\}$ with $1 < 2 < 3$. Note that the admissible ordering coincides with the flow order.

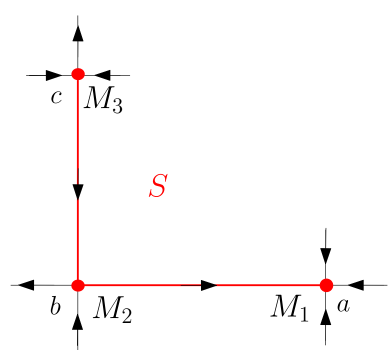


Figure 1.3: S is an isolated invariant set.

The homotopy Conley index of the Morse sets M_I, with $I \in \mathcal{I}(<)$ are $I(M_1) = \Sigma^0$, $I(M_2) = I(M_3) = I(M_{123}) = \Sigma^1$, $I(M_{12}) = \bar{0}$, $I(M_{23}) = \Sigma^1 \vee \Sigma^1$. Consider the module

$G = \mathbb{Z}_2$. Let $\Delta : C\Delta(P) \to C\Delta(P)$ be a strictly upper triangular boundary map

$$\Delta = \begin{pmatrix} 0 & \Delta(2,1) & \Delta(3,1) \\ 0 & 0 & \Delta(3,2) \\ 0 & 0 & 0 \end{pmatrix},$$

where the homomorphisms $\Delta(2,1) : H(M_2) \to H(M_1)$, $\Delta(3,1) : H(M_3) \to H(M_1)$ and $\Delta(3,2) : H(M_3) \to H(M_2)$ are of degree -1. Of course $\Delta(3,2)$ is the null map, since $H(M_3)$ and $H(M_2)$ are non zero only in dimension 1.

Since the Conley indices are computed in this example over $\mathbb{Z}_2$, then $\Delta(2,1)$ is an isomorphism or2 the null map. Analogously for $\Delta(3,1)$.

In order for Δ to be a connection matrix for $\mathcal{D}(S)$, the graded module braid $\mathcal{H}\Delta$ generated by Δ must be isomorphic to the homology index braid. From this fact, one can obtain information about the maps $\Delta(2,1)$ and $\Delta(3,1)$. In this way, the homology of the complexes $(C\Delta(I), \Delta(I))$, where $I \in \mathcal{I}$ and $\Delta(I)$ is the restriction of Δ to the interval I, are:

$$H_n\Delta(i) = \frac{\ker\, \Delta_n(i)}{\mathrm{im}\, \Delta_{n+1}(i)} = \frac{H_n(M_i)}{0} \cong H_n(M_i), \quad \text{for } i = 1, 2, 3,$$

$$H_n\Delta(12) = \frac{\ker\, \Delta_n(2,1)}{\mathrm{im}\, \Delta_{n+1}(2,1)} = \frac{H_n(M_1) \oplus \ker\, \Delta_n(2,1)}{\mathrm{im}\, \Delta_{n+1}(2,1) \oplus 0}, \tag{1.2}$$

$$H_n\Delta(23) = \frac{\ker\, \Delta_n(3,2)}{\mathrm{im}\, \Delta_{n+1}(3,2)} = \frac{H_n(M_2) \oplus H_n(M_3)}{0} \cong H_n(M_2) \oplus H_n(M_3).$$

Observe that the homology of $(C\Delta(I), \Delta(I))$ is isomorphic to the homology Conley index of M_I, for all interval I, except when $I = \{1,3\}$. Hence, one needs to guarantees that $H\Delta(12) \cong H(M_{12})$. Remember that $\Delta(2,1)$ can be a null map or an isomorphism; since (1.2) must hold, then if $\Delta(2,1) = 0$ one has $H\Delta(12) \cong H(M_1) \oplus H(M_2) \ncong H(M_{12})$. But, if $\Delta(2,1)$ is an isomorphism, then $H\Delta(12) \cong H(M_{12})$.

Note that there are no restrictions on the map $\Delta(3,1)$ in order for Δ to be a connection matrix for the Morse decomposition $\mathcal{D}(S)$. Therefore, the connections matrices for this example are the maps

$$\Delta = \begin{pmatrix} 0 & \approx & \approx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} 0 & \approx & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

◀

1.2 Spectral Sequence Sweeping Algorithm

In this section, one presents the main algebraic tools that will be used throughout this thesis. The first one is a spectral sequence associated to a chain complex and the basic references are [15, 41]. The second one is the Spectral Sequence Sweeping algorithm which, when applied to a filtered chain complex (C_*, ∂_*), recovers from it the spectral sequence associated to (C_*, ∂_*). This algorithm was introduced in [13] and in [30] for the case of a chain complex over $\mathbb{Z}$ and $\mathbb{F}$, respectively.

1.2.1 Spectral Sequence for a Chain Complex

Let R be a principal ideal domain. A *k-spectral sequence* E over R is a sequence $\{E^r, \partial^r\}$, for $r \geq k$, such that

1. E^r is a bigraded module over R, i.e., an indexed collection of R-modules $E^r_{p,q}$, for all $p, q \in \mathbb{Z}$;
2. d^r is a differential of bidegree $(-r, r-1)$ on E^r, i.e., an indexed collection of homomorphisms $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, for all $p, q \in \mathbb{Z}$, and $(d^r)^2 = 0$;
3. for all $r \geq k$, there exists an isomorphism $H(E^r) \approx E^{r+1}$, where

$$H_{p,q}(E^r) = \frac{\mathrm{Ker} d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\mathrm{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

Observe that if $E^r_{p,q} = 0$ for a fixed pair of integers (p, q), then $E^{r+a}_{p,q} = 0$, for all integers $a \geq 0$. Moreover, defining $E^r_q = \bigoplus_{s+t=q} E^r_{s,t}$, the differential d^r induces a homomorphism $\partial^r : E^r_q \to E^r_{q-1}$ such that $\{E^r, \partial^r\}$ is a chain complex with q-th homology module equal to $\bigoplus_{s+t=q} H(E)_{s,t}$.

Let $Z^k_{p,q} = Ker(d^k_{p,q} : E^k_{p,q} \to E^k_{p,q-1})$ and $B^k_{p,q} = Im(d^k_{p,q+1} : E^k_{p,q+1} \to E^k_{p,q})$, then $B^k \subseteq Z^k$ and $E^{k+1} = Z^k/B^k$. Now, define $Z(E^{k+1})_{p,q} = Ker(d^{k+1}_{p,q} : E^{k+1}_{p,q} \to E^{k+1}_{p-1,q})$ and $B(E^{k+1})_{p,q} = Im(d^{k+1}_{p+1,q} : E^{k+1}_{p+1,q} \to E^{k+1}_{p,q})$. By the Noether Isomorphism Theorem, there exist bigraded modules Z^{k+1} and B^{k+1} of Z^k containing B^k such that $Z(E^{k+1})_{p,q} = Z^{k+1}_{p,q}/B^k_{p,q}$ and $B(E^{k+1})_{p,q} = B^{k+1}_{p,q}/B^k_{p,q}$, for all $p, q \in \mathbb{Z}$. Hence $B^k \subseteq B^{k+1} \subseteq Z^{k+1} \subseteq Z^k$. By induction, one obtains submodules

$$B^k \subseteq B^{k+1} \subseteq \ldots \subseteq B^r \subseteq \ldots \subseteq Z^r \subseteq \ldots \subseteq Z^{k+1} \subseteq Z^k,$$

such that $E^{r+1} = Z^r/B^r$.

Consider the bigraded modules $Z^\infty = \cap_r Z^r$, $B^\infty = \cup_r B^r$ and $E^\infty = Z^\infty/B^\infty$. The latter module is called the *limit of the spectral sequence*. A spectral sequence $E = \{E^r, \partial^r\}$ is *convergent* if given p, q there is $r(p,q) \geq k$ such that for all $r \geq r(p,q)$, $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is trivial. A spectral sequence $E = \{E^r, \partial^r\}$ is *convergent in the strong sense* if given $p, q \in \mathbb{Z}$ there is $r(p,q) \geq k$ such that $E^r_{p,q} \approx E^\infty_{p,q}$, for all $r \geq r(p,q)$.

Let (C, ∂) be a chain complex. An *increasing filtration* F on (C, ∂) is a sequence of submodules F_pC of C such that:

1. $F_pC \subset F_{p+1}C$, for all integer p;

2. the filtration is compatible with the gradation of C, i.e. F_pC is a chain subcomplex of C consisting of $\{F_pC_q\}$.

Compatible with the gradation

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_{p-1}C_{p+q+1} & \longrightarrow & F_pC_{p+q+1} & \longrightarrow & F_{p+1}\ C_{p+q+1} & \longrightarrow & F_{p+2}\ C_{p+q+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_{p-1}C_{p+q} & \longrightarrow & F_pC_{p+q} & \xrightarrow{i} & F_{p+1}\ C_{p+q} & \longrightarrow & F_{p+2}\ C_{p+q} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \partial & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_{p-1}C_{p+q-1} & \longrightarrow & F_pC_{p+q-1} & \longrightarrow & F_{p+1}\ C_{p+q-1} & \longrightarrow & F_{p+2}\ C_{p+q-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Increasing filtration

A filtration F on C is called *convergent* if $\cap_p F_pC = 0$ and $\cup_p F_pC = C$. It is called *finite* if there are $p, p' \in \mathbb{Z}$ such that $F_pC = 0$ and $F_{p'}C = C$. Also, it is said to be *bounded below* if for any q there is $p(q)$ such that $F_{p(q)}C_q = 0$.

Given a filtration on C, the associated bigraded module $G(C)$ is defined as

$$G(C)_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}.$$

A filtration F on C induces a filtration F on $H_*(C)$ defined by

$$F_pH_*(C) = \text{Im}\ [H_*(F_pC) \to H_*(C)].$$

If the filtration F on C is convergent and bounded below then the same holds for the induced filtration on $H_*(C)$.

The following theorem (see [41]) shows that one can associate a spectral sequence to a filtered chain complex whenever the filtration is convergent and bounded below.

Theorem 1.1. *Let F be a convergent and bounded below filtration on a chain complex C. There is a convergent spectral sequence with*

$$E^0_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} = G(C)_{p,q} \qquad \text{and} \qquad E^1_{p,q} \approx H_{p+q}\left(\frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}\right)$$

and E^∞ is isomorphic to the bigraded module $GH_*(C)$ associated to the induced filtration on $H_*(C)$.

The proof of this theorem provides algebraic formulas for the modules E^r, which are

$$E^r_{p,q} = \frac{Z^r_{p,q}}{Z^{r-1}_{p-1,q+1} + \partial Z^{r-1}_{p+r-1,q-r+2}},$$

where

$$Z^r_{p,q} = \{c \in F_pC_{p+q} \mid \partial c \in F_{p-r}C_{p+q-1}\}.$$

Note that, E^∞ does not determine $H_*(C)$ completely, but

$$E^\infty_{p,q} \approx GH_*(C)_{p,q} = \frac{F_pH_{p+q}(C)}{F_{p-1}H_{p+q}(C)}.$$

However, it is a well known fact [15] that whenever $GH_*(C)_{p,q}$ is free and the filtration is bounded,

$$\bigoplus_{p+q=k} GH_*(C)_{p,q} \approx H_{p+q}(C). \tag{1.3}$$

1.2.2 Spectral Sequence Sweeping Algorithm

In [13], a sweeping algorithm was introduced from which a spectral sequence associated to a finite chain complex over $\mathbb{Z}$ with a special filtration is recovered. More specifically, let (C, ∂) be a finite chain complex such that each module C_k is finite generated. Denote the generators of the C_k chain module by $h^1_k, \cdots, h^{c_k}_k$. One can reorder the set of the generators of C_* as

$$\{h^1_0, \cdots, h^{\ell_0}_0, h^{\ell_0+1}_1, \cdots, h^{\ell_1}_1, \cdots, h^{\ell_{k-1}+1}_k, \cdots, h^{\ell_k}_k, \cdots\},$$

where $\ell_k = c_0 + \cdots + c_k$ [2]. Let F be a finest filtration on C defined by

$$F_pC_k = \bigoplus_{h_k^\ell,\ \ell \leq p+1} \mathbb{Z}\langle h_k^\ell \rangle,$$

for $p \in \mathbb{N}$. The spectral sequence associated to (C_*, ∂_*) with this finest filtration has a special property: the only q for which $E^r_{p,q}$ is non-zero is $q = k - p$, where k is the index of the chain in $F_pC \setminus F_{p-1}C$. Hence, in this case, we omit reference to q. It is understood that E^r_p is in fact $E^r_{p,k-p}$. The sweeping algorithm presented below, provides an alternative way to obtain such modules as well as the differentials d^r's.

For this purpose, we can view the differential boundary map ∂ of the chain complex C as the matrix Δ:

$$\Delta \; = \; \begin{array}{cc}
 & \begin{matrix} C_0 & \cdots & C_{k-1} & C_k & C_{k+1} & C_{k+2} & \cdots & C_n \end{matrix} \\
\begin{matrix} C_0 \\ \vdots \\ C_{k-1} \\ C_k \\ C_{k+1} \\ C_{k+2} \\ \vdots \\ C_n \end{matrix} &
\begin{pmatrix}
0 & & & & & & & \\
 & & & & & & & \\
0 & & \boxed{\Delta_{k-1}} & & & 0 & & \\
0 & & 0 & \boxed{\Delta_k} & & & & \\
0 & & & 0 & \boxed{\Delta_{k+1}} & & & \\
0 & & & & 0 & \boxed{\Delta_{k+2}} & & \\
 & & 0 & & & & & \\
0 & & & & & & & 0
\end{pmatrix}
\end{array}$$

where Δ_k is the map ∂_k and the order of the columns of Δ follows the order determined on the generators of C_*. From now on, the boundary operator ∂ and the matrix Δ will be used interchangeably. Note that the numbering on the columns of Δ is shifted by one with respect to the subindex p of the filtration F_p.

Remember that the term h_k^l denotes an elementary k-chain of the module C_k and this k-chain is associated to the column l of the matrix Δ. Moreover, $F_{l-1} \setminus F_{l-2} = \mathbb{Z}\langle h_k^l \rangle$.

We now present the algorithm which when computed on Δ recovers at each stage the modules and differentials of the spectral sequence $\{E^r, d^r\}$. For more details see Theorems 4.4 and 5.7 in [13].

Spectral Sequence Sweeping Algorithm - SSSA

[2] In order to simplify notation, we use the index f_k to denote the first column of Δ associated to a k-chain. Hence $f_k = \ell_{k-1} + 1$. Moreover, ℓ_k denotes the latter column associated to a k-chain.

For a fixed diagonal r parallel and to the right of the main diagonal, the method described below must be applied simultaneously for all k.

Initial Step.

(1) Let ξ_1 be the first diagonal of Δ that contains non-zero entries $\Delta_{i,j}$ in Δ_k, which will be called index k primary pivots. Define Δ^{ξ_1} to be Δ with the k- index *primary pivots* marked on the ξ_1-th diagonal.

(2) Consider the matrix Δ^{ξ_1}. Let ξ_2 be the first diagonal greater than ξ_1 which contains non-zero entries $\Delta^{\xi_1}_{i,j}$. The construction of Δ^{ξ_2} follows the procedure below. Given a non-zero entry $\Delta^{\xi_1}_{i,j}$ on the ξ_2-th diagonal of Δ^{ξ_1}:
If $\Delta^{\xi_1}_{s,j}$ contains an index k primary pivot for $s > i$, **then** the numerical value of the given entry remains the same, $\Delta^{\xi_2}_{i,j} = \Delta^{\xi_1}_{i,j}$, and the entry is left unmarked.
If $\Delta^{\xi_1}_{s,j}$ does not contain a primary pivot for $s > i$:
then if $\Delta^{\xi_1}_{i,t}$ contains a primary pivot, for $t < j$,
then define $\Delta^{\xi_2}_{i,j} = \Delta^{\xi_1}_{i,j}$ and mark the entry $\Delta^{\xi_2}_{i,j}$ as a *change-of-basis pivot.*
Else, define $\Delta^{\xi_2}_{i,j} = \Delta^{\xi_1}_{i,j}$ and permanently mark $\Delta^{\xi_2}_{i,j}$ as an index k *primary pivot.*

Intermediate Step.

Suppose by induction that Δ^{ξ} is defined for all $\xi \leq r$ with the primary and change-of-basis pivots marked on the diagonals smaller or equal to ξ. In what follows it will be shown how Δ^{r+1} is defined. Without loss of generality, one can assume that there is at least one change-of-basis pivot on the r-th diagonal of Δ^r. If it is not the case, define $\Delta^{r+1} = \Delta^r$ with primary pivots and change-of-basis pivots marked as in step (2) below.

(1) **Change of basis.** Let $\Delta^r_{i,j}$ be a change-of-basis pivot in Δ^r_k. Perform the change of basis on Δ^r by adding a linear combination over $\mathbb{Q}$ of all the h^l_k columns of Δ^r with $\ell < j$ to a positive integer multiple $u \neq 0$ of column j of Δ^r, in order to zero out the entry $\Delta^r_{i,j}$ without introducing non-zero entries in $\Delta^r_{s,j}$ for $s > i$. Moreover, the resulting linear combination should be of the form $\beta^{f_k} h^{\kappa}_k + \cdots + \beta^{j-1} h^{j-1}_k + \beta^j h^j_k$, where f_k is the first column of Δ^r associated to a k-chain and $\beta^\ell \in \mathbb{Z}$, for all $j = \kappa, \cdots, j$.

The integer u is called the leading coefficient of the change of basis. If more than one linear combination is possible, one must choose the one which minimizes u. One can define a matrix T^r which performs all the change of basis on all of the r-th diagonal.

Define $\Delta^{r+1} = (T^r)^{-1}\Delta^r T^r$ and mark the entries of the $(r+1)$-th diagonal of Δ^{r+1} as follows.

(2) **Markup.** Given a non-zero entry $\Delta^{r+1}_{i,j}$ on the $(r+1)$-th diagonal of Δ^{r+1}_k:
If $\Delta^{r+1}_{s,j}$ contains a primary pivot for $s > i$, **then** leave the entry $\Delta^{r+1}_{i,j}$ unmarked.
If $\Delta^{r+1}_{s,j}$ does not contain a primary pivot for $s > i$:
then if $\Delta^r_{i,t}$ contains a primary pivot, for $t < j$,
then mark $\Delta^r_{i,j}$ as a change-of-basis pivot.
Else permanently mark $\Delta^r_{i,j}$ as a primary pivot.

Final Step.

Repeat the above procedure until all diagonals have been considered.

According to the algorithm, if $\Delta^r_{i,j}$ is a change-of-basis pivot on the r-th diagonal of Δ^r_k, then once the corresponding change of basis has been performed, one obtains a new k-chain associated to column j of Δ^{r+1}, which will be denoted by $\sigma_k^{j,r+1}$. Observe that $\sigma_k^{j,r+1}$ is a linear combination over $\mathbb{Q}$ of columns ℓ of Δ^r with $f_k \leq \ell \leq j$ such that $\Delta^{r+1}_{i,j} = 0$, i.e., $\sigma_k^{j,r+1}$ is a linear combination over $\mathbb{Q}$ of $\sigma_k^{f_k,r}, \cdots, \sigma_k^{j,r}$. Also, $\sigma_k^{j,r+1}$ is a linear combination over $\mathbb{Z}$ of the columns $f_k, \cdots, j$ of Δ^r, i.e., of $h_k^{f_k}, \cdots, h_k^j$. Hence,

$$\begin{aligned}
\sigma_k^{j,r+1} &= u \underbrace{\sum_{\ell=f_k}^{j} c_\ell^{j,r} h_k^\ell}_{\sigma_k^{j,r}} + q_{j-1} \underbrace{\sum_{\ell=f_k}^{j-1} c_\ell^{j-1,r} h_k^\ell}_{\sigma_k^{j-1,r}} + \cdots \\
&\quad + q_{f_k+1} \underbrace{c_{f_k}^{f_k+1,r} h_k^{f_k} + c_{f_k+1}^{f_k+1,r} h_k^{f_k+1}}_{\sigma_k^{f_k+1,r}} + q_{f_k} \underbrace{c_{f_k}^{f_k,r} h_k^{f_k}}_{\sigma_k^{f_k,r}} \qquad (1.4) \\
&= \mathsf{c}_\mathsf{k}^{\mathsf{j},\mathsf{r}+1} h_k^j + \mathsf{c}_\mathsf{k}^{\mathsf{j}-1,\mathsf{r}+1} h_k^{j-1} + \cdots + \mathsf{c}_\mathsf{k}^{\mathsf{f_k},\mathsf{r}+1} h_k^{f_k}
\end{aligned}$$

where $\mathsf{c}_\mathsf{k}^{\ell,\mathsf{r}+1} \in \mathbb{Z}$, for $\ell = f_k, \cdots, j$. If Δ^r contains an index k primary pivot in the entry $\Delta^r_{s,\bar{\ell}}$ with $s > i$ and $\bar{\ell} < j$, then $q_{\bar{\ell}} = 0$. Of course, the first column of any Δ_k cannot undergo changes of basis, since there is no column to its left associated to a k-chain.

The family of matrices $\{\Delta^r\}$ produced by the Spectral Sequence Sweeping Algorithm has several properties, which are proven in [13, 30], such as:

(a) Δ^r is a strictly upper triangular boundary map, for each r.

(b) It is not possible to have more than one primary pivot in a fixed row or column.

(c) If the entry $\Delta^r_{j-r,j}$ is a primary pivot or a change-of-basis pivot, then $\Delta^r_{s,j} = 0$ for all $s > j - r$.

(d) If $\Delta^r_{j-r,j}$ is a primary or a change-of-basis pivot, then $\Delta^r_{j-r,j}$ is an integer.

(e) Let Δ^L be the last matrix produced by the SSSA. Then, the primary pivots are non-null and each non-null entry is located above a unique primary pivot.

(f) If column j of Δ^L is non-null, then row j is null. The matrix Δ^L is integral.

Example 1.2. To illustrate the SSSA over $\mathbb{Z}$, consider the graded group C_* defined by $C_0 = \mathbb{Z}\langle h_0^1\rangle$, $C_2 = \mathbb{Z}\langle h_2^2\rangle$, $C_3 = \mathbb{Z}\langle h_3^3\rangle \oplus \mathbb{Z}\langle h_3^4\rangle \oplus \mathbb{Z}\langle h_3^5\rangle$, $C_4 = \mathbb{Z}\langle h_4^6\rangle \oplus \mathbb{Z}\langle h_4^7\rangle$, $C_6 = \mathbb{Z}\langle h_6^8\rangle$ and $C_k = 0$, for $k \in \mathbb{N} \setminus \{0, 2, 3, 4, 6\}$. Also, consider the differential $\partial_k : C_k \to C_{k-1}$, defined on the generators of C_k by $\partial_3(h_3^3) = 5h_2^2$, $\partial_3(h_3^4) = 3h_2^2$, $\partial_3(h_3^5) = h_2^2$, $\partial_4(h_4^6) = 2h_3^3 - 4h_3^4 + 2h_3^5$, $\partial_4(h_4^7) = 1h_3^3 - 5h_3^5$, and the other ∂_k are the null map. The pair (C_*, ∂_*) is a finite chain complex.

Applying the SSSA to this complex, one obtains the sequence of matrices Δ^r shown in Figures 1.4 to 1.9. In these figures, the primary pivots entries are indicated by means of a red background and darker edge, the change-of-basis pivots are indicated by blue background and dashed edges, null entries are left blank and the diagonal being swept is indicated with a gray line. ◀

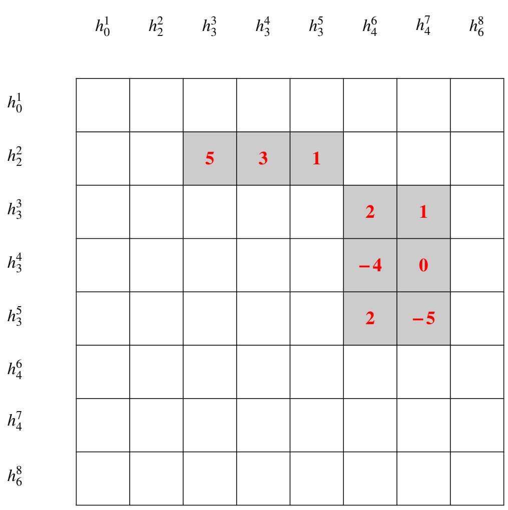


Figure 1.4: Initial matrix Δ.

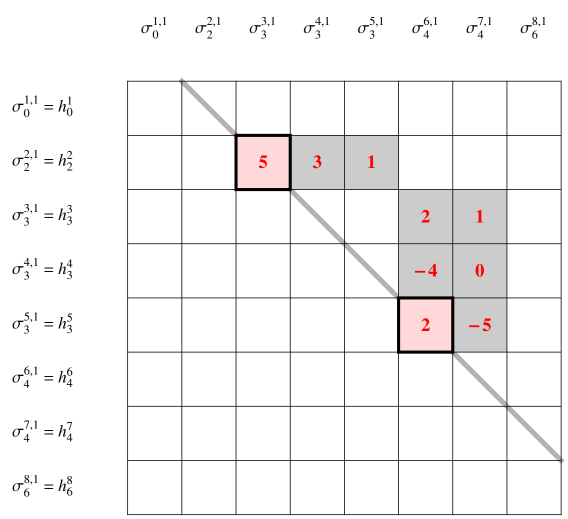


Figure 1.5: Δ^1; marking primary pivots.

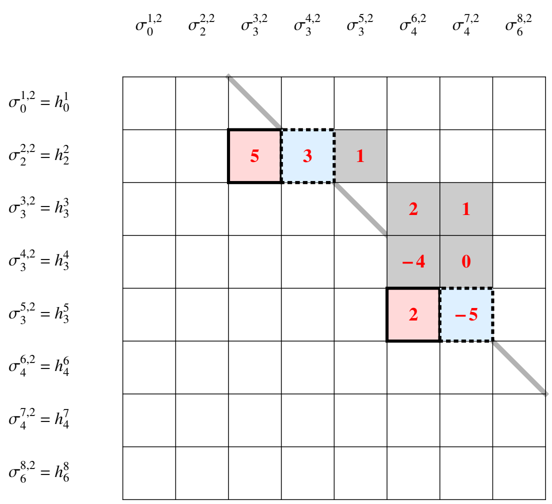


Figure 1.6: Δ^2; marking pivots.

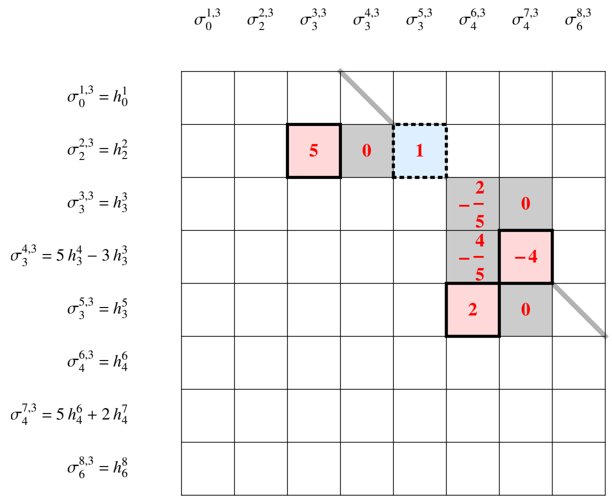


Figure 1.7: Δ^3; marking pivots.

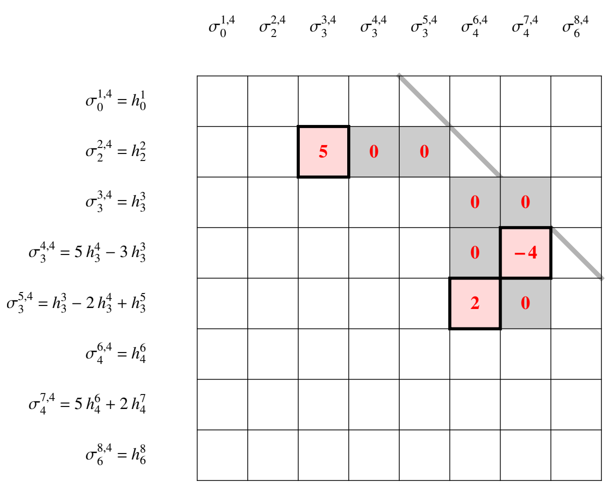


Figure 1.8: Δ^4; sweep 4-th diagonal.

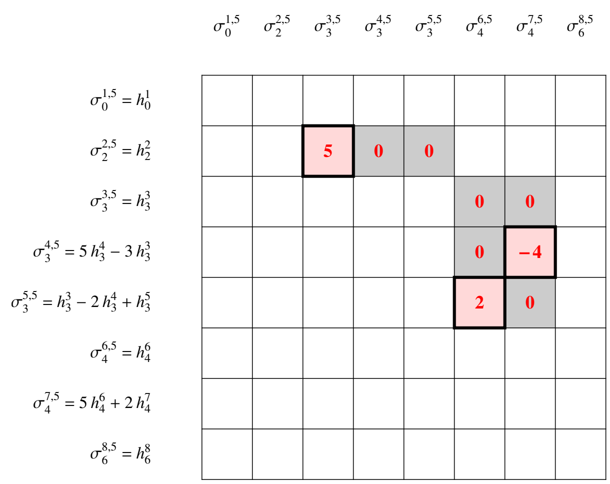


Figure 1.9: Final matrix Δ^5.

In [13] it is proved that the Spectral Sequence Sweeping Algorithm provides a system that spans the modules E^r in terms of the original basis of C_* and identifies all differentials $d_p^r : E_p^r \to E_{p-r}^r$ with primary and change-of -basis pivots on the r-th diagonal. In fact, the matrix Δ^r obtained in the r-th step of the SSSA determines the bigraded $\mathbb{Z}$-module E^r and the differential d^r. The primary and change-of-basis pivots in Δ^r have important roles in determining the generators of Z_p^r. A formula for the module $Z_{p,k-p}^r$ in terms of the chains σ_k's is

$$Z_{p,k-p}^r = \mathbb{Z}\Big[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \cdots, \mu^{f_k,r-p-1+f_k}\sigma_k^{f_k,r-p-1+f_k}\Big], \tag{1.5}$$

where f_k is the first column of Δ associated to a k-chain, and $\mu^{j,\xi} = 0$ whenever there is a primary pivot on column j below row $(p-r+1)$ and $\mu^{j,\xi} = 1$ otherwise. In [13], it is proved that the modules $E^r_{p,k-p}$ are generated by certain σ_k's associated to the SSSA. Moreover, if E^r_p and E^r_{p-r} are both non-zero, then the differential $d^r : E^r_p \to E^r_{p-r}$ is induced by multiplication by $\Delta^r_{p-r+1,p+1}$, whenever this entry is either a primary pivot, change-of-basis pivot or a zero with a column of zero entries below it.

Example 1.3. Returning to Example 1.2, the spectral sequence associated to the chain complex (C_*, ∂_*) is presented below. Since one works with a finest filtration on (C_*, ∂_*), the only q for which $E^r_{p,q}$ is non-zero is $q = k-p$, where k is the index of the chain in $F_pC \setminus F_{p-1}C$. Hence, we omit reference to q and represent the E^r-page of the spectral sequence as a line.

$$\begin{tikzcd}
E^1: & \mathbb{Z}[h_0^1] & \mathbb{Z}[h_2^2] & \mathbb{Z}[h_3^3] \arrow[l, bend right, "d_2^1 = 5"'] & \mathbb{Z}[h_3^4] & \mathbb{Z}[h_3^5] & \mathbb{Z}[h_4^1] \arrow[l, bend right, "d_5^1 = 2"'] & \mathbb{Z}[h_4^7] & \mathbb{Z}[h_6^8] \\
E^2: & \mathbb{Z}[h_0^1] & \mathbb{Z}_5[h_2^2] & 0 & \mathbb{Z}[h_3^4] \arrow[ll, bend right, "d_3^2 = 3"'] & \mathbb{Z}_2[h_3^5] & 0 & \mathbb{Z}[h_4^7] \arrow[ll, bend right, "d_6^2 = -5"'] & \mathbb{Z}[h_6^8] \\
E^3: & \mathbb{Z}[h_0^1] & 0 & 0 & \mathbb{Z}[5h_3^4 - h_3^3] & 0 & 0 & \mathbb{Z}[2h_4^7 + 5h_4^6] \arrow[lll, bend right, "d_6^3 = -4"'] & \mathbb{Z}[h_6^8] \\
E^4: & \mathbb{Z}[h_0^1] & 0 & 0 & \mathbb{Z}[5h_3^4 - h_3^3] & 0 & 0 & 0 & \mathbb{Z}[h_6^8]
\end{tikzcd}$$

The differentials d_2^1, d_5^1 and d_6^3 are induced by the primary pivots $\Delta^1_{2,3}$, $\Delta^1_{5,6}$ and $\Delta^3_{4,7}$. On the other hand, the differentials d_3^2 and d_5^2 are induced by the change-of-basis pivots $\Delta^2_{2,4}$ and $\Delta^2_{5,7}$. ◀

Remark 1.1. Let (C_*, ∂) be a finite chain complex such that each module C_k finite generated by c_k k-chains $\{h_k\}$'s. In this section, the set of generators h_*^j's of C_* is ordered respecting the grading and increasing in j, i.e,

$$\{h_0^1, \cdots, h_0^{\ell_0}, h_1^{\ell_0+1}, \cdots, h_1^{\ell_1}, \cdots, h_k^{\ell_{k-1}+1}, \cdots, h_k^{\ell_k}, \cdots\},$$

where $\ell_k = c_0 + \cdots + c_k$. This order was used to define the filtration F considered in (C_*, ∂), which is given by

$$F_pC_k = \bigoplus_{h_k^\ell,\ \ell \leq p+1} \mathbb{Z}\langle h_k^\ell \rangle.$$

However, this assumption is not necessary, although it simplifies notation. One can freely reorder the generators of C_* by respecting the condition that if ah^i_{k-1} for $a \in \mathbb{Z} \setminus \{0\}$ is a component of $\partial_k(h^j_k)$, then $i < j$. In this case, we also define the filtration as the one which respects the order considered on the set of generators C_*. In this case, the square matrix which represents the boundary operator ∂ will also have its columns ordered accordingly. All the results still holds in this case.

Note the the Spectral Sequence Sweeping Algorithm previously defined is applicable to the differential Δ of a chain complex C_* over $\mathbb{Z}$, with some additional structure, as stated above. With a small adjustment, one can use the same algorithm for chain complexes over a field $\mathbb{F}$. This is accomplish by modifying the step "change of basis" as follows: if $\Delta^r_{i,j}$ is a change-of-basis pivot, perform the change of basis on Δ^r by adding column p multiplied by $-\Delta^r_{i,j}(\Delta^r_{i,p})^{-1}$ to column j of Δ^r, where $\Delta^r_{i,p}$ is a primary pivot. This primary pivot exists, otherwise $\Delta^r_{i,j}$ would not be a change-of-basis pivot and the change of basis is well defined since the operation is done over a field. The properties of the SSSA over $\mathbb{Z}$ also hold for the SSSA over a field $\mathbb{F}$. For more details on SSSA over a field, see [30].

In [24], the dynamical implications of the SSSA over $\mathbb{Z}_2$ was investigated. Therein, the SSSA over $\mathbb{Z}_2$ can also be viewed as a schematic continuation that undergoes bifurcation, through saddle-saddle connections which are coded in the off diagonal non-zero entries of the transition matrices.

1.3 Lyapunov Graph Theory

In this section, the necessary background from Lyapunov Graph Theory is introduced. The references for this section are [6, 7, 8].

A *directed graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set of ordered pairs of V. The set V is the set of vertices and E is the set of edges; an edge e that join the vertices u and v is denoted by $e = (u, v)$. In this thesis, only finite digraphs are considered, that is, V and E are always finite sets.

A *directed semi-graph* G' is a pair of disjoint sets (V', E'), with $V' = V \cup \{\infty\}$ and $E' \subset V' \times V'$. As usual, the elements of V' are called vertices and the elements of E' are called edges. Furthermore the edges of the form (∞, v) and (v, ∞) are called semi-edges.

As the terminology suggests, we do not usually think of a directed graph as an ordered pair of sets, but as a collection of vertices some of which are joined by edges. Unless it is explicitly stated otherwise, throughout the remainder of this thesis the term "graph" will be used to mean a directed graph.

Let v be a vertex of a graph, then the number of incoming edges of v is called the *indegree* of v and is denoted by $e^+(v)$, the number of outgoing edges of v is called *outdegree* of v and is denoted by $e^-(v)$, and the sum of the indegree and the outdegree of v is called the *degree* of v and is denoted by $e(v)$.

A *path* between two vertices v_0 and v_n in a graph is an alternating sequence of vertices and edges, $v_0, e_1, v_1, \ldots, v_{n-1}, e_n, v_n$ such that, for each $i = 1, \ldots, n$, one has $e_i = (v_i, v_{i-1})$ or $e_i = (v_{i-1}, v_i)$. A graph is *connected* if there is a path between any two vertices in the graph. A path is *oriented* if for each i, $e_i = (v_{i-1}, v_i)$. A path is a *cycle* if the edges are distinct and $v_0 = v_n$.

Given a continuous flow $\phi_t : M \to M$ on a closed n-manifold M, there exists a continuous function $f : M \to \mathbb{R}$ which decreases along the orbits outside the chain recurrent set R of ϕ_t, i.e., if $x \notin R$ then $f(\phi_t(x)) < f(\phi_s(x))$ if $t > s$, and it is constant on the chain recurrent components of R. See [12, 19].

Given a Lyapunov function $f : M \to \mathbb{R}$, consider the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. Consider the quotient $M/\sim_f$. This space determine a graph L when a point of $x \in M/\sim_f$ is identified as a vertex of L if and only if the level set $\in f^{-1}(c)$ containing x also contains a chain recurrent component; and all the other points of $M/\sim_f$ are identified as edges of L. The graph L is called the *Lyapunov graph* of f. See Figure 1.10.

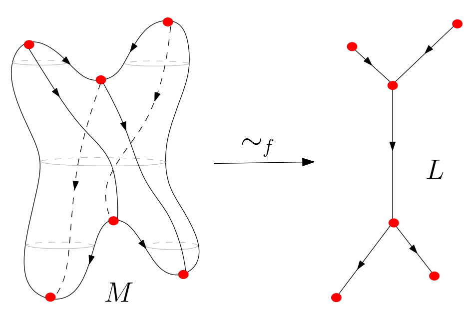


Figure 1.10: Lyapunov graph.

In addition, L can be oriented according to the gradient-like flow and hence it has no oriented cycles. Moreover, since each vertex represent a component of R, it can be labelled with dynamical invariants component as in [6]. On the other hand, since each edge represents a level set times an interval, it can be labelled with topological invariants of the level sets.

Since we consider only flows admitting a finite number of chain recurrent components, we obtain a graph with a finite number of vertices. One can do the same process when the underlying manifold has boundary. In this case, we obtain a Lyapunov semi-graph.

An *n-abstract Lyapunov graph* is a finite and connected directed graph Γ with no oriented cycles such that each vertex v of Γ is labelled with a list of non-negative integers, denoted herein by $\{h_0(v), \cdots, h_n(v), \kappa_v\}$, and each incoming (outgoing) edge of v is labelled with a list $\{\beta_0^+ = 1, \beta_1^+, \cdots, \beta_{n-2}^+, \beta_{n-1}^+ = 1\}$ $(\{\beta_0^- = 1, \beta_1^-, \cdots, \beta_{n-2}^-, \beta_{n-1}^- = 1\})$ of non-negative integers satisfying Poincaré duality[3]. The number $\kappa(v)$ is called the *cycle number* of the vertex v.

Let ν be a vertex in an n-abstract Lyapunov graph with $e^+(\nu)$ incoming edges denoted by e_ℓ^+ and labelled with $(\beta_0^{e_\ell^+}, \ldots, \beta_{n-1}^{e_\ell^+})$, for $\ell = 1, \cdots, e^+(\nu)$ and with $e^-(\nu)$ outgoing edges denoted by e_ℓ^- and labelled with $(\beta_0^{e_\ell^-}, \ldots, \beta_{n-1}^{e_\ell^-})$, for $\ell = 1, \cdots, e^-(\nu)$ (see Figure 1.11). Define

$$B_j^-(\nu) = \sum_{\ell=1}^{e^-(\nu)} \beta_j^{e_\ell^-} \quad \text{and} \quad B_j^+(\nu) = \sum_{\ell=1}^{e^+(\nu)} \beta_j^{e_\ell^+}.$$

Given an abstract Lyapunov graph Γ, the *cycle number* κ of Γ is defined as $\kappa = \kappa_L + \kappa_V$, where κ_V is the sum of the cycle numbers of all vertices of Γ and κ_L is the *cycle rank* of Γ, i.e., the maximum number of edges that can be removed without disconnecting Γ. We will denote the Lyapunov graph Γ by $\Gamma(h_0, \cdots, h_n; \kappa)$ where $h_j = \sum_{v_k} h_j(v_k)$ and the sum is over all vertexes of Γ.

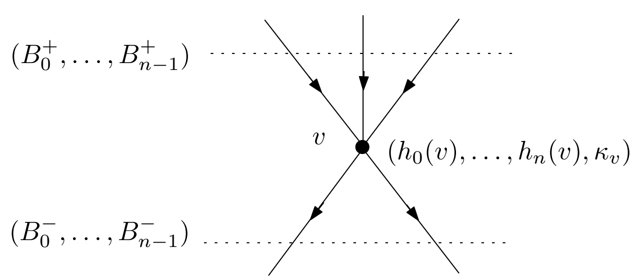


Figure 1.11: A vertex of a Lyapunov graph.

An *n-abstract Lyapunov graph of Morse type* is an n-abstract Lyapunov graph such that

(1) Each vertex v is labelled with $h_j = 1$ for some $j = 0, \cdots, n$ and the cycle number of each vertex is equal to zero.

[3] A list of non-negative integers $\{\beta_0 = 1, \beta_1, \cdots, \beta_{n-2}, \beta_{n-1} = 1\}$ is said to satisfies Poincaré duality if $\beta_j = \beta_{n-1-j}$ for all $j = 0, \cdots, n-1$.

(2) The number of incoming edges $e^+(v)$ and the number of outgoing edges $e^-(v)$ of a vertex v must satisfy:

   (a) if $h_j = 1$ for $j \neq 0, 1, n-1, n$ then $e^+(v) = 1$ and $e^-(v) = 1$;

   (b) if $h_1 = 1$ then $e^+(v) = 1$ and $0 < e^- \leq 2$; if $h_{n-1} = 1$ then $e^-(v) = 1$ and $0 < e^+(v) \leq 2$;

   (c) if $h_0 = 1$ then $e^-(v) = 0$ and $e^+ = 1$; if $h_n = 1$ then $e^+(v) = 0$ and $e^- = 1$. The label in each edge incident to a vertex v label with h_0 or $h_n = 1$ must be $(1, 0, \cdots, 0, 1)$.

(3) Each vertex labelled with $h_l = 1$ must be of type l-disconnecting or $(l-1)$-connecting. If the labels in which incoming and outgoing edges of v satisfy $B_l^- + 1 = B_l^+$ and $B_j^- = B_j^+$ for $j \neq l$, then the vertex v is of type l-disconnecting. If the labels in which incoming and outgoing edge of v satisfy $B_{l-1}^- = B_{l-1}^+ - 1$ and $B_j^- = B_j^+$ for $j \neq l-1$, then the vertex v is of type $(l-1)$-connecting. Furthermore, if $n = 2i \equiv 0 \pmod 4$ and $h_i = 1$ then, v may be labelled with β_{inv} (β-invariant) in the case $B_j^- = B_j^+$ for all j. See Figure 1.12.

Note that the cycle number of an abstract Lyapunov graph of Morse type is equal to its cycle rank.

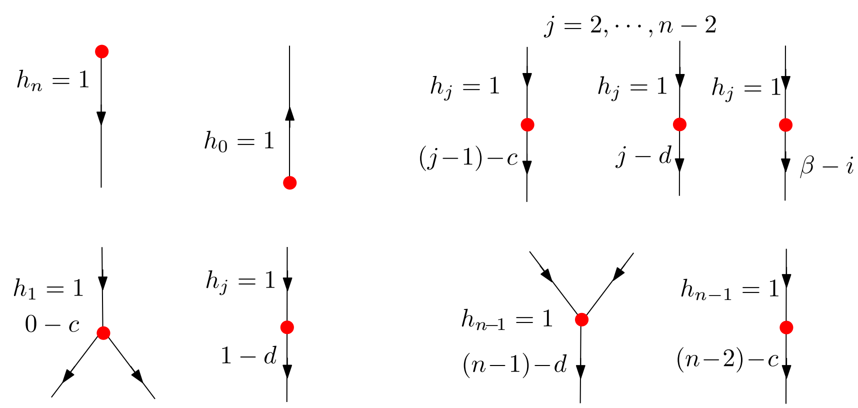


Figure 1.12: Local conditions on an abstact Lyapunov graph of Morse type.

Similarly, one can define abstract Lyapunov semi-graph and abstract Lyapunov semi-graph of Morse type.

In order to define continuation of abstract Lyapunov graphs, we will introduce the notion of *vertex explosion.* A vertex v labelled with $(h_0(v), \ldots, h_n(v); \kappa_v)$ in an n-abstract Lyapunov graph Γ can be *exploded* if v can be removed from Γ and replaced by an n-abstract Lyapunov semi-graph of Morse type Γ_M with cycle rank greater than or equal to κ_v and with the labels on incoming (outgoing) semi-edges of Γ_M matching the labels on the outgoing (incoming) semi-edges of $\Gamma\backslash\{v\}$. Moreover, for all η,

$$h_\eta(v) = \sum_j h_{\eta+\lambda_i}(v_j),$$

where the sum is over all vertices v_j of Γ_M.

An abstract Lyapunov graph Γ admits a continuation to an abstract Lyapunov graph of Morse type Γ_M if each vertex can be exploded such that Γ_M has cycle rank greater or equal to κ.

In [6] and [7] the *Poincaré-Hopf inequalites were introduced*:

$$\begin{cases} h_j & \geq & -(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + \cdots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\ & & -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) \pm \cdots \pm (h_{n-1} - h_1) \\ & & \pm[(h_n - h_0) + (e^+ - e^-)] \\ & & \\ h_{n-j} & \geq & -[-(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + \cdots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \\ & & -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) \pm \cdots \pm (h_{n-1} - h_1) \\ & & \pm[(h_n - h_0) + (e^+ - e^-)]] \end{cases}$$

$$\vdots$$

$$\begin{cases} h_2 & \geq & -(B^+_1 - B^-_1) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \\ h_{n-2} & \geq & -[-(B^+_1 - B^-_1) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-)] \end{cases}$$

$$\begin{cases} h_1 & \geq & h_0 - 1 + e^- + \kappa_v \\ h_{n-1} & \geq & h_n - 1 + e^+ + \kappa_v \end{cases}$$

where $0 \leq j \leq n$.

$$\begin{cases} \text{If } n = 2i+1, \text{ then} \quad \mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \displaystyle\sum_{j=0}^{2i+1} (-1)^j h_j, \\ \text{where } \mathcal{B}^+ = \dfrac{(-1)^i}{2} B^+_i \pm B^+_{i-1} \pm \cdots - B^+_1, \qquad \mathcal{B}^- = \dfrac{(-1)^i}{2} B^-_i \pm B^-_{i-1} \pm \cdots - B^-_1. \\ \\ \text{If } n = 2i \equiv 2 (\text{mod } 4), \quad \text{then} \\ \qquad h_i - \displaystyle\sum_{j=1}^{i-1} (-1)^{j+1} (B^+_j - B^-_j) - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) + (e^+ - e^-) \text{ must be even.} \end{cases}$$

The main Theorem of [6, 7] establishes that: an abstract Lyapunov graph $\Gamma(h_0, \cdots, h_n; \kappa)$ admits a continuation to an abstract Lyapunov graph of Morse type with cycle rank greater or equal to κ if and only if it satisfies the Poincaré-Hopf inequalities at each vertex, where $\kappa \leq min\{h_1 - (h_0 - 1), h_{n-1} - (h_n - 1)\}$. Of course, there exists an analogous result for Lyapunov semi-graphs.

Example 1.4. Consider the 7-abstract Lyapunov graph Γ in Figure 1.13. Recall that the label on each edge is a list of seven non-negative integers satisfying Poincaré duality $(\beta_0, \cdots, \beta_6)$, hence $\beta_0 = \beta_7 = 1$ and $\beta_j = \beta_{6-j}$. Therefore, in Figures 1.13 and 1.14, the label on each edge is $(\beta_1, \beta_2, \beta_3)$. Moreover, if the label of a vertex contains many zeros, e.g. $(0, 1, 0, 0, 0, 0, 0, 0, \kappa = 0)$, we adopt the alternative notation $h_1 = 1$. Figure 1.14 contains two possible continuations for the Lyapunov graph Γ.

Questions regarding the realization of abstract Lyapunov graphs were investigated in [9, 10].

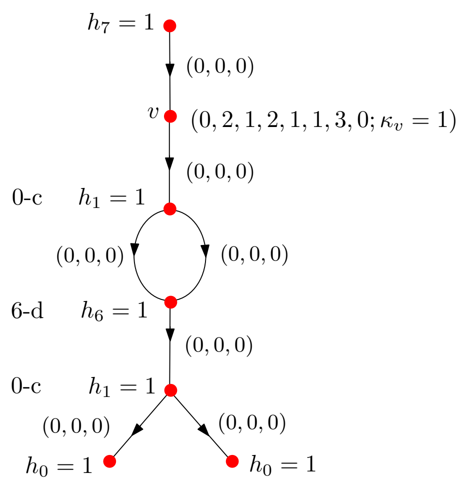


Figure 1.13: Abstract Lyapunov graph $\Gamma(2, 4, 1, 2, 1, 1, 4, 1; \kappa = 2)$.

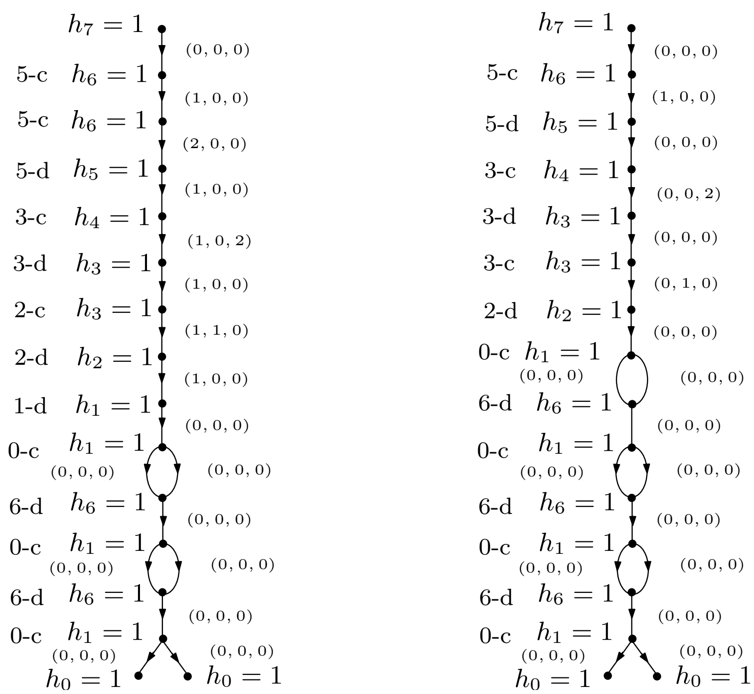


Figure 1.14: Continuations of the graph Γ in Figure 1.13, $\Gamma_1(2,4,1,2,1,1,4,1;\kappa = 2)$ and $\Gamma_2(2,4,1,2,1,1,4,1;\kappa = 3)$, respectively.

Chapter 2

Spectral Sequences for two dimensional Morse Complexes

The computation of a spectral sequence of a filtered Morse chain complex (C, ∂) developed in [13] led to the question of how closely the dynamics follows the spectral sequence. More specifically, the Spectral Sequence Sweeping Algorithm defined in [13] produces a sequence of connection matrices starting with the matrix of the boundary operator ∂, from which the modules and differentials (E^r, d^r) of the spectral sequence may be retrieved. As one "turns the pages" of the spectral sequence, i.e. considers progressively modules E^r, one observes algebraic cancellation occurring within the E^r's.

In this chapter, we wish to understand the dynamical meaning of these algebraic cancellations. In other words, given the dynamics that has been converted to a filtered chain complex description, what homological conclusions can be drawn from the spectral sequence? On the other hand, how much dynamical information can be recovered or even gained from the spectral sequence analysis? Herein, we answer these questions in the setting of flows on smooth closed 2-dimensional orientable manifolds.

In Section 2.1, an extremely concise background on Morse Theory, designed for our goals, is presented. See [3, 39, 42] for more details.

In Section 2.2, one finds the analysis of the relation between the Morse differentials associated to a Morse function on a manifold M when the set of orientations of the unstable manifolds undergo changes. This analysis makes it possible to characterize the set of connection matrices for a Morse flow on a surface.

In Section 2.3, one finds the investigation of the implications that the characterization of the connection matrices for orientable surfaces has on the SSSA. Several properties of this algorithm are proved therein which have homological implications for the spectral sequence. More specifically, one proves that all the differentials d^r in the spectral sequence are isomorphisms and hence torsion does not appear in the modules E^r. We also keep track of all the algebraic cancellations.

A new algorithm, called Smale's Cancellation Sweeping Algorithm (SCSA), is defined in Section 2.4. Moreover, one proves that the primary pivots identified in the r-th step of the SSSA coincide with the primary pivots identified in the r-th step of the SCSA.

In Section 2.5, the algebraic cancellations of the spectral sequence of the filtered Morse chain complex are dynamically interpreted as the history of birth and death of connecting orbits of φ_f caused by the cancellation of consecutive critical points. Theorem 2.4 associates the algebraic cancellation in the spectral sequence to the dynamical cancellation of critical points in the flow via the Spectral Sequence Sweeping Algorithm. Theorem 2.3 constructs a family of flows associated to the spectral sequence which also defines a continuation to the minimal flow.

2.1 Morse Chain Complex

Let M be a smooth manifold of finite dimension n and $f : M \to \mathbb{R}$ a smooth function. A point $p \in M$ is a *critical point* of f if df_p is the null map. In this case $f(p)$ is a critical value. The set of all critical points of f is denoted by $Crit(f)$. A critical point p is said to be *nondegenerate* if the matrix of second partial derivatives at p (the Hessian matrix H_p^f) is non-singular. Otherwise, p is a degenerate critical point. Nondegenerate critical points are isolated.

A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if each critical point of f is nondegenerate. The Morse index of a critical point p is the dimension of the maximal subspace where H_p^f is negative definite, and it will be denoted by $ind_f(p)$. The set of Morse functions on a manifold M is dense in $C^1(M, \mathbb{R})$. Moreover, if M is a closed manifold, then the set of critical points of a Morse function is finite.

Let $f : M \to \mathbb{R}$ be a smooth function on a smooth closed Riemannian manifold M of dimension n. Fix a Riemannian metric g on M. The identity

$$g(\nabla(f), \cdot) = df(\cdot)$$

uniquely determines a gradient vector field $\nabla(f)$ on M. Denote the flow associated to $-\nabla(f)$ by φ_f, which is called of *negative gradient flow.* To simplify notation, we write φ for φ_f whenever emphasis of f need not be given. The singularities of the vector field $-\nabla(f)$ corresponds to the critical points of f.

The negative gradient flow φ_f has special properties when f is a Morse function such as:

(1) Given $x \in Crit(f)$, the Morse index of x corresponds to the dimension of the unstable manifold of φ_f at x, $W^u(x)$.

(2) The function f decreases along nonsingular orbits of φ_f and possesses no closed orbits.

(3) Each regular orbit intersects a regular level set at most once and this intersection is orthogonal to the level set with respect to the metric g.

(4) Given $x \in M$ such that $x \notin Crit(f)$, $\omega(x)$ and $\alpha(x)$ consist of one singularity of φ_f.

These properties are well known in Morse Theory. More details can be found in [3] and [31].

Given $x, y \in Crit(f)$, the *connecting manifold* of x and y is given by

$$\mathcal{M}_{xy} := W^u(x) \cap W^s(y).$$

The connecting manifold $\mathcal{M}_{xy}$ is the set containing all points $p \in M$ such that $\omega(p) = y$ and $\alpha(p) = x$. The *moduli space* between x and y is defined by

$$\mathcal{M}_y^x(a) := \mathcal{M}_{xy} \cap f^{-1}(a),$$

where a is a regular value between $f(x)$ and $f(y)$. See Figure 2.1.

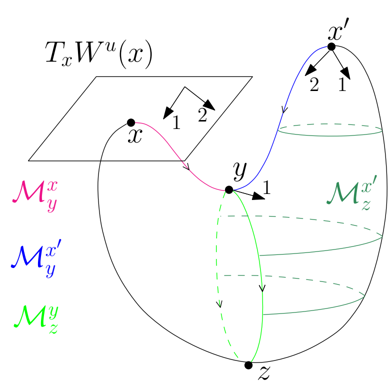


Figure 2.1: Connecting manifolds in a negative gradient flow on a sphere.

The space $\mathcal{M}_y^x(a)$ is the set of all orbits running from x to y. For different choices of regular values a_1, a_2 there is a natural identification between $\mathcal{M}_y^x(a_1)$ and $\mathcal{M}_y^x(a_1)$ given by the flow. Hence, one use the notation $\mathcal{M}_y^x$ for the moduli space.

A Morse function f is called a *Morse-Smale function* if, for each $x, y \in Crit(f)$, the unstable manifold of φ_f at x, $W^u(x)$, and the stable manifold of φ_f at y, $W^s(y)$, intersect transversally.

Whenever f is a Morse-Smale function, the connecting manifolds and the moduli spaces are closed submanifolds of M. Moreover, one has that their dimensions are given by

$$\dim(\mathcal{M}_{xy}) = ind_f(x) - ind_f(y), \quad \dim(\mathcal{M}_y^x) = ind_f(x) - ind_f(y) - 1.$$

Hereafter, in this chapter, assume that f is a Morse-Smale function, unless stated otherwise. In this case, the negative gradient flow φ_f is also called *Morse flow*.

Given $x, y \in Crit(f)$, the connecting manifold $\mathcal{M}_{xy}$ and the moduli space $\mathcal{M}_y^x$ are orientable manifolds. This follows since, once orientations are chosen for $W^u(x)$ and $W^u(y)$, these induce an orientation on $\mathcal{M}_{xy}$ denoted by $[\mathcal{M}_{xy}]_{ind}$. The procedure given by Weber in [42] to obtain this orientation is:

(1) If $ind_f(y) > 0$, then

(a) Let $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$ be the normal bundle of $W^s(y)$ restricted to $\mathcal{M}_{xy}$. Consider the fiber $\mathcal{V}_y W^s(y)$ with an orientation given by the isomorphism

$$T_y W^u(y) \oplus T_y W^s(y) \simeq T_y M \simeq \mathcal{V}_y W^s(y) \oplus T_y W^s(y).$$

The orientation on the fiber at y determines an orientation on the normal bundle $\mathcal{V}_{\mathcal{M}_{xy}} W^s(y)$ restricted to the submanifold $\mathcal{M}_{xy}$.

(b) The orientation on $\mathcal{M}_{xy}$ is determined by the isomorphism

$$T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy} \oplus \mathcal{V}_{\mathcal{M}_{xy}} W^s(y). \tag{2.1}$$

(2) If $ind_f(y) = 0$, then $\mathcal{V}_y W^s(y) = 0$. Hence, $T_{\mathcal{M}_{xy}} W^u(x) \simeq T\mathcal{M}_{xy}$.

Note that there are no restrictions about the orientability of the manifold M.

Example 2.1. Consider the Morse flow on S^2 and the orientations on the unstable manifolds illustrated in Figure 2.1. In Figure 2.2 we use a planar representation of the flow in order to obtain the induced orientations on the connecting manifolds. We also represent these orientations in Figure 2.3. ◀

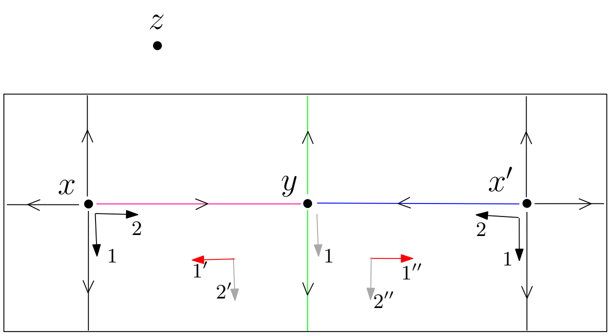


Figure 2.2: Planar representation of the flow in S^2.

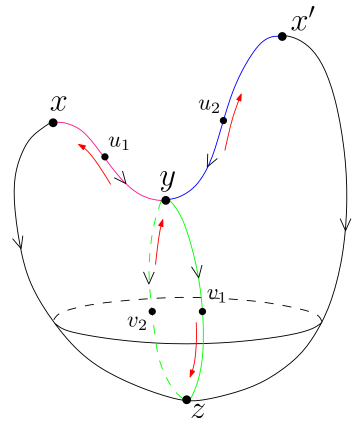


Figure 2.3: Induced orientations on the connecting manifolds.

Given $x, y \in Crit(f)$ with $ind_f(x) - ind_f(y) = 1$, let $u \in \mathcal{M}_y^x$. The *characteristic sign* n_u of the orbit $\mathcal{O}(u)$ through u is defined via the identity $[\mathcal{O}(u)]_{ind} = n_u[\dot{u}]$, where $[\dot{u}]$ and $[\mathcal{O}(u)]_{ind}$ denote the orientations on $\mathcal{O}(u)$ induced by the flow and by $\mathcal{M}_{xy}$, respectively. The *intersection number* of x and y is defined by

$$n(x, y) = \sum_{u \in \mathcal{M}_y^x} n_u.$$

The intersection number between x and y counts the flow lines form x to y with sign. In the literature there are other ways to count such flow lines with orientations, for example, see [3].

Fix an arbitrary orientation for the unstable manifolds $W^u(x)$, for each $x \in Crit(f)$, and denote by Or the set of these choices. Denote by $\langle x \rangle$ the pair consisting of the critical point x of f and the orientation chosen on $W^u(x)$.

The *Morse graded group* $C = \{C_k(f)\}$ is defined as the free abelian groups generated by the critical points of f and graded by their Morse index, i.e.

$$C_k(f) := \bigoplus_{x \in Crit_k(f)} \mathbb{Z}\langle x \rangle.$$

The *Morse boundary operator* $\partial_k(x) : C_k(f) \longrightarrow C_{k-1}(f)$ is given on a generator x of $C_k(f)$ by

$$\partial_k\langle x\rangle := \sum_{y\in Crit_{k-1}(f)} n(x,y)\langle y\rangle, \tag{2.2}$$

and it is extended by linearity to general chains.

The pair $(C_*(f), \partial_*)$ is a chain complex, that is, ∂ is of degree -1 and $\partial \circ \partial = 0$. This chain complex is called a *Morse chain complex*. Observe that $(C_*(f), \partial_*)$ depends on the function f on M, the metric g and the set Or of orientations. The *Morse homology groups* with integer coefficients are defined by

$$HM_k(M, f, g, Or; \mathbb{Z}) = \frac{\text{Ker } \partial_k}{\text{Im } \partial_{k+1}}, \quad \forall k \in \mathbb{Z}.$$

In [42], it was proved that, for two choice of Morse-Smale pairs (f^1, g^1) and (f^2, g^2) and orientations Or^1 and Or^2 of all unstable manifolds, the associated Morse homology groups $HM_k(M, f^1, g^1, Or^1; \mathbb{Z})$ and $HM_k(M, f^1, g^1, Or^1; \mathbb{Z})$ are naturally isomorphic for all $k \in \mathbb{Z}$. Hence, this homology will be denoted by $HM_*(M, \mathbb{Z})$. Moreover, one has that

$$HM_*(M; \mathbb{Z}) \cong H^{sing}(M; \mathbb{Z}),$$

i.e., the Morse homology of M is isomorphic to the singular homology of M.

Example 2.2. Returning to Example 2.1, the Morse chain groups are $C_2(f) = \mathbb{Z}\langle x\rangle \oplus \mathbb{Z}\langle x'\rangle$, $C_1(f) = \mathbb{Z}\langle y\rangle$, $C_2(f) = \mathbb{Z}\langle z\rangle$ and $C_k(f) = 0$ for all $k \neq 0, 1, 2$. From Figure 2.3, the characteristic signs of orbits connecting consecutive critical points are: $n_{u_1} = -1$, $n_{u_2} = -1$, $n_{v_1} = +1$ and $n_{v_2} = -1$; which implies that

$$n(x,y) = n_{u_1} = -1, \quad n(x',y) = n_{u_2} = -1, \quad n(y,z) = n_{v_1} + n_{v_2} = 1 - 1 = 0.$$

The Morse operators $\partial_2 : C_2 \to C_1$, $\partial_1 : C_1 \to C_0$ and $\partial_0 : C_0 \to 0$ are defined on generators by:

$$\partial_2(x) = -\langle y\rangle, \quad \partial_2(x') = -\langle y\rangle, \quad \partial_1(y) = \langle z\rangle - \langle z\rangle, \quad \partial_0(z) = 0.$$

Hence, the integral Morse homology is given by:

$$HM_0(M; \mathbb{Z}) \cong \mathbb{Z}, \quad HM_1(M; \mathbb{Z}) \cong 0, \quad HM_2(M; \mathbb{Z}) \cong \mathbb{Z}. \qquad \blacktriangleleft$$

One can also consider the Morse chain complex over $\mathbb{Z}_2$. In this case, the Morse graded group is defined in the same way. However, the intersection number $n(x, y)$ between critical points x, y with $ind_f(x) - ind_f(y) = 1$ is defined as the number of orbits of φ_f from x to y modulo 2. In this case, $HM_*(M; \mathbb{Z}_2) \cong H^{sing}(M; \mathbb{Z}_2)$.

2.2 Characterization of Surface Connection Matrices

To define the Morse chain complex (C, ∂), a set of orientations Or for the unstable manifold $W^u(x)$ of all critical points $x \in Crit(f)$ is chosen. Given two different set of such orientations, namely Or^1 and Or^2, one has two Morse chain complexes (C^1_*, ∂^1_*) and (C^2_*, ∂^2_*) associated to the Morse function f and to the sets Or^1 and Or^2, respectively. The Morse groups C^1_k and C^2_k may differ on generators $\langle x_k \rangle$ by the choices of orientations of $W^u(x_k)$. In fact, the Morse graded groups C^1_* and C^2_* have the same generators possibly with different signs. However, the relation between the Morse boundary operators ∂^1 and ∂^2 is not immediate to attain. The next proposition provides a relation between the intersection numbers $n^1(x, y)$ and $n^2(x, y)$ obtained considering the sets Or^1 and Or^2, respectively.

Proposition 2.1. *Let M be a smooth closed manifold and $f : M \to \mathbb{R}$ a Morse-Smale function. Consider two sets Or^1 and Or^2 of orientations for $W^u(x)$, $\forall x \in Crit(f)$ with $ind_f(x) > 0$. Suppose that these sets differ only by orientations of the critical points $a_1, \dots, a_l$, then*

$$\begin{cases} n(x,y) &= -\bar{n}(x,y) \quad \text{if } x = a_j, \text{ for some } j, \text{ and } y \neq a_i \ \forall i; \\ n(x,y) &= -\bar{n}(x,y) \quad \text{if } y = a_j, \text{ for some } j, \text{ and } x \neq a_i \ \forall i; \\ n(x,y) &= \bar{n}(x,y) \quad \text{otherwise.} \end{cases}$$

where $n(x, y)$ and $\bar{n}(x, y)$ denote the intersection number between x and y considering the orientations of Or^1 and Or^2, respectively.

Proof. As $Crit(f) < \infty$, then Or^1 and Or^2 are finite. Hence, it is enough to prove the case when Or^1 and Or^2 do not coincide by only one orientation. In this sense, let p be the only critical point whose orientations ξ_1 and ξ_2 for $W^u(p)$ given by the sets Or^1 and Or^2, respectively, are opposite.

Let k be the index of the critical point p, i.e. $ind_f(p) = k$. The proof is divide in two cases: when $ind_f(p) > 1$ and when p is a saddle of index 1.

• $ind_f(p) = 1$:

Let $u \in \mathcal{M}_y^p$, where $y \in Crit_{k-1}(f)$. Since, $k = 1$, the orientation induced on the orbit $\mathcal{O}(u)$ is given directly by the pre-set orientation of $W^u(x)$. By hypothesis, ξ_1 and ξ_2 are opposite, hence they induce opposite orientations in $\mathcal{O}(u)$. Therefore, one of them coincides with the orientation given by the flow and the other is opposite to this orientation, i.e, $n_u = 1$ and $\bar{n}_u = -1$ or $n_u = -1$ and $\bar{n}_u = +1$. Since u was taken arbitrarily, it follows that $n(p, y) = -\bar{n}(p, y)$.

Now, let $v \in \mathcal{M}_p^x$, where $x \in Crit_{k+1}(f)$. The orientation induced by Or^1 and Or^2 on the orbit $\mathcal{O}(v)$ is given by the isomorphism:

$$T_{\mathcal{O}(v)}W^u(x) \approx T\mathcal{O}(v) \oplus \mathcal{V}_{\mathcal{O}(v)}W^s(p). \tag{2.3}$$

By hypothesis, the orientations of $T_{\mathcal{O}(v)}W^u(x)$ given by Or^1 and Or^2 coincide. However, these sets induce opposite orientations ξ_1 and ξ_2 on the unstable manifold $W^u(p)$. Consequently, they induce opposite orientations on the normal bundle $\mathcal{V}_{\mathcal{O}(v)}W^s(p)$, since the orientation induced in this bundle is compatible with the orientation on $W^u(p)$. In this way, the isomorphism given in (2.3) guarantees that the orientations induced on $\mathcal{O}(v)$ are also opposite. Since v is arbitrary, $n(x, p) = -\bar{n}(x, p)$.

• $ind_f(p) > 1$:

Let $u \in \mathcal{M}_y^p$, where $y \in Crit_{k-1}(f)$. The orientation induced on the orbit $\mathcal{O}(u)$, is given by the isomorphism:

$$T_{\mathcal{O}(u)}W^u(p) \approx T\mathcal{O}(u) \oplus \mathcal{V}_{\mathcal{O}(u)}W^s(y). \tag{2.4}$$

The orientations given by Or^1 and Or^2 on $W^u(y)$ are the same, hence these two sets induce the same orientation on the bundle $\mathcal{V}_{\mathcal{O}(u)}W^s(y)$. On the other hand, the orientations ξ_1 and ξ_2 on $T_{\mathcal{O}(u)}W^u(p)$ given by Or^1 and Or^2 are opposite. From these observations and the isomorphism (2.4) it follows that the induced orientations on the orbit $\mathcal{O}(u)$ by Or^1 and Or^2 are opposites. Therefore, $n(p, y) = -\bar{n}(p, y)$, since u was chosen arbitrarily.

Given $v \in \mathcal{M}_p^x$, where $v \in Crit_{k+1}(f)$, the proof that $n(x, p) = -\bar{n}(x, p)$ is analogous to the previous case. ■

As immediate consequence of Proposition 2.1, one has that the Morse boundary operators ∂^1 and ∂^2 represent the same operator in different bases.

A result due to Salamon, see [39], establishes a relation between the boundary operator ∂_*

of a Morse chain complex with connection matrices. Considering M as an isolated invariant set relative to the flow φ_f, then the set $\mathcal{D}(M) = \{M_x = \{x\} \mid x \in Crit(f)\}$ is a finest Morse decomposition of M, i.e., each Morse set M_π contains only one singularity of φ_f. Salamon proved in [39] that the boundary operator ∂_* of a Morse chain complex associated to f is a connection matrix of the Morse decomposition $\mathcal{D}(M)$, by identifying the free abelian module $\mathbb{Z}\langle x\rangle$ with the homology index $CH(M_x)$ of M_x. Therefore, the Morse boundary operator, represented as a matrix, is called a connection matrix. For more details, see [3] and [39].

Denote by $\Delta(M, f, Or)$ the connection matrix for $\mathcal{D}(M)$ obtained considering the Morse function f and the set Or of orientations chosen on the unstable manifolds of the singularities of the gradient flow $-\nabla f$. One has the following corollary of Proposition 2.1:

Corollary 2.1. *Let (M, g) be a smooth closed Riemannian manifold and $f : M \to \mathbb{R}$ a Morse-Smale function. If two sets Or^1 and Or^2 of orientations for $W^u(x)$, $\forall x \in Crit(f)$ with $ind_f(x) > 0$, differ only by the orientations of the critical points $a_1, \ldots, a_l$, then the connection matrix $\Delta(M, f, Or^2)$ is obtained from $\Delta(M, f, Or^1)$ by multiplying by -1 the rows and columns corresponding to the points $a_1, \ldots, a_l$. In particular, $\Delta(M, f, Or^1)$ and $\Delta(M, f, Or^2)$ are similar matrices.*

Moreover, an entry in the matrix $\Delta(M, f, Or^1)$ is non-zero if and only if, it is non-zero in $\Delta(M, f, Or^2)$, i.e., given a smooth closed manifold (M, g) and a Morse-Smale function on M, the connection matrices are equal modulo 2. This result in the modulo 2 case is a well known fact, see [28]. Therein, it is shown that for a Morse-Smale flow on a smooth Riemannian closed n-manifold M, the entries of the connection matrix corresponding to critical points $x \in Crit_k(f)$ and $y \in Crit_{k-1}(f)$ counts the number of connecting orbits from x to y modulo 2. On the other hand, if we count orbits over $\mathbb{Z}$ considering orientations, the difference between $\Delta(M, f, Or^1)$ and $\Delta(M, f, Or^2)$ is the choice of the generators that compose the basis where the matrices are represented and hence they represent the same operator.

From now on, in this Chapter, the algebraic properties of the SSSA applied to a connection matrix will be investigated, with the underling motivation of obtaining dynamical information from this algorithm. As a first attempt, to obtain results in this direction, we will restrict our attention to 2-dimensional manifolds. In this section, a characterization of surface connection matrices is established and some algebraic implications of this characterization are proved.

Let M be a two dimensional Riemannian manifold and $f : M \to \mathbb{R}$ be a Morse-Smale function. Given $x, y \in Crit(f)$, if x and y are consecutive critical points, i.e. $ind_f(x) - ind_f(y) = 1$, then $\#\mathcal{M}_y^x \leq 2$. In fact, there are at most two connecting orbits joining a source to a saddle or a saddle to a sink. This fact implies that the non zero entries of a connection matrix for an orientable surface are ± 1, as one proves in the next result.

Proposition 2.2. *Let M be an orientable closed surface and φ a Morse-Smale flow on M generated by a Morse-Smale function f. Then the entries of a connection matrix for M belong to the set $\{-1, 0, 1\}$.*

Proof. We need to show that $n(x, y) \in \{-1, 0, 1\}$, for all critical points of f with relative index 1 and with $0 \leq ind_f(y) < ind_f(x) \leq 2$.

Since $dim(M) = 2$, it follows that $\#\mathcal{M}_y^x \leq 2$. If $\#\mathcal{M}_y^x = 1$ then $\mathcal{M}_y^x = \{u\}$ and therefore $n(x, y) = n_u = \pm 1$. Now if $\#\mathcal{M}_y^x = 2$, say $\mathcal{M}_y^x = \{u, v\}$, then there are two cases to consider:

Case 1: x is a saddle and y is a sink. Since the flow is a gradient flow, it induces the same orientation on the orbits $\mathcal{O}(u_1)$ and $\mathcal{O}(u_2)$. However, the orientation that $\mathcal{M}_{xy}$ induces on $\mathcal{O}(u)$ is opposite to the orientation induced by $\mathcal{M}_{xy}$ on $\mathcal{O}(v)$, see Figure 2.4. Hence, $n_u = 1$ and $n_v = -1$ or $n_u = -1$ and $n_v = 1$. In both cases, $n(x, y) = 0$.

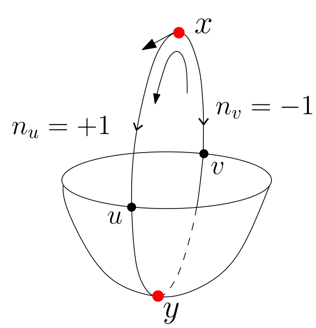


Figure 2.4: Determining characteristic signs for the saddle sink moduli spaces.

Case 2: x is a source and y is a saddle. Since M is an orientable surface, given an orientation $\{\xi_1(p), \xi_2(p)\}$ for $W^u(x)$, it follows that the ordered basis $\{\xi_1(u), \xi_2(u)\}$ and $\{\xi_1(v), \xi_2(v)\}$ are either both equivalent or both non-equivalent to the standard basis of $\mathbb{R}^2$. Once again by the orientability of M, it follows that the basis $\mathcal{V}_u W^s(y)$ and $\mathcal{V}_v W^s(y)$ are either both equivalent or both non-equivalent to the standard basis of $\mathbb{R}$. By the isomorphism (2.1), the orientations on the orbits $\mathcal{O}(u)$ and $\mathcal{O}(v)$ are opposite. Hence, on only one of the orbits, the orientation induced by the flow coincides with the orientation induced by the connecting manifold, see Figure 2.5. Hence, $n(x, y) = 0$. ■

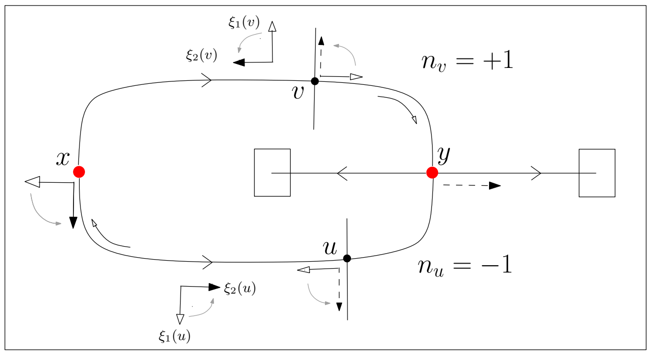


Figure 2.5: Determining characteristic sign for the source saddle moduli spaces.

The orientability hypothesis in the previous proposition is necessary. For instance, consider the minimal flow in the projective plane $\mathbb{R}^2$, which is viewed as the unit disc in $\mathbb{R}^2$ with opposite boundary points identified as in Figure 2.6. The orbits connecting the source x to the saddle y are represented in the connection matrix by the integer entry $+2$. See Figure 2.7.

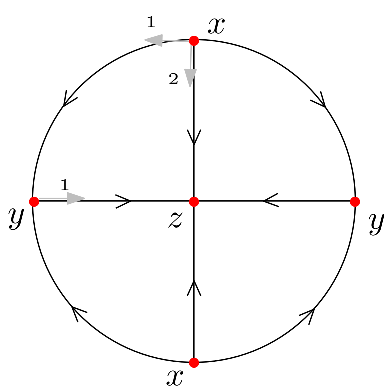


Figure 2.6: Morse flow on $\mathbb{R}P^2$.

	z	y	x
z	0		
y		0	2
x			0

Figure 2.7: Connection matrix for $\mathbb{R}P^2$.

The columns and rows of a connection matrix Δ for $\mathcal{D}(M)$ may be partitioned into three groups, namely S_0, S_1 and S_2, the first S_0 associated with sinks (h_0's), the second S_1 with saddles (h_1's) and the third S_2 with sources (h_2's). Block $\Delta_{S_0S_1}$ contains information on the connections from saddles to sinks, while block $\Delta_{S_1S_2}$ contains information on the connections from sources to saddles. Figure 2.8 illustrates a possible structure for a surface connection matrix with columns ordered with respect to the index. The columns of the matrix Δ need not be ordered with respect to the index.

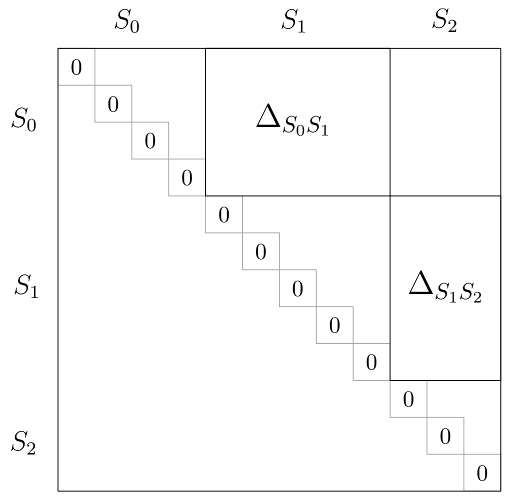


Figure 2.8: Surface connection matrix with $S_0 = \{1, 2, 3, 4\}$, $S_1 = \{5, 6, 7, 8, 9\}$ and $S_2 = \{10, 11, 12\}$.

Note that, each column $j \in S_1$ in Δ has either two non-zero entries, namely $+1$ and -1, or is a column of zeros. This follows easily since a saddle either connects to two sinks, in the first case, or it connects to only one sink, in the latter case. Each row $i \in S_1$ in Δ either has two non-zero entries or is a row of zeros. The signs of these non-zeros entries are determined by the set Or. If we choose for instance the orientation of all $W^u(h_2)$ to be the same, it follows that the non-zero entries in the row $i \in S_1$ of Δ have opposite signs. By Proposition 2.1, for any other choice of orientation, the new matrix $\overline{\Delta}$ obtained is similar to Δ.

Corollary 2.2 below summarizes this observation and gives a characterization of the connection matrices associated to orientable closed surfaces.

Corollary 2.2 (Characterization of connection matrices for orientable closed surfaces)**.** *The connection matrix Δ for an orientable closed surface either has the following properties:*

(1) $\Delta_{ij} \in \{0, 1, -1\}$*;*

(2) each column in $\Delta_{S_0S_1}$ contains either two non-zero elements, namely 1 *and* -1*, or none;*

(3) each row in $\Delta_{S_1S_2}$ contains either two non-zero elements, namely 1 *and* -1*, or none;*

or is obtained from a matrix with the properties above by multiplying a subset of rows and/or columns by -1.

Proof. Let M be an orientable surface. Fix an orientation on M. Let Δ be a surface connection matrix for M when we consider the unstable manifold $W^u(x)$ of each source endowed with orientation compatible with the orientation of M. Doing the same analysis in the proof of Proposition 2.2, one obtains that the rows and columns of Δ satisfies item $1-3$. Now, let $\tilde{\Delta}$ be a surface connection matrix considering an arbitrary set of orientations Or for the unstable manifolds. By Proposition 2.1, $\tilde{\Delta}$ is obtained form Δ by multiplying a subset of rows and/or columns by -1. ■

Example 2.3. Let M be a 2-sphere as in Figure 2.9 and let f be a Morse-Smale function on M such that the negative gradient flow associated to $-\nabla f$ is as shown in Figure 2.9. The Morse chain complex $(C_*(f), \partial_*)$, determined by the function f and by considering the orientation of $W^u(h_k^j)$ as in the Figure 2.9, is presented below.

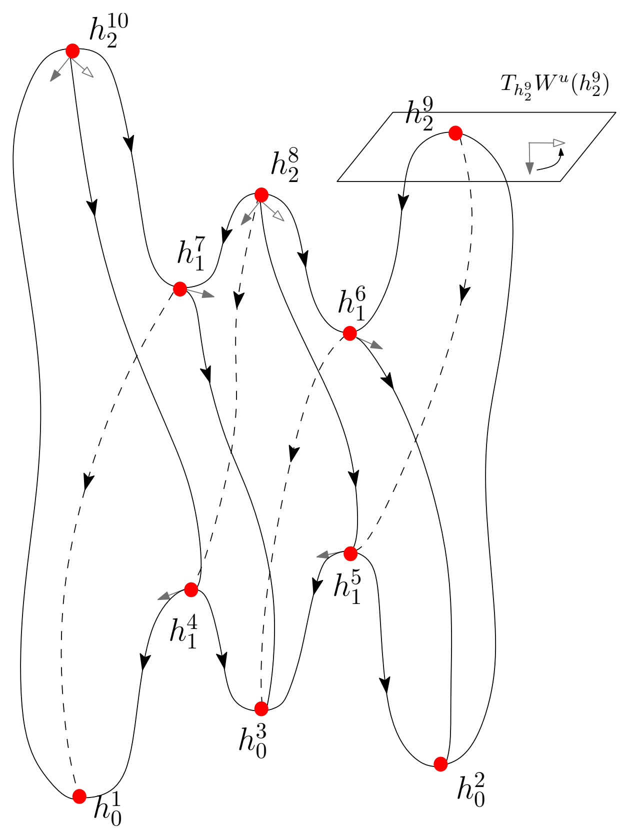


Figure 2.9: Morse-Smale flow in the 2-sphere.

Morse graded groups:
$C_0(f) = \mathbb{Z}\langle h_0^1\rangle \oplus \mathbb{Z}\langle h_0^2\rangle \oplus \mathbb{Z}\langle h_0^3\rangle$,
$C_1(f) = \mathbb{Z}\langle h_1^4\rangle \oplus \mathbb{Z}\langle h_1^5\rangle \oplus \mathbb{Z}\langle h_1^6\rangle \oplus \mathbb{Z}\langle h_1^7\rangle$,
$C_2(f) = \mathbb{Z}\langle h_2^8\rangle \oplus \mathbb{Z}\langle h_2^9\rangle \oplus \mathbb{Z}\langle h_2^{10}\rangle$
and $C_k = 0$ for $k > 2$.

Morse boundary operators:
$\partial_1(h_1^4) = h_0^1 - h_0^3$,
$\partial_1(h_1^5) = h_0^3 - h_0^2$,
$\partial_1(h_1^6) = h_0^2 - h_0^3$,
$\partial_1(h_1^7) = h_0^3 - h_0^1$,
$\partial_2(h_2^8) = h_1^4 - h_1^5 - h_1^6 + h_1^7$,
$\partial_2(h_2^9) = h_1^5 + h_1^6$,
$\partial_2(h_2^{10}) = -h_1^4 - h_1^7$,
$\partial_k(h_k^j) = 0$ for all $k > 2$.

The connection matrix with respect to the flow ordering associated to the finest Morse decomposition of M is as in Figure 2.10. Applying the SSSA over $\mathbb{Z}$ in Δ, one obtains the sequence of matrices illustrated in Figures 2.11 to 2.15. In these figures, the primary pivot entries are indicated by means of a light red background and darker edge, the change-of-basis pivots are indicated by blue background and dashed edges, null entries are left blank and the diagonal being swept is indicated with a gray line. ◀

	h_0^1	h_0^2	h_0^3	h_1^4	h_1^5	h_1^6	h_1^7	h_2^8	h_2^9	h_2^{10}
h_0^1				1			−1			
h_0^2					−1	1				
h_0^3				−1	1	−1	1			
h_1^4								1		−1
h_1^5								−1	1	
h_1^6								−1	1	
h_1^7								1		−1
h_2^8										
h_2^9										
h_2^{10}										

Figure 2.10: Δ^0

	$\sigma_0^{1,1}$	$\sigma_0^{2,1}$	$\sigma_0^{3,1}$	$\sigma_1^{4,1}$	$\sigma_1^{5,1}$	$\sigma_1^{6,1}$	$\sigma_1^{7,1}$	$\sigma_2^{8,1}$	$\sigma_2^{9,1}$	$\sigma_2^{10,1}$
$\sigma_0^{1,1} = h_0^1$				1			−1			
$\sigma_0^{2,1} = h_0^2$					−1	1				
$\sigma_0^{3,1} = h_0^3$				−1	1	−1	1			
$\sigma_1^{4,1} = h_1^4$								1		−1
$\sigma_1^{5,1} = h_1^5$								−1	1	
$\sigma_1^{6,1} = h_1^6$								−1	1	
$\sigma_1^{7,1} = h_1^7$								1		−1
$\sigma_2^{8,1} = h_2^8$										
$\sigma_2^{9,1} = h_2^9$										
$\sigma_2^{10,1} = h_2^{10}$										

Figure 2.11: Δ^1

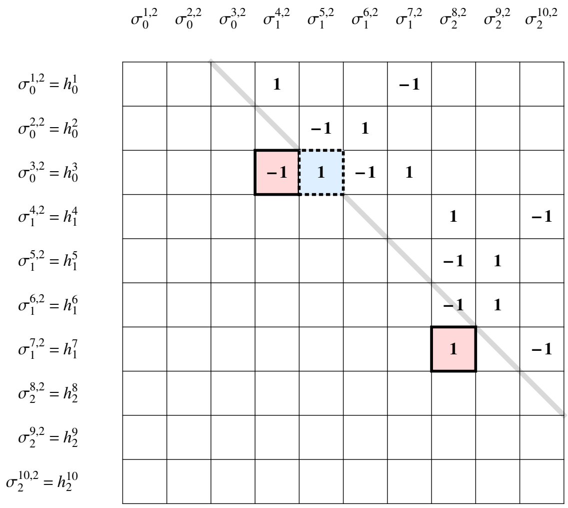


Figure 2.12: Δ^2

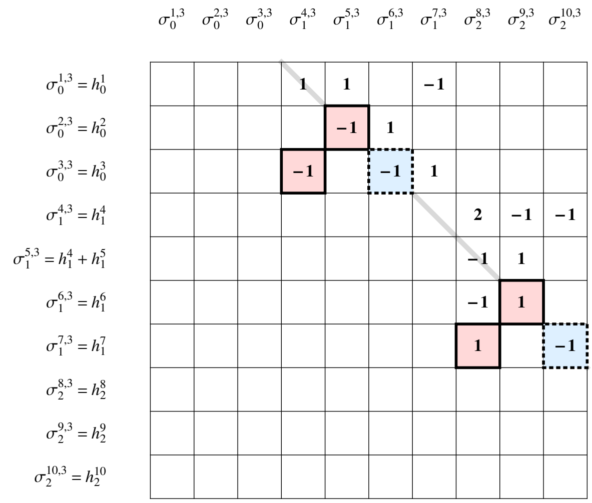


Figure 2.13: Δ^3

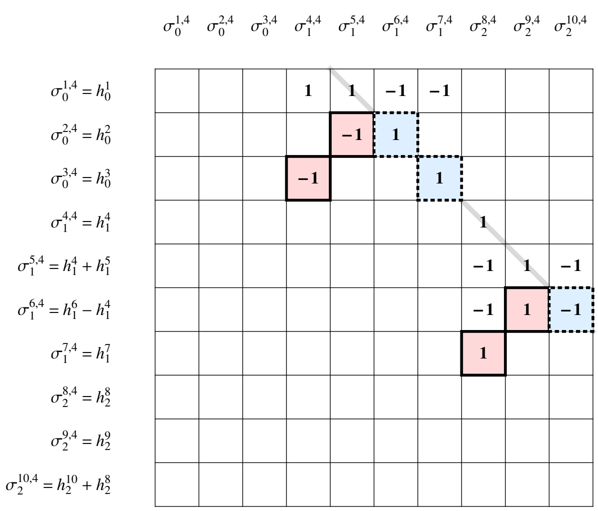


Figure 2.14: Δ^4

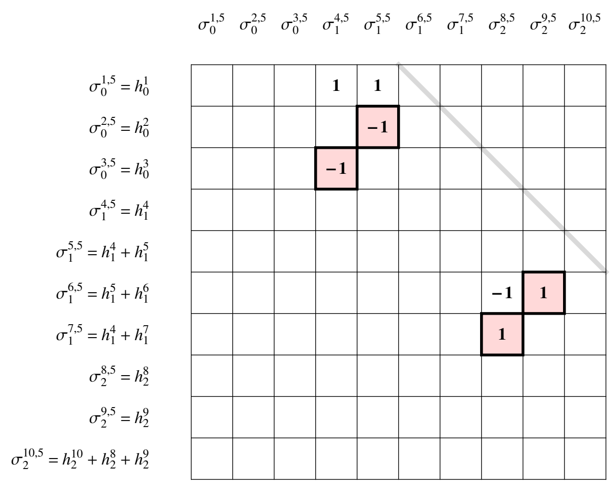


Figure 2.15: Δ^5

2.3 SSSA applied to surface connection matrices

In this section, we investigate properties of the SSSA which can be extracted from the characterization of surface connection matrices. Note that, in Example 2.3, the primary and change-of-basis pivots identified during all steps of the SSSA are ± 1. However, in Δ^3 the entry $\Delta^3_{4,8}$ has numerical value equal to 2. In this section, we prove that the primary pivots, obtained by the applying the SSSA over $\mathbb{Z}$ to a surface connection matrix Δ, have values ± 1.

We prove the characterization of the primary pivots for orientable surfaces in Theorem 2.1. This is done by presenting some auxiliary algorithms over a field $\mathbb{F}$ that produce the same primary pivots as the SSSA. The characterization of the primary pivots for orientable surfaces will follow immediately from these results.

This section contains the technical details which make it possible to prove Theorem 2.1. The reader can skip this section in a first reading without any loss of the posterior development of the results herein. Denote by Δ^L the last connection matrix produced by the SSSA.

Lemma 2.1. *Given a connection matrix Δ over $\mathbb{F}$, with column/row partition $J_0, \cdots, J_b$, the update from Δ^r to Δ^{r+1} may be accomplished blockwise, according to the individual updates*

$$\Delta^r_{J_{k-1}J_k} = (T^{r-1}_{J_{k-1}J_{k-1}})^{-1}\Delta^{r-1}_{J_{k-1}J_k}T^{r-1}_{J_kJ_k}, \quad \text{for } k = 1, \ldots, b. \tag{2.5}$$

Consequently, only columns containing change-of-basis pivots are subjected to elementary column operations, and only rows with the same index as the column of the primary pivots used for canceling out the change-of-basis pivots suffer elementary row operations.

Proof. Let Δ be a connection matrix with column/row partition given by $J_0, \cdots, J_b$. Algebraically, the post-multiplication of Δ^r by T^r consists of t_r elementary column operations on the change-of-basis columns of Δ^r. Of the three possible elementary column operations on a column j, only one is used in the SSSA over $\mathbb{F}$: "add to column j a multiple of another column". In keeping with its counterpart, the pre-multiplication of $\Delta^r T^r$ by $(T^r)^{-1}$ consists of t_r elementary row operations. Amongst the ones available, the only row operation on row i considered herein is of the type "add to row i a multiple of another row". Column operations are due to the post-multiplication, and by construction of $T^{r-1}_{J_kJ_k}$, affect only columns of J_k that contain change-of-basis pivots. The pre-multiplication by $(T^{r-1}_{J_{k-1}J_{k-1}})^{-1}$ affects only rows with same index as the columns that contain the primary pivots used for cancelling out the change-of-basis pivots. ■

Denote by $\mathcal{J}_k$ the subset of columns in J_k which contains primary pivot entries in Δ^L. Let $\overline{J}_k = J_k \backslash \mathcal{J}_k$, for all k. The markings on entries in columns belonging to J_k and the construction of $T^r_{J_kJ_k}$ are completely determined by the values of the change-of-basis and primary pivots in $\Delta^r_{\overline{J}_{k-1}J_k}$. In fact, suppose there is a change in entry in position $(i, j) \in J_{k-1} \times J_k$ from Δ^r to Δ^{r+1}. Lemma 2.1 implies this may be due to an elementary column operation on column j, due to a change-of-basis mark on entry $(j-r, j)$ on this column, and/or to an elementary row operation, due to a primary pivot in column $i \in J_{k-1}$ that is being used to cancel out a change-of-basis pivot in position $(q-r, q)$ in some column $q \in J_{k-1}$, where $q > i$. Moreover, since the rows of Δ^L in $\mathcal{J}_k$ must be zero, thus they cannot contain primary and change-of-basis pivots.

This discussion implies that, if we give up keeping track of the evolution of the rows in $\cup_{k=1}^{b} \mathcal{J}_k$, then one must only do the post-multiplication of Δ^r by T^r. Moreover, one can do this sequentially blockwise as follows. At step k , for $k = 1, \cdots, b$, let $\Delta(k)$ be the matrix obtained from Δ by zeroing rows in $\mathcal{J}_{k-1}$ and entries outside positions in $J_{k-1} \times J_k$. The diagonals of this matrix are swept as in the SSSA over $\mathbb{F}$, and the same rules are used for marking up the entries, as well as building the transition matrix $T(k)^r$, which will contain nonzero off-diagonal entries only in positions $J_k \times J_k$, by virtue of the construction of the connection matrix used as input. Applying the SSSA over $\mathbb{F}$ to $\Delta(k)$, one obtains sequences $(\Delta(k)^0, \ldots, \Delta(k)^L)$ and $(T(k)^0, \ldots, T(k)^{L-1})$, and the set $\mathcal{J}_k$ of the column indices of $\Delta(k)^L$ containing primary pivots, for $k = 1, \cdots, b$. This new formulation of the SSSA is called *Block Sequential Sweeping Algorithm over* $\mathbb{F}$. See the Appendix A for the algorithm, per se.

The difference between the SSSA and the Block Sequential Sweeping Algorithm described above is that the latter does not keep track of the evolution of the rows in $\mathcal{J}_k$, for $k = 1, \cdots, b$. Hence, one has the following lemma.

Lemma 2.2 (Uncoupling). *$\Delta^L_{J_{k-1}J_k} = \Delta(k)^L_{J_{k-1}J_k}$, for all k and the collection of change-of-basis and primary pivots encountered in the application of the SSSA over $\mathbb{F}$ to $\Delta(k)$ coincides with the change-of-basis and primary pivots found when it is applied to Δ.*

The proof of the Uncoupling Lemma can be found in the Appendix A, Lemma A.1.

The Uncoupling Lemma implies that we may restrict our attention to connection matrices containing at most one nonzero block when studying the SSSA over $\mathbb{F}$, provided we lose track of the evolution of rows in $\cup_{k=1}^{b} \mathcal{J}_k$, which will end up zero and will not contain neither primary nor change-of-basis pivots. To ease the discussion that follows, we henceforth call this special case the *1-Block Incremental Sweeping Algorithm over* $\mathbb{F}$.

Now, we propose a reengineered version of the 1-Block Incremental Sweeping Algorithm therefor, in which, once a primary pivot is identified, all cancellations it is responsible for in the 1-block Incremental Sweeping Algorithm over $\mathbb{F}$ are performed. To arrive at the same final matrix as in the original algorithm, the primary pivots must also be identified in an upward order, from the bottom up. The second and last important aspect for the identification of primary pivots, is the left-to-right order of the sweeping. The algorithm which incorporates both of these conditions is called *Revised 1-Block Incremental Sweeping Algorithm over* $\mathbb{F}$. See the Appendix A for the algorithm per se.

The next Lemma establishes the equality between the final matrices produced by the Revised 1-block Incremental Sweeping over $\mathbb{F}$ and by the 1-block Incremental Sweeping Algorithm over $\mathbb{F}$. There is of course no sense in looking for equality between other matrices in the sequence produced by the algorithm, since the order of cancellation is in all likelihood quite different in the two algorithms.

Lemma 2.3. *Let Δ be a connection matrix with column/row partition J_0, J_1. Let $\tilde{\Delta}^{t^*+1}$ and Δ^L be the matrices obtained by applying the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$ and the 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$ to Δ, respectively. Then $\tilde{\Delta}^{t^*+1} = \Delta^L$ and their primary pivots coincide.*

The proof of Lemma 2.3 can be found in the Appendix A, Lemma A.2.

A matrix is totally unimodular (TU) if all its square submatrices have determinant 0, 1 or -1, see, for instance, [34]. This property is invariant under transposition, multiplying a row or column by $0, \pm 1$, adding or removing zero rows/columns or unit rows/columns. If each column of a $0, \pm 1$ matrix has at most two nonzero entries of opposite signs, then it is TU. It follows that the submatrices $\Delta_{J_0 J_1}$ and $\Delta_{J_1 J_2}$ of a surface connection matrix Δ are TU.

The set $\{0, \pm 1\}$ is not closed under addition, so the appropriate algorithm to apply to surface connection matrices is the SSSA over $\mathbb{Z}$. Nevertheless, using the Block Sequential Sweeping Algorithm over $\mathbb{F}$, the Revised 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$, Lemma 2.2 and the total unimodularity property, we will show the sequence of matrices and bases obtained when applying the SSSA over $\mathbb{F}$ to a surface connection matrix is compatible with the corresponding ones produced by the application of SSSA over $\mathbb{Z}$ to Δ, in the following sense. The change from Δ^r to Δ^{r+1} results from replacing basis element σ_j^r, of each column j containing a change-of-basis entry, with an integral linear combination of elements of h with same chain index and associated with columns of index less than or equal to j, determined so as to zero out in Δ^{r+1} the entry in the change-of-basis position, while maintaining the pattern of trailing zeros below it. Amongst the integral linear combinations that accomplish this, one must choose one with the smallest possible positive leading coefficient.

Even with this condition, there may be more than one optimal integral linear combination. The bases constructed by the SSSA over $\mathbb{F}$ trivially satisfy the conditions pertaining the zero patterns. We show that, for every r and j, the leading coefficient in the integral linear combination that is σ_j^r, produced by the application of the SSSA over $\mathbb{F}$ to Δ, is 1, so it satisfies the optimality criterium, and the choice is thus compatible with the rules of the SSSA over $\mathbb{Z}$.

Lemma 2.4. *Let Δ be a surface connection matrix with column/row partition J_0, J_1 and J_2. If we apply the SSSA over $\mathbb{F}$ to Δ, then all primary pivots value are either* 1 *or* -1.

Proof. Lemma 2.2 (Uncoupling) justifies the application of the Block Sequential Sweeping Algorithm over $\mathbb{F}$ to Δ. Both $\Delta_{J_0 J_1}$ and $\Delta_{J_1 J_2}$ are TU and this property is maintained if one adds zero rows and/or columns to a matrix. Thus each $\Delta(k)$, for $k = 1, 2$, in the application of the Block Sequential Algorithm over $\mathbb{F}$ to Δ is TU. Then to prove the claim one can consider matrices Δ which are totally unimodular with column/row partition J_0, J_1.

First we analyze the application of the Revised 1-Block Sweeping Algorithm over $\mathbb{F}$ to Δ with column/row partition J_0, J_1. Let (i_1, j_1), $\ldots$, (i_{t^*}, j_{t^*}) be the positions of the primary pivots marked.

Let C^t be the set of index columns of Δ^t which do not have a primary pivot. We affirm that $\tilde{\Delta}^t_{\bullet C^t}$ is totally unimodular, for $t = 1, \ldots, t^* + 1$. This is trivially true for $t = 1$, by hypothesis, and it can be proved by induction to hold for all t, see Lemma A.3 in the Appendix A. This implies that the entries in $\tilde{\Delta}^t_{\bullet C^t}$ are -1, $+1$ or zero, for all t. Hence all primary pivots marked in the application of the Revised 1-Block Sweeping Algorithm over $\mathbb{F}$ to Δ value 1 or -1 when marked, since they are, by choice, nonzero. Finally, they do not change once marked.

Lemma 2.3 implies $\Delta^{t^*+1} = \Delta^L$, the last matrix produced by the application of the SSSA over $\mathbb{F}$ to Δ. Moreover, their primary pivots coincide in position and value. ■

Theorem 2.1 (Primary pivots for orientable surfaces)**.** *Given a surface connection matrix Δ, let $\{\Delta^1, \ldots, \Delta^L\}$ be the sequence of connection matrices produced by the SSSA applied to Δ. The primary pivots identified in the r-th diagonal of Δ^r are ± 1, for all $r \in \{1, \ldots, L\}$.*

Proof. Let Δ be a surface connection matrix (over $\mathbb{Z}$). By Lemma 2.4, if we apply the SSSA over $\mathbb{F}$ to Δ, then all primary pivots value are either 1 or -1.

Let σ^r be the basis associated with the r-th matrix in the sequence produced by the SSSA over $\mathbb{F}$, Δ^r. We will prove, by induction in r, that each basis element σ_j^r, for all r and

j, is an integral linear combination of the elements of h associated with columns to the left of, and including, column j, whose leading coefficient is 1.

First of all, from the definition of T^r in the SSSA over $\mathbb{F}$ and the hypotheses that Δ is integral and the primary pivots value ± 1, we obtain that T^1 and $(T^1)^{-1}$ are also integral. Then Δ^2 is integral and, applying induction, we may conclude that Δ^r is integral, for all r.

Initially $\sigma^1 = h$, so the property is true for $r = 1$. Suppose it is true for σ^r. If $\Delta^r_{\bullet j}$ does not contain a change-of-basis pivot, then $\sigma^{r+1}_j = \sigma^r_j$ and the result is true by the induction hypothesis. Suppose it does contain a change-of-basis pivot. It is sufficient to consider it in a generic fixed position, say $(j - r, j)$. Lemma 2.4 implies that the primary pivot to its left, say in position $(j - r, p)$, is ± 1. Then, by the rules of the algorithm,

$$\begin{aligned}
\sigma^{r+1}_j &= \sigma^r_j - \frac{\Delta^r_{j-r,j}}{\Delta^r_{j-r,p}} \sigma^r_p \\
&= \sigma^r_j \pm \Delta^r_{j-r,j}\, \sigma^r_p \\
&= h^j_k + \sum_{j'<j} c^{r,j}_{j'} h^{j'}_k \pm \Delta^r_{j-r,j} \sum_{j'<p} c^{r,p}_{j'} h^{j'}_k,
\end{aligned}$$

where the induction hypothesis is used in the last equality. This proves the result for $r + 1$, since $p < j$, $\Delta^r_{j-r,j}$ and, by induction, all coefficients $c^{r,\bullet}_{\bullet}$ are integer. Using induction, the result if true for all r.

Therefore, the sequence produced by the application of the SSSA over $\mathbb{F}$ to a surface connection matrix is compatible with the application of the SSSA over $\mathbb{Z}$ thereto. ■

Theorem 2.1 has an important dynamical consequence which will be addressed in Section 2.5. Moreover, an algebraic consequence of this result ensures that the spectral sequence associated to a two-dimensional Morse chain complex converges to the corresponding Morse homology.

Corollary 2.3. *If M is a smooth closed orientable 2-dimensional manifold, $f : M \to \mathbb{R}$ a Morse function, (C_*, Δ) a filtered Morse chain complex with the finest filtration, then the modules $E^\infty_{p,q} \approx GH_*(C)_{p,q}$ of the associated spectral sequence are free for all p and q.*

Proof. As presented in Section 1.2, it was proved in [13] that, if E^r_p and E^r_{p-r} are both non-zero, the differential $d^r : E^r_p \to E^r_{p-r}$ is induced by multiplication by $\Delta^r_{p-r+1,p+1}$, whenever this entry is either a primary pivot, change-of-basis pivot or a zero with a column of zero entries below it. On the other hand, all primary pivots are ± 1 by Theorem 2.1, implying that differentials d^r induced by primary pivots are isomorphisms. Moreover, if $d^r : E^r_p \to E^r_{p-1}$ is an isomorphism then $E^{r+1}_p = E^{r+1}_{p-1} = 0$.

Note that, if the differential $d^r_p : E^r_p \to E^r_{p-r}$ corresponds to a change-of-basis pivots, then there is a primary pivot in row $p - r$ and thus $E^r_{p-r} = 0$. Consequently, the non-zero differentials are the ones induced by primary pivots, hence isomorphisms.

Since $E^{r+1}_p \cong \text{Ker } d^r_p / \text{Im } d^r_{p-r}$, it follows that the modules E^r_p are free for all $r \geq 0$ and $p \geq 0$. Then $E^\infty_{p,q} \cong GH_*(C)_{p,q} \cong HM_*(M, f)$, by equation (1.3). ■

In the more general setting treated in [13], where one was interested in spectral sequences computed over $\mathbb{Z}$ of a Morse chain complex associated to a gradient flow on an n-dimensional manifold, the spectral sequence need not converge to the homology of the complex, i.e. (1.3) need not be true. However, as a consequence of the Primary Pivots for Orientable Surfaces Theorem (Theorem 2.1), in the 2-dimensional setting, $GH_*(C)_{p,q}$ is free for all p and q and thus (1.3) holds.

2.4 Smale's Cancellation Sweeping Algorithm

In Corollary 2.2, one characterizes the connection matrices associated to orientable closed surfaces. The relevance of this results resides in the fact that it guarantees that the SSSA does not determine a flow continuation of the initial flow, which has its dynamics coded in Δ^0, to a flow having the dynamics coded in the last connection matrix Δ^L produced by the SSSA applied to Δ^0. In fact, not all last matrices produced by the SSSA is a connection matrix associated to a Morse flow on a orientable closed surface. Returning to Example 2.3, observe that the matrix Figure 2.15 does not satisfy condition (3) of Corollary 2.2, which implies that one can not realise this matrix as a connection matrix associated to a Morse flow on a surface.

In this section, we present an adaptation of the SSSA, called the Smale's Cancellation Sweeping Algorithm (SCSA). This algorithm is an attempt to modify the SSSA in order to obtain an algorithm which provides a flow continuation where the dynamics is coded by the matrices produced by the SCSA. More specifically, our approach herein is to interpret the algebraic cancellation of the modules of the spectral sequence, which has been coded by the Spectral Sequence Sweeping Algorithm, as dynamical cancellations. In fact, whenever we mark a primary pivot $\Delta^r_{j-r,j} = \pm 1$ on the r-th diagonal of Δ^r, the next step of the spectral sequence produces algebraic cancellations of the modules E^{r+1}_p and E^{r+1}_{p-r}, i.e. $E^{r+1}_p = E^{r+1}_{p-r} =$

0. We wish to interpret these algebraic cancellations dynamically as cancellation of a pair of consecutive index singularities.

Note that the changes of basis caused by pivots in row $j-r$ reflect all the changes in connecting orbits caused by the cancellation of h_k^j and h_{k-1}^{j-r}. However, when we remove the pair of critical points h_k^j and h_{k-1}^{j-r}, all the connecting orbits between index k critical points and h_{k-1}^{j-r} and also all the ones between h_k^j and index $k-1$ critical points are immediately removed and new ones take their place. Hence, in order to interpret dynamically the Spectral Sequence Sweeping Algorithm, we have to perform the changes of basis that occur therein in a different order to reflect the death and birth of connections. More specifically, if $\Delta^r_{j-r,j} = \pm 1$ is a primary pivot marked in step r of the Spectral Sequence Sweeping Algorithm, all changes of basis caused by $\Delta^r_{j-r,j}$ must be performed in step $r+1$. This new algorithm will be called Smale's Cancellation Sweeping Algorithm.

In the Smale's Cancellation Sweeping Algorithm we keep the same order of sweeping along the diagonals and the criteria for marking an entry as a primary pivot. But the upper triangular unit-diagonal transition matrix is calculated so that in the next matrix all entries to the right of the primary pivots are zeroed by means of elementary column operations using exclusively columns to the left of the column in question.

Smale's Cancellation Sweeping Algorithm - SCSA

For a fixed diagonal r parallel and to the right of the main diagonal, the method described below must be applied simultaneously for all blocks J_k.

Initial Step.

(1) Let ξ_1 be the first diagonal of $\widetilde{\Delta}$ that contains non-zero entries $\widetilde{\Delta}_{i,j} \in \Delta_{J_{k-1}J_k}$, which will be called index k primary pivots. Define $\widetilde{\Delta}^{\xi_1}$ to be $\widetilde{\Delta}$ with the k- index primary pivots on the ξ_1-th diagonal marked.

(2) Consider the matrix $\widetilde{\Delta}^{\xi_1}$. Let $\xi_2 = \xi_1 + 1$. The construction of $\widetilde{\Delta}^{\xi_2}$ follows the procedure below. Let $\widetilde{\Delta}_{i,j}$ be a k-index primary pivot in the ξ_1-th diagonal.
Given a non-zero entry $\widetilde{\Delta}^{\xi_1}_{i,l}$ on row i of $\widetilde{\Delta}^{\xi_1}$ and $l > j$, perform a change of basis on $\widetilde{\Delta}^{\xi_1}$ as in SSSA in order to zero out this entry. Moreover, if there are more than one non zero entry in row i, the procedure to zero out these entries is to zero out the entries in increasing order with respect columns.
Given a non zero entry $\widetilde{\Delta}^{\xi_2}_{i,j}$ on the ξ_2-th diagonal of $\widetilde{\Delta}^{\xi_2}$:

If $\widetilde{\Delta}^{\xi_2}_{s,j}$ does not contain a primary pivot for $s > i$,
then permanently mark $\widetilde{\Delta}^{\xi_2}_{i,j}$ as a primary pivot.

Intermediated Step.

Suppose by induction that $\widetilde{\Delta}^{\xi}$ is defined for all $\xi \leq r$ with the primary pivots marked on the diagonals smaller or equal to ξ. In what follows it will be shown how $\widetilde{\Delta}^{r+1}$ is defined. Without loss of generality, one can assume that there is at least one primary pivot on the r-th diagonal of $\widetilde{\Delta}^{r}$. If this is not the case, define $\widetilde{\Delta}^{r+1} = \widetilde{\Delta}^{r}$ with primary pivots marked as in step (2) below.

(1) **Change of basis.** Let $\widetilde{\Delta}^{r}_{i,j}$ be a primary pivot.

For each non-zero entry in row i, perform a change of basis on $\widetilde{\Delta}^{r}$ as in SSSA in order to zero out this entry. Moreover, if there is more than one non zero entry in row i, the procedure to zero them out these entries is to zero out in increasing order with respect to the columns.

If there is more than one primary pivot in the r-th diagonal of $\widetilde{\Delta}^{r}$, perform this step for each primary pivot in decreasing order with respect to the rows.

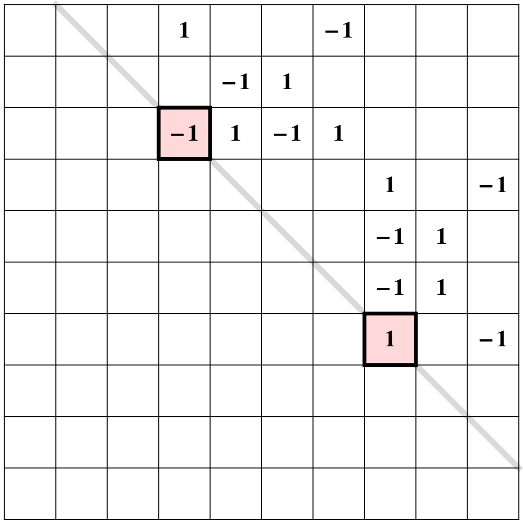
(2) **Markup.** Given a non-zero entry $\widetilde{\Delta}^{r+1}_{i,j}$ on the $(r+1)$-th diagonal of $\widetilde{\Delta}^{r+1}$:
If $\widetilde{\Delta}^{r+1}_{s,j}$ does not contain a primary pivot for $s > i$,
then permanently mark $\widetilde{\Delta}^{r}_{i,j}$ as a primary pivot.

Final Step.

Repeat the above procedure until all diagonals have been swept.

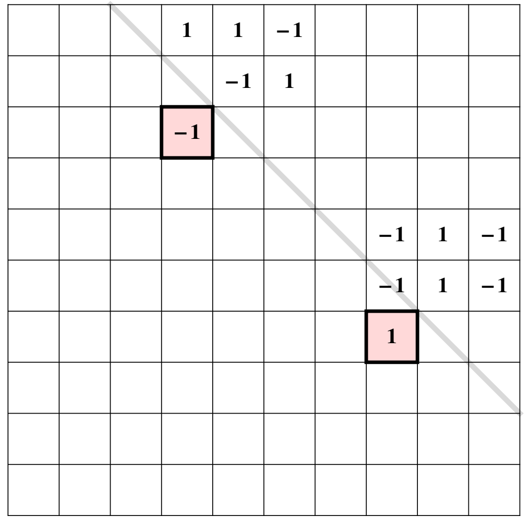
The above algorithm can be applied over $\mathbb{F}$ with the provision that in order to zero out a change-of-basis pivot, one uses only the column of the corresponding primary pivot. In this case, we refer to this algorithm over $\mathbb{F}$ as the Row Cancellation Algorithm.

Example 2.4. Consider the Morse chain complex (C, Δ) presented in Example 2.3. Applying the SCSA for the connection matrix Δ presented in that example, one obtains the matrices in Figures 2.16 to 2.19. In these figures, the primary pivot entries are indicated by means of a light red background and darker edge, null entries are left blank and the diagonal being swept is indicated with a gray line. ◀



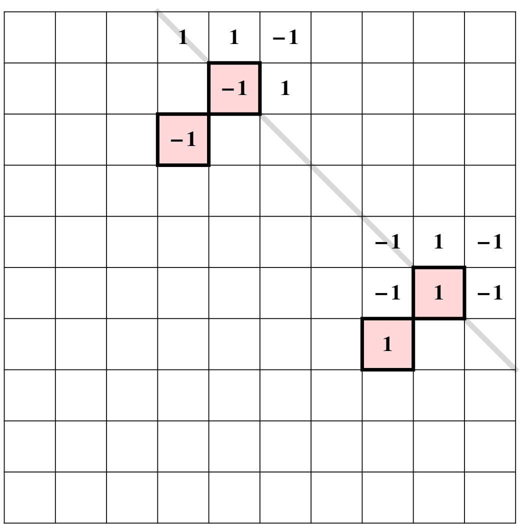
$\sigma_0^{j,1} = h_0^j$ for $j \in J_0 = \{1,2,3\}$;
$\sigma_1^{j,1} = h_1^j$ for $j \in J_1 = \{4,5,6,7\}$;
$\sigma_2^{j,1} = h_2^j$ for $j \in J_2 = \{8,9,10\}$.

Figure 2.16: $\widetilde{\Delta}^1$; marking primary pivots.



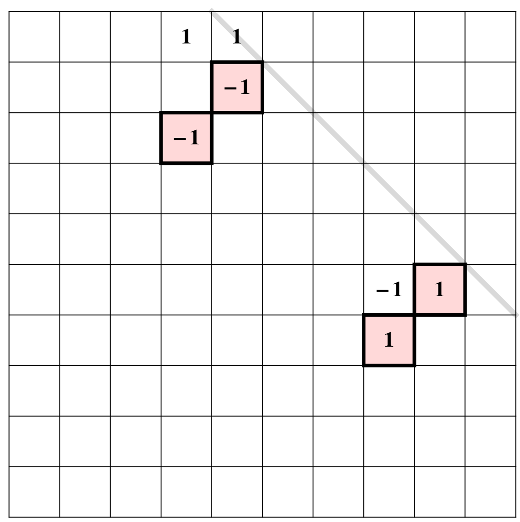
$\sigma_1^{5,2} = h_1^4 + h_1^5$; $\quad \sigma_1^{6,2} = h_1^6 - h_1^4$;
$\sigma_1^{7,2} = h_1^4 + h_1^7$; $\quad \sigma_2^{10,2} = h_2^{10} + h_2^8$;
$\sigma_k^{j,2} = \sigma_k^{j,1}$ for all the remaining σ's.

Figure 2.17: $\widetilde{\Delta}^2$; sweeping diagonal 2.



$\sigma_k^{j,3} = \sigma_k^{j,2}$ for all the σ's.

Figure 2.18: $\widetilde{\Delta}^3$; marking primary pivots.



$\sigma_1^{6,4} = h_1^5 + h_1^6$; $\quad \sigma_2^{10,4} = h_2^{10} + h_2^8 + h_2^9$;
$\sigma_k^{j,4} = \sigma_k^{j,3}$ for all the remaining σ's.

Figure 2.19: $\widetilde{\Delta}^4$; sweeping diagonal 4.

Observe that the primary pivots identified by the SSSA in Example 2.3 coincide in values and positions with the primary pivots identified by the SCSA in Example 2.4. In fact, the next theorem states that, given a surface connection matrix Δ, the primary pivots identified by the SSSA applied to Δ coincides in position and value with the primary pivots identified by the SCSA applied to Δ.

Theorem 2.2 (Equality of primary pivots)**.** *Let Δ be a surface connection matrix with column/row partition J_0, J_1, J_2. The primary pivots marked in the application of the SSSA over $\mathbb{Z}$ to Δ coincide in value and position with the primary pivots marked in the application of Smale's Cancellation Sweeping Algorithm thereto.*

Proof. As in the proof of Theorem 2.1, we will first investigate if this theorem holds over $\mathbb{F}$. Then we will show how to apply it over $\mathbb{Z}$.

Let $(\widehat{\Delta}^0 = \Delta, \widehat{\Delta}^1, \ldots, \widehat{\Delta}^{m-1})$ and $(\widehat{T}^0 = I, \widehat{T}^1, \ldots, \widehat{T}^{m-2})$ be the sequences of connection and transition matrices produced by the application of the Row Cancellation Algorithm over $\mathbb{F}$ to the connection matrix Δ with column/row partition J_0, J_1, J_2. One has the following properties:

1. The nonzero entries of $\widehat{\Delta}^r$ strictly below the r-th diagonal are either primary pivots (always nonzero) or lie above a unique primary pivot.
2. If $\widehat{\Delta}^r_{p-r,p}$ is marked as a primary pivot, then $\widehat{\Delta}^s_{p\bullet} = 0$, for $s \geq r+1$.
3. If $\widehat{\Delta}^r_{p-r,p}$ is marked as a primary pivot, then $\widehat{\Delta}^s_{p-r,q} = 0$, for $s \geq r+1$ and $q = p{+}1..m$.
4. Each row of $\widehat{\Delta}^{m-1}$ may contain at most one primary pivot.
5. Let $\widehat{\Delta}^{m-1}$ be the last connection matrix in the sequence produced by the application of the Row Cancellation Algorithm over $\mathbb{F}$ to the connection matrix Δ . Then

$$\widehat{\Delta}^{m-1}_{\bullet j}\widehat{\Delta}^{m-1}_{j\bullet} = 0, \quad \text{for all } j. \tag{2.6}$$

The matrix updates in the application of the Row Cancellation Algorithm over $\mathbb{F}$ to the connection matrix Δ with column/row partition J_0, J_1, J_b can be done in a blockwise fashion as follows.

$$\widehat{\Delta}^r_{J_{k-1}J_k} = (\widehat{T}^{r-1}_{J_{k-1}J_{k-1}})^{-1}\widehat{\Delta}^{r-1}_{J_{k-1}J_k}\widehat{T}^{r-1}_{J_kJ_k}, \quad \text{for } k = 1, 2. \tag{2.7}$$

We let $\widehat{\mathcal{J}}_k$ be the set of columns in J_k that contain primary pivot entries in $\widehat{\Delta}^{m-1}$, for $k = 1, 2$. Additionally, $\widehat{\widehat{J}}_k = J_k \backslash \widehat{\mathcal{J}}_k$, for $k = 1, 2$. Suppose the Row Cancellation Algorithm over $\mathbb{F}$ is applied to the connection matrix Δ with column/row partition J_0, J_1, J_2. Then the markings during the sweeping of the r-th diagonal of $\widehat{\Delta}^r$ on entries in columns belonging to J_k and the construction of $\widehat{T}^r_{J_k J_k}$ are completely determined by the values of the entries in $\widehat{\Delta}^r_{\widehat{\overline{J}}_{k-1} J_k}$.

The validity of the block update established in equation (2.7) and the complementarity relationship between a column with a primary pivot and the row of same index in equation (2.6), give rise to a simplified version of the Row Cancellation Algorithm over $\mathbb{F}$. The Block Sequential Row Cancellation Algorithm over $\mathbb{F}$ is a straightforward adaptation of the Block Sequential Sweeping Algorithm over $\mathbb{F}$, where, at step k, instead of applying the SSSA over $\mathbb{F}$ to $\Delta(k)$, one applies the Row Cancellation Algorithm over $\mathbb{F}$ thereto. The proof of the corresponding Uncoupling Lemma is a straightforward adaptation of the original one and it states that:

Row Cancellation Uncoupling Lemma: Let Δ be a connection matrix with row/column partition J_0, J_1, J_2. Let $\widehat{\Delta}^{m-1}$ be the matrix produced by the application of the Row Cancellation Algorithm over $\mathbb{F}$ to Δ, and let $\widehat{\Delta}(k)^{m-1}$, for $k = 1, 2$, be the matrices obtained in the Block Sequential Row Cancellation Algorithm over $\mathbb{F}$ applied to Δ. Then

$$\widehat{\Delta}^{m-1}_{J_{k-1} J_k} = \widehat{\Delta}(k)^{m-1}_{J_{k-1} J_k},$$

for $k = 1, 2$ and the collection of primary pivots encountered in the application of the Row Cancellation Algorithm over $\mathbb{F}$ to $\Delta(k)$, for $k = 1, 2$, coincides with the primary pivots found when it is applied to Δ.

Row Cancellation Uncoupling Lemma significantly simplifies the next results, since it allows us to consider connection matrices with only one block, which means only elementary column operations need be performed in the matrix update step. This special instance of the Row Cancellation Algorithm over $\mathbb{F}$ will be called 1-Block Row Cancellation Algorithm over $\mathbb{F}$. Notice that, although the proposition guarantees the equalities of the primary pivots up to $r = m - 1$, this is sufficient, since the primary pivots of Δ^{m-1} and of Δ^m are equal.

Let Δ be a connection matrix with row/column partition J_0, J_1, J_2. Let $\Delta^1, \ldots, \Delta^m$ and $T^1, \ldots, T^{m-1}$ (resp., $\widehat{\Delta}^1, \ldots, \widehat{\Delta}^{m-1}$ and $\widehat{T}^1, \ldots, \widehat{T}^{m-2}$) be the matrices produced in the application of the SSSA over $\mathbb{F}$ (resp., Row Cancellation Algorithm over $\mathbb{F}$) to Δ. Then the primary pivots of Δ^r and $\widehat{\Delta}^r$ coincide in position and value, for $r = 1, \ldots, m - 1$.

This fact, Theorem 2.1 and the specific change of basis done in the proof of this theorem imply that the primary pivots of $\widehat{\Delta}^{m-1}$, the last matrix produced by the application of the Row Cancellation Algorithm over $\mathbb{F}$ to the order m surface connection matrix Δ, are either 1 or -1. This implies the transition matrices (and their inverses) produced by the application of the Row Cancellation Algorithm over $\mathbb{F}$ to a surface connection matrix are all integral. This justifies the application of the Row Cancellation Algorithm over $\mathbb{F}$ to surface connection matrices. When the input to the Row Cancellation Algorithm over $\mathbb{F}$ is restricted to the special class of surface connection matrices, we obtain *Smale's Cancellation Sweeping Algorithm.* ■

2.5 Cancellation of Critical Points: Birth and Death of Connections

In this section, the dynamical meaning of the Spectral Sequence Sweeping Algorithm is described via Smale's Cancellation Sweeping Algorithm. First, one shows in Theorem 2.3 that the SCSA determines a continuation of the initial flow by cancelling pair of consecutive critical points. Then one establishes in Theorem 2.4 a correspondence between algebraic cancellations in SSSA with these dynamical cancellations of critical points in SCSA. To achieve our goal, from now on, we will consider Morse chain complexes $(C_*(f), \partial)$ where f is a Morse function with one critical point per critical level set, which is a generic condition. Moreover, the filtration F on $(C_*(f), \partial)$ will be the one determined by the function f, i.e., if $Crit(f) = \{h^1, \cdots, h^m\}$ and $f(h^j) = c_j$, then $F = \{F_p\}_{p=1}^m$, where $F_p = f^{-1}(-\infty, c_p + \epsilon)$ and $\epsilon > 0$ is sufficiently small.

Before going into the proof of Theorem 2.3, an example is presented where one sees the interplay between the dynamics and the algebra codified in the spectral sequence.

Example 2.5. Consider the Morse chain complex (C, Δ) previously considered in Example 2.3 and the family of matrices produced by the SCSA when applied to Δ in Example 2.4. One can associate the primary pivots identified during the algorithm with dynamical cancellations of critical points as follows: a primary pivot $\widetilde{\Delta}^r_{i,j}$ identified in the r-th step of the SCSA indicates that the pair of critical points (h^i_{k-1}, h^j_k) is cancelled in the $(r+1)$-th step. See Figure 2.20, where the flows in the continuation are represented on the 2-spheres.

Moreover, the matrices produces by the SCSA contain the connection matrices of the flows in the continuation. The matrix $\widetilde{\Delta}^1$ is a connection matrix of φ_1. The submatrix of

$\widetilde{\Delta}^2$ obtained by eliminating columns and rows $3, 4, 7, 8$ is a connection matrix of the flow φ_2. The submatrix of $\widetilde{\Delta}^4$ obtained by eliminating columns and rows $2, 3, 4, 5, 6, 7, 8, 9$ is a connection matrix of the flow φ_4. The birth and death of connections are registered in the sequence of connection matrices produced by the SCSA.

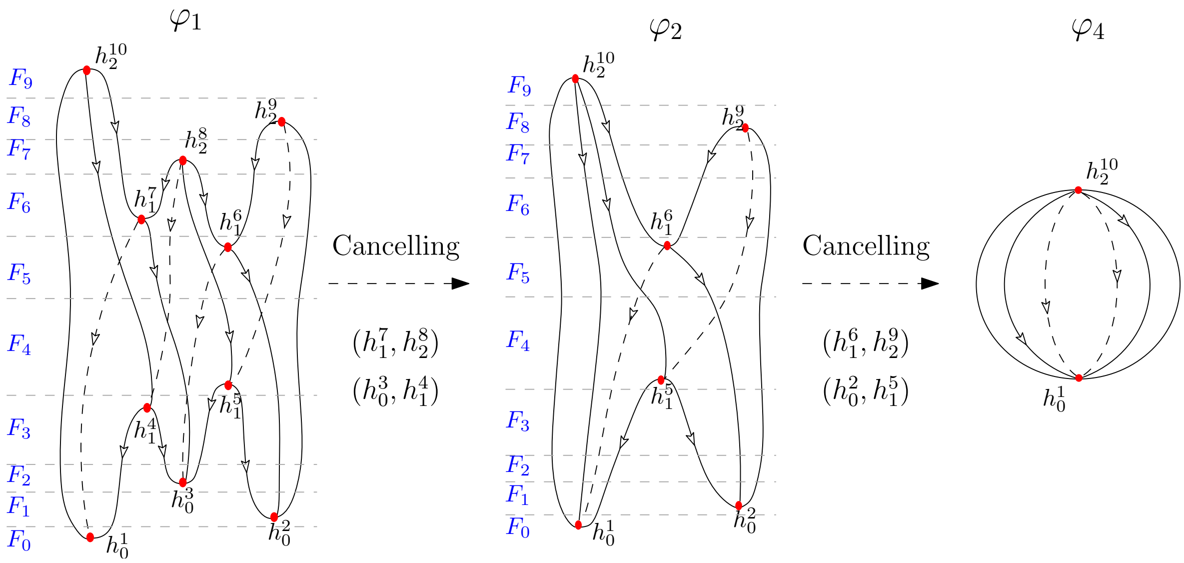


Figure 2.20: Continuation via cancellation of critical points.

On the other hand, consider the spectral sequence associated to the Morse chain complex (C_*, ∂_*) presented below.

$$
\begin{array}{lcccccccccc}
E^0: & [h_0^1] & [h_0^2] & [h_0^3] & [h_1^4] & [h_1^5] & [h_1^6] & [h_1^7] & [h_2^8] & [h_2^9] & [h_2^{10}] \\
E^1: & E_0^1 & E_1^1 & E_2^1 & E_3^1 & E_4^1 & E_5^1 & E_6^1 & E_7^1 & E_8^1 & E_9^1 \\
E^2: & E_0^2 & E_1^2 & 0 & 0 & E_4^2 & E_5^2 & 0 & 0 & E_8^2 & E_9^2 \\
E^3: & E_0^3 & E_1^3 & 0 & 0 & E_4^3 & E_5^3 & 0 & 0 & E_8^3 & E_9^3 \\
E^4: & E_0^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_9^3
\end{array}
$$

The primary pivots identified in the SSSA determine algebraic cancellations of modules in the spectral sequence. More specifically, if $\Delta_{p-r+1,p+1}^r$ is a primary pivot marked in the r-th step, then the modules E_p^{r+1} and E_{p-r}^{r+1} are null. In the example above, two entries, namely $\Delta_{3,4}^1$ and $\Delta_{7,8}^1$, are marked in the first step of the SSSA as primary pivots. This pivots induce the differentials d_3^1 and d_7^1, respectively, of the first page of the spectral sequence (E^r, d^r). Hence, in the second page of (E^r, d^r) the modules E_3^2, E_2^2, E_7^2 and E_6^2 are null.

By Theorem 2.2, the primary pivots marked in the r-th step of the SSSA are identical to those marked in the r-th step of the SCSA. Therefore, one can associate the algebraic cancellations in (E^r, d^r) with the dynamical cancellations of critical points. ◀

For the remainder of this section, we formalize the ideas presented in Example 2.5 by proving in Theorem 2.3 that the SCSA determines a continuation of the Morse flow in question, and in Theorem 2.4 we establish the correspondence between algebraic cancellations in the spectral sequence with dynamical cancellations of critical points.

The proof of Theorem 2.3 constructs a family of flows $\{\varphi^r\}$, $r = 1, \ldots, \omega$ recursively, where φ^{r+1} is obtained from φ^r by removing the pairs of critical points corresponding to the cancelled spaces of r-th page of the spectral sequence, i.e. the pairs associated to the primary pivots on the r-th diagonal of Δ^r. The main point is to prove that each algebraic cancellation of the spectral sequence can in fact be associated to a dynamical cancellation. In order to do this, we have to prove that whenever a primary pivot $\Delta^r_{j-r,j}$ on the r-th diagonal of Δ^r is marked, it is actually an intersection number between two consecutive singularities h_k^j and h_{k-1}^{j-r} of a flow φ^r. This intersection number must be ± 1 by Theorem 2.1 and hence we can use Smale's Cancellation Theorem to realise the dynamical cancellation. See [40] for the classical Cancellation Theorem referred here as the Smale's Cancellation Theorem.

The *filtration length* with respect to a filtration F of the orbit $\mathcal{O}_{h_k h_{k-1}}$ that connects h_k to h_{k-1} is defined as being the natural number r whenever $h_k \in F_p C$ and $h_{k-1} \in F_{p-r} C$. The number r is also called the *gap* between the singularities h_k and h_{k-1}.

Theorem 2.3. *Smale's Cancellation Sweeping Algorithm for the connection matrix $\Delta(M, \varphi_f)$ produces a family of Morse-Smale flows $\{\varphi^1 = \varphi_f, \varphi^2, \ldots, \varphi^\omega\}$ where φ^r continues to φ^{r+1} by cancelling all pairs of critical points of gap r with respect to the filtration F.*

Proof. The proof is divided in three steps. The first considers the local effect a cancellation of a pair of critical points has on a connection matrix $\Delta(M, \varphi')$ of the new flow φ'. The second step analyzes the global effect of this cancellation on $\Delta(M, \varphi')$. The third step constructs a family of Morse-Smale flows $\{\varphi^1 = \varphi_f, \varphi^2, \ldots, \varphi^\omega\}$ via the Smale's Cancellation Sweeping Algorithm.

Throughout the proof, we adopt the loose terminology that a critical point h_k^j connects with a critical point h_{k-1}^i if the moduli space $\mathcal{M}_{h_k^j}^{h_{k-1}^i}$ is non-zero.

Step 1: Without loss of generality, the connection matrix $\Delta(M, f, Or)$ associated to f is considered to be the one where all orientations of $W^u(h_2)$ are chosen to be the same. With this choice, the orbits in $W^s(h_1)\backslash\{h_1\}$ have opposite characteristic signs. On the other hand, by definition, the orbits in $W^u(h_1)\backslash\{h_1\}$ always have opposite characteristic signs.

Denote by $n(h_k, h_{k-1}, \varphi)$ the intersection number of h_k and h_{k-1} with respect to the flow φ.

Let h_k^j and h_{k-1}^{j-r} be consecutive critical points of a Morse-Smale function f. For a gradient flow φ_f , if $n(h_k^j, h_{k-1}^{j-r}, \varphi_f) = \pm 1$ then by Smale's Cancellation Theorem these critical points can be cancelled, i.e. there is a gradient flow φ' which coincides with φ_f outside a neighborhood of $\{h_k^j, h_{k-1}^{j-r}\} \cup \mathcal{O}(u)$, where $\mathcal{M}_{h_k^j}^{h_{k-1}^{j-r}} = \{u\}$. Let h_1^j be a saddle which connects with the sinks h_0^{j-r} and h_0^i. If h_1^j cancels with h_0^{j-r}, then each saddle h_1^p which connects with h_0^{j-r} in φ will connect with h_0^i in φ'. Since the old and new connections have the same characteristic signs, then $n(h_1^p, h_0^i, \varphi') = n(h_1^p, h_0^{j-r}, \varphi) + n(h_1^p, h_0^i, \varphi)$.

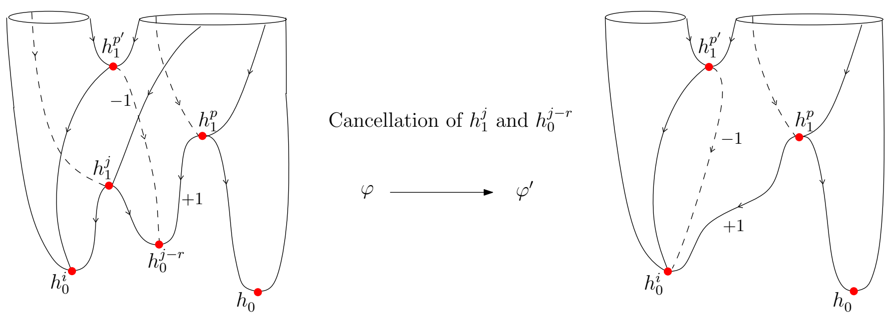


Figure 2.21: Birth and death of connections.

Let h_2^j and h_2^p be sources which connect with a saddle h_1^{j-r} and assume that h_2^j cancels with h_1^{j-r}. Then each saddle h_1^i which connects with h_2^j in φ will connect with h_2^p in φ'. Since the old and new connections have the same characteristic signs, then $n(h_2^p, h_1^i, \varphi') = n(h_2^p, h_1^{j-r}, \varphi) + n(h_2^p, h_1^i, \varphi)$.

Step 2: Since the flow φ' coincides with the flow φ outside a neighborhood U of $\{h_k^j, h_{k-1}^{j-r}\} \cup \mathcal{O}(u)$, then connections between $h_q^{\ell_2}$ and $h_{q-1}^{\ell_1}$, where $\ell_1, \ell_2 \notin \{j, j-r\}$, which do not intersect U are not changed. Also, their characteristic signs remain the same after the cancellation in U occurs.

Therefore the intersection number of $h_q^{\ell_2}$ and $h_{q-1}^{\ell_1}$ remain the same after cancellation, whenever

1. $k \neq q$, that is, $k = 1$ and $q = 2$ or $k = 2$ and $q = 1$,

2. $\mathcal{M}_{h_q^{\ell_2}}^{h_{k-1}^{j-r}} = \emptyset$ when $k = q = 1$,

3. $\mathcal{M}_{h_q^{\ell_2}}^{h_{k-1}^{j-r}} = \{u\}$ and $h_{q-1}^{\ell_1} \neq h_0^i$, when $k = q = 1$,

4. $\mathcal{M}_{h_k^j}^{h_{q-1}^{\ell_1}} = \emptyset$ when $k = q = 2$,

5. $\mathcal{M}_{h_k^j}^{h_{q-1}^{\ell_1}} = \{v\}$ and $h_q^{\ell_2} \neq h_2^p$, when $k = q = 2$,

since these are the only cases where connecting orbits from $h_q^{\ell_2}$ to $h_{q-1}^{\ell_1}$ are not born and do not die during the cancellation. Hence, the only intersection numbers that are modified after cancellation are those $n(h_1^p, h_0^i)$, where h_1^p is such that $\mathcal{M}_{h_1^p}^{h_0^{j-r}} \neq \emptyset$ in the case of saddle-sink cancellation, and those $n(h_2^p, h_1^i)$, where h_2^p is such that $\mathcal{M}_{h_2^p}^{h_1^{j-r}} \neq \emptyset$, in the case of source-saddle cancellation.

In order to understand how $\Delta(M, f_{\varphi'})$ is obtained from $\Delta(M, f_\varphi)$ one must analyze the effect that a cancellation of critical points in φ has on $\Delta(M, f_{\varphi'})$:

1. If a saddle h_1^j is cancelled with a sink h_0^{j-r}, then define the matrix $\widetilde{\Delta}$ to be the matrix obtained from $\Delta(M, f_\varphi)$ by replacing row i by the sum of row $(j - r)$ to row i. Then $\Delta(M, f_{\varphi'})$ is the submatrix of $\widetilde{\Delta}$ which does not contain rows $j - r$, j and neither column $j - r$, j.

2. If a source h_2^j is cancelled with a saddle h_1^{j-r}, then define the matrix $\widetilde{\Delta}$ to be the matrix obtained from $\Delta(M, \varphi)$ by replacing column p by the sum of column $(j - r)$ to column p. Then $\Delta(M, f_{\varphi'})$ is the submatrix of $\widetilde{\Delta}$ which does not contain rows $j - r$, j rows and columns $j - r$, j.

This corresponds to the row operations in the Smale's Cancellation Sweeping Algorithm.

Step 3: Let $\{\widetilde{\Delta}^r\}$ be the matrices produced by the Smale's Cancellation Sweeping Algorithm. Define $\varphi^1 = \varphi$ and define φ^{r+1} to be a flow obtained from φ^r by cancelling all pairs of critical points corresponding to primary pivots on the r-th diagonal of $\widetilde{\Delta}^r$. In order to show that these flows are well defined, we have to prove that whenever a primary pivot $\Delta_{j-r,j}^r$ on the r-th diagonal of $\widetilde{\Delta}^r$ is marked, it is actually an intersection number between two consecutive singularities h_k^j and h_{k-1}^{j-r} of the flow φ^r and hence they can be cancelled by Smale's Cancellation Theorem.

Since $\varphi^1 = \varphi$, the connection matrix $\Delta(M, \varphi^1)$ is $\widetilde{\Delta}^1$. Let $\Delta^1_{j-1,j} = \pm 1$ be a primary pivot on the first diagonal of $\widetilde{\Delta}^1$. By definition, this primary pivot represents the intersection number between two singularities of the flow φ^1, namely h^j_k and h^{j-1}_{k-1}, which are consecutive since the gap between them is one. Using Smale's Cancellation Theorem, we can define a flow φ^2 by cancelling all pairs of critical points corresponding to primary pivots on the first diagonal of $\widetilde{\Delta}^1$. Moreover, by step 2, the connection matrix $\Delta(M, \varphi^2)$ is the submatrix obtained from $\widetilde{\Delta}^2$ which does not contain the columns and rows corresponding to the cancelled singularities. Because of this and the fact that all non-zero entries of $\widetilde{\Delta}^2$ belong to $\Delta(M, \varphi^2)$, each non-zero entry of $\widetilde{\Delta}^2$ represents an intersection number between two singularities of φ^2. Observe that two singularities h^j_k and h^{j-2}_{k-1} of φ^2 with gap two in the filtration F are consecutive in the flow φ^2 since all the gap 1 singularities have been cancelled in the previous stage.

Suppose that φ^r is well defined, that is, each primary pivot $\Delta^{r-1}_{j-(r-1),j}$ on the diagonal $(r-1)$ of $\widetilde{\Delta}^{r-1}$ corresponds to the intersection number of consecutive singularities h^j_k and $h^{j-(r-1)}_{k-1}$ of φ^{r-1} and the connection matrix $\Delta(M, \varphi^r)$ is a submatrix of $\widetilde{\Delta}^r$ which does not contain columns and rows of $\widetilde{\Delta}^r$ corresponding to all primary pivots marked until the diagonal $r-1$. These correspond to all singularities of φ of gap less than or equal to $r-1$. Under these hypothesis singularities h^i_k and h^{i-r}_{k-1} of φ^r with gap r with respect to the filtration F are consecutive in the flow φ^r. Hence two singularities corresponding to a primary pivot on the diagonal r of $\widetilde{\Delta}^r$ can be cancelled, by Smale's Cancellation Theorem. Therefore, φ^{r+1} is a well defined flow obtained from φ^r by cancelling all pairs of critical points corresponding to primary pivots on the diagonal r of $\widetilde{\Delta}^r$. Moreover, the connection matrix $\Delta(M, \varphi^{r+1})$ is a submatrix of $\widetilde{\Delta}^{r+1}$ which does not contain columns and rows of $\widetilde{\Delta}^{r+1}$ corresponding to all primary pivots marked until step r. The flow φ continues to φ^r for all r. ■

Corollary 2.4. *There is a continuation from φ_f to the minimal flow φ_{min}.*

Proof. By Theorem 2.3, following the cancellation of pairs of critical points determine by the SSSA, one obtain a continuation of the initial flow φ to a flow φ^ω, The connection matrices associated to some flows of this continuation is produced by the Smale's Cancellation Sweeping Algorithm, as seen in the proof of Theorem 2.3.

Since, non zero entries in the last matrix $\tilde{\Delta}^L$ produced by the SCSA must be above primary pivots, then the submatrix of $\tilde{\Delta}^L$ which is a connection matrix for the last flow φ^ω is the null matrix. Therefore, φ^ω corresponds to the minimal flow. ■

The following theorem establishes a correspondence between the algebraic cancellations of modules on the spectral sequence with dynamical cancellations of critical points.

Theorem 2.4 (Ordered Smale's Cancellation Theorem via Spectral Sequence)**.** *Let (C, Δ) be the Morse chain complex associated to a Morse-Smale function f. Let (E^r, d^r) be the associated spectral sequence for the finest filtration $F = \{F_pC\}$ defined by f. The algebraic cancellation of the modules E^r of the spectral sequence are in one-to-one correspondence with dynamical cancellations of critical points of f. Moreover, the order of cancellation occurs as gap r increases.*

Proof. As it was proved in [13], the non zero differentials of the spectral sequence are induced by the pivots. When working on surfaces, the connection matrices are under more limiting conditions than the connection matrices for k-manifolds with $k > 2$. It follows from the Primary Pivots for Orientable Surfaces Theorem (Theorem 2.1) that the primary pivots are always equal to ± 1. Hence, the differentials $d^r_p : E^r_p \to E^r_{p-r}$ associated to primary pivots are isomorphisms and the ones associated to change-of-basis pivots always correspond to zero maps. In fact, if a differential $d^r_p : E^r_p \to E^r_{p-r}$ corresponds to a change-of-basis pivot, then there is a primary pivot in row $p - r$ and thus $E^r_{p-r} = 0$. Consequently, the non-zero differentials are isomorphisms and this implies that at the next stage of the spectral sequence they produce algebraic cancellations, i.e. if a primary pivot $\Delta^r_{p-r+1,p+1}$ is marked in step r then $E^{r+1}_p = E^{r+1}_{p-r} = 0$.

Note that the algebraic cancellations $E^{r+1}_p = E^{r+1}_{p-r} = 0$ are associated to the primary pivots $\Delta^r_{p-r+1,p+1} = \pm 1$ on the r-th diagonal of Δ^r in the Spectral Sequence Sweeping Algorithm, row $p - r + 1$ is associated to $h^{p-r+1}_{k-1} \in F_{p-r}C_{k-1} \setminus F_{p-r-1}C_{k-1}$ and column $p + 1$ is associated to $h^{p+1}_k \in F_pC_k \setminus F_{p-1}C_k$ in a gradient flow φ associated to f. By the Primary Pivots Equality Theorem (Theorem 2.2), the primary pivot $\Delta^r_{p-r+1,p+1} = \pm 1$ is also a primary pivot $\widetilde{\Delta}^r_{p-r+1,p+1} = \pm 1$ of the Smale's Cancellation Sweeping Algorithm. By Theorem 2.3 the primary pivot $\widetilde{\Delta}^r_{p-r+1,p+1}$ is an intersection number of two consecutive singularities h^{p+1}_k and h^{p-r+1}_{k-1} of the flow φ^r. By Smale's Cancellation Theorem, there is a dynamical cancellation of this pair of critical points.

Note that E^r_p and E^r_{p-r} correspond to a saddle and a sink or a source and a saddle, respectively, with gap r with respect to the filtration F. Hence, dynamically and algebraically, the cancellations occur with increasing gap. ■

In summary, the spectral sequence cancellation policy follows a proximity algorithm, where proximity is measured by closeness within the filtration determined by a Morse (height) function, i.e in increasing order of filtration length. In other words, the spectral sequence starts by cancelling all consecutive indices (i.e saddle-sink or source-saddle) within gap one. This is done by considering the modules and differentials (E^1, d^1). Next, the cancellation occurs by considering the modules and differentials (E^2, d^2) and cancelling all consecutive index critical points within gap 2. And so forth, cancellation occurs by considering the modules and differentials (E^r, d^r) and consecutive index critical points within gap r.

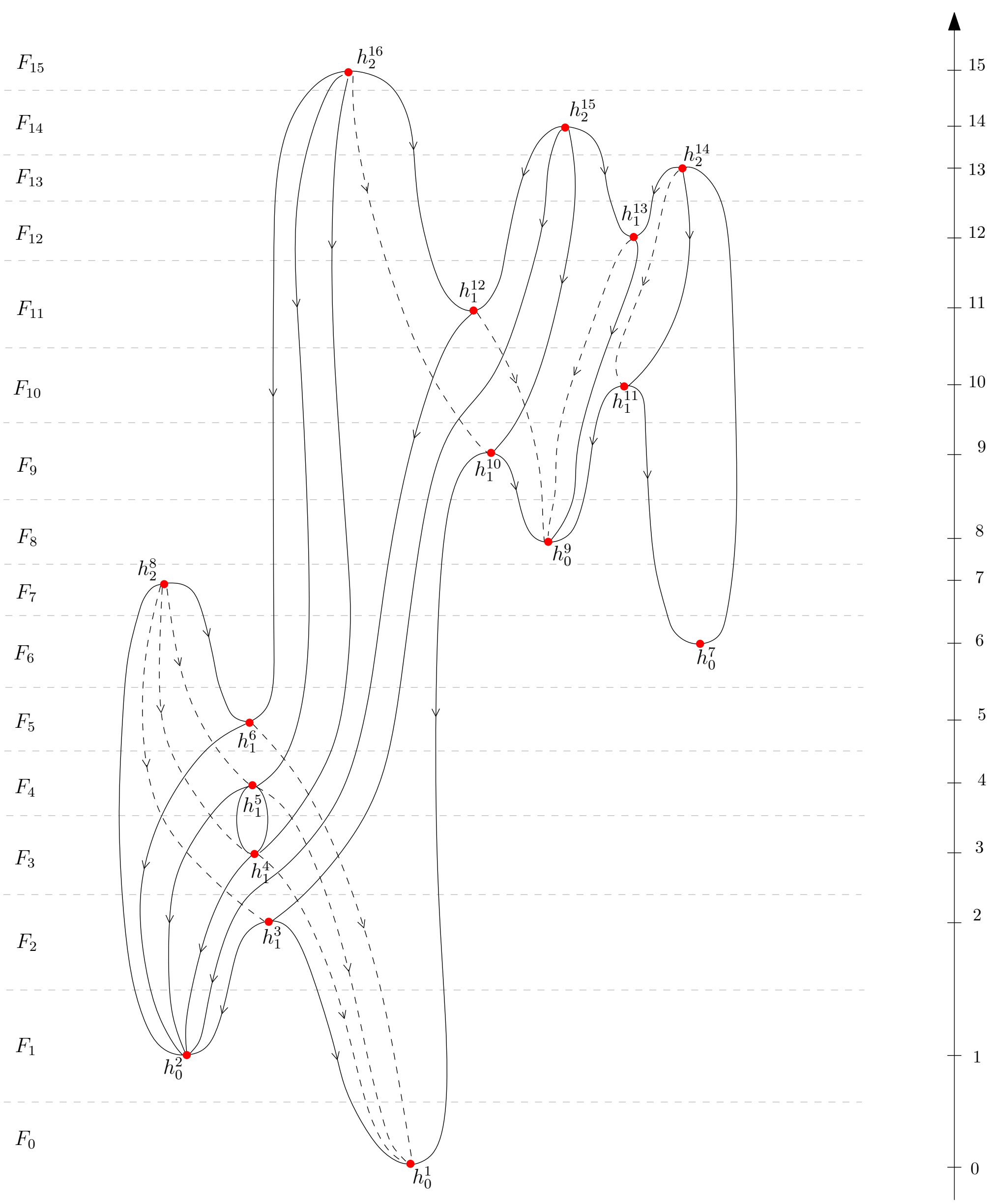


Figure 2.22: Morse flow on a torus and the finest filtration.

Example 2.6. Consider the flow on the torus as in Figure 2.22, which is associated to a Morse-Smale function f. Observe that each critical level set of f contains only one critical point of f. Consider the filtration on the Morse chain complex $(C_*(f), \Delta)$ to be the one determined by the function f, as illustrated in Figure 2.22. The connection matrix Δ associated to this flow and filtration is given by the matrix presented in Figure 2.23.

Applying the SSSA to Δ, we obtain the matrices $\Delta^1, \ldots, \Delta^{11}$ given by Figures 2.24,..., 2.34, respectively. In these figures, the primary pivot entries are indicated by means of a light red background and darker edge, the change-of-basis pivots are indicated by blue background and dashed edges, null entries are left blank and the diagonal being swept is indicated with a gray line.

Figure 2.35 illustrates the cancellations of pairs of critical points of f following the SSSA, i.e., the cancellations occur in order by gap proximity, where the filtration is given by the height function as in Figure 2.22. In Figure 2.35, we make use of Lyapunov graphs (Reeb graphs with labels) Γ^r associated to the flows φ^r to represent what happens dynamically when critical points are cancelled. ◀

		-1	-1	-1	-1				1						
		1	1	1	1						1				
							1							-1	
							-1								1
							1								-1
							-1								1
										-1					
									-1	1	-1				
														-1	1
														1	-1
													1	-1	

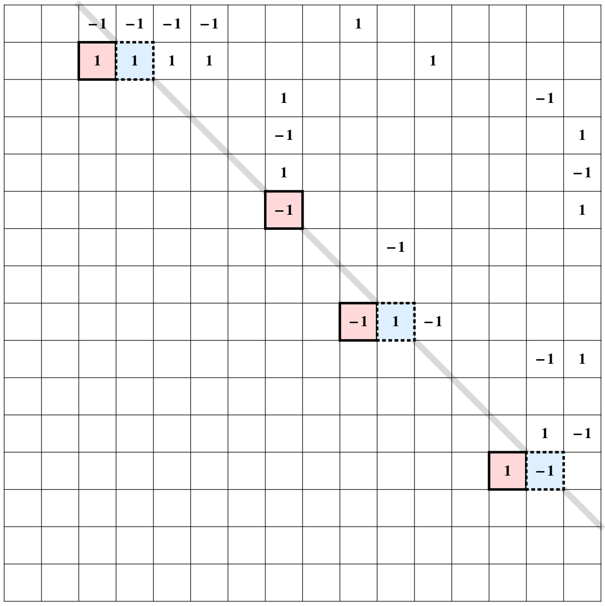
$J_0 = \{1,2,7,9\}$, $J_1 = \{3,4,5,6,10,11,12,13\}$, $J_2=\{8,14,15,16\}$; $\sigma_0^{j,0} = h_0^j$ for $j \in J_0$; $\sigma_1^{j,0} = h_1^j$ for $j \in J_1$; $\sigma_2^{j,0} = h_2^j$ for $j \in J_2$.

Figure 2.23: Δ, connection matrix.

		-1	-1	-1	-1				1						
		$\boxed{1}$	1	1	1						1				
							1							-1	
							-1								1
							1								-1
							-1								1
										-1					
									$\boxed{-1}$	1	-1				
														-1	1
														1	-1
													$\boxed{1}$	-1	

$\sigma_0^{j,1} = h_0^j$ for $j \in J_0$;
$\sigma_1^{j,1} = h_1^j$ for $j \in J_1$;
$\sigma_2^{j,1} = h_2^j$ for $j \in J_2$.

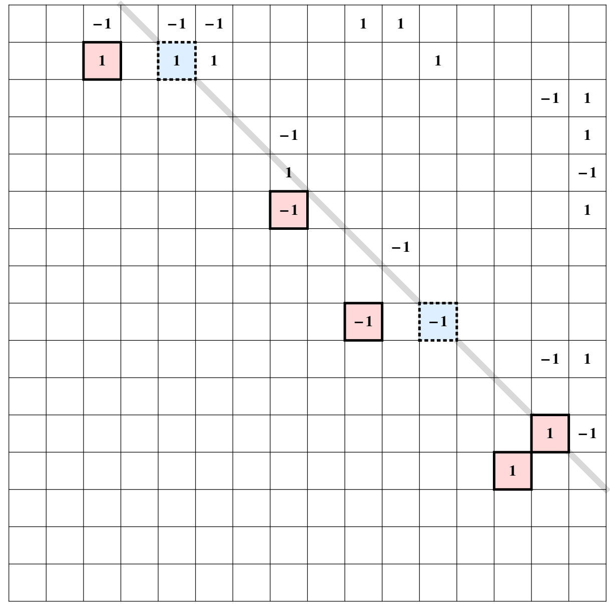
Figure 2.24: Δ^1, marking primary pivots.



$\sigma_0^{j,2} = h_0^j$ for $j \in J_0$; $\sigma_1^{j,2} = h_1^j$ for $j \in J_1$;

$\sigma_2^{j,2} = h_2^j$ for $j \in J_2$.

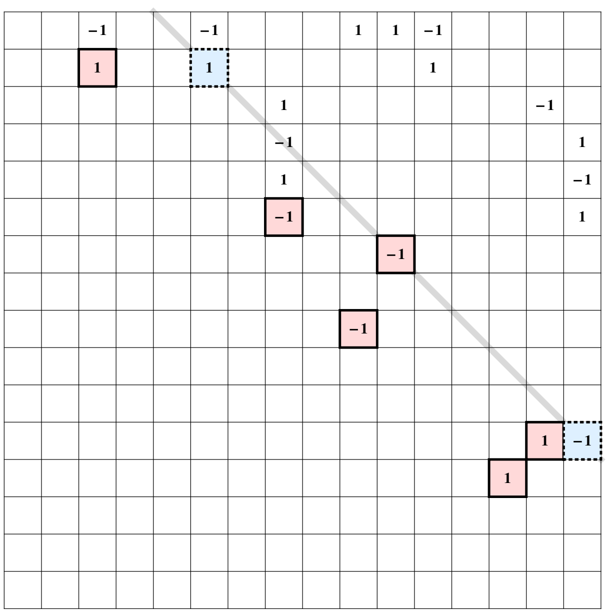
Figure 2.25: Δ^2, marking pivots.



$\sigma_1^{4,3} = h_1^4 - h_1^3$; $\sigma_1^{11,3} = h_1^{10} - h_1^{11}$; $\sigma_2^{15,3} = h_2^{14} - h_2^{15}$.

$\sigma_k^{j,3} = \sigma_k^{j,2}$ for all the remaining σ's.

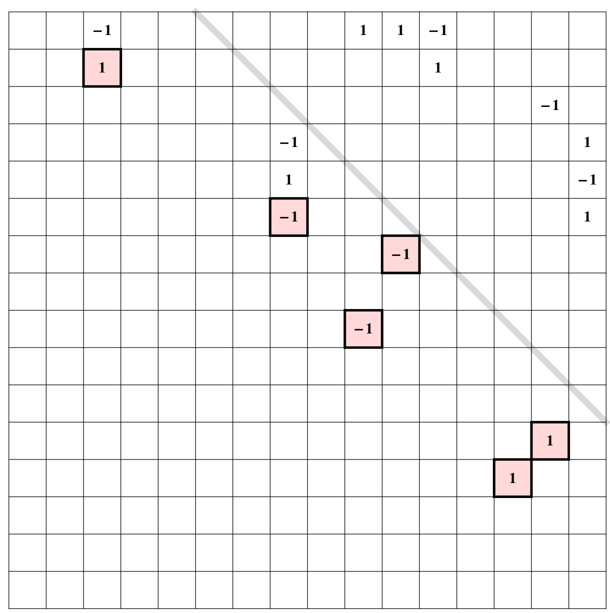
Figure 2.26: Δ^3, marking pivots.



$\sigma_1^{5,4} = h_1^5 - h_1^3$; $\sigma_1^{12,4} = h_1^{12} - h_1^{10}$.

$\sigma_k^{j,4} = \sigma_k^{j,3}$ for all the remaining σ's.

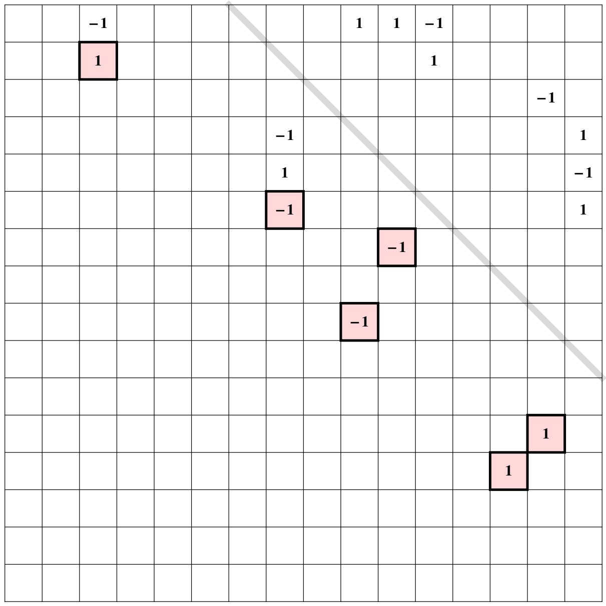
Figure 2.27: Δ^4, marking pivots.



$\sigma_1^{6,5} = h_1^6 - h_1^3$; $\sigma_2^{16,5} = h_2^{14} + h_2^{15} + h_2^{16}$.

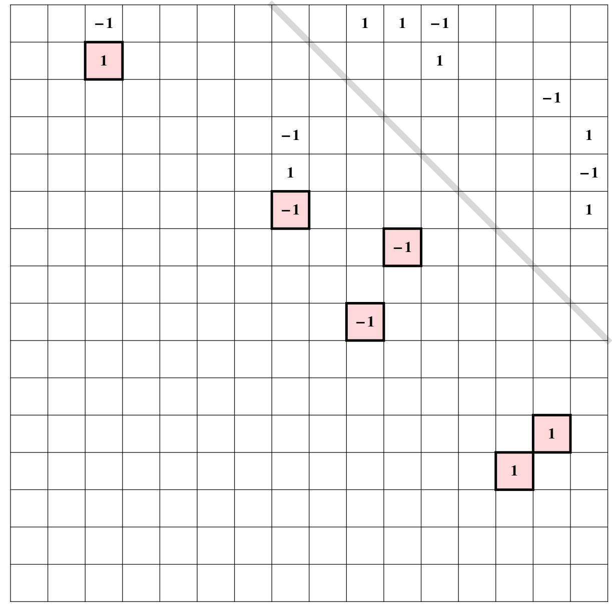
$\sigma_k^{j,5} = \sigma_k^{j,4}$ for all the remaining σ's.

Figure 2.28: Δ^5, sweeping 5-th diagonal.



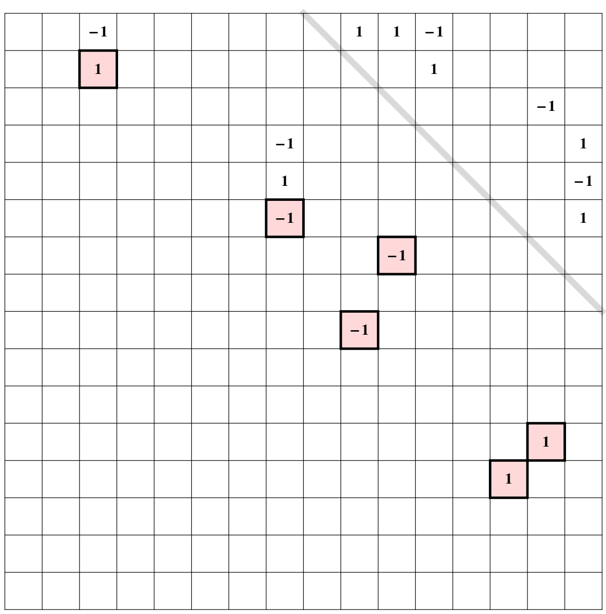
$\sigma_k^{j,6} = \sigma_k^{j,5}$ for all σ's.

Figure 2.29: Δ^6, sweeping 6-th diagonal.



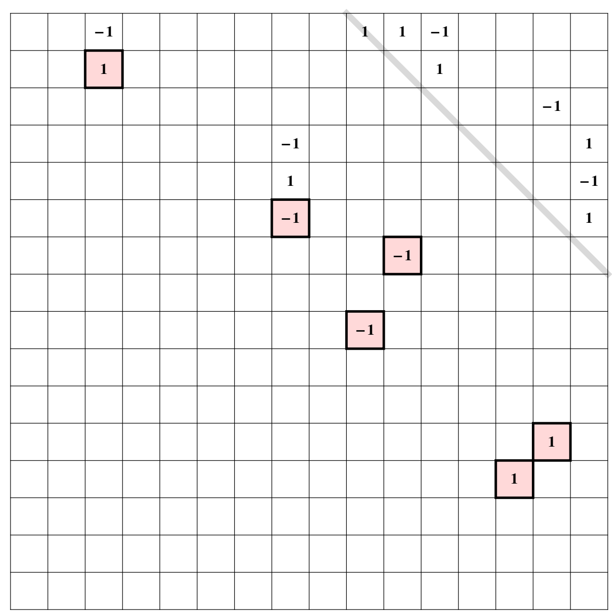
$\sigma_k^{j,7} = \sigma_k^{j,6}$ for all σ's.

Figure 2.30: Δ^7, sweeping 7-th diagonal.



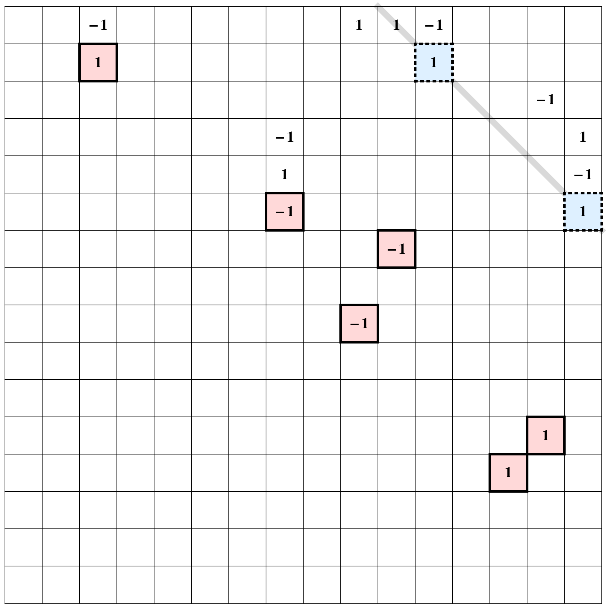
$\sigma_k^{j,8} = \sigma_k^{j,7}$ for all σ's.

Figure 2.31: Δ^8, sweeping 8-th diagonal.



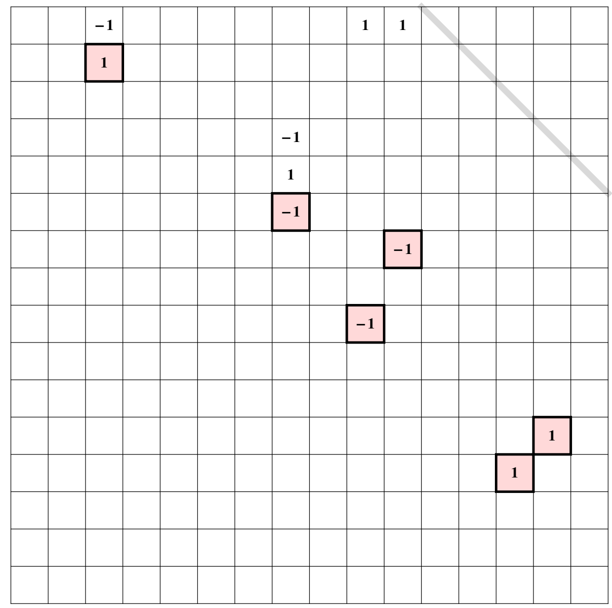
$\sigma_k^{j,9} = \sigma_k^{j,8}$ for all σ's.

Figure 2.32: Δ^9, sweeping 9-th diagonal.



$\sigma_k^{j,10} = \sigma_k^{j,9}$ for all σ's.

Figure 2.33: Δ^{10}, marking pivots.



$\sigma_1^{12,11} = -h_1^{10} + h_1^{12} - h_1^3$; $\sigma_2^{16,11} = h_2^{14} + h_2^{15} + h_2^{16} + h_2^8$.

$\sigma_k^{j,5} = \sigma_k^{j,4}$ for all the remaining σ's.

Figure 2.34: Δ^{11}, sweeping 11-th diagonal.

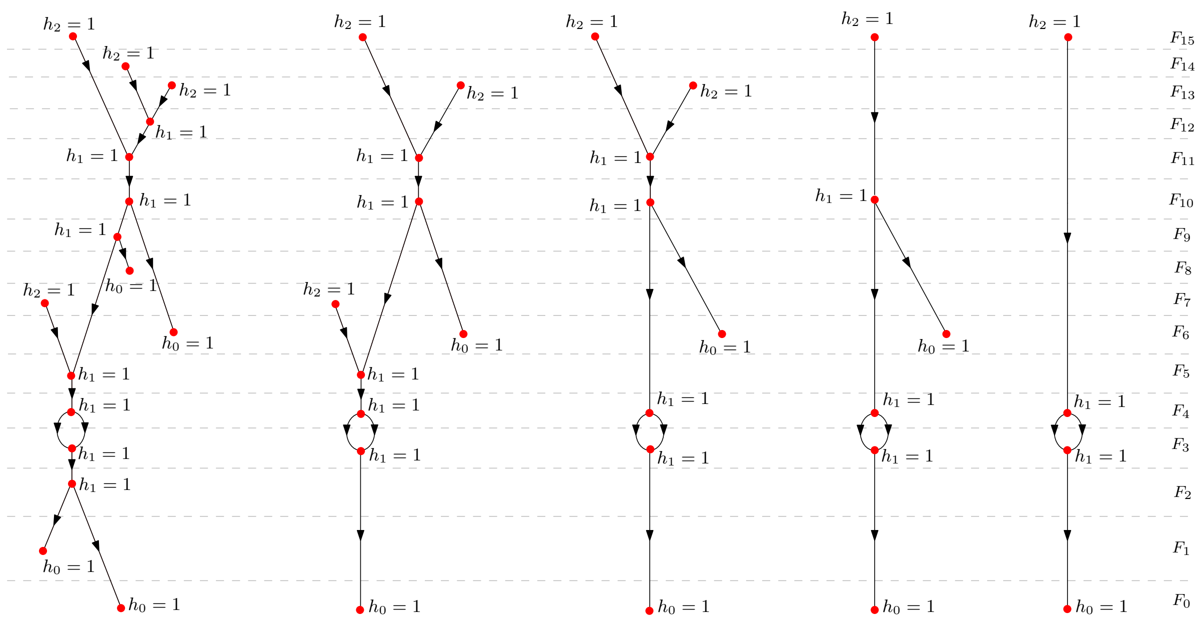


Figure 2.35: Representation of the SCSA by means of Lyapunov graphs.

Final Remarks

The field of computation topology has been interested in algorithms which solve problems anywhere from reconstruction of surfaces in computer graphics to the treatment of noise in data input. For example, to gather information about a surface using a computer, it is necessary to make use of a combinatorial representation. This information can be provided by making use of simplicial complexes. Data structures are then constructed with the purpose of storing cell complex information. To retrieve topological connectivity information from this data, homology is used. Persistent homology is an invariant that records the "homological history of a space that is undergoing growth" (see [43]). Algorithms were developed initially to compute the persistent homology of simplicial complexes but have been extended recently to general classes of filtered cell complexes. Computational topology will only prove useful to deal with massive data sets, if both theoretical results and practical algorithms are adequately elaborated.

More specifically, the computational geometric topology has been interested in the study of Morse-Smale complexes that arise from large data sets in order to extract significant topological features which can be measured in some sense by using persistent homology theory. In this theory there is a well known critical point cancellation rule referred to as the Elder's Rule [16], which cancels critical points with respect to the level sets determined by a height function as one sweeps from bottom to top, i.e. from lower level sets to higher level sets.

We approached this problem from a dynamical systems point of view. Our interest resides in understanding the bifurcation behaviour, i.e. birth and death of critical points, that parametrized families of flows on surfaces undergo. There are many techniques that may be used to achieve such an endeavour. However, the underlying approach used here has been to bridge the algebraic-topological and dynamical realms. As we have shown in this chapter, the SSSA determines a collection of connection matrices $\{\Delta^r\}$ which record the history of the birth and death of connecting orbits of φ_f as one calculates the spectral sequence of the filtered Morse chain complex $(C(f), \Delta)$. The major role in the birth of new connecting orbits is played by the primary and change of basis pivots. As one traverses the diagonals of the matrix via the Spectral Sequence Sweeping Algorithm, longer and longer connecting orbits are produced (birth) while the connecting orbits corresponding to the change of basis pivots are eliminated (death). In summary, longer orbits are born due to the death of shorter ones caused by the cancellation of critical points.

Chapter 3

Spectral Sequences for two dimensional Novikov Complexes

Spectral sequence analysis has proven to be a useful algebraic tool in detecting bifurcation phenomenon within a parametrized family of flows. This was initially explored in [13, 24, 30] for a filtered Morse chain complex where the filtration is determined by a Morse function. For instance, as presented in the Background Section 1.2, a spectral sequence of a filtered Morse chain complex associated to a flow can be retrieved from its differential, which is a connection matrix, via the SSSA.

In Chapter 2, we obtained strong results on the interconnection between algebraic and dynamical information in this setting. Since the spectral sequence analysis in Chapter 2 was realised for a two dimensional filtered Morse chain complex over $\mathbb{Z}$, we were able to prove a global dynamical cancellation theorem (Theorem 2.3) as well as a continuation result presented in Theorem 2.4 that keeps track of birth and death of connections between singularities in a Morse flow.

With this motivation in mind, we undertake a new dynamical setup, namely Morse-Novikov flows which arise within Novikov's theory as gradients of circle-valued Morse functions $f : M \to S^1$ on a surface M.

We consider, in this chapter, Novikov chain complexes $(\mathcal{N}_*, \partial)$ associated to circle-valued Morse functions f on surfaces. The Novikov modules $\mathcal{N}_*$ are the $\mathbb{Z}((t))$-modules freely generated by the critical points of f. The Novikov differential ∂ "counts" over $\mathbb{Z}((t))$ orbits with signs connecting consecutive critical points. In this context, we will prove that the SSSA is well defined and it also recovers the spectral sequence associated to the Novikov chain complex as in the Morse case.

The difficulty herein is that we lose the notion of how to minimize a leading coefficient of a change of basis when we are working with the ring $\mathbb{Z}((t))$. Our first step in generalizing the spectral sequence analysis for Novikov complexes will be done on orientable surfaces, since in this case one can always zero out a change-of-basis by using only the column of the corresponding primary pivot.

This chapter is organized as follows. The Novikov chain complex is introduced in Section 3.1, where we also characterize the Novikov incidence coefficients between consecutive critical points.

In Section 3.2, we present the Spectral Sequence Sweeping Algorithm (SSSA) for a two dimensional Novikov complex over $\mathbb{Z}((t))$. This algorithm produces a collection of Novikov matrices generated from the Novikov differential.

In Section 3.3, we prove in Theorem 3.2 that the SSSA for a Novikov chain complex is well defined. Also, the Novikov matrices which appear in SSSA are characterized in Theorems 3.3 and 3.4. In Theorem 3.5, we prove the surprising result that the last matrix produced by the SSSA has polynomial entries in $\mathbb{Z}((t))$, although the intermediate Novikov matrices exhibit entries which are infinite series.

In Section 3.4, we prove in Theorems 3.6 and 3.7 that from the sequence of Novikov matrices produced by the SSSA, the modules and differentials (E^r, d^r) of the spectral sequence may be retrieved. More specifically, the SSSA provides a system which spans E^r in terms of the original basis of $\mathcal{N}$ as well as identifies all differentials $d_p^r : E_p^r \to E_{p-r}^r$.

3.1 Novikov Complex

In this section, some background material on circle-valued functions and on Novikov complexes over $\mathbb{Z}((t))$ are presented. Further details can be found in [35]. Moreover, a characterization of the Novikov differential is proven in the case of orientable surfaces.

Denote by $\mathbb{Z}[t, t^{-1}]$ the Laurent polynomial ring. Let $\mathbb{Z}((t))$ be the set consisting of all Laurent series

$$\lambda = \sum_{i \in \mathbb{Z}} a_i t^i$$

in one variable with coefficients $a_i \in \mathbb{Z}$, such that the negative part of λ is finite, i.e., there is $n = n(\lambda)$ such that $a_k = 0$ if $k < n(\lambda)$. In fact, $\mathbb{Z}((t))$ has a natural ring structure such that the inclusion $\mathbb{Z}[t,t^{-1}] \subset \mathbb{Z}((t))$ is a homomorphism. Moreover, $\mathbb{Z}((t))$ is a Euclidean ring.

Let M be a closed connected manifold and $f : M \to S^1$ be a smooth map from M to the one-dimensional sphere[1]. Given a point $x \in M$ and a neighbourhood V of $f(x)$ in S^1 diffeomorphic to an open interval of $\mathbb{R}$, the map $f|_{f^{-1}(V)}$ is identified to a smooth map from $f^{-1}(V)$ to $\mathbb{R}$. Therefore, in this context one can define non-degenerate critical points and Morse index as in the classical case of smooth real valued map. A smooth map $f : M \to S^1$ is called a circle-valued Morse function if its critical points are non-degenerate. The set of critical points of f will be denoted by $Crit(f)$ and, more specifically, $Crit_k(f)$ is the set of critical points of f of index k.

Considering the exponential function $Exp : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$ and a covering $E : \overline{M} \to M$ such that $E_\pi(\pi_1(\overline{M})) \subset \mathrm{Ker} f_\pi$, where E_π and f_π are the induced maps in π_1 by E and f respectively, there exists a map $F : \overline{M} \to \mathbb{R}$ which makes the following diagram commutative:

$$\begin{tikzcd} \overline{M} \arrow[r, "F"] \arrow[d, "E"'] & \mathbb{R} \arrow[d, "Exp"] \\ M \arrow[r, "f"] & \mathbb{S}^1 \end{tikzcd}$$

Moreover, f is a *circle-valued Morse function* if and only if F is a real valued Morse function. Observe that if $Crit(F)$ is non empty then it has infinite cardinality. If $\overline{M}$ is non compact, one can not apply the classical Morse theory to study F, however, one can restrict F to a fundamental cobordism W of $\overline{M}$, which is compact, and apply the techniques of Morse theory. The *fundamental cobordism* W is defined as $W = F^{-1}([a-1, a])$, where a is a regular value of F. The cobordism W can be viewed as the manifold M obtained by cutting along the submanifold $V = f^{-1}(\alpha)$, where $\alpha = Exp(a)$. Hence, W is a cobordism with both boundary components diffeomorphic to V.

Given a circle-valued Morse function f, consider the vector field $v = -\nabla f$. One says that f satisfies the transversality condition if the lift of v to $\overline{M}$ satisfies the classical transversality condition on the unstable and stable manifolds.

From now on, consider circle-valued Morse functions f such that $v = -\nabla f$ satisfies the transversality condition. Denote by $\overline{v}$ the lift of v to $\overline{M}$ and arbitrarily choose orientations for all unstable manifolds $W^u(p)$ of critical points of f.

Given $p \in Crit_k(f)$ and $q \in Crit_{k-1}(f)$, *the Novikov incidence coefficient* between p and

[1] S^1 is viewed as the submanifold $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and is endowed with the corresponding smooth structure.

q is defined as the Laurent series

$$N(p,q;v) = \sum_{\ell \in \mathbb{Z}} n(p, t^{\ell} q; \overline{v}) t^{\ell},$$

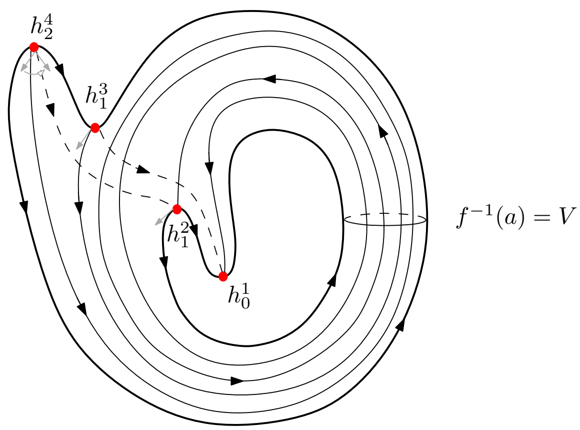
where $n(p, t^{\ell}q; \overline{v})$ is the intersection number between the critical points p and $t^{\ell}q$ of F, i.e., the number obtained by counting with signs the flow lines of $\overline{v}$ from p to $t^{l}q$, when one considers the orientations on the unstable manifolds $W^u(p)$ and $W^u(t^{\ell}q)$ according to the previous fixed orientations in $W^u(p)$ and $W^u(q)$. See [3] and [35].

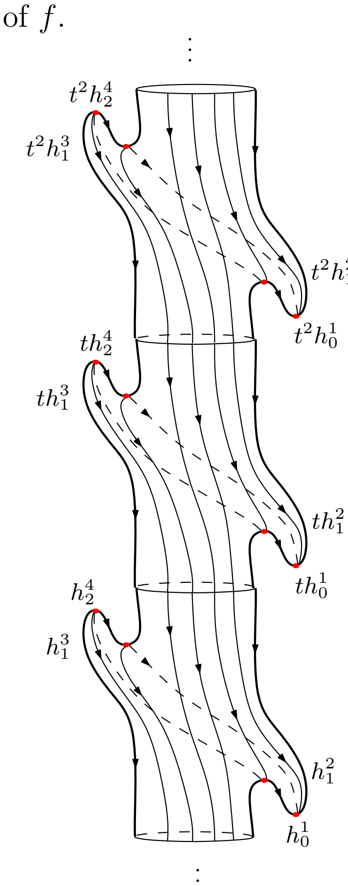
Let $\mathcal{N}_k$ be the $\mathbb{Z}((t))$-module freely generated by the critical points of f of index k. Consider the k-th boundary operator $\partial_k : \mathcal{N}_k \to \mathcal{N}_{k-1}$ which is defined on a generator $p \in Crit_k(f)$ by

$$\partial_k(p) = \sum_{q \in Crit_{k-1}(f)} N(p,q;v) q$$

and extended to all chains by linearity. In [35] it is proved that $\partial_k \circ \partial_{k+1} = 0$, hence $(\mathcal{N}_*, \partial_*)$ is a chain complex which is called the *Novikov complex* associated to the pair (f, v).

Example 3.1. As a first example, consider the flow on a torus T associated to a circle-valued Morse function f, as in figure below, where a is a regular value of f.





The Novikov chain groups are $\mathcal{N}_0 = \mathbb{Z}((t))\{h_0^1\}$, $\mathcal{N}_1 = \mathbb{Z}\{h_1^2, h_1^3\}$ and $\mathcal{N}_2 = \mathbb{Z}\{h_2^4\}$. The figure on the right represents a covering space of T. Choosing the orientations for the unstable manifolds of the critical points of f as indicated in the figure above, one can compute the intersection number between consecutive critical points with respect to the Morse flow in the cobordism.

In this case, the Novikov incidence coefficient is a polynomial $N(p, q; v) = \sum_{l \in \mathbb{Z}} a_l t^l$, where a_l is the intersection number between p and $t^l q$ with respect to the Morse flow in the cobordism, i.e., a_l is the number of flows lines from p to q, counted with signs, intersecting l times the regular level set $f^{-1}(a)$. Therefore, the Novikov incidence coefficients for this example are given by:

1. $N(h_1^2, h_0^1; \overline{v}) = t - 1$ indicating two flow lines from h_1^2 to h_0^1, one of which intersects $f^{-1}(a)$ once and the other does not intersect;

2. $N(h_1^3, h_0^1; \overline{v}) = t^2 - 1$, indicating two flow lines from h_1^3 to h_0^1, one of which intersects $f^{-1}(a)$ twice and the other does not intersect;

3. Analogously, $N(h_2^4, h_1^2; \overline{v}) = t^2 - 1$ and $N(h_2^4, h_1^3; \overline{v}) = t - 1$.

The Novikov boundary operator is defined on the generators by: $\partial_1(h_1^2) = (t-1)h_0^1$, $\partial_1(h_1^3) = (t^2 - 1)h_0^1$ and $\partial_2(h_2^4) = (t^2 - 1)h_1^2 + (t-1)h_1^3$. ◀

One can consider the Novikov differential ∂ as a matrix Δ where each column corresponds to generators $p, q \in Crit(f)$ and the entries are the coefficients $N(p, q; v)$ of the Novikov differential ∂. Moreover, one assumes that the columns of Δ are ordered with respect to the Morse indices of the critical points, e.g., in increasing order with respect to the Morse index. See Figure 3.1.

$$
\Delta \quad = \quad
\begin{array}{cc}
 & \begin{array}{cccccccc} \mathcal{N}_0 & \cdots & \mathcal{N}_{k-1} & \mathcal{N}_k & \mathcal{N}_{k+1} & \mathcal{N}_{k+2} & \cdots & \mathcal{N}_n \end{array} \\
\begin{array}{c} \mathcal{N}_0 \\ \vdots \\ \mathcal{N}_{k-1} \\ \mathcal{N}_k \\ \mathcal{N}_{k+1} \\ \mathcal{N}_{k+2} \\ \vdots \\ \mathcal{N}_n \end{array} &
\left(\begin{array}{cccccccc}
0 & & & & & & & \\
 & & & & & & & \\
0 & & \boxed{\Delta_{k-1}} & & & 0 & & \\
0 & & 0 & \boxed{\Delta_k} & & & & \\
0 & & & 0 & \boxed{\Delta_{k+1}} & & & \\
0 & & & & 0 & \boxed{\Delta_{k+2}} & & \\
 & & 0 & & & & & \\
0 & & & & & & & 0
\end{array}\right)
\end{array}
$$

Figure 3.1: Novikov differential viewed as a matrix, where Δ_k is the matrix representation of ∂_k.

In the case when M is a surface, the columns of Δ may be partitioned into subsets J_0, J_1, J_2 such that J_s are the columns associated with critical points of index s, i.e., the generators of $\mathcal{N}_s$. Hence, the non-zero entries of the matrix Δ are in the block $J_0 \times J_1$, which corresponds to connections from saddles to sinks, and in the block $J_1 \times J_2$, which corresponds to connections from sources to saddles. The block $J_0 \times J_1$ (respectively, $J_1 \times J_2$) is referred to as the first block (respectively, second block) of the matrix Δ. Figure 3.2 illustrates a possible structure for a Novikov differential associated to a circle-valued Morse function on a surface.

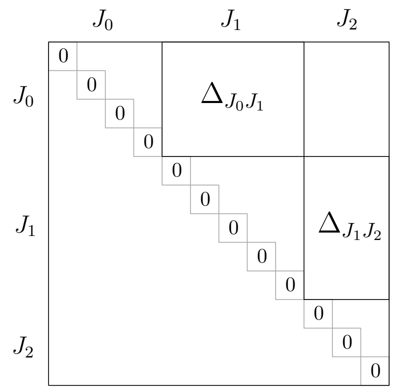


Figure 3.2: Novikov differential, with $J_0 = \{1, 2, 3, 4\}$, $J_1 = \{5, 6, 7, 8, 9\}$ and $J_0 = \{10, 11, 12\}$.

Since we work with a matrix representation for the Novikov differential ∂, we will switch the notation, from this point on, to the matrix Δ, hereupon referred to as a *Novikov matrix associated to* Δ.

The following theorem will describe special characteristics of the Novikov matrix associated to Δ by describing the Novikov incidence coefficients. In order to do this, we must define a fundamental domain. A cobordism $W = F^{-1}([a-1; a])$, where a is a regular value of F and $\lambda \in \mathbb{N}$, is said to be a *fundamental domain* for $(M; f)$ if the following property is satisfied: given $p \in Crit_k(f)$ and $q \in Crit_{k-1}(f)$, W contains a lift of each orbit of the flow v from p to q.

Theorem 3.1. *Let M be an orientable surface and $(\mathcal{N}_*, \Delta)$ be the Novikov complex associated to a circle-valued Morse function $f : M \to S^1$. The Novikov incidence coefficient $N(p, q; v)$ is either zero, a monomial $\pm t^{\ell}$ or a binomial $t^{\ell_1} - t^{\ell_2}$.*

Proof. Given an orientable surface M, let $W_\lambda = F^{-1}([a-1, a+\lambda])$ be a fundamental domain for (M, f). Then W_λ is an orientable compact surface with boundary ∂W_λ, possibly empty. If $\partial W_\lambda = \emptyset$, define $\widetilde{W}_\lambda = W_\lambda$. In the case that the boundary ∂W_λ is non empty, it is the disjoint union of $\partial W_\lambda^- = F^{-1}(a-1)$ and $\partial W_\lambda^+ = F^{-1}(a+\lambda)$. Let $\widetilde{W}_\lambda$ be the closed surface obtained from W_λ by gluing 2-dimensional disk along its boundary to each connected component of ∂W_λ. Moreover, one can assume that each of its disks contains a singularity, more specifically, a source if the disk is glued to ∂W_λ^+ and a sink if the disk is glued to ∂W_λ^-. This procedure extends the Morse function $F : W_\lambda \to \mathbb{R}$ to a classical Morse function $\widetilde{F} : \widetilde{W}_\lambda \to \mathbb{R}$ on a closed surface.

Given $p \in Crit_k(f)$ and $q \in Crit_{k-1}(f)$, the Novikov incidence coefficient $N(p, q; v)$ counts the number of flow lines from p to q with signs. Since, each of these flow lines has a lift in W_λ, then $N(p, q; v)$ can be obtained by analysing W_λ. On the other hand, the intersection number $n(p, t^\ell q; \overline{v})$ between critical points p and $t^\ell q$ in W_λ is the same as when considered in the surface $\widetilde{W}_\lambda$. Since $\widetilde{W}_\lambda$ is closed, $n(p, t^\ell q; \overline{v})$ is zero, when there are two flow lines from p to $t^\ell q$. It is -1 or $+1$, when there is one flow line from p to $t^\ell q$. See [4]. Therefore, the Novikov incidence coefficient $N(p, q; v)$ is:

(a) 0, if there are two flow lines from p to q in v which intersect the level set $f^{-1}(a)$ the same number of times;

(b) $\pm t^\ell$, if there is only one flow line from p to q which intersects ℓ times the level set $f^{-1}(a)$;

(c) $t^{\ell_1} - t^{\ell_2}$, if there are two flow lines from p to q in v, one intersecting ℓ_1 times and the other intersecting ℓ_2 times the level set $f^{-1}(a)$. ■

Corollary 3.1 (Characterization of the Novikov differential on orientable surfaces)**.** *Let M be an orientable surface and $(\mathcal{N}_*, \Delta)$ be the Novikov complex associated to a circle-valued Morse function $f : M \to S^1$ such that the chosen orientation on the unstable manifold of each critical point of index 2 is the same. Then there are three possibilities for either a column or a row $j \in J_1$ of Δ:*

(1) all entries are null;

(2) exactly one non zero entry which is a binomial $t^{\ell_1} - t^{\ell_2}$, for some $\ell_1, \ell_2 \in \mathbb{Z}$;

(3) exactly two non zero entries which are monomials t^{ℓ_1} and $-t^{\ell_2}$, for some $\ell_1, \ell_2 \in \mathbb{Z}$.

Proof. If f does not have a critical point of index 1, then the Novikov matrix is null. Suppose that f has at least one critical point of index 1, i.e., a saddle. In this case, it is clear that, given a row (respectively, column) $j \in J_1$ of Δ, there are at most two non zero entries in this row (respectively, column). In fact, there are exactly two flow lines whose ω-limit (respectively, α-limite) sets are the same saddle. By Corollary 2.2, choosing the same orientation for each unstable manifold of critical points of index 2, the signs on flow lines associated to the stable (respectively, unstable) manifold of a saddle are opposite. Therefore, if there are two flow lines from a source (respectively, saddle) p to a saddle (respectively, sink) q intersecting a regular level set $f^{-1}(a)$ ℓ times, then $N(p,q;v) = t^{\ell} - t^{\ell} = 0$. On the other hand, if there is a flow line from a source (respectively, saddle) p to a saddle (respectively, sink) q intersecting ℓ_1 times a regular level set $f^{-1}(a)$ and a flow line from a source (respectively, saddle) p' to the saddle (respectively, sink) q intersecting ℓ_2 times the same regular level set, then $N(p,q;v) = \pm t^{\ell_1}$ and $N(p',q;v) = \mp t^{\ell_2}$. ■

Example 3.2. Figure 3.3 illustrates a flow on the torus T^2 associated to a circle-valued Morse function f defined on T^2, where a is a regular value of f. The Novikov chain groups are $\mathcal{N}_0 = \mathbb{Z}((t))\{h_0^1, h_0^2\}$, $\mathcal{N}_1 = \mathbb{Z}((t))\{h_1^3, h_1^4, h_1^5, h_1^6\}$ and $\mathcal{N}_2 = \mathbb{Z}((t))\{h_2^7, h_2^8\}$. Choosing the orientations for the unstable manifolds of the critical points of f as indicated in Figure 3.3, the Novikov matrix associated to ∂ is presented in Figure 3.4. Hence, for example, $\partial(h_1^3) = (t-1)h_0^2$ meaning that there are two flow lines from h_1^3 to h_0^2 such that one intersects $f^{-1}(a)$ once and the other does not intersects this level set. ◀

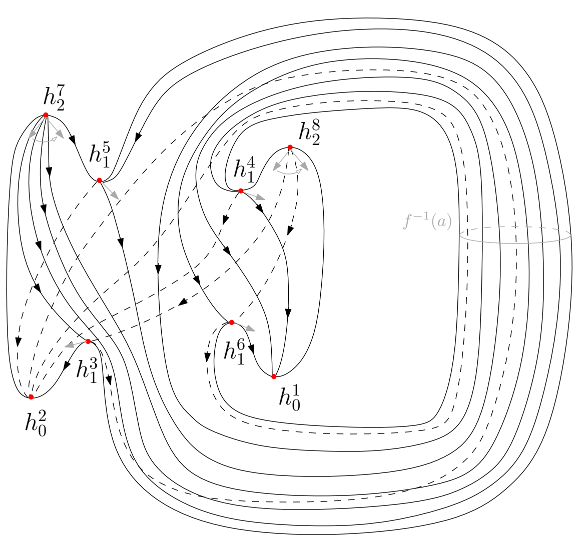


Figure 3.3: Circle-valued Morse function.

	h_0^1	h_0^2	h_1^3	h_1^4	h_1^5	h_1^6	h_2^7	h_2^8
h_0^1	**0**	**0**	**0**	**1**	$\boldsymbol{t}$	**1**	**0**	**0**
h_0^2	**0**	**0**	$\boldsymbol{t-1}$	$\boldsymbol{-1}$	$\boldsymbol{-1}$	$\boldsymbol{-t}$	**0**	**0**
h_1^3	**0**	**0**	**0**	**0**	**0**	**0**	**1**	$\boldsymbol{-1}$
h_1^4	**0**	**0**	**0**	**0**	**0**	**0**	$\boldsymbol{-t^2}$	**1**
h_1^5	**0**	**0**	**0**	**0**	**0**	**0**	$\boldsymbol{t-1}$	**0**
h_1^6	**0**	**0**	**0**	**0**	**0**	**0**	$\boldsymbol{t}$	$\boldsymbol{-1}$
h_2^7	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
h_2^8	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Figure 3.4: Novikov matrix.

3.2 Spectral Sequence Sweeping Algorithm for a Novikov Complex

A square matrix will be called a Novikov matrix if it is a strictly upper triangular matrix with square zero and entries in the ring $\mathbb{Z}((t))$.

In this section, we define the Spectral Sequence Sweeping Algorithm (SSSA) for a Novikov matrix associated to a Novikov complex $(\mathcal{N}_*, \Delta)$ on an orientable surface. The SSSA constructs a family of Novikov matrices $\{\Delta^r, r \geq 0\}$ recursively, where $\Delta^0 = \Delta$, by considering at each stage the r-th diagonal.

Spectral Sequence Sweeping Algorithm - SSSA

For a fixed r-th diagonal the method described below must be applied for all Δ_k for $k = 0, 1, 2$ simultaneously.

A - Initial step

1. Without loss of generality, we assume that the first diagonal[2] of Δ contains non-zero entries $\Delta_{i,j}$ where $j \in J_k$ and $i \in J_{k-1}$. Whenever the first diagonal contains only zero entries, we define $\Delta^1 = \Delta$ and we repeat this step until we reach a diagonal of Δ which contains non-zero entries.

   The non-zero entries $\Delta_{i,j}$ of the first diagonal are called *index k primary pivots*. It follows that the entries $\Delta_{s,j}$ for $s > i$ are all zero.

   We end this first step by defining Δ^1 as Δ with the index k primary pivots on the first diagonal marked.

2. Consider the matrix Δ^1 and let $\Delta^1_{i,j}$ be the entries in Δ^1 where the $i \in J_{k-1}$ and $j \in J_k$. Analogously to step one, we assume without loss of generality that the second diagonal contains non-zero entries $\Delta^1_{i,j}$. We now construct a matrix Δ^2 following the procedure:

   Given a non-zero entry $\Delta^1_{i,j}$ on the second diagonal of Δ^1

   (a) if there are no primary pivots in row i and column j, mark it as an index k primary pivot and the numerical value of the entry remains the same, i.e. $\Delta^2_{i,j} = \Delta^1_{i,j}$.

[2]By r-th diagonal one means the collection of entries Δ_{ij} of Δ such that $j - i = r$.

(b) if this is not the case, consider the entries in column j and in a row s with $s > i$ in Δ^1.

(b1) If there is an index k primary pivot in an entry in column j and in a row s, with $s > i$, then the numerical value remains the same and the entry is left unmarked, i.e. $\Delta^2_{i,j} = \Delta^1_{i,j}$.

(b2) If there are no primary pivots in column j below $\Delta^1_{i,j}$ then there is an index k primary pivot in row i, say a column u of Δ^1, with $u < j$. In this case, we define $\Delta^2_{i,j} = \Delta^1_{i,j}$ and this entry marked as a *change-of-basis pivot*.

Note that we have defined a matrix Δ^2 which is actually equal to Δ^1 except that the second diagonal is marked with primary and change of basis pivots.

B - Intermediate step

In this step we consider a matrix Δ^r with the primary and change of basis pivots marked on the ξ-th diagonal for all $\xi \leq r$. We now describe how Δ^{r+1} is defined. If there does not exist a change of basis pivot on the r-th diagonal we go directly to step B.2, that is, we define $\Delta^{r+1} = \Delta^r$ with the $(r+1)$-th diagonal marked with primary and change of basis pivots as in B.2.

B.1 - Change of basis

Suppose $\Delta^r_{i,j}$ is a change of basis pivot on the r-th diagonal. Since we have a change of basis pivot in row i, there is a column, namely u-th column, associated to a k-chain such that $\Delta^r_{i,u}$ is a primary pivot. Then, perform a change of basis on Δ^r in order to zero out the entry $\Delta^r_{i,j}$ without introducing non-zero entries in $\Delta^r_{s,j}$ for $s > i$. We will prove in Theorem 3.2 that all the entries in Δ^r which are primary pivots are equal to $\pm t^{l_1} \pm t^{l_2}$ and, since these entries are invertible in $\mathbb{Z}((t))$, it is always possible choosing a particular change of basis using just column j and u of Δ^r.

Once this is done, we obtain a k-chain associated to column j of Δ^{r+1}. It is a linear combination over $\mathbb{Z}((t))$ of column u of Δ^r and column j of Δ^r such that $\Delta^{r+1}_{i,j} = 0$. It is also a particular linear combination of the columns of Δ in J_k on and to the left of column j.

Let k_1 be the column of Δ^r which is associated to a k-chain. We denote by $\sigma_k^{j,r}$ to indicate the Morse index k and the column j of Δ^r. We have

$$\sigma_k^{j,r} = \sum_{\ell=k_1}^{j} c_\ell^{j,r} h_k^\ell,$$

and the column j of Δ^{r+1} is

$$\sigma_k^{j,r+1} = \sigma_k^{j,r} - \Delta_{i,j}^r (\Delta_{i,u}^r)^{-1} \sigma_k^{u,r} = c_{k_1}^{j,r+1} h_k^{k_1} + \cdots + c_{j-1}^{j,r+1} h_k^{j-1} + c_j^{j,r+1} h_k^j \tag{3.1}$$

where $c_\ell^{j,r+1} \in \mathbb{Z}((t))$ and $c_j^{j,r+1} = 1$.

Therefore the matrix Δ^{r+1} has entries determined by a change of basis over $\mathbb{Z}((t))$ of Δ^r. In particular, all the change-of-basis pivots on the r-th diagonal Δ^r are zero in Δ^{r+1}.

Once the above procedure is done for all change-of-basis pivots of the r-th diagonal of Δ^r we can define a change-of-basis matrix T^r, and let $\Delta^{r+1} = (T^r)^{-1}\Delta^r T^r$.

B.2 - Marking the $(r+1)$-th diagonal of Δ^{r+1}

Consider the matrix Δ^{r+1} defined in the previous step. We mark the $(r+1)$-th diagonal with primary and change of basis pivots as follows:

Given a non-zero entry $\Delta_{i,j}^{r+1}$

1. If there are no primary pivots in row i and column j, mark it as an index k primary pivot.
2. If this is not the case, consider the entries in column j and in a row s with $s > i$ in Δ^{r+1}.
   - (b1) If there is an index k primary pivot in the entries in column j below $\Delta_{i,j}^{r+1}$ then leave the entry unmarked.
   - (b2) If there are no primary pivots in column j below $\Delta_{i,j}^{r+1}$ then there is an index k primary pivot in row i, say in the column u of Δ^{r+1}, with $u < j$. In this case, mark it as a change of basis pivot.

C - Final step

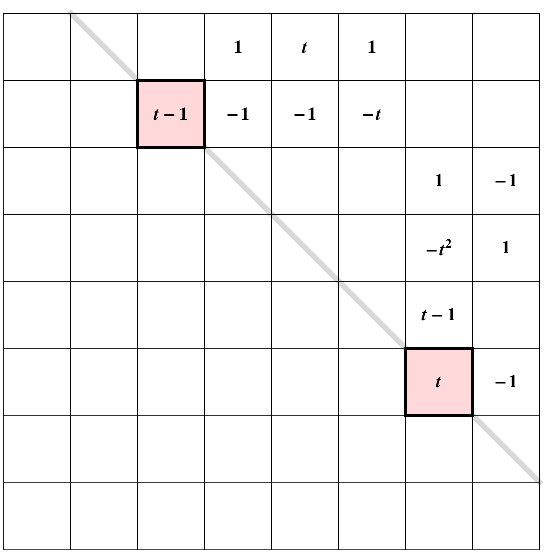
We repeat the above procedure until all diagonals have been considered.

Note that in the Spectral Sequence Sweeping Algorithm the columns of the matrix Δ are not necessarily ordered with respect to k, or equivalently, that the singularities h_k are not ordered with respect to the filtration. In this chapter, without loss of generality, we consider the singularities to be ordered with respect to the Morse index for the sole reason of simplifying notation.

In order to perform the particular change of basis (3.1) in step $B.1$ of the Spectral Sequence Sweeping Algorithm, the primary pivots must be invertible polynomials in the ring $\mathbb{Z}((t))$. Otherwise, the change of basis in (3.1) is not well defined. The example below shows a Novikov differential for which the SSSA is well defined.

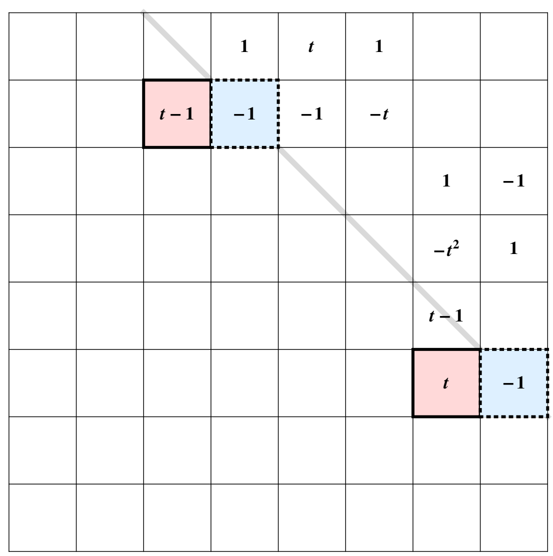
Example 3.3. Applying the SSSA to the Novikov matrix Δ in Figure 3.4, one obtains the sequence of Novikov matrices $\Delta^1, \cdots, \Delta^6$ presented in Figures 3.5, $\cdots$, 3.10, respectively. In these figures, the markup process at the r-th iteration is done as follows: primary pivots are encircled and change-of-basis pivots are encased in boxes.

Note that, in this example, each marked primary pivot in Δ^r is invertible in the ring $\mathbb{Z}((t))$, making it possible to apply the SSSA and obtain the next Novikov matrix Δ^{r+1}.



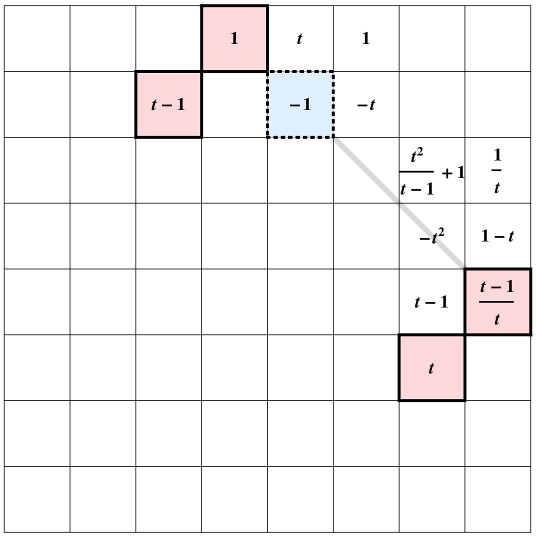
$\sigma_0^{j,1} = h_0^j$ for $j \in J_0$; $\sigma_1^{j,1} = h_1^j$ for $j \in J_1$;
$\sigma_2^{j,1} = h_2^j$ for $j \in J_2$.

Figure 3.5: Δ^1 for Example 3.3.



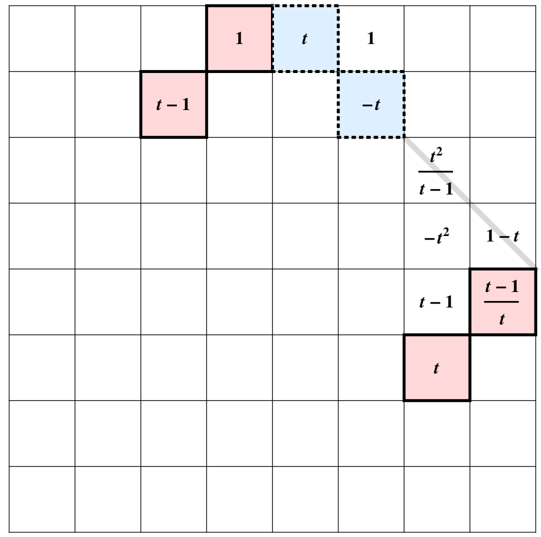
$\sigma_0^{j,2} = h_0^j$ for $j \in J_0$; $\sigma_1^{j,2} = h_1^j$ for $j \in J_1$;
$\sigma_2^{j,2} = h_2^j$ for $j \in J_2$.

Figure 3.6: Δ^2 for Example 3.3.



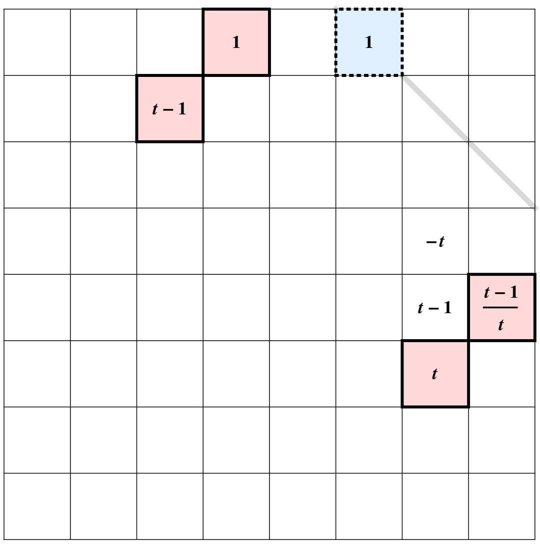
$\sigma_1^{4,3} = h_1^4 + (t-1)^{-1}h_1^3$; $\sigma_1^{8,3} = h_2^8 + t^{-1}h_2^7$.
$\sigma_k^{j,3} = \sigma_k^{j,2}$ for all the remaining σ's.

Figure 3.7: Δ^3 for Example 3.3.



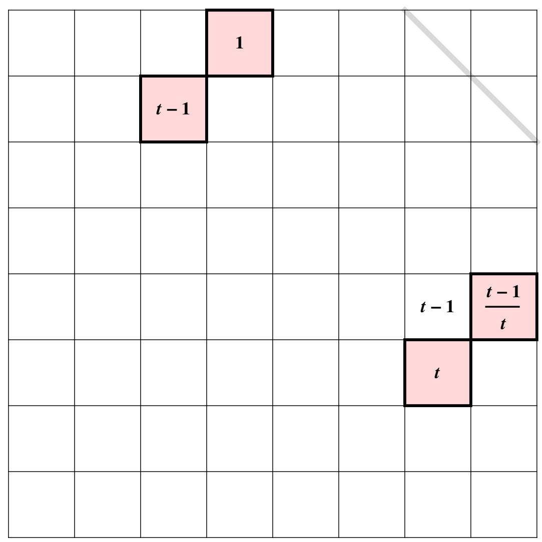
$\sigma_1^{5,4} = h_1^5 + (t-1)^{-1}h_1^3$.
$\sigma_k^{j,4} = \sigma_k^{j,3}$ for all the remaining σ's.

Figure 3.8: Δ^4 for Example 3.3.



$\sigma_1^{5,5} = h_1^5 - t\ h_1^4$; $\sigma_1^{6,5} = h_1^6 + t(t-1)^{-1}h_1^3$;
$\sigma_k^{j,5} = \sigma_k^{j,4}$ for all the remaining σ's.

Figure 3.9: Δ^5 for Example 3.3.



$\sigma_1^{6,6} = h_1^6 - \ h_1^4$;
$\sigma_k^{j,6} = \sigma_k^{j,5}$ for all the remaining σ's.

Figure 3.10: Δ^6 for Example 3.3.

3.3 Characterization of the Novikov Matrices

The primordial aim in this section is to show that the SSSA is well defined for all Novikov differentials of a 2-dimensional Novikov complex. This is done by showing that, given a Novikov differential Δ, all primary pivots determined by the SSSA are invertible polynomials in the ring $\mathbb{Z}((t))$. In fact, they are monomials with coefficient ± 1 or binomials with coefficients ± 1. Throughout this section, the term monomial (respectively, binomial) will be used to refer to polynomials in $\mathbb{Z}((t))$ of the form $\pm t^{\ell}$ (respectively, $t^{\ell_1} - t^{\ell_2}$), where $\ell \in \mathbb{Z}$ (respectively, $\ell_1, \ell_2 \in \mathbb{Z}$).

Theorem 3.2. *Given a Novikov differential Δ, the primary pivots and the change of basis pivots in the Spectral Sequence Sweeping Algorithm are polynomials of the form t^{ℓ} or $t^{\ell_1} - t^{\ell_2}$, where $\ell, \ell_1, \ell_2 \in \mathbb{Z}$.*

The proof of Theorem 3.2 is an immediate consequence of Theorem 3.3, Lemma 3.2 and Theorem 3.4.

Lemma 3.1 asserts that we cannot have more than one primary pivot in a fixed row or column. Moreover, if there is a primary pivot in row i, then there is no primary pivot in column i.

Lemma 3.1. *Let Δ be a Novikov differential for which the SSSA is well defined up to step R. Let $\Delta^1, \cdots, \Delta^R$ be the family of Novikov matrices produced by the SSSA until step R. Given two primary pivots, the ij- th entry $\Delta^r_{i,j}$ and the ml-th entry $\Delta^r_{m,l}$, then $\{i,j\} \cap \{m,l\} = \emptyset$.*

We omit the proof of Lemma 3.1, since it is similar in nature to proof of Proposition 3.2 in [13], where the SSSA was defined for a Morse chain complex over $\mathbb{Z}$. The next lemma implies that, in order to know the pivots which will appear during the execution of the SSSA, one can apply this algorithm separately in block $J_0 \times J_1$ and $J_1 \times J_2$.

Lemma 3.2. *Let Δ be a Novikov differential for which the SSSA is well defined up to step R. Then, the change of basis caused by change-of-basis pivots in block $J_0 \times J_1$ do not affect the pivots in block $J_1 \times J_2$. In other words, multiplication by $(T^r)^{-1}$ does not change the primary and change-of-basis pivots in block $J_1 \times J_2$.*

Proof. Without loss of generality, suppose that there is only one change-of-basis pivot $\Delta^r_{i,j}$ in Δ^r with $j \in J_1$. The change of basis matrix T^r has unit diagonal and the only non zero entry off the diagonal is $T^r_{u,j} = -\Delta^r_{i,j}(\Delta^r_{i,u})^{-1}$. Hence, $(T^r)^{-1}$ has unit diagonal and the only non zero entry off the diagonal is $(T^r)^{-1}_{u,j} = -T^r_{u,j} = \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}$. Therefore, multiplication by $(T^r)^{-1}$ will only affect row u of Δ^r. By Lemma 3.1, there are no primary pivots in row u and hence there are no change-of-basis pivot in row u as well. ■

Before proving Theorem 3.2, we introduce the notation and terminology that will be used in the proof. From now on, we consider the SSSA without realising the pre-multiplication by $(T^r)^{-1}$, unless mention otherwise.

Let $\Delta^r_{i,j}$ be a change-of-basis pivot caused by a primary pivot $\Delta^r_{i,u}$. Suppose a change of basis determined by $\Delta^r_{i,j}$ is performed by the SSSA in the matrix Δ^r, i.e., in the step $(r+1)$, one has

$$\sigma_k^{j,r+1} = \sigma_k^{j,r} - \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\sigma_k^{u,r}.$$

Hence, whenever this change-of-basis occurs, only column j of the matrix Δ^r is modified, in fact, for each $s = 0, \ldots, i$, the entry $\Delta^r_{s,j}$ is added to a multiple of the entry $\Delta^r_{s,u}$, where u is the column of the primary pivot in row i. In other words, $\Delta^{r+1}_{s,j} = \Delta^r_{s,j} - \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\Delta^r_{s,u}$. See the matrix in Figure 3.11 (this figure shows part of the block associated with index k, as the r-th diagonal is swept).

$$\begin{array}{cc}
 & \begin{array}{ccccc} & u & & j & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \left(\begin{array}{ccccc}
 & \vdots & & \vdots & \\
\cdots & \Delta^r_{s,u} & \cdots & \Delta^r_{s,j} & \cdots \\
 & \vdots & \ddots & \vdots & \\
\cdots & \Delta^r_{i,u} & \cdots & \boxed{\Delta^r_{i,j}} & \cdots \\
 & \vdots & & \vdots &
\end{array}\right)
\end{array}$$

Figure 3.11: $\Delta^r_{J_{k-1}J_k}$; marking change-of-basis pivot.

Definition 3.1. *In the situation described above and represented in Figure 3.11, we assert that the entry $\Delta^r_{s,u}$ in column u generates the entry $\Delta^{r+1}_{s,j}$ in column j in Δ^{r+1}, whenever $\Delta^r_{s,u} \neq 0$.* [3]

Note that if an entry in a column j generates another entry in a column t then $t > j$, i.e, $\Delta^r_{s,j}$ generates an entry in a column on the right of the column j.

Example 3.4. It is helpful to keep in mind some configurations that allow an entry $\Delta^r_{s,u} \neq 0$ to generate another entry $\Delta^{r+1}_{s,j}$. Consider for instance that $\Delta^r_{i,u} = t^{\ell}$ and $\Delta^r_{i,j} = -t^{\tilde{\ell}}$. We list some of the possibilities for the entries in positions (s,u) and (s,j) of Δ^r:

1. $\Delta^r_{s,u} = t^l$ and $\Delta^r_{s,j} = 0$. In this case, $\Delta^{r+1}_{s,j} = \Delta^r_{s,j} - \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\Delta^r_{s,u} = t^{\tilde{\ell}}t^{-\ell}t^l$, see Figure 3.12.

[3]If an entry $\Delta^{\xi}_{s,u}$ with $\xi < r$ does not change until step r, i.e. $\Delta^{\xi}_{s,u} = \Delta^{\xi+1}_{s,u} = \cdots = \Delta^r_{s,u}$, and $\Delta^r_{s,u}$ generates $\Delta^{r+1}_{s,j}$, we say that $\Delta^{\xi}_{s,u}$ generates $\Delta^{r+1}_{s,j}$.

2. $\Delta^r_{s,u} = -t^l$ and $\Delta^r_{s,j} = t^{\tilde{l}}$. In this case, $\Delta^{r+1}_{s,j} = \Delta^r_{s,j} - \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\Delta^r_{s,u} = t^{\tilde{l}} - t^{\tilde{\ell}}t^{-\ell}t^l$, see Figure 3.13.

3. $\Delta^r_{s,u} = t^l - t^{\tilde{l}}$ and $\Delta^r_{s,j} = 0$. In this case, $\Delta^{r+1}_{s,j} = \Delta^r_{s,j} - \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\Delta^r_{s,u} = t^{\tilde{\ell}}t^{-\ell}(t^l - t^{\tilde{l}})$, see Figure 3.14.

Figures 3.12, 3.13 and 3.14 show part of the block associated with index k, as the r-th diagonal is swept.

$$\begin{array}{cc} & \begin{array}{ccccc} & u & & j & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & t^l & \cdots & 0 & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & t^\ell & \cdots & \boxed{-t^{\tilde{\ell}}} & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \xrightarrow{\text{SSSA}} \begin{array}{cc} & \begin{array}{ccccc} & u & & j & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & t^l & \cdots & t^{\tilde{\ell}-\ell+l} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & t^\ell & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

Figure 3.12: $\Delta^r_{J_{k-1}J_k}$ and $\Delta^{r+1}_{J_{k-1}J_k}$, respectively.

$$\begin{array}{cc} & \begin{array}{ccccc} & u & & j & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & -t^l & \cdots & t^{\tilde{l}} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & t^\ell & \cdots & \boxed{-t^{\tilde{\ell}}} & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \xrightarrow{\text{SSSA}} \begin{array}{cc} & \begin{array}{ccccc} & u & & j & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & -t^l & \cdots & t^{\tilde{l}} - t^{\tilde{\ell}-\ell+l} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & t^\ell & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

Figure 3.13: $\Delta^r_{J_{k-1}J_k}$ and $\Delta^{r+1}_{J_{k-1}J_k}$, respectively.

$$\begin{array}{cc} & \begin{array}{ccccc} & u & & j & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & t^l - t^{\tilde{l}} & \cdots & 0 & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & t^\ell & \cdots & \boxed{-t^{\tilde{\ell}}} & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array} \xrightarrow{\text{SSSA}} \begin{array}{cc} & \begin{array}{ccccc} & u & & j & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & t^l - t^{\tilde{l}} & \cdots & t^{\tilde{\ell}-\ell+l} - t^{\tilde{\ell}-\ell+\tilde{l}} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & t^\ell & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

Figure 3.14: $\Delta^r_{J_{k-1}J_k}$ and $\Delta^{r+1}_{J_{k-1}J_k}$, respectively.

In fact, we will see as a consequence of Theorem 3.4, the cases shown in the previous example are the only possibilities up to sign of generating entries under a change of basis in block $J_1 \times J_2$. Of course, in block $J_0 \times J_1$ there are more possibilities.

Lemma 3.3. *An entry which is or will be marked as a primary or a change-of-basis pivot never generates entries.*

Proof. Observe that, if an entry $\Delta^r_{s,u}$ generates an entry in Δ^{r+1}, then there must be a primary pivot in column u and row $i > s$, which was marked in step $\xi < r$. Hence, $\Delta^r_{s,u}$ can not be marked as a pivot in any given step r. ■

Definition 3.2. *Let Δ be a Novikov differential.*

(a) When an entry $\Delta^r_{s,u}$ generates another one $\Delta^{r+1}_{s,j}$, we say that $\Delta^{r+1}_{s,j}$ is an immediate successor of $\Delta^{r+1}_{s,u}$.

(b) A sequence of entries $\{\Delta^{\xi_1}_{s,j_1}, \Delta^{\xi_2}_{s,j_2}, \cdots, \Delta^{\xi_f}_{s,j_f}\}$ such that each entry is an immediate successor of the previous one is called a generation sequence.

(c) Given an entry $\Delta_{i,j}$ of Δ, the $\Delta_{i,j}$-lineage is defined to be the set of all generation sequences whose first element is $\Delta_{i,j}$.

We will say that all the elements in these sequences are in the same lineage or in $\Delta^{\xi}_{i,j}$-lineage. Also an element of a generation sequence is said to be successor of every element of this sequence which is to its left.

Lemma 3.4. *Let Δ be a Novikov differential for which the SSSA is well defined up to step R. If Δ has the property that at most one change-of-basis pivot is marked in a row during the SSSA until step R, then every lineage is formed by a unique generation sequence.*

Proof. By hypothesis, one has that in each row i at most one change-of-basis pivot is marked through out the algorithm and if so the mark up is done in step $2 \leq \xi \leq m-1$, where m is the order of Δ. Then an entry $\Delta^{\xi}_{i,j}$ in row i generates at most one entry $\Delta^{\xi_1}_{i,j_1}$ through out the algorithm, and this entry will necessarily be in a column $j_1 > j$ and $\xi < \xi_1 \leq m-1$. In fact, if $\Delta^{\xi}_{i,j}$ generates two entries, then either there would be two change-of-basis pivots in row i, which contradicts our initial hypothesis, or two primary pivots in column j, which can not occur by the definition of primary pivots.

Now, if $\xi_1 \leq m-1$, then $\Delta^{\xi_1}_{i,j_1}$ can generate at most a unique entry $\Delta^{\xi_2}_{i,j_2}$ where $\xi_1 < \xi_2 \leq m-1$ and $j_2 > j_1$ and this can be done successively. More specifically, the entry $\Delta^{\xi}_{i,j}$ is a generator, i.e. it is responsible for generating a unique immediate successor and this successor can in turn generate a unique immediate successor and thus it determines a full lineage of entries represented in one finite sequence $\{\Delta^{\xi}_{i,j}, \Delta^{\xi_1}_{i,j_1}, \Delta^{\xi_2}_{i,j_2}, \cdots\}$, where $2 \leq \xi < \xi_1 < \xi_2 < \cdots \leq m-1$, and $j < j_1 < j_2 < \cdots \leq m$. ∎

Corollary 3.2. *Let Δ be a matrix for which the SSSA is well defined up to step R. Suppose that the second block of Δ has the property that at most one change-of-basis pivot is marked in each row from the beginning until the end of the SSSA. Therefore, each $\Delta_{i,j}$-lineage with $i \in J_1$ is formed by a unique generation sequence.*

Proof. The SSSA applied to the first block $J_0 \times J_1$ of Δ does not interfere in the number of change-of-basis pivots identified in block $J_1 \times J_2$. ∎

Consider a $\Delta_{i,j}$-lineage which is formed by a unique generation sequence. If this generation sequence contains only monomials (binomials, resp.) then one says that $\Delta_{i,j}$-lineage is a monomial (binomial, resp.) lineage. However, if $\Delta_{i,j}$ is a monomial, then the $\Delta_{i,j}$-lineage could eventually contain binomials. One way that this can occur is when row s is of type 3 in Δ, $\Delta_{s,u}$ and $\Delta_{s,j}$ are monomials and the lineage determined by these entries merge giving rise to a binomial. More specifically, suppose that the first binomial in row s appears in $\Delta^{\varsigma+1}$ then one has two monomial lineages $\{\Delta_{s,u}, \Delta^{\xi_1}_{s,u_1}, \Delta^{\xi_2}_{s,u_2}, \cdots, \Delta^{\xi_f}_{s,u_f}\}$ and $\{\Delta_{s,j}, \Delta^{\zeta_1}_{s,j_1}, \Delta^{\zeta_2}_{s,j_2}, \cdots, \Delta^{\zeta_f}_{s,j_f}\}$, where $\xi_f, \zeta_f \leq \varsigma$; observe that $\Delta^{\xi_f}_{s,u_f} = \Delta^{\varsigma}_{s,u_f}$ and $\Delta^{\zeta_f}_{s,j_f} = \Delta^{\varsigma}_{s,j_f}$. The binomial will appear in $\Delta^{\varsigma+1}$ as a consequence of a change of basis caused by a change-of-basis pivot $\Delta^{\varsigma}_{i,j_f}$ and a primary pivot $\Delta^{\varsigma}_{i,u_j}$ in a row $i > s$, as in Figure 3.15. In this case, $\Delta^{\varsigma}_{s,u_f}$ is the generator of the binomial $\Delta^{\varsigma+1}_{s,j_f}$. Hence, we say that the $\Delta_{s,j}$-lineage ceases, i.e, this lineage remains the same until Δ^{r+1} and the $\Delta_{s,j}$-lineage is an eventual binomial lineage. From this point on, this lineage contains only binomials.

$$\Delta^{\varsigma}_{J_a J_{a+1}} = \begin{array}{c c} & \begin{array}{cccccc} & u_f & & j_f & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \left(\begin{array}{cccccc} & \vdots & & \vdots & \\ \cdots & \Delta^{\varsigma}_{s,u_f} & \cdots & \Delta^{\varsigma}_{s,j_f} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & \Delta^{\varsigma}_{i,u_f} & \cdots & \boxed{\Delta^{\varsigma}_{i,j_f}} & \cdots \\ & \vdots & & \vdots & \end{array} \right) \end{array}$$

Figure 3.15: Generating a binomial from two monomial lineages.

Once an element of a lineage is marked as a pivot, this lineage ceases, since pivots do not generate entries, by Lemma 3.3.

The next theorem provides a characterization of columns in the first block $J_0 \times J_1$ of a Novikov matrix as the r-th diagonal is swept.

Theorem 3.3 (First Block Characterization)**.** *Let Δ be a Novikov differential for which the SSSA is well defined up to step R. Then we have the following possibilities for a column j, with $j \in J_1$, of the matrix Δ^r produced by the SSSA in the step $r \leq R$:*

1. *all the entries in column j are equal to zero, .i.e, $\Delta^r_{\bullet,j} = 0$.*
2. *there is only one non-zero entry in column j and it is a binomial $t^\ell - t^{\tilde{\ell}}$, where $\ell, \tilde{\ell} \in \mathbb{Z}$.*
3. *there are exactly two non-zero entries in column j and they are monomials t^ℓ and $-t^{\tilde{\ell}}$, where $\ell, \tilde{\ell} \in \mathbb{Z}$.*
4. *there is only one non-zero entry in column j and it is a monomial t^ℓ, where $\ell \in \mathbb{Z}$.*

Proof. The proof is done by induction. Note that the result is trivial for Δ^1 and Δ^2. In fact, the first change-of-basis pivot can be only detected from the second diagonal of Δ, this implies that the entries of Δ may change from $r = 3$ onwards. Because of that the base of the induction is $r = 3$.

r=3:

To prove that the rows of Δ^3 are of type 1-4, we will analyze the effect a change-of-basis pivot marked in Δ^2 has on Δ^3. Suppose, without loss of generality, that there is a change-of-basis pivot $\Delta^2_{i,i+2}$ on the second diagonal. Consequently, $\Delta^2_{i,i+1}$ is a primary pivot marked in Δ^1. Recall that the columns in Δ^2 satisfy Proposition 3.1. A primary pivot or a change-of-basis pivot can only occur in a column of Δ^2 if this column is of type 2 or 3. Hence, one has the following possibilities:

1. column $i+1$ is of type 2:
   (a) If the column $i+2$ is of type 2, then the primary pivot $\Delta^2_{i,i+1}$ and the change-of-basis pivot $\Delta^2_{i,i+2}$ are binomials. In this case, $\Delta^3_{i,i+2} = 0$ and all the other entries in column $i+2$ remain the same. Hence column $i+2$ turns into a column of type 1.
   (b) If the column $i+2$ is of type 3, then $\Delta^2_{i,i+2}$ is a monomial and $\Delta^2_{i,i+1}$ is a binomial. In this case, $\Delta^3_{i,i+2} = 0$ and all the other entries remain the same. Hence column $i+2$ turns into a column of type 4.

2. column $i+1$ is of type 3. Then $\Delta^2_{i,i+1}$ is a monomial and there is $s<i$ such that $\Delta^2_{s,i+1}$ is also a monomial.

   (a) If the column $i+2$ is of type 2, then the change-of-basis pivot $\Delta^2_{i,i+2}$ is a binomial. In this case, $\Delta^3_{i,i+2}=0$ and $\Delta^3_{s,i+2}=-\Delta^2_{s,i+1}(\Delta^2_{i,i+1})^{-1}\Delta^2_{i,i+2}$, which is a binomial. Hence, the column $i+2$ remains of type 2.

   (b) If the column $i+2$ is of type 3, then change-of-basis pivot $\Delta^2_{i,i+2}$ is a monomial and there is $\bar{s}<i$ such that $\Delta^2_{\bar{s},i+2}$ is also a monomial. If $s=\bar{s}$, then $\Delta^3_{i,i+2}=0$ and $\Delta^3_{s,i+2}=\Delta^2_{s,i+2}-\Delta^2_{s,i+1}(\Delta^2_{i,i+1})^{-1}\Delta^2_{i,i+2}$, which is either a binomial or zero. Hence, column $i+2$ turns into a column of type 2 or 1, respectively. On the other hand, if $s\neq\bar{s}$, then $\Delta^3_{i,i+2}=0$, $\Delta^3_{s,i+2}=-\Delta^2_{s,i+1}(\Delta^2_{i,i+1})^{-1}\Delta^2_{i,i+2}$, which is a monomial, and the other entries of column $i+2$ remain the same. Hence, column $i+2$ remains of type 3.

Induction hypothesis: Suppose that the conclusion of the Theorem holds for $3\leq r<R$. We will show, that it also holds for $r+1$.

Suppose that $\Delta^r_{i,j}$ is a change-of-basis pivot in the r-th diagonal. Then there is a primary pivot $\Delta^r_{i,u}$ in a column $u<j$.

1. if column u is of type 2 then $\Delta^r_{i,u}$ is a binomial and we have one of the possibilities:

   (a) If the column j is of type 2, then the change-of-basis pivot $\Delta^r_{i,j}$ is a binomial. In this case, $\Delta^{r+1}_{i,j}=0$ and all the other entries in column $r+1$ remain the same. Hence, column $r+1$ turns into a column of type 1.

   (b) If the column j is of type 3 (type 4, resp.), then $\Delta^r_{i,j}$ is a monomial. In this case, $\Delta^{r+1}_{i,j}=0$ and all the other entries remain the same. Hence column j turns into a column of type 4 (type 1, resp.).

2. column u is of type 3. Then $\Delta^r_{i,u}$ is a monomial and there is $s<i$ such that $\Delta^r_{s,u}$ is also a monomial.

   (a) If the column j is of type 2, then the change-of-basis pivot $\Delta^r_{i,j}$ is a binomial. In this case, $\Delta^{r+1}_{i,j}=0$ and $\Delta^{r+1}_{s,j}=-\Delta^r_{s,u}(\Delta^r_{i,u})^{-1}\Delta^r_{i,j}$, which is a binomial. Hence, the column j remains of type 2.

   (b) If the column j is of type 3, then the change-of-basis pivot $\Delta^r_{i,j}$ is a monomial and there exists $\bar{s}<i$ such that $\Delta^r_{\bar{s},j}$ is also a monomial. If $s=\bar{s}$, then $\Delta^{r+1}_{i,j}=0$

and $\Delta^{r+1}_{s,j} = \Delta^r_{s,j} - \Delta^r_{s,u}(\Delta^r_{i,u})^{-1}\Delta^r_{i,j}$, which is either a binomial or zero. Hence, column j turns into a column of type 2 or 1, respectively. On the other hand, if $s \neq \bar{s}$, then $\Delta^{r+1}_{i,j} = 0$, $\Delta^{r+1}_{s,j} = -\Delta^r_{s,u}(\Delta^r_{i,u})^{-1}\Delta^r_{i,j}$, which is a monomial, and the other entries of column j remain the same. Hence, column j remains of type 3.

(c) If the column j is of type 4, then the change-of-basis pivot $\Delta^r_{i,j}$ is a monomial. Hence, $\Delta^{r+1}_{i,j} = 0$ and $\Delta^{r+1}_{s,j} = -\Delta^r_{s,u}(\Delta^r_{i,u})^{-1}\Delta^r_{i,j}$, which is a monomial. Hence the column j remains of type 4.

3. column u is of type 4. The only non-zero entry in column u is the primary pivot $\Delta^r_{i,u}$ which is a monomial. Hence, in column j all the entries remain the same besides the change-of-basis $\Delta^{r+1}_{i,j} = 0$. If j is a column of type 2 or 4 it turns into a column of type 1; if j is of type 3 it turns into a column of type 4.

■

Theorem 3.4 (Second Block Characterization)**.** *Let Δ be a Novikov differential for which the SSSA is well defined up to step R. Then we have the following possibilities for a non-zero row s, with $s \in J_1$, of the matrix Δ^r produced by the SSSA in step $r \leq R$ without realising the pre-multiplication by $(T^r)^{-1}$:*

(A) all non null entries are binomials of the form $t^\ell - t^{\tilde{\ell}}$, where $\ell, \tilde{\ell} \in \mathbb{Z}$;

(B) all non null entries are monomials of the form t^ℓ, where $\ell \in \mathbb{Z}$;

(C) all non null entries are either monomials t^ℓ or binomials $t^\ell - t^{\tilde{\ell}}$. Moreover, if a column $j \in J_2$ contains a binomial, then there are no monomials in columns $j' \in J_2$ with $j' > j$.

Proof. We will prove this theorem by induction in $r \leq R$. In the course of the proof, we will also prove the following set of statements:

(i) If an entry t^ℓ is a primary pivot in row i then at most one entry will be marked as a change-of-basis pivot in row i.

(ii) An entry $t^\ell - t^{\tilde{\ell}}$ is never marked as a change-of-basis pivot, i.e. all change-of-basis pivots are monomials t^ℓ, for some $\ell \in \mathbb{Z}$.

(*iii*) If $\Delta^r_{s,j} = t^\ell - t^{\tilde{\ell}}$, $\Delta^r_{i,j}$ is a change-of-basis pivot in row $i > s$ and $\Delta^r_{i,u}$ is the primary pivot in row i with $u < j$, then $\Delta^r_{s,u}$ is zero.

(*iv*) A primary pivot $\Delta^\xi_{i,u}$ and a change-of-basis pivot $\Delta^\xi_{i,j}$ in row i are always monomials with opposite signs, i.e., $\Delta^r_{i,j} = \pm t^\ell$ and $\Delta^r_{i,u} = \mp t^{\tilde{\ell}}$, for $\ell, \tilde{\ell} \in \mathbb{Z}$.

(*v*) A monomial $\Delta^\xi_{s,u}$ above a primary pivot $\Delta^r_{i,u}$ and a monomial $\Delta^\xi_{s,j}$ above a change-of-basis pivot $\Delta^r_{i,j}$ always have opposite signs, i.e., $\Delta^r_{s,j} = \pm t^\ell$ and $\Delta^r_{s,u} = \mp t^{\tilde{\ell}}$, for $\ell, \tilde{\ell} \in \mathbb{Z}$.

Observe that the matrices Δ^1 and Δ^2 differ from the initial matrix Δ only in the mark-ups of primary and change-of-basis pivots, since the entries can only change as of 3-th diagonal.
Base case r=3: In order to prove that the rows of Δ^3 satisfies conditions (A), (B) and (C) of the theorem, we must analyze the effect on a row of Δ^3 caused by a change-of-basis pivot marked in the second step $r = 2$ of the SSSA.

$$\Delta^2_{J_1 J_2} = \begin{array}{cc} & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\ \begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & \vdots & \\ \cdots & \Delta^2_{s,i+1} & \Delta^2_{s,i+2} & \cdots \\ & \vdots & \vdots & \\ \cdots & \Delta^2_{i,i+1} & \boxed{\Delta^2_{i,i+2}} & \cdots \\ & \vdots & \vdots & \end{pmatrix} \end{array}$$

Figure 3.16: Primary and change-of-basis pivots in the first and second diagonals of Δ^2, respectively.

Suppose, without loss of generality, that there is a change-of-basis pivot $\Delta^2_{i,i+2}$ on the second diagonal. Consequently, $\Delta^2_{i,i+1}$ is a primary pivot marked in the first step of the SSSA, see Figure 3.16. Recall that the rows in Δ^2 satisfy Proposition 3.1. A change-of-basis pivot only occurs in row i and column $i + 1$ if this row is of type 3. In what follows, we analyze the effect of this change of basis on a row s with $s < i$:

1. If row s is null (i.e., of type 1) then only row i is altered and becomes a row of type B.

2. Suppose that row s is of type 2. If the only non-zero entry in row s is in a column different from $i + 1$, then row s remains unaltered and row i turns into a row of type B. On the other hand, if this non zero entry is in the column $i + 1$, which is the same column as that of the primary pivot, then row i turns into a row of type B and row s turns into a row of type A, as one can see in Figure 3.17.

$$
\begin{array}{cc}
 & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \\
\cdots & t^{l}-t^{\tilde{l}} & 0 & \cdots \\
 & \vdots & \vdots & \\
\cdots & \pm t^{\ell} & \boxed{\mp t^{\tilde{\ell}}} & \cdots \\
 & \vdots & \vdots &
\end{pmatrix}
\end{array}
\xrightarrow{\text{SSSA}}
\begin{array}{cc}
 & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \\
\cdots & t^{l}-t^{\tilde{l}} & t^{-\tilde{\ell}\ell l}-t^{-\tilde{\ell}\ell\tilde{l}} & \cdots \\
 & \vdots & \vdots & \\
\cdots & \pm t^{\ell} & 0 & \cdots \\
 & \vdots & \vdots &
\end{pmatrix}
\end{array}
$$

Figure 3.17: $\Delta^2_{J_1J_2}$ and $\Delta^3_{J_1J_2}$, respectively.

3. Suppose that row s is of type 3. If $\Delta_{s,i+1}$ is zero, then row s remains unaltered. If $\Delta^2_{s,i+1} = t^l$, one has two possibilities for $\Delta^2_{s,i+2}$, namely, 0 or $t^{\tilde{l}}$. In the first case, after performing the change of basis, row s turns into a row of type B (see Figure 3.18), and in the second case it turns into a row of type C (see Figure 3.19).

$$
\begin{array}{cc}
 & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \\
\cdots & \pm t^{l} & 0 & \cdots \\
 & \vdots & \vdots & \\
\cdots & \pm t^{\ell} & \boxed{\mp t^{\tilde{\ell}}} & \cdots \\
 & \vdots & \vdots &
\end{pmatrix}
\end{array}
\xrightarrow{\text{SSSA}}
\begin{array}{cc}
 & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \\
\cdots & \pm t^{l} & \pm t^{\tilde{\ell}-\ell+l} & \cdots \\
 & \vdots & \vdots & \\
\cdots & \pm t^{\ell} & 0 & \cdots \\
 & \vdots & \vdots &
\end{pmatrix}
\end{array}
$$

Figure 3.18: $\Delta^2_{J_1J_2}$ and $\Delta^3_{J_1J_2}$, respectively.

$$
\begin{array}{cc}
 & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \\
\cdots & \pm t^{l} & \mp t^{\tilde{l}} & \cdots \\
 & \vdots & \vdots & \\
\cdots & \pm t^{\ell} & \boxed{\mp t^{\tilde{\ell}}} & \cdots \\
 & \vdots & \vdots &
\end{pmatrix}
\end{array}
\xrightarrow{\text{SSSA}}
\begin{array}{cc}
 & \begin{array}{cccc} & i+1 & i+2 & \end{array} \\
\begin{array}{c} \\ s \\ \\ i \\ \\ \end{array} &
\begin{pmatrix}
 & \vdots & \vdots & \\
\cdots & \pm t^{l} & \mp t^{\tilde{l}} \pm t^{\tilde{\ell}-\ell+l} & \cdots \\
 & \vdots & \vdots & \\
\cdots & \pm t^{\ell} & 0 & \cdots \\
 & \vdots & \vdots &
\end{pmatrix}
\end{array}
$$

Figure 3.19: $\Delta^2_{J_1J_2}$ and $\Delta^3_{J_1J_2}$, respectively.

In the base case, it is ease to see that (i) through (v) hold.

In order to prove (i) note that, the only case that needs to be analyzed is when $\Delta^2_{i,i+1}$ is a primary pivot and $\Delta^2_{i,i+2}$ is a change-of-basis pivot. Observe that $\Delta^2_{i,i+3} = 0$, since rows of Δ^2 have at most two non zero entries. As pivots do not generate entries, by Lemma 3.3, the entry $\Delta^2_{i,i+3}$ is not altered by change-of-basis pivots marked in step 2. Hence, $\Delta^3_{i,i+3} = \Delta^2_{i,i+3} = 0$ and it is not a change-of-basis pivot.

In order to prove (ii), we must consider each row s where the entry $\Delta^3_{s,s+3} = t^l - t^{\tilde{l}}$ was generated in Δ^3, otherwise this entry would be in a row of type 2 and hence, could not be a change-of-basis pivot. There are exactly two ways that this entry can be generated: when row s was of type 2 and of type 3 in Δ^2. Suppose by contradiction that $\Delta^3_{s,s+3}$ is a change-of-basis pivot. Since primary pivots do not generate entries and there exists at most one change-of-basis pivots in a row then row s could not be of type 2 in Δ^2, see Figure 3.13. If row s was of type 3 in Δ^2 then the primary pivot would generate $\Delta^3_{s,s+3}$ which also contradicts Lemma 3.3, see Figure 3.14. Consequently, $\Delta^3_{s,s+3}$ is not a change-of-basis pivot.

In order to prove (iii), suppose by contradiction that $\Delta^3_{s,u} \neq 0$. Since rows of the initial matrix Δ do not admit two non zero entries with one of them being a binomial, see Proposition 3.1, then $\Delta^2_{s,u} = \Delta^3_{s,u}$ generates $\Delta^3_{s,i+3}$ in Δ^3. Therefore, by Definition 3.1, there must exist a change-of-basis pivot in the second diagonal in a row $\tilde{i} > i$ and a primary pivot in row $\tilde{i}$ and column u, which contradicts the fact that each column has at most one primary pivot.

Items (iv) and (v) are trivially true.

We now prove Theorem 3.4 and item (i) through (v) by induction.

Induction hypothesis: Suppose that Theorem 3.4 and item (i) through (v) hold for $\xi \leq r < R$. We will show that they also hold for $r+1$. First, note that by the induction hypothesis, at most one entry in a fixed row in J_1 is marked as change-of-basis pivot up to step r. Hence, by Corollary 3.2, given an entry $\Delta_{s,u}$, the $\Delta_{s,u}$-lineage is formed by a unique generation sequence until Δ^{r+1}, which is either a binomial lineage, if $\Delta_{s,u}$ is a binomial; or a monomial lineage or an eventual binomial lineage, if $\Delta_{s,u}$ is a monomial[4]. More specifically, if row s is of type 2 in Δ, where $\Delta_{s,u} \neq 0$ is a binomial, then the $\Delta_{s,u}$-lineage is a binomial lineage. If row s is of type 3, where $\Delta_{s,u}$ and $\Delta_{s,j}$ are monomials, then each monomial determines a lineage, which are either both monomial lineages or one monomial lineage which ceases and merges with the other to create an eventual binomial lineage. It is important to keep in mind that, if there is at most one change-of basis per row up to step r, then the $\Delta_{i,u}$-lineage is formed by a unique generation sequence until Δ^{r+1}. This follows since entries in Δ^{r+1} can only be generated by change of basis determined in step r.

By the induction hypothesis that characterizes the rows of Δ^r as being of type $(A), (B)$ and (C) and item (ii), it follows that if $\Delta^r_{i,j}$ is a change-of-basis pivot on the r-th diagonal and $\Delta^r_{i,u}$ is the primary pivot of row i, then these entries must be monomials. Moreover, by item (iv), $\Delta^r_{i,j} = \pm t^{\ell}$ and $\Delta^r_{i,u} = \mp t^{\tilde{\ell}}$, for $\ell, \tilde{\ell} \in \mathbb{Z}$.

[4]Note that in this proof, whenever we assume the induction hypothesis for r, the index r is shifted by one, i.e., $r+1$ when referring to the lineages.

Now we will prove the statement of the Theorem 3.4 and item (i) through (v) hold for $r+1$.

• We will first show that the non-zero rows in Δ^{r+1} are of type A, B or C.

In order to prove this fact, we will perform all the possible change of basis that could occur due to a change-of-basis pivot in the r-th diagonal. Let the monomial $\Delta^r_{i,u}$ be the primary pivot in row i and the monomial $\Delta^r_{i,j}$ be a change-of-basis pivot in diagonal r. Observe that after the change of basis, row i will remain of the same type. Each row $s < i$ in Δ^r is of type A, B or C, and thus one has the following cases to analyze the effect a change of basis causes in row s (bear in mind the configuration of the matrix in Figure 3.11):

1. If $\Delta^r_{s,u} = 0$, then row s remains unaltered after performing the change of basis.

2. If $\Delta^r_{s,u} \neq 0$ and $\Delta^r_{s,j} = 0$, then $\Delta^{r+1}_{s,j} = -\Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\Delta^r_{s,u}$, which is a monomial if $\Delta^r_{s,u}$ is a monomial, or a binomial if $\Delta^r_{s,u}$ is a binomial. Hence, row s remains of the same type.

3. If $\Delta^r_{s,u} \neq 0$ and $\Delta^r_{s,j} \neq 0$:

   (a) If $\Delta^r_{s,u} = \pm t^l$ and $\Delta^r_{s,j} = \mp t^{\tilde{l}}$, then $\Delta^{r+1}_{s,j} = \Delta^r_{s,j} - \Delta^r_{i,j}(\Delta^r_{i,u})^{-1}\Delta^r_{s,u}$. Item (v) applied to Δ^r ensures that $\Delta^{r+1}_{s,j}$ is a zero entry or a binomial with coefficients equal to ± 1. Hence, row s turns into a row of type B or C, respectively.

   (b) The case $\Delta^r_{s,u} = \pm t^l$ and $\Delta^r_{s,j} = \pm t^{\tilde{l}}$ can not occur by the induction hypothesis (v).

   (c) The case where $\Delta^r_{s,u} \neq 0$ and $\Delta^r_{s,j} = t^l - t^{\tilde{l}}$ can not occur, by the induction hypothesis (iii).

   (d) Note that the case where $\Delta^r_{s,u} = t^l - t^{\tilde{l}}$ is a binomial and $\Delta^r_{s,j} = t^{\ell}$ is a monomial can not occur, by the induction hypothesis on rows of Δ^r.

   Hence, every row $s \in J_1$ of Δ^{r+1} is also of type A, B or C.

• We will now show that item (i) holds for Δ^{r+1}.

Let $\Delta^{r+1}_{i,j}$ be a change-of-basis pivot marked in the $(r+1)$-th diagonal. Suppose by contradiction that an entry $\Delta^{\xi}_{i,t}$, where $t < j$, was marked as a change-of-basis pivot in an earlier step $\xi < r+1$. Consequently, the primary pivot in row i, which is in a column $u < t$, was marked in a previous step $< \xi$. Thus, by item (ii) of the induction hypothesis, this primary pivot is a monomial and it can not generate entries, which implies that $\Delta^{\xi}_{i,t}$ and $\Delta^{r+1}_{i,j}$ are not in the lineage of this primary pivot. Therefore, $\Delta^{\xi}_{i,t}$ and $\Delta^{r+1}_{i,j}$ must be in the same lineage, since these entries were generated up to step $r+1$ by the change-of-basis pivots up to step r, and by the induction hypothesis, there is only one change-of-basis pivots per row up to step r. This is a contradiction, since by Lemma 3.3 the change-of-basis pivot $\Delta^{\xi}_{i,t}$ can not generate entries.

• We will prove item (ii) for $r+1$:

Let $\Delta^{r+1}_{i,j}$ be a binomial in the $(r+1)$-th diagonal. Suppose by contradiction that this entry is a change-of-basis pivot marked in step $r+1$. Let u be the column of the primary pivot in row i, hence $u < j$.

If row i was originally of type 2 in Δ^2, then we have seen that in row i there is only one lineage until Δ^{r+1}, which is a binomial lineage. Hence, the primary pivot must generate another entry since $\Delta^{r+1}_{i,j}$ is a successor of the primary pivot, contradicting Lemma 3.3.

If row i was of type 3 in Δ^2, then originally there were two lineages that merged in order to create a binomial. Note that the primary pivot in row i can not be a monomial, i.e., it can not be marked before the two sequences have merged, since pivots do not generates entries. Hence the primary pivot must be a binomial. As we have seen in the previous paragraph, this contradicts Lemma 3.3.

• We will prove item (iii) for $r+1$:

Let $\Delta^{r+1}_{i,j}$ be a change-of-basis pivot marked in the $(r+1)$-th diagonal, $\Delta^{r+1}_{s,j}$ be a binomial and let u be the column of the primary pivot $\Delta^{r+1}_{i,u}$ in row i. Suppose by contradiction that the entry $\Delta^{r+1}_{s,u}$ is non-zero.

1. If row s is of type 2 in Δ^2, then $\Delta^{r+1}_{s,u}$ and $\Delta^{r+1}_{s,j}$ are in the same lineage, i.e., $\Delta^{r+1}_{s,j}$ must be a succesor of $\Delta^{r+1}_{s,u}$, which is a contradiction. In fact, $\Delta^{r+1}_{s,j}$ is not an immediate successor of $\Delta^{r+1}_{s,u}$, since in this case it would imply the existence of two primary pivots in column u. Moreover, $\Delta^{r+1}_{s,j}$ is not an eventual successor of $\Delta^{r+1}_{s,u}$, since it would imply the existence of two change-of-basis pivots in row i marked up to step r.

2. If row s is of type 3 in Δ^2 and if $\Delta^{r+1}_{s,u}$ is a binomial then the argument is the same as the one above. However, if $\Delta^{r+1}_{s,u}$ is a monomial, one has two cases to consider: $\Delta^{r+1}_{s,u}$ and $\Delta^{r+1}_{s,j}$ are in the same lineage or in different lineages. If they are in the same lineage, which is an eventual binomial lineage, then $\Delta^{r+1}_{s,j}$ is a successor of $\Delta^{r+1}_{s,u}$, hence $\Delta^{\xi}_{s,u}$ generated an entry in $\Delta^{\xi+1}$ with $\xi < r+1$. Now, if they are in different lineages,

let $\xi \in \mathbb{N}$ such that the first binomial in row s is generated in $\Delta^{\xi+1}$ with $\xi < r+1$, hence $\Delta^{\xi}_{s,u}$ generates the entry $\Delta^{\xi+1}_{s,j}$. By item (i), which was already proved to hold for Δ^{r+1}, $\Delta^{r+1}_{s,u}$ generates a unique immediate successor in Δ^{r+2}, which must then be $\Delta^{r+2}_{s,j}$ and this is a contradiction in both cases.

• We will prove item for (iv) for $r+1$, i.e. that a primary pivot $\Delta^{\xi}_{i,u}$ and a change-of-basis pivot $\Delta^{\xi}_{i,j}$ in row i are always monomials with opposite signs.

As consequence of the induction hypothesis (iv) and (v) for r, all elements in a monomial lineage have the same sign, up to step $\xi \leq r+1$. Moreover, if row i was of type 3 in Δ, then the two monomial lineages of this row have opposite signs. Now, suppose that there is a change-of-basis pivot $\Delta^{r+1}_{i,j}$ in $(r+1)$-th diagonal and let $\Delta^{r+1}_{i,u}$ be the primary pivot in row i with $u < j$. Since pivots do not generate entries by Lemma 3.3, the entries $\Delta^{r+1}_{i,j}$ and $\Delta^{r+1}_{i,u}$ are clearly in different lineages, therefore they have opposite signs.

• We will prove item (v) for $r+1$, i.e, that a monomial $\Delta^{\xi}_{s,u}$ above a primary pivot $\Delta^{r}_{i,u}$ and a monomial $\Delta^{\xi}_{s,j}$ above a change-of-basis pivot $\Delta^{r}_{i,j}$ always have opposite signs.

As a consequence of the induction hypothesis (iv) and (v), until Δ^{r+1}, all elements in a monomial lineage have the same coefficient, which is either $+1$ or -1. Moreover, if row i was of type 2 in Δ, then the two monomial lineages of this row have opposite signs. Let $\Delta^{r+1}_{s,u}$ be a monomial above a primary pivot $\Delta^{r+1}_{i,u}$ and $\Delta^{r+1}_{s,j}$ be a monomial above a change-of-basis pivot $\Delta^{r+1}_{i,j}$. Observe that, by the induction hypothesis, $\Delta^{r+1}_{s,u}$ and $\Delta^{r+1}_{s,j}$ are not in the same lineage. Hence, they have opposite signs.

■

We now proceed with the proof Theorem 3.2.

Proof of Theorem 3.2: Let Δ be a Novikov differential. By the characterization of the initial matrix, see Proposition 3.1, the entries of Δ are invertible in $\mathbb{Z}((t))$; hence, one can apply the SSSA in Δ. Since the first change of basis can only occur from step 2 to step 3, then the SSSA is well defined until step 2, and $\Delta^1 = \Delta^2$. Now, using Lemma 3.2, one can apply the SSSA to each block of Δ. Theorem 3.3 and 3.4 imply that the pivots in Δ^3 are invertible. Hence, the SSSA is also well defined for Δ^3. By an induction argument, one can suppose that the SSSA is well defined until step r. Theorems 3.3 and 3.4 also imply that the SSSA is well defined for Δ^{r+1}. Therefore, Theorem 3.2 is proved. ■

Observe that, if Δ^L is the last matrix produced by the SSSA, then the non null columns of Δ^L are the columns containing primary pivots. The primary pivots are non-zero and are unalterable after being identified. Moreover, $\Delta^L \circ \Delta^L = 0$.

Returning to Example 3.3, observe that the entry $\Delta^3_{4,7}$ is an infinite series. See Figure 3.7. However, the last matrix produced by the SSSA in Exameple 3.3 does not contain entries which are infinite serie.

In fact, the next two results imply that the last matrix Δ^L produced by the SSSA never contains an entry which is an infinite series.

The proof of Theorem 3.5 follows the same steps of the proof of its analogous version in [30], where the SSSA is done over a field $\mathbb{F}$.

Theorem 3.5. *Given a Novikov complex $(\mathcal{N}_*, \Delta)$ on surfaces, let Δ^L be the last matrix produced by the SSSA over $\mathbb{Z}((t))$. If column j of Δ^L is non null then row j is null.*

Proof. The statement of the lemma is equivalent to say that $\Delta^L_{j\bullet}\Delta^L_{\bullet j} = 0$ for all j. If $\Delta^L_{\bullet j} = 0$ then it is trivial that $\Delta^L_{j\bullet}\Delta^L_{\bullet j} = 0$. Suppose that $\Delta^L_{\bullet j} \neq 0$. Let s be an integer such that $j \in J_s$. Labelling the primary pivots in block J_s such that, if $\Delta^L_{i_1,j_1}, \cdots, \Delta^L_{i_a,j_a}$ are the primary pivots in block J_s, then $i_1 < i_2 < \cdots < i_a$, one has that $j_1, \cdots, j_a$ are the non null columns of J_s. Moreover, $\Delta^L_{i_a,j_a}$ is the unique non zero entry in row i_a. Row i_{a-1} has non zero entry in column j_{a-1} and may have another one non zero entry in column j_a, and so on. Since $\Delta^L \circ \Delta^L = 0$, one has

$$0 = \Delta^L_{i_a\bullet}\Delta^L_{\bullet j'} = \Delta^L_{i_aj_a}\Delta^L_{j_aj'},$$

for all j'. Since $\Delta^L_{i_aj_a}$ is a primary pivot, hence non null, then $\Delta^L_{j_aj'} = 0$ for all j', i.e., $\Delta^L_{j_a\bullet} = 0$. Analogously, one has

$$0 = \Delta^L_{i_{a-1}\bullet}\Delta^L_{\bullet j'} = \Delta^L_{i_{a-1}j_{a-1}}\Delta^L_{j_{a-1}j'} + \Delta^L_{i_{a-1}j_a}\Delta^L_{j_aj'},$$

for all j'. Since $\Delta^L_{i_{a-1}j_{a-1}} \neq 0$ and $\Delta^L_{j_aj'} = 0$, it follows that $\Delta^L_{j_{a-1}j'} = 0$ for all j', i.e., $\Delta^L_{j_{a-1}\bullet} = 0$. Proceeding in this way, one can show the nullity of rows $j_{a-2}, \cdots, j_1$. ■

Corollary 3.3. *Given a Novikov complex $(\mathcal{N}_*, \Delta)$ on a surface, let Δ^L be the last matrix produced by the SSSA over $\mathbb{Z}((t))$. Then, the entries of Δ^L are monomials t^ℓ or binomials $t^\ell - t^{\tilde{\ell}}$.*

Proof. By Theorems 3.3 and 3.4, without performing the pre-multiplication by $(T^r)^{-1}$, the entries of Δ^r are monomials t^ℓ or binomials $t^{\ell_1} - t^{\ell_2}$. Moreover, the pre-multiplication by

$(T^r)^{-1}$ only affects row j if column j contains a primary pivot. However, by Lemma 3.5, these rows will be zeroed out by the time SSSA reaches the last matrix Δ^L. ■

3.4 Spectral Sequence (E^r, d^r) associated to SSSA

As presented in Section 1.2, given a filtered chain complex over a principal ideal domain R, there is a spectral sequence over R associated to it whenever the filtration is convergent and bounded below.

Given a circle-valued Morse function f, let $(\mathcal{N}_*, \Delta)$ be the Novikov chain complex generated by f. Denote the generator of the $\mathcal{N}_k$ chain module by $h_k^1, \cdots, h_k^{c_k}$. One can reorder the set of critical points of f as

$$\{h_0^1, \cdots, h_0^{\ell_0}, h_1^{\ell_0+1}, \cdots, h_1^{\ell_1}, \cdots, h_k^{\ell_{k-1}+1}, \cdots, h_k^{\ell_k}, \cdots\},$$

where $\ell_k = c_0 + \cdots + c_k$. Consider the filtration $F = \{F_p\mathcal{N}\}$ on this complex defined by

$$F_p\mathcal{N}_k = \bigoplus_{h_k^\ell,\ \ell \leq p+1} \mathbb{Z}((t))\langle h_k^\ell \rangle.$$

Note that for each $p \in \mathbb{Z}$ there is only one singularity in $F_p\mathcal{N} \setminus F_{p-1}\mathcal{N}$, hence the filtration F is called a finest filtration. The filtration F is convergent, i.e. $\cap_p F_p\mathcal{N} = 0$ and $\cup F_p\mathcal{N} = \mathcal{N}$. In fact, F is finite, that is, $F_p\mathcal{N} = 0$ for some p and $F_{p'}\mathcal{N} = \mathcal{N}$ for some p'. Moreover, the filtration F is bounded below. Therefore, there exists a convergent spectral sequence with

$$E^0_{p,q} = F_p\mathcal{N}_{p+q}/F_{p-1}\mathcal{N}_{p+q} = G(\mathcal{N})_{p,q}\ , \qquad E^1_{p,q} \approx H_{(p+q)}(F_p\mathcal{N}_{p+q}/F_{p-1}\mathcal{N}_{p+q})$$

and E^∞ is isomorphic to the module $GH_*(\mathcal{N})$. The algebraic formulas for the modules $E^r_{p,q}$ of the spectral sequence are shown in Section 1.2.

Whenever the filtration considered is a finest filtration F, the only q such that $E^r_{p,q}$ is non-zero is $q = k - p$. Hence, we omit reference to q, i.e. E^r_p is in fact $E^r_{p,k-p}$.

Note that, E^∞ does not completely determine the Novikov homology $H_*^{Nov}(\mathcal{N}) = H_*(\mathcal{N}, \partial)$ of M, but

$$E^\infty_{p,q} \approx GH_*(\mathcal{N})_{p,q} = \frac{F_pH_{p+q}(\mathcal{N})}{F_{p-1}H_{p+q}(\mathcal{N})}.$$

However, it is a well known fact [15] that whenever $GH_*(\mathcal{N})_{p,q}$ is free and the filtration is bounded,

$$\bigoplus_{p+q=k} GH_*(\mathcal{N})_{p,q} \approx H^{Nov}_{p+q}(\mathcal{N}). \tag{3.2}$$

In Corollary 3.5, we prove that the isomorphism in (3.2) holds for Novikov Complexes over orientable surfaces when we consider the filtration F defined above.

In this section, we show that the SSSA provides a mechanism to recover the modules E^r and differentials d^r of the spectral sequence (E^r, d^r). More specifically, the SSSA provides a system to detect the generators of E^r in terms of the original basis of $\mathcal{N}_*$ and to identify the differentials d^r with the primary pivots in the r-th diagonal. The results in this section are similar in nature to the ones obtained in [13] for the SSSA computed for a Morse chain complex over $\mathbb{Z}$.

The next proposition establishes a formula for the modules $Z^r_{p,k-p}$ via the chains $\sigma_k^{i,j}$'s determined by the SSSA applied to Δ.

Proposition 3.1. *Let $r, p \geq 0$ be integers and κ be the first column in Δ associated to a k-chain. Consider $\mu^{j,\zeta} = 0$ whenever the primary pivot of column j is below row $(p - r + 1)$ and $\mu^{j,\zeta} = 1$ otherwise. Then*

$$Z^r_p = \mathbb{Z}((t))[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \dots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}].$$

Proof. By definition, $\sigma_k^{p+1-\xi,r-\xi}$ is associated to column $(p + 1 - \xi)$ of the matrix $\Delta^{r-\xi}$, for $\xi \in \{0, \dots, p + 1 - \kappa\}$, and $\mu^{p+1-\xi,r-\xi} = 1$ if and only if the primary pivot on column $(p + 1 - \xi)$ is in or above row $(p + 1 - \xi) - (r - \xi) = p - r + 1$ or if this column does not have a primary pivot. If $\sigma_k^{p+1-\xi,r-\xi}$ is such that $\mu^{p+1-\xi,r-\xi} = 1$, one can show that the k-chain $\sigma_k^{p+1-\xi,r-\xi}$ corresponds to a generator of Z^r_p. In fact, $\sigma_k^{p+1-\xi,r-\xi}$ is in $F_p\mathcal{N}_k$ for $\xi \geq 0$. Furthermore, all nonzero entries of column $(p+1-\xi)$ of $\Delta^{r-\xi}$ are in or above row $(p-r+1)$, since the $(r - \xi)$-th step of SSSA has zeroed out all entries below the $(r - \xi)$-th diagonal. Hence, the boundary of $\sigma_k^{p+1-\xi,r-\xi}$ is in $F_{p-r}\mathcal{N}_{k-1}$. Hence,

$$\mathbb{Z}((t))[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \dots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}] \subset Z^r_p.$$

Below we prove by multiple induction in p and r that

$$Z^r_p \subset \mathbb{Z}((t))[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \dots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}]. \tag{3.3}$$

Base case:

- Consider $F_{\kappa-1}$, where κ is the first column of Δ associated to a k-chain. Let ξ be such that the boundary of h_k^κ is in $F_{\kappa-1-\xi}\mathcal{N}_k$ but it is not in $F_{\kappa-1-\xi-1}\mathcal{N}_k$. We will show that $Z_{\kappa-1}^r = \mathbb{Z}((t))[\mu^{\kappa,r}\sigma_k^{\kappa,r}]$, for all $r \geq 0$.

  Since $Z_{\kappa-1}^r$ is generated by k-chain in $F_{\kappa-1}\mathcal{N}_k$ with boundaries in $F_{\kappa-1-r}\mathcal{N}_{k-1}$ and $F_{\kappa-1}\mathcal{N}_k$ is generated by only one chain h_k^κ, then:

  (a) If $\xi < r$ then $\partial h_k^\kappa \notin F_{\kappa-1-r}\mathcal{N}_{k-1}$. Thus, $Z_{\kappa-1}^r = 0$.

  (b) If $\xi \geq r$ then $\partial h_k^\kappa \in F_{\kappa-1-r}\mathcal{N}_{k-1}$. Thus, $Z_{\kappa-1}^r = \mathbb{Z}((t))[h_k^\kappa]$.

  On the other hand, since there is no change of basis caused by the SSSA that affects the first column of Δ_k, $\sigma_k^{\kappa,r} = h_k^\kappa$, where $\sigma_k^{\kappa,r}$ is a k-chain associated to the column κ of Δ^r. Furthermore, $\mu^{\kappa,r} = 1$ if and only if the boundary of $h_k^\kappa = \sigma_k^{\kappa,r}$ is in or above the r-th diagonal. Hence

  (a) If $\xi < r$ then $\mu^{\kappa,r} = 0$. Thus $\mathbb{Z}((t))[\mu^{\kappa,r}\sigma_k^{\kappa,r}] = 0$

  (b) If $\xi \geq r$ then $\mu^{\kappa,r} = 1$. Thus $\mathbb{Z}((t))[\mu^{\kappa,r}\sigma_k^{\kappa,r}] = \mathbb{Z}((t))[\sigma_k^{\kappa,r}] = \mathbb{Z}((t))[h_k^\kappa]$.

  It follows that $Z_{\kappa-1}^r = \mathbb{Z}((t))[\mu^{\kappa,r}\sigma_k^{\kappa,r}]$ in both cases, for all $r \geq 0$.

- Let the ξ_1-th diagonal be the first diagonal in Δ that intersects Δ_k. All the columns of Δ corresponding to the chains $h_k^{p+1}, \ldots, h_k^\kappa$ have nonzero entries above the ξ_1-th diagonal, thus, above the $(p-\xi_1+1)$-st row of Δ. We will show that

  $$Z_p^{\xi_1} = \mathbb{Z}((t))[\mu^{p+1,\xi_1}\sigma_k^{p+1,r}, \ldots, \mu^{\kappa,\kappa-p+1+\xi_1}\sigma_k^{\kappa,\kappa-p+1+\xi_1}].$$

  Since $Z_p^{\xi_1}$ is generated by k-chains contained in $F_p\mathcal{N}_k$ with boundary in $F_{p-\xi_1}\mathcal{N}_{k-1}$ and the columns of Δ associated to the chains $h_k^{p+1}, \ldots, h_k^\kappa$ have nonzero entries above the $(p-\xi_1+1)$-st row, then the boundaries are in $F_{p-\xi_1}\mathcal{N}_{k-1}$, i.e.,

  $$Z_p^{\xi_1} = \mathbb{Z}((t))[h_k^{p+1}, \ldots, h_k^\kappa].$$

  On the other hand, nonzero entries in the columns of Δ associated to the chains $h_k^{p+1}, \ldots, h_k^\kappa$ are all above the ξ_1-th diagonal, then $\sigma_k^{j,\xi_1} = h_k^j$, $j = \kappa, \ldots p+1$ and

$\mu^{j,\xi_1} = 1$, $j = \kappa, \ldots p+1$. Hence,

$$\mathbb{Z}((t))[\mu^{p+1,\xi_1}\sigma_k^{p+1,r}, \ldots, \mu^{\kappa,\kappa-p+1+\xi_1}\sigma_k^{\kappa,\kappa-p+1+\xi_1}] = \mathbb{Z}((t))[h_k^{p+1}, \ldots, h_k^{\kappa}].$$

Therefore, $Z_p^{\xi_1} = \mathbb{Z}((t))[\mu^{p+1,\xi_1}\sigma_k^{p+1,r}, \ldots, \mu^{\kappa,\kappa-p+1+\xi_1}\sigma_k^{\kappa,\kappa-p+1+\xi_1}]$.

Induction Hypothesis: Assume that the generators of Z_{p-1}^{r-1} correspond to k-chains associated to $\sigma_k^{p+1-\xi,r-\xi}$, $\xi = 1, \ldots, p+1-\kappa$, whenever the primary pivot of the column $(p+1-\xi)$ is above the row $(p-r+1)$. We will prove that (3.3) holds for Z_p^r.

If the primary pivot of the column $(p+1)$ is below row $(p-r+1)$, then $Z_p^r = Z_{p-1}^{r-1}$ and this is the case when $\mu^{(p+1),r} = 0$. Suppose now that the primary pivot of the column $(p+1)$ is on and above the row $(p-r+1)$ and let $\mathfrak{h}_k = b^{p+1}h_k^{p+1} + \cdots + b^{\kappa}h_k^{\kappa} \in Z_{p,k-p}^r$. We know that $\mathfrak{h}_k$ is in F_p and its boundary is on and above row $(p-r+1)$. If $b^{p+1} = 0$ then $\mathfrak{h}_k \in Z_{p-1}^{r-1}$ and the result follows by the induction hypothesis. If $b^{p+1} \neq 0$, we can rewrite $\mathfrak{h}_k$ as

$$\mathfrak{h}_k = b^{p+1}\sigma_k^{p+1,r} + (b^p - b^{p+1}c_p^{p+1,r})h_k^p + \cdots + (b^{\kappa} - b^{p+1}c_{\kappa}^{p+1,r})h_k^{\kappa},$$

since by the definition of the SSSA

$$\sigma_k^{p+1,r} = \sum_{\ell=\kappa}^{p+1} c_\ell^{j,r} h_k^\ell.$$

Note that $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} = (b^p - b^{p+1}c_p^{p+1,r})h_k^p + \cdots + (b^{\kappa} - b^{p+1}c_{\kappa}^{p+1,r})h_k^{\kappa} \in F_{p-1}$. Moreover, since $\mathfrak{h}_k$ and $\sigma_k^{p+1,r}$ have their boundaries on and above row $(p-r+1)$ then the boundary of $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r}$ is on and above row $(p-r+1)$. Hence $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} \in Z_{p-1}^{r-1}$. By the induction hypotheses, we have that $\mathfrak{h}_k - b^{p+1}\sigma_k^{p+1,r} = \alpha_p\mu^{p,r-1}\sigma_k^{p,r-1} + \cdots + \alpha_\kappa\mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}$. Therefore,

$$\mathfrak{h}_k = b^{p+1}\sigma_k^{p+1,r} + \alpha_p\mu^{p,r-1}\sigma_k^{p,r-1} + \cdots + \alpha_\kappa\mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa},$$

as required. ■

Lemma 3.5. *Given integer $r, p \geq 0$, if $\partial Z_{p+r-1}^{r-1} \not\subseteq Z_{p-1}^{r-1}$, then $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1} = \mathbb{Z}_p^r$.*

Proof. Denote by κ the first column associated to a k-chain. By hypothesis $\partial Z_{p+r-1}^{r-1} \not\subseteq Z_{p-1}^{r-1}$, which implies that $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ is a submodule of

$$Z_p^r = \mathbb{Z}((t))[\mu^{p+1,r}\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \ldots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}]$$

but it is not a submodule of

$$Z_{p-1}^{r-1} = \mathbb{Z}((t))[\mu^{p,r-1}\sigma_k^{p,r-1}, \mu^{p-1,r-2}\sigma_k^{p-1,r-2}, \ldots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}].$$

Then $\mu^{p+1,r} = 1$ and $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ contains a multiple of $\sigma_k^{p+1,r}$ over $\mathbb{Z}((t))$. We will show that $\sigma_k^{p+1,r} \in Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$. Note that

$$\partial Z_{p+r-1}^{r-1} = \mathbb{Z}((t))[\mu^{p+r,r-1}\partial\sigma_{k+1}^{p+r,r-1}, \mu^{p+r-1,r-2}\partial\sigma_{k+1}^{p+r-1,r-2}, \ldots, \mu^{\overline{\kappa},\overline{\kappa}-p-1}\partial\sigma_{k+1}^{\overline{\kappa},\overline{\kappa}-p-1}]. \qquad (3.4)$$

where $\overline{\kappa}$ is the first column associated to a $(k+1)$-chain. For $\xi = 0, \ldots, p+r-\overline{\kappa}$ with $\mu^{p+r-\xi,r-1-\xi} = 1$ we have $\Delta_{i,p+r-\xi}^{r-1-\xi} = 0$ for all $i > p+1$ and hence

$$\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi} = \Delta_{p+1,p+r-\xi}^{r-1-\xi}\sigma_k^{p+1,r-1-\xi} + \cdots + \Delta_{\kappa,p+r-\xi}^{r-1-\xi}\sigma_k^{\kappa,r-1-\xi}.$$

In fact, the boundaries $\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi}$ with $\Delta_{i,p+r-\xi}^{r-1-\xi} \neq 0$ for some $i > p+1$ correspond exactly to the columns which have the primary pivots below the $(p+1)$-st row and therefore $\mu^{p+r-\xi,r-1-\xi} = 0$.

Hence, for $\xi = 0, \ldots, p+r-\overline{\kappa}$, when $\mu^{p+r-\xi,r-1-\xi} = 1$ we have

$$Z_{p-1}^{r-1} + [\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi}] = Z_{p-1}^{r-1} + [\Delta_{p+1,p+r-\xi}^{r-1-\xi}\sigma_k^{p+1,r-1-\xi} + \cdots + \Delta_{\kappa,p+r-\xi}^{r-1-\xi}\sigma_k^{\kappa,r-1-\xi}]. \qquad (3.5)$$

On the other hand, since $Z_{p-1}^{r-1} + [\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi}] \subset Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ then

$$Z_{p-1}^{r-1} + [\partial\sigma_{k+1}^{p+r-\xi,r-1-\xi}] = [\ell_\xi\sigma_k^{p+1,r}, \mu^{p,r-1}\sigma_k^{p,r-1}, \mu^{p-1,r-2}\sigma_k^{p-1,r-2}, \ldots, \mu^{\kappa,r-p-1+\kappa}\sigma_k^{\kappa,r-p-1+\kappa}]. \qquad (3.6)$$

The coefficient of h_k^{p+1} on the set of generators of the $\mathbb{Z}((t))$-module in (3.5) is $\Delta_{p+1,p+r-\xi}^{r-1-\xi}$. On the other hand, the coefficient of h_k^{p+1} on the set of the generators of the $\mathbb{Z}((t))$-module in (3.6) is ℓ_ξ. Since for each $\xi = 0, \ldots, p+r-\overline{\kappa}$, $\Delta_{i,p+r-\xi}^{r-1-\xi} = 0$ for all $i > p+1$ then $\Delta_{p+1,p+r-\xi}^{r-1-\xi}$ is either a pivot or a zero entry. Note that the entries $\Delta_{p+1,p+r-\xi}^{r-1-\xi}$ can not be all zeros, since it would contradict the hipothesis of $\partial Z_{p+r-1}^{r-1} \nsubseteq Z_{p-1}^{r-1}$. It follows from Theorem 3.2 that if $\Delta_{p+1,p+r-\xi}^{r-1-\xi}$ is nonzero then it is invertible in $\mathbb{Z}((t))$. Then $\sigma_k^{p+1,r} \in Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}$ and hence $Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1} = Z_p^r$. ■

Theorem 3.6. *The matrix Δ^r obtained from the sweeping method applied to Δ determines*

$$E_p^r = \frac{Z_p^r}{Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}}.$$

More specifically, E_p^r is either zero or a finitely generated $\mathbb{Z}((t))$-module whose generator corresponds to a k-chain associated to column $(p+1)$ of Δ^r.

Proof. The entry $\Delta^r_{p-r+1,p+1}$ on the r-th diagonal plays a crucial role in determining the generators of E_p^r. If the entry $\Delta^r_{p-r+1,p+1}$ is nonzero then it can be either a primary pivot, a change-of-basis pivot or it is above a primary pivot. If the entry $\Delta^r_{p-r+1,p+1}$ is zero then it can be in a column above a primary pivot or all entries below it are also zero. We will analyze each possibility for the entry $\Delta^r_{p-r+1,p+1}$ and show how to determine E_p^r for each case. The proof is a consequence of formulas obtained in Proposition 3.1 and Lemma 3.5.

1. Suppose that the entry $\Delta^r_{p-r+1,p+1}$ is identified by the SSSA as a primary pivot, a change-of-basis pivot or a zero entry with column of zeros below it.

   In these cases $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$ and hence the generator $\sigma_k^{p+1,r}$ corresponding to the k-chain associated to column $(p+1)$ in Δ^r is a generator of Z_p^r. Thus we must analyze row $(p+1)$. We have the following possibilities:

   (a) $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$, i.e, all the boundaries of the elements in Z_{p+r-1}^{r-1} are above row p. In this case, as before, by Proposition 3.1 $E_p^r = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$.

   (b) $\partial Z_{p+r-1}^{r-1} \nsubseteq Z_{p-1}^{r-1}$, i.e, there exist elements in Z_{p+r-1}^{r-1} whose boundary has a nonzero entry in row $(p+1)$. By Proposition 3.1 and Lemma 3.5 $E_p^r = 0$.

   Note that if $\Delta^r_{p-r+1,p+1}$ has been identified by the SSSA as a primary pivot then $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$. In fact, the generators of Z_{p+r-1}^{r-1} must correspond to $(k+1)$-chains associated to h_{k+1} columns with the property that their boundaries are above row $(p+1)$ and consequently all entries below row $(p+1)$ are zero. Hence the entries of these h_{k+1} columns on row $(p+1)$ must, by SSSA, either be a primary pivot or a zero entry. On the other hand, by Lemma 3.1, row $(p+1)$ cannot contain a primary pivot since we have assumed that column $(p+1)$ has a primary pivot. Therefore, the entries in row $(p+1)$ of the h_{k+1} columns in Z_{p+r-1}^{r-1} must be zeroes. It follows that ∂Z_{p+r-1}^{r-1} does not contain in its set of generators the generator $\sigma_k^{p+1,r}$. Hence, $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$.

2. Suppose that the entry $\Delta^r_{p-r+1,p+1}$ is an entry above a primary pivot, i.e. there exists $s > p-r+1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. In this case, the generator $\sigma_k^{p+1,r}$ corresponding to the k-chain associated to column $(p+1)$ is not a generator of Z^r_p and hence $Z^{r-1}_{p-1} = Z^r_p$. It follows that $E^r_p = 0$.

3. Suppose that the entry $\Delta^r_{p-r+1,p+1}$ is not in Δ^r_k. This includes the case where $p-r+1 < 0$, i.e, $\Delta^r_{p-r+1,p+1}$ is not on the matrix Δ^r.

   The analyzes of E^r_p is very similar to the previous one and we have two possibilities:

   (a) There is a primary pivot in column $(p+1)$ in a diagonal $\overline{r} < r$. In this case the generator corresponding to the k-chain associated to column $(p+1)$, $\sigma_k^{p+1,r}$ is not a generator of Z^r_p. Hence $Z^{r-1}_{p-1} = Z^r_p$ and $E^r_p = 0$.

   (b) All the entries in Δ^r in column $(p+1)$ in diagonals lower than r are zero, i.e, the generator corresponding to the k-chain associated to column $(p+1)$, $\sigma_k^{p+1,r}$ in Δ^r is a generator of Z^r_p. Then we have to analyze row $(p+1)$.

      i. If $\partial Z^{r-1}_{p+r-1} \subseteq Z^{r-1}_{p-1}$ then, by Proposition 3.1, $E^r_p = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$.

      ii. If $\partial Z^{r-1}_{p+r-1} \nsubseteq Z^{r-1}_{p-1}$ then, by Proposition 3.1 and Lemma 3.5, $E^r_p = 0$.

■

We will describe how the SSSA applied to Δ induces the differentials $d^r_p : E^r_p \to E^r_{p-r}$ of the spectral sequence.

Theorem 3.7. *If E^r_p and E^r_{p-r} are both nonzero, then the map $d^r_p : E^r_p \to E^r_{p-r}$ is induced by the multiplication by the entry $\Delta^r_{p-r+1,p+1}$.*

Proof. Suppose that E^r_p and E^r_{p-r} are both nonzero. Let $\delta^r_p : \mathbb{Z}((t))[\sigma_k^{p+1,r}] \to \mathbb{Z}((t))[\sigma_{k-1}^{p-r+1,r}]$ be the multiplication by the entry $\Delta^r_{p-r+1,p+1}$ and $\tilde{\delta}^r_p$ the induced map in E^r_p. We must show that

$$\frac{\text{Ker } \tilde{\delta}^r_p}{\text{Im } \tilde{\delta}^r_{p+r}} \cong E^{r+1}_p.$$

Since we are considering E^r_p nonzero, it follows from Theorem 3.6, that we must consider three cases for the entry $\Delta^r_{p-r+1,p+1}$: primary pivot, change-of-basis pivot and zero with a column of zeroes below it. However, if $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot then there exists a primary pivot in row $(p-r+1)$ on a diagonal below the r-th diagonal and hence

$E^r_{p-r} = 0$. Hence, whenever E^r_p and E^r_{p-r} are both nonzero, the entry $\Delta^r_{p-r+1,p+1}$ in Δ^r is either a primary pivot or a zero with a column of zero entries below it.

In this case $E^r_p = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$ and $E^r_{p-r} = \mathbb{Z}((t))[\sigma_{k-1}^{p-r+1,r}]$.

1. Suppose $\Delta^r_{p-r+1,p+1}$ is a primary pivot.

   Since $\delta^r_p : \mathbb{Z}((t))[\sigma_k^{p+1,r}] \to \mathbb{Z}((t))[\sigma_{k-1}^{p-r+1,r}]$ is multiplication by $\Delta^r_{p-r+1,p+1}$, which is invertible in $\mathbb{Z}((t))$, then Ker $\delta^r_p = 0$. Since $\tilde{\delta}^r_p = \delta^r_p$, then $\dfrac{\text{Ker } \tilde{\delta}^r_p}{\text{Im } \tilde{\delta}^r_{p+r}} = 0$. On the other hand, since $\Delta^r_{p-r+1,p+1}$ is a primary pivot, then $\sigma_k^{p+1,r+1} = \sigma_k^{p+1,r} \notin Z_p^{r+1}$. Thus $Z_p^{r+1} = Z^r_{p-1}$ and $E_p^{r+1} = 0$.

2. Suppose $\Delta^r_{p-r+1,p+1} = 0$ with a column of zeroes below it. In this case Ker $\delta^r_p \cong E^r_p =$ Ker $\tilde{\delta}^r_p$ and $\sigma_k^{p+1,r} = \sigma_k^{p+1,r+1}$. There are three cases to consider:

   (a) If $\Delta^r_{p+1,p+r+1}$ is an entry above a primary pivot then we have $E^r_{p+r} = 0$ and hence $\text{Im } \tilde{\delta}^r_{p+r} = 0$. Thus,

   $$\frac{\text{Ker } \tilde{\delta}^r_p}{\text{Im } \tilde{\delta}^r_{p+r}} = E^r_p.$$

   On the other hand, since $\mu^{p+r+1,r} = 0$ then $E_p^{r+1} = E^r_p$.

   (b) If $\Delta^r_{p+1,p+r+1}$ is a primary pivot then $E^r_{p+r} = \mathbb{Z}((t))[\sigma_k^{p+r+1,r}]$. Therefore δ^r_{p+r} is an isomorphism and hence

   $$\frac{\text{Ker } \tilde{\delta}^r_p}{\text{Im } \tilde{\delta}^r_{p+r}} \cong \frac{\mathbb{Z}((t))[\sigma_k^{p+1,r}]}{\mathbb{Z}((t))[\sigma_k^{p+1,r}]} = 0.$$

   On the other hand, since $\Delta^r_{p-r+1,p+1}$ is zero with a column of zero entries below it then $\sigma_k^{p+1,r+1} \in Z^{r+1}_{p,k-p}$ and hence $Z^r_{p-1} \subsetneq Z^{r+1}_{p,k-p}$. Moreover, since $E^r_p = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$ then $\partial Z^{r-1}_{p+r-1} \subseteq Z^{r-1}_{p-1}$. But the difference between ∂Z^{r-1}_{p+r-1} and ∂Z^r_{p+r} is that the last one includes the boundary of column $(p+r+1)$. The element in column $(p+r+1)$ and row $(p+1)$ is $\Delta^r_{p+1,p+r+1}$. Since $\Delta^r_{p+1,p+r+1}$ is a primary pivot then $\partial Z^r_{p+r} \nsubseteq Z^r_{p-1}$ and $E_p^{r+1} = 0$.

   (c) If $\Delta^r_{p+1,p+r+1} = 0$ with a column of zero entries below it then $\text{Im } \delta^r_{p+r} = 0$ and

   $$\frac{\text{Ker } \tilde{\delta}^r_p}{\text{Im } \tilde{\delta}^r_{p+r}} = E^r_p.$$

Analogously to the previous case, $\sigma_k^{p+1,r+1} \in Z_{p,k-p}^{r+1}$ and hence $Z_{p-1}^r \subsetneq Z_p^{r+1}$. Moreover, $\partial Z_{p+r-1}^{r-1} \subseteq Z_{p-1}^{r-1}$ and the only difference between ∂Z_{p+r-1}^{r-1} and ∂Z_{p+r}^r is that the last one includes the boundary of column $(p+r+1)$. Since the element in column $(p+r+1)$ and row $(p+1)$ is $\Delta_{p+1,p+r+1}^r = 0$ then $\partial Z_{p+r}^r \subseteq Z_{p-1}^r$ and $E_p^{r+1} = \mathbb{Z}((t))[\sigma_k^{p+1,r}]$.

Note that the case where $\Delta_{p+1,p+r+1}^r$ is a change-of-basis pivot does not have to be considered, since in this case $E_p^r = 0$.

Therefore, in all cases

$$\frac{\text{Ker } d_p^r}{\text{Im } d_{p+r}^r} = E_p^{r+1} = \frac{\text{Ker } \tilde{\delta}_p^r}{\text{Im } \tilde{\delta}_{p+r}^r}.$$

■

Corollary 3.4. *Each non zero differential d_p^r of the spectral sequence (E^r, d^r) is an isomorphism.*

Proof. In fact, by the proof of Theorem 3.7, non zero differentials of (E^r, d^r) are induced by primary pivots. Theorem 3.2 states that each primary pivot produced by the SSSA is an invertible polynomial, hence each induced non zero differential is an isomorphism. ■

By Corollary 3.4, if $d^r : E_p^r \to E_{p-r}^r$ is a non zero differential, then algebraic cancellation occurs in the $(r+1)$-th page of (E^r, d^r), i.e., the modules E_p^{r+1} and E_{p-r}^{r+1} are zero.

The next corollary states that the spectral sequence (E^r, d^r) converges to the Novikov homology of $(\mathcal{N}_*, \Delta)$.

Corollary 3.5. *If M is a smooth closed orientable 2-dimensional manifold, $f : M \to S^1$ a circle-valued Morse function, $(\mathcal{N}_*, \Delta)$ a filtered Novikov chain complex with a finest filtration, then the modules $E_{p,q}^\infty$ of the associated spectral sequence are free for all p and q. Moreover,*

$$E_{p,q}^\infty \approx GH_*(\mathcal{N})_{p,q} = \frac{F_p H_{p+q}(\mathcal{N})}{F_{p-1} H_{p+q}(\mathcal{N})} \approx H_*^{Nov}(M, f).$$

Proof. By Corollary 3.4, the non zero differentials $d^r : E_p^r \to E_{p-r}^r$ of the spectral sequence are isomorphisms. Since, $E_p^{r+1} \cong \text{Ker } d_p^r / \text{Im } d_{p-r}^r$, it follows that the modules E_p^r's are free for all $r \geq 0$ and $p \geq 0$. Therefore, $E_{p,q}^\infty \cong GH_*(\mathcal{N})_{p,q} \cong H_*^{Nov}(M, f)$, by equation (1.3). ■

Example 3.5. Consider the Novikov complex $(\mathcal{N}_*, \Delta)$ presented in Example 3.2. The matrices produced by the SSSA are illustrated in Figures 3.5 to 3.10.

The Novikov homology of $(\mathcal{N}_*, \Delta)$ is given by

$$H_0^{Nov}(M, f) = 0, \quad H_1^{Nov}(M, f) = 0, \quad H_2^{Nov}(M, f) = 0.$$

Consider the filtration F on $(\mathcal{N}_*, \Delta)$ defined by

$$F_p\mathcal{N}_k = \bigoplus_{h_k^\ell,\ \ell \le p+1} \mathbb{Z}((t))\langle h_k^\ell \rangle.$$

The spectral sequence associated to $(\mathcal{N}_*, \Delta)$ endowed with this filtration F is given by:

$$\begin{tikzcd}[column sep=small]
E^0: & \mathbb{Z}((t))\,[h_0^1] & \mathbb{Z}((t))\,[h_0^2] & \mathbb{Z}((t))\,[h_1^3] & \mathbb{Z}((t))\,[h_1^4] & \mathbb{Z}((t))\,[h_1^5] & \mathbb{Z}((t))\,[h_1^6] & \mathbb{Z}((t))\,[h_2^7] & \mathbb{Z}((t))\,[h_2^8] \\
E^1: & \mathbb{Z}((t))\,[h_0^1] & \mathbb{Z}((t))\,[h_0^2] & \mathbb{Z}((t))\,[h_1^3] \arrow[l, bend right, "d_2^1"'] & \mathbb{Z}((t))\,[h_1^4] & \mathbb{Z}((t))\,[h_1^5] & \mathbb{Z}((t))\,[h_1^6] & \mathbb{Z}((t))\,[h_2^7] \arrow[l, bend right, "d_6^1"'] & \mathbb{Z}((t))\,[h_2^8] \\
E^2: & \mathbb{Z}((t))\,[h_0^1] & 0 & 0 & \mathbb{Z}((t))\,[h_1^4] & \mathbb{Z}((t))\,[h_1^5] & 0 & 0 & \mathbb{Z}((t))\,[h_2^8] \\
E^3: & \mathbb{Z}((t))\,[h_0^1] & 0 & 0 & \mathbb{Z}((t))\,[*] \arrow[lll, bend right, "d_3^3"'] & \mathbb{Z}((t))\,[h_1^5] & 0 & 0 & \mathbb{Z}((t))\,[**] \arrow[lll, bend right, "d_7^3"'] \\
E^4: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{tikzcd}$$

where $[*] = [h_1^4 + (t-1)^{-1}h_1^3]$ and $[**] = [h_2^8 + t^{-1}h_2^7]$. The primary pivots $\Delta_{2,3}^1$ and $\Delta_{6,7}^1$ induce the differentials on the first page d_2^1 and d_6^1, respectively. The primary pivots $\Delta_{1,4}^3$ and $\Delta_{5,8}^3$ induce the differentials on the third page d_3^3 and d_7^1, respectively. On the other hand, the change-of-basis pivots $\Delta_{2,4}^2$ and $\Delta_{6,8}^2$, marked in the second step of the SSSA, determine change of generators in E_3^3 and E_7^3. Observe that the spectral sequence (E^r, d^r) converges to the Novikov homology of $(\mathcal{N}_*, \Delta)$. ◀

Final Remarks

In this chapter, the computation of a spectral sequence of a filtered Novikov chain complex leads to the question of how closely the dynamics follows the spectral sequence. Herein we proved that the SSSA produces a sequence of Novikov matrices from which the modules and differentials may be retrieved. Analogously to the Morse case seen in Chapter 2, as one "turns the pages" of the spectral sequence, i.e. considers progressively modules E^r, one observes algebraic cancellations occurring within the E^r's, as proved in Corollary 3.4. Several open questions arise at this point: What is the dynamical significance of these algebraic cancellations? How much dynamical information can be recovered or even gained from the spectral sequence analysis? Do the Novikov matrices provide dynamical information on the birth and death of connections due to cancellations of consecutive critical points? Does the SSSA determine a continuation result to the minimal flow, as it does in the Morse setting? Of course, the investigation of these questions in higher dimensions will, without a doubt, constitute a challenging line of research.

The results obtained in this chapter provide a solid foundation on which we hope to establish results which answer these questions.

Chapter 4

Generalized Morse-Bott Inequalities

The study of the interplay between topology and dynamics dates back to Poincaré. Morse related the topology of the closed n-manifold M to its dynamical data by establishing relations between the number of nondegenerate critical points c_k of Morse index k of a smooth real valued function $f : M \to M$ and the Betti numbers of M. See [33].

In [12], Conley generalized these results to a theory with a more topological flavor and independent of the differentiable nature of the flow, allowing much richer dynamics than only singularities. Moreover, Conley proves the existence of a Lyapunov function associated to a flow on a manifold such that the flow maintains an underlying gradient-like behaviour with respect to this function.

In the setting of continuous flows on manifolds, the Poincaré-Hopf inequalities are introduced in [6, 8], which imposes constraints on the dynamics of a continuous flows without reference to the Betti numbers of the manifold M. These inequalities generalize the classical Morse inequalities.

The concept of abstract Lyapunov graphs was introduced in [6]. The realizability of an abstract Lyapunov graphs Γ is considered in [10], where the authors proved that, under certain assumptions, Γ is realizable as a flow on a manifold if and only if it satisfies the Poincaré-Hopf inequalities. In Section 1.3, we summarized the results obtained on this topic thus far.

In this chapter, we restrict out attention to gradient flows associated to Morse-Bott functions and present the results of Lyapunov graphs and Poincaré-Hopf inequalities in this setting.

The background material on Morse-Bott theory is presented in Section 4.1. In Section

4.2, we characterize the rank of the Conley index of critical manifolds over $\mathbb{Z}_2$. In Section 4.3, we define the abstract Morse-Bott graphs and the generalized Morse-Bott inequalities.

4.1 Morse-Bott Functions

In this section, we present some definitions and results from Morse-Bott theory that will be required subsequently in this chapter. The references for this section are [3], [2] and [17].

Let $f : M \to \mathbb{R}$ be a smooth function on a smooth compact n-manifold with boundary, possibly empty in which case it is defined as closed. Throughout this chapter, it will be assumed that the set of critical points $Crit(f)$ of f lies in the interior of M and that M is smooth. Suppose that the set $Crit(f)$ contains a closed k-submanifold S of M. Choosing a Riemannian metric on M, the tangent space of M restricted to S splits as

$$T_*M|_S = T_*S \oplus \nu_*S,$$

where T_*S and and ν_*S are the tangent and the normal bundles of S, respectively.

Let $Hess_p(f)$ be the Hessian of f at $p \in S \subset Crit(f)$. Given $v \in T_pS$ and $w \in T_pM$, then

$$Hess_p(f)(v, w) = V_p \cdot (W \cdot f) = 0,$$

since $V_p \in T_pS$ and any extension of w to a vector field W satisfies $df(W)|_S = 0$. Therefore, the Hessian $Hess_p(f)$ induces a symmetric bilinear form on the normal space ν_pS, denoted by $Hess_p^\nu(f)$.

Definition 4.1. *A smooth function $f : M \to \mathbb{R}$ on a compact manifold with boundary M is called a Morse-Bott function if the the set of critical points $Crit(f)$ is a disjoint union of connected closed submanifolds contained in the interior of M, which are called critical manifolds of f, and for each critical manifold S, the bilinear form $Hess_p^\nu(f)$ is non-degenerate for all $p \in S$.*

The second condition of Definition 4.1 means that, given $p \in S$, for each $v \in \nu_pS$ there exists $w \in \nu_pS$ such that $Hess_p^\nu(f)(v, w) \neq 0$. One says that the Hessian is non-degenerate in the normal direction to the critical manifolds.

Given $p \in Crit(f)$, where f is a Morse-Bott function, the *Morse-Bott index* of p is defined to be the maximal dimension of a subspace of ν_pS on which $Hess_p^\nu(f)$ is negative definite.

The Morse-Bott index of a critical point p will be denoted by λ_p.

For the following lemma see [2].

Lemma 4.1 (Morse-Bott Lemma). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on an n-dimensional manifold M and $S \subset Crit(f)$ a critical manifold. For each $p \in S$, there exists a local chart ϕ of M around p and a local splitting of the normal bundle of S, $\nu_* S = \nu_*^+ S \oplus \nu_*^- S$, which identifies a point x in the domain of ϕ to (u, v, w), where $u \in S$, $v \in \nu_*^- S$ and $w \in \nu_*^+ S$, such that $f \circ \phi^{-1}(u, v, w) = f(u) - |v|^2 + |w|^2$.*

Note that, by the Morse-Bott Lemma, if S is a connected critical manifold then λ_p is constant throughout S, that is, $\lambda_p = \lambda_q$, for all $p, q \in S$. Hence, one can refer to λ_p as the Morse-Bott index λ_S of the connected critical manifold S. Moreover, Lemma 4.1 shows that, at a critical point $p \in S$, the tangent space splits in the following way

$$T_p M = T_p S \oplus \nu_*^+ S \oplus \nu_*^- S,$$

where $\lambda_p = \dim\,(\nu_p^- S)$. If $k = \dim\, S$ and $\lambda_p^* = \dim\,(\nu_p^+ S)$, then one has the relation $n = k + \lambda_p + \lambda_p^*$.

Given a closed n-manifold M, the Poincaré polynomial of M is defined to be

$$P_t(M) = \sum_{k=0}^{n} \beta_k(M) t^k,$$

where $\beta_k(M)$ is the k^{th} Betti number[1] of M.

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional closed manifold M. Assume that

$$Crit(f) = \coprod_{i=1}^{l} S_i,$$

where each S_i is a connected critical manifold of f with finite dimension k_i and Morse-Bott index λ_i, for $i = 1, \cdots, l$. Under these assumptions, the Morse-Bott polynomial of f is defined to be

$$MB_t(f) = \sum_{i=1}^{l} P_t(S_i) t^{\lambda_i}$$

where $P_t(S_i)$ is the Poincaré polynomial of S_i.

For the following theorem see [3] and [2].

[1] Homology is computed using $\mathbb{Z}_2$ coefficients in the non-orientable case.

Theorem 4.1 (Polynomial form of the Morse-Bott inequalities). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional oriented closed manifold and assume that all critical manifolds of f are orientable. Then there exists a polynomial $R(t)$ with non-negative integer coefficients such that $MB_t(f) = P_t(M) + (1+t)R(t)$.*

As in the Morse case, one has the following result.

Proposition 4.1. *The polynomial form $MB_t(f) = P_t(M) + (1+t)R(t)$ of the Morse-Bott inequalities are equivalent to the Morse-Bott inequalities*

$$\begin{cases} \mathsf{b}_m - \mathsf{b}_{m-1} + \cdots + (-1)^m \mathsf{b}_0 \geq \beta_m - \beta_{m-1} + \cdots + (-1)^m \beta_0, \ \forall m = 0, \cdots, n-1 \\ \mathsf{b}_n - \mathsf{b}_{n-1} + \cdots + (-1)^n \mathsf{b}_0 = \beta_n - \beta_{n-1} + \cdots + (-1)^n \beta_0. \end{cases} \tag{4.1}$$

where β_j represents the j^{th} Betti number $\beta_j(M)$ of M, and $\mathsf{b}_k = \sum_{i=1}^{l} \beta_{s(i,k)}(S_i)$, where $s(i,k) = k - \lambda_i$ and $\beta_\eta(S_i) = 0$ when $\eta \notin [0, k_i]$.

Proof. In order to show that the polynomial form of the Morse-Bott inequalities imply the inequalities in (4.1), observe that the Morse-Bott polynomial of f can be expanded as follows:

$$\begin{aligned} MB_t(f) &= \sum_{i=1}^{l} P_t(S_i) t^{\lambda_i} = \sum_{k=0}^{n} \sum_{i=1}^{l} \beta_{s(i,k)}(S_i) t^k \\ &= \sum_{i=1}^{l} \beta_{s(i,n)}(S_i) t^n + \sum_{i=1}^{l} \beta_{s(i,n-1)}(S_i) t^{n-1} + \cdots + \sum_{i=1}^{l} \beta_{s(i,1)}(S_i) t^1 + \sum_{i=1}^{l} \beta_{s(i,0)}(S_i), \end{aligned}$$

where $k = s(i,k) + \lambda_i$ and $\beta_\eta(S_i) = 0$ when $\eta \notin [0, k_i]$.

By the polynomial form of the Morse-Bott inequalities,

$$\sum_{k=0}^{n} (-1)^k \sum_{i=1}^{l} \beta_{s(i,k)}(S_i) = MB_{-1}(f) = P_{-1}(M) = \sum_{k=0}^{n} (-1)^k \beta_k. \tag{4.2}$$

Rewriting (4.2) with the notation $\mathsf{b}_k = \sum_{i=1}^{l} \beta_{s(i,k)}(S_i)$,

$$\sum_{k=0}^{n} (-1)^k \mathsf{b}_k = \sum_{k=0}^{n} (-1)^k \beta_k. \tag{4.3}$$

Note that $MB_t(f) = P_t(M) + (1+t)R(t)$, implies that $\mathsf{b}_0 = \beta_0 + r_0$. Hence, $\mathsf{b}_1 = \beta_1 + r_1 + r_0$, i.e., $\mathsf{b}_1 - \mathsf{b}_0 = \beta_1 - \beta_0 + r_1$. Thereby continuing in this manner, one obtains

$$\mathsf{b}_m - \mathsf{b}_{m-1} + \cdots + (-1)^m \mathsf{b}_0 = \beta_m - \beta_{m-1} + \cdots + (-1)^m \beta_0 + r_m,$$

for all $m = 0, \cdots, n-1$. Since $r_m \geq 0$, for all $m = 0, \cdots, n-1$, the Morse-Bott inequalities follow as in (4.1).

On the other hand, the equality in (4.1) implies

$$MB_{-1}(f) = \sum_{k=0}^{n} (-1)^k \sum_{i=1}^{l} \beta_{s(i,k)}(S_i) = \sum_{k=0}^{n} (-1)^k \beta_k = P_{-1}(M).$$

Therefore, the polynomial $MB_t(f) - P_t(M)$ is divisibly by $(1+t)$ and hence, one has that $MB_t(f) = P_t(M) + (1+t)R(t)$, for some polynomial $R(t) = \sum_{k=0}^{n-1} r_k t^k$. As both polynomials $MB_t(f)$ and $P_t(M)$ have integer coefficients then the coefficients of $R(t)$ are integer. It remains to show that $r_m \geq 0$, for all $m = 0, \cdots, n-1$. Note that

$$\mathsf{b}_m - \mathsf{b}_{m-1} + \cdots + (-1)^m \mathsf{b}_0 = \beta_m - \beta_{m-1} + \cdots + (-1)^m \beta_0 + r_m,$$

for all $m = 0, \cdots, n-1$. Therefore, $r_m \geq 0$. ■

4.2 Conley Index for Critical Manifolds

In the background we presented the definition of Conley index for an isolated invariant set. In this section, we presented an equivalent definition of the Conley index for critical manifolds via Thom spaces.

There is an interesting formulation of the Conley index in terms of a Thom space, TE, of a vector bundle $\pi : E \to M$. Choosing a bundle metric g on E, denote by D_gE and S_gE the unit ball bundle and the unit sphere bundle over M, respectively. The *Thom space* of E is defined as the pointed topological space

$$T_gE := D_gE/S_gE := ((B_gE \backslash S_gE \cup [S_gE]), [S_gE])$$

obtained by collapsing S_gE to a single point denoted by $[S_gE]$. The pointed isomorphism class of T_gE is independent of the choice of the bundle metric g and is denoted by TE.

Given a Morse-Bott function $f : M \to \mathbb{R}$, let $\varphi_t : M \to M$ denote the flow associated to $-\nabla f$, i.e., $\varphi_t(x) = \gamma(t)$ where $\gamma'(t) = -(\nabla f)(\gamma(t))$ and $\gamma(0) = x$. Note that S is an isolated invariant set with respect to φ_t and to the reverse flow of φ_t, denoted by φ'_t. The Conley homotopical index of S, $I(S, \varphi_t)$, with respect to φ_t, is the Thom space of $\nu_*^-(S)$, i.e.,

$$I(S, \varphi_t) = T(v_*^- S). \tag{4.4}$$

For more details see [18].

From now on, we will work with homology with $\mathbb{Z}_2$ coefficients to avoid complications arising from orientability.

The next two proposition provides some properties of the homological Conley index of a critical manifold.

Proposition 4.2. *Let $f : M \to \mathbb{R}$ be a Morse-Bott function and S a connected critical k-manifold of f. The ranks h_* of the homological Conley indices $CH_*(S)$ with respect to the flow φ_t are given by*

$$\begin{cases} h_j = \beta_{j-\lambda}(S), & \text{if } \ \lambda \le j \le \lambda + k \\ h_j = 0, & \text{if } j < \lambda, \quad \text{or if } \ j > \lambda + k \end{cases}$$

where $\beta_i(S)$ is the i^{th} Betti number of S.

Proof. According to Thom's Isomorphism Theorem (see reference [11]), the reduced homology of the Thom space $T(v_*^- S)$ is given by $\widetilde{H}_{i+\lambda}(T(v_*^- S)) = H_i(S)$, for $i = 0, \cdots, k$ and is null otherwise. By (4.4), the homological Conley index of S coincides with the reduced homology of $T(v_*^- S)$. Then, $h_{i+\lambda} = \beta_i(S)$, where $i = 0, \cdots, k$. Rewriting this,

$$\begin{cases} h_j = \beta_{j-\lambda}(S), & \text{if } \ \lambda \le j \le \lambda + k \\ h_j = 0, & \text{if } j < \lambda, \quad \text{or if } \ j > \lambda + k. \end{cases}$$

■

Proposition 4.3. *Let S be a connected critical k-manifold of f with Morse-Bott index λ. Then the ranks of the homological Conley indices with respect to the flow φ_t and its reverse flow φ'_t satisfy $h_j(S, \varphi_t) = h_{n-j}(S, \varphi'_t)$, for $j = 0, \cdots, n$.*

Proof. Firstly, by Lemma 4.1, $n = k + \lambda + \lambda^*$, where λ is the Morse-Bott index of S with respect to the flow φ and λ^* is the Morse-Bott index of S with respect to the reverse flow φ'.

Suppose that $\lambda > 0$ and $\lambda + k < n$. By Proposition 4.2,

$$\begin{cases} h_j(S, \varphi_t) = 0, & \text{if } \ j = 0, \cdots, \lambda - 1 \\ h_j(S, \varphi_t) = \beta_{j-\lambda}, & \text{if } \ j = \lambda, \cdots, \lambda + k \\ h_j(S, \varphi_t) = 0, & \text{if } \ j = \lambda + k + 1, \cdots, n. \end{cases}$$

Note that, if $j \in \{0, \cdots, \lambda - 1\}$ then $n - j \in \{\lambda^* + k + 1, \cdots, n\}$; if $j \in \{\lambda, \cdots, \lambda + k\}$ then $n - j \in \{\lambda^*, \cdots, \lambda^* + k\}$; and if $j \in \{\lambda + k + 1, \cdots, n\}$ then $n - j \in \{0, \cdots, \lambda^* - 1\}$. Therefore, once again by Proposition 4.2, it follows that

$$\begin{cases} h_{n-j}(S, \varphi'_t) = 0, & \text{if } \ j = 0, \cdots, \lambda - 1 \\ h_{n-j}(S, \varphi'_t) = \beta_{n-j-\lambda^*}, & \text{if } \ j = \lambda, \cdots, \lambda + k \\ h_{n-j}(S, \varphi'_t) = 0, & \text{if } \ j = \lambda + k + 1, \cdots, n. \end{cases}$$

It remains to check that $\beta_{j-\lambda} = \beta_{n-j-\lambda^*}$ when $j = \lambda, \cdots, \lambda + k$. Indeed, as $n - \lambda^* = \lambda + k$, then $\beta_{j-\lambda} = \beta_{k-(j-\lambda)} = \beta_{n-\lambda^*-j}$. Hence, $h_j(S, \varphi) = h_{n-j}(S, \varphi')$.

The proof is analogous if $\lambda = 0$ or $\lambda + k = n$. ■

4.3 Generalized Morse-Bott Inequalities

In this section, we introduce the generalized Morse-Bott inequalities for manifolds with boundary. The novelty is that these inequalities depend solely on the Betti numbers of the critical manifolds and of the codimension one closed submanifolds of the boundary.

4.3.1 Morse-Bott Graphs

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a finite dimensional compact manifold with boundary (possibly empty). Consider the following equivalence relation on M: $x \sim_f y$ if and only if y belongs to the same connected component of a level set of f. Denote by $M/\sim_f$ the quotient space of M under this equivalence relation. A point on $M/\sim_f$ is a *vertex point* if under the equivalence relation $\sim_f$ it corresponds to a connected component of a level set containing a critical set of f. All other points are *edge points*. In this way, $M/\sim_f$ is a finite graph and each edge represents a codimension one submanifold Q of M times an open bounded interval I, $Q \times I$. In order to retain some information of $Q \times I$, the edges can be labeled with some topological invariant such as the Betti numbers of Q.

For simplicity, it will henceforth be assumed that the Morse-Bott function $f : M \to \mathbb{R}$ has at most one critical manifold on each level set, unless otherwise stated. With this assumption, there is a one-to-one correspondence between the vertices of the graph $M/\sim_f$ and the critical manifolds of f.

Definition 4.2. *Let f be a Morse-Bott function on a closed manifold M with critical manifolds $S_1, \cdots, S_l$. Define the Morse-Bott graph Γ_f associated to f as the graph $M/\sim_f$ on which each vertex S_i is labelled with $(\beta_0(S_i), \ldots, \beta_{k_i}(S_i); \lambda_i)$, where $\beta_j(S_i)$ is the j^{th} Betti number of S_i and λ_i is the Morse-Bott index of S_i. Also, each edge is labelled with the Betti numbers of the level sets associated with the given edge.*

In the case of a manifold with boundary $\partial M = N^+ \sqcup N^-$, one can define a Morse-Bott semi-graph similarly where the labels in the incoming (outgoing) semi-edges are the Betti numbers of the entering set N^+ (exiting set N^-) of the flow.

Example 4.1. Consider a Morse-Bott function f defined on the torus having four critical manifolds, all homeomorphic to the 1-sphere, such that, considering the flow φ_f associated to the vector field $-\nabla f$, two critical manifolds are repellers and the other two critical manifolds are attractors. In this way, the repellers have Morse-Bott indices equal to 1 and the attractors have Morse-Bott indices equal to 0. In Figure 4.1, one can see a representation of the flow φ_f and the Morse-Bott graph associated to f.

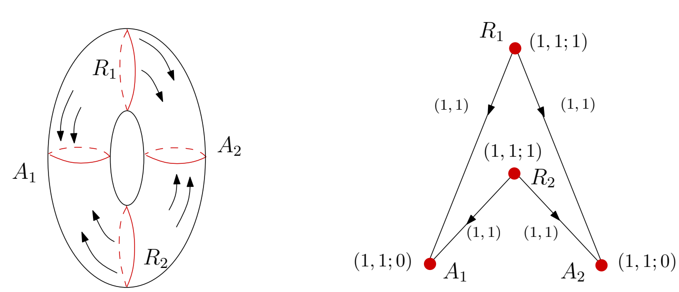


Figure 4.1: The Morse-Bott graph associated to f defined in Example 4.1.

Example 4.2. Consider S^3 as the manifold obtained from gluing two solid tori together by a homeomorphism of their boundaries which identifies a parallel of one torus T_1 to the meridian of the other torus T_2. Let f be a Morse-Bott function on S^3 such that the critical manifolds of f are the torus T_1 as a repeller and two 1-spheres S_1 and S_2 as attractors, where S_i is in the interior of T_i, $i = 1, 2$. Their Morse-Bott indices are $\lambda_{T_1} = 1$, $\lambda_{S_1} = 0$ and $\lambda_{S_2} = 0$. The Morse-Bott graph associated to f is shown in Figure 4.2.

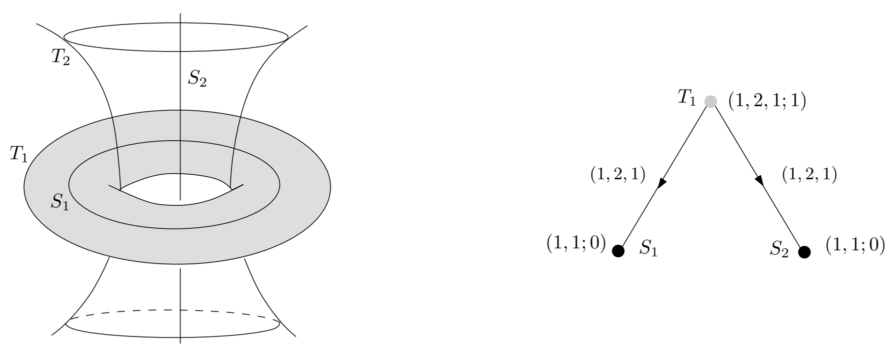


Figure 4.2: The Morse-Bott graph associated to f defined in Example 4.2.

Definition 4.3. *An n-abstract Morse-Bott graph is a finite, connected directed graph Γ with no oriented cycles, such that*

(1) *each vertex ν_i is labelled with a list of non-negative integers $(b_0^i, \cdots, b_{k_i}^i; \lambda_i, \kappa_i)$, where $k_i \leq n$, $\lambda_i \leq n - k_i$ and $\{b_0^i = 1, b_1^i, \cdots, b_{k_i-1}^i, b_{k_i}^i = 1\}$ satisfies Poincaré duality in dimension k_i;*

(2) *each incoming edge e_ℓ^+ incident to a vertex ν_i is labelled with a list of non-negative integers $\beta_0^{e_\ell^+}, \ldots, \beta_{n-1}^{e_\ell^+}$ which satisfies Poincaré duality and $\beta_0^{e_\ell^+} = \beta_{n-1}^{e_\ell^+} = 1$. If $n = 2p + 1$ is odd then $\beta_p^{e_\ell^+}$ must be even;*

(3) *each outgoing edge e_ℓ^- incident to a vertex ν_i is labelled with a list of non-negative integers $\beta_0^{e_\ell^-}, \ldots, \beta_{n-1}^{e_\ell^-}$ which satisfies Poincaré duality and $\beta_0^{e_\ell^-} = \beta_{n-1}^{e_\ell^-} = 1$. If $n = 2p + 1$ is odd then $\beta_p^{e_\ell^-}$ must be even;*

(4) *the number of incoming edges $e^+(\nu_i)$ and the number of outgoing edges $e^-(\nu_i)$ of a vertex ν_i satisfy the following conditions*

(a) *if $\lambda_i = 0$ then $e^-(\nu_i) = 0$ and $e^+(\nu_i) > 0$;*

(b) *if* $\lambda_i = n - k_i$ *then* $e^+(\nu_i) = 0$ *and* $e^-(\nu_i) > 0$*;*

(c) *if* $0 < \lambda_i < n - k_i$ *then* $e^-(\nu_i) > 0$ *and* $e^+(\nu_i) > 0$.

The *cycle number* κ of an abstract Morse-Bott graph Γ is defined as the cycle rank κ_L of Γ plus the cycle number κ_ν of all vertices of Γ.

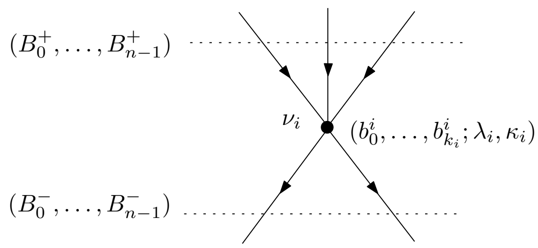


Figure 4.3: A typical vertex of an abstract Morse-Bott graph.

Definition 4.4. *A vertex* ν_i *labelled with* $(b_0^i, \ldots, b_{k_i}^i; \lambda_i, \kappa_i)$ *in an* n*-abstract Morse-Bott graph* Γ *can be exploded if* ν_i *can be removed from* Γ *and replaced by an* n*-abstract Lyapunov semi-graph of Morse type* Γ_M *with cycle rank greater than or equal to* κ_i *and with the labels on incoming (outgoing) semi-edges of* Γ_M *matching the labels on the outgoing (incoming) semi-edges of* $\Gamma \backslash \{v_i\}$*. Moreover, for all* η*,*

$$b_\eta(\nu_i) = \sum_j h_{\eta+\lambda_i}(v_j),$$

where the sum is over all vertices v_j *of* Γ_M*.*

Definition 4.5. *An abstract Morse-Bott graph* Γ *with cycle number* κ *admits a continuation to an abstract Lyapunov graph of Morse type* Γ_M *if each vertex can be exploded such that* Γ_M *has cycle rank greater than or equal to* κ*.*

Example 4.3. An example of a 7-abstract Morse-Bott graph Γ with $\kappa = 1$ is given in Figure 4.4. A possible continuation of Γ is presented in Figure 4.5, in this picture h_j^d (h_j^c) denotes a vertex labelled with a singularity $h_j = 1$ of type j-disconnecting (($j-1$)-connecting).

The vertex S_i in Figure 4.4 may represent a 5-dimensional critical manifold with Betti numbers $b_0 = b_5 = 1$, $b_2 = b_3 = 1$, $b_1 = b_4 = 0$ of Morse-Bott index $\lambda_1 = 1$ in a 7-dimensional basic block with an incoming (respectively, outgoing) boundary component with Betti numbers $(1, 0, 0, 0, 0, 0, 1)$ (respectively, $(1, 0, 1, 2, 1, 0, 1)$) and of Cornea genus greater than or equal to one, see [14]. The subgraph indicated on the right represents a Morse

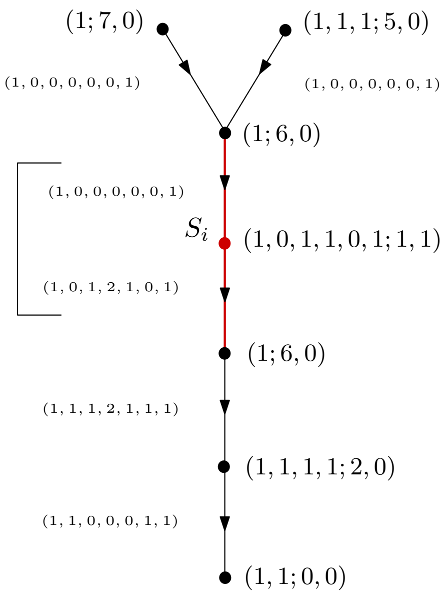


Figure 4.4: Abstract Morse-Bott graph.

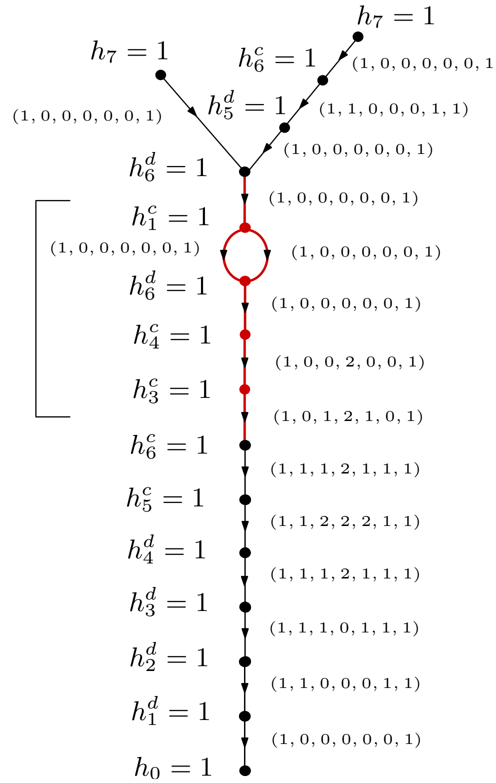


Figure 4.5: Explosion of graph in Figure 4.4.

flow on the same basic block with one index 1, one index 4, one index 3 and one index 6 singularities. ◀

4.3.2 Morse-Bott Graph Continuation

The Theorem 4.2 below describes sufficient and necessary conditions for an abstract Morse-Bott semi-graph can be continued to an abstract Lyapunov semi-graph of Morse type. These conditions are the generalized Morse-Bott inequalities presented in the statement of the following theorem.

Theorem 4.2. *Let Γ_v be an n-abstract Morse-Bott semi-graph containing only one vertex ν labelled with $(b_0, \cdots, b_k; \lambda, \kappa_\nu)$. Then Γ_ν admits a continuation to an n-abstract Lyapunov semi-graph Γ_M of Morse type with cycle rank greater than or equal to κ_ν, where $\kappa_\nu \leq min\{b_{1-\lambda} - (b_{-\lambda} - 1), b_{n-1-\lambda} - (b_{n-\lambda} - 1)\}$, if and only if ν satisfies the generalized Morse-Bott inequalities:*

$$\begin{cases} b_{j-\lambda} & \geq & -(B_{j-1}^{+} - B_{j-1}^{-}) + (B_{j-2}^{+} - B_{j-2}^{-}) + \cdots \pm (B_{2}^{+} - B_{2}^{-}) \pm (B_{1}^{+} - B_{1}^{-}) \\ & & -(b_{n-(j-1)-\lambda} - b_{j-1-\lambda}) + (b_{n-(j-2)-\lambda} - b_{j-2-\lambda}) \pm \cdots \pm (b_{n-1-\lambda} - b_{1-\lambda}) \\ & & \pm[(b_{n-\lambda} - b_{-\lambda}) + (e^{+} - e^{-})] \\ b_{n-j-\lambda} & \geq & -[-(B_{j-1}^{+} - B_{j-1}^{-}) + (B_{j-2}^{+} - B_{j-2}^{-}) + \cdots \pm (B_{2}^{+} - B_{2}^{-}) \pm (B_{1}^{+} - B_{1}^{-}) \\ & & -(b_{n-(j-1)-\lambda} - b_{j-1-\lambda}) + (b_{n-(j-2)-\lambda} - b_{j-2-\lambda}) \pm \cdots \pm (b_{n-1-\lambda} - b_{1-\lambda}) \\ & & \pm[(b_{n-\lambda} - b_{-\lambda}) + (e^{+} - e^{-})]] \end{cases}$$

$$\vdots$$

$$\begin{cases} b_{2-\lambda} & \geq & -(B_{1}^{+} - B_{1}^{-}) - (b_{n-1-\lambda} - b_{1-\lambda}) + (b_{n-\lambda} - b_{-\lambda}) + (e^{+} - e^{-}) \\ b_{n-2-\lambda} & \geq & -[-(B_{1}^{+} - B_{1}^{-}) - (b_{n-1-\lambda} - b_{1-\lambda}) + (b_{n-\lambda} - b_{-\lambda}) + (e^{+} - e^{-})] \end{cases}$$

$$\begin{cases} b_{1-\lambda} & \geq & b_{-\lambda} - 1 + e^{-} + \kappa_{\nu} \\ b_{n-1-\lambda} & \geq & b_{n-\lambda} - 1 + e^{+} + \kappa_{\nu} \end{cases}$$

where $0 \leq j \leq n$ *and* $b_{\eta} = 0$ *when* $\eta \notin [0, k]$.

$$\begin{cases} \textit{If } n = 2i+1, \textit{ then} \quad \mathcal{B}^{+} - \mathcal{B}^{-} = e^{-} - e^{+} + \sum_{j=0}^{2i+1} (-1)^{j} b_{j-\lambda}, \\ \textit{where } \mathcal{B}^{+} = \dfrac{(-1)^{i}}{2} B_{i}^{+} \pm B_{i-1}^{+} \pm \cdots - B_{1}^{+}, \qquad \mathcal{B}^{-} = \dfrac{(-1)^{i}}{2} B_{i}^{-} \pm B_{i-1}^{-} \pm \cdots - B_{1}^{-}. \\ \\ \textit{If } n = 2i \equiv 2 (\textit{mod } 4), \quad \textit{then} \\ b_{i-\lambda} - \sum_{j=1}^{i-1} (-1)^{j+1} (B_{j}^{+} - B_{j}^{-}) - \sum_{j=0}^{i-1} (-1)^{j} (b_{2i-j-\lambda} - b_{j-\lambda}) + (e^{+} - e^{-}) \quad \textit{must be even.} \end{cases}$$

Proof. The generalized Morse-Bott inequalities for the vertex ν of Γ_{ν} are equivalent to the Poincaré-Hopf inequalities for ν labelled with the data $(B_0^{\pm}, \ldots, B_{n-1}^{\pm})$; $h_{\lambda+j} = b_j$ if $j = 0, \cdots, k$ and $h_{\eta} = 0$ if $\eta < \lambda$ or $\eta > \lambda + k$. Therefore, by Theorem 1.1 on [7], the semi-graph Γ_{ν} admits a continuation to a n-abstract Lyapunov semi-graph of Morse type with cycle rank greater than or equal to κ_{ν} if and only if ν satisfies the generalized Morse-Bott inequalities, where $\kappa_{\nu} \leq \min\{b_{1-\lambda} - (b_{-\lambda} - 1), b_{n-1-\lambda} - (b_{n-\lambda} - 1)\}$. A explosion algorithm for a vertex can be found in [7]. ∎

An abstract Morse-Bott graph Γ is said to satisfies the generalized Morse-Bott inequalities if each vertex of Γ satisfies these inequalities.

Corollary 4.1. *Let Γ be an n-abstract Morse-Bott graph with cycle number κ. Then Γ admits a continuation to an n-abstract Lyapunov graph Γ_M of Morse type with cycle rank* *greater than or equal to κ if and only if Γ satisfies the generalized Morse-Bott inequalities.*

Example 4.4. The abstract Morse-Bott graph Γ presented in Example 4.3 satisfies the generalized Morse-Bott inequalities since each vertex of Γ satisfies these inequalities. For instance, one can look at the vertex S_i in Γ labelled with $(1,0,1,1,0,1;\lambda_v = 1, \kappa_v = 1)$ and verify that this vertex satisfies the generalized Morse-Bott inequalities, which have the following form:

$$\begin{array}{lcl}
b_2 & \geq & -(B_2^+ - B_2^-) + (B_1^+ - B_1^-) - (b_4 - b_1) + (b_5 - b_0) - (b_6 - b_{-1}) - (e^+ - e^-) \\
b_3 & \geq & -[-(B_2^+ - B_2^-) + (B_1^+ - B_1^-) - (b_4 - b_1) + (b_5 - b_0) - (b_6 - b_{-1}) - (e^+ - e^-)] \\
b_1 & \geq & -(B_1^+ - B_1^-) - (b_5 - b_0) + (b_6 - b_{-1}) + (e^+ - e^-) \\
b_4 & \geq & -[-(B_1^+ - B_1^-) - (b_5 - b_0) + (b_6 - b_{-1}) + (e^+ - e^-)] \\
b_0 & \geq & b_{-1} - 1 + e^- + \kappa_\nu \\
b_5 & \geq & b_6 - 1 + e^+ + \kappa_\nu \\
\mathcal{B}^+ & - & \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{7} (-1)^j b_{j-\lambda}
\end{array}$$

◀

4.3.3 A New Layout for the Morse-Bott Inequalities

In this subsection we use the generalized Morse-Bott inequalities to obtain a new version of the classical Morse-Bott inequalities in (4.1).

Theorem 4.3. *A Morse-Bott function f on a compact manifold with boundary given by $\partial M = N^+ \sqcup N^-$ satisfies the generalized Morse-Bott inequalities with $\kappa = 0$ and $e^\pm$ equal to the zeroeth Betti number of $N^\pm$.*

Proof. Let Γ be the Morse-Bott graph associated to f. According to the results in [7], a Lyapunov graph associated to a continuous flow ϕ_t on a manifold such that $h_i(\Lambda, \phi_t) = h_{n-i}(\Lambda, \phi_t')$, for all isolated invariant set Λ, satisfies the Poincaré-Hopf inequalities. Now, given a vertex S of the Morse-Bott graph Γ labelled with $(\beta_0, \cdots, \beta_k; \lambda)$, by Proposition 4.2, the rank of the homological Conley index of S is $h_{\lambda+j} = \beta_j(S)$, for $j = 0, \cdots, k$, and it is null otherwise. In this way, Γ can be visualized as a Lyapunov graph of a continuous flow. Since the flow φ_t generated by f satisfies $h_i(S, \varphi_t) = h_{n-i}(S, \varphi_t')$, (see Proposition 4.3), then Γ satisfies the Poincaré-Hopf inequalities for the data set $(B_0^\pm, \ldots, B_{n-1}^\pm)$, $\kappa = 0$, $e^\pm = N^\pm$, $h_{\lambda+j} = \beta_j(S)$ for $j = 0, \cdots, n$ and $h_\eta = 0$ otherwise. As the generalized Morse-Bott inequalities are equivalent to the Poincaré-Hopf inequalities for this data set, one concludes that any Morse-Bott flow satisfies the generalized Morse-Bott inequalities. ■

Corollary 4.2. *If an abstract Morse-Bott graph (semi-graph) Γ does not admit a continuation to an abstract Lyapunov graph (semi-graph) of Morse type, then Γ is not realizable as a flow of a Morse-Bott function on any closed n-manifold (compact manifold with boundary).*

Proof. Indeed, if Γ were realizable as a flow of a Morse-Bott function on an closed n-manifold (compact manifold with boundary), then it would admit continuation to an abstract Lyapunov graph (semi-graph) of Morse type, since every Morse-Bott graph (semi-graph) can be continued. ■

Theorem 4.4. *Given an n-abstract Morse-Bott graph Γ with cycle number κ and with l vertices $\nu_1, \cdots, \nu_l$ labelled with $(b_0^i, \cdots, b_{k_i}^i; \lambda_i, \kappa_i)$, define $\mathsf{b}_\eta = \sum_{i=1}^{l} b^i_{s(i,\eta)}$, where $\eta = s(i,\eta) + \lambda_i$ for $\eta = 0, \cdots, n$. If Γ satisfies the generalized Morse-Bott inequalities then the set of data $\{\mathsf{b}_0, \cdots, \mathsf{b}_n\}$ satisfies the following inequalities*

$$
\left\{
\begin{array}{l}
\left\{
\begin{array}{l}
-\mathsf{b}_j \leq (\mathsf{b}_{n-(j-1)} - \mathsf{b}_{j-1}) - (\mathsf{b}_{n-(j-2)} - \mathsf{b}_{j-2}) \pm \cdots \\
\qquad\qquad \pm(\mathsf{b}_{n-1} - \mathsf{b}_1) \pm (\mathsf{b}_n - \mathsf{b}_0) \leq \mathsf{b}_{n-j} \\
\qquad\qquad\qquad\qquad \vdots \\
-\mathsf{b}_2 \leq (\mathsf{b}_{n-1} - \mathsf{b}_1) - (\mathsf{b}_n - \mathsf{b}_0) \leq \mathsf{b}_{n-2}
\end{array}
\right.
\\[2em]
\left\{
\begin{array}{lcl}
\mathsf{b}_1 & \geq & \mathsf{b}_0 - 1 + \kappa \\
\mathsf{b}_{n-1} & \geq & \mathsf{b}_n - 1 + \kappa
\end{array}
\right.
\\[2em]
\textit{If } n = 2i+1, \displaystyle\sum_{j=0}^{2i+1} (-1)^j \mathsf{b}_j = 0. \\
\textit{If } n = 2i \equiv 2 \ (mod\ 4),\ \ \mathsf{b}_i - \displaystyle\sum_{j=0}^{i-1} (-1)^j (\mathsf{b}_{2i-j} - \mathsf{b}_j) \ \ \textit{must be even.}
\end{array}
\right.
\tag{4.5}
$$

Proof. By Theorem 4.1, Γ admits a continuation to an n-abstract Lyapunov graph of Morse type Γ_M. Each vertex ν_i of Γ is exploded to a collection of vertices $v_1^i, \cdots, v_{K_i}^i$ and the labels on each graph satisfy

$$b^i_{\eta-\lambda_i} = \sum_{j=0}^{K_i} h_\eta(v_j^i),$$

for all $i = 1, \cdots, l$. Moreover, Γ_M has b_n vertices labels with $h_n = 1$ and with outgoing edges labelled with $(1, 0, \cdots, 0, 1)$. Also Γ_M has b_0 vertices labelled with $h_0 = 1$ with incoming edges labelled with $(1, 0, \cdots, 0, 1)$.

Let Γ_I be the implosion of Γ, that is, Γ_I is an n-abstract graph with one vertex ν, called saddle vertex, labelled with $(0, \mathsf{b}_1, \cdots, \mathsf{b}_{n-1}, 0; 0, \kappa)$; the vertex ν has $b_n = e_\nu^+$ incoming edges and $b_0 = e_\nu^-$ outgoing edges; the incoming edges of ν are outgoing edges of e_ν^+ vertices labelled with $(1; n)$ and the outgoing edges of ν are incoming edges of e_ν^- vertices labelled with $(1; 0)$; the labels of all edges are equal to $(1, 0, \cdots, 0, 1)$. As

$$\mathsf{b}_\eta = \sum_{i=1}^{l} b^i_{\eta-\lambda_i} = \sum_{i=1}^{l}\sum_{j=0}^{K_i} h_\eta(v^i_j),$$

the saddle vertex of Γ_I can be exploded to the semi-graph obtained from Γ_M by disregarding the vertices $h_0's$ and $h_n's$.

Note that Γ_I is not a Morse-Bott graph since the saddle vertex ν does not have the property of duality. On the other hand, Theorem 4.1 does not require that the label on a vertex of an abstract Morse-Bott graph satisfies the Poincaré duality, then one can use Theorem 4.1 and conclude that the vertex ν satisfies the generalized Morse-Bott inequalities. Therefore, $\{\mathsf{b}_0, \cdots, \mathsf{b}_n\}$ satisfies the inequalities in (4.5) since these inequalities are the generalized Morse-Bott inequalities for a saddle vertex $(0, \mathsf{b}_1, \cdots, \mathsf{b}_{n-1}, 0; 0, \kappa)$. ■

The next result shows that the inequalities in (4.5) are equivalent to the classical Morse-Bott inequalities. More specifically, whenever the data $(\mathsf{b}_0, \cdots, \mathsf{b}_n, \kappa)$ satisfies the inequalities in (4.5), there exits a collection of Betti numbers that satisfies the Morse-Bott inequalities with this same data. Conversely, if $(\mathsf{b}_0, \cdots, \mathsf{b}_n)$ and $(\beta_0, \cdots, \beta_n)$, where $\beta_1 \geq \kappa$, satisfies the Morse-Bott inequalities then $(\mathsf{b}_0, \cdots, \mathsf{b}_n, \kappa)$ satisfies the inequalities in (4.5). This result proves that the generalized Morse-Bott inequalities defined in Theorem 4.2 is in fact a generalization of the Morse-Bott inequalities, since they are well defined for manifolds with boundary.

A list $(\beta_0, \cdots, \beta_n)$ of non-negative integral numbers satisfying $\beta_{n-k} = \beta_k$, for $k = 0, \cdots, n$, $\beta_0 = \beta_n = 1$ and β_i is even if $n = 2i \equiv 2 \pmod 4$ is called a *Betti number vector.*

Theorem 4.5. *A set of non-negative numbers* $(\mathsf{b}_0, \cdots, \mathsf{b}_n, \kappa)$ *satisfies the inequalities in* (4.5) *if and only if it satisfies the Morse-Bott inequalities in* (4.1) *for some Betti number vector* $(\beta_0, \cdots, \beta_n)$ *such that* $\beta_1 \geq \kappa$.

Proof. Suppose that the set of non-negative numbers $(\mathsf{b}_0, \cdots, \mathsf{b}_n, \kappa)$ satisfies the inequalities in (4.1) for a Betti number vector $(\beta_0, \cdots, \beta_n)$ with $\beta_1 \geq \kappa$. Then, by the first inequality in

(4.1),

$$\mathsf{b}_1 \;\geq\; \mathsf{b}_0 + \beta_1 - \beta_0 \;\geq\; \mathsf{b}_0 - 1 + \kappa.$$

By the equality in (4.1) and the duality between β_{n-1} and β_1,

$$\mathsf{b}_{n-1} \;=\; \mathsf{b}_n - \beta_n + \beta_{n-1} \underbrace{-\big[\beta_{n-2} + \cdots + (-1)^n\beta_0\big] + \mathsf{b}_2 + \cdots + (-1)^n\mathsf{b}_0}_{\geq 0} \;\geq\; \mathsf{b}_n - 1 + \kappa.$$

Let $n = 2i+1$ if n is odd and $n = 2i$ if n is even. For $j = 2, \cdots, i$, one has from the equality in (4.1) that

$$\begin{aligned}(-1)^j\mathsf{b}_{n-j} \;=\; & -\Big[\mathsf{b}_n - \mathsf{b}_{n-1} + \cdots + (-1)^{j-1}\mathsf{b}_{n-(j-1)}\Big] \\ & + \underbrace{\beta_n - \beta_{n-1} + \cdots + (-1)^{j-1}\beta_{n-(j-1)}}_{*_1} \\ & + (-1)^j\beta_{n-j} + \underbrace{(-1)^{j+1}\beta_{n-(j+1)} + \cdots + (-1)^n\beta_0}_{*_2} \\ & -\Big[\underbrace{(-1)^{j+1}\mathsf{b}_{n-(j+1)} + \cdots + (-1)^n\mathsf{b}_0}_{*_3}\Big].\end{aligned}$$

There are two cases to consider: $j+1$ even and $j+1$ odd.

Case 1. If $j+1$ is even, one has $(-1)^{j-1}\mathsf{b}_{j-1} + (-1)^{j-2}\mathsf{b}_{j-2} + \cdots + \mathsf{b}_0 \geq (-1)^{j-1}\beta_{j-1} + \cdots + \beta_0$, $*_2 - *_3 \leq 0$ and $(-1)^j\beta_{n-j} \leq 0$. Then, using these fact and the duality of the Betti number vector $(\beta_0, \cdots, \beta_n)$, one has

$$(-1)^j\mathsf{b}_{n-j} \leq -\Big[\mathsf{b}_n - \mathsf{b}_{n-1} + \cdots + (-1)^{j-1}\mathsf{b}_{n-(j-1)}\Big] + (-1)^{j-1}\mathsf{b}_{j-1} + (-1)^{j-2}\mathsf{b}_{j-2} + \cdots + \mathsf{b}_0.$$

Case 2. If $j+1$ is odd, one has $(-1)^{j-1}\mathsf{b}_{j-1} + (-1)^{j-2}\mathsf{b}_{j-2} + \cdots + \mathsf{b}_0 \leq (-1)^{j-1}\beta_{j-1} + \cdots + \beta_0$, $*_2 - *_3 \geq 0$ and $(-1)^j\beta_{n-1} \geq 0$. Then

$$(-1)^j\mathsf{b}_{n-j} \geq -\Big[\mathsf{b}_n - \mathsf{b}_{n-1} + \cdots + (-1)^{j-1}\mathsf{b}_{n-(j-1)}\Big] + (-1)^{j-1}\mathsf{b}_{j-1} + (-1)^{j-2}\mathsf{b}_{j-2} + \cdots + \mathsf{b}_0.$$

In both cases it follows that

$$\mathsf{b}_{n-j} \geq (\mathsf{b}_{n-(j-1)} - \mathsf{b}_{j-1}) - (\mathsf{b}_{n-(j-2)} - \mathsf{b}_{j-2}) \pm \cdots \pm (\mathsf{b}_{n-1} - \mathsf{b}_1) \pm (\mathsf{b}_n - \mathsf{b}_0).$$

If $n = 2i+1$ then $\sum_{j=0}^{n}(-1)^j \mathsf{b}_j = \sum_{j=0}^{n}(-1)^j \beta_j = 0$, the last equality follows from the duality of the list $(\beta_0, \cdots, \beta_n)$ and the fact the n is odd. If $n = 2i \equiv 2 \pmod 4$, then

$$\begin{aligned} &\mathsf{b}_i - \sum_{j=0}^{i-1}(-1)^j(\mathsf{b}_{n-j} - \mathsf{b}_j) = \\ &\mathsf{b}_i - \Big[\mathsf{b}_n - \mathsf{b}_{n-1} + \cdots - \mathsf{b}_{n-(i-2)} + \mathsf{b}_{n-(i-1)}\Big] + \mathsf{b}_0 - \mathsf{b}_1 + \cdots - \mathsf{b}_{i-2} + \mathsf{b}_{i-1}. \end{aligned} \tag{4.6}$$

Since $\mathsf{b}_i = \mathsf{b}_{n-i}$, from the equality in (4.1), it follows that

$$\mathsf{b}_n - \mathsf{b}_{n-1} + \cdots + \mathsf{b}_{n-(i-1)} - \mathsf{b}_i = \beta_n - \beta_{n-1} + \cdots - \beta_1 + \beta_0 - \Big[\mathsf{b}_{i-1} - \mathsf{b}_{i-2} + \cdots + \mathsf{b}_2 - \mathsf{b}_1 + \mathsf{b}_0\Big].$$

Substituting this expression in (4.6), one obtains

$$\begin{aligned} &\mathsf{b}_i - \sum_{j=0}^{i-1}(\mathsf{b}_{n-j} - \mathsf{b}_j) \quad = \\ &-\Big[\beta_n - \beta_{n-1} + \cdots - \beta_1 + \beta_0\Big] + 2\Big[\mathsf{b}_0 - \mathsf{b}_1 + \cdots - \mathsf{b}_{i-2} + \mathsf{b}_{i-1}\Big] \quad = \\ &-2\Big[\beta_{i-1} - \beta_{i-2} + \cdots - \beta_1 + \beta_0\Big] - \beta_i + 2\Big[\mathsf{b}_0 - \mathsf{b}_1 + \cdots - \mathsf{b}_{i-2} + \mathsf{b}_{i-1}\Big], \end{aligned}$$

which is even. This concludes the proof that de Morse-Bott inequalities imply the inequalities in (4.5).

Conversely, in order to prove that the inequalities in (4.5) imply the Morse-Bott inequalities in (4.1), consider an Lyapunov graph Γ with cycle number κ, with one saddle vertex label with $(\mathsf{b}_1, \cdots, \mathsf{b}_{n-1}; 0, \kappa)$, b_0 vertices labels with $(1; 0)$ and b_n vertices labels with $(1; n)$. Then Γ admits a continuation to an abstract Lyapunov graph of Morse type Γ_M with cycle rank greater than or equal to κ and with h_l^c vertices labels with $h_l = 1$ of type $(l-1)$-connecting and h_l^d vertices labels with $h_l = 1$ of type l-disconnecting, such that $h_l^c + h_l^d = \mathsf{b}_l$, for all $l = 1, \cdots, n$. According to [6] and [7], the distributions of $\mathsf{b}_l = h_l^c + h_l^d$ between the types c and d must satisfy a network flow. We illustrate such a network in Example 4.5.

Hence, defining $\beta_0 = \beta_n = 1$ and

$$n = 2i+1$$

$$\beta_j = \begin{cases} h_1^d - h_2^c & \text{if } \ j=1 \\ h_j^d - h_{j+1}^c & \text{if } \ 2 \leq j < i \\ h_i^d & \text{if } \ j = i \\ h_{i+1}^c & \text{if } \ j = i+1 \\ -h_{j-1}^d + h_j^c & \text{if } \ i+2 \leq j \leq 2i-1 \\ -h_{2i-1}^d + h_{2i}^c & \text{if } \ j = 2i \end{cases} \tag{4.7}$$

$$n = 2i$$

$$\beta_j = \begin{cases} h_1^d - h_2^c & \text{if } \ j=1 \\ h_j^d - h_{j+1}^c & \text{if } \ 2 \leq j < i-1 \\ \beta_{inv} & \text{if } \ j = i, \ 2i \equiv 0 \pmod 4 \\ 0 & \text{if } \ j = i, \ 2i \equiv 2 \pmod 4 \\ -h_{j-1}^d + h_j^c & \text{if } \ i+1 \leq j \leq 2i-2 \\ -h_{2i-2}^d + h_{2i-1}^c & \text{if } \ j = 2i-1, \end{cases}$$

the Proposition 6.1 in [7] guarantees that the data $(\mathsf{b}_0, \cdots, \mathsf{b}_n)$ and $(\beta_0, \cdots, \beta_n)$ satisfy the inequalities in (4.1). ■

Example 4.5. Consider the abstract Morse-Bott graph presented in Example 4.3 which satisfies the generalized Morse-Bott inequalities. Note that, for the graph in Figure 4.4, one has $\mathsf{b}_0 = 1, \mathsf{b}_1 = 2, \mathsf{b}_2 = 1, \mathsf{b}_3 = 2, \mathsf{b}_4 = 2, \mathsf{b}_5 = 2, \mathsf{b}_6 = 4, \mathsf{b}_7 = 2$ and the inequalities in (4.5) are $-\mathsf{b}_3 \leq (\mathsf{b}_5 - \mathsf{b}_2) - (\mathsf{b}_6 - \mathsf{b}_1) + (\mathsf{b}_7 - \mathsf{b}_0) \leq \mathsf{b}_4$; $-\mathsf{b}_2 \leq (\mathsf{b}_6 - \mathsf{b}_1) - (\mathsf{b}_7 - \mathsf{b}_0) \leq \mathsf{b}_5$; $\mathsf{b}_1 \geq \mathsf{b}_0 - 1 + \kappa$; $\mathsf{b}_6 \geq \mathsf{b}_7 - 1 + \kappa$. It is easy to see that these inequalities are verified for the graph in consideration. Also, $\sum_{j=0}^{7} (-1)^j \mathsf{b}_j = 1 - 2 + 1 - 2 + 2 - 2 + 4 - 2 = 0$.

In order to construct a Betti number vector $(\beta_0, \cdots, \beta_7)$ which, together with the list $(1, 1, 1, 1, 1, 2, 3, 2)$, satisfies the Morse-Bott inequalities, one can make use of the network-flow in Figure 4.6 and define $(\beta_0, \cdots, \beta_7)$ as in (4.7). There are some possibilities for this Betti vector number corresponding to the possibilities of choice of the list $\{h_1^c, h_1^d, h_2^c, h_2^d, \cdots, h_6^c, h_6^d\}$ that satisfies the network flow in Figure 4.6. One option for this list is the one made in Figure 4.5 which provides the Betti vector number $(1, 1, 0, 1, 1, 0, 1, 1)$. ◀

Theorem 4.5 can be used to show that if $(\mathsf{b}_0, \cdots, \mathsf{b}_n, \kappa)$ does not satisfies the inequalities

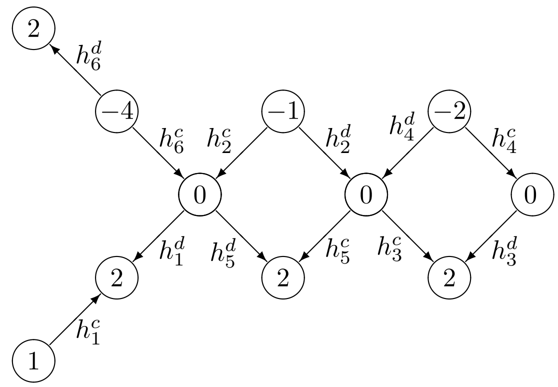


Figure 4.6: Network flow for the list $(1, 2, 1, 2, 2, 2, 4, 2)$.

in (4.5) then there is no closed n-manifold M with $\beta_1(M) \geq \kappa$ which admits a Morse-Bott function with the data $(\mathsf{b}_0, \cdots, \mathsf{b}_n)$. In fact, a necessary condition for the realizability of an abstract Morse-Bott graph with the data $(\mathsf{b}_0, \cdots, \mathsf{b}_n, \kappa)$ is that this same data must satisfy the inequalities in (4.5). Of course, one may ask if an abstract data satisfying these inequalities is sufficient to guarantee the existence of a Morse-Bott flow.

Final Remarks

Realization questions have been addressed in [9] and [10]. In these articles, abstract Lyapunov graphs with vertices labelled either with singularities or periodic orbits have been realized as Morse-Smale flows on n-manifolds. A natural question then arises in regard to the realizability of an abstract Morse-Bott graph as a Morse-Bott flow.

The realization of Morse-Bott graphs on surfaces, i.e. when $n = 2$, is a consequence of the results in [23]. Each basic set of a Morse-Bott flow is either a singularity or a critical manifold isomorphic to the 1-sphere. Each singularity is either attracting, repelling or a saddle and each critical manifold is either attracting or repelling. The next result characterizes the basic blocks associated to these basic sets. As in Figure 4.7, the basic blocks for attracting and repelling singularities are discs; and for saddles and S^1 critical manifold the basic blocks can be orientable and non orientable. In this case, one says that the corresponding vertex in the graph is orientable or non orientable.

The following theorem gives necessary and sufficient conditions for an abstract Morse-Bott graph to be associated with a Morse-Bott flow on a closed manifold. In this theorem, we consider the cycle number κ_v of a vertex v as being 0 and we omit reference to it on the label of the vertex v.

Theorem 4.6. *Let M be a closed surface. An abstract Morse-Bott graph Γ is associated to a Morse-Bott flow φ_t on M satisfying the transversality condition*[2] *if and only if the following statements are satisfied:*

(1) (Local conditions) If the vertex v is labelled with

(a) $(1;2)$ (resp., $(1;0)$) the the number of exiting (entering) edges e_v^- (resp., e_v^+) is equal to one.

(b) $(1;1)$ then $1 \leq e_v^+ \leq 2$, $1 \leq e_v^- \leq 2$ and $e_v^+ + e_v^- \leq 3$. The vertex is orientable if and only if $e_v^+ + e_v^- = 3$.

(c) $(1,1;1)$ (resp., $(1,1;0)$) then $e_v^- \leq 2$ (resp., $e_v^+ \leq 2$). The vertex is orientable if and only if $e_v^- = 2$ (resp., $e_v^+ = 2$).

(2) (Global conditions)

(a) If M is orientable, the cycle rank of Γ must be equal to the genus of M.

(b) If M is non orientable, twice the cycle rank of Γ plus the number of non orientable vertices must be equal to the genus of M.

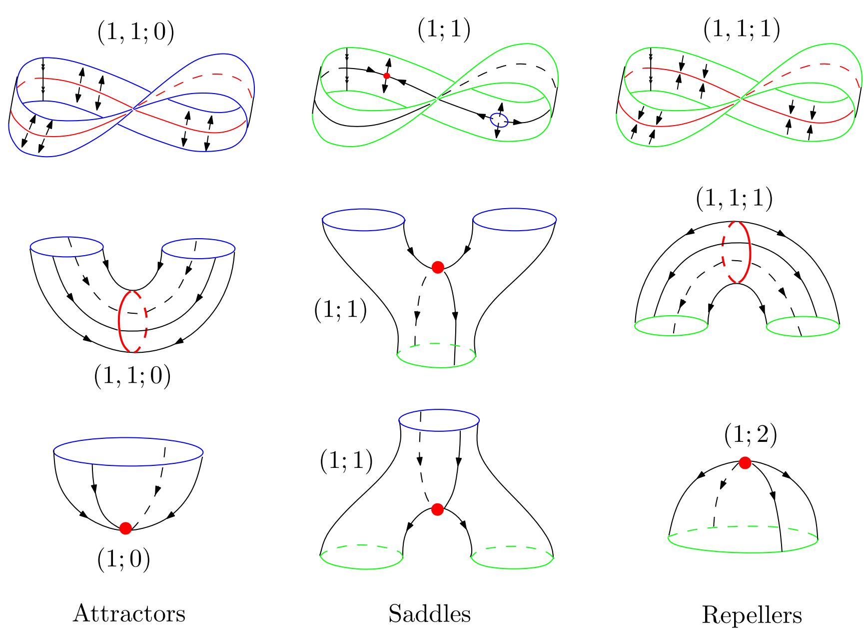


Figure 4.7: Basic blocks for Morse-Bott flows on surfaces.

[2] In the two dimensional case, the transvesality condition means that there is no saddle-saddle connections.

In order to prove the necessity of the local and global conditions, one must show that given a Morse-Bott graph associated to a Morse-Bott flow on a surface, these conditions are verified. All possible basic blocks in this context are illustrated in Figure 4.7. To prove that a given abstract Morse-Bott graph Γ is realizable on a surface, the basic blocks are glued together appropriately to define a flow on M that realizes Γ. The proof follows the same steps of Theorem 1 in [23].

Chapter 5

Connection Matrices for Morse-Bott Flows on Closed Manifolds

In this chapter, our goal is to make use of Conley index theory, [12], to study Morse-Bott flows on a smooth closed n-manifold M with the underlying motivation of obtaining dynamical information from homotopical invariants. We wish to introduce a connection matrix theory approach for Morse-Bott flows. The motivation for this resides in the fact that connection matrices, as defined by Franzosa in [20], [21] and [22], were introduced to study the behaviour of connecting orbits in a flow which undergoes perturbation. Eventual bifurcations were captured by transition matrices, see also [24], [29] and [36]. On the other hand, in the setting of Morse theory, connection matrices can be viewed as differentials of Morse complexes (see [39]), making it possible to translate topological data into dynamical data and vice versa.

Considering a Morse-Bott function $f : M \to \mathbb{R}$ on a smooth closed n-manifold M, a Morse-Bott complex associated to f is constructed in [2] by means of a Morse-Smale perturbation $h : M \to \mathbb{R}$ of f. A natural question to consider is whether the differential of a Morse-Bott complex can be interpreted as a connection matrix. The answer to this is affirmative. However, a necessary step to accomplish this endeavour is to explore more deeply the connection matrix theory for Morse-Bott flows, which is our focus in this chapter.

The idea is to obtain a characterization of the set of connection matrices for a Morse-Bott flow φ_f on M using the set of connection matrices for a Morse-Smale flow φ_h on M, where h is a Morse perturbation of f and φ_f (resp., φ_h) is a flow associated to the vector field $-\nabla f$ (resp., $-\nabla h$). As a result of this characterization, proved in Section 5.2, one can define a

Morse-Bott complex with differential being a connection matrix which opens the possibility to the use of spectral sequence techniques, such as in [13], [30] and [24], to obtain further dynamical information, e.g., bifurcating orbits.

Our approach to study connection matrices for Morse-Bott flows and obtain the required characterization is to initially consider connection matrices of Morse decompositions in a general setting. Given an isolated invariant set S, we analyze what properties remain on the sets of connection matrices when both Morse decompositions of S and partial orderings undergo changes.

This chapter is organized as follows. In Section 5.1 we consider connection matrices in a general framework. More specifically, given an isolated invariant set S and a $<$-ordered Morse decomposition $\mathcal{D}(S)$, we measure the effect a change in the Morse decomposition $\mathcal{D}(S)$, caused by a modification on the partial order $<$, has on the respective connection matrices, as shown in Proposition 5.4 and Theorem 5.3.

In Section 5.2 we apply the results obtained in Section 5.1 to Morse-Bott flows φ_f on a closed n-manifold M. For instance, in Theorem 5.4 we prove that the set of connection matrices for Morse-Bott flows on M coincides with the set of connection matrices for perturbations that give rise to Morse-Smale flows on M. Theorem 5.5 is more constructive in nature, since we show how a connection matrix of $\mathcal{D}(M, \varphi_h)$, where φ_h is a perturbation of φ_f, induces a connection matrix of the Morse-Bott flow φ_f.

In Section 5.2, we also answer a natural question that arises in this context, which examines if for a given connection matrix Δ for a Morse-Bott flow φ_f there exists a Morse-Smale perturbation φ_h of φ_f such that Δ is induced from a connection matrix of φ_h.

It is in due course to present the following well known result.

Proposition 5.1. *Let f be a perfect Morse function on a closed manifold M. Consider a flow φ_f associated to the vector field $-\nabla f$ and the finest Morse decomposition $\mathcal{D}(M)$ of M with respect to this flow. Then, the set of connection matrices of $\mathcal{D}(M)$ contains only the null map.*

Proof. Fisrt, note that a connection matrix Δ for the finest Morse decomposition of M encodes the weak Morse inequalities $c_k \geq \beta_k$, where $c_k = \#Crit_k(f)$ and β_k is the k-th Betti number of M, i.e., the rank of $H_k(M;\mathbb{Z})$[1]. Indeed, one has that $H\Delta(P) \cong CH(M) = H_*(M;\mathbb{Z})$, since Δ is a connection matrix; and $C_k\Delta(P) = \oplus CH_k(M_x)$, where the sum is

[1]If the manifold is nonorientable, the homology is computed over $\mathbb{Z}_2$.

over all critical points of f. Therefore,

$$\beta_k = \text{rank } H_k\Delta(P) = \text{rank } \frac{\text{Ker } \Delta_k(P)}{\text{Im } \Delta_{k+1}(P)} \leq \text{rank } \Big[\text{Ker } \Delta_k(P)\Big] = c_k,$$

which is precisely the classic Morse inequalities. Furthermore, the equalities hold if and only if $\Delta \equiv 0$. On the other hand, if the Morse function f is a perfect Morse function, then the equality $\beta_k = c_k$ holds for all $k = 1, \ldots, n = dim(M)$, implying that $\Delta \equiv 0$. ■

This result will be used in Section 5.2. Moreover, it is interesting to note as a consequence of this proposition that each connection matrix of the finest Morse decomposition of M, $\mathcal{D}(M)$, encodes the weak Morse inequalities.

5.1 Connection Matrices for Coarser Morse Decompositions

The motivation for this section can be seen in the following hypothetical situation. Given an isolated invariant set S and a Morse decomposition $\mathcal{D}(S)$, one can consider a coarser Morse decomposition $\widetilde{\mathcal{D}}(S)$ of S relative to $\mathcal{D}(S)$. An important question is if there exists any relation between connection matrices of $\mathcal{D}(S)$ and connection matrices of $\widetilde{\mathcal{D}}(S)$. Our goal is to describe the relationship among connection matrices of $\mathcal{D}(S)$ and $\widetilde{\mathcal{D}}(S)$. For instance, let $\mathcal{D}(S) = \{M_1, \ldots, M_7\}$ and let $\widetilde{\mathcal{D}}(S)$ be the Morse decomposition obtained when one groups M_3, M_4 and M_5 as well as their connections in one Morse set M_I, i.e., $\widetilde{\mathcal{D}}(S) = \{M_1, M_2, M_I, M_6, M_7\}$. This grouping will be described in more detail subsequently. Assume that there is an isomorphism F_I from $CH(M_I)$ to $CH(M_3) \oplus CH(M_4) \oplus CH(M_5)$ and let

$$\Delta = \begin{bmatrix} 0 & \Delta(2,1) & \Delta(3,1) & \Delta(4,1) & \Delta(5,1) & \Delta(6,1) & \Delta(7,1) \\ 0 & 0 & \Delta(3,2) & \Delta(4,2) & \Delta(5,2) & \Delta(6,2) & \Delta(7,2) \\ 0 & 0 & 0 & \Delta(4,3) & \Delta(5,3) & \Delta(6,3) & \Delta(7,3) \\ 0 & 0 & 0 & 0 & \Delta(5,4) & \Delta(6,4) & \Delta(7,4) \\ 0 & 0 & 0 & 0 & 0 & \Delta(6,5) & \Delta(7,5) \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta(7,6) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a connection matrix of $\mathcal{D}(S)$. One can induce a connection matrix $\widetilde{\Delta}$ of the coarser Morse decomposition $\widetilde{\mathcal{D}}(S)$ from Δ, as follows:

$$\widetilde{\Delta} = \begin{bmatrix} 0 & \Delta(2,1) & \widetilde{\Delta}(I,1) & \Delta(6,1) & \Delta(7,1) \\ 0 & 0 & \widetilde{\Delta}(I,2) & \Delta(6,2) & \Delta(7,2) \\ 0 & 0 & 0 & \widetilde{\Delta}(6,I) & \widetilde{\Delta}(7,I) \\ 0 & 0 & 0 & 0 & \widetilde{\Delta}(7,6) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the maps $\widetilde{\Delta}(\cdot,\cdot)$ are given by:

$$\widetilde{\Delta}(I,j) = \begin{bmatrix} \Delta(3,j) & \Delta(4,j) & \Delta(5,j) \end{bmatrix} \circ F_I \quad \text{and} \quad \widetilde{\Delta}(k,I) = F_I \circ \begin{bmatrix} \Delta(k,3) \\ \Delta(k,4) \\ \Delta(k,5) \end{bmatrix}$$

for $j = 1,2$ and $k = 6,7$.

In what follows we intend to formalize this approach.

Consider a partially ordered set $(P,<)$. Let $I_1,\ldots,I_k$ be a collection of mutually disjoint intervals with respect to $(P,<)$, such that, if $i<j$ then there are no elements $\pi \in I_j$ and $\pi' \in I_i$ with $\pi < \pi'$. In what follows, a subset $\widetilde{P}$ of $\mathcal{I}(P,<)$ is defined to be $\widetilde{P}_1 \cup \widetilde{P}_2$, where $\widetilde{P}_2 = \{\widetilde{\pi} \mid \widetilde{\pi} = I_j, j = 1,\cdots,k\}$ and $\widetilde{P}_1 = \{\widetilde{\pi} \mid \ \widetilde{\pi} = \{\pi\}$ such that $\pi \in P \setminus (I_1 \cup \cdots \cup I_k)\}$. Hence, the set $\widetilde{P} = \widetilde{P}_1 \cup \widetilde{P}_2$ is composed by intervals of $(P,<)$. It is important to note that $\widetilde{\pi}$ is an element of $\widetilde{P}$ and is not an element of P, but an interval in $(P,<)$. From now on, an element of $\widetilde{P}$ will be denoted by $\widetilde{\pi}$. Although $\widetilde{P}$ is a subset of $\mathcal{I}(<)$ and the elements of $\mathcal{I}(<)$ are usually denoted by I,J,K, we will denote elements of $\widetilde{P}$ by $\widetilde{\pi}$ and intervals of $\widetilde{P}$ by $\widetilde{I},\widetilde{J},\widetilde{K}$. We also adopt a loose notation $\pi \in \widetilde{\pi}$ to indicate that the element $\pi \in P$ belongs either to the interval I_j for $j = 1,\cdots,k$ or to the singleton interval in $(P,<)$ composed by itself.

Consider the transitive closure of the relation $\prec$ in $\widetilde{P}$ given by:

$$\widetilde{\pi}_1 \prec \widetilde{\pi}_2, \text{ if there are } \pi_1 \in \widetilde{\pi}_1 \text{ and } \pi_2 \in \widetilde{\pi}_2 \text{ such that } \pi_1 < \pi_2$$

where $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \widetilde{P}$ and $\widetilde{\pi}_1 \neq \widetilde{\pi}_2$.

Proposition 5.2. *The pair $(\widetilde{P},\prec)$ is a partially ordered set.*

Proof. One needs to prove that $\widetilde{\pi} \nprec \widetilde{\pi}$ for all $\widetilde{\pi} \in \widetilde{P}$. Suppose by contradiction that $\widetilde{\pi} \prec \widetilde{\pi}$. Hence, by definition of the relation $\prec$, there exists $\widetilde{\pi}' \in \widetilde{P}$ such that $\widetilde{\pi} \prec \widetilde{\pi}' \prec \widetilde{\pi}$. If $\widetilde{\pi}'$ is a singleton, i.e. $\widetilde{\pi}' = \{\pi'\} \in \widetilde{P}_1$, then there must exist $\pi_1, \pi_2 \in \widetilde{\pi}$ such that $\pi_1 < \pi' < \pi_2$. But $\widetilde{\pi} \in \mathcal{I}(P, <)$, then $\pi' \in \widetilde{\pi}$, which contradicts the fact that $\widetilde{\pi}' \in \widetilde{P}_1$. Therefore, $\widetilde{\pi}'$ is not a singleton, that is, $\widetilde{\pi}' = I_j \in \widetilde{P}_2$, for some $j = 1, \cdots, k$. Hence, one has that $\widetilde{\pi} \prec \widetilde{\pi}' = I_j \prec \widetilde{\pi}$, which contradicts the choices of the intervals $I_1, \ldots, I_k$. ■

The next proposition relates intervals in $(\widetilde{P}, \prec)$ with intervals in $(P, <)$. Given an interval $\widetilde{J}$ in $(\widetilde{P}, \prec)$, define J to be the subset of P such that $\pi \in J$ iff $\pi \in \widetilde{\pi}$ for some $\widetilde{\pi} \in \widetilde{J}$. The set J is well defined since each element $\widetilde{\pi}$ of $\widetilde{J}$ is an interval of $(P, <)$.

Proposition 5.3. **(a)** *If $\widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)$ then $J \in \mathcal{I}(P, <)$.*

(b) *If $(\widetilde{J}, \widetilde{K}) \in \mathcal{I}_2(\widetilde{P}, \prec)$ then $(J, K) \in \mathcal{I}_2(P, <)$.*

Proof.

(a) Let $\widetilde{J}$ be an interval in $(\widetilde{P}, \prec)$. Given $\pi_1, \pi_2 \in J$ and $\pi \in P$ such that $\pi_1 < \pi < \pi_2$, one must show that $\pi \in J$. In fact, there must exist $\widetilde{\pi}_1, \widetilde{\pi}_2, \widetilde{\pi} \in \widetilde{P}$ such that $\pi_1 \in \widetilde{\pi}_1, \pi_2 \in \widetilde{\pi}_2$ and $\pi \in \widetilde{\pi}$. Hence, $\widetilde{\pi}_1 \prec \widetilde{\pi} \prec \widetilde{\pi}_2$, which implies that $\widetilde{\pi} \in \widetilde{J}$. Therefore, $\pi \in J$.

(b) As $\widetilde{J} \cup \widetilde{K}$ is an interval in $(\widetilde{P}, \prec)$, by the previous item , $J \cup K$ is an interval in $(P, <)$. Suppose by contradiction that $\pi_1 \in J$, $\pi_2 \in K$ and $\pi_2 < \pi_1$. By definition of J, K, there exist $\widetilde{\pi}_1 \in \widetilde{J}$ and $\widetilde{\pi}_2 \in \widetilde{K}$ such that $\pi_1 \in \widetilde{\pi}_1$ and $\pi_2 \in \widetilde{\pi}_2$. Hence, $\widetilde{\pi}_2 \prec \widetilde{\pi}_1$, which contradicts the fact that $(\widetilde{J}, \widetilde{K})$ is an adjacent pair of intervals. ■

Indeed, one can show that if $(\widetilde{J}_1, \ldots, \widetilde{J}_n) \in \mathcal{I}_n(\widetilde{I}, \prec)$ then $(J_1, \ldots, J_n) \in \mathcal{I}_n(I, \prec)$. The proof follows the same ideas of the proof of Proposition 5.3, (b).

Let S be an isolated invariant set and $\mathcal{D}(S) = \{M_\pi : \pi \in P\}$ be a Morse decomposition of S with admissible ordering $<$. The purpose of this section is to study connection matrices with coarser Morse decompositions of S, $\widetilde{\mathcal{D}}(S)$. This is done by considering as Morse sets of $\widetilde{\mathcal{D}}(S)$ the union of some M_π of $\mathcal{D}(S)$ and their connections. More specifically, consider the set $\widetilde{\mathcal{D}}(S) = \{\widetilde{M_{\widetilde{\pi}}} : \widetilde{\pi} \in \widetilde{P}\}$, where

$$\widetilde{M_{\widetilde{\pi}}} = M_{\widetilde{\pi}} = \left(\bigcup_{\pi \in \widetilde{\pi}} M_\pi \right) \cup \left(\bigcup_{\pi, \pi' \in \widetilde{\pi}} C(M_{\pi'}, M_\pi) \right).$$

$\widetilde{\mathcal{D}}(S)$ is not just a collection of some Morse sets of P with respect to $<$, it is an $\prec$-ordered Morse decomposition of S. See Proposition 5.4 below. Before proving this, we present a characterization of the set of orbits connecting two Morse sets in the coarser Morse decomposition $\widetilde{\mathcal{D}}(S)$ by means of the set of orbits connecting two isolated invariant set in the original Morse decomposition $\mathcal{D}(S)$. Let $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \widetilde{P}$, the set of orbits connecting $\widetilde{M}_{\widetilde{\pi}_2}$ to $\widetilde{M}_{\widetilde{\pi}_1}$, $C(\widetilde{M}_{\widetilde{\pi}_2}, \widetilde{M}_{\widetilde{\pi}_1})$, is given by

$$C(\widetilde{M}_{\widetilde{\pi}_2}, \widetilde{M}_{\widetilde{\pi}_1}) = \bigcup_{\pi' \in \widetilde{\pi}_2, \pi \in \widetilde{\pi}_1} C(M_{\pi'}, M_{\pi}).$$

This characterization is essentially the proof of the following two propositions.

Proposition 5.4. *The set $\widetilde{\mathcal{D}}(S)$ is a Morse decomposition of S with admissible ordering $\prec$.*

Proof. The sets $\widetilde{M}_{\widetilde{\pi}}$ are isolated, invariant, disjoint and compact, by definition. If $\gamma \in S$ and $\gamma \notin \cup_{\widetilde{\pi} \in \widetilde{P}} \widetilde{M}_{\widetilde{\pi}}$ one must prove that $\gamma \in C(\widetilde{M}_{\widetilde{\pi}}, \widetilde{M}_{\widetilde{\pi}'})$ with $\widetilde{\pi}' \prec \widetilde{\pi}$. Observe that $\gamma \notin \cup_{\pi \in P} M_{\pi}$ and, as $\mathcal{D}(S)$ is a Morse decomposition of S, there exist $\pi_1, \pi_2 \in P$ such that $\pi_1 < \pi_2$ and $\gamma \in C(M_{\pi_2}, M_{\pi_1})$. Therefore, $\gamma \in C(\widetilde{M}_{\widetilde{\pi}}, \widetilde{M}_{\widetilde{\pi}'})$, for $\widetilde{\pi}', \widetilde{\pi} \in \widetilde{P}$ such that $\pi_1 \in \widetilde{\pi}$ and $\pi_2 \in \widetilde{\pi}'$. ∎

Proposition 5.5. *If $<$ induces the flow ordering of the Morse decomposition $\mathcal{D}(S)$, then $\prec$ induces the flow ordering of the coarser Morse decomposition $\widetilde{\mathcal{D}}(S)$.*

Proof. Given $\widetilde{\pi}, \widetilde{\pi}' \in \widetilde{P}$ with $\widetilde{\pi} \prec \widetilde{\pi}'$, one needs to show that there exist a sequence in $\widetilde{P}$ $\widetilde{\pi} = \widetilde{\pi}_0, \widetilde{\pi}_1, \ldots, \widetilde{\pi}_{n-1}, \widetilde{\pi}_n = \widetilde{\pi}'$ such that $C(\widetilde{M}_{\widetilde{\pi}_i}, \widetilde{M}_{\widetilde{\pi}_{i-1}}) \neq \emptyset$. This proof is straightforward and is done by analysing the possibilities of $\pi, \pi' \in \widetilde{P}$ as elements of $\widetilde{P}_1$ or $\widetilde{P}_2$ and using the characterization of the set of orbits connecting $\widetilde{M}_{\pi'}$ to $\widetilde{M}_{\pi}$. ∎

Having defined the Morse decomposition $\widetilde{D}(S)$ with admissible ordering $\prec$, one can now define connection matrices in this setting.

Let $\mathcal{H}(<) = \mathcal{H}(<; G)$ be the homology index braid of $<$ with coefficients in G. Let $C = \{C\Delta(\pi)\}_{\pi \in P}$ be a collection of free chain complexes with trivial boundary operator, where $C\Delta(\pi) = CH(M_{\pi})$ is the Conley homological index of M_{π} with coefficients in G, for all $\pi \in P$. Therefore, the collection of connection matrices of $\mathcal{H}(<)$, $\mathcal{CM}(\mathcal{H}(<))$, is non empty, since the graded module braid $\mathcal{H}(<)$ is a chain complex generated.

From a given connection matrix $\Delta : C\Delta(P) \to C\Delta(P)$ of $\mathcal{H}(<)$, we will construct a connection matrix $\widetilde{\Delta} : C\widetilde{\Delta}(\widetilde{P}) \to C\widetilde{\Delta}(\widetilde{P})$ of the homological index braid of $\prec$ with coefficients in G, $\mathcal{H}(\prec) = \mathcal{H}(\prec; G)$. In order to do this, we will henceforth assume that, for each interval $I_1, \ldots, I_k$ previously set in $(P, <)$, $\Delta(I_j) = 0$. Note that, for other intervals in $(P, <)$ different from $I_1, \ldots, I_k$, this assumption is not required. Using this assumption, it follows that

$$H\Delta(I_j) = \frac{\text{Ker } \Delta(I_j)}{\text{Im } \Delta(I_j)} = \frac{\oplus_{\pi \in I_j} CH(M_\pi)}{0} \cong \bigoplus_{\pi \in I_j} CH(M_\pi)$$

for all $j = 1, \ldots, k$. As Δ is a connection matrix of $\mathcal{H}(<)$ then $CH(M_{I_j}) \cong H\Delta(I_j)$, for all $j = 1, \ldots, k$, and one has

$$CH(M_{I_j}) \cong \bigoplus_{\pi \in I_j} CH(M_\pi).$$

Now, let $\widetilde{C} = \{C\widetilde{\Delta}(\widetilde{\pi})\}_{\widetilde{\pi} \in \widetilde{P}}$ be a chain complex braid over $\prec$, where $C\widetilde{\Delta}(\widetilde{\pi}) = CH(\widetilde{M}_{\widetilde{\pi}})$, for each $\widetilde{\pi} \in \widetilde{P}$, and the boundary operator is trivial. Observe that the chain complex braids $C = \{C\Delta(\pi)\}_{\pi \in P}$ and $\widetilde{C} = \{C\widetilde{\Delta}(\widetilde{\pi})\}_{\widetilde{\pi} \in \widetilde{P}}$ are related to each other by:

- $C\widetilde{\Delta}(\{\pi\}) = CH(\widetilde{M}_{\{\pi\}}) = CH(M_\pi) = C\Delta(\pi)$;
- $C\widetilde{\Delta}(I_j) = CH(\widetilde{M}_{I_j}) = CH(M_{I_j}) \cong \bigoplus_{\pi' \in I_j} CH(M_{\pi'}) = \bigoplus_{\pi' \in I_j} C\Delta(\pi') = C\Delta(I_j)$.

In short, $C\widetilde{\Delta}(\widetilde{\pi}) = C\Delta(\widetilde{\pi})$.

For each $\widetilde{\pi} \in \widetilde{P}$, let

$$F_{\widetilde{\pi}} : C\widetilde{\Delta}(\widetilde{\pi}) \longrightarrow C\Delta(\widetilde{\pi}) \tag{5.1}$$

be an isomorphism. If $\widetilde{\pi} \in \widetilde{P}_1$, i.e. $\widetilde{\pi}$ is a singleton, we consider $F_{\widetilde{\pi}}$ as be the identity map.

The set of connection matrices of $\mathcal{H}(\prec, G)$ is non empty, since this graded module braid is chain complex generated. Now, from Δ we will make explicit a connection matrix of $\mathcal{H}(\prec, G)$. Let $\widetilde{\Delta} : C\widetilde{\Delta}(\widetilde{P}) \to C\widetilde{\Delta}(\widetilde{P})$ be the map regarded as a matrix

$$\widetilde{\Delta} = \left(\quad \widetilde{\Delta}(\widetilde{\pi}', \widetilde{\pi}) \quad \right)_{\widetilde{\pi}, \widetilde{\pi}' \in \widetilde{P}},$$

where each $\widetilde{\Delta}(\widetilde{\pi}', \widetilde{\pi})$ is the map from $C\widetilde{\Delta}(\widetilde{\pi}')$ to $C\widetilde{\Delta}(\widetilde{\pi})$ defined as follows:

$$\widetilde{\Delta}_{\widetilde{\pi}', \widetilde{\pi}} = F_{\widetilde{\pi}}^{-1} \circ \Delta(\widetilde{\pi}', \widetilde{\pi}) \circ F_{\widetilde{\pi}'}. \tag{5.2}$$

The goal in this section is to show that $\widetilde{\Delta} = \widetilde{\Delta}(\widetilde{P})$ is a connection matrix of $\mathcal{H}(\prec)$, which will be proved in Theorem 5.3. The next result describes a relation between the maps $\widetilde{\Delta}(\widetilde{K}, \widetilde{J})$ and $\Delta(K, J)$, for each $(\widetilde{J}, \widetilde{K})$ adjacent pair of intervals in $(\widetilde{P}, \prec)$. This result is essential to the proof of the main theorem.

Theorem 5.1. *Given a pair of adjacent intervals $(\widetilde{J}, \widetilde{K}) \in \mathcal{I}_2(\widetilde{P}, \prec)$, the map $\widetilde{\Delta}(\widetilde{K}, \widetilde{J})$ is conjugated to $\Delta(K, J)$, that is, there exist isomorphisms $R_J : C\widetilde{\Delta}(\widetilde{J}) \to C\Delta(J)$ and $R_K : C\widetilde{\Delta}(\widetilde{K}) \to C\Delta(K)$ such that*

$$\widetilde{\Delta}(\widetilde{K}, \widetilde{J}) = R_J^{-1} \circ \Delta(K, J) \circ R_K.$$

In particular, for each $\widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)$,

$$\widetilde{\Delta}(\widetilde{J}) = R_J^{-1} \circ \Delta(J) \circ R_J.$$

Proof. Let $\widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)$ and $\{I_{j_1}, \dots, I_{j_p}\}$ the elements in the intersection $\widetilde{J} \cap \widetilde{P}_2$. To simplify notation, renumber these elements as $I_1, \dots, I_p$, such that, if $I_i \prec I_j$ then $i < j$. Define the following subsets of $\widetilde{J}$:

$$\begin{aligned} B_\ell &= \{\widetilde{\pi} \in \widetilde{J} : \widetilde{\pi} \prec I_{\ell+1}\} \setminus (B_0 \cup \dots \cup B_\ell), \quad \text{for } \ell = 0, \cdots, p-1 \\ B_p &= \{\widetilde{\pi} \in \widetilde{J} : I_p \prec \widetilde{\pi}\} \cup \{\widetilde{\pi} \in \widetilde{J} : \widetilde{\pi} \text{ and } I_i \text{ are noncomparable } \ \forall i = 1, \dots, p\}. \end{aligned}$$

Note that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $(\cup_{i=0}^p B_i) \cup (\cup_{i=1}^p I_i) = \widetilde{J}$. This collection of subsets of $\widetilde{J}$ has the following properties:

(A) $B_i \in \mathcal{I}(\widetilde{P}, \prec)$, for $i = 0, \dots, p$;

(B) $(B_0, I_1), (B_k, I_{k+1}), (I_k, B_k) \in \mathcal{I}_2(\widetilde{P}, \prec)$, for all $k = 1, \dots, p$;

(C) $(B_0, I_1, B_1, \dots, I_p, B_p) \in \mathcal{I}_{2p+1}(\widetilde{P}, \prec)$.

Given $(\widetilde{J}, \widetilde{K}) \in \mathcal{I}_2(\widetilde{P}, \prec)$, let $I_{i_1}^J, \dots, I_{i_p}^J$[2] $\in \widetilde{J}$ and $I_{i_1}^K, \dots, I_{i_q}^K \in \widetilde{K}$ be the only elements in $\widetilde{J} \cap \widetilde{P}_2$ and $\widetilde{K} \cap \widetilde{P}_2$, respectively. Renumber these elements as $I_1^J, \dots, I_p^J$ and $I_1^K, \dots, I_q^K$ such that, if $I_i^J \prec I_j^J$ then $i < j$, and if $I_i^K \prec I_j^K$ then $i < j$. Consider the decompositions $(B_0, I_1^J, B_1, \dots, I_p^J, B_p)$ of $\widetilde{J}$ and $(A_0, I_1^K, A_1, \dots, I_q^K, A_q)$ of $\widetilde{K}$, as described above. Using

[2]The superscript J (resp., K) is in order to clarify the relation of elements belonging to $\widetilde{J}$ (resp., $\widetilde{K}$).

these decompositions, one can view $\widetilde{\Delta}$ as a map given by the matrix

$$\widetilde{\Delta}(\widetilde{K},\widetilde{J}) = \begin{bmatrix} \widetilde{\Delta}(A_0,B_0) & \widetilde{\Delta}(I_1^K,B_0) & \widetilde{\Delta}(A_1,B_0) & \cdots & \widetilde{\Delta}(I_q^K,B_0) & \widetilde{\Delta}(A_q,B_0) \\ \widetilde{\Delta}(A_0,I_1^J) & \widetilde{\Delta}(I_1^K,I_1^J) & \widetilde{\Delta}(A_1,I_1^J) & \cdots & \widetilde{\Delta}(I_q^K,I_1^J) & \widetilde{\Delta}(A_q,I_1^J) \\ \widetilde{\Delta}(A_0,B_1) & \widetilde{\Delta}(I_1^K,B_1) & \widetilde{\Delta}(A_1,B_1) & \cdots & \widetilde{\Delta}(I_q^K,B_1) & \widetilde{\Delta}(A_q,B_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{\Delta}(A_0,I_p^J) & \widetilde{\Delta}(I_1^K,I_p^J) & \widetilde{\Delta}(A_1,I_p^J) & \cdots & \widetilde{\Delta}(I_q^K,I_p^J) & \widetilde{\Delta}(A_q,I_p^J) \\ \widetilde{\Delta}(A_0,B_p) & \widetilde{\Delta}(I_1^K,B_p) & \widetilde{\Delta}(A_1,B_p) & \cdots & \widetilde{\Delta}(I_q^K,B_p) & \widetilde{\Delta}(A_q,B_p) \end{bmatrix}.$$

For each $\widetilde{J} \in \mathcal{I}(\widetilde{P},\prec)$, consider the isomorphism R_J from

$$C\widetilde{\Delta}(\widetilde{J}) = \bigoplus_{\pi\in B_0} C\widetilde{\Delta}(\pi) \oplus C\widetilde{\Delta}(I_1) \oplus \cdots \oplus \bigoplus_{\pi\in B_{p-1}} C\widetilde{\Delta}(\pi) \oplus C\widetilde{\Delta}(I_p) \bigoplus_{\pi\in B_p} C\widetilde{\Delta}(\pi)$$

to

$$C\Delta(J) = \bigoplus_{\pi\in B_0} C\Delta(\pi) \bigoplus_{\pi\in I_1} C\Delta(\pi) \oplus \cdots \oplus \bigoplus_{\pi\in B_{p-1}} C\Delta(\pi) \bigoplus_{\pi\in I_p} C\Delta(\pi) \bigoplus_{\pi\in B_p} C\Delta(\pi),$$

given by

$$R_J = (id, F_{I_1}, id, F_{I_2}, \ldots, id, F_{I_p}, id), \tag{5.3}$$

where F_{I_j} is the isomorphisms defined in (5.1). Using this matrix notation, it is not difficult to see that $\widetilde{\Delta}(\widetilde{K},\widetilde{J}) = R_J^{-1} \circ \Delta(K,J) \circ R_K$.

If $\widetilde{J}$ (resp.,$\widetilde{K}$) contains no elements of $\widetilde{P}_2$, it is also true that $\widetilde{\Delta}(\widetilde{K},\widetilde{J}) = R_J^{-1} \circ \Delta(K,J) \circ R_K$, where $R_J = id$ ($R_K = id$, respectively).

To see that $\widetilde{\Delta}(\widetilde{J}) = R_J^{-1} \circ \Delta(J) \circ R_J$, for a given $\widetilde{J} \in \mathcal{I}(\widetilde{P},\prec)$, just consider the decomposition $(B_0, I_1, B_1, \ldots, I_p, B_p)$ of $\widetilde{J}$, as described above, and visualize the map $\widetilde{\Delta}(\widetilde{J})$ as

$$\widetilde{\Delta}(\widetilde{J}) = \begin{bmatrix} \widetilde{\Delta}(B_0) & \widetilde{\Delta}(I_1,B_0) & \widetilde{\Delta}(B_1,B_0) & \cdots & \widetilde{\Delta}(I_p,B_0) & \widetilde{\Delta}(B_p,B_0) \\ 0 & \widetilde{\Delta}(I_1) & \widetilde{\Delta}(B_1,I_1) & \cdots & \widetilde{\Delta}(I_p,I_1) & \widetilde{\Delta}(B_p,I_1) \\ 0 & 0 & \widetilde{\Delta}(B_1) & \cdots & \widetilde{\Delta}(I_p,B_1) & \widetilde{\Delta}(B_p,B_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \widetilde{\Delta}(I_p) & \widetilde{\Delta}(B_p,I_p) \\ 0 & 0 & 0 & \cdots & 0 & \widetilde{\Delta}(B_p) \end{bmatrix}.$$

If $\widetilde{J}$ contains none of the elements $I_1, \ldots, I_k \in \widetilde{P}_2$, then $\widetilde{\Delta}(\widetilde{J}) = \Delta(J)$, by definition of the map $\widetilde{\Delta}$.

■

The first step to show that $\widetilde{\Delta}$ is a connection matrix of $\mathcal{H}(\prec)$ is to check if $\widetilde{\Delta}$ is a strictly upper triangular boundary map, since all connection matrices have this property.

Theorem 5.2. **(a)** *The map $\widetilde{\Delta}$ is strictly upper triangular, i.e., $\widetilde{\Delta}(\widetilde{\pi}', \widetilde{\pi}) \neq 0$ implies $\widetilde{\pi} \prec \widetilde{\pi}'$;*

(b) *The map $\widetilde{\Delta}$ is a boundary map, i.e., $\widetilde{\Delta}(\widetilde{\pi}', \widetilde{\pi})$ is of degree -1 and $\widetilde{\Delta} \circ \widetilde{\Delta} = 0$.*

Proof.

(a) Suppose that $\widetilde{\Delta}(\widetilde{\pi}', \widetilde{\pi}) \neq 0$. By definition of $\widetilde{\Delta}$ in (5.2), one has that $\Delta(\widetilde{\pi}', \widetilde{\pi}) \neq 0$. Hence, there are $\pi_1 \in \widetilde{\pi}, \pi_2 \in \widetilde{\pi}'$ such that $\Delta(\pi_2, \pi_1) \neq 0$. Therefore, $\pi_1 < \pi_2$, which implies $\widetilde{\pi} \prec \widetilde{\pi}'$.

(b) By definition of $\widetilde{\Delta}$ in (5.2), it follows that $\widetilde{\Delta}(\widetilde{\pi}', \widetilde{\pi})$ is of degree -1, for all $\widetilde{\pi}, \widetilde{\pi}' \in \widetilde{P}$. By Theorem 5.1, one has that for each interval $\widetilde{J} \in (\widetilde{P}, \prec)$:

$$\widetilde{\Delta}(\widetilde{J})^2 = (R_J^{-1} \circ \Delta(J) \circ R_J)(R_J^{-1} \circ \Delta(J) \circ R_J) = (R_J^{-1} \circ \Delta(J)^2 \circ R_J),$$

which is zero, since $\Delta(J)$ is a boundary map. In particular, $\widetilde{\Delta}(\widetilde{P})^2 = 0$. ■

We are now able to prove our main theorem.

Theorem 5.3. *The map $\widetilde{\Delta} : C\widetilde{P} \to C\widetilde{P}$ is a connection matrix of $\mathcal{H}(\prec)$.*

Proof. By Theorem 5.2, $\widetilde{\Delta}$ is a strictly upper triangular boundary map. To show that this map is a connection matrix of $\mathcal{H}(\prec)$, one needs to guarantee that the graded module braid $\mathcal{H}\widetilde{\Delta}$ is isomorphic to the homology index braid $\mathcal{H}(\prec)$.

As Δ is a connection matrix, then the graded module braid $\mathcal{H}\Delta$ is isomorphic to the homology index braid $\mathcal{H}(<)$. Moreover, $CH(\widetilde{M}_{\widetilde{J}}) \cong CH(M_J)$ for each $\widetilde{J} \in \mathcal{I}_2(\widetilde{P}, \prec)$. Hence, to prove this theorem, it is sufficient to show that $\mathcal{H}\widetilde{\Delta}$ is isomorphic to $\mathcal{H}\Delta$, i.e., that there exists a collection of isomorphisms $\Psi(\widetilde{J}) : H\widetilde{\Delta}(\widetilde{J}) \to H\Delta(J)$, $\widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)$, such that, for each $(\widetilde{J}, \widetilde{K}) \in \mathcal{I}_2(\widetilde{P}, \prec)$ the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "\widetilde{\Delta}_*(\widetilde{K},\widetilde{J})"] & H\widetilde{\Delta}(\widetilde{J}) \arrow[r, "\widetilde{i}_*"] \arrow[d, "\Psi(\widetilde{J})"] & H\widetilde{\Delta}(\widetilde{J}\widetilde{K}) \arrow[r, "\widetilde{p}_*"] \arrow[d, "\Psi(\widetilde{J}\widetilde{K})"] & H\widetilde{\Delta}(\widetilde{K}) \arrow[r, "\widetilde{\Delta}_*(\widetilde{K},\widetilde{J})"] \arrow[d, "\Psi(\widetilde{K})"] & H\widetilde{\Delta}(\widetilde{J}) \arrow[r, "\widetilde{i}_*"] \arrow[d, "\Psi(\widetilde{J})"] & \cdots \\
\cdots \arrow[r, "\Delta_*(K,J)"] & H\Delta(J) \arrow[r, "i_*"] & H\Delta(JK) \arrow[r, "p_*"] & H\Delta(K) \arrow[r, "\Delta_*(K,J)"] & H\Delta(J) \arrow[r, "i_*"] & \cdots
\end{tikzcd} \tag{5.4}$$

We will start by showing that, for each $\widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)$, $H\widetilde{\Delta}(\widetilde{J})$ and $H\Delta(J)$ are isomorphic by constructing an isomorphism $\Psi(\widetilde{J})$. Afterwards, we will use this collection of isomorphisms to prove that the diagram in (5.4) is commutative.

For each $\widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)$, let $\Psi(\widetilde{J}) : H\widetilde{\Delta}(\widetilde{J}) \to H\Delta(J)$ be the map induced from the collection $\{F_\pi\}_{\pi\in\widetilde{P}}$ on the quotient modules $H\widetilde{\Delta}(\widetilde{J})$ and $H\Delta(J)$. More specifically, considering a decomposition of $\widetilde{J}$ as $(B_0, I_1, B_1, \ldots, I_p, B_p)$ (as in the proof of Theorem 5.2), define

$$\begin{array}{rcrcl} \Psi(\widetilde{J}) & : & H\widetilde{\Delta}(\widetilde{J}) = \dfrac{\text{Ker } \widetilde{\Delta}(\widetilde{J})}{\text{Im } \widetilde{\Delta}(\widetilde{J})} & \longrightarrow & H\Delta(J) = \dfrac{\text{Ker } \Delta(J)}{\text{Im } \Delta(J)} \\ & & [a] & \longmapsto & [R_J(a)] \end{array}$$

where, $R_J : C\widetilde{\Delta}(\widetilde{J}) \to C\Delta(J)$ is the isomorphism defined in (5.3). Observe that, if $\widetilde{J}$ does not contain elements of $\widetilde{P}_2$, then $\Psi(\widetilde{J})$ is the identity map. The following two claims show that $\Psi(\widetilde{J})$ is well defined and is an isomorphism of modules.

Claim 1. *The map $\Psi(\widetilde{J})$ is well defined.*

We need to show that if $[a] \in H\widetilde{\Delta}(\widetilde{J})$ then $[R_J(a)] \in H\Delta(J)$, and that $\Psi(\widetilde{J})$ does not depend on the particular choice of representatives. Firstly, one has that

$$\begin{array}{rcl} [a] \in H\widetilde{\Delta}(\widetilde{J}) & \Rightarrow & a \in \text{Ker } \widetilde{\Delta}(\widetilde{J}) \\ & \Rightarrow & R_J(a) \in \text{Ker } \Delta(J), \text{ since } \widetilde{\Delta}(\widetilde{J}) = R_J^{-1} \circ \Delta(J) \circ R_J \\ & \Rightarrow & [R_J(a)] \in H\Delta(J). \end{array}$$

This proves that if $[a] \in H\widetilde{\Delta}(\widetilde{J})$ then $[R_J(a)] \in H\Delta(J)$. On the other hand,

$$\begin{array}{rcl} [a] = [b] \in H\widetilde{\Delta}(\widetilde{J}) & \Rightarrow & [a-b] = 0 \\ & \Rightarrow & a - b \in \text{Im } \widetilde{\Delta}(\widetilde{J}) \\ & \Rightarrow & R_J(a-b) \in \text{Im } \Delta(J), \text{ since } \widetilde{\Delta}(\widetilde{J}) = R_J^{-1} \circ \Delta(J) \circ R_J \\ & \Rightarrow & R_J(a) - R_J(b) \in \text{Im } \Delta(J) \\ & \Rightarrow & [R_J(a)] = [R_J(b)]. \end{array}$$

Hence, $\Psi(\widetilde{J})$ does not depend on the particular choice of representatives. $\triangleleft$

Claim 2. *The map $\Psi(\widetilde{J})$ is an isomorphism of modules.*

It is not difficult to see that $\Psi(\widetilde{J})$ is a homomorphism of modules. We will prove that it is bijective. Observe that $\Psi(\widetilde{J})$ is injective, since

$$\begin{aligned}
\Psi(\widetilde{J})[a] &= \Psi(\widetilde{J})[b] \in H\Delta(J) \\
&\Rightarrow [R_J(a-b)] = 0, \text{ i.e, } R_J(a-b) \in \text{Im } \Delta(J) \\
&\Rightarrow a-b \in \text{Im } R_J^{-1} \circ \Delta(J) \\
&\Rightarrow a-b \in \text{Im } \widetilde{\Delta}(\widetilde{J}), \text{ since } \widetilde{\Delta}(\widetilde{J}) \circ R_J^{-1} = R_J^{-1} \circ \Delta(J) \\
&\Rightarrow [a] = [b].
\end{aligned}$$

Now, given $[b] \in H\Delta(J)$, let $a = R_J^{-1}(b)$. Observe that

$$\widetilde{\Delta}(\widetilde{J})(a) = \widetilde{\Delta}(\widetilde{J}) \circ R_J^{-1}(b) = R_J^{-1} \circ \Delta(J)(b) = 0,$$

which implies $[a] \in H\widetilde{\Delta}(\widetilde{J})$. Moreover, $\Psi(\widetilde{J})[a] = \Psi(\widetilde{J})[R_J^{-1}(b)] = [b]$, proving that $\Psi(\widetilde{J})$ is surjective.

$\lhd$

Therefore, $\Psi(\widetilde{J})$ is an isomorphism between $H\widetilde{\Delta}(\widetilde{J})$ and $H\Delta(J)$, for all intervals $\widetilde{J}$ in $(\widetilde{P}, \prec)$. Now, considering the family $\{\Psi(\widetilde{J}) : \widetilde{J} \in \mathcal{I}(\widetilde{P}, \prec)\}$, we will show that the diagram in (5.4) is commutative, for all pair of adjacent intervals $(\widetilde{J}, \widetilde{K}) \in (\widetilde{P}, \prec)$. Indeed,

- $\Psi(\widetilde{J}) \circ \widetilde{\Delta}_*(\widetilde{K}, \widetilde{J}) = \Delta_*(K, J) \circ \Psi(\widetilde{K})$:

By Theorem 5.1, $\widetilde{\Delta}(\widetilde{K}, \widetilde{J}) = R_J^{-1} \circ \Delta(K, J) \circ R_K$. Therefore, given $[a] \in H\widetilde{\Delta}(\widetilde{K})$, one has

$$\begin{aligned}
\Psi(\widetilde{J}) \circ \widetilde{\Delta}_*(\widetilde{K}, \widetilde{J})[a] &= \Psi(\widetilde{J})[\widetilde{\Delta}(\widetilde{K}, \widetilde{J})a] = [R_J \circ \widetilde{\Delta}(\widetilde{K}, \widetilde{J})a] \\
&= [\Delta(K, J) \circ R_K a] = \Delta_*(K, J)[R_K a] \\
&= \Delta_*(K, J) \circ \Psi(\widetilde{K})[a].
\end{aligned}$$

- $\Psi(\widetilde{J}\widetilde{K}) \circ \widetilde{i}_* = i_* \circ \Psi(\widetilde{J})$:

The isomorphism R_{JK} restricted to the first component of $C\widetilde{\Delta}(\widetilde{J}\widetilde{K}) = C\widetilde{\Delta}(\widetilde{J}) \oplus C\widetilde{\Delta}(\widetilde{K})$ behaves as the isomorphism R_J, i.e., $R_{JK}|_{C\widetilde{\Delta}(\widetilde{J})} = R_J$. Thus, $R_{JK} \circ \widetilde{i} = i \circ R_J$. Given $[a] \in H\widetilde{\Delta}(\widetilde{J})$, one has

$$\begin{aligned}
\Psi(\widetilde{J}\widetilde{K}) \circ \widetilde{i}_*[a] &= \Psi(\widetilde{J}\widetilde{K})[\widetilde{i}(a)] = [R_{JK} \circ \widetilde{i}(a)] \\
&= [i \circ R_J(a)] = i_*[R_J(a)] \\
&= i_* \circ \Psi(\widetilde{J})[a].
\end{aligned}$$

- $\Psi(\widetilde{K}) \circ \widetilde{p}_* = p_* \circ \Psi(\widetilde{J}\widetilde{K})$:

Similarly, $R_{JK}|_{C\widetilde{\Delta}(\widetilde{K})} = R_K$ and $R_K \circ \widetilde{p} = p \circ R_{JK}$. Given $[a] \in H\widetilde{\Delta}(\widetilde{J}\widetilde{K})$, one has

$$\begin{aligned} \Psi(\widetilde{K}) \circ \widetilde{p}_*[a] &= \Psi(\widetilde{K})[\widetilde{p}(a)] \;=\; [R_K \circ \widetilde{p}(a)] \\ &= [p \circ R_{JK}(a)] \;=\; p_*[R_{JK}(a)] \\ &= p_* \circ \Psi(\widetilde{J}\widetilde{K})[a]. \end{aligned}$$

Hence, we have shown that the diagram in (5.4) is commutative and, by our initial considerations, this suffice to prove that the graded module braid $\mathcal{H}\widetilde{\Delta}$ is isomorphic to the homology index braid $\mathcal{H}(\prec)$. This in turn, proves that the map $\widetilde{\Delta} : C\widetilde{P} \to C\widetilde{P}$ is a connection matrix of $\mathcal{H}(\prec)$. ■

5.2 Application to Morse-Bott Flows

In this section, our goal is to obtain an explicit connection matrix of the finest Morse decomposition of a Morse-Bott flow in M. In order to do this, we will make use of the results in the previous section as well as a specific perturbation of f described below.

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a smooth closed manifold M of finite dimension n. In [2], a perturbation technique of Morse-Bott functions to Morse-Smale functions is presented. The perturbation defined therein produces an explicit Morse-Smale function $h : M \to \mathbb{R}$ which is arbitrarily close to a given Morse-Bott function f, such that, $h = f$ outside of a neighborhood of the critical set of f. More specifically, if f has l disjoint connected critical manifolds, namely $S_1, \ldots, S_l$, then h is given by the expression

$$h = f + \epsilon \left(\sum_{j=1}^{l} \rho_j f_j \right),$$

where f_j is a Morse-Smale function on a tubular neighborhood T_j of the critical manifold S_j and ρ_j is a bump function which is identically 1 near S_j and identically zero outside T_j, for each $j = 1, \ldots, l$. The critical points of h are exactly the union of the critical points of f_j, for all $j = 1, \ldots, l$. Moreover, if p is a critical point of f_j of index λ_p^j, then p is a critical point of h of index $\lambda_p^h = \lambda_j + \lambda_p^j$, where λ_j denotes the Morse-Bott index of the critical manifold S_j. For more details see [2].

Consider a flow φ_f in M generated by f, that is, a flow associated to the vector field $-\nabla f$. Throughout this chapter, flows generated by a Morse-Bott function f will be call Morse-Bott flows. Let h be a Morse-Smale perturbation of f, as described above, and denote by φ_h a Morse-Smale flow associated to the vector field $-\nabla h$.

The next result provides a relation between the connection matrices for a Morse-Bott flow φ_f and the connection matrices for a perturbed Morse-Smale flow φ_h of φ_f. More specifically, we prove that the set of connection matrices of a Morse decomposition of M relative to φ_f is equal to the set of connection matrices of the induced coarser Morse decomposition of M relative to φ_h .

Theorem 5.4. *Let f be a Morse-Bott function on M and h a Morse-Smale perturbation of f. If $\mathcal{D}(M)$ is a $\prec$-ordered Morse decomposition of M relative to the flow φ_f, then $\mathcal{D}(M)$ is also a $\prec$-ordered Morse decomposition of M relative to the flow φ_h and the sets of connection matrices relative to the both flows are equal, i.e.,*

$$\mathcal{CM}(\prec;\varphi_f) = \mathcal{CM}(\prec;\varphi_h).$$

Proof. Let $\{S_1,\ldots,S_l\}$ be the critical manifolds of f and N_j small isolating neighborhoods of S_j, for each $j=1,\ldots,l$. Without loss of generality, one can consider h as a perturbation of f such that $f=h$ in $M\setminus N$, where $N=\cup_{j=1}^{l} N_j$. Then φ_f and φ_h coincide in $M\setminus N$, since $\nabla f=\nabla h$ in $M\setminus N$.

Let $\mathcal{D}(M;\varphi_f)$ be a ($\prec$-ordered) Morse decomposition of M relative to φ_f. Since φ_f and φ_h coincide outside N, this set is also a Morse decomposition of M relative to the perturbed flow φ_h, which will be denoted by $\widetilde{\mathcal{D}}(M;\varphi_h)$. Moreover, an admissible ordering of $\mathcal{D}(M;\varphi_f)$ is also an admissible ordering of $\widetilde{\mathcal{D}}(M;\varphi_h)$ and conversely; flow orderings of both Morse decompositions coincide. Therefore, for each interval J of $\prec$, the Conley index of M_J as Morse set of $\mathcal{D}(M;\varphi_f)$ is equal to the Conley index of M_J as Morse set of $\widetilde{\mathcal{D}}(M;\varphi_h)$. Furthermore, the homology index braid of the admissible ordering $\prec$ of $\mathcal{D}(M;\varphi_f)$ coincides with the homology index braid of the admissible ordering $\prec$ of $\widetilde{\mathcal{D}}(M;\varphi_h)$. Hence, the collection $\mathcal{CM}(\prec;\varphi_f)$ of connection matrices of the admissible ordering $\prec$ of $\mathcal{D}(M;\varphi_f)$ is equal to the collection $\mathcal{CM}(\prec;\varphi_h)$ of connection matrices of the admissible ordering $\prec$ of $\widetilde{\mathcal{D}}(M;\varphi_h)$. ■

Note that the set $\{S_1,\ldots,S_l\}$ of all critical manifolds of f is a Morse decomposition of M with respect to the flow φ_f. Denote this Morse decomposition, which is the finest one, by $\mathcal{D}(M;\varphi_f)$. On the other hand, if h is a Morse-Smale perturbation of f, denote the finest

Morse decomposition of M relative to φ_h by $\mathcal{D}(M;\varphi_h)$. The next theorem provides a relation between the connections matrices of both Morse decompositions. More specifically, we prove that each connection matrix of $\mathcal{D}(M;\varphi_h)$ induces a connection matrix of $\mathcal{D}(M;\varphi_f)$.

For the next result, the homology is computed over $\mathbb{Z}$ or over a field.

Theorem 5.5. *Let f be a Morse-Bott function on M and h be a Morse-Smale perturbation of f such that h restricted to each critical manifold of f is a perfect Morse function. Then a connection matrix of $\mathcal{D}(M;\varphi_h)$ induces a connection matrix of $\mathcal{D}(M;\varphi_f)$.*

Proof. By Theorem 5.4, the Morse decomposition $\mathcal{D}(M;\varphi_f) = \{S_1, \ldots, S_l\}$ is also a Morse decomposition of M relative to φ_h and it will be denoted by $\widetilde{\mathcal{D}}(M;\varphi_h)$. Both Morse decompositions are $\prec_F$-ordered, where $\prec_F$ is the admissible flow ordering. Observe that $\mathcal{D}(M;\varphi_f)$ is the finest Morse decomposition of M in φ_f, but it is not the case of $\widetilde{\mathcal{D}}(M;\varphi_h)$. Denote by $\mathcal{D}(M;\varphi_h)$ the finest Morse decomposition of M relative to φ_h, that is, each critical point of h corresponds to a Morse set of $\mathcal{D}(M;\varphi_h)$.

Now, using the main result of Section 5.1, we will induce a connection matrix of $\widetilde{\mathcal{D}}(M;\varphi_h)$ from $\mathcal{D}(M;\varphi_h)$. Since, by Theorem 5.4, $\mathcal{CM}(\prec_F;\varphi_f) = \mathcal{CM}(\prec_F;\varphi_h)$, we will obtain the required connection matrix of $\mathcal{D}(M;\varphi_f)$.

Observe that the finest Morse decomposition $\mathcal{D}(M;\varphi_h) = \{M_\pi\}_{\pi\in P}$ of M is $<_F$-ordered, where $<_F$ denotes the flow ordering. Considering the intervals $I_1, \ldots, I_l$, where $I_j = \{\pi \in P : M_\pi \in S_j\}$, then $\widetilde{\mathcal{D}}(M;\varphi_h)$ is obtained from $\mathcal{D}(M;\varphi_h)$ by defining $\widetilde{M}_{\{\pi\}} = M_\pi$, if $\pi \notin I_1, \ldots, I_l$ and $\widetilde{M}_{I_j} = M_{I_j}$, for $j = 1, \ldots, l$. Given a connection matrix Δ of $\mathcal{D}(M;\varphi_h)$, observe that each submatrix $\Delta(I_j)$, for $j = 1, \ldots, l$, corresponds to a connection matrix of the finest Morse decomposition of S_j under the flow restricted to S_j, since each non null map in $\Delta(I_j)$ is flow defined. By hypothesis, $h|_{S_j}$ is a perfect Morse function. Hence, by Proposition 5.1, one has that $\Delta(I_j) = 0$, for all j. Therefore, we are able to apply Theorem 5.3 which provides a connection matrix $\widetilde{\Delta}$ of $\widetilde{\mathcal{D}}(M;\varphi_h)$. ■

Example 5.1. Let f be a Morse-Bott function on S^2 having three isolated critical points of indices 2, namely x_1, x_2 and x_3, one isolated critical point of index 1, namely, y and $B = S^1$ as critical manifold of index 0, as depicted in Figure 5.1. Let h be a perturbation of f when one considers a perfect Morse function on the critical manifold S^1, as in Figure 5.2. Denote by φ_f (resp., φ_h) a Morse-Bott flow (resp., Morse flow) on S^2 associated to the vector field $-\nabla f$ (resp., $-\nabla h$).

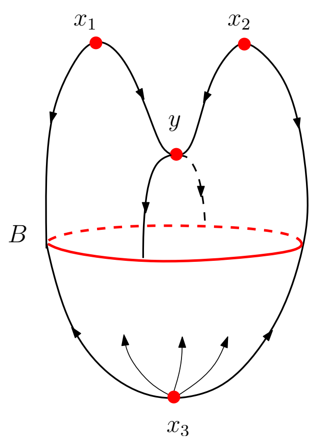


Figure 5.1: A Morse-Bott flow on S^2 associated to the function f.

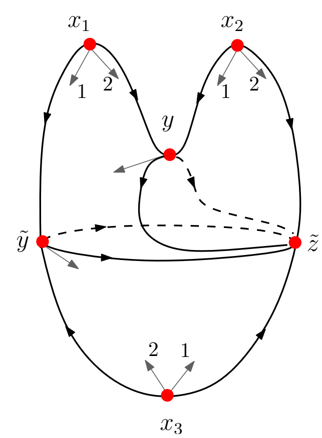


Figure 5.2: A Morse-Smale perturbation of the Morse-Bott function f.

The objective herein is to obtain a connection matrix of the finest Morse decomposition of S^2 with respect to the Morse-Bott flow φ_f by means of a connection matrix of the finest Morse decomposition of S^2 with respect to the Morse flow φ_h. Firstly, we will compute a connection matrix of a Morse decomposition of S^2 with respect to φ_h. Then, using the tools proved in Section 5.1, we will obtain the required matrix.

Consider the set $P = \{1, 2, 3, 4, 5, 6\}$ with partial order $<$ given by $[1 < 2, 3]$; $[2 < 4, 5]$; and $[1 < 4, 6]$. Let $M_1 = \tilde{z}$, $M_2 = \tilde{y}$, $M_3 = y$, $M_4 = x_1$, $M_5 = x_2$ and $M_6 = x_3$. The set $\mathcal{D}(S^2; \varphi_h) = \{M_i : i \in P\}$ is the finest $<$-ordered Morse decomposition of S^2 with respect to the flow φ_h. Moreover, $<$ is the flow ordering. As proved by Salamon in [39], the differential of a Morse-Witten complex of h is a connection matrix of the Morse decomposition $\mathcal{D}(S^2; \varphi_h)$. In order to obtain a connection matrix of $\mathcal{D}(S^2; \varphi_h)$, we will compute the Morse-Witten complex $(C_*(h), \partial_*)$ of h, considering the orientations on the unstable manifolds of the critical points as the ones illustrated in Figure 5.2. For this choice of orientations, we have that the Morse chain groups are $C_0(f) = \mathbb{Z}\langle \tilde{z} \rangle$, $C_1(f) = \mathbb{Z}\langle \tilde{y} \rangle \oplus \mathbb{Z}\langle y \rangle$ and $C_2(f) = \mathbb{Z}\langle x_1 \rangle \oplus \mathbb{Z}\langle x_2 \rangle \oplus \mathbb{Z}\langle x_3 \rangle$, where $\langle x \rangle$ denotes both the critical point x as well as its orientation. The differential ∂_*^c is defined on the generators according to the matrix:

$$\partial_* \;=\; \begin{array}{c} \\ \tilde{z} \\ \tilde{y} \\ y \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cccccc} \tilde{z} & \tilde{y} & y & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array},$$

which is a connection matrix of $\mathcal{D}(S^2, \varphi_h)$ by identifying the number ± 1 with an isomorphism between the Conley homology indices in question.

Now, note that $I = \{1,2\}$ is an interval in $(P,<)$ and, defining $\widetilde{P} = \{I, \{3\}, \{4\}, \{5\}, \{6\}\}$ and the order $[I \prec \{3\} \prec \{4\}, \{5\}]$; $[I \prec \{6\}]$, then $(\widetilde{P}, \prec)$ is a partially ordered set, by Proposition 5.2. Let $\widetilde{\mathcal{D}}(S^2; \varphi_h) = \{\widetilde{M_{\widetilde{\pi}}} : \widetilde{\pi} \in \widetilde{P}\}$, where $\widetilde{M_I} = \tilde{y} \cup C(M_2, M_1) \cup \tilde{z}$ and $\widetilde{M_{\{i\}}} = M_i$, for $i \in \{3,4,5,6\}$. By Proposition 5.4, this set is an $\prec$-ordered Morse decomposition of S^2 with respect to the flow φ_h. Moreover, by Proposition 5.5, $\prec$ is the flow ordering. Finally, by Theorem 5.3, Δ induces a connection matrix $\widetilde{\Delta}$ of $\widetilde{\mathcal{D}}(S^2, \varphi_h)$, which is given by the following map of degree -1 from $CH(B) \oplus CH(y) \oplus CH(x_1) \oplus CH(x_2) \oplus CH(x_3)$ to itself:

$$\widetilde{\Delta} \;=\; \begin{array}{c} \\ B \\ y \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} B & y & x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccccc} 0 & 0 & \approx & 0 & \approx \\ 0 & 0 & \approx & \approx & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}. \tag{5.5}$$

Consider the $\prec$-ordered Morse decomposition $\widetilde{\mathcal{D}}(S^2; \varphi_f) = \{M_{\widetilde{\pi}} : \widetilde{\pi} \in \widetilde{P}\}$ relative to the Morse-Bott flow φ_f, where $M_I = B$, $M_{\{3\}} = y$, $M_{\{4\}} = x_1$, $M_{\{5\}} = x_2$ and $M_{\{6\}} = x_3$. The partial order $\prec$ is the flow ordering. By Theorem 5.5, the map in (5.5) is a connection matrix of $\mathcal{D}(S^2; \varphi_f)$. ◀

In the previous example, one could have chosen a different perturbation $\tilde{h}$ of f. For instance, using the perturbation shown in Figure 5.3, one obtains the following map from

$CH(B) \oplus CH(y) \oplus CH(x_1) \oplus CH(x_2) \oplus CH(x_3)$ to itself:

$$\Delta = \begin{array}{c} \\ B \\ y \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccc} B & y & x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \approx & \approx \\ 0 & 0 & \approx & \approx & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

which is a connection matrix of the finest Morse decomposition of S^2 relative to the flow φ_f. Therefore, we do not have the uniqueness of connection matrices in Morse-Bott flows, even if the homology were computed over a field.

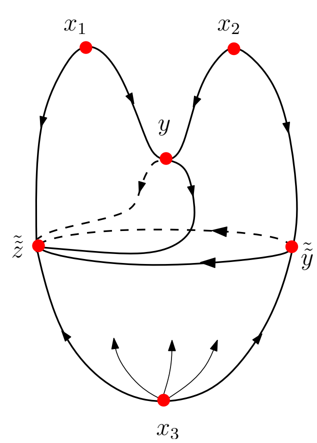


Figure 5.3: A Morse perturbation of the Morse-Bott function f.

At this point, a natural question arises: are all connection matrices of the finest Morse decomposition of a Morse-Bott flow φ_f obtained via Theorem 5.5? In other words, for each connection matrix Δ of the finest Morse decomposition of a Morse-Bott flow is there a Morse-Smale perturbation φ_h of φ_f such that Δ is induced from a connection matrix of φ_h?

In general, this is not the case as seen in Example 5.3. On the other hand, if some additional structure is assumed it may hold true, as can be verified in Examples 5.1 and 5.2.

Example 5.2. Consider S^3 as the manifold obtained from gluing two solid tori $\mathcal{T}_1$ and $\mathcal{T}_2$ by a homeomorphism of their boundaries, the tori T_1 and T_2, which identifies a parallel of T_1 to the meridian of T_2. Let f be a Morse-Bott function on S^3 such that the critical manifolds of f are the torus T_1 as a repeller and two 1-spheres S_1 and S_2 as attractors, where S_i lies in the interior of $\mathcal{T}_i$, $i = 1, 2$. Their Morse-Bott indices are $\lambda_{T_1} = 1, \lambda_{S_1} = 0$ and $\lambda_{S_2} = 0$.

Considering the finest Morse decomposition $\mathcal{D}(S^3) = \{S_1, S_2, T_1\}$ of S^3 with the admissible flow ordering and a connection matrix $\widetilde{\Delta}$ of $\mathcal{D}(S^3)$, then the only possible non null maps are $\widetilde{\Delta}(T_1, S_1)$ and $\widetilde{\Delta}(T_1, S_2)$, which are flow defined. Hence, there is a unique connection matrix of $\mathcal{D}(S^3)$, namely,

$$\widetilde{\Delta} = \begin{array}{cc} & \begin{array}{ccc} S_1 & S_2 & T_1 \end{array} \\ \begin{array}{c} S_1 \\ S_2 \\ T_1 \end{array} & \begin{bmatrix} 0 & 0 & \widetilde{\Delta}_1 \\ 0 & 0 & \widetilde{\Delta}_2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} .$$

We will show that this map can be obtained from a connection matrix of a perturbation of the function f, as discussed in Section 5.2. In this sense, let h be a Morse-Smale perturbation of f, such that it is a perfect Morse function when restricted to the the critical manifolds, S_1, S_2, T_1, of f. Denote the critical points of h by $z_1, y_1 \in S_1$, $z_2, y_2 \in S_2$ and $x, v_1, v_2, y_3 \in T_1$, where $z_i's$ have indices zero, $y's$ have indices one, $v's$ have indices two and x has index three. The differential of the Morse complex $(C_*(f), \partial_*^c)$ over $\mathbb{Z}_2$ associated to h is given by the matrix:

$$\partial_*^c = \begin{array}{cc} & \begin{array}{cccccccc} z_1 & z_2 & y_1 & y_2 & y_3 & v_1 & v_2 & x \end{array} \\ \begin{array}{c} z_1 \\ z_2 \\ y_1 \\ y_2 \\ y_3 \\ w_1 \\ w_2 \\ x \end{array} & \begin{bmatrix} 0 & & & & 1 & & & \\ & 0 & & & 1 & & & \\ & & 0 & & & 1 & 1 & \\ & & & 0 & & 1 & 1 & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix} \end{array} .$$

Note that the Conley homological indices of the critical manifolds are as follows:

$$CH_n(T_1) = \begin{cases} \mathbb{Z}_2, & n = 1, 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & n = 2 \\ 0, & n \neq 1, 2, 3 \end{cases} \qquad CH_n(S_i) = \begin{cases} \mathbb{Z}, & n = 0, 1 \\ 0, & n \neq 0, 1 \end{cases} ,$$

where $i = 1, 2$. Hence, there exist isomorphisms $CH(S_i) \cong CH(z_i) \oplus CH(y_i)$, for $i = 1, 2$, and $CH(T_1) \cong CH(y_3) \oplus CH(v_1) \oplus CH(v_2) \oplus CH(x)$, which will be denoted by F_i and F_{T_1}, respectively. Now, observe that the submatrix of ∂_*^c

$$\Delta_i \;=\; \begin{array}{c} \\ z_i \\ y_i \end{array}\!\!\begin{array}{c} \begin{array}{cccc} y_3 & v_1 & v_2 & x \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

induces a map $\widetilde{\Delta}_i : CH(T_1) \to CH(S_i)$, for $i = 1, 2$, by composing Δ_i with the isomorphisms F_{T_1} and F_i^{-1}. Therefore, the induced connection matrix of the finest Morse decomposition of S^3 relative to the Morse-Bott flow φ_f is

$$\widetilde{\Delta} = \begin{array}{c} \\ S_1 \\ S_2 \\ T_1 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} S_1 & S_2 & T_1 \end{array} \\ \left[\begin{array}{ccc} 0 & 0 & \widetilde{\Delta}_1 \\ 0 & 0 & \widetilde{\Delta}_2 \\ 0 & 0 & 0 \end{array}\right] \end{array} .$$

◀

The following is an adaptation to our context of Reineck's example in [37] and illustrates that there are connection matrices that do not arise from Morse-Smale perturbation.

Example 5.3. Consider a flow in $\mathbb{R}^2$ as in Figure 5.4 having seven singularities and one critical manifold diffeomorphic to S^1. By taking the one point compactification $\mathbb{R}^2 \cup \{\infty\}$ of $\mathbb{R}^2$ and letting ∞ be an attractor point, one obtains a Morse-Bott flow on $M = S^2$.

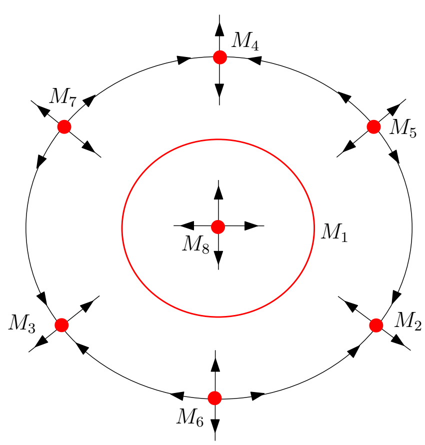


Figure 5.4: Morse-Bott flow on $\mathbb{R}^2$.

The critical manifold together with the eight isolated singularities form a Morse decomposition $\mathcal{D}(S^2)$ of M, with flow ordering $[0 < 2, 3, 4]$; $[1 < 2, 3, 4, 8]$; $[2 < 5, 6]$; $[3 < 6, 7]$ and

$[4 < 5, 7]$, in $P = \{1, \ldots, 8\}$. Let Δ be a connection matrix of $\mathcal{D}(S^2)$. All maps $\Delta(\pi', \pi)$ are flow defined, except for $\Delta(5,1)$, $\Delta(6,1)$ and $\Delta(7,1)$. It is easy to compute the boundary flow defined maps using index triples. In this example they are all isomorphism. The connection matrix Δ is as follows:

$$\Delta = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & & 1 & 1 & 1 & & & & \\ & 0 & 1 & 1 & 1 & a & b & c & 1 \\ & & 0 & & & 1 & & 1 & \\ & & & 0 & & 1 & 1 & 0 & \\ & & & & 0 & & 1 & 1 & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix} \end{array} .$$

The homology with $\mathbb{Z}_2$-coefficients was used in order to simplify computations, however one could as well have used $\mathbb{Z}$-coefficients.

The maps a, b and c in Δ are not flow defined. As Δ is a connection matrix, then

$$\frac{\text{Ker } \Delta(P)}{\text{Im } \Delta(P)} = H\Delta(P) \cong CH(M) = H_*(M; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & n = 2, 0 \\ 0, & c.c. \end{cases},$$

which implies that:

- $\dfrac{\text{Ker } \Delta_0}{\text{Im } \Delta_1} \cong \mathbb{Z}_2$, which implies that the rank of Δ_1 must be 1;
- $\dfrac{\text{Ker } \Delta_1}{\text{Im } \Delta_2} \cong 0$, which implies that the rank of Δ_2 must be equal to 3;
- $\dfrac{\text{Ker } \Delta_2}{\text{Im } \Delta_3} \cong \mathbb{Z}_2$, which implies that the rank of the kernel of Δ_2 must be equal to 1.

By the last item, we must have $a + b + c \neq 0 \pmod 2$. In other orders, either one of these entries is one or all of these entries are one. Therefore, combining possibilities, one has four connection matrices of $\mathcal{D}(M)$. Three of these connection matrices are obtained from connection matrices of a Morse-Smale perturbation, namely the ones where only one of the entries a, b, c is one. On the other hand, the connection matrix where $a = b = c = 1$ can not be obtained from a Morse flow by substituting the critical manifolds for two singularities, since the saddle has only two orbits in its stable manifold. ◀

Therefore not all connection matrices for Morse-Bott flows arise from connection matrices for Morse-Smale perturbations.

5.3 Spectral Sequences for Morse-Bott Complexes

Given a Morse-Bott function $f : M \to \mathbb{R}$ on an orientable closed manifold, there are many ways to define a Morse-Bott complex and prove that its homology coincides with the singular homology of M with integers coefficients. We briefly describe three of these approaches below. The Austin-Braam approach in [1] uses differential forms to construct a comulticomplex which computes the de Rham cohomology of the manifold with real coefficients. The approach in [2] is to consider a Morse-Smale perturbation of f and use the Morse chain complex of the perturbed function. Banyaga and Hurtubise developed in [3] the Morse-Bott multicomplex by using singular cubical chains and fibered product constructions. The Morse- Bott multicomplex is fundamentally different from other approaches to Morse-Bott homology. It provides a common framework for singular cubical chains and Morse chains, making it possible to interpolate singular cubical chain complexes and Morse chain complexes.

For our purpose herein, we consider the simplest definition of Morse-Bott complex as in [2]. Assume that the Morse-Bott function f satisfies the transversality condition and let $h : M \to \mathbb{R}$ be a Morse-Smale perturbation of f, as in Section 5.2. One can define the Morse-Bott complex associated to f as the Morse complex $(C_*(h), \partial_*^h)$ generated by the perturbation h.

Moreover, if one assumes that the critical manifolds of f admit a perfect Morse function and that h restricted to each critical manifold of f is a perfect Morse function, then the differential ∂^h is a special case of a connection matrix for the finest Morse decomposition of M with respect to the flow φ_h. By Theorem 5.5, the differential of the Morse-Bott complex induces a connection matrix for a finest Morse decomposition[3] of M for the Morse-Bott flow φ_f.

Consider a Morse-Bott function f and its associated Morse-Bott flow φ_f with a finest Morse decomposition. For a perturbation h of f, consider the Morse chain complex $(C_*(h), \partial)$

[3]Each Morse set of the Morse decomposition contains only one critical manifold.

with a coarser filtration $\mathcal{F}$ compatible with the Morse decomposition for M with respect to the Morse-Bott flow φ_f.

In what follows, we consider this Morse chain complex with coarser filtration and show how to obtain from the SSSA the modules and differentials associated to this spectral sequence. Our motivation in studying dynamical spectral sequence for Morse chain complexes with a coarser filtration resides in the fact that this coarser filtration provides the finest filtration for more general flows, in particular, for Morse-Bott flows as discussed above.

Spectral sequence with coarser filtration

Consider a Morse chain complex (C_*, Δ) endowed with a finest filtration F, not necessarily respecting the order of increasing Morse index. By Remark 1.1, one can apply the SSSA for Δ. In what follows, the formulas for the change of basis in (1.4) and for the $\mathbb{Z}$-modules Z_p^r in (1.5) are rewritten for arbitrary filtration which do not necessary respect the order given by the Morse index.

Let $k_1, k_2, \ldots$ be the columns of Δ^r which are associated to k-chains. Denote by $\sigma_k^{k_\ell, r}$ the k-chain represented in the k_ℓ-th column of Δ^r. Hence, the k_j-th column of Δ^{r+1} are

$$\begin{aligned} \sigma_k^{k_j, r+1} &= \underbrace{\sum_{\ell=1}^{j} c_\ell^{k_j, r} h_k^{k_\ell}}_{\sigma_k^{k_j, r}} \pm \underbrace{\sum_{\ell=1}^{t} c_\ell^{k_t, r} h_k^{k_\ell}}_{\sigma_k^{k_t, r}} \\ &= c_1^{k_j, r+1} h_k^{k_1} + c_2^{k_j, r+1} h_k^{k_2} + \cdots + c_{j-1}^{k_j, r+1} h_k^{k_{j-1}} + c_j^{k_j, r+1} h_k^{k_j} \end{aligned} \tag{5.6}$$

where $c^{k_\ell, r} \in \mathbb{Z}$.

In Section 1.2, it is shown how the $\mathbb{Z}$-modules E_p^r are determined by the connection matrix Δ. In order to do this, a formula for the module $Z_{p,k-p}^r$ was established in terms of the $\sigma_k^{r,j}$ determined by the SSSA. Let k_{ℓ_p} be the rightmost h_k column such that $k_{\ell_p} \leq p+1$, i.e. the rightmost h_k column in F_pC. Then $Z_{p,k-p}^r$ is given by

$$\mathbb{Z}[\mu^{k_{\ell_p}, r-p-1+k_{\ell_p}} \sigma_k^{k_{\ell_p}, r-p-1+k_{\ell_p}}, \mu^{k_{\ell_p}-1, r-p-1+k_{\ell_p}-1} \sigma_k^{k_{\ell_p}-1, r-p-1+k_{\ell_p}-1}, \ldots, \mu^{k_1, r-p-1+k_1} \sigma_k^{k_1, r-p-1+k_1}]$$

where $\mu^{j,\zeta} = 0$ whenever the primary pivot of the j-th column is below the $(p-r+1)$-th row and $\mu^{j,\zeta} = 1$ otherwise. Moreover, through the SSSA, Δ induces the differentials d_p^r in the spectral sequence. In fact, whenever E_p^r and E_{p-r}^r are both non-zero, the map $d_p^r : E_p^r \to E_{p-r}^r$ is multiplication by the entry $\Delta_{p-r+1,p+1}^r$ which is either a primary pivot or a zero with a column of zero entries below it. Otherwise d_p^r is zero. In the case of Morse chain complexes on orientable surfaces, the primary pivots are always equal to ± 1 by Theorem 2.1, then the nonzero d^r's are always isomorphisms induced by primary pivots.

Now, consider a coarser filtration $\mathcal{F} = \{\mathcal{F}_p\}$ in the Morse chain complex (C_*, ∂) which can always be obtained from a finest filtration by grouping some F_p's into only one $\mathcal{F}_p$.

Let $(\mathcal{E}_P^R, d_P^R)$ be the spectral sequence for $(C_*(h), \partial)$ endowed with the filtration $\mathcal{F}_P$. One has that

$$\mathcal{E}_P^R = \mathcal{Z}_P^R / (\mathcal{Z}_{P-1}^{R-1} + \partial \mathcal{Z}_{P+R-1}^{R-1}) \quad \text{where,} \quad \mathcal{Z}_P^R = \{c \in \mathcal{F}_P C \mid \partial c \in \mathcal{F}_{P-R} C\}.$$

As in Section 1.2, the modules of the spectral sequence (E_p^r, d_p^r) associated to the finest filtration are determined by the SSSA. Using this fact, one can show that, more generally, the modules $\mathcal{E}_P^R$ of the spectral sequence associated to the filtration $\mathcal{F}$ can also be determined by the SSSA.

In fact, note that the module $\mathcal{Z}_P^R$ consists of chains in $\mathcal{F}_P C$ with boundary in $\mathcal{F}_{P-R} C$. These chains are associated to all the columns of the connection matrix Δ to the left of and including the column ℓ_P, where ℓ_P denote the rightmost column of Δ associated to a chain in $\mathcal{F}_P C$. Furthermore, since the boundary of the chains must be in $\mathcal{F}_{P-R} C$ we must consider columns or their linear combinations that have the property that the entries in rows $i > \ell_{P-R}$ are all zeroes. Hence, in terms of the finest filtration $\{F_p\}$, one has

$$\mathcal{Z}_P^R = \{c \in F_{\ell_P - 1} C \mid \partial c \in F_{\ell_{P-R} - 1} C\} = Z_{\ell_P - 1}^{\ell_P - \ell_{P-R}}. \tag{5.7}$$

Analogously,

$$\mathcal{Z}_{P-1}^{R-1} = \{c \in F_{\ell_{P-1} - 1} C \mid \partial c \in F_{\ell_{P-R} - 1} C\} = Z_{\ell_{P-1} - 1}^{\ell_{P-1} - \ell_{P-R}}$$

and

$$\partial \mathcal{Z}_{P+R-1}^{R-1} = \partial \{c \in F_{\ell_{P+R-1} - 1} C \mid \partial c \in F_{\ell_P - 1} C\} = \partial Z_{\ell_{P+R-1} - 1}^{\ell_{P+R-1} - \ell_P}.$$

Recall that $\{k_i\}$ are the columns of Δ associated to k-chains, let k_{ℓ_P} be the rightmost column of Δ associated to a k-chain in $\mathcal{F}_P$, i.e. k_{ℓ_P} is the rightmost h_k column such that $k_{\ell_P} \leq \ell_P$. Using the formula for $Z_{p,k-p}^r$ and (5.7), the module $\mathcal{Z}_{P,k-P}^R$ can be described in terms of the basis determined by the connection matrices in the SSSA as follows:

$$\mathbb{Z}[\mu^{k_{\ell_P}, k_{\ell_P} - \ell_{P-R}} \sigma_k^{k_{\ell_P}, k_{\ell_P} - \ell_{P-R}}, \mu^{k_{\ell_P - 1}, k_{\ell_P - 1} - \ell_{P-R}} \sigma_k^{k_{\ell_P - 1}, k_{\ell_P - 1} - \ell_{P-R}}, \ldots, \mu^{k_1, k_1 - \ell_{P-R}} \sigma_k^{k_1, k_1 - \ell_{P-R}}].$$

Analogously, one can describe the modules $\mathcal{Z}_{P-1}^{R-1}$ and $\partial \mathcal{Z}_{P+R-1}^{R-1}$ in terms of σ's, providing a way to recover from the SSSA the module $\mathcal{E}^R_{P,k-P}$.

Denote by k_{f_P} the leftmost h_h column such that $\ell_{P-1} < k_{f_P}$. Now, consider the following matrix

$$\begin{pmatrix} \Delta^{k_{f_P}-\ell_{P-R}}_{(k-1)_{f_{P-R}},k_{f_P}} & \cdots & \Delta^{k_{\ell_P}-\ell_{P-R}}_{(k-1)_{f_{P-R}},k_{\ell_P}} \\ \vdots & & \vdots \\ \Delta^{k_{f_P}-\ell_{P-R}}_{(k-1)_{\ell_{P-R}},k_{f_P}} & \cdots & \Delta^{k_{\ell_P}-\ell_{P-R}}_{(k-1)_{\ell_{P-R}},k_{\ell_P}} \end{pmatrix}.$$

Note that each column of this matrix is taken from a different Δ^r, and the columns correspond to h_k starting with the k_{f_P} column and ending with the k_{ℓ_P} column.

Given a h_k column $k_i \in \{k_{f_P}, \cdots, k_{\ell_P}\}$ of $\Delta^{k_i-\ell_{P-R}}$, there are two possibilities:

1. The non-zero entries of this column are below the row ℓ_{P-R}, i.e. it has a primary pivot below the row ℓ_{P-R} and hence $\sigma_k^{k_i,k_i-\ell_{P-R}}$ is not a generator of $\mathcal{Z}^R_{P,k-P}$ (i.e. $\mu^{k_i,k_i-\ell_{P-R}} = 0$).

2. The non-zero entries of this column are above the row ℓ_{P-R}, i.e. $\sigma_k^{k_i,k_i-\ell_{P-R}}$ is a generator of $\mathcal{Z}^R_{P,k-P}$ (i.e. $\mu^{k_i,k_i-\ell_{P-R}} = 1$).

Let $\Delta_{k\,P}^{\,R}$ be the submatrix of the above matrix composed only by the h_k-columns k_i such that $\mu^{k_i,k_i-\ell_{P-R}} = 1$, i.e. the columns which correspond to generators of $\mathcal{Z}^R_{P,k-P}$.

Conjecture: *Whenever the modules $\mathcal{E}^R_{P,k-P}$ and $\mathcal{E}^R_{P-R,k-1-(P-R)}$ are non-zero, the differential $d^R_{P,k-P} : \mathcal{E}^R_{P,k-P} \to \mathcal{E}^R_{P-R,k-1-(P-R)}$ is induced by $\Delta_{k\,P}^{\,R}$. In other words,*

$$\mathcal{E}^{R+1}_{P,k-P} \cong \frac{\text{Ker } \Delta_{k\,P}^{\,R}}{\text{Im } \Delta_{k+1\,P+R}^{\;\;\,R}}.$$

We believe that this conjecture is true. An example which illustrate it follows.

Example 5.4. Let f be a Morse function on S^2 such that the flow associated to $-\nabla f$ is as shown in Figure 5.5. The Morse chain complex $(C_*(f), \partial)$ determined by f was presented in Example 2.3, where we considered a finest filtration in $(C_*(f), \partial)$. The collection of matrices obtained applying the SSSA for Δ is the one illustrated in Figures 2.10 through 2.15.

Now we endow this chain complex with the following coarser filtration $\mathcal{F}$:

$\mathcal{F}_0 C_0 = \mathbb{Z}\langle h_0^1\rangle \oplus \mathbb{Z}\langle h_0^2\rangle \oplus \mathbb{Z}\langle h_0^3\rangle$, $\mathcal{F}_0 C_k = 0$ for $k > 0$;

$\mathcal{F}_1 C_0 = \mathbb{Z}\langle h_0^1\rangle \oplus \mathbb{Z}\langle h_0^2\rangle \oplus \mathbb{Z}\langle h_0^3\rangle$, $\mathcal{F}_1 C_1 = \mathbb{Z}\langle h_1^4\rangle \oplus \mathbb{Z}\langle h_1^5\rangle$ and $\mathcal{F}_1 C_k = 0$ for $k > 1$;

$$\mathcal{F}_2C_0 = \mathbb{Z}\langle h_0^1\rangle \oplus \mathbb{Z}\langle h_0^2\rangle \oplus \mathbb{Z}\langle h_0^3\rangle,\ \ \mathcal{F}_2C_1 = \mathbb{Z}\langle h_1^4\rangle \oplus \mathbb{Z}\langle h_1^5\rangle \oplus \mathbb{Z}\langle h_1^6\rangle \oplus \mathbb{Z}\langle h_1^7\rangle \quad \mathcal{F}_2C_2 = 0;$$

$$\mathcal{F}_3C_0 = \mathcal{F}_2C_0,\ \ \mathcal{F}_3C_1 = \mathbb{Z}\langle h_1^4\rangle \oplus \mathbb{Z}\langle h_1^5\rangle \oplus \mathbb{Z}\langle h_1^6\rangle \oplus \mathbb{Z}\langle h_1^7\rangle \quad \mathcal{F}_3C_2 = \mathbb{Z}\langle h_2^8\rangle \oplus \mathbb{Z}\langle h_2^9\rangle \oplus \mathbb{Z}\langle h_2^{10}\rangle;$$

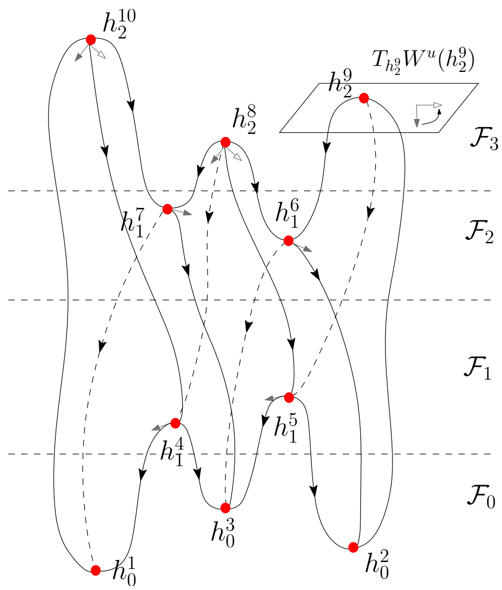


Figure 5.5: Morse-Smale flow in S^2 with a coarser filtration.

The spectral sequence associated to $(C_*(h), \partial)$ endowed with this filtration $\mathcal{F}$ is given by:

$\mathcal{E}^0$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
E^0_{0,1}=0 & E^0_{1,1}=0 & E^0_{2,1}=0 \arrow[d, "d^0"] & E^0_{3,1}=0 & \cdots \\
E^0_{0,0}=[h_0^1,h_0^2,h_0^3] & E^0_{1,0}=[h_1^4,h_1^5] & E^0_{2,0}=0 & E^0_{3,0}=0 & \cdots \\
E^0_{0,-1}=0 & E^0_{1,-1}=0 & E^0_{2,-1}=[h_1^6,h_1^7] & E^0_{3,-1}=[h_2^8,h_2^9,h_2^{10}] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$\mathcal{E}^1$

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
E^1_{0,1}=0 & E^1_{1,1}=0 & E^1_{2,1}=0 & E^1_{3,1}=0 & \cdots \\
E^1_{0,0}=[h_0^1,h_0^2,h_0^3] & E^1_{1,0}=[h_1^4,h_1^5] \arrow[l, "d_1^1"'] & E^1_{2,0}=0 & E^1_{3,0}=0 & \cdots \\
E^1_{0,-1}=0 & E^1_{1,-1}=0 & E^1_{2,-1}=[h_1^6,h_1^7] & E^1_{3,-1}=[h_2^8,h_2^9,h_2^{10}] \arrow[l, "d_3^1"'] & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$\begin{array}{llllll}
\mathcal{E}^2 & \vdots & \vdots & \vdots & \vdots & \\
& E^2_{0,1} = 0 & E^2_{1,1} = 0 & E^2_{2,1} = 0 & E^2_{3,1} = 0 & \cdots \\
& E^2_{0,0} = [h^1_0 + h^2_0 + h^3_0] & E^2_{1,0} = 0 & E^2_{2,0} = 0 & E^2_{3,0} = 0 & \cdots \\
& E^2_{0,-1} = 0 & E^2_{1,-1} = 0 & E^2_{2,-1} = 0 & E^2_{3,-1} = [h^8_2 + h^9_2 + h^{10}_2] & \cdots \\
& \vdots & \vdots & \vdots & \vdots &
\end{array}$$

The differential $d^1_1 : E^1_{1,0} \to E^1_{0,0}$ is induced by $\Delta_{k=1}{}^{R=1}_{P=1}$ and the differential $d^1_3 : E^1_{3,-1} \to E^1_{2,-1}$ is induced by the matrix $\Delta_{k=2}{}^{R=1}_{P=3}$, where:

$$\Delta_{k=1}{}^{R=1}_{P=1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \Delta_{k=2}{}^{R=1}_{P=3} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

◀

Final Remarks

The work reported in this thesis gives continuity to the systematic study of the dynamical implications associated to the algebraic behavior of a spectral sequence in [13, 24, 30].

The SSSA provides a way to recover the modules and differentials (E^r, d^r) of a spectral sequence associated to a Morse chain complex $(C_*(f), \Delta)$. As we apply the SSSA to Δ important entries in the r-th diagonal of Δ^r are singled out in order to determine Δ^{r+1}. These entries are the primary and change-of-basis pivots which induce the differentials d^r of the spectral sequence. Moreover, as r increases, the $\mathbb{Z}$-modules E_p^r undergo change of generators. The SSSA relates this change in generators of E_p^r to change of basis over $\mathbb{Q}$ of the connection matrix Δ.

Considering Morse chain complexes on surfaces M, we have shown in Theorem 2.4 that as r increases, the $\mathbb{Z}$-modules E_p^r's undergo algebraic cancellations which reflect dynamical cancellations of pair of consecutive critical points of a given Morse flow on M. In Theorem 2.4, we have proven that the primary pivots marked through the SSSA determine a continuation of the initial flow to the minimal flow on M. In higher dimensions, the integrality of the last matrix in the SSSA over $\mathbb{Z}$, raises the question of whether this procedure can be related to a continuation as in [22] of a flow associated to the initial connection matrix. Some examples indicate this might be true. The dynamical interpretation of the intermediary matrices produced by the SSSA over $\mathbb{Z}$ is yet not well understood, since many entries are non integers.

We intend to investigate the appearance of torsion in a spectral sequence in higher dimensions. Our goal is to search for properties in the connection matrix which either make this torsion disappear or permit it to remain in the stabilization of the unfolding of a spectral sequence.

The same type of dynamical problems exposed above in the Morse setting can be reformulated to the contexts of Morse-Novikov and Morse-Bott flows, which constitute a challenging line of research.

In the Novikov setting, we have given the first step in order to extend the spectral sequence analysis to Novikov complexes. In this work, we restricted our attention to orientable surfaces, where we have proven that the SSSA is well defined and that the non zero differentials of the spectral sequence are isomorphisms induced by primary pivots. The difficulty to define a SSSA over $\mathbb{Z}((t))$ in higher dimension is that we lose the notion of how to minimize leading coefficients in a change of basis.

A natural question in this context is whether the Novikov differential is a connection matrix. To answer the question, we intend to make a deeper study of Novikov complexes in all their generality. Also, we propose to use the Conley index and connection matrix theories for Novikov flows in order to obtain dynamical informations on the connections between the isolated invariants sets in this type of flows.

We envision to generalize the spectral sequence analysis for more general Morse decompositions than ones that have only critical points. This is the central stimulus in studying spectral sequences for Morse chain complexes with coarser filtration, since this provides a finest filtration for a more general flows. It is natural that our first attempt be in the context of Morse-Bott flows, where a finest Morse decomposition admits critical manifolds.

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Appendix A

Some Technical Results

This appendix is dedicated to prove some technical lemmas which are necessary for the prove of the Primary Pivots for Orientable Surfaces Theorem 2.1 in Section 2.3.

Below we present the Block Sequential Sweeping Algorithm over $\mathbb{F}$, which is a version of the SSSA where there are no elementary row operations, only elementary column operations in the execution of the SSSA over $\mathbb{F}$, since we give up keeping track the evolution of the rows in $\cup_{k=1}^{n} \mathcal{J}_k$.

Block Sequential Sweeping Algorithm over $\mathbb{F}$

Input: nilpotent $L \times L$ upper triangular matrix Δ with column/row partition $J_0, \cdots, J_b$.

Initialization Step: $\mathcal{J}_0 = \emptyset$

Iterative Step:

For $k = 1, \cdots, b$ **do**

Let $\Delta(k)$ be the matrix obtained from Δ by zeroing rows in $\mathcal{J}_{k-1}$ and entries outside positions in $J_{k-1} \times J_k$.

Apply the SSSA over $\mathbb{F}$ to $\Delta(k)$, with column/row partition $J_0, \cdots, J_b$, obtaining $\Delta(k)^L$

$\mathcal{J}_k =$ indices of columns of $\Delta(k)^L$ containing primary pivots

Output: $(\Delta(k)^0, \ldots, \Delta(k)^L)$ and $(T(k)^0, \ldots, T(k)^{L-1})$, for $k = 1, \cdots, b$.

Lemma A.1 (Uncoupling)**.** *Let Δ be a connection matrix with row/column partition $J_0, \cdots, J_b$. Let Δ^L be the matrix produced by the SSSA over $\mathbb{F}$ applied to Δ, and let $\Delta(k)^L$, for all k, be the matrices obtained in the Block Sequential Sweeping Algorithm over $\mathbb{F}$ applied to Δ.*

Then $\Delta^m_{J_{k-1}J_k} = \Delta(k)^m_{J_{k-1}J_k}$, *for all* k *and the collection of change-of-basis and primary pivots encountered in the application of the SSSA over* $\mathbb{F}$ *to* $\Delta(k)$, *for all* k, *coincides with the change-of-basis and primary pivots found when it is applied to* Δ.

Proof. Let $T(k)^0$, $T(k)^1$, ..., $T(k)^L$ be the transition matrices constructed when applying the SSSA over $\mathbb{F}$ to $\Delta(k)$. Given that $\Delta(k)$ has at most one nonzero block, $\Delta(k)_{J_{k-1}J_k}$, Proposition 6 , Lemma 1 and induction imply that $T(k)^r_{J_iJ_i}$ is an identity matrix, for all $i \neq k$. Now, using Lemma 2.1, the update of $\Delta(k)$ is reduced to the following update

$$\Delta(k)^r_{J_{k-1}J_k} = (T(k)^{r-1}_{J_{k-1}J_{k-1}})^{-1}\Delta(k)^{r-1}_{J_{k-1}J_k}T(k)^{r-1}_{J_kJ_k} = \Delta(k)^{r-1}_{J_{k-1}J_k}T(k)^{r-1}_{J_kJ_k},$$

for all $r \geq 1$. Since the update of $\Delta(k)$ involves only the post-multiplication step, only elementary column operations are performed.

Lemma 10 implies that $\Delta(1)^L_{J_0J_1} = \Delta^L_{J_0J_1}$ and the change-of-basis and primary pivots marked during the application of the algorithm to $\Delta(1)$ coincide with the ones marked in columns in J_1 when the algorithm is applied to Δ.

Assume by induction that $\Delta(k-1)^L_{J_{k-2}J_{k-1}} = \Delta^L_{J_{k-2}J_{k-1}}$ and change-of-basis and primary pivots of $\Delta(k-1)$ agree with the ones in columns in J_{k-1} marked when Δ is swept. Observe that rows of Δ^L in $\mathcal{J}_{k-1}$ are zero. By construction, rows of $\Delta(k)$ in $\mathcal{J}_{k-1}$ are also zero, and, since $\Delta(k)$ suffers only elementary column operations during the application of the SSSA over $\mathbb{F}$, these rows are not changed. So $\Delta^L_{\mathcal{J}_{k-1}J_k} = \Delta(k)^L_{\mathcal{J}_{k-1}J_k}$.

By Lemma 2.1, entries in the rows of Δ in $\overline{J}_{k-1}$ are not subjected to elementary row operations during the application of the SSSA over $\mathbb{F}$ thereto. Furthermore, all change-of-basis and primary pivots in columns in J_k occur in positions in $\overline{J}_{k-1} \times J_k$. Hence changes to entries in these rows are only due to elementary column operations, as also happens when the SSSA is applied to $\Delta(k)$. Since $\Delta_{\overline{J}_{k-1}J_k} = \Delta(k)_{\overline{J}_{k-1}J_k}$ and the elementary column operations on columns in J_k are solely dependent on the entries in this submatrix, it follows that the change-of-basis and primary pivots marked in columns in J_k during execution of the SSSA over $\mathbb{F}$ to both Δ and $\Delta(k)$ coincide, and so $\Delta^L_{\overline{J}_{k-1}J_k} = \Delta(k)^L_{\overline{J}_{k-1}J_k}$. ■

The Uncoupling Theorem implies that we may restrict our attention to connection matrices containing at most one nonzero block when studying the Incremental Sweeping Algorithm over $\mathbb{F}$, if we accept to miss the evolution of rows in $\cup_{k=1}^b \mathcal{J}_k$, which we know will end up zero and will not contain neither primary nor change-of-basis pivots. To ease the discussion that follows, we henceforth call this special case the *1-Block Incremental Sweeping Algorithm* *over* $\mathbb{F}$.

The analysis of the 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$ is significantly simpler than that of its more general counterpart. If the columns of the connection matrix $\Delta \in \mathbb{F}^{L \times L}$ are partitioned into two subsets J_0 and J_1, only rows in J_1 are altered in the pre-multiplication by $(T^{r-1})^{-1}$, in the matrix update step. But $\Delta_{J_1} = 0$ and this zero pattern is invariant under elementary column operations. So the post-multiplication part of the update doesn't change the nullity of rows in J_1 and the elementary row operations performed during the pre-multiplication part of the update step involve only the zero rows in J_1. This implies $\Delta^r_{J_1} = 0$ for all r, and we may eliminate the pre-multiplication part of the update step, so that only elementary column operations need be performed.

During the execution of the 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$, columns of Δ^r may be classified as *active* (resp., *passive*), if they contain (resp., do not contain) a primary pivot mark. The active columns effect change upon the passive columns. The passive columns suffer changes caused by active columns. At the beginning of the algorithm all columns are passive and before changes are allowed to happen, at least one column must become active. Once a column reaches the active state, it doesn't leave it, since primary pivot marks are permanent. Passive columns undergo a (possibly empty) sequence of elementary column operations and either reach an active state or become zero, since columns without primary pivots must be zero. If a column reaches an active state, it does so when the lowest nonzero entry in the column is marked as a primary pivot. The order of sweeping implies that the change-of-basis pivots that occur in a fixed column, say j, are marked in an upward fashion. If the entry in position $(j-r, j)$ is marked as a change-of-basis pivot, and the entry in position $(j-r, p)$ contains the primary pivot to its left, columns p and j exhibit a sequence of trailing of zeros from row $j-r+1$ to the last row. The elementary column operation that eliminates this change-of-basis pivot changes only the entries in rows 1 through $j-r$, the actual operation being determined by the values of the two pivots. By construction, each operation increases the number of trailing zeros by at least one.

In the 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$, the passive columns dictate the cancellations, which are done only once a change-of-basis pivot is marked. We propose a reengineered version therefor, in which this role is transferred to the active columns. Once an primary pivot is identified, all cancellations it is responsible for in the 1-block Incremental Sweeping Algorithm over $\mathbb{F}$ are performed. To arrive at the same final matrix as in the original algorithm, the primary pivots must also be identified in a upward order, from the bottom up. The second and last important aspect for the identification of primary pivots, is the left-to-right order of the sweeping. The algorithm below incorporates both of these. In the algorithm we adopt the usual convention that if S is an empty set and A is a real matrix, then $A_{S\bullet} = 0$. Since we are considering connection matrices with at most one nonzero block, we may assume, without loss of generality, that the row/column partition has two subsets.

Revised 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$

Input: nilpotent $L \times L$ upper triangular matrix Δ with column/row partition J_0, J_1.

Initialization Step: $C^1 = \{1, \ldots, L\}$, $\tilde{\Delta}^1 = \Delta$, $t = 1$.

Iterative Step:

While $\tilde{\Delta}^t_{\bullet C^t} \neq 0$ **do**

Let $i_t = \max\{i \mid \tilde{\Delta}^t_{iC^t} \neq 0\}$

Let $j_t = \min\{j \in C^t \mid \tilde{\Delta}^t_{i_t j} \neq 0\}$

Permanently mark $\Delta^t_{i_t j_t}$ as a primary pivot

Update Matrix Construction

$$\tilde{T}^t \leftarrow I - \sum_{\substack{j \in C^t \\ j > j_t}} \frac{\tilde{\Delta}^t_{i_t j}}{\tilde{\Delta}^t_{i_t j_t}} U^{j_t j}$$

Simplified Matrix Δ update

$$\tilde{\Delta}^{t+1} = \tilde{\Delta}^t \tilde{T}^t$$

$C^{t+1} \leftarrow C^t \backslash \{j_t\}$

$t \leftarrow t + 1$

Output: $(\tilde{\Delta}^0, \ldots)$ and $(\tilde{T}^0, \ldots)$

Let (i_1, j_1), $\ldots$, (i_{t^*}, j_{t^*}) be the positions of primary pivots marked in the application of the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$ to the connection matrix $\Delta \in \mathbb{F}^{L \times L}$ with row/column partition J_0, J_1, in the order in which they were marked. Then the following are true:

(i) once a column receives a primary pivot mark, it remains invariant until the end of the algorithm,

(ii) $j_t > i_t$, for $t = 1, \ldots, t^*$,

(iii) $i_1 > i_2 > \cdots > i_{t^*}$,

(iv) $\tilde{\Delta}^{t+1}_{\{i_t,\ldots,L\}C^{t+1}} = 0$, for $t = 1, \ldots, t^*$,

(v) the number of consecutive zero entries at the bottom of each column never decreases.

Corollary A.1. *Let $\tilde{\Delta}^{t^*+1}$ be the last matrix obtained by the application of the Revised 1-block Incremental Sweeping Algorithm to the connection matrix $\Delta \in \mathbb{F}^{L\times L}$ with row/column partition J_0, J_1. Then the primary pivot entries are nonzero and each nonzero entry of $\tilde{\Delta}^{t^*+1}$ lies above a primary pivot.*

Proof. Let (i_1, j_1), $\ldots$, (i_{t^*}, j_{t^*}) be the positions of primary pivots. A simple induction shows that C^{t^*+1} is the set of indices of columns of $\tilde{\Delta}^{t^*+1}$ without primary pivots. The stopping criterium implies $\tilde{\Delta}^{t^*+1}_{\bullet C^{t^*+1}} = 0$. Finally, primary pivots, when marked, are, by the rules of the algorithm, the lowest nonzero entry of the column, and columns do not change after receiving a primary pivot mark. ■

The next lemma establishes the equality between the final matrices produced by the Revised 1-block Incremental Sweeping over $\mathbb{F}$ and the 1-block Incremental Sweeping Algorithm over $\mathbb{F}$. There is of course no sense in looking for equality between other matrices in the sequence produced by the algorithm, since the order of cancellation is in all likelihood quite different in the two algorithms.

Lemma A.2. *Let Δ be a connection matrix with column/row partition J_0, J_1. Let $\tilde{\Delta}^{t^*+1}$ and Δ^L be the matrices obtained by applying the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$ and the 1-Block Incremental Sweeping Algorithm over $\mathbb{F}$ to Δ, respectively. Then $\tilde{\Delta}^{t^*+1} = \Delta^L$ and their primary pivots coincide.*

Proof. If $\tilde{\Delta}^{t^*+1} = \Delta^L$, then their primary pivots coincide in position and value, since the primary pivots entries are nonzero and each nonzero entry is located above a unique primary pivot in both algorithms.

In both algorithms columns may suffer elementary column operations until they either reach zero or receive a primary pivot mark. Additionally, a column may suffer an elementary column operation only from another column with a primary pivot mark on its left and successive operations on a column may only increase the range of trailing zeros in that column, that is, the set of successive rows, ending in row L, containing zero entries. The order in which the primary pivots are identified probably differs between algorithms, but the important thing is that when an entry is eligible for receiving a primary pivot in either algorithm, the column it belongs to will have been subjected to the same changes in both of them. In order for these changes to be the same in both algorithms, the active columns acting on them must be the same and the changes they provoke in a fixed column must occur in the correct order, from bottom up.

The proof is by induction on the number of nonzero columns of Δ. If there is but one nonzero column, the nonzero entry at the bottom of this column will be marked by both algorithms as the unique primary pivot of Δ, so $t^* = 1$, there will be no elementary column operations and in fact $\tilde{\Delta}^{t^*+1} = \Delta^L = \Delta$.

Admit by induction that the last matrix of both algorithms coincide, when the number of nonzero columns is smaller than k. Suppose Δ has k nonzero columns. Let (i_1, j_1) be the first position to receive a primary pivot mark in the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$. Then the entry $\Delta_{i_1 j_1}^{j_1 - i_1}$ must also receive a primary pivot mark in the SSSA over $\mathbb{F}$. To see that, note that, by definition of (i_1, j_1), the entries in $\Delta_{\{i_1,\ldots,L\}\{1,\ldots,j_1-1\}}$ and $\Delta_{\{i_1+1,\ldots,L\}\{1,\ldots,L\}}$ are zero, and the number of trailing zeros can only increase. Consequently no entries in these submatrices may have been marked as a primary pivot before the sweeping of the $j_1 - i_1$ diagonal, so $\Delta_{i_1 j_1}^{j_1 - i_1}$ is nonzero, with no primary pivot marks on its left or below it. So, in this case, $\Delta_{\bullet j_1}^{t^*+1} = \Delta_{\bullet j_1}^{L} = \Delta_{\bullet j_1}$. Furthermore, notice that, entries in positions $(i_1, j_1 + 1), \ldots (i_1, L)$ will be swept after this and be marked, if nonzero, as change-of-basis entries in the Incremental Sweeping Algorithm over $\mathbb{F}$, since rows $i_1 + 1, \ldots, L$ of Δ are zero and the entry in position (i_1, j_1) has a primary pivot. The changes possibly effected on columns due to these markings in the SSSA over $\mathbb{F}$ on the corresponding iterations are precisely the changes done in the first iteration of the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$.

Let Δ' be defined as follows:

$$\Delta'_{\bullet j} = \begin{cases} \tilde{\Delta}^2_{\bullet j}, & \text{if } j \neq j_1, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix Δ' agrees with the matrix obtained from Δ after the first iteration of the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$, except for column j_1, which is zero. So Δ' encompasses the changes to columns due to change-of-basis entries in row i_1, and has the j_1-th column equal to zero. Thus if we apply the Incremental Sweeping Algorithm over $\mathbb{F}$ to Δ', the matrix ${\Delta'}^L$ obtained agrees with Δ^L, except for column j_1, which would remain zero throughout the algorithm. Analogously, Δ' encompasses changes made to Δ in the first iteration of the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$, but differs from it in column j_1, which has been zeroed. Thus the application of the Revised 1-block Incremental Sweeping Algorithm over $\mathbb{F}$ to Δ' produces a matrix $\tilde{\Delta}'^{t^*}$ whose columns coincide with the corresponding ones from $\tilde{\Delta}^{t^*+1}$, except for the j_1-th column. Since Δ' has at most $k-1$ nonzero columns, induction implies that $\Delta'^L = \tilde{\Delta}'^{t^*+1}$. This implies

$$\Delta^L_{\bullet j} = \Delta'^L_{\bullet j} = \tilde{\Delta}'^{t^*}_{\bullet j} = \tilde{\Delta}^{t^*+1}_{\bullet j}, \quad \text{for } j \neq j_1.$$

But since we already have that $\Delta^L_{\bullet j_1} = \tilde{\Delta}^{t^*+1}_{\bullet j_1}$, we conclude that $\Delta^L = \tilde{\Delta}^{t^*+1}$. ■

Lemma A.3. *Let $\Delta \in \{0, \pm 1\}^{L \times L}$ be a totally unimodular connection matrix with column/row partition J_0, J_1. Let C^t be the set of index columns of $\tilde{\Delta}^t$ which do not have a primary pivot. The submatrix $\tilde{\Delta}^t_{\bullet C^t}$ is totally unimodular, for $t = 1, \ldots, t^* + 1$.*

Proof. Let (i_1, j_1), …, (i_{t^*}, j_{t^*}) be the positions of the primary pivots marked.

We claim that $\tilde{\Delta}^t_{\bullet C^t}$ is totally unimodular, for $t = 1, \ldots, t^* + 1$. This is trivially true for $t = 1$, by hypothesis. Assume it is true for t. Since

$$(\tilde{\Delta}^t T^t)_{\bullet j} = \begin{cases} \tilde{\Delta}^t_{\bullet j}, & \text{if } j \notin C^t, \\ \tilde{\Delta}^t_{\bullet j}, & \text{if } j \in C^t \text{ and } j \leq j_t, \\ \tilde{\Delta}^t_{\bullet j} - \dfrac{\tilde{\Delta}^t_{i_t j}}{\tilde{\Delta}^t_{i_t j_t}} \tilde{\Delta}^t_{\bullet j_t}, & \text{if } j \in C^t \text{ and } j > j_t, \end{cases}$$

the marking of an entry in position (i_t, j_t) of $\tilde{\Delta}^t_{\bullet C^t}$ as a primary pivot implies the cancellation, in $\tilde{\Delta}^{t+1}$, of entries on row i_t, in columns in C^t other than j_t (although the update matrix construction provides the cancellation of entries to the right of column j_t, entries to its left are zero, since $\tilde{\Delta}^t_{i_t j_t}$ is the leftmost nonzero entry in row i_t of $\tilde{\Delta}^t_{\bullet C^t}$). Thus $\tilde{\Delta}^{t+1}_{\bullet C^t} = (\tilde{\Delta}^t T^t)_{\bullet C^t} = \tilde{\Delta}^t_{\bullet C^t} T^t_{C^t C^t}$. Notice that the cancellations are achieved by adding to column $j \in C^t$, $j \neq j_t$, the appropriate multiple of column j_t. But this is simply a transposed version of the variant of the linear programming pivoting described above, with $\tilde{\Delta}^t_{\bullet C^t} = A^T$ and $T^t_{C^t C^t} = \widehat{B}^T$. Therefore, if $\tilde{\Delta}^t_{\bullet C^t}$ is totally unimodular, then $\tilde{\Delta}^{t+1}_{\bullet C^t}$ is also totally unimodular. Since this property is, by definition, inherited by submatrices, and $C^{t+1} \subset C^t$, we conclude $\tilde{\Delta}^{t+1}_{\bullet C^{t+1}}$ is also totally unimodular. By induction, $\tilde{\Delta}^t_{\bullet C^t}$ is totally unimodular for all t. ■