FERNANDA DE ANDRADE PEREIRA

CLASSIFICATION OF THE TYPE *D* IRREGULAR MINIMAL AFFINIZATIONS

CLASSIFICAÇÃO DAS AFINIZAÇÕES MINIMAIS IRREGULARES DE TIPO D

CAMPINAS
2014

UNICAMP

UNIVERSIDADE ESTADUAL DE CAMPINAS
INSTITUTO DE MATEMÁTICA, ESTATÍSTICA
E COMPUTAÇÃO CIENTÍFICA

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CLASSIFICAÇÃO DAS AFINIZAÇÕES MINIMAIS IRREGULARES DE TIPO *D*

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em Matemática

Orientador(a): Prof. dr. Adriano Adrega de Moura

Coorientador(a): Prof. Dr. David Hernandez

ESTE EXEMPLAR CORRESPONDE À VERSÃO FINAL DA TESE DEFENDIDA PELA ALUNA **FERNANDA DE ANDRADE PEREIRA** E ORIENTADA PELO PROF. DR. ADRIANO ADREGA DE MOURA.

Moura

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CAMPINAS
2014

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Maria Fabiana Bezerra Muller - CRB 8/6162

P414c Pereira, Fernanda de Andrade, 1986-
Classification of the type D irregular minimal affinizations / Fernanda de Andrade Pereira. – Campinas, SP : [s.n.], 2014.

Orientador: Adriano Adrega de Moura.
Coorientador: David Hernandez.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Grupos quânticos. 2. Representações de álgebras. I. Moura, Adriano Adrega de,1975-. II. Hernandez, David. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações para Biblioteca Digital

Título em outro idioma: Classificação das afinizações minimais irregulares de tipo D
Palavras-chave em inglês:
Quantum groups
Representations of algebras
Área de concentração: Matemática
Titulação: Doutora em Matemática
Banca examinadora:
Adriano Adrega de Moura [Orientador]
Marcos Benevenuto Jardim
Vyacheslav Futorny
Bárbara Seelig Pogorelsky
Vitor de Oliveira Ferreira
Data de defesa: 07-04-2014
Programa de Pós-Graduação: Matemática

Tese de Doutorado defendida em 07 de abril de 2014 e aprovada

Pela Banca Examinadora composta pelos Profs. Drs.

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Abstract

The concept of minimal affinization, introduced by Chari and Pressley, arose from the impossibility to extend, in general, a representation of the quantum group associated to a simple Lie algebra for the quantum group associated to its loop algebra, which is always possible in the classical context. A special class of minimal affinizations is that of the Kirillov-Reshetikhin modules, which are minimal affinizations of the irreducible modules with highest weight multiple of a fundamental weight. These modules are central objects in the study of integrable lattices in mechanical statistics. In the past two decades it has been intense the scientific research in the direction of understanding the minimal affinizations, not only by their potential applications in mathematical physics, but also for being a very rich theory for itself, in addition to having strong interaction with combinatorics. There exists an almost complete classification of the equivalence classes of the minimal affinizations in terms of Drinfeld polynomials due to Chari and Pressley. The classification is completed in the case where the support of the highest weight does not enclose a subdiagram of type D_4, and in this case there is only one equivalence class. In the case where the support encloses a subdiagram of type D_4 the situation depends essentially if support contains the trivalent node of the diagram or not. If it contains, the classification is also completed and there are three equivalence classes. Otherwise the classification is not completed. In this work we present the classification of the equivalence classes for algebras of type D. The main technique used was the combinatorial manipulation of qcharacters through mainly its description via tableaux and sometimes using the Frenkel-Mukhin algorithm.

Keywords: Minimal affinizations, qcharacters, quantum groups, representations of algebras.

Resumo

O conceito de afinização minimal, introduzido por Chari e Pressley, surgiu a partir da impossibilidade de se estender, em geral, uma representação do grupo quântico associado a uma álgebra de Lie simples para o grupo quântico associado à sua álgebra de laços, o que sempre é possível no contexto clássico. Uma classe especial de afinizações minimais é a dos módulos de Kirillov-Reshetikhin, que são afinizações minimais dos módulos irredutíveis quando os pesos máximos são múltiplos dos pesos fundamentais. Esses módulos são objetos centrais no estudo de reticulados integráveis em mecânica estatística. Nas últimas duas décadas, tem sido intensa a investigação científica na direção de se entender as afinizações minimais, devido não só às suas potenciais aplicações em física-matemática, mas também por ser uma teoria muito rica por si só, além de ter forte interação com combinatória. Existe uma classificação quase completa das classes de equivalências de afinizações minimais em termos de polinômios de Drinfeld, devido a Chari e Pressley. A classificação está completa no caso em que o suporte do peso máximo não engloba um subdiagrama de tipo D_4, e neste caso existe uma única classe de equivalência. No caso em que o suporte engloba um subdiagrama de tipo D_4 a situação depende essencialmente se o suporte contém o vértice trivalente do diagrama ou não. Se ele o contém, a classificação também está completa e existem três classes de equivalências. Caso contrário a classificação não está completa. Neste trabalho apresentamos a classificação das classes de equivalências para álgebras de tipo D. A principal técnica empregada foi a manipulação combinatória de qcaráteres através principalmente de sua descrição via tableaux e, algumas vezes, utilizando-se o algoritmo de Frenkel-Mukhin.

Palavras-chave: Afinizações minimais, qcaráteres, grupos quânticos, representações de álgebras.

Contents

"A educação é a arma mais poderosa com a qual se pode mudar o mundo."

Nelson Mandela

Agradecimentos

Primeiramente quero agradecer a Deus, por tantas oportunidades que tem colocado em meu caminho, e por tantas realizações em minha vida.

Agradeço aos meus pais, Rosa Maria e Junior, por tudo o que me proporcionaram. Em especial a minha mãe, por todo apoio, amizade, carinho e amor que está sempre disposta a dar incondicionalmente, e por tudo o que me ensinou.

Agradeço a todos os meus familiares que também sempre me apoiaram, especialmente meus irmãos Rafael e Débora, meus avós Nair e Dyonisio e minha tia Rita, que me deram um enorme suporte durante a minha caminhada.

Agradeço a todos os amigos e colegas, que tornaram essa jornada muito mais prazerosa. Em especial minhas amigas Cintya e Grasiele, com quem tive a oportunidade de dividir república durante o doutorado, com quem compartilhei muitos momentos de alegria, e que serão sempre grandes amigas minhas.

Agradeço ao meu marido Luís Felipe, por ter estado ao meu lado durante essa jornada, sempre me apoiando, sendo meu amigo e companheiro. Agradeço por ser essa pessoa maravilhosa que me acompanha no dia a dia, e que torna minha vida muito mais colorida! E agradeço por ter me proporcionado a realização de um dos meus maiores desejos na vida, o de poder ser mãe! Agradeço a nossa filhinha, que ainda nem nasceu, mas já nos dá tanta alegria. Amo vocês!

Agradeço aos professores e funcionários do IMECC-UNICAMP, por todo apoio e ensinamento durantes esses anos do doutorado. Em especial, agradeço ao meu orientador Adriano, por todo o ensinamento, pela excelente orientação, pela dedicação, pela paciência, e por todo apoio em minhas decisões. Eu o levarei como um exemplo em que me espelho para seguir minha vida profissional.

I thank my co-advisor David Hernandez for having me in Paris for six months, and for the tremendous help that he gave to the development of this work.

Agradeço aos Profs. Drs. Marcos Benevenuto Jardim (IMECC-UNICAMP), Vyacheslav Futorny (IME-USP), Bárbara Seelig Pogorelsky (UFRGS) e Vitor de Oliveira Ferreira (IME-USP), por terem participado da banca examinadora da defesa desta tese, pelas críticas, elogios e pelas sugestões oferecidas para melhorar este trabalho.

Agradeço a FAPESP pelo apoio financeiro.

Agradeço a todos que passaram em minha vida e me deixaram algum aprendizado ou alguma experiência e a todos que de alguma forma contribuiram para que eu estivesse aqui. Peço que me perdoem os que deixei de citar.

List of symbols

$\mathbb{C}$: field of complex numbers;
$\mathfrak{g}$: finite dimensional semisimple Lie algebra over $\mathbb{C}$;
I: set of vertices of a finite-type Dynkin diagram;
α_i: simple root;
h_i: co-root;
ω_i: fundamental weight;
R: set of roots;
R^+: set of positive roots;
$\mathrm{ht}(\alpha)$: height of the root α;
Q: root lattice;
Q^+: positive cone of Q^+;
P: weight lattice;
P^+: set of dominant weights;
$\mathrm{supp}(\mu)$: support of a weight μ;
$\overline{\mathrm{supp}}(\mu)$: connected closure of the support of μ;
$\tilde{\mathfrak{g}}$: loop algebra of $\mathfrak{g}$;
q: nonzero complex number, not root of unity;
$U_q(\mathfrak{g})$: quantum group associated to $\mathfrak{g}$;
$U_q(\tilde{\mathfrak{g}})$: quantum group associated to $\tilde{\mathfrak{g}}$;
$\mathcal{P}$: set of ℓ-weights;
$\mathcal{P}^+$: set of dominant ℓ-weights (or Drinfeld polynomials);
$\mathcal{Q}$: ℓ-root lattice;
$\mathcal{Q}^+$: positive cone of $\mathcal{Q}$;
$\boldsymbol{\omega}_{i,a}$: fundamental ℓ-weight;
$\boldsymbol{\alpha}_{i,a}$: simple ℓ-root;
wt: weight map from $\mathcal{P}$ to P;
$\boldsymbol{\omega}_{i,a,m} = \prod_{j=0}^{m-1} \boldsymbol{\omega}_{i,aq^{m-1-2j}}$;
$V_q(\lambda)$: irreducible finite-dimensional $U_q(\mathfrak{g})$-module of highest weight λ;
$V_q(\boldsymbol{\lambda})$: irreducible finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module of highest ℓ-weight $\boldsymbol{\lambda}$;
$m_\mu(V)$: multiplicity of the irreducible factor $V_q(\mu)$ in V;
$|\lambda| = \sum_{i\in I} \lambda(h_i)$; $\;{}_i|\lambda|_j = \sum_{k=i}^{j} \lambda(h_k)$; $\;|\lambda|_j = {}_1|\lambda|_j$;
$A = \{i \in I : i < n-2\}$;
$i_\lambda = \min\{i \in I : \lambda(h_i) \neq 0\}$;
$f_\lambda = \max\{i \in A : \lambda(h_i) \neq 0\}$;
$\mathrm{ch}(V)$: character of V;
$\mathrm{qch}(V)$: qcharacter of V;
$\mathrm{wt}_\ell(V)$: set of ℓ-weights of V;
$Y_{i,s} = \boldsymbol{\omega}_{i,q^s}$;
$A_{i,s}^{-1} = \boldsymbol{\alpha}_{i,q^{s-1}}$;

$Y_{i,r,m} = \prod_{k=0}^{m-1} Y_{i,r+2k}$;
$A_{i,j,r} = \prod_{k=i}^{j} A_{k,r+k-i+1}$ if $i \leq j$; $\quad A_{j,i,r} = \prod_{k=i}^{j} A_{j+i-k,r+k-i+1}$ if $i \leq j$;
$r(\boldsymbol{\omega}) = \max\{s \in \mathbb{Z} : Y_{i,s}^{\pm 1}$ appears in $\boldsymbol{\omega}$ for some $i \in I\}$;
$\boldsymbol{\omega}_J$: J-tuple of rational functions associated to $\boldsymbol{\omega}$ $(J \subseteq I)$;
$\boldsymbol{\omega}^J$: $(\boldsymbol{\omega}^J)_j = \boldsymbol{\omega}_j(u)$ if $j \in J$ and $(\boldsymbol{\omega}^J)_j = 1$ otherwise;
π_J: projection map from $\mathcal{P}$ to $\mathcal{P}_J = \{\boldsymbol{\omega}_J : \boldsymbol{\omega} \in \mathcal{P}\}$;
$\iota_J : \mathbb{Z}(\mathcal{Q}_J) \to \mathbb{Z}(\mathcal{Q})$ ring homomorphism such that $\iota(\boldsymbol{\alpha}_{j,a}) = \boldsymbol{\alpha}_{j,a}$ for all $j \in J$, $a \in \mathbb{C}^\times$;
$\chi_J(\boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \iota_J(\boldsymbol{\omega}_J^{-1} \mathrm{qch}(V_q(\boldsymbol{\omega}_J)))$;
$\mathrm{ht}(\boldsymbol{\alpha}) = \mathrm{ht}(\mathrm{wt}(\boldsymbol{\alpha}))$;
$\mathrm{STab}(T)$: set of the semi-standard tableaux with the same shape of T;
$\boldsymbol{\omega}^T$: ℓ-weight associated to the tableau T.

Introduction

The theory of representations of Kac-Moody algebras and their quantizations is of great interest in many areas of mathematics and physics. These algebras were proposed independently in the 1960's by Kac and Moody (see [**29**] and [**34**]) as a generalization of the successful concept of semisimple Lie algebra. Around 1985, influenced by the works in statistical-physics related to the so-called Yang-Baxter equation, Drinfeld [**14, 15**] and Jimbo [**27, 28**] introduced deformations of the universal enveloping algebras of Kac-Moody algebras, which are known as Quantum Groups. The classical Kac-Moody algebras (or rather their universal enveloping algebras) are recovered by taking the limit when the parameter of deformation (quantization) goes to 1. Later the theory of quantum groups showed itself to be useful in many subareas of mathematics, establishing relations among a priori seemingly highly unrelated works. Some important results on the classical Kac-Moody algebras, such as the existence of "canonical" bases for certain important classes of their representations, were only possible to obtain after the advent of quantum groups through the works of Kashiwara [**30**] and Lusztig [**33**].

Almost three decades after quantum groups were introduced, many relevant problems, both in classic and quantum contexts, remain open, maintaining a very intense research activity in its various subareas. In particular, the theory of finite-dimensional representations of quantum groups associated to affine Kac-Moody algebras (or simply quantum affine algebras) is a very relevant research topic nowadays (see for example [**1, 3, 5, 7, 13, 17, 18, 26, 35, 46**]). The underlying categories of modules are Jordan-Hölder abelian tensor categories. Despite the classification of the simple objects being known since [**8**] and the great development of the theory since then, several other basic questions about the structure of these categories remain essentially unanswered.

The central problem investigated in this work concerns the subclass of simple objects known as minimal affinizations defined by Chari in [**1**]. These representations appear naturally in the study of integrable lattices in statistical-mechanics. In the past two decades, there has been intense scientific work in the direction of understanding the minimal affinizations, not only because of their potential applications in mathematical-physics, but also for being a very rich theory by itself in addition to having a strong interaction with combinatorics. Important examples of minimal affinizations are the Kirillov-Reshetikhin modules initially studied in [**31**], where it was conjectured that the characters of their tensor products satisfied certain fermionic formulas. This conjecture (whose proof was finished in [**23**]) was motivated by an essential tool for the study of such integrable lattices, called Bethe Ansatz. In fact, the fermionic formulas provide an algorithm to calculate the character of any Kirillov-Reshetikhin module as a polynomial on the characters of the so-called fundamental representations. Despite the characters of the fundamental representations being known, the task of obtaining "closed" formulas from this algorithm does not seem feasible. However, also motivated by the Bethe Ansatz, the authors of [**19, 20**] formulated conjectures for characters of Kirillov-Reshetikhin modules directly, which are now proved as a byproduct of the results of [**23**] and references therein.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over the complex numbers and $\tilde{\mathfrak{g}}$ be the associated (non twisted) affine Kac-Moody algebra. Denote by $U_q(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$ the corresponding quantum groups at a generic value of the quantization parameter q. Recall that the simple finite-dimensional $U_q(\mathfrak{g})$-modules are determined by their highest weight, which must be a dominant integral weight. Given such a weight λ, denote by $V_q(\lambda)$ any element of the associated isomorphism class of simple modules. Recall also that every finite-dimensional $U_q(\mathfrak{g})$-module is completely reducible. Given such a module V and a dominant weight μ, we denote by $m_\mu(V)$ the multiplicity of $V_q(\mu)$ as a simple factor of V. Hence, we can write

$$V \cong \bigoplus_{\mu} V_q(\mu)^{\oplus m_\mu(V)}$$

and the numbers $m_\mu(V)$ determine the isomorphism class of V. Since $U_q(\mathfrak{g})$ is a subalgebra of $U_q(\tilde{\mathfrak{g}})$, any $U_q(\tilde{\mathfrak{g}})$-module can be regarded as a $U_q(\mathfrak{g})$-module. We shall say that two finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules are equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. Following [**1**], a finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module V is said to be an affinization of $V_q(\lambda)$ if

$$m_\lambda(V) = 1 \quad \text{and} \quad m_\mu(V) \neq 0 \quad \text{only if} \quad \mu \leq \lambda,$$

where $\leq$ denotes the usual partial order on the weight lattice of $\mathfrak{g}$. This partial order also induces in a very natural way a partial order in the set of (equivalence classes of) affinizations of $V_q(\lambda)$. The corresponding minimal elements are called minimal affinizations. Evidently, a minimal affinization is necessarily an irreducible $U_q(\tilde{\mathfrak{g}})$-module.

The simple objects of the category of finite-dimensional representation of $U_q(\tilde{\mathfrak{g}})$ are classified by Drinfeld polynomials. Denote by $V_q(\boldsymbol{\omega})$ any element of the isomorphism class of simple modules associated the Drinfeld polynomial $\boldsymbol{\omega}$. To each Drinfeld polynomial $\boldsymbol{\omega}$ it is associated a dominant integral weight $\mathrm{wt}(\boldsymbol{\omega})$. It is immediate that $V_q(\boldsymbol{\omega})$ is an affinziation of $V_q(\lambda)$ with $\lambda = \mathrm{wt}(\boldsymbol{\omega})$. It is then natural to ask about the classification of minimal affinizations, i.e., the characterization of the Drinfeld polynomials corresponding to minimal affinizations. This classification is complete only when the support of λ does not enclose a subdiagram of type D_4 (see [**10, 11**]). Indeed, in that case, there exists only one equivalence class of minimal affinizations. If the support of λ encloses a subdiagram of type D_4, the situation is more complicated and it depends essentially on whether the support contains the trivalent node or not. In the first case the classification is also complete [**10**] and there are three equivalence classes. If the trivalent node is not in the support of λ, essentially nothing is known except when $\mathfrak{g}$ is of type D_4, in which case Chari and Pressley almost finished the classification in [**12**].

The goal of the present work is to complete the classification for $\mathfrak{g}$ of type D_n. In particular, it follows from our results that a conjecture left in [**12**] is false. Namely, three families of Drinfeld polynomials were presented in [**12**] and it was proved that the Drinfeld polynomials corresponding to minimal affinizations must lie in one those families. It was then conjectured that two of the families parameterized the same equivalence classes and, hence, there should be only two families of minimal affinizations. We prove here that the Drinfeld polynomials in one of the three families do not actually correspond to minimal affinizations: the corresponding modules are actually smaller than those that they were conjectured to be equivalent to. It is interesting to mention that the number of non equivalent elements in one of the families is not uniformly bounded, i.e., the number of equivalence classes of minimal affinizations of $V_q(\lambda)$ grows as λ "grows". For $n > 4$ we have more families: beside the two analogous families to those in the case $n = 4$, we have two extra families in the case that the support of λ intersects the "type A" part of the Dynkin diagram in a single node and four extra families otherwise.

Beside the techniques developed by Chari and Pressley in [**10, 11, 12**], we used another tool, not available when those papers were published, in a very crucial way in the proof of our classification theorems: the notion of qcharacters introduced by Frenkel and Resehtikhin in [**18**] in their study of W-algebras. The qcharacters are analogues of the usual concept of characters for $U_q(\mathfrak{g})$-modules in the sense that they describe the dimension of the so-called ℓ-weight spaces. Despite all the progress related to theory of qcharacters, there does not exist any general formula for computing them (see however, [**42, 43, 47**] for formulas in terms of Jacobi-Trudi determinants for $\mathfrak{g}$ of classical type). In [**44**], Nakajima described the qcharacters of fundamental representations in the case that $\mathfrak{g}$ is of type A and D in terms of tableaux. A similar description for the qcharacters of minimal affinizations of type A in terms of semi-standard tableaux is known (see the paragraph preceding Proposition 3.2.1) and we use it to obtain partial information on the simple factors of certain tensor products of simple modules. In particular, we characterize their irreducibility in some cases and compute special simple factors in other cases. This study of tensor products is then applied to compare certain affinizations. For type D, we obtain a similar characterization in terms of semi-standard tableaux for studying tensor products of Kirillov-Reshetikhin modules associated to the spin nodes. In some cases we also employed the Frenkel-Mukhin algorithm [**17**] and some lemmas obtained in [**22**] for obtaining partial information on the qcharacters. These results are then brought together in the proof of the classification theorems for showing that the Drinfeld polynomial of any minimal affinization must belong to one of the presented families as well as for showing that the modules corresponding to different elements of these families are not comparable affinizations and, hence, must be minimal.

As we have seen in the previous paragraph, the proof of the classification theorems provides some partial information about the structure of minimal affinizations, specially about their qcharacters. However, most of their structure remains unveiled. In particular, it is natural to ask about the structure of the minimal affinizations as $U_q(\mathfrak{g})$-modules. More precisely, given a minimal affinization V, how to compute the numbers $m_\mu(V)$? In type A, it is well-known that $m_\mu(V) = 0$ unless μ is the highest weight (this is a consequence of the fact that, in type A, there exist quantum analogues of the evaluation maps $\tilde{\mathfrak{g}} \to \mathfrak{g}$). In the case of Kirillov-Reshetikhin modules, most of these numbers can computed from the formulas given in the aforementioned [**19, 20**] as well as from the ones given in [**5, 6, 16**]. The methods of the latter references are quite different from the former and consist of considering the so-called graded limits of the minimal affinizations. Moreover, the results of [**5, 6**] also give generators and relations for these graded limits while [**16**] was the first to explore the role of Demazure modules in the study of Kirillov-Reshetikhin and other finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules. An extension of the methods of [**5, 6**] to more general minimal affinizations was proposed in [**35**], where it was conjectured generators and relations for the graded limits of minimal affinizations when the highest weight is regular. Here, by regular we mean that, if its support encloses a subdiagram of type D, then the trivalent node must be in the support. Small examples of the validity of this conjecture for the orthogonal types were given still in [**35**] while [**38**] partially proved the conjecture for type E_6. In the last two years, the conjecture was proved by Naoi in [**39, 40**] by further exploring the role of Demazure modules in the theory. In particular, a character formula for minimal affinizations in terms of interactions of Demazure operators was obtained. Thus, there are two natural problems to investigate after the present work: complete the classification for type E and study the role of Demazure modules for understanding the structure of the minimal affinizations we have classified here.

The text is divided in five chapters. In the first chapter we briefly review the affine classical and quantum algebras and their finite-dimensional representations. We also present the formal definition of minimal affinizations and state the Chari and Pressley classification of the minimal affinizations with regular highest weight. The main theorems of the present work are presented in Section 1.7. In the second chapter we present a few concepts and results that will be needed throughout the text, such as the notion of qcharacters. In the third chapter we develop the theory qcharacters for minimal affinizations of type A in terms of tableaux. In particular, we prove the aforementioned results on tensor products and ordering of affinizations. In the fourth chapter we prove similar results to that of Chapter 3, but for Kirillov-Reshetikhin modules of type D related to the spins nodes. In Section 4.6 we also prove a result (Proposition 4.6.4) comparing two affinizations having highest weight supported at the three extreme nodes of the Dynking diagram. This proposition together with the results of [**12**] completes the classification in type D_4 (in particular, the proposition shows that the conjecture left in [**12**] is false). The proofs of the main theorems are presented in Chapter 5.

CHAPTER 1

Definitions, notation, and the main result

Throughout the text, let $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{Z}_{\geq m}$ denote the sets of complex numbers, reals, integers, and integers bigger or equal m, respectively. Given a ring $\mathbb{A}$, the underlying multiplicative group of units is denoted by $\mathbb{A}^\times$. The dual of a vector space V is denoted by V^*. The symbol $\cong$ means "isomorphic to".

1.1. Classical algebras

Let $I = \{1, \ldots, n\}$ be the set of vertices of a finite-type simply laced Dynkin diagram (labeled as in [**25**]) and let $\mathfrak{g}$ be the associated semisimple Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h}$. Fix a set of positive roots R^+ and let

$$\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^+} \mathfrak{g}_{\pm\alpha} \quad \text{where} \quad \mathfrak{g}_{\pm\alpha} = \{x \in \mathfrak{g} : [h, x] = \pm\alpha(h)x, \ \forall\ h \in \mathfrak{h}\}.$$

The simple roots will be denoted by α_i and the fundamental weights by ω_i, $i \in I$. Q, P, Q^+, P^+ will denote the root and weight lattices with corresponding positive cones, respectively. Let also $h_i \in \mathfrak{h}$, be the co-root associated to $\alpha_i, i \in I$. We equip $\mathfrak{h}^*$ with the partial order $\lambda \leq \mu$ iff $\mu - \lambda \in Q^+$. Let $C = (c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$, i.e., $c_{ij} = \alpha_j(h_i)$. The Weyl group is denoted by $\mathcal{W}$ and its longest element by w_0.

The subalgebras $\mathfrak{g}_{\pm\alpha}, \alpha \in R^+$, are one-dimensional and $[\mathfrak{g}_{\pm\alpha}, \mathfrak{g}_{\pm\beta}] = \mathfrak{g}_{\pm\alpha\pm\beta}$ for every $\alpha, \beta \in R^+$. We denote by $x^\pm_\alpha$ any generator of $\mathfrak{g}_{\pm\alpha}$ and, in case $\alpha = \alpha_i$ for some $i \in I$, we may also use the notation $x^\pm_i$ in place of $x^\pm_{\alpha_i}$. In particular, if $\alpha + \beta \in R^+$, $[x^\pm_\alpha, x^\pm_\beta]$ is a nonzero generator of $\mathfrak{g}_{\pm\alpha\pm\beta}$ and we simply write $[x^\pm_\alpha, x^\pm_\beta] = x^\pm_{\alpha+\beta}$.

The support of $\mu \in P$ is defined by

$$\mathrm{supp}(\mu) = \{i \in I : \mu(h_i) \neq 0\}.$$

Let also $\overline{\mathrm{supp}}(\mu)$ be the minimal subset of I containing $\mathrm{supp}(\mu)$ such that the corresponding subdiagram of the Dynkin diagram of $\mathfrak{g}$ is connected.

If $\mathfrak{a}$ is a Lie algebra over $\mathbb{C}$, define its loop algebra to be $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ with bracket given by $[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s}$. Clearly $\mathfrak{a} \otimes 1$ is a subalgebra of $\tilde{\mathfrak{a}}$ isomorphic to $\mathfrak{a}$ and, by abuse of notation, we will continue denoting its elements by x instead of $x \otimes 1$. Then $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra. The elements $x^\pm_\alpha \otimes t^r, x^\pm_i \otimes t^r$, and $h_i \otimes t^r$ will be denoted by $x^\pm_{\alpha,r}, x^\pm_{i,r}$, and $h_{i,r}$, respectively.

1.2. Quantum algebras

Let $q \in \mathbb{C}^\times$ not a root of unity. Set

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]! = [m][m-1]\ldots[2][1], \qquad \left[\begin{smallmatrix} m \\ r \end{smallmatrix}\right] = \frac{[m]!}{[r]![m-r]!},$$

for $r, m \in \mathbb{Z}_{\geq 0}$, $m \geq r$.

The quantum loop algebra $U_q(\tilde{\mathfrak{g}})$ of $\mathfrak{g}$ is the associative $\mathbb{C}$-algebra with generators $x_{i,r}^{\pm}$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z}\backslash\{0\}$) and the following defining relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i,$$
$$k_i h_{j,r} = h_{j,r} k_i,$$
$$k_i x_{j,r}^{\pm} k_i^{-1} = q^{\pm c_{ij}} x_{j,r}^{\pm},$$
$$[h_{i,r}, h_{j,s}] = 0, \ [h_{i,r}, x_{j,s}^{\pm}] = \pm \frac{1}{r}[rc_{ij}] x_{j,r+s}^{\pm},$$
$$x_{i,r+1}^{\pm} x_{j,s}^{\pm} - q^{\pm c_{ij}} x_{j,s}^{\pm} x_{i,r+1}^{\pm} = q^{\pm c_{ij}} x_{i,r}^{\pm} x_{j,s+1}^{\pm} - x_{j,s+1}^{\pm} x_{i,r}^{\pm},$$
$$[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{i,j} \frac{\psi_{i,r+s}^{+} - \psi_{i,r+s}^{-}}{q - q^{-1}},$$
$$\sum_{\sigma \in S_m} \sum_{k=0}^{m} (-1)^k \left[{m \atop k} \right] x_{i,r_{\sigma(1)}}^{\pm} \dots x_{i,r_{\sigma(k)}}^{\pm} x_{j,s}^{\pm} x_{i,r_{\sigma(k+1)}}^{\pm} \dots x_{i,r_{\sigma(m)}}^{\pm} = 0, \quad \text{if } i \neq j,$$

for all sequences of integers $r_1, \dots, r_m$, where $m = 1 - c_{ij}$, S_m is the symmetric group on m letters, and the $\psi_{i,r}^{\pm}$ are determined by equating powers of u in the formal power series

$$\Psi_i^{\pm}(u) = \sum_{r=0}^{\infty} \psi_{i,\pm r}^{\pm} u^r = k_i^{\pm 1} \exp\left(\pm (q - q^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^s \right).$$

Denote by $U_q(\tilde{\mathfrak{n}}^{\pm}), U_q(\tilde{\mathfrak{h}})$ the subalgebras of $U_q(\tilde{\mathfrak{g}})$ generated by $\{x_{i,r}^{\pm}\}, \{k_i^{\pm 1}, h_{i,s}\}$, respectively. Let $U_q(\mathfrak{g})$ be the subalgebra generated by $x_i^{\pm} := x_{i,0}^{\pm}, k_i^{\pm 1}, i \in I$, and define $U_q(\mathfrak{n}^{\pm}), U_q(\mathfrak{h})$ in the obvious way. $U_q(\mathfrak{g})$ is a subalgebra of $U_q(\tilde{\mathfrak{g}})$ and multiplication establishes isomorphisms of $\mathbb{C}(q)$-vectors spaces:

$$U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \qquad \text{and} \qquad U_q(\tilde{\mathfrak{g}}) \cong U_q(\tilde{\mathfrak{n}}^-) \otimes U_q(\tilde{\mathfrak{h}}) \otimes U_q(\tilde{\mathfrak{n}}^+).$$

For $i \in I, r \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$, define $(x_{i,r}^{\pm})^{(k)} = \frac{(x_{i,r}^{\pm})^k}{[k]!}$. Define also elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$ by

$$\sum_{r=0}^{\infty} \Lambda_{i,\pm r} u^r = \exp\left(- \sum_{s=1}^{\infty} \frac{h_{i,\pm s}}{[s]} u^s \right).$$

The elements $\Lambda_{i,\pm r}$ together with $k_i^{\pm 1}$, $i \in I$, $r \in \mathbb{Z}$, generate $U_q(\tilde{\mathfrak{h}})$ as an algebra.

Remark 1.2.1. The algebra $U_q(\tilde{\mathfrak{g}})$ can also be described by the Chevalley-Kac generators $x_i^{\pm}, k_i^{\pm 1}, i \in I \cup \{0\}$. We shall note use this presentation here.

1.3. The ℓ-weight lattice

Consider the multiplicative group $\mathcal{P}$ of n-tuples of rational functions $\boldsymbol{\mu} = (\boldsymbol{\mu}_1(u), \dots, \boldsymbol{\mu}_n(u))$ with values in $\mathbb{C}$ such that $\boldsymbol{\mu}_i(0) = 1$ for all $i \in I$, which is called the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}})$. Given $a \in \mathbb{C}^{\times}$ and $i \in I$, let $\boldsymbol{\omega}_{i,a} \in \mathcal{P}$ be defined by

$$(\boldsymbol{\omega}_{i,a})_j(u) = 1 - \delta_{i,j} a u.$$

Clearly, $\mathcal{P}$ is the free abelian group generated by these elements which are called fundamental ℓ-weights. If

$$\boldsymbol{\mu} = \prod_{(i,a)\in I\times\mathbb{C}^\times} \boldsymbol{\omega}_{i,a}^{p_{i,a}} \tag{1.3.1}$$

we shall say that $\boldsymbol{\omega}_{i,a}$ (respectively, $\boldsymbol{\omega}_{i,a}^{-1}$) appears in $\boldsymbol{\mu}$ if $p_{i,a} > 0$ (respectively, $p_{i,a} < 0$).

Consider the group homomorphism (weight map) $\mathrm{wt} : \mathcal{P} \to P$ by setting $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$. The submonoid $\mathcal{P}^+$ of $\mathcal{P}$ consisting of n-tuples of polynomials is called the set of dominant ℓ-weights or Drinfeld polynomials of $U_q(\tilde{\mathfrak{g}})$. Given $\boldsymbol{\omega} \in \mathcal{P}^+$ with $\boldsymbol{\omega}_i(u) = \prod_j (1 - a_{i,j}u)$, where $a_{i,j} \in \mathbb{C}$, let $\boldsymbol{\omega}^- \in \mathcal{P}^+$ be defined by $\boldsymbol{\lambda}_i^-(u) = \prod_j (1 - a_{i,j}^{-1}u)$. We will also use the notation $\boldsymbol{\omega}^+ = \boldsymbol{\omega}$. Given $\boldsymbol{\mu} \in \mathcal{P}$, say $\boldsymbol{\mu} = \boldsymbol{\omega}\boldsymbol{\varpi}^{-1}$ with $\boldsymbol{\omega}, \boldsymbol{\varpi} \in \mathcal{P}^+$, define a $\mathbb{C}$-algebra homomorphism $\boldsymbol{\Psi}_{\boldsymbol{\mu}} : U_q(\tilde{\mathfrak{h}}) \to \mathbb{C}$ by setting $\boldsymbol{\Psi}_{\boldsymbol{\mu}}(k_i^{\pm 1}) = q_i^{\pm \mathrm{wt}(\boldsymbol{\mu})(h_i)}$ and

$$\sum_{r\geq 0} \boldsymbol{\Psi}_{\boldsymbol{\mu}}(\Lambda_{i,\pm r})u^r = \frac{(\boldsymbol{\omega}^\pm)_i(u)}{(\boldsymbol{\varpi}^\pm)_i(u)}.$$

One easily checks that the map $\boldsymbol{\Psi} : \mathcal{P} \to (U_q(\tilde{\mathfrak{h}}))^*$ given by $\boldsymbol{\mu} \mapsto \boldsymbol{\Psi}_{\boldsymbol{\mu}}$ is injective. From now on we will identify $\mathcal{P}$ with its image in $(U_q(\tilde{\mathfrak{h}}))^*$ under $\boldsymbol{\Psi}$.

It will be convenient to introduce the following notation. Given $i \in I, a \in \mathbb{C}^\times, m \in \mathbb{Z}_{\geq 0}$, define

$$\boldsymbol{\omega}_{i,a,m} = \prod_{j=0}^{m-1} \boldsymbol{\omega}_{i,aq^{m-1-2j}}.$$

1.4. Finite-dimensional representation of quantum simple Lie algebras

For the sake of fixing notation, we now review some basic facts about the representation theory of $U_q(\mathfrak{g})$. For the details see [**9**] for instance.

Given a $U_q(\mathfrak{g})$-module V and $\mu \in P$, let

$$V_\mu = \{v \in V : k_i v = q^{\mu(h_i)}v \text{ for all } i \in I\}.$$

A nonzero vector $v \in V_\mu$ is called a weight vector of weight μ. If v is a weight vector such that $x_i^+ v = 0$ for all $i \in I$, then v is called a highest-weight vector. If V is generated by a highest-weight vector of weight λ, then V is said to be a highest-weight module of highest weight λ. A $U_q(\mathfrak{g})$-module V is said to be a weight module if $V = \bigoplus_{\mu\in P} V_\mu$. Denote by $\mathcal{C}_q$ be the category of all finite-dimensional weight modules of $U_q(\mathfrak{g})$. The following theorem summarizes the basic facts about $\mathcal{C}_q$.

THEOREM 1.4.1. *Let V be an object of $\mathcal{C}_q$. Then:*

(a) $\dim V_\mu = \dim V_{w\mu}$ *for all* $w \in \mathcal{W}$.
(b) V *is completely reducible.*
(c) *For each* $\lambda \in P^+$ *the* $U_q(\mathfrak{g})$*-module* $V_q(\lambda)$ *generated by a vector* v *satisfying*

$$x_i^+ v = 0, \qquad k_i v = q^{\lambda(h_i)}v, \qquad (x_i^-)^{\lambda(h_i)+1}v = 0, \quad \forall\, i \in I,$$

is irreducible and finite-dimensional. If $V \in \mathcal{C}_q$ *is irreducible, then* V *is isomorphic to* $V_q(\lambda)$ *for some* $\lambda \in P^+$. $\square$

1.5. Finite-dimensional representations of quantum loop algebras

Let V be a $U_q(\tilde{\mathfrak{g}})$-module. We say that a nonzero vector $v \in V$ is an ℓ-weight vector if there exists $\boldsymbol{\omega} \in \mathcal{P}$ and $k \in \mathbb{Z}_{>0}$ such that $(\eta - \boldsymbol{\Psi}_{\boldsymbol{\omega}}(\eta))^k v = 0$ for all $\eta \in U_q(\tilde{\mathfrak{h}})$. In that case, $\boldsymbol{\omega}$ is said to be the ℓ-weight of v. V is said to be an ℓ-weight module if every vector of V is a linear combination of ℓ-weight vectors. In that case, let $V_{\boldsymbol{\omega}}$ denote the subspace spanned by all ℓ-weight vectors of ℓ-weight $\boldsymbol{\omega}$. An ℓ-weight vector v is said to be a highest-ℓ-weight vector if $\eta v = \boldsymbol{\Psi}_{\boldsymbol{\omega}}(\eta) v$ for every $\eta \in U_q(\tilde{\mathfrak{h}})$ and $x_{i,r}^+ v = 0$ for all $i \in I$ and all $r \in \mathbb{Z}$. V is said to be a highest-ℓ-weight module if it is generated by a highest-ℓ-weight vector. Denote by $\widetilde{\mathcal{C}}_q$ the category of all finite-dimensional ℓ-weight modules of $U_q(\tilde{\mathfrak{g}})$. Quite clearly $\widetilde{\mathcal{C}}_q$ is an abelian category.

Observe that if $V \in \widetilde{\mathcal{C}}_q$, then $V \in \mathcal{C}_q$ and

$$V_\lambda = \bigoplus_{\boldsymbol{\omega}:\mathrm{wt}(\boldsymbol{\omega})=\lambda} V_{\boldsymbol{\omega}}. \tag{1.5.1}$$

Moreover, if V is a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega}$, then

$$\dim(V_{\mathrm{wt}(\boldsymbol{\omega})}) = 1 \qquad \text{and} \qquad V_\mu \neq 0 \Rightarrow \mu \leq \mathrm{wt}(\boldsymbol{\omega}). \tag{1.5.2}$$

The next proposition is easily established using (1.5.2).

Proposition 1.5.1. If V is a highest-ℓ-weight module, then it has a unique proper submodule and, hence, a unique irreducible quotient. □

Given $\boldsymbol{\omega} \in \mathcal{P}^+$, the Weyl module $W_q(\boldsymbol{\omega})$ is the $U_q(\tilde{\mathfrak{g}})$-module defined by the quotient of $U_q(\tilde{\mathfrak{g}})$ by the left ideal generated by the elements $x_{i,r}^+, (x_{i,r}^-)^{\mathrm{wt}(\boldsymbol{\omega})(h_i)+1}$, and $\eta - \boldsymbol{\Psi}_{\boldsymbol{\omega}}(\eta)$ for every $i \in I, r \in \mathbb{Z}$, and $\eta \in U_q(\tilde{\mathfrak{h}})$. In particular, it is a highest-ℓ-weight module. Denote by $V_q(\boldsymbol{\omega})$ the irreducible quotient of $W_q(\boldsymbol{\omega})$. The next theorem was proved in [**13**] and recovers the classification of the simple objects of $\widetilde{\mathcal{C}}_q$ obtained previously in [**8**].

Theorem 1.5.2. For every $\boldsymbol{\omega} \in \mathcal{P}^+$ the module $W_q(\boldsymbol{\omega})$ is nonzero and, moreover, it is the universal finite-dimensional $U_q(\tilde{\mathfrak{g}})$-module with highest ℓ-weight $\boldsymbol{\omega}$. Every simple object of $\widetilde{\mathcal{C}}_q$ is highest-ℓ-weight. □

1.6. Minimal affinizations

We now review the notion of minimal affinizations of an irreducible $U_q(\mathfrak{g})$-module introduced in [**1**].

Given $\lambda \in P^+$, a $U_q(\tilde{\mathfrak{g}})$-module V is said to be an affinization of $V_q(\lambda)$ if there exists an isomorphism of $U_q(\mathfrak{g})$-module,

$$V \cong V_q(\lambda) \oplus \bigoplus_{\mu < \lambda} V_q(\mu)^{\oplus m_\mu(V)} \tag{1.6.1}$$

for some $m_\mu(V) \in \mathbb{Z}_{\geq 0}$. Two affinizations of $V_q(\lambda)$ are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. Notice that a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega} \in \mathcal{P}^+$ is an affinization of $V_q(\lambda)$ if and only if $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$.

The partial order on P^+ induces a natural partial order on the set of (equivalence classes of) affinizations of $V_q(\lambda)$. Namely, if V and W are affinizations of $V_q(\lambda)$, say that $V \leq W$ if one of the following conditions hold:

(i) $m_\mu(V) \leq m_\mu(W)$ for all $\mu \in P^+$;
(ii) for all $\mu \in P^+$ such that $m_\mu(V) > m_\mu(W)$ there exists $\nu > \mu$ such that $m_\nu(V) < m_\nu(W)$.

A minimal element of this partial order is said to be a minimal affinization. Clearly, a minimal affinization of $V_q(\lambda)$ must be irreducible as a $U_q(\tilde{\mathfrak{g}})$-module and, hence, is of the form $V_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in \mathcal{P}^+$ such that $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$.

Given $i, j \in I, i \leq j$, and $\lambda \in P$, set

$$_i|\lambda|_j = \sum_{k=i}^{j} \lambda(h_k).$$

If $i = 1$, we simplify notation and write $|\lambda|_j$ instead of $_1|\lambda|_j$ and similarly if $j = n$. For $i > j$ we set $_i|\lambda|_j = 0$. Set also

$$p_{i,j}(\lambda) = \; _{i+1}|\lambda|_j + \; _i|\lambda|_{j-1} + j - i \quad \text{if} \quad i \leq j$$

and $p_{i,j}(\lambda) = p_{j,i}(\lambda)$ if $j < i$. Notice that, if $i = j$, then $p_{i,j}(\lambda) = 0$ and

$$p_{i,j}(\lambda) = \lambda(h_i) + \lambda(h_j) + 2 \; _{i+1}|\lambda|_{j-1} + j - i \qquad \text{if} \quad i < j.$$

THEOREM 1.6.1. [**10**] Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ and let $\lambda \in P^+$. Then, $V_q(\lambda)$ has a unique class of minimal affinization. Moreover, $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$ if and only if there exist $a_i \in \mathbb{C}^\times, i \in I$, and $\epsilon = \pm 1$ such that

$$\boldsymbol{\omega} = \prod_{i \in I} \boldsymbol{\omega}_{i, a_i, \lambda(h_i)} \qquad \text{with} \qquad \frac{a_i}{a_j} = q^{\epsilon p_{i,j}(\lambda)} \qquad \text{for all} \quad i < j. \tag{1.6.2}$$

In that case, $V_q(\boldsymbol{\omega}) \cong V_q(\lambda)$ as representations of $U_q(\mathfrak{g})$. □

Notice that (1.6.2) is equivalent to saying that there exist $a \in \mathbb{C}^\times$ and $\epsilon = \pm 1$ such that

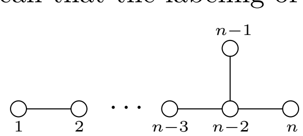
$$\boldsymbol{\omega} = \prod_{i \in I} \boldsymbol{\omega}_{i, a_i, \lambda(h_i)} \qquad \text{with} \qquad a_i = a q^{\epsilon p_{i,n}(\lambda)} \qquad \text{for all} \quad i \in I. \tag{1.6.3}$$

Notice that if $\#\ \mathrm{supp}(\lambda) > 1$, the pair (a, ϵ) in (1.6.3) is unique. In that case, If $\boldsymbol{\omega}$ satisfies (1.6.2) with $\epsilon = 1$, we say that $V_q(\boldsymbol{\omega})$ is a decreasing minimal affinization (because the powers of q in (1.6.3) decrease as i increases). Otherwise, we say $V_q(\boldsymbol{\omega})$ is a increasing minimal affinization. However, if $\#\ \mathrm{supp}(\lambda) = 1$, $\boldsymbol{\omega}$ can be represented in the form (1.6.3) by two choices of pairs (a, ϵ), one for each value of ϵ. We do not fix a preferred presentation in that case. The minimal affinizations (for $\mathfrak{g}$ of any type) satisfying $\#\ \mathrm{supp}(\lambda) \leq 1$ are called Kirillov-Reshetikhin modules.

The classification of the elements $\boldsymbol{\omega} \in \mathcal{P}^+$ such that $V_q(\boldsymbol{\omega})$ is a minimal affinization is also complete in types B, C, F, and G [**11**]. In particular, in these cases, for every $\lambda \in P^+$, there exists exactly one equivalence class of minimal affinizations. For types D and E the classification is not complete and we recall what is known about the classification in Theorem 2.6.5 below (see also Remark 1.7.2). In the next section, we state our main results which complete the classification of minimal affinizations for type D.

1.7. The main theorems

Assume $\mathfrak{g}$ is of type D_n and recall that the labeling of the diagram is given by:



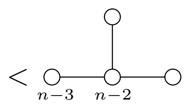
We need to prepare some notation for stating our main results. Fix $\lambda \in P^+$ such that $\lambda(h_{n-2}) = 0$ and $\overline{\operatorname{supp}}(\lambda)$ contains the subdiagram of type D_4. Fix also $\epsilon = \pm 1$. To simplify notation, set $p_{i,j}(\lambda) = p_{i,j}$, $i, j \in I$,

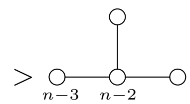
$$A = \{i \in I : i < n-2\}, \quad i_\lambda = \min\{i \in I : \lambda(h_i) \neq 0\}, \quad \text{and} \quad f_\lambda = \max\{i \in A : \lambda(h_i) \neq 0\}.$$

Let $a, a_{n-1}, a_n \in \mathbb{C}^\times$ and

$$\boldsymbol{\omega} = \prod_{i \in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)} \qquad \text{with} \qquad a_i = a q^{\epsilon p_{i,n-3}} \qquad \text{for all} \quad i \leq n-2. \tag{1.7.1}$$

We shall graphically denote the value of ϵ in (1.7.1) by the pictures



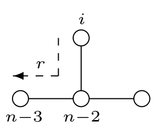


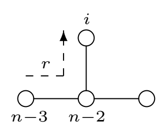
according to whether $\epsilon = -1$ or $\epsilon = 1$, respectively. Notice that, if $\#(\operatorname{supp}(\lambda) \cap A) = 1$, then the same $\boldsymbol{\omega}$ can be represented by either choices of ϵ (with different values of a if $n-3 \notin \operatorname{supp}(\lambda)$).

We now state the relevant conditions on a/a_i for $i = n, n-1$. The condition

$$a = a_i q^{\pm(\lambda(h_i) + \lambda(h_{n-3}) + 4 - 2r)} \qquad \text{for some} \qquad 1 \leq r \leq \min\{|\lambda|_{n-2}, \lambda(h_i)\} \tag{1.7.2}$$

will be indicated graphically by the following pictures





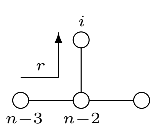
where the first is for the $-$ sign and the second for the $+$ sign. The condition

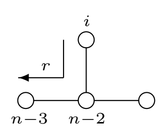
$$a_i = a q^{\pm(\lambda(h_i) + \lambda(h_{n-3}) + 2(n-1-i_\lambda - r + {}_{j}|\lambda|_{n-4}))} \tag{1.7.3}$$

for some pair (r, j) such that

$$1 \leq r - (j - i_\lambda + |\lambda|_{j-1}) \leq \min\{\lambda(h_j), \lambda(h_i)\} \qquad \text{and} \qquad i_\lambda \leq j < n-2,$$

will be indicated graphically by the following pictures





where the first is for the $-$ sign and the second for the $+$ sign. Notice that j is uniquely determined from the value of r (and it is necessarily an element of $\operatorname{supp}(\lambda)$). Therefore, we shall denote by i_r the value of j determined by r in (1.7.3). Also, if $i_\lambda = n-3$, (1.7.3) is equivalent to (1.7.2). More generally, if $\#(\operatorname{supp}(\lambda) \cap A) = 1$ and the sign of ϵ in (1.7.1) is the same one as on the right hand side of (1.7.3), then (1.7.3) becomes

$$a_{i_\lambda} = a_i q^{\pm(\lambda(h_{i_\lambda}) + \lambda(h_i) + n + 1 - i_\lambda - 2r)} \qquad \text{for some} \qquad 1 \leq r \leq \min\{\lambda(h_{i_\lambda}), \lambda(h_i)\}. \tag{1.7.4}$$

Similarly, if the sign of ϵ is the same one as on the right hand side of (1.7.2), then (1.7.2) becomes (1.7.4).

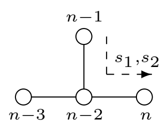
We also consider the following condition on a_n/a_{n-1}:

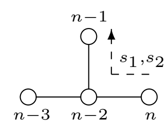
$$a_n = a_{n-1}q^{\pm(\lambda(h_{n-1})+\lambda(h_n)+4s_1-2s_2)} \tag{1.7.5}$$

for some $s_1, s_2 \in \mathbb{Z}$ such that

$$2 \le 2s_1 \le n - f_\lambda - 1 \qquad \text{and} \qquad 1 \le s_2 \le \min\{\lambda(h_{n-1}), \lambda(h_n)\}.$$

Condition (1.7.5) will be indicated graphically by the following pictures



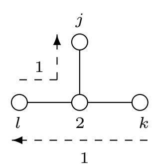


where the first is for the $-$ sign and the second for the $+$ sign. Notice that, if $s_1 = 1$ (which is always the case if $n = 4$), then (1.7.5) is the analogue of (1.7.2) for the spin nodes. We shall often omit s_1 from the above pictures in the case it is 1. If $n = 4$, we shall regard all three extreme nodes as spin nodes and we use the picture associated to (1.7.5) to express the ration a_i/a_j for $i, j \in \{1, 3, 4\}$.

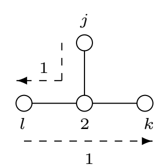
We are ready to state the main theorems of the thesis.

THEOREM 1.7.1. Let $\mathfrak{g}$ be of type D_4. Then, $\boldsymbol{\omega} \in \mathcal{P}^+$ is such that $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$ if and only if $\boldsymbol{\omega}$ is of the form (1.7.1) with the parameters a_i related by one of the conditions.

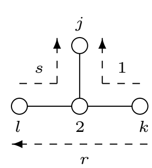
(a)$_l$ The parameters are related by either of the following pictures where $\{j, k, l\} = \{1, 3, 4\}$:



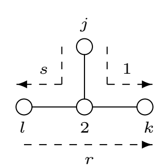
or



(b)$_l^r$ The parameters are related by either of the following pictures where $\{j, k, l\} = \{1, 3, 4\}$ with $j > k$ and $s = \lambda(h_l) + 3 - r$:

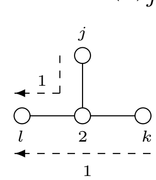


or



Moreover, these conditions parameterize the distinct equivalence classes of minimal affizations. In particular, the two pictures listed for each condition give rise to equivalent affinizations.

REMARK 1.7.2. Theorem 1.7.1 was mostly proved in [**12**]. However, it was conjectured there that an affinization satisfying condition (a)$_l$ (denoted (a)$_{j,k}$ there) was equivalent to an affinization corresponding to

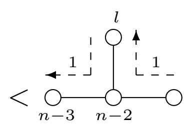


This is actually false and we prove that the affinizations satisfying the latter conditions are not minimal (see Proposition 4.6.4).

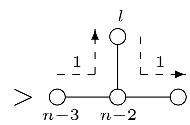
For the next two theorems we assume $n > 4$.

THEOREM 1.7.3. Suppose $\#(\mathrm{supp}(\lambda) \cap A) = 1$. Then, $\boldsymbol{\omega} \in \mathcal{P}^+$ is such that $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$ if and only if $\boldsymbol{\omega}$ is of the form (1.7.1) with the parameters a_i related by one of the following conditions.

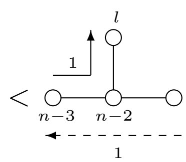
$(\mathrm{a})_l$ The parameters are related by either of the following pictures where l is a spin node:



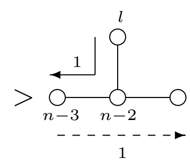
or



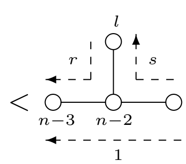
$(\mathrm{b})_l$ The parameters are related by either of the following pictures where l is a spin node:



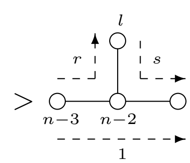
or



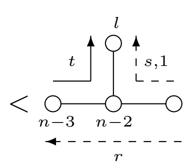
$(\mathrm{c})_l^r$ The parameters are related by either of the following pictures where l is a spin node and $s = \lambda(h_l) + 3 - r$:



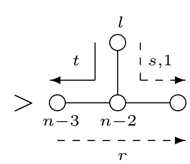
or



$(\mathrm{d})_l^{r,s}$ The parameters are related by either of the following pictures:



or



where l is a spin node and $t = \lambda(h_{i_\lambda}) + n - i_\lambda + 2 - 2s - r$, and, if $r \le t$ and $\{l, l'\} = \{n-1, n\}$, then

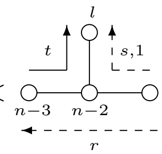
$$\lambda(h_l) + 3 - r > \min\{l, \lambda(h_{l'})\}.$$

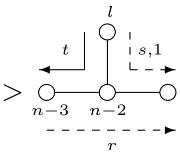
Moreover, these conditions parameterize the distinct equivalence classes of minimal affizations. In particular, the two pictures listed for each condition give rise to equivalent affinizations.

REMARK 1.7.4. Notice that condition $(\mathrm{a})_l$ of Theorem 1.7.3 degenerates to that of Theorem 1.7.1 when $n = 4$. Similarly, condition $(\mathrm{b})_l$ degenerates to condition $(\mathrm{a})_1$, condition $(\mathrm{c})_l^r$ degenerates to condition $(\mathrm{b})_l^r$, and condition $(\mathrm{d})_l^{r,s}$ degenerates to condition $(\mathrm{b})_1^t$ if $l = n-1$ and to condition $(\mathrm{b})_1^r$ if $l = n$.

THEOREM 1.7.5. Suppose $\#(\mathrm{supp}(\lambda) \cap A) > 1$. Then, $\boldsymbol{\omega} \in \mathcal{P}^+$ is such that $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$ if and only if $\boldsymbol{\omega}$ is of the form (1.7.1) with the parameters a_i related by either one of the conditions $(\mathrm{a})_l$, $(\mathrm{b})_l$, $(\mathrm{c})_l^r$ in Theorem 1.7.3 or by the following conditions.

$(\mathrm{d})_l^{r,s}$ The parameters are related by either of the following pictures:

$<$ 

or
$>$ 

where l is a spin node and $k = \lambda(h_{i_\lambda}) - |\lambda|_{f_\lambda - 1} + n - f_\lambda + 2 - 2s - r$, $t = |\lambda|_{f_\lambda-1} + f_\lambda - i_1 + k$ and, letting $= \{l, l'\} = \{n-1, n\}$, the parameters satisfy the following conditions:

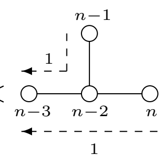
(i) If $r \leq k$, then $\lambda(h_l) + 3 - r > \min\{l, \lambda(h_{l'})\}$.

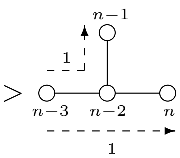
(ii) If $p \in \mathrm{supp}(\lambda) \cap A \setminus \{f_\lambda\}$, $0 \leq k' < k$ and $1 \leq s' \leq s$ are such that

$$1 \leq k'' := {}_{p}|\lambda|_{f_\lambda} + n - p + 2 - r - 2s - 2s' - k' - \lambda(h_l) \leq \min\{\lambda(h_p), \lambda(h_{l'})\},$$

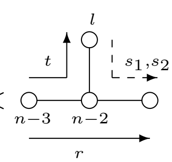
then $r - {}_{p}|\lambda|_{f_\lambda} \leq k''$.

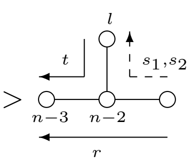
(e) The parameters are related by either of the following pictures:

$<$ 

or
$>$ 

(f)$_l^{r,s_1,s_2}$ The parameters are related by either of the following pictures

$<$ 

or
$>$ 

where l is a spin node, $t = \lambda(h_l) + 2s_1 + r - s_2$, and, letting $\{l, l'\} = \{n-1, n\}$, $k = t - (i_t - i_\lambda + |\lambda|_{i_t - 1})$ and $\bar{k} = r - (i_r - i_\lambda + |\lambda|_{i_r-1})$, the parameters satisfy:

(i) $\bar{k} = 1$, $s_1 > 1$, $s_2 \leq 2$, $k \leq 2$, with $k = 1$ if $s_2 = 2$. Besides, if $i_r < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < \max\{k, s_2\}$ such that $i_r < \bar{i} \leq i \leq f_\lambda$, $i_t \leq i$ and $0 \leq \bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_t - {}_{i_t}|\lambda|_{i-1} + k'$ is even, then

$$s_1 < \frac{1}{2}(\bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_t - {}_{i_t}|\lambda|_{i-1} + k') + 1.$$

Moreover, if $i_t < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < \max\{k, s_2\}$ such that $i_t \leq i < \bar{i} \leq f_\lambda$ and $\bar{i} - i + {}_{i}|\lambda|_{\bar{i}-1} + 2s_1 - k + k' + 2 - \lambda(h_{l'})$ is even then

either $\frac{1}{2}(\bar{i} - i + {}_{i}|\lambda|_{\bar{i}-1} + 2s_1 - k + k' + 2 - \lambda(h_{l'})) < s_1$ or $\frac{1}{2}(\bar{i} - i + {}_{i}|\lambda|_{\bar{i}-1} + 2s_1 - k + k' + 2 - \lambda(h_{l'})) > 2s_1$.

(ii) $\bar{k} = s_1 = 1$, $i_r > i_r$ and

$$k < \lambda(h_{f_\lambda}) + i_r - i_r + n - f_\lambda - 1.$$

Besides, if $i_r < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < k$ such that $i_r \leq \bar{i} < i \leq f_\lambda$; $i_r < i$ and $\bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} \leq i - i_r + {}_{i_r}|\lambda|_{i-1} - k'$, then

$$s_2 < i - i_r + {}_{i_r}|\lambda|_{i-1} - k' - \bar{i} + i_r - {}_{i_r}|\lambda|_{\bar{i}-1} + 1.$$

Moreover, if $i_r < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < k$ such that $i_r \leq i < \bar{i} \leq f_\lambda$ then

either $\lambda(h_n) + 1 + k - k' - \bar{i} + i - {}_{i}|\lambda|_{\bar{i}-1} < 1$ or $\lambda(h_n) + 1 + k - k' - \bar{i} + i - {}_{i}|\lambda|_{\bar{i}-1} > s_2$.

(iii) $\bar{k} > 1$, $k = s_2 = 1$. Besides, if $i_r < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $0 \le s_1' < s_1$ such that $i_r < \bar{i} \le i \le f_\lambda$; $i_r \le i$ with either $l' > 0$ or $i_r < i$, and $i - i_r + {}_{i_r}|\lambda|_{i-1} + 2s_1' \le \bar{i} - i_r + {}_{\bar{i}}|\lambda|_{i_r-1}$, then

either $\bar{k} < \bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_r - {}_{i_r}|\lambda|_{i-1} - 2s_1' + 1$ or $\bar{k} > \bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_r - {}_{i_r}|\lambda|_{i-1} - 2s_1' + \lambda(h_{\bar{i}})$.

Moreover, if $i_r < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, \bar{k} - \lambda(h_{\bar{i}})\} \le \bar{k}' < \bar{k}$ such that $i_r \le i \le \bar{i} \le f_\lambda$, $\bar{i} > i_r$ and $-\lambda(h_n) + \bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_r - {}_{i_r}|\lambda|_{i-1} + k'$ is even, then

$$\text{either} \quad s_1 > \frac{1}{2}(-\lambda(h_n) + \bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_r - {}_{i_r}|\lambda|_{i-1} + k')$$

$$\text{or} \quad \frac{1}{2}(-\lambda(h_n) + \bar{i} - i_r + {}_{i_r}|\lambda|_{\bar{i}-1} - i + i_r - {}_{i_r}|\lambda|_{i-1} + k') > 2s_1 - 1.$$

Moreover, these conditions parameterize the distinct equivalence classes of minimal affinizations. In particular, the two pictures listed for each condition give rise to equivalent affinizations.

REMARK 1.7.6. The extra conditions (i) and (ii) listed within family (d) assure that the corresponding modules $V_q(\boldsymbol{\omega})$ are not larger (in the sense of affinizations) than any of the modules corresponding to families (c) and (f), respectively. The three extra conditions listed within family (f) assure that there is no other element in the same family which is smaller.

CHAPTER 2

Further notation and concepts

In this chapter we collect several previously proved results which will be used in the proofs of our main results. In particular, we recall several results concerning the theory of qcharacters which will be our main tool later on.

2.1. Hopf algebra structure and duality

The following assignments,

$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+, \quad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-, \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$
$$S(x_i^+) = -k_i^{-1} x_i^+, \quad S(x_i^-) = -x_i^- k_i, \quad S(k_i^{\pm 1}) = k_i^{\mp 1},$$
$$\varepsilon(x_i^{\pm}) = 0, \quad \varepsilon(k_i^{\pm 1}) = 1,$$

for all $i \in I$, define a structure of Hopf algebra in $U_q(\mathfrak{g})$, where Δ is the co-multiplication, ε is the co-unity and S is the antipode. The algebra $U_q(\tilde{\mathfrak{g}})$ is also a Hopf algebra and the structure maps can be described exactly as above using the Chevalley-Kac generators. However, a precise expression for the comultiplication in terms of the generators $x_{i,r}^{\pm}, h_{i,r}, k_i^{\pm 1}$ is not known (see [**2**] and references therein).

Given a finite-dimensional $U_q(\mathfrak{g})$-module V, let V^* denote the dual module defined using the antipode as usual, and similarly for $U_q(\tilde{\mathfrak{g}}^*)$. Given $\lambda \in P^+$ and $\boldsymbol{\omega} \in \mathcal{P}_q^+$, we have:

$$V_q(\lambda)^* \cong V_q(\lambda^*) \qquad \text{and} \qquad V_q(\boldsymbol{\omega})^* \cong V_q(\boldsymbol{\omega}^*) \tag{2.1.1}$$

where

$$\lambda^* = -w_0\lambda \qquad \text{and} \qquad (\boldsymbol{\omega}^*)_i(u) = \boldsymbol{\omega}_{w_0 \cdot i}(q^{-h^\vee} u). \tag{2.1.2}$$

Here, $w_0 \cdot i = j$ iff $w_0\omega_i = -\omega_j$ and $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. For a proof of the second isomorphism in (2.1.1) see [**17**].

We will also need a different kind of duality established by using the Cartan involution: the unique algebra automorphism ω of $U_q(\mathfrak{g})$ such that $x_i^{\pm} \mapsto -x_i^{\mp}$, $k_i^{\pm 1} \mapsto k_i^{\mp 1}$ for all $i \in I$. Given a $U_q(\mathfrak{g})$-module V, denote by V^ω the pull-back of V by ω. It is not difficult to see that a highest-weight vector of $V_q(\lambda)$ is a lowest-weight vector of $V_q(\lambda)^\omega$ and, hence, $V_q(\lambda)^\omega \cong V_q(\lambda)^*$. The situation is more interesting in the affine case. The following proposition is easily established.

Proposition 2.1.1.

(a) There is a unique algebra automorphism $\hat{\omega}$ of $U_q(\tilde{\mathfrak{g}})$ given by

$$\hat{\omega}(x_{i,r}^{\pm}) = -x_{i,-r}^{\mp}, \qquad \hat{\omega}(h_{i,r}) = -h_{i,r}, \qquad \hat{\omega}(\phi_{i,r}^{\pm}) = \phi_{i,-r}^{\mp}, \qquad \hat{\omega}(k_i^{\pm 1}) = k_i^{\mp 1}.$$

Moreover, we have

$$(\hat{\omega} \otimes \hat{\omega}) \circ \Delta = \Delta^{\mathrm{op}} \circ \hat{\omega},$$

where Δ^{op} is the opposite comultiplication of $U_q(\tilde{\mathfrak{g}})$.

(b) For all $a \in \mathbb{C}^\times$, there exists a Hopf algebra automorphism τ_a of $U_q(\tilde{\mathfrak{g}})$ such that

$$\tau_a(x_{i,r}^\pm) = a^r x_{i,r}^\pm, \qquad \tau_a(h_{i,r}) = a^r h_{i,r}, \qquad \tau_a(k_i^{\pm 1}) = k_i^{\pm 1}.$$

□

Given a $U_q(\tilde{\mathfrak{g}})$-module V, denote by $V^{\hat{\omega}}$ the pull-back of V by $\hat{\omega}$ and by $V(a)$ the pull-back of V by τ_a. It is easy to see that, if V is highest-ℓ-weight module with highest ℓ-weight $\boldsymbol{\omega}$, then $V(a)$ is also a highest-ℓ-weight module with the highest ℓ-weight $\boldsymbol{\omega}^a$ determined by

$$\boldsymbol{\omega}_i^a = \boldsymbol{\omega}_i(au). \tag{2.1.3}$$

Moreover, a highest-ℓ-weight vector of $V_q(\boldsymbol{\omega})$ is a lowest-ℓ-weight vector of $V_q(\boldsymbol{\omega})^{\hat{\omega}}$ and

$$\hat{\omega}(\Lambda_i^\pm(u)) = (\Lambda_i^\mp(u))^{-1} \quad \text{for all} \quad i \in I. \tag{2.1.4}$$

It follows from (2.1.1) and usual duality arguments for Hopf algebras that, the lowest ℓ-weight of $V_q(\boldsymbol{\omega})$ is $(\boldsymbol{\omega}^a)^{-1}$ with $a = q^{h^\vee}$. In particular,

$$V_q(\boldsymbol{\omega})^{\hat{\omega}} \cong V_q({}^*\boldsymbol{\omega}) \tag{2.1.5}$$

where ${}^*\boldsymbol{\omega}$ is defined by requiring that

$$({}^*\boldsymbol{\omega})^+ = ((\boldsymbol{\omega}^*)^{2h^\vee})^-. \tag{2.1.6}$$

The following proposition is well-known and is easily proved by considering pull-backs by the morphisms given by Proposition 2.1.1.

Proposition 2.1.2. *Let $\boldsymbol{\omega} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,m_i}$ and $\boldsymbol{\varpi} = \prod_{i \in I} \boldsymbol{\omega}_{i,b_i,m_i}$, with $a_i, b_i \in \mathbb{C}^\times$ and $m_i \in \mathbb{Z}_{\geq 0}$. If*

$$\frac{a_i}{a_j} = \frac{b_j}{b_i} \qquad \text{for all} \qquad i, j \in I,$$

then $V_q(\boldsymbol{\omega}) \cong_{U_q(\mathfrak{g})} V_q(\boldsymbol{\varpi})$. □

2.2. The ℓ-root lattice

Given $i \in I, a \in \mathbb{C}^\times, m \in \mathbb{Z}_{\geq 0}$, following [**4**], define

$$\boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,aq,2} \prod_{j \neq i} \boldsymbol{\omega}_{j,aq,-c_{j,i}}^{-1}.$$

We shall refer to $\boldsymbol{\alpha}_{i,a}$ as a simple ℓ-root. The subgroup of $\mathcal{P}$ generated by the simple ℓ-roots is called the ℓ-root lattice of $U_q(\tilde{\mathfrak{g}})$ and will be denoted by $\mathcal{Q}$. Let also $\mathcal{Q}^+$ be the submonoid generated by the simple ℓ-roots. Quite clearly $\mathrm{wt}(\boldsymbol{\alpha}_{i,a}) = \alpha_i$. Define a partial order on $\mathcal{P}$ by

$$\boldsymbol{\mu} \leq \boldsymbol{\omega} \qquad if \qquad \boldsymbol{\omega}\boldsymbol{\mu}^{-1} \in \mathcal{Q}^+.$$

It is well-known that the elements $\boldsymbol{\alpha}_{i,a}$ are multiplicatively independent, i.e., if $(i_j, a_j), j = 1, \ldots, m$, is a family of distinct elements of $I \times \mathbb{C}^\times$, then

$$\prod_{j=1}^m \boldsymbol{\alpha}_{i_j,a_j}^{k_j} = 1 \;\Leftrightarrow\; k_j = 0 \text{ for all } j = 1, \ldots, m. \tag{2.2.1}$$

2.3. Diagram subalgebras and sublattices

By abuse of language, we will refer to any subset J of I as a subdiagram of the Dynkin diagram of $\mathfrak{g}$. Let $\mathfrak{g}_J$ be the Lie subalgebra of $\mathfrak{g}$ generated by $x^{\pm}_{\alpha_j}, j \in J$, and define $\mathfrak{n}^{\pm}_J, \mathfrak{h}_J$ in the obvious way. Let also Q_J be the subgroup of Q generated by $\alpha_j, j \in J$, and $R^+_J = R^+ \cap Q_J$. Given $\lambda \in P$, λ_J is the restriction of λ to $\mathfrak{h}^*_J$ and let $\lambda^J \in P$ be such that $\lambda^J(h_j) = \lambda(h_j)$ if $j \in J$ and $\lambda^J(h_j) = 0$ otherwise. Diagram subalgebras $\tilde{\mathfrak{g}}_J$ are defined in the obvious way.

Consider also the subalgebra $U_q(\tilde{\mathfrak{g}}_J)$ generated by $k_j^{\pm 1}, h_{j,r}, x^{\pm}_{j,s}$ for all $j \in J, r, s \in \mathbb{Z}, r \neq 0$. If $J = \{j\}$, the algebra $U_q(\tilde{\mathfrak{g}}_j) := U_q(\tilde{\mathfrak{g}}_J)$ is isomorphic to $U_q(\tilde{\mathfrak{sl}}_2)$. Similarly we define the subalgebra $U_q(\mathfrak{g}_J)$, etc.

For $\boldsymbol{\omega} \in \mathcal{P}$, let $\boldsymbol{\omega}_J$ be the associated J-tuple of rational functions and let $\mathcal{P}_J = \{\boldsymbol{\omega}_J : \boldsymbol{\omega} \in \mathcal{P}\}$. Similarly define $\mathcal{P}^+_J$. Notice that $\boldsymbol{\omega}_J$ can be regarded as an element of the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}}_J)$. Let $\pi_J : \mathcal{P} \to \mathcal{P}_J$ denote the map $\boldsymbol{\omega} \mapsto \boldsymbol{\omega}_J$. If $J = \{j\}$ is a singleton, we write π_j instead of π_J. An ℓ-weight $\boldsymbol{\omega} \in \mathcal{P}$ is said to be J-dominant if $\boldsymbol{\omega}_J \in \mathcal{P}^+_J$. Let also $\mathcal{Q}_J \subset \mathcal{P}_J$ be the subgroup generated by $\pi_J(\boldsymbol{\alpha}_{j,a}), j \in J, a \in \mathbb{C}^\times$. When no confusion arises, we shall simply write $\boldsymbol{\alpha}_{j,a}$ for its image in $\mathcal{P}_J$ under π_J. Let

$$\iota_J : \mathbb{Z}[\mathcal{Q}_J] \to \mathbb{Z}[\mathcal{Q}],$$

be the ring homomorphism such that $\iota_J(\boldsymbol{\alpha}_{j,a}) = \boldsymbol{\alpha}_{j,a}$ for all $j \in J, a \in \mathbb{C}^\times$. We shall often abuse of notation and identify $\mathcal{Q}_J$ with its image under ι_J. In particular, given $\boldsymbol{\mu} \in \mathcal{P}$, we set

$$\boldsymbol{\mu}\mathcal{Q}_J = \{\boldsymbol{\mu}\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \iota_J(\mathcal{Q}_J)\}.$$

It will also be useful to introduce the element $\boldsymbol{\omega}^J \in \mathcal{P}$ defined by

$$(\boldsymbol{\omega}^J)_j(u) = \boldsymbol{\omega}_j(u) \quad \text{if} \quad j \in J \quad \text{and} \quad (\boldsymbol{\omega}^J)_j(u) = 1 \quad \text{otherwise.}$$

2.4. Characters and qcharacters

Let $\mathbb{Z}[P]$ be the integral group ring over P and denote by $e : P \to \mathbb{Z}[P], \lambda \mapsto e^\lambda$, the inclusion of P in $\mathbb{Z}[P]$ so that $e^\lambda e^\mu = e^{\lambda+\mu}$. The character of an object V from $\mathcal{C}_q$ is defined by

$$\mathrm{ch}(V) = \sum_{\mu \in P} \dim(V_\mu) e^\mu.$$

For $\lambda \in P^+$, let $m_\lambda(V)$ be the multiplicity of $V_q(\lambda)$ as a simple factor of V. It is well-known that the numbers $m_\mu(V)$ can be computed from $\mathrm{ch}(V)$ and vice-versa.

Similarly, for an object V from $\widetilde{\mathcal{C}}_q$ and $\boldsymbol{\omega} \in \mathcal{P}^+$, let $m_{\boldsymbol{\omega}}(V)$ be the multiplicity of $V_q(\boldsymbol{\omega})$ as a simple factor of V. We now turn to the concept which plays a role analogous to character for the category $\widetilde{\mathcal{C}}_q$. It was introduced in [**18**] under the name of qcharacter. In particular, one can compute the multiplicities $m_{\boldsymbol{\omega}}(V)$ from the qcharacter of V.

Let $\mathbb{Z}[\mathcal{P}]$ be the integral group ring over $\mathcal{P}$. Given $\chi \in \mathbb{Z}[\mathcal{P}]$, say

$$\chi = \sum_{\boldsymbol{\mu} \in \mathcal{P}} \chi(\boldsymbol{\mu})\ \boldsymbol{\mu},$$

we identify it with the function $\mathcal{P} \to \mathbb{Z}, \boldsymbol{\mu} \to \chi(\boldsymbol{\mu})$. Conversely, any function $\mathcal{P} \to \mathbb{Z}$ with finite support can be identified with an element of $\mathbb{Z}[\mathcal{P}]$. The qcharacter of $V \in \widetilde{\mathcal{C}}_q$ is the element $\mathrm{qch}(V)$ corresponding to the function

$$\boldsymbol{\mu} \mapsto \dim(V_{\boldsymbol{\mu}}).$$

We shall denote by $\mathrm{wt}_\ell(\chi)$ the support of $\chi \in \mathbb{Z}[\mathcal{P}]$. In particular, we set

$$\mathrm{wt}_\ell(V) = \mathrm{wt}_\ell(\mathrm{qch}(V)) = \{\boldsymbol{\mu} \in \mathcal{P} : V_{\boldsymbol{\mu}} \neq 0\}.$$

Given an ℓ-weight module V and a vector subspace W of V, let $W_{\boldsymbol{\mu}} = W \cap V_{\boldsymbol{\mu}}$. We shall say that W is an ℓ-weight subspace of V if

$$W = \bigoplus_{\boldsymbol{\mu} \in \mathcal{P}} W \cap V_{\boldsymbol{\mu}}.$$

In that case, we set

$$\mathrm{qch}(W) = \sum_{\boldsymbol{\mu}\in\mathcal{P}} \dim W_{\boldsymbol{\mu}} \qquad \text{and} \qquad \mathrm{wt}_\ell(W) = \mathrm{wt}_\ell(\mathrm{qch}(W)).$$

LEMMA 2.4.1. [**22**, Lemma 5.4] Let V be an object of $\tilde{\mathcal{C}}_q$, $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V)$ and $v \in V_{\boldsymbol{\mu}}$. Then, for each $j \in I$, $U_q(\tilde{\mathfrak{g}}_j)v$ is a sub-$U_q(\tilde{\mathfrak{h}})$-module of V and $\mathrm{wt}_\ell(U_q(\tilde{\mathfrak{g}}_j)v) \subseteq \boldsymbol{\mu}\mathcal{Q}_{\{j\}}$. ☐

The next theorem was conjectured in [**18**] and proved in [**17**].

THEOREM 2.4.2. Let V be a quotient of $W_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in \mathcal{P}^+$. If $V_{\boldsymbol{\mu}} \neq 0$, then $\boldsymbol{\mu} \leq \boldsymbol{\omega}$. ☐

THEOREM 2.4.3. [**18**, Lemma 2] Let $V, W \in \widetilde{\mathcal{C}}_q$. Then $\mathrm{qch}(V \otimes W) = \mathrm{qch}(V)\mathrm{qch}(W)$. ☐

Frenkel and Mukhin proposed in [**17**] an algorithm that associates an element of $\mathbb{Z}[\mathcal{P}]$ to each $\boldsymbol{\omega} \in \mathcal{P}^+$ and conjectured that this element should be $\mathrm{qch}(V_q(\boldsymbol{\omega}))$. We shall refer to this algorithm as the FM algorithm. It is now known that the conjecture is not true for any $\boldsymbol{\omega} \in \mathcal{P}^+$. However, the following theorem proved in [**17**] has been shown to be very useful.

THEOREM 2.4.4. If $\boldsymbol{\omega} \in \mathcal{P}^+$ is such that $V_q(\boldsymbol{\omega})$ is ℓ-minuscule, then $\mathrm{qch}(V_q(\boldsymbol{\omega}))$ can be computed using the FM algorithm. ☐

REMARK 2.4.5. We refer the reader to [**36**, Section 4] for more details on the FM algorithm, including a more detailed explanation of its definition, a connection with the theory of blocks and elliptic characters, a proof of Theorem 2.4.4, and examples of its usage.

2.5. Right negative ℓ-weights

The following proposition is well-known, but, to the best of our knowledge, there is no proper proof in the literature other than the one given in what follows.

PROPOSITION 2.5.1. Let $\lambda \in P^+$ and suppose $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$. Then, there exist $a_i \in \mathbb{C}^\times, i \in I$, such that $a_i/a_j \in q^{\mathbb{Z}}$ for all $i, j \in I$ and $\boldsymbol{\omega} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}$. ☐

PROOF. After Theorems 1.6.1 and 2.6.5, it remains to consider the case that $\mathfrak{g}$ is of type D or E, $\overline{\mathrm{supp}}(\lambda)$ contains a subdiagram of type D, and the trivalent node i_0 is not in $\mathrm{supp}(\lambda)$. Let I_1, I_2, I_3 be the 3 connected components of $I \setminus \{i_0\}$ and set $J_k = I_k \cup \{i_0\} \cup I_3$ for $k = 1, 2$. It follows from Proposition 2.6.4 and Theorem 1.6.1 that there exist $a_i \in \mathbb{C}^\times, i \in I$, such that $\boldsymbol{\omega} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}$ and, moreover, $a_i/a_j \in q^{\mathbb{Z}}$ if $i, j \in I_k$ for some k. By contradiction, assume $a_i/a_j \notin q^{\mathbb{Z}}$ for $i \in I_1$ and $j \in I_2$. It is known (from the main result of [**2**], for instance) that

$$V_q(\boldsymbol{\omega}) \cong V_q(\boldsymbol{\mu}) \otimes V_q(\boldsymbol{\nu}),$$

where

$$\boldsymbol{\mu} = \begin{cases} \boldsymbol{\omega}^{I_1}, & \text{if } a_i/a_j \notin q^{\mathbb{Z}} \text{ for } i \in I_1, j \in I_3; \\ \boldsymbol{\omega}^{J_1}, & \text{otherwise.} \end{cases}$$

and $\boldsymbol{\nu}$ is such that $\boldsymbol{\mu}\boldsymbol{\nu} = \boldsymbol{\omega}$. In particular, $V_q(\boldsymbol{\omega})$ is not a prime representation. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$ and

$$V_q(\boldsymbol{\varpi}_{J_k}) \quad \text{is a minimal affinization for} \quad k = 1, 2.$$

Such $\boldsymbol{\varpi}$ clearly exists and satisfies condition (ii) in Theorem 2 of [**7**] which, in combination with [**7**, Theorem 1], implies that $V_q(\boldsymbol{\varpi})$ is prime. In particular,

$$V_q(\boldsymbol{\varpi}) \ncong V_q(\boldsymbol{\mu}') \otimes V_q(\boldsymbol{\nu}'),$$

where

$$\boldsymbol{\mu}' = \begin{cases} \boldsymbol{\varpi}^{I_1}, & \text{if } a_i/a_j \notin q^{\mathbb{Z}} \text{ for } i \in I_1, j \in I_3; \\ \boldsymbol{\varpi}^{J_1}, & \text{otherwise.} \end{cases}$$

and $\boldsymbol{\nu}$ is such that $\boldsymbol{\mu}'\boldsymbol{\nu}' = \boldsymbol{\varpi}$. Since $V_q(\boldsymbol{\mu}') \otimes V_q(\boldsymbol{\nu}') \cong V_q(\boldsymbol{\omega})$ as $U_q(\mathfrak{g})$-modules, it clearly follows that $[V_q(\boldsymbol{\varpi})] < [V_q(\boldsymbol{\omega})]$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. □

Set

$$Y_{i,s} := \boldsymbol{\omega}_{i,q^s} \qquad \text{and} \qquad A_{i,s} := \boldsymbol{\alpha}_{i,q^{s-1}}, \quad i \in I, s \in \mathbb{Z}. \tag{2.5.1}$$

We shall denote by $\mathcal{P}_{\mathbb{Z}}$ the subgroup of $\mathcal{P}$ generated by $Y_{i,s}, i \in I, s \in \mathbb{Z}$, and we similarly define the subgroup $\mathcal{Q}_{\mathbb{Z}}$ of $\mathcal{Q}$ and the monoids $\mathcal{P}_{\mathbb{Z}}^+$ and $\mathcal{Q}_{\mathbb{Z}}^+$.

Corollary 2.5.2. If $\boldsymbol{\omega} \in \mathcal{P}^+$ is such that $V_q(\boldsymbol{\omega})$ is a minimal affinization, there exists $\boldsymbol{\varpi} \in \mathcal{P}_{\mathbb{Z}}^+$ such that $V_q(\boldsymbol{\varpi})$ is an affinization equivalent to $V_q(\boldsymbol{\omega})$. □

Remark 2.5.3. The reason for defining $A_{i,s} := \boldsymbol{\alpha}_{i,q^{s-1}}$ instead of simply $A_{i,s} := \boldsymbol{\alpha}_{i,q^s}$ is to match with the notation of [**18**].

We also introduce the following notation. Given $i, j \in I, i \leq j, r \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}$, define

$$Y_{i,r,m} = \prod_{k=0}^{m-1} Y_{i,r+2k} = \boldsymbol{\omega}_{i,q^{m+r-1},m},$$

$$A_{i,j,r} = \prod_{k=i}^{j} A_{k,r+k-i+1} \quad \text{and} \quad A_{j,i,r} = \prod_{k=i}^{j} A_{j+i-k,r+k-i+1}.$$

Definition 2.5.4. [**17**] Let $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{Z}} \setminus \{\mathbf{1}\}$. Define

$$r(\boldsymbol{\omega}) := \max\{s \in \mathbb{Z} : Y_{i,s}^{\pm 1} \text{ appears in } \boldsymbol{\omega} \text{ for some } i \in I\}. \tag{2.5.2}$$

$\boldsymbol{\omega}$ is said to be right negative if $Y_{i,r(\boldsymbol{\omega})}$ does not appear in $\boldsymbol{\omega}$ for all $i \in I$.

Observe that the product of right negative ℓ-weights is a right negative ℓ-weight and a dominant ℓ-weight is not right negative.

Proposition 2.5.5. [**45**, Theorem 3.2] Given, $i \in I$, $r \in \mathbb{Z}$, and $k \in \mathbb{Z}_{\geq 0}$, all the elements of $\mathrm{wt}_\ell(V_q(Y_{i,r,k})) \setminus \{Y_{i,r,k}\}$ are right negative. Moreover, if $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(Y_{i,r,k})) \setminus \{Y_{i,r,k}\}$ is such that $r(\boldsymbol{\mu}) \leq r + 2k$, then

$$\boldsymbol{\mu} = Y_{i,r,s} Y_{i,r+2(s+1),k-s}^{-1} \prod_{j\,:\,c_{ij}=-1} Y_{j,r+2s+1,k-s} \qquad \text{for some} \quad s = 0, \ldots, k-1.$$

In particular, $r(\boldsymbol{\mu}) = r + 2k > r + 2(k-1) = r(Y_{i,r,k})$. □

2.6. Reduction to diagram subalgebras

If $J \subseteq I$ we shall denote by $V_q(\lambda_J)$ the simple $U_q(\mathfrak{g}_J)$-module of highest weight λ_J. Since $\mathcal{C}_q$ is semisimple, it is easy to see that, if $\lambda \in P^+$ and $v \in V_q(\lambda)_\lambda$ is nonzero, then $U_q(\mathfrak{g}_J)v \cong V_q(\lambda_J)$.

Lemma 2.6.1. [**10**, Lemma 2.4] Suppose $\emptyset \neq J \subseteq I$ defines a connected subdiagram of the Dynkin diagram of $\mathfrak{g}$, let V be a highest-ℓ-weight module with highest-ℓ-weight $\boldsymbol{\omega} \in \mathcal{P}^+, \lambda = \mathrm{wt}(\boldsymbol{\omega})$, $v \in V_\lambda \setminus \{0\}$, and $V_J = U_q(\tilde{\mathfrak{g}}_J)v$. Then, $m_\mu(V) = m_{\mu_J}(V_J)$ for all $\mu \in \lambda - Q_J^+$. $\square$

Keeping the notation of Lemma 2.6.1, notice that if V is irreducible, then

$$V_J \cong V_q(\boldsymbol{\omega}_J). \tag{2.6.1}$$

Hence,

$$\boldsymbol{\nu} \in \boldsymbol{\omega}\mathcal{Q}_J \;\Rightarrow\; \dim(V_q(\boldsymbol{\omega})_{\boldsymbol{\nu}}) = \dim(V_q(\boldsymbol{\omega}_J)_{\boldsymbol{\nu}_J}). \tag{2.6.2}$$

Lemma 2.6.2. [**10**, Lemma 2.6] Let $i_0 \in I$ be such that

$$I = J_1 \sqcup \{i_0\} \sqcup J_2 \quad \text{(disjoint union)}$$

where J_1 is of type A, $J_2 \sqcup \{i_0\}$ is connected and $c_{jk} = 0$ for all $j \in J_1$, $k \in J_2$. Let $\boldsymbol{\omega} \in \mathcal{P}^+, \lambda = \mathrm{wt}(\boldsymbol{\omega})$, and suppose $V_q(\boldsymbol{\omega}_{J_1})$ is a minimal affinization of $V_q(\lambda_{J_1})$. Let also

$$\mu = \lambda - \sum_{j \in I\setminus\{i_0\}} s_j\alpha_j \quad \text{with} \quad s_j \in \mathbb{Z}_{\geq 0} \quad \text{for all} \quad j \in I \setminus i_0.$$

If $m_\mu(V_q(\boldsymbol{\omega})) > 0$, then $s_j = 0$ for all $j \in J_1$. $\square$

Definition 2.6.3. Suppose $\mathfrak{g}$ is of type D or E and let $i_0 \in I$ be the trivalent node. A connected subdiagram $J \subseteq I$ is said to be admissible if J is of type A and $J\setminus\{i_0\}$ is connected. If $\mathfrak{g}$ is of type A, then any connected subdiagram is admissible.

Proposition 2.6.4. [**10**, Proposition 4.2] Let $\emptyset \neq J \subseteq I$ define an admissible subdiagram, $\boldsymbol{\omega} \in \mathcal{P}^+$, and $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$. If $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$, then $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization of $V_q(\lambda_J)$. $\square$

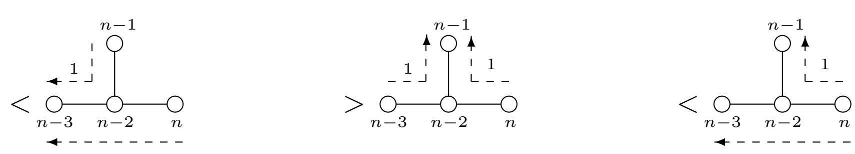
For information and comparison purposes, we recall the partial classification of the minimal affinizations for $\mathfrak{g}$ of types D or E proved in [**10**].

Theorem 2.6.5. Let $\mathfrak{g}$ be of type D or E, $\lambda \in P^+$, and $J = \overline{\mathrm{supp}}(\lambda)$. Let also $i_0 \in I$ be the trivalent node, and $I_j, j = 1, 2, 3$, be an enumeration of the 3 connected components of $I \setminus \{i_0\}$.

(i) If J is of type A, then $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$ iff $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization of $V_q(\lambda_J)$.

(ii) If J is not of type A and $i_0 \in \mathrm{supp}(\lambda)$, then $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$ iff there exist distinct $r, s \in \{1, 2, 3\}$ such that $V_q(\boldsymbol{\omega}_{I\setminus I_r})$ and $V_q(\boldsymbol{\omega}_{I\setminus I_s})$ are minimal affinizations of $V_q(\lambda_{I\setminus I_r})$ and $V_q(\lambda_{I\setminus I_s})$, respectively. $\square$

Remark 2.6.6. We can express part (ii) of this theorem in terms of the pictorial notation of Section 1.7. Namely, the 3 equivalence classes of minimal affinizations correspond to the pictures



or, equivalently, the ones with all directions inverted. Notice that such pictures do not show up in Theorems 1.7.1 and 1.7.3 and that the second and third do not show up in Theorem 1.7.5 as well.

If $\boldsymbol{\omega} \in \mathcal{P}$ is J-dominant for some subdiagram J, set

$$\chi_J(\boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \iota_J(\boldsymbol{\omega}_J^{-1} \mathrm{qch}(V_q(\boldsymbol{\omega}_J))).$$

Proposition 2.6.7. [**22**, Corollary 3.15] Let $J \subset I$, $\boldsymbol{\omega} \in \mathcal{P}^+$ and suppose $\boldsymbol{\mu} \in \mathcal{P}$ satisfies:

(i) $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$,
(ii) $\boldsymbol{\mu} \in \mathcal{P}_J^+$,
(iii) there is no $\boldsymbol{\varpi} > \boldsymbol{\mu}$ satisfying $\boldsymbol{\varpi} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(\chi_J(\boldsymbol{\varpi}))$.

Then $\mathrm{wt}_\ell(\chi_J(\boldsymbol{\mu})) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. ☐

Remark 2.6.8. Notice that taking $\boldsymbol{\mu} = \boldsymbol{\omega}$ in Proposition 2.6.7, it follows that $\mathrm{wt}_\ell(\chi_J(\boldsymbol{\omega})) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$.

Let ht denote the usual height function on the root lattice Q, i.e., $\mathrm{ht}(\sum_i m_i\alpha_i) := \sum_i m_i$ and, given $\boldsymbol{\alpha} \in \mathcal{Q}$, set

$$\mathrm{ht}(\boldsymbol{\alpha}) = \mathrm{ht}(\mathrm{wt}(\boldsymbol{\alpha})).$$

Lemma 2.6.9. [**22**, Theorem 5.1] Let $\boldsymbol{\lambda} \in \mathcal{P}^+, V = V_q(\boldsymbol{\lambda})$, and $\boldsymbol{\mu} \in \mathcal{P}$. Assume $\boldsymbol{\mu} < \boldsymbol{\lambda}$ and that there exists $i \in I$ such that the following conditions are satisfied:

(i) there exists a unique i-dominant ℓ-weight $\boldsymbol{\nu} \in (\mathrm{wt}_\ell(V) \cap \boldsymbol{\mu}\, \iota_i(\mathcal{Q}_{\{i\}}^+)) \setminus \{\boldsymbol{\mu}\}$ and $\dim V_{\boldsymbol{\nu}} = 1$;
(ii) $x_{i,r}^+ V_{\boldsymbol{\nu}} = 0$ for all $r \in \mathbb{Z}$;
(iii) $\boldsymbol{\mu} \notin \chi_{\{i\}}(\boldsymbol{\nu})$;
(iv) if $\boldsymbol{\nu}' \in \mathrm{wt}_\ell(U_q(\tilde{\mathfrak{g}}_i)V_{\boldsymbol{\nu}})$ is i-dominant, then $\mathrm{ht}(\boldsymbol{\nu}'\boldsymbol{\lambda}^{-1}) \leq \mathrm{ht}(\boldsymbol{\nu}\boldsymbol{\lambda}^{-1})$;
(v) for all $j \neq i$, $\{\boldsymbol{\nu}' \in \mathrm{wt}_\ell(V) : \mathrm{ht}(\boldsymbol{\nu}'\boldsymbol{\lambda}^{-1}) > \mathrm{ht}(\boldsymbol{\mu}\boldsymbol{\lambda}^{-1})\} \cap \boldsymbol{\mu}\, \mathcal{Q}_{\{j\}} = \emptyset$.

Then, $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V)$. ☐

Proposition 2.6.10. [**41**, Theorem 3.4] Let $\boldsymbol{\omega} \in \mathcal{P}^+$. Suppose that $\mathcal{M} \subseteq \mathcal{P}$ is a finite set of distinct ℓ-weights such that:

(i) $\mathcal{P}^+ \cap \mathcal{M} = \{\boldsymbol{\omega}\}$;
(ii) for all $\boldsymbol{\mu} \in \mathcal{M}$ and $(i,a) \in I \times \mathbb{C}^\times$, if $\boldsymbol{\mu}\boldsymbol{\alpha}_{i,a}^{-1} \notin \mathcal{M}$, then $\boldsymbol{\mu}\boldsymbol{\alpha}_{j,b}\boldsymbol{\alpha}_{i,a}^{-1} \notin \mathcal{M}$ unless $(j,b) = (i,a)$;
(iii) for all $\boldsymbol{\mu} \in \mathcal{M}$ and $i \in I$, there exists $\boldsymbol{\nu} \in \mathcal{M}$, i-dominant, such that

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \sum_{\boldsymbol{\eta} \in \boldsymbol{\mu}\mathcal{Q}_{\{i\}} \cap \mathcal{M}} \pi_i(\boldsymbol{\eta}).$$

Then,

$$\mathrm{qch}(V_q(\boldsymbol{\omega})) = \sum_{\boldsymbol{\mu} \in \mathcal{M}} \boldsymbol{\mu}.$$

☐

2.7. Assorted results for type A

We now collect several known results for $\mathfrak{g}$ of type A_n which will be relevant for us.

Given $\boldsymbol{\omega} \in \mathcal{P}^+$, it is not difficult to see that there exist unique $m_i \in \mathbb{Z}_{\geq 0}$, $a_{ik} \in \mathbb{C}^\times$ and $r_{ik} \in \mathbb{Z}_{\geq 1}$ such that

$$\boldsymbol{\omega} = \prod_{i\in I} \prod_{k=1}^{m_i} \boldsymbol{\omega}_{i,a_{ik},r_{ik}}$$

with

$$\frac{a_{ij}}{a_{il}} \neq q_i^{\pm(r_{ij}+r_{il}-2p)} \quad \text{and} \quad \sum_{k=1}^{m_i} r_{ik} = \mathrm{wt}(\boldsymbol{\omega})(h_i)$$

for all $i \in I$, $j \neq l$ and $0 \leq p < \min\{r_{ij}, r_{il}\}$. This decomposition is called the q-factorization of $\boldsymbol{\omega}$.

THEOREM 2.7.1. [**8**, Theorem 4.11] Let $\mathfrak{g} = \mathfrak{sl}_2$ and $\boldsymbol{\omega} = \prod_{j=1}^m \boldsymbol{\omega}_{i,a_j,r_j}$ be the q-factorization of $\boldsymbol{\omega} \in \mathcal{P}^+$, where i is the unique element of I. Then, $V_q(\boldsymbol{\omega}) \cong V_q(\boldsymbol{\omega}_{i,a_1,r_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i,a_m,r_m})$. □

COROLLARY 2.7.2. Let $\mathfrak{g} = \mathfrak{sl}_2$ and $\boldsymbol{\omega}, \boldsymbol{\varpi} \in \mathcal{P}^+$. If the set of q-factors of $\boldsymbol{\omega\varpi}$, counted with multiplicities, is equal to the union of the sets of q-factors of $\boldsymbol{\omega}$ and $\boldsymbol{\varpi}$, then

$$V_q(\boldsymbol{\omega\varpi}) \cong V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}).$$

□

For $\mathfrak{g} = \mathfrak{sl}_2$, the qcharacters of the Kirillov-Reshetikhin modules are known (see [**18**, formula (4.3)]):

$$\mathrm{qch}(V_q(\boldsymbol{\omega}_{i,a,r})) = \boldsymbol{\omega}_{i,a,r} \left(1 + \sum_{j=1}^{r} \prod_{m=0}^{j-1} (\boldsymbol{\alpha}_{i,aq^{r-1-2m}})^{-1} \right). \tag{2.7.1}$$

Combined with Theorems 2.4.3 and 2.7.1, this describes the qcharacters of every finite-dimensional simple $U_q(\tilde{\mathfrak{sl}}_2)$-module.

THEOREM 2.7.3. [**21**, Theorem 3.10] If $V_q(\boldsymbol{\omega})$ is a minimal affinization, then $V_q(\boldsymbol{\omega})$ is thin, i.e, $\dim(V_q(\boldsymbol{\omega})_{\boldsymbol{\mu}}) = 1$ for all $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. □

PROPOSITION 2.7.4. [**10**, Proposition 3.3] Let $\boldsymbol{\omega} \in \mathcal{P}^+$ and $\lambda = \mathrm{wt}(\boldsymbol{\omega})$. Suppose:

(i) $V_q(\boldsymbol{\omega})$ is not a minimal affinization of $V_q(\lambda)$;
(ii) $V_q(\boldsymbol{\omega}_{I\setminus\{i\}})$ is a minimal affinization of $V_q(\lambda_{I\setminus\{i\}})$ for $i = 1$ and $i = n$.

Then, $m_{\lambda-\theta}(V_q(\boldsymbol{\omega})) > 0$ where θ is the maximal root of R. □

PROPOSITION 2.7.5. [**12**, Proposition 3.4] Assume $n \geq 2$. Let $m_1, m_n \in \mathbb{Z}_{\geq 0}, a_1, a_n \in \mathbb{C}^\times$, and $\boldsymbol{\omega} = \boldsymbol{\omega}_{1,a_1,m_1}\boldsymbol{\omega}_{n,a_n,m_n}$. Then, we have an isomorphism of $U_q(\mathfrak{g})$-modules:

$$V_q(\boldsymbol{\omega}) \cong \bigoplus_{t=0}^{s} V_q((m_1 - t)\omega_1 + (m_n - t)\omega_n),$$

where

$$s = \begin{cases} p, & \text{if } a_1/a_n = q^{\pm(m_1+m_n+n-1-2p)} \text{ for some } 0 \leq p < \min\{m_1, m_n\}, \\ \min\{m_1, m_n\}, & \text{otherwise.} \end{cases}$$

□

CHAPTER 3

On qcharacters and tensor products for type A

In this chapter we study several results about the category $\widetilde{\mathcal{C}}_q$ in the case that $\mathfrak{g}$ is of type A_n which will be crucial in the proof of the theorems stated in Section 1.7. Thus, throughout the chapter $\mathfrak{g}$ is of type A_n. The main results of this chapter are Propositions 3.5.1 and 3.6.1.

3.1. Tableaux and ℓ-weights

In this section we review Nakajima's description of ℓ-weights in terms of tableaux [**44**]. Recall that

$$A_{i,s} = Y_{i,s-1}Y_{i,s+1}Y_{i-1,s}^{-1}Y_{i+1,s}^{-1},$$

where we set $Y_{0,s} = 1 = Y_{n+1,r} = 1$ for convenience.

Consider the fundamental representation $V_q(Y_{1,s})$. Its qcharacter is given by

$$\mathrm{qch}(V_q(Y_{1,s})) = Y_{1,s}\left(1+\sum_{j=1}^{n} A_{1,j,s}^{-1}\right) = \sum_{i=0}^{n} Y_{i,s+i+1}^{-1}Y_{i+1,s+i}. \tag{3.1.1}$$

Represent the element $Y_{i-1,s+i}^{-1}Y_{i,s+i-1}, i=1,\dots,n+1$, by the picture $\boxed{i}_s$. Given such a box $\boxed{i}_s$, we shall refer to i as the content of the box and to s as its support. Thus, $\mathrm{wt}_\ell(V_q(Y_{1,s}))$ can be described by the following graph

$$\boxed{1}_s \xrightarrow{1,s+1} \boxed{2}_s \xrightarrow{2,s+2} \cdots \xrightarrow{n,s+n} \boxed{n{+}1}_s$$

where the label $(i, s{+}i)$ on the i-th arrow indicates that $\boxed{i{+}1}_s$ is obtained from $\boxed{i}_s$ by multiplication by $A_{i,s+i}^{-1}$ and (3.1.1) becomes:

$$\mathrm{qch}(V_q(Y_{1,s})) = \sum_{i=1}^{n+1} \boxed{i}_s. \tag{3.1.2}$$

Let $\mathbf{B} = \{1,\dots,n{+}1\}$ equipped with the usual ordering $<$ coming from $\mathbb{Z}$. Given $k, s \in \mathbb{Z}, k>0$, a column tableau T of length k with support starting in s is a map

$$T : \{1,\dots,k\} \to \mathbf{B}\times\mathbb{Z}$$

such that, if we denote by $T(j)_2$ the $\mathbb{Z}$-component of $T(j)$, then

$$T(j)_2 = s + 2(k-j) \qquad \text{for all} \qquad j = 1,\dots,k. \tag{3.1.3}$$

We represent T by the picture:

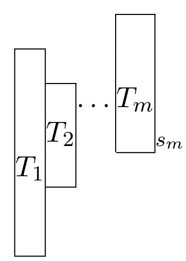
$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_k \\ \hline \end{array}_{\,s} \qquad \text{where} \qquad i_j = T(j)_1 \tag{3.1.4}$$

and $T(j)_1$ denotes the $\mathbf{B}$-component of $T(j)$. For notational convenience, we set $T(0)_1 = 0$. Notice that we can think of this picture as a vertical juxtaposition of the boxes $\boxed{i_j}_{s+2(k-j)}$ with explicit mention of the support of the k-th box only since the others are recovered from it. Given such a tableau, we associate to it an element $\boldsymbol{\omega}^T \in \mathcal{P}$ given by

$$\boldsymbol{\omega}^T = \prod_{j=1}^{k} \boxed{i_j}_{s+2(k-j)}.$$

Remark 3.1.1. Nakajima's original definition regards T as a map $\mathbb{Z} \to \mathbf{B} \cup \{0\}$ such that $T(a) = 0$ if and only if $a \notin \{s, s+2, \dots, s+2(k-1)\}$. Thus, in our notation, $T(j)_2$ corresponds to the j-th element of the support of T in Nakajima's notation while $T(j)_1$ is the value it assumes at that element.

A tableau T is a finite sequence of column tableaux $T = (T_1, T_2, \dots, T_m)$. The shape of T is the sequence of lengths and beginnings of the support of the elements of the sequence. Thus, if T_j has length k_j and support starting at s_j, the shape of T is $((k_1, s_1), (k_2, s_2), \dots, (k_m, s_m))$. We represent T graphically by horizontal juxtaposition of the associated pictures (3.1.4) in such a way that the edge of a given box of T_j touches the edge of at most one box of T_{j+1} and such touching occur if and only if the two boxes have the same support:



In particular, if the picture is connected, the supports of all boxes have the same parity and can be recovered from the support s_m of the last box of T_m. We associate to a tableau T the element $\boldsymbol{\omega}^T \in \mathcal{P}$ given by

$$\boldsymbol{\omega}^T = \prod_{j=1}^{m} \boldsymbol{\omega}^{T_j}.$$

Henceforth, we shall only consider tableaux whose associated picture is connected and will not explicitly mention this again.

A tableau T is said to be column-increasing (or simply increasing) if the contents in each column strictly increase from top to bottom. Note that a column tableau is increasing of length equal to the content of its last box if and only if $T(j)_1 = j$ for all j. In pictures, T it is of the form

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline i \\ \hline \end{array}_{\,s}$$

for some $i \in \{1, 2, \dots, n+1\}$, $s \in \mathbb{Z}$. Notice that if T is such a column tableau, then

$$\boldsymbol{\omega}^T = Y_{i,s+i-1}. \tag{3.1.5}$$

In particular, if T is an increasing column tableau of length $n+1$, i.e., if T has the form

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline n{+}1\\\hline\end{array}_{\,s}$$

for some $s \in \mathbb{Z}$, $\boldsymbol{\omega}^T = 1$. Hence, adding increasing columns of length $n+1$ to a tableau T does not change $\boldsymbol{\omega}^T$.

Two tableaux T and T' are said to be equivalent if, for all $(i,a) \in \mathbf{B} \times \mathbb{Z}$, we have

$$\#\{j : (i,a) \in \mathcal{I}m(T_j)\} \;=\; \#\{j : (i,a) \in \mathcal{I}m(T'_j)\}.$$

In terms of pictures, T' is obtained from T by permuting the contents of the boxes in the same row. It is easy to see that $\boldsymbol{\omega}^T = \boldsymbol{\omega}^{T'}$ if T and T' are equivalent. The converse is not true, but "almost":

Lemma 3.1.2. [**44**, Lemma 4.4] Let T and T' be tableaux. The elements $\boldsymbol{\omega}^T$ and $\boldsymbol{\omega}^{T'}$ are equal if and only if T and T' become equivalent after adding several increasing column-tableaux of length $n+1$ to T and T'. $\square$

Lemma 3.1.3. [**44**, Lemma 4.5] Let T be a tableau. Then, $\boldsymbol{\omega}^T \in \mathcal{P}^+$ if and only if T is equivalent to a tableau T' whose every column is increasing of length equal to the content of the last box. $\square$

Lemma 3.1.4. Let T be a tableau and $J = I \setminus \{n\}$. Then, $\boldsymbol{\omega}^T$ is J-dominant if and only if, after adding increasing columns of length $n+1$ to T, it becomes equivalent to a tableau T' whose columns satisfy some of the following conditions:

(i) is increasing of length equal to the content of the last box;
(ii) is increasing, the content of the last box is $n+1$ and the column tableau obtained by removing the last box is increasing of length equal to the content of its box;
(iii) has length 1 and $n+1$ is the content of its only box.

In pictures, each column of T' is of one of the following:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline i\\\hline\end{array}_{\,s} \qquad \text{or} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline \vdots\\\hline i\\\hline n{+}1\\\hline\end{array}_{\,s} \qquad \text{or} \qquad \begin{array}{|c|}\hline n{+}1\\\hline\end{array}_{\,s}$$

for some $i \in \{1, 2, \ldots, n\}$ and $s \in \mathbb{Z}$.

Proof. Suppose T is equivalent to a tableau T' as above after adding increasing columns of length $n+1$ and recall that

$$\begin{array}{|c|}\hline n{+}1\\\hline\end{array}_{\,s} = Y^{-1}_{n,s+n+1}.$$

This and (3.1.5) imply that $\boldsymbol{\omega}^{T'} = \boldsymbol{\omega}^T$ is J-dominant. Conversely, suppose $\boldsymbol{\omega}^T$ is J-dominant and observe that (3.1.5) implies that any J-dominant ℓ-weight can be constructed from a tableau whose columns satisfy the listed conditions. Thus, let T'' be a tableau whose columns satisfy the listed conditions and $\boldsymbol{\omega}^T = \boldsymbol{\omega}^{T''}$. An application of Lemma 3.1.2 gives that T and T'' are equivalent after adding increasing columns of length $n+1$. Let T' be the tableau obtained from T'' after adding these columns and notice that the columns of T' also satisfy the listed conditions. $\square$

Remark 3.1.5. Notice that the above lemma remains valid if we do not list condition (ii).

Let T be a tableaux of shape (k, s) and suppose $j \in \{1, \dots, k\}$ is such that $i_j = T(j)_1 \in I$. Then, given $m \geq 0$ such that $i'_j := i_j + m \leq n$, we have $\boldsymbol{\omega}^T A^{-1}_{i_j, i'_j, s+2(k-j)+i_j-1} = \boldsymbol{\omega}^{T'}$ where T' is obtained from T by replacing the content of the j-th box by $i'_j + 1$. In pictures:

$$\begin{array}{|c|}\hline \vdots \\ \hline i_j \\ \hline \vdots \\ \hline \end{array}_{\,s} \; A^{-1}_{i_j, i'_j, s+2(k-j)+i_j-1} = \begin{array}{|c|}\hline \vdots \\ \hline i'_j{+}1 \\ \hline \vdots \\ \hline \end{array}_{\,s} \tag{3.1.6}$$

Suppose T is an increasing column tableau. We say that T has a gap at the j-th row if

$$T(j)_1 - T(j-1)_1 > 1.$$

The number $T(j)_1 - T(j-1)_1 - 1$ will be referred to as the size of the gap. In particular, for $j = 1$, T has a gap of size $i - 1$ at the first row iff $T(1)_1 = i > 1$.

Lemma 3.1.6. Let T be a column increasing tableau of shape (k, s) with a gap. More precisely, suppose $T(j)_1 = l_1$ and $T(j+1)_1 = l_2$ with $1 \leq l_1 < l_2 - 1 \leq n$, for some $j \in \{1, \dots, k-1\}$. Then $Y_{l_1, s+2(k-j)+l_1-1}$ and $Y^{-1}_{l_2-1, s+2(k-(j+1))+l_2}$ appear in $\boldsymbol{\omega}^T$.

Proof. By hypothesis, T contains the boxes

$$\boxed{l_1}_{\,s+2(k-j)} = Y^{-1}_{l_1-1, s+2(k-j)+l_1} Y_{l_1, s+2(k-j)+l_1-1}$$

and

$$\boxed{l_2}_{\,s+2(k-j)-2} = Y^{-1}_{l_2-1, s+2(k-j)+l_2-2} Y_{l_2, s+2(k-j)+l_1-3}.$$

Since $l_1 < l_2 - 1$, the negative power produced by $\boxed{l_2}_{\,s+2(k-j)-2}$ cannot be canceled with the positive power produced by $\boxed{l_1}_{\,s+2(k-j)}$ (the box immediately above it). Also, since T is increasing, $T(j')_1 < l_1$ for all $j' < j$ and $T(j'')_1 > l_2$ for all $j'' > j+1$. Thus, there is no other possibility for canceling $Y^{-1}_{l_2-1, s+2(k-j)+l_2-2}$ implying that $Y^{-1}_{l_2-1, s+2(k-j)+l_2-2}$ appears in $\boldsymbol{\omega}^T$. The proof that $Y_{l_1, s+2(k-j)+l_1-1}$ appears in $\boldsymbol{\omega}^T$ is similar. □

3.2. The qcharacters of minimal affinizations

We now study the qcharacters of minimal affinizations in terms of tableaux.

A tableau T with shape $((k_1, s_1), (k_2, s_2), \dots, (k_m, s_m))$ is said to be semi-standard if it is column-increasing and satisfies:

(i) $s_1 \geq s_2 \geq \cdots \geq s_m$;
(ii) $(i, s) \in \mathcal{I}m(T_j)$ and $(i', s-2) \in \mathcal{I}m(T_{j+1}) \Rightarrow i \geq i'$.

In terms of pictures, the sequences of diagonal contents from left to right and top to bottom are decreasing (not necessarily strictly). Given a tableau T, we will denote by $\mathrm{STab}(T)$ the set of semi-standard tableaux with the same shape as T.

Recall the definition of increasing and decreasing minimal affinizations given after Theorem 1.6.1. Let $\lambda \in P^+$ and $\boldsymbol{\omega} \in \mathcal{P}^+$ be the highest-ℓ-weight of a minimal affinization of $V_q(\lambda)$, say

$$\boldsymbol{\omega} = \prod_{i \in I} Y_{i, r_i, \lambda(h_i)}. \tag{3.2.1}$$

Then $\boldsymbol{\omega} = \boldsymbol{\omega}^T$, where $T = (T^n, T^{n-1}, \ldots, T^1)$ if $V_q(\lambda)$ is increasing and $T = (T^1, T^2, \ldots, T^n)$ if $V_q(\lambda)$ is decreasing, with T^i omitted if $\lambda(h_i) = 0$ and, otherwise, $T^i = (T^i_1, \ldots, T^i_{\lambda(h_i)})$ with T^i_j column-increasing with length equal the content of its last box and support starting at $r_i+2(\lambda(h_i)-j)-i+1$:

$$T^i_j = \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline i \\ \hline \end{array}_{\,r_i+2(\lambda(h_i)-j)-i+1} \tag{3.2.2}$$

Notice that T has $|\lambda|$ columns and

$$\boldsymbol{\omega}^{T^i_j} = Y_{i,r_i+2(\lambda(h_i)-j)}.$$

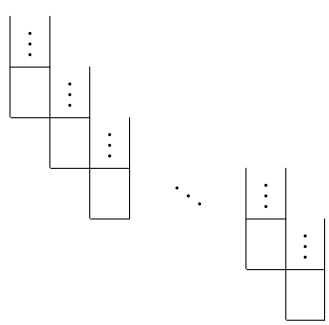
One easily checks using the formulas for $p_{i,j}(\lambda)$ that, if $\boldsymbol{\omega}$ corresponds to an increasing minimal affinization, then, for every $i \in \operatorname{supp}(\lambda)$, we have

$$r_i = r_{i'} + 2(\lambda(h_{i'}) - 1) + i - i' + 2 \qquad \text{with} \qquad i' = \max\{1, \ldots, i-1\} \cap \operatorname{supp}(\lambda). \tag{3.2.3}$$

Moreover, if the support of the j-th column of T starts at s, then that of the $(j+1)$-th column starts at $s-2$. Indeed, if they are both columns of T^i, this is obvious. Otherwise, consider the last column of T^i and suppose the next column is the first one of T^k. Then,

$$\begin{array}{rcl} (r_i - i + 1) - (r_k + 2(\lambda(h_k) - 1) - k + 1) & = & (r_i - (r_k + 2(\lambda(h_k) - 1))) - i + k \\ & \overset{(3.2.3)}{=} & (i - k + 2) - i + k \\ & = & 2. \end{array}$$

In pictures, T has the form

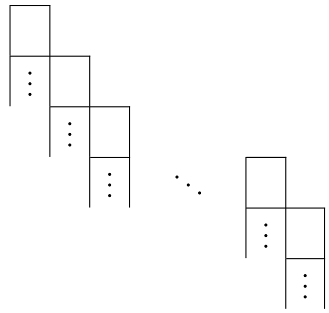
(3.2.4) 

and, since the minimal affinization is increasing, the top of each column is in a row below the top of the previous column. Similarly, if $\boldsymbol{\omega} \in \mathcal{P}^+$ corresponds to a decreasing minimal affinization of $V_q(\lambda)$, for every $i \in \operatorname{supp}(\lambda)$, we have

$$r_i = r_{i'} + 2(\lambda(h_{i'}) - 1) + i - i' + 2 \qquad \text{with} \qquad i' = \min\{i+1, \ldots, n\} \cap \operatorname{supp}(\lambda). \tag{3.2.5}$$

This time, if the support of the first box j-th column of T is s, then the support of the first box of the $(j+1)$-th column is $s-2$. In pictures, T has the form

(3.2.6)



and the bottom of each column is in a row below the bottom of the previous column.

For the next theorem, see [**41**, Corollary 7.6 and Remark 7.4 (i)] and references therein (cf. Theorem 2.7.3).

THEOREM 3.2.1. Let T be as in (3.2.2). Then, $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) = \{\boldsymbol{\omega}^{T'} : T' \in \mathrm{STab}(T)\}$. $\square$

We shall need the following constructions associated to a semi-standard tableau T as in (3.2.2). Denote by l_t the length of the t-th column of T. Given $0 \leq t \leq |\lambda|$ and $1 \leq p \leq l_t + 1$, consider the tableau $T_{t,p}$ which is the unique tableau having the shape of T and such that

(1) if $t' > t$, the t'-th column of $T_{t,p}$ does not have gaps;
(2) each of the first t columns of $T_{t,p}$ have exactly one gap;
(3) all gaps have size 1 and occur at the the p-th row of the corresponding column.

In particular, $T_{0,p} = T_{t,l_t+1} = T$. Also, given $1 \leq c \leq f \leq |\lambda|$ and $0 \leq j \leq n+1$, consider the tableau $T_{c,f,j}$ which is the unique tableau having the shape of T and such that

(1) each column has at most one gap;
(2) if $t < c$ or $t > f$, the t-th column of $T_{c,f,j}$ does not have gaps;
(3) if $c \leq t \leq f$, the t-th column has a gap at its last row whose size is j minus the length of the column, if this number is positive, and there is no gap in the t-th column if this number is negative or zero.

For notational convenience, we set $T_{c,f,j} = T_{c,|\lambda|,j}$ for $f > |\lambda|$ and $T_{c,f,j} = T$ for $f < c$ as well as for $c > |\lambda|$ and $j = 0$, and $T_{t,p} = T_{|\lambda|,p}$ for $t > |\lambda|$.

EXAMPLE 3.2.2. Suppose that T is the semi-standard tableau such that $\boldsymbol{\omega}^T = Y_{n,s,k}$. We now describe the qcharacter of $V = V_q(Y_{n,s,k})$. Note that $T = (T_1, \ldots, T_k)$ is the semi-standard tableau where each column T_j has length n, the content of the last box is n, and the shape of T is

$$((n, s+1+2(k-1)-n), \ldots, (n, s+3-n), (n, s+1-n)).$$

By Theorem 3.2.1, the ℓ-weights of V are represented by semi-standard tableaux with shape of T. Thus, we have the highest ℓ-weight $\boldsymbol{\omega}^T$ and all the ℓ-weights in $\mathrm{wt}_\ell(V) \setminus \{Y_{n,s,k}\}$ are obtained from T by changing the contents of the boxes of T without breaking condition of being semi-standard.

Consider the case $k = 1$ first. Then, the corresponding semi-standard tableaux are $T_{1,j}, 1 \leq j \leq n$:

$$\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n{-}2 \\ \hline n{-}1 \\ \hline n \\ \hline \end{array}_{s+1-n} = Y_{n,s} \xrightarrow{n,s+1} \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n{-}2 \\ \hline n{-}1 \\ \hline n{+}1 \\ \hline \end{array}_{s+1-n} = \boldsymbol{\omega}^{T_{1,n}} \xrightarrow{n-1,s+2} \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n{-}2 \\ \hline n \\ \hline n{+}1 \\ \hline \end{array}_{s+1-n} = \boldsymbol{\omega}^{T_{1,n-1}}$$

$$\xrightarrow{n-2,s+3} \cdots \xrightarrow{2,s+n-1} \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \vdots \\ \hline n{+}1 \\ \hline \end{array}_{s+1-n} = \boldsymbol{\omega}^{T_{1,2}} \xrightarrow{1,s+n} \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \vdots \\ \hline n{+}1 \\ \hline \end{array}_{s+1-n} = \boldsymbol{\omega}^{T_{1,1}}.$$

In other words

$$\mathrm{qch}(V) = \sum_{p=1}^{n+1} \boldsymbol{\omega}^{T_{1,p}} = Y_{n,s}\left(1 + \sum_{p=1}^{n} A_{n,p,s}^{-1}\right).$$

For $k > 1$, we first notice that we can do the same sequence of changes on the first column. Suppose we have done j changes on the first column. Then we can do the same type of changes on the second column up to the j-th change and so on. In other words, the ℓ-weights of $V_q(\boldsymbol{\varpi})$ are parameterized by the set of partitions $J(n,k) = \{\boldsymbol{j} = (j_1, j_2, \dots, j_k) : 0 \leq j_k \leq \cdots \leq j_2 \leq j_1 \leq n\}$ and the ℓ-weight associated to $\boldsymbol{j} \in J$ is

$$\boldsymbol{\varpi}_{\boldsymbol{j}} = Y_{n,s,k} \prod_{l=1}^{k} A^{-1}_{n,n+1-j_l,s+2(k-l)}. \tag{3.2.7}$$

Moreover, it is known that the multiplicities are all 1 and, hence,

$$\mathrm{qch}(V) = \sum_{\boldsymbol{j} \in J(n,k)} \boldsymbol{\varpi}_{\boldsymbol{j}}.$$

In terms of fundamental ℓ-weights, we have

$$\boldsymbol{\varpi}_{\boldsymbol{j}} = Y_{n,s,k} \prod_{l=1}^{k} \left(Y^{-1}_{n,s+2(k-l)} Y^{-1}_{n-j_l+1,s+2(k-l)+j_l+1} Y_{n-j_l,s+2(k-l)+j_l}\right)^{1-\delta_{0,j_l}}. \tag{3.2.8}$$

Notice that $j_l > 0$ means that there exists a gap of size 1 at the $(n - j_l + 1)$-th row of the l-th column of T. Moreover, if $\boldsymbol{\varpi}_{\boldsymbol{j}} \neq Y_{n,s,k}$, then $j_1 > 0$ which implies $Y^{-1}_{n+1-j_1,s+2(k-1)+j_1+1} Y_{n-j_1,s+2(k-1)+j_1}$ appears in $\boldsymbol{\varpi}_{\boldsymbol{j}}$ and (see (2.5.2)):

$$r(\boldsymbol{\varpi}_{\boldsymbol{j}}) = s + 2(k-1) + j_1 + 1. \tag{3.2.9}$$

Indeed, since $l_1 < l_2$ implies $j_{l_1} \geq j_{l_2}$, we have $s + 2(k - l_1) + j_{l_1} > s + 2(k - l_2) + j_{l_2}$.

The elements $T_{t,p}$ will play an important role later on. Note that

$$T_{t,p} \quad \text{corresponds to the partition } \boldsymbol{j} \text{ given by} \quad j_i = \begin{cases} p, & \text{if } i \leq t, \\ 0, & \text{if } i > t. \end{cases}$$

We also note that (3.1.6) implies

$$\boldsymbol{\omega}^{T_{t+1,p}} = \boldsymbol{\omega}^{T_{t,p}}\ A^{-1}_{n,p,s+2(k-t)} \qquad \text{for all} \qquad 0 \leq t < k,\ 1 \leq p \leq n. \tag{3.2.10}$$

Iterating we get

$$\boldsymbol{\omega}^{T_{t,p}} = Y_{n,s,k} \prod_{l=1}^{t} A^{-1}_{n,p,s+2(k-l)} \qquad \text{for all} \qquad 1 \le t \le k,\ 1 \le p \le n. \tag{3.2.11}$$

We end this example by specializing it to the case $n = 1$. In that case, $J(n,k)$ is in bijection with $\{T_{t,1} : 0 \le t \le k\}$ and we have

$$\mathrm{qch}(V) = \sum_{t=0}^{k} \boldsymbol{\omega}^{T_{t,1}} = Y_{1,s,k} \sum_{t=0}^{k} \prod_{l=1}^{t} A^{-1}_{1,s+2(k-l)+1}. \tag{3.2.12}$$

We end this section with the following lemma on the combinatorics of semi-standard the tableau which will be used systematically in the Section 3.4.

Lemma 3.2.3. Let T be a semi-standard tableau with shape as in (3.2.4) and $(i,s) \in \mathbf{B} \times \mathbb{Z}$. Suppose the box $\boxed{i}_s$ is part of the the j-th column of T. Then:

(a) The box $\boxed{i}_s$ is not in any other column of T.
(b) If $\boxed{i-1}_{s+2}$ is a box in T, it must be in the j-th column.
(c) if $\boxed{i+1}_{s-2}$ is a box in T, it must be in the j-th column.

Proof. We write down the proof of (b) only since the other items are similar. Suppose $\boxed{i-1}_{s+2}$ appears in the $(j+m)$-th column, $m \ge 1$. Since T' is as in (3.2.4), this column has a box supported at $s-2m$. Since T' is columns increasing, the content c of the box supported at $s-2m$ is at least $i+m > i$. This contradicts the assumption that T' is semi-standard because the box $\boxed{i}_s$ in column j and the box $\boxed{c}_{s-2m}$ in column $j+m$ are in the same diagonal from left to right and top to bottom. Suppose now that $\boxed{i-1}_{s+2}$ is in the $(j-m)$-th column, $m \ge 1$. This time (3.2.4) implies that this column has a box supported at $s+2m$. Since all columns are increasing, the content c of the box supported at $s+2m$ is at most $i-m < i$. This contradicts the assumption that T' is semi-standard because the box $\boxed{c}_{s+2m}$ in column $j-m$ and the box $\boxed{i}_s$ in column j are in the same diagonal from left to right and top to bottom. $\square$

3.3. J-dominant ℓ-weights of certain minimal affinizations

For the remainder of Chapter 3, we fix $\lambda \in P^+$ and $\boldsymbol{\omega} \in \mathcal{P}^+$ such that $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$ as in (3.2.1), with $V_q(\boldsymbol{\omega})$ an increasing minimal affinization. Set

$$i_0 = \max(\mathrm{supp}(\lambda)).$$

It follows from (3.2.3) that

$$r_i = r_{i_0} - i_0 + i - 2\,{}_i|\lambda|_{i_0-1} \quad \text{for all} \quad i \in \mathrm{supp}(\lambda). \tag{3.3.1}$$

For notational convenience, we define r_i by (3.3.1) for all $1 \le i \le i_0$. Then, for all $1 \le i \le j$ it holds:

$$r_i = r_j - j + i - 2\,{}_i|\lambda|_{j-1}. \tag{3.3.2}$$

Let S be the semi-standard tableau such that $\boldsymbol{\omega}^S = \boldsymbol{\omega}$ as given by (3.2.2) and recall Definition 2.5.4.

Lemma 3.3.1. If $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ is not right negative, then $r(\boldsymbol{\nu}) = r_{i_0} + 2(\lambda(h_{i_0}) - 1)$ and $\pi_{i_0}(\boldsymbol{\nu}) = Y_{i_0, r_{i_0}, \lambda(h_{i_0})}$.

PROOF. The statement is clear if $\boldsymbol{\nu}=\boldsymbol{\omega}$. Suppose $\boldsymbol{\nu}\neq\boldsymbol{\omega}$ and write $\boldsymbol{\nu}=\boldsymbol{\omega}^{S'}$ where $S'\in \mathrm{STab}(S)$. Since $S'\neq S$, there exists $1\le l\le|\lambda|$ such that the l-th column S'_l of S' has a gap. Assume l is the smallest such index. It follows from Lemma 3.1.6 that $Y^{-1}_{i,r(\boldsymbol{\omega}^{S'_l})}$ appears in $\boldsymbol{\omega}^{S'_l}$ for some $i\in I$. By Lemma 3.2.3, $Y_{i,r(\boldsymbol{\omega}^{S'_l})}$ does not appear in $\boldsymbol{\omega}^{S'\setminus S'_l}$, where $S'\setminus S'_l$ is the tableaux obtained from S' by removing its l-th column. Hence, $Y^{-1}_{i,r(\boldsymbol{\omega}^{S'_l})}$ appears in $\boldsymbol{\nu}$. Since $\boldsymbol{\nu}$ is not right negative, there must exist $1\le l'\le|\lambda|$ such that $S'_{l'}$ is gap-free and $r(\boldsymbol{\omega}^{S'_{l'}})>r(\boldsymbol{\omega}^{S'_l})$. One easily checks that, if $l'>l$ and $S'_{l'}$ has no gaps, then $r(\boldsymbol{\omega}^{S'_{l'}})<r(\boldsymbol{\omega}^{S'_l})$. Therefore, we must have $1\le l'<l$ and, since both S'_1 and $S'_{l'}$ are gap-free, it follows that $r(\boldsymbol{\omega}^{S'_1})>r(\boldsymbol{\omega}^{S'_{l'}})$ which proves the first statement of the lemma. Since S' is semi-standard and the first column is gap-free, all columns of length i_0 must also be gap-fee which implies the second statement. $\square$

Henceforth, assume

$$\lambda(h_n)=0.$$

Recall the definition of the tableaux $S_{c,f,j}$ in the paragraph preceding Example 3.2.2.

PROPOSITION 3.3.2. Let $J=I\setminus\{n\}$. Then, the J-dominant elements of $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ are $\boldsymbol{\omega}^{S_{1,j,n+1}}$, $0\le j\le|\lambda|$.

PROOF. It is clear from the definition of $S_{1,j,n+1}$ that $\boldsymbol{\omega}^{S_{1,j,n+1}}$ is J-dominant for all $0\le j\le|\lambda|$.

For proving the converse, we start by showing that the J-dominant ℓ-weights of $V_q(\boldsymbol{\omega})$ can be represented by elements of $\mathrm{STab}(S)$ whose columns are of the form listed in Lemma 3.1.4. Indeed, by Theorem 3.2.1, any element of $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ can be represented by an element of $\mathrm{STab}(S)$. Let $S'\in\mathrm{STab}(S)$ and suppose S' has a column which is not of the form listed in Lemma 3.1.4. Look at the first such column (say it is the j-th column) and consider the first box whose content $i<n+1$ is not equal to its position l in the column. In other words, this column contains $\boxed{i}_s$ for some s and $\boxed{l-1}_{s+2}$ with $l<i$. If $\boldsymbol{\omega}^{S'}$ were J-dominant, there would exist a column containing the box $\boxed{i-1}_{s+2}$. By Lemma 3.2.3, this box would also be in the j-th column contradicting $l<i$.

It now remains to check that, if $S'\in\mathrm{STab}(S)$ has columns as listed in Lemma 3.1.4 (which implies $\boldsymbol{\omega}^{S'}$ is J-dominant), then $S'=S_{1,j,n+1}$ for some $0\le j\le|\lambda|$. If there is no box having $n+1$ as content, then $S'=S=S_{1,0,n+1}$. Otherwise, we claim $S'=S_{1,j,n+1}$ where j is the number of the right-most column having $n+1$ as the content of its last box. Indeed, since S' is semi-standard, it follows from (3.2.4) that the last box of all the previous columns must be $n+1$. By the choice of j and the fact that all columns of S' are as listed in Lemma 3.1.4, all columns to the right are increasing having length equal to the content of the last box and, hence, coincide with corresponding columns of S. $\square$

Given $1\le j\le|\lambda|$, let l_j be the length of the j-th column of S and $d_j:=j-\sum_{i>l_j}\lambda(h_i)$ (this means that the j-th column is the d_j-th one of length l_j). It follows from (3.1.6) that

$$\boldsymbol{\omega}^{S_{1,j,n+1}}=\boldsymbol{\omega}^{S_{1,j-1,n+1}}A^{-1}_{l_j,n,r_{l_j}+2(\lambda(h_{l_j})-d_j)}\,. \tag{3.3.3}$$

This implies that

$$\begin{aligned}\pi_n(\boldsymbol{\omega}^{S_{1,j,n+1}}) &= Y^{-1}_{n,r_{l_j}+2(\lambda(h_{l_j})-d_j)+n-l_j+2,d_j}\prod_{m>l_j}Y^{-1}_{n,r_m+n-m+2,\lambda(h_m)}\\ &\overset{(3.3.1)}{=} Y^{-1}_{n,r_{i_0}+2\lambda(h_{i_0})+n-i_0-2j+2,j}\,.\end{aligned} \tag{3.3.4}$$

We shall also need the following information. By definition of $S_{1,j,n+1}$ and (3.1.6), we have

$$\boldsymbol{\omega}^{S_{1,j,n+1}} = \boldsymbol{\omega}\left(\prod_{i=l_j+1}^{i_0}\prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,n,r_i+2(\lambda(h_i)-m)}\right)\left(\prod_{m=1}^{d_j} A^{-1}_{l_j,n,r_{l_j}+2(\lambda(h_{l_j})-m)}\right) \tag{3.3.5}$$

and

$$\boldsymbol{\omega}^{S_{1,j,n}} = \boldsymbol{\omega}\left(\prod_{i=l_j+1}^{i_0}\prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,n-1,r_i+2(\lambda(h_i)-m)}\right)\left(\prod_{m=1}^{d_j} A^{-1}_{l_j,n-1,r_{l_j}+2(\lambda(h_{l_j})-m)}\right). \tag{3.3.6}$$

This, together with (3.3.1), imply that

$$\begin{aligned}\pi_n(\boldsymbol{\omega}^{S_{1,j,n}}) &= \left(\prod_{i=l_j+1}^{i_0}\prod_{m=1}^{\lambda(h_i)} Y_{n,r_l+2(\lambda(h_i)-m)+n-l}\right)\left(\prod_{m=1}^{d_j} Y_{n,r_{l_j}+2(\lambda(h_{l_j})-m)+n-l_j}\right)\\ &= Y_{n,r_{i_0}+2\lambda(h_{i_0})+n-i_0-2j,j}.\end{aligned} \tag{3.3.7}$$

Notice also that

$$\boldsymbol{\omega}^{S_{c,f+1,p}} = \boldsymbol{\omega}^{S_{c,f,p}}\ A^{-1}_{l_f,p-1,r_{l_f}+2(\lambda(h_{l_f})-d_f)} \tag{3.3.8}$$

for all $0\le c-1\le f\le|\lambda|, 1\le p<n$. Iterating this, we get

$$\boldsymbol{\omega}^{S_{c,f,p}} = \boldsymbol{\omega}\left(\prod_{i=l_f+1}^{p-1}\prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,p-1,r_i+2(\lambda(h_i)-m)}\right)\left(\prod_{m=1}^{d_f} A^{-1}_{l_f,p-1,r_{l_f}+2(\lambda(h_{l_f})-m)}\right). \tag{3.3.9}$$

Observe that, by (2.2.1), there exists a unique $\Xi=\Xi_{c,f,p}\subseteq I\times\mathbb{Z}$ such that (3.3.9) can be written in the form

$$\boldsymbol{\omega}^{S_{c,f,p}} = \boldsymbol{\omega}\prod_{\xi\in\Xi} A_\xi^{-1}$$

Notice also that, if $l_f<p$ and $r=\min\{r':(i,r')\in\Xi \text{ for some } i\}$ (such minimum is reached by the pair $(l_f, r_{l_f}+2(\lambda(h_f)-d_f)+1))$, then $r-l_f$ is the support of the last box of the f-th column of S,

$$\max\{i:(i,s)\in\Xi \text{ for some } s\}=p-1, \qquad (p-1,r+p-l_f-1)\in\Xi, \tag{3.3.10}$$

and $(i,r+p-l_f-1)\notin\Xi$ for $i\neq p-1$. We shall need the following combinatorial lemma in Section 3.4.

LEMMA 3.3.3. Assume $p\in\operatorname{supp}(\lambda)$, $c=\min\{j: l_j<p\}, c\le f\le|\lambda|$, let Ξ and r be defined as above and set $\xi_0=(p-1,r+p-l_f-1)$. If $S'\in\operatorname{STab}(S)$ is such that $\boldsymbol{\omega}^{S'}=\boldsymbol{\omega}\prod_{\xi\in\Xi'}A_\xi^{-1}$ for some Ξ' with $\xi_0\in\Xi'$, then $\Xi\subseteq\Xi'$. In particular, if $\Xi'\subseteq\Xi$, we have $S'=S_{c,f,p}$.

PROOF. Equation (3.1.6) implies that, for obtaining S' by modifications in S, the element ξ_0 corresponds to the modification

$$\boxed{p-1}_{\,r-l_f}\to\boxed{\;p\;}_{\,r-l_f}.$$

Since the f-th column of S is the first column which has a box supported at $r-l_f$, if this modification occurs in the l-th column, we must have $l\ge f$. In particular, if j is the content of the last box of the l-th column of S', then $j\ge p$ with equality holding only for $l=f$. In particular, if $l>f$, since S' is semi-standard of the form (3.2.4), it follows that all the columns to the left of the l-th one have the last box with content bigger or equal to $p+1$, contradicting the first statement in (3.3.10). Thus, we must have $l=f$ and, using that S' is semi-standard with the form (3.2.4) once more, it follows that all the columns to the left of f-th one have the last box with content at least p. One now easily checks using (3.1.6) that, even if Ξ' would have only the elements corresponding to modifying the last boxes of the columns between the c-th and the f-th, we must have $\Xi \subseteq \Xi'$. Since the equality $\Xi = \Xi'$ implies that $\boldsymbol{\omega}^{S'} = \boldsymbol{\omega}^{S_{c,f,p}}$, the last statement follows from the fact that $V_q(\boldsymbol{\omega})$ is thin. $\square$

3.4. Dominant ℓ-weights on certain tensor products

Throughout this and the next sections, we also fix: $\boldsymbol{\varpi} = Y_{n,r_n,k}$ and let T be the semi-standard tableau such that $\boldsymbol{\omega}^T = \boldsymbol{\varpi}$ as in Example 3.2.2. Our present goal is to describe the elements of

$$D := \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})) \cap \mathcal{P}^+$$

explicitly in terms of tableaux. The next proposition will follow as a byproduct and will be crucial for our next goal (Section 3.5).

Proposition 3.4.1. The partial order on $\mathcal{P}$ induces a total order on D and

$$\dim((V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}))_{\boldsymbol{\mu}}) = 1 \qquad \text{for all} \qquad \boldsymbol{\mu} \in D.$$

We will consider separately the following two subcases

$$r_{i_0} + 2\lambda(h_{i_0}) \leq r_n + 2k \tag{3.4.1}$$

and

$$r_n + 2k \leq r_{i_0} + 2\lambda(h_{i_0}). \tag{3.4.2}$$

Assume first that (3.4.1) holds.

Lemma 3.4.2. The elements of D are of the form $\boldsymbol{\nu}\boldsymbol{\varpi}$ with $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$.

Proof. Let $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ be such that $\boldsymbol{\nu}\boldsymbol{\mu} \in \mathcal{P}^+$. In particular, $\boldsymbol{\nu}\boldsymbol{\mu}$ is not right negative. Suppose by contradiction that $\boldsymbol{\mu} \neq \boldsymbol{\varpi}$. Then, by Proposition 2.5.5, $\boldsymbol{\mu}$ is right negative and it follows from (3.2.9) that

$$r(\boldsymbol{\mu}) = r_n + 2(k-1) + j + 1$$

for some $1 \leq j \leq n$. Since the product of right negative elements is again right negative, $\boldsymbol{\nu}$ is not right negative and Lemma 3.3.1 implies that

$$r(\boldsymbol{\nu}) = r_{i_0} + 2(\lambda(h_{i_0}) - 1).$$

Together with (3.4.1), this implies that $r(\boldsymbol{\nu}) < r(\boldsymbol{\mu})$. It then follows that $\boldsymbol{\nu}\boldsymbol{\mu}$ is right negative, yielding the desired contradiction. $\square$

Note that, together with Theorem 2.7.3, Lemma 3.4.2 implies the second statement of Proposition 3.4.1 in the present case. We shall now prove that

$$D = \{\boldsymbol{\omega}^{S_{1,j,n+1}}\boldsymbol{\omega}^T : j = 0, 1, \ldots, k'\} \tag{3.4.3}$$

where $0 \leq k' \leq k$ is either zero or given by the condition:

$$r_{i_0} + 2(\lambda(h_{i_0}) - 1) + n - i_0 + 2 = r_n + 2(k' - 1). \tag{3.4.4}$$

Indeed, since the elements in D are of the form $\boldsymbol{\nu}\boldsymbol{\varpi}$ with $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$, it follows that $\boldsymbol{\nu}$ must be $(I \setminus \{n\})$-dominant and, hence, by Proposition 3.3.2, we must have $\boldsymbol{\nu} = \boldsymbol{\omega}^{S_{1,j,n+1}}$ for some $0 \leq j \leq |\lambda|$. It now easily follows from (3.3.4) that $\boldsymbol{\omega}^{S_{1,j,n+1}}\boldsymbol{\varpi} \in \mathcal{P}^+$ if and only if $0 \leq j \leq k'$.

Notice that the first statement of Proposition 3.4.1 follows easily from (3.4.3) and (3.3.8). Thus, henceforth, assume (3.4.2) holds.

Lemma 3.4.3. If $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ is such that $\boldsymbol{\nu}\boldsymbol{\varpi} \in D$, then $\boldsymbol{\nu} = \boldsymbol{\omega}$.

PROOF. Obviously, if $\boldsymbol{\nu\varpi} \in D$, $\boldsymbol{\nu}$ must be J-dominant. Let $s = r_{i_0} + 2\lambda(h_{i_0}) + n - i_0$. By (3.3.4), $Y_{n,s}^{-1}$ appears in all J-dominant ℓ-weights of $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ except $\boldsymbol{\omega}$. We claim that, if $\boldsymbol{\nu} \neq \boldsymbol{\omega}$, then $Y_{n,s}^{-1}$ appears in $\boldsymbol{\nu\varpi}$, which proves the lemma. By definition, $Y_{n,r}$ appears in $\boldsymbol{\varpi}$ iff $r = r_n + 2(j-1)$ for some $j = 1, \dots, k$. Since

$$r_n + 2(k-1) \overset{(3.4.2)}{\leq} r_{i_0} + 2(\lambda(h_{i_0}) - 1) < s,$$

the claim follows. $\square$

Suppose there exists $p \in \mathrm{supp}(\lambda)$ and $k' \in \{1, \dots, \lambda(h_p)\}$ satisfying

$$r_n + 2(k-1) + n - p + 2 = r_p + 2(k'-1). \tag{3.4.5}$$

Observe that the pair (p, k') is unique, if it exists. Indeed, assume (p, k') and (p', k'') satisfy (3.4.5). If $p = p'$ we must obviously have $k' = k''$. Otherwise, without loss of generality, assume that

$$p - p' < 0.$$

To obtain a contradiction, observe that, since $V_q(\boldsymbol{\omega})$ is an increasing minimal affinization, we must have

$$r_{p'} - (r_p + 2(\lambda(h_p) - 1)) \geq p' - p + 2$$

(by (3.2.3), the equality holds if $p' = \min\{i \in \mathrm{supp}(\lambda) : i > p\}$). It follows that,

$$\begin{aligned} p - p' \quad &= \quad r_n + 2(k-1) + n - p' + 2 - (r_n + 2(k-1) + n - p + 2) \\ &\overset{(3.4.5)}{=} \quad r_{p'} + 2(k''-1) - (r_p + 2(k'-1)) \;\geq\; r_{p'} + 2(1-1) - (r_p + 2(\lambda(h_p) - 1)) \\ &\geq \quad p' - p + 2 \;>\; 0. \end{aligned}$$

If a pair (p, k') satisfying (3.4.5) does not exist, we set $p = k' = 0$. Set also

$$c = 1 + \sum_{i=p}^{n-1} \lambda(h_i)$$

and recall the definition of the tableaux $T_{t,j}$ at the end of Section 3.1. Consider the subset D' of $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}))$ defined as follows.

$$\boldsymbol{\mu} \in D' \;\Leftrightarrow\; \boldsymbol{\mu} = \begin{cases} \boldsymbol{\omega}\, \boldsymbol{\omega}^{T_{t,p}}, & \text{for } 0 \leq t \leq k' \\ \boldsymbol{\omega}^{S_{c,f,p}} \boldsymbol{\omega}^{T_{t,p}}, & \text{for } k' \leq t \leq k, f \leq |\lambda|, f = c + t - k' - \epsilon \text{ with } \epsilon \in \{0,1\}. \end{cases}$$

We now show that

$$D = D'. \tag{3.4.6}$$

Together with (3.2.10) and (3.3.8), (3.4.6) easily implies the first statement of Proposition 3.4.1. Moreover, if $r_n + 2(k-1) + n - i + 2 \neq r_i + 2(k'-1)$ for all $i \in \mathrm{supp}(\lambda)$ and $k' = 1, \dots, \lambda(h_i)$, then $p = k' = 0$ and, hence, $D = \{\boldsymbol{\omega\varpi}\}$. However, we will prove that this is true independently of the proof of (3.4.6) (see Proposition 3.4.5).

We begin the proof of (3.4.6) by investigating the elements in D of the form $\boldsymbol{\omega\nu}$ with $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$. Recall from Example 3.2.2 that the elements of $\mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ are in bijection with the set $J(n,k) = \{\boldsymbol{j} = (j_1, j_2, \dots, j_k) : 0 \leq j_k \leq \cdots \leq j_2 \leq j_1 \leq n\}$. Let $\boldsymbol{j}$ and T' be the tuple and tableaux associated to $\boldsymbol{\nu}$, respectively. In particular, if $\boldsymbol{\nu} \neq \boldsymbol{\varpi}$, we have $j_1 > 0$ and Lemma 3.1.6 implies that the first column of T' contributes with the appearance of the factor $Y_{i_1, r_n+2(k-1)+n-i_1+2}^{-1}$ in $\boldsymbol{\nu}$, where

$$i_1 := n - j_1 + 1. \tag{3.4.7}$$

Recall that

$$\boldsymbol{\omega} = \prod_{i\in I} Y_{i,r_i,\lambda(h_i)} = \prod_{i\in I}\prod_{k'=1}^{\lambda(h_i)} Y_{i,r_i+2(k'-1)}.$$

This implies that $Y_{i_1,r_n+2(k-1)+n-i+2}$ must appear in $\boldsymbol{\omega}$ and it follows that there exists $1 \le k' \le \lambda(h_{i_1})$ such that

$$r_n + 2(k-1) + n - i + 2 = r_{i_1} + 2(k'-1).$$

In other words, the pair (i_1, k') satisfies (3.4.5) and, hence, $i_1 = p$. In particular, we have shown the following lemma.

LEMMA 3.4.4. Suppose $r_n + 2(k-1) + n - i + 2 \neq r_i + 2(k'-1)$ for all $i \in \mathrm{supp}(\lambda)$ and $k' = 1, \dots, \lambda(h_i)$. If $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ is such that $\boldsymbol{\omega\nu} \in D$, then $\boldsymbol{\nu} = \boldsymbol{\varpi}$. $\square$

PROPOSITION 3.4.5. If $r_n + 2(k-1) + n - i + 2 \neq r_i + 2(k'-1)$ for all $i \in \mathrm{supp}(\lambda)$ and $k' = 1, \dots, \lambda(h_i)$, then $D = \{\boldsymbol{\omega\varpi}\}$.

PROOF. By Lemma 3.4.3, there is no element in D of the form $\boldsymbol{\mu\varpi}$ with $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \setminus \{\boldsymbol{\omega}\}$. Suppose $\boldsymbol{\mu\nu} \in D$ with $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ and $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})) \setminus \{\boldsymbol{\varpi}\}$. By (3.2.8), the term

$$Y^{-1}_{i,r_n+2(k-1)+n-i+2} Y_{i-1,r_n+2(k-1)+n-i+1}, \tag{3.4.8}$$

where $i = i_1$ as defined in (3.4.7). In particular, it follows from (3.2.8) that

$$\text{If} \quad Y_{j,r} \quad \text{appears in} \quad \boldsymbol{\nu}, \quad \text{then} \quad j \ge i-1 \quad \text{and} \quad r \le r_n + 2(k-1) + n - i + 1. \tag{3.4.9}$$

We show that the negative power in (3.4.8) cannot be canceled with a factor of $\boldsymbol{\mu}$. Let S' be a semi-standard tableau with shape of S such that $\boldsymbol{\omega}^{S'} = \boldsymbol{\mu}$. To cancel the negative power in (3.4.8), S' must have $\boxed{i}$ supported at $r_n + 2(k-1) + n - 2i + 3$ and, if there exists a box immediately below it, this box does not contain $i+1$. Suppose this occurs at the l-th column of S'.

Assume first that $\boxed{i}_{r_n+2(k-1)+n-2i+3}$ is the last box of the l-th column of S'. We have two cases:

(i) the l-th column of S' has length i;
(ii) the l-th column of S' has length strictly smaller than i.

In both cases we will get a contradiction.

Case (i). Since S' has the same shape of S, the columns of length i have their last box supported at $r_i + 2(v-1) - i + 1$, $v = 1, \dots, \lambda(h_i)$. Thus, we have

$$r_n + 2(k-1) + n - 2i + 3 = r_i + 2(v-1) - i + 1 \Rightarrow r_n + 2(k-1) + n - i + 2 = r_i + 2(v-1)$$

contradicting the hypothesis of the proposition.

Case (ii). Since the l-th column of S' has the last box $\boxed{i}$ and length strictly smaller than i, it follows that this column has a gap. Suppose that $\boxed{j}_r$, $j \le i$, is a box of the l-th column of S' such that $\boxed{j-1}_{r+2}$ is not a box of this column. Observe that $r \ge r_n + 2(k-1) + n - 2i + 3$. By Lemmas 3.1.6 and 3.2.3, $Y^{-1}_{j-1,r+j}$ appears in $\boldsymbol{\omega}^{S'}$. Since $\boldsymbol{\mu\nu} \in D$, $Y_{j-1,r+j}$ must appear in $\boldsymbol{\nu}$ and, hence, $j = i$ by (3.4.9). This implies

$$r + j = r + i = r_n + 2(k-1) + n - i + 3 \ > \ r_n + 2(k-1) + n - i + 2,$$

contradicting the second claim in (3.4.9).

Finally, assume that $\boxed{i}_{r_n+2(k-1)+n-2i+3}$ is not the last box of the l-th column of S'. This implies that there exists $j > i+1$ such that $\boxed{j}_{r_n+2(k-1)+n-2i+1}$ is a box of this column. By Lemmas 3.1.6 and 3.2.3, $Y^{-1}_{j,r_n+2(k-1)+n-2i+j+1}$ appears in $\boldsymbol{\omega}^{S'}$ and, hence, $Y_{j,r_n+2(k-1)+n-2i+j+1}$ must appear in $\boldsymbol{\nu}$. Since $j > i+1$, we get

$$r_n + 2(k-1) + n - 2i + j + 1 > r_n + 2(k-1) + n - i + 2$$

contradicting the second claim in (3.4.9) again. □

Now we suppose condition (3.4.5) is satisfied by (p, k'), for some $p \in \operatorname{supp}(\lambda)$ and $k' \in \{1, \dots, \lambda(h_p)\}$. Set

$$b = c - k' \tag{3.4.10}$$

and observe that (3.4.5) implies

$$r_n + 2(k-1) + n - 2p + 1 + 2 = r_n + 2(k-1) + n - 2p + 3 = r_p + 2(k'-1) - p + 1.$$

This means that, writing

$$s := r_n + 2(k-1) + n - 2p + 1, \tag{3.4.11}$$

then,

$$s \text{ is the support of the } p\text{-th box of the first column of } T, \tag{3.4.12}$$

and

$$s+2 \ \text{ is the support of the the last box of the } b\text{-th column of } S. \tag{3.4.13}$$

Since S has the form (3.2.4), it follows that, for all $1 \le t \le k$, if s' is the support of a box in the l-th column S, then

$$s' \le s - 2(t-1) + 2 \quad \Rightarrow \quad l \ge b + t - 1. \tag{3.4.14}$$

Observe that $s - 2(t-1)$ is the support of the p-th box of the t-th column of T. Observe also that, the b-th column of S is its $(1 + \lambda(h_p) - k')$-th column of length p, or equivalently, the k'-th counted from right to left. In particular,

$$\text{if } l \ge c, \text{ the } l\text{-th column of } S \text{ has at most } p-1 \text{ boxes.}$$

We also record that, for all $0 \le t \le k$, we have

$$\begin{aligned} \boldsymbol{\omega}^{T_{t,p}} &= \left(\prod_{l=1}^{t} Y_{p-1,s+p-2(l-1)} \ Y^{-1}_{p,s+p-2(l-1)+1} \right) \left(\prod_{l=1}^{k-t} Y_{n,s-2(t+l-p)-n+1} \right) \\ &= Y_{p-1,s+p-2(t-1),t} \ Y^{-1}_{p,s+p-2t+3,t} \ Y_{n,s-2(k-p)-n+1,k-t}. \end{aligned} \tag{3.4.15}$$

Lemma 3.4.6. Every element of D is of the form $\boldsymbol{\omega}^{S'}\boldsymbol{\omega}^{T_{t,p}}$ for some $0 \le t \le k$ and $S' \in \operatorname{STab}(S)$.

Proof. We will prove that, if $T' \in \operatorname{STab}(T)$, then

$$T' \ne T_{t,p} \ \Rightarrow \ \boldsymbol{\omega}^{T'}\boldsymbol{\omega}^{S'} \notin \mathcal{P}^+ \text{ for all } S' \in \operatorname{STab}(S). \tag{3.4.16}$$

Thus, suppose $T' \ne T_{t,p}$ for all $0 \le t \le k$ and, by contradiction, suppose there exists S' such that $\boldsymbol{\omega}^{T'}\boldsymbol{\omega}^{S'} \in \mathcal{P}^+$. Since $T_{0,p} = T$, we must have $T' \ne T$ and, hence, the first column of T' must contain a gap (necessarily of size 1). The hypothesis $T' \ne T_{t,p}$ implies that T' has a column containing a gap not located at its p-box. Suppose the t_0-th column of T' is the first such column and assume the gap occurs at the j-th box. Then, Lemmas 3.1.6 and 3.2.3 imply that $Y^{-1}_{j,r_n+2(k-t_0)+n-j+2}$ appears in $\boldsymbol{\omega}^{T'}$ and, hence, $Y_{j,r_n+2(k-t_0)+n-j+2}$ must appear in $\boldsymbol{\omega}^{S'}$. This means that $\boxed{j}_{r_n+2(k-t_0)+n-2j+3}$

must be a box in S'. Since this box does not appear at any other column of S' by Lemma 3.2.3, it follows that

$$\boxed{j{+}1}_{r_n+2(k-t_0)+n-2j+1} \quad \text{is not a box in} \quad S' \tag{3.4.17}$$

(otherwise, it would cancel the $Y_{j,r_n+2(k-t_0)+n-j+2}$ coming from $\boxed{j}_{r_n+2(k-t_0)+n-2j+3}$).

Suppose $j > p$ and notice that this implies that,

$$\text{if } \boxed{i}_{s'} \text{ is a box in } T' \text{ with } s' > s, \text{ then it is the } i\text{-th box of its column.} \tag{3.4.18}$$

Indeed, the condition $s' > s$ implies that this box is among the first $p-1$ boxes of its column. The assumption on j implies that all gaps in T' occur in or after the p-th box of each column. Hence, the content of the boxes of T' supported at s' must be equal to its position in the column.

Say $\boxed{j}_{r_n+2(k-t_0)+n-2j+3}$ is in the l-th column of S'. Since

$$r_n + 2(k-t_0) + n - 2j + 3 < r_n + 2(k-t_0) + n - 2p + 3 = s - 2(t_0-1) + 2,$$

(3.4.14) implies that $l \geq b + t_0 - 1 \geq b$. S' being semi-standard, its b-th column must have a box whose content is at least j. Since the length of the b-th column is p, this implies that the b-th column of S' has a gap. Suppose the j'-th box of the b-th column of S' has a gap and let d be the content of this box. In particular, since the columns are increasing, we have

$$d > j' \tag{3.4.19}$$

Thus, $\boxed{d}_{s+2(p-j')}$ is a box in the b-th column of S' while $\boxed{d{-}1}_{s+2(p-j')+2}$ is not. This implies that $Y^{-1}_{d-1,s+2(p-j')+d}$ appears in $\boldsymbol{\omega}^{S'}$ and hence, $\boxed{d{-}1}_{s+2(p-j')+2}$ must be a box in T', say, at its l'-th column. Observe that, $s + 2(p-j') + 2 = s + 2(p-(j'-1))$ is the support of the $(j'-1)$-th box of the first column of T' and, hence, it is also the support of the $(j'-l')$-th box of the l'-th column of T'. Moreover, since $s + 2(p-j') + 2 > s$, (3.4.18) implies that $d - 1 = j' - l'$. Hence,

$$d = j' - (l'-1) \leq j' \tag{3.4.20}$$

yielding the desired contradiction.

Suppose now that $j < p$. Since T' is semi-standard, this implies that $t_0 = 1$ and, hence,

$$r_n + 2(k-t_0) + n - 2j + 3 = r_n + 2(k-1) + n - 2p + 1 + 2(p-j) + 2 = s + 2 + 2(p-j)$$

is the support of the j-th box of the b-th column of S' whose content is at least j (because the columns of S' are increasing). Since S' is semi-standard, this implies that $\boxed{j}_{r_n+2(k-1)+n-2j+3}$ is a box in the l-th column of S' with $l \geq b$. We have to analyze the cases $l = b$ and $l > b$ separately.

If $l = b$, $\boxed{j}_{r_n+2(k-1)+n-2j+3}$ is a box in the b-th column of S' and $\boxed{j{+}1}_{r_n+2(k-1)+n-2j+1}$ is not a box in S'. Thus, the b-th column of S' has $\boxed{d}_{r_n+2(k-1)+n-2j+1}$ with $d > j+1$. This gap generates the negative power $Y^{-1}_{d-1,r_n+2(k-1)+n-2j+1+d}$, forcing $\boxed{d{-}1}_{r_n+2(k-1)+n-2j+3}$ to be a box in T'. Observe that $r_n + 2(k-1) + n - 2j + 3 = s + 2(p-(j-1))$ is the support of the $(j-1)$-th row of the first column of T' and recall that, since this column has a gap in the j-th box, all boxes above it have have content equal to its position in the column. In particular, the content of the box supported at $r_n + 2(k-1) + n - 2j + 3$ is $j - 1 < d - 1$. Since T' is semi-standard, the content of the boxes of the remaining columns supported at $r_n + 2(k-1) + n - 2j + 3$ must be at most $j - 1$ and we have a contradiction.

Suppose now $l > b$. Using that S' is semi-standard, this implies that the b-th column of S' contains a box $\boxed{j'}_{r_n+2(k-1)+n-2j+3+2(l-b)}$ with $j' \geq j$. Since

$$r_n + 2(k-1) + n - 2j + 3 + 2(l-b) = s + 2 + 2(p - (j - (l-b)))$$

is the support of the $(j-(l-b))$-th box in the b-th column of S' and $j - (l-b) < j$, it follows that there exists a gap in the b-th column of S' at the j''-th box for some $j'' \leq j - (l-b) < j$. Let d be the content of this box, in particular, $d > j''$. This generates the negative power $Y^{-1}_{d-1,r_n+2(k-1)+n-2j''+3+d}$. To be canceled, it is necessary a box $\boxed{d-1}_{r_n+2(k-1)+n-2j''+3}$ in T (this box cannot exist in S' by Lemma 3.2.3). Look at the first column of S'. Since

$$r_n + 2(k-1) + n - 2j'' + 3 = s + 2 + 2(p - j'') = s + 2(p - (j''-1)),$$

$r_n + 2(k-1) + n - 2j'' + 3$ is the support of the $(j''-1)$-th row of the first column of T' (recall $j'' < j$), and this column has a gap at the j-th row, thus the boxes above the j-th have their contents equal to their position in the column. Hence T' has $\boxed{j''-1}_{r_n+2(k-1)+n-2j''+3}$ in the first column, and $d-1 > j''-1$ which implies $j''-1 \neq d-1$. At the other columns of T' we also cannot have $\boxed{d-1}_{r_n+2(k-1)+n-2j''+3}$, because the contents of the boxes supported at $r_n + 2(k-1) + n - 2j'' + 3$ are smaller or equal to $j''-2$ (because T' is semi-standard). This completes the proof of part (i) of the lemma. □

LEMMA 3.4.7. *Every element of D is of the form $\boldsymbol{\omega}^{S_{c,f,p}}\boldsymbol{\omega}^{T_{t,p}}$ for some $t = 0, 1, \dots, k$ and some $f \leq |\lambda|$.*

PROOF. By the first part of Lemma 3.4.6, we need to show that, if $\boldsymbol{\omega}^{S'}\boldsymbol{\omega}^{T_{t,p}} \in D$ for some $S' \in \mathrm{STab}(S)$ and some $t = 0, 1, \dots, k$, then $S' = S_{c,f,p}$ for some $f \leq |\lambda|$. If, $S' = S$, then $S' = S_{c,f,p}$ for any $f < c = \sum_{i=p}^{n-1} \lambda(h_i) + 1$ and there is nothing to do. Thus, assume $S' \neq S$. It follows from Lemma 3.4.3 that we must also have $T_{t,p} \neq T$ or, equivalently, $t \geq 1$. Observe that, if $p = 1$, then $S_{c,f,p} = S$ since, in this case, $f \leq |\lambda| < c$.

We need to study the structure of the gaps in S'. Since each gap contributes with the appearance of a term of the form $Y^{-1}_{i,r}$ in $\boldsymbol{\omega}^{S'}$, it follows that $Y_{i,r}$ must appear in $\boldsymbol{\omega}^{T_{t,p}}$. By (3.4.15), $Y_{i,r}$ must be among the following elements:

$$Y_{p-1,r_n+2(k-t')+n-p+1},\ 1 \leq t' \leq t, \quad \text{and} \quad Y_{n,r_n+2t''},\ 0 \leq t'' < k-t. \tag{3.4.21}$$

Hence, we must have $i \in \{p-1, n\}$.

We start showing that we cannot have $i = n$ (in particular, it follows that each column of S' can have at most one gap). Indeed, suppose the m-th column of S' contains such a gap which necessarily occurs at the last box whose content is $n+1$. Since S' is semi-standard of the form (3.2.4), this implies that the contents of all the last boxes of the columns to the left of the m-th column are also equal to $n+1$. In particular, if $l \leq \min\{m, b\}$, the content of the last box of the l-th column of S' is $n+1$. Moreover, (3.4.13) and (3.2.4) imply that the support of this box is $s + 2(b-l+1)$. Using Lemma 3.1.6, this implies that

$$Y_{i,r} = Y_{n,r_n+2(k+b-l+n-p+1)}$$

which contradicts (3.4.21) since $k + (b-l) + (n-p) + 1 \geq k + 1 > k - t$.

Since, as we have observed, we already know that each column of S' has at most one gap, it remains to show that there exists $f \leq \sum_{l=1}^{n-1} \lambda(h_l)$ such that the l-th column of S' has a gap iff $c \leq l \leq f$, the gap occurs at the last box and its size is p minus the length of that column.

Assume the l-th column of S' has a gap and let r be the support of the box where the gap occurs. In particular, since the support of the last box of this column is $s + 2(b-l+1)$, we have

$r \geq s + 2(b - l + 1)$. Let us show, by contradiction, that $l \geq c$. Assume first that $l \leq b$. Lemmas 3.1.6 and 3.2.3 imply that $Y_{p-1,r+p}^{-1}$ appears in $\boldsymbol{\omega}^{S'}$. But

$$r + p \geq s + 2(b - l + 1) + p \overset{(3.4.11)}{=} r_n + 2(k + b - l) + n - p + 1 > r_n + 2(k - 1) + n - p + 1,$$

which contradicts (3.4.21). Next, suppose $b < l < c$ which implies that the l-th column of S' has p boxes (see comment preceding Lemma 3.4.6). Since, S' is semi-standard, this implies the b-th column also has a gap, which is a contradiction by the case $l \leq b$.

Next, we show that if the l-th column has a gap, it must occur at its last box. Indeed, Lemma 3.1.6 and (3.4.21) imply that the content of this box of the gap must be p. Hence, if this were not the last box, it would follow that the content of the last box is at least $p+1$. Since S' is semi-standard of the form (3.2.4), this would imply that the last box of the b-th column is at least $p+1$ implying that the b-th column would have a gap (because it has p boxes), yielding a contradiction. The same reasoning implies that, if $c \leq l' \leq l$, the last box of the l'-th column has content p and, hence, has a gap (because its length is at most $p-1$). Notice also that, since there is no other gap in the l-th column, all the boxes but the last must have content equal to their position in the column. In particular, if the l-th column has m boxes, the content of its $(m-1)$-th box is $m-1$ and, hence, the size of the gap is $p - m$. $\square$

Recall (3.4.11) and the definitions of l_j and d_j in the paragraph preceding (3.3.3) and that, for $f \in \{c, \ldots, |\lambda|\}$, we have $l_f \leq p - 1$. Then, using Lemma 3.1.6, we see that

$$\begin{aligned}\boldsymbol{\omega}^{S_{c,f,p}} &= \left(\prod_{m=0}^{f-c} Y_{p-1,s+p-2(k'-1+m)}^{-1}\, Y_{p,s+p-2(k'-1+m)-1}\, Y_{l_{c+m}-1,s-2(k'-1+m)+l_{c+m}}\right) \\ &\quad \times \left(\prod_{\substack{m>f: \\ l_m = l_f}} Y_{l_f, r_{l_f}+2(m-(f+1))}\right) \left(\prod_{i=p}^{n-1} Y_{i,r_i,\lambda(h_i)}\right) \left(\prod_{i=1}^{l_f - 1} Y_{i,r_i,\lambda(h_i)}\right).\end{aligned} \tag{3.4.22}$$

The terms in the first line of the right-hand-side of (3.4.22) are the ones corresponding to the columns of $S_{c,f,p}$ which are not equal to those of S, while the ones in the second line come from the columns which were not modified. Observe that there is no cancelation in (3.4.22). Indeed, this is clear if $l_f < p - 1$ and can be easily checked if $l_f = p - 1$ (it is also guaranteed by Lemma 3.2.3). Now, a simple comparison of (3.4.22) with (3.4.15) completes the proof of (3.4.6).

It remains to prove the second statement of Proposition 3.4.1, which is clear in the case that $D = \{\boldsymbol{\omega}\boldsymbol{\varpi}\}$. Thus, we can assume (3.4.5) holds. Fix $\boldsymbol{\mu} \in D$, say

$$\boldsymbol{\mu} = \boldsymbol{\omega}^{S_{c,f,p}} \boldsymbol{\omega}^{T_{t,p}}$$

with $0 \leq t \leq \min\{k, \sum_{i=1}^{p-1} \lambda(h_i) + k'\}$ and $f = c + t - k' - \epsilon$ for some $\epsilon \in \{0, 1\}$. Suppose $S' \in \mathrm{STab}(S), T' \in \mathrm{STab}(T)$ are such that

$$\boldsymbol{\omega}^{S'} \boldsymbol{\omega}^{T'} = \boldsymbol{\mu}.$$

Since $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ are thin, we are left to show that $S' = S_{c,f,p}$ and $T' = T_{t,p}$. For doing that, notice that (3.2.11) and (3.3.9) have the form

$$\boldsymbol{\omega}^{S_{c,f,p}} = \boldsymbol{\omega} \prod_{\xi \in \Xi_1} A_\xi^{-1} \qquad \text{and} \qquad \boldsymbol{\omega}^{T_{t,p}} = \boldsymbol{\varpi} \prod_{\xi \in \Xi_2} A_\xi^{-1}$$

with $\Xi_1, \Xi_2 \subseteq I \times \mathbb{Z}$ satisfying

$$(i,r) \in \Xi_1 \ \Rightarrow\ i < p \quad \text{and} \quad r \leq s - 2(k'-1) + i \tag{3.4.23}$$

while

$$(i,r) \in \Xi_2 \ \Rightarrow\ i \geq p \quad \text{and} \quad r \leq s + p, \tag{3.4.24}$$

where s is given by (3.4.11). Setting $\Xi = \Xi_1 \cup \Xi_2$, it then follows from (2.2.1) that there must exist a partition $\Xi = \Xi_1' \cup \Xi_2'$ such that

$$\boldsymbol{\omega}^{S'} = \boldsymbol{\omega} \prod_{\xi \in \Xi_1'} A_\xi^{-1} \qquad \text{and} \qquad \boldsymbol{\omega}^{T'} = \boldsymbol{\varpi} \prod_{\xi \in \Xi_2'} A_\xi^{-1}.$$

We shall show that

$$\Xi_j \subseteq \Xi_j' \quad \text{for} \quad j = 1, 2, \tag{3.4.25}$$

which clearly completes the proof of Proposition 3.4.1.

We first show (3.4.25) for $j = 2$. By contradiction, suppose there exists $(i,r) \in \Xi_2 \cap \Xi_1'$. By (3.1.6), this implies that, for obtaining S' from S, a modification $\boxed{i}_{\,r-i} \to \boxed{i{+}1}_{\,r-i}$ was performed in some column of S. Since $r - i \leq s$ by (3.4.24), (3.4.13) implies that this modification can only be performed in columns to the right of the b-th column of S. In particular, the content of the last box of b-th column of S' is equal to p. We claim that the above modification also implies that the content of the last box of $(b+1)$-th column of S' is at least $p+1$, contradicting the fact that S' is semi-standard. Indeed, since $i \geq p$ by (3.4.24), the last box of the modified column has content larger or equal to $p+1$. Since S' is semi-standard, the same holds for the last box of the $(b+1)$-th column of S' as claimed.

Next, notice that, if either $t < k'$ or $t = k'$ and $f = c - 1$, then $\Xi_1 = \emptyset$ and we are done. In particular, we can assume $t \geq k'$. Notice also that (3.4.25) with $j = 2$ implies that T' is obtained from $T_{t,p}$ by modifications corresponding to elements of $\Xi_1 \cap \Xi_2'$. Again by contradiction, suppose that there exists $(i,r) \in \Xi_1 \cap \Xi_2'$. By (3.4.23), we have $i < p$ and $r \leq s - 2(k'-1) + i$. This implies that, for obtaining T' from $T_{t,p}$, a modification of the form $\boxed{i}_{\,r'} \to \boxed{i{+}1}_{\,r'}$ was performed in some column of $T_{t,p}$ with $i < p$ and $r' \leq s - 2(k'-1)$. Since $t \geq k'$, this together with (3.4.12), implies that the first $p-1$ boxes of the first t columns of T' coincide with those of $T_{t,p}$ and, hence, also coincide with those of T. In particular, if $1 \leq l \leq k$, then the l-th column of T' has a box supported at $s - 2(l-1) + 2$ and, if j is its content, we must have

$$j \leq p - 1 \tag{3.4.26}$$

with equality holding for $l \leq t$.

As observed in the paragraph preceding Lemma 3.3.3, we have

$$\xi_0 := (p-1, \min\{r : (p-1,r) \in \Xi_1\}) = (p-1, s - 2(t-1) + p - 1) \in \Xi_1.$$

Since (3.4.25) for $j = 2$ implies that $\Xi_1' \subseteq \Xi_1$, it follows from Lemma 3.3.3 that, in order to show (3.4.25) for $j = 1$, it suffices to show that we cannot have $\xi_0 \in \Xi_2'$. Since ξ_0 corresponds to the modification

$$\boxed{p{-}1}_{\,s-2(t-1)} \to \boxed{p}_{\,s-2(t-1)},$$

if it were $\xi_0 \in \Xi_2'$, the previous paragraph implies that such modification would be performed after the t-th column of $T_{t,p}$. Say that the modification is in the l-th column with $l > t$. Then, the content of the box of the l-th column of T' supported at $s - 2(l-1) + 2$ is at least p, contradicting (3.4.26). This completes the proof of Proposition 3.4.1.

3.5. Simple factors of certain tensor products

We keep the notation fixed in the previous sections and set

$$\boldsymbol{\lambda} = \boldsymbol{\omega}\boldsymbol{\varpi} \qquad \text{and} \qquad V = V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}).$$

Our next goal is to prove the following proposition.

Proposition 3.5.1. V is reducible if and only if either

$$\text{(3.4.4) holds with } k' \le |\lambda| \tag{3.5.1}$$

or

$$\text{(3.4.5) holds with } k' \le k. \tag{3.5.2}$$

If (3.5.1) holds, the simple factors of V are $V_q(\boldsymbol{\lambda})$ and $V_q(\boldsymbol{\mu})$ where

$$\boldsymbol{\mu} = \boldsymbol{\omega}^{S_{1,k',n+1}}\boldsymbol{\omega}^T.$$

If (3.5.2) holds, the simple factors of V are $V_q(\boldsymbol{\lambda})$ and $V_q(\boldsymbol{\mu})$ where

$$\boldsymbol{\mu} = \boldsymbol{\omega}^S\boldsymbol{\omega}^{T_{k',p}}.$$

Remark 3.5.2. Let $i_1 = \min\operatorname{supp}(\lambda), m_i = \lambda(h_i), i < n$, and $m_n = k$. Define also $a_i = q^{r_i+m_i-1}, i \in I$, so that $\boldsymbol{\lambda} = \prod_{i\in I}\boldsymbol{\omega}_{i,a_i,m_i}$. One easily checks that condition (3.5.1) is equivalent to

$$\frac{a_{i_1}q^{|\lambda|_{i_0}}}{a_n} = q^{-(|\lambda|_{i_0}+m_n\omega_n+n-i_1+2-2r)},$$

for some $r \in \{1,\dots,\min\{|\lambda|, m_n\}\}$. Indeed, this follows from (3.4.4) and (3.3.1) putting $r = k'$. Similarly, condition (3.5.2) is equivalent to

$$\frac{a_{i_1}q^{|\lambda|_{i_0}}}{a_n} = q^{|\lambda|_{i_0}+m_n\omega_n+n-i_1+2-2r},$$

with $r - (|\lambda|_{p-1} + p - i_1) \in \{1,\dots,\min\{\lambda(h_p), m_n\}\}$ for some $p \in \{1,\dots,i_0\}$. Indeed, this follows from (3.4.5) and (3.3.2) (with $(i,j) = (i_1,p)$) letting $r = |\lambda|_{p-1} + p - i_1 + k'$.

We prove Proposition 3.5.1 in the remainder of this section. Evidently, if neither (3.4.4) nor (3.4.5) hold, then D is a singleton and V must be irreducible.

Assume first that (3.4.4) holds, in which case $\boldsymbol{\mu}$ is the minimum of $D = \{\boldsymbol{\varpi}\boldsymbol{\omega}^{S_{1,j,n+1}} : j = 0,1,\dots,k'\}$. Set

$$\boldsymbol{\mu}_j = \boldsymbol{\varpi}\boldsymbol{\omega}^{S_{1,j,n+1}}$$

and recall that $S_{1,j,n+1} = S_{1,|\lambda|,n+1}$ for all $j \ge |\lambda|$. Proposition 3.5.1 clearly follows from Proposition 3.4.1 together with the following lemma.

Lemma 3.5.3. We have:

(a) $\boldsymbol{\mu}_j \in \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}))$ for all $0 \le j \le \min\{|\lambda|, k'-1\}$.
(b) $\boldsymbol{\mu}_{k'} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}))$ if $k' \le |\lambda|$.

Proof. Since $\boldsymbol{\mu}_0 = \boldsymbol{\lambda}$, part (a) clearly holds for $j = 0$. For $j > 0$, consider

$$\boldsymbol{\mu}'_j := \boldsymbol{\omega}^{S_{1,j,n}}\boldsymbol{\varpi}.$$

We claim that

$$\boldsymbol{\mu}'_j \in \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda})) \quad \text{for all} \quad j > 0.$$

Indeed, using Remark 2.6.8 with $\boldsymbol\lambda$ in place of $\boldsymbol\omega$ and $J = I\setminus\{n\}$, we have $\mathrm{wt}_\ell(\chi_J(\boldsymbol\lambda)) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol\lambda))$ and, hence, we are left to show that

$$\boldsymbol\mu'_j \in \mathrm{wt}_\ell(\chi_J(V_q(\boldsymbol\lambda))).$$

Note that $\boldsymbol\lambda_J = \boldsymbol\omega$ and, therefore, $V_q(\boldsymbol\lambda_J)$ is an increasing minimal affinization whose highest weight is determined by S. Proposition 3.2.1 then implies that $(\boldsymbol\omega^{S_{1,j,n}})_J \in \mathrm{wt}_\ell(V_q(\boldsymbol\lambda_J))$. One easily checks that

$$\boldsymbol\lambda \cdot \iota_J\left((\boldsymbol\omega^{-1}\boldsymbol\omega^{S_{1,j,n}})_J\right) = \boldsymbol\mu'_j$$

completing the proof of the claim. In particular, it follows that $\boldsymbol\mu'_j$ satisfies condition (i) of Proposition 2.6.7 with $\boldsymbol\lambda$ in place of $\boldsymbol\omega$ and $J = \{n\}$. Equation (3.3.7) implies that condition (ii) is also satisfied. As for condition (iii), (3.3.6) implies that

$$\boldsymbol\mu'_j = \boldsymbol\lambda\left(\prod_{i=l_j+1}^{i_0}\prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,n-1,r_i+2(\lambda(h_i)-m)}\right)\left(\prod_{m=1}^{d_j} A^{-1}_{l_j,n-1,r_{l_j}+2(\lambda(h_{l_j})-m)}\right).$$

Hence, there is no $\boldsymbol\nu \in \mathrm{wt}_\ell(V_q(\boldsymbol\lambda))$, $\boldsymbol\nu > \boldsymbol\mu'_j$ such that $\boldsymbol\mu'_j \in \mathrm{wt}_\ell(\chi_{\{n\}}(\boldsymbol\nu))$, since the elements of $\mathrm{wt}_\ell(\chi_{\{n\}}(\boldsymbol\nu))$ are of the form $\boldsymbol\nu\prod_{r\in\mathbb Z} A_{n,r}^{-m_r}$ with $m_r \in \mathbb Z_{\ge 0}$ (see (2.2.1)). It then follows from Proposition 2.6.7 that

$$\mathrm{wt}_\ell(\chi_{\{n\}}(\boldsymbol\mu'_j)) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol\lambda)).$$

Part (a) of the lemma now follows if we show that

$$\boldsymbol\mu_j \in \mathrm{wt}_\ell(\chi_{\{n\}}(\boldsymbol\mu'_j)) \quad \text{for all} \quad 1 \le j \le \min\{|\lambda|, k'-1\}. \tag{3.5.3}$$

Observe that (3.3.5), (3.3.6), (3.3.1) and (3.4.4) imply that

$$\boldsymbol\mu_j = \boldsymbol\mu'_j\left(\prod_{l=1}^{j} A^{-1}_{n,r_n+2(k'-1-l)+1}\right). \tag{3.5.4}$$

Moreover, (3.3.7) and (3.4.4) imply that

$$\pi_n(\boldsymbol\omega^{S_{1,j,n}}) = Y_{n,r_n+2(k'-1-j),j}$$

and, hence,

$$\pi_n(\boldsymbol\mu'_j) = Y_{n,r_n+2(k'-1-j),j}\; Y_{n,r_n,k}.$$

One easily checks that, if $j \le \min\{|\lambda|, k'-1\}$, the above is the q-factorization of $\pi_n(\boldsymbol\mu'_j)$. Thus, Corollary 2.7.2 implies that

$$V_q(\pi_n(\boldsymbol\mu'_j)) \cong V_q(Y_{n,r_n+2(k'-1-j),j}) \otimes V_q(Y_{n,r_n,k}).$$

Applying (3.2.12) to the first factor of the above tensor product and comparing with (3.5.4) one easily deduces (3.5.3).

To prove part (b), consider the tableau S' formed by the first $k'-1$ columns of S and S'' be the one obtained by juxtaposing T and $S\setminus S'$. Note that the hypothesis $k' \le |\lambda|$ implies that $S\setminus S' \ne \emptyset$. Then,

$$\boldsymbol\omega^{S'} = Y_{l_{k'},r_{k'}+2(\lambda(h_{l_{k'}})-d_{k'}+1),d_{k'}-1}\prod_{i=l_{k'}+1}^{i_0} Y_{i,r_i,\lambda(h_i)},$$

$$\boldsymbol\omega^{S''} = \left(\prod_{i=1}^{l_{k'}-1} Y_{i,r_i,\lambda(h_i)}\right) Y_{l_{k'},r_{l_{k'}},\lambda(h_{l_{k'}})-d_{k'}+1}\; Y_{n,r_n,k},$$

and $\boldsymbol{\omega}^{S'}\boldsymbol{\omega}^{S''} = \boldsymbol{\lambda}$. In particular, $V_q(\boldsymbol{\lambda})$ is the simple quotient of the submodule of $V_q(\boldsymbol{\omega}^{S'}) \otimes V_q(\boldsymbol{\omega}^{S''})$ generated by the top weight space and part (b) follows if we show that

$$\boldsymbol{\mu}_{k'} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{S'}) \otimes V_q(\boldsymbol{\omega}^{S''})). \tag{3.5.5}$$

It is clear from the construction of S' that $V_q(\boldsymbol{\omega}^{S'})$ is an increasing minimal affinization. We claim that $V_q(\boldsymbol{\omega}^{S''})$ is also an increasing minimal affinization, i.e., that

$$r_n = r_{l_{k'}} + 2(\lambda(h_{l'_k}) - d_{k'}) + n - l_{k'} + 2. \tag{3.5.6}$$

To prove this claim, set

$$s := r_{i_0} + 2(\lambda(h_{i_0}) - 1) - i_0 + 1. \tag{3.5.7}$$

Then, (3.2.3) implies that

$$s \text{ is the support of the last box of the first column of } S,$$

and it follows from (3.2.3), together with condition (3.4.4), that

$$s+2 \ \text{ is the support of the last box of the } (k-k'+1)\text{-th column of } T \tag{3.5.8}$$

(the $(k-k'+1)$-th column of T is the k'-th one counted from right to left). Therefore, the support of the last box of the k'-th column of S is

$$s - 2(k'-1) = r_{l_{k'}} + 2(\lambda(h_{l_{k'}}) - d_{k'}) - l_{k'} + 1. \tag{3.5.9}$$

Together with (3.4.4) this proves (3.5.6).

By (3.3.5),

$$\boldsymbol{\mu}_{k'} = \boldsymbol{\lambda} \left(\prod_{i=l_{k'}+1}^{i_0} \prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,n,r_i+2(\lambda(h_i)-m)} \right) \left(\prod_{m=1}^{d_{k'}} A^{-1}_{l_{k'},n,r_{l_{k'}}+2(\lambda(h_{l_{k'}})-m)} \right). \tag{3.5.10}$$

This gives rise to an expression for $\boldsymbol{\lambda}^{-1}\boldsymbol{\mu}_{k'}$ as a product of distinct simple ℓ-roots:

$$\boldsymbol{\lambda}^{-1}\boldsymbol{\mu}_{k'} = \prod_{\xi \in \Xi} A_\xi^{-1}$$

where $\Xi \subseteq I \times \mathbb{Z}$. Equation (3.5.5) follows if we show that there is no partition $\Xi = \Xi' \cup \Xi''$ such that

$$\boldsymbol{\omega}^{S'} \prod_{\xi \in \Xi'} A_\xi^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{S'})) \qquad \text{and} \qquad \boldsymbol{\omega}^{S''} \prod_{\xi \in \Xi''} A_\xi^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{S''})).$$

By contradiction, assume such a partition exists. It follows from (2.2.1) and (3.1.6), that each element $A_\xi^{-1}, \xi \in \Xi$, corresponds to adding 1 to the content of a particular box of either S' or S''. More precisely, if $\xi = (i,r)$, then (3.1.6) implies that the modification associated to A_ξ^{-1} is of the form

$$\boxed{i}_{\,r-i} \longrightarrow \boxed{i{+}1}_{\,r-i}.$$

Inspecting (3.5.10), one checks that

$$\max\{r - i : (i,r) \in \Xi\} = s.$$

Let $\xi = (i,r)$ be such that $r - i = s$. This means that the boxes of S' and S'' supported above s are not being modified. Together with (3.5.8), it follows that the first $k-k'+1$ columns of S'' are left unmodified. In particular, the content of the last box of the first column of S'' is n, since this column comes from T. Knowing that $V_q(\boldsymbol{\omega}^{S''})$ is a minimal affinization, this implies, together with

Proposition 3.2.1, that all first k columns of S'' are left unmodified. On the other hand, another inspection of (3.5.10) (cf. (3.5.9)) shows that

$$(n, s-2(k'-1)+n) \in \Xi.$$

Since this corresponds to a modification of the form

$$\boxed{n}_{\,s-2(k'-1)} \longrightarrow \boxed{n+1}_{\,s-2(k'-1)},$$

this box must be the last box of its column. Since both $V_q(\boldsymbol{\omega}^{S'})$ and $V_q(\boldsymbol{\omega}^{S''})$ are minimal affinizations, the modified tableau must be semi-standard and, hence, the last box of each column to the left must also have $n+1$ as content. Therefore, by the previous discussion, $(n, s-2(k'-1)+n)$ cannot be in Ξ''. Observing that, by construction, the box of S' having the lowest support is supported at $s-2(k'-1)+2$, it follows that $(n, s-2(k'-1)+n)$ cannot be in Ξ' as well, yielding the desired contradiction. $\square$

Henceforth, assume (3.4.5) holds. Then, by (3.4.6),

$$D = \left\{ \boldsymbol{\omega}^{S_{c,c+t-k'-1,p}} \boldsymbol{\omega}^{T_{t,p}},\ \boldsymbol{\omega}^{S_{c,c+t-k',p}} \boldsymbol{\omega}^{T_{t,p}} \quad : \quad 0 \le t \le \min\left\{ k, \sum_{l=1}^{p-1} \lambda(h_l) + k' \right\} \right\},$$

where $c = 1 + \sum_{l=p}^{i_0} \lambda(h_l)$ (recall that $S_{c,f,p} = S$ for $f < c$ and for $c > |\lambda|$, $S_{c,f,p} = S_{c,|\lambda|,p}$ for $f > |\lambda|$, and $T_{t,p} = T_{k,p}$ for $t > k$). One easily checks that, if either $k' > k$ or $p = \min \operatorname{supp}(\lambda)$, then $\boldsymbol{\mu} = \boldsymbol{\omega}\boldsymbol{\omega}^{T_{k,p}}$ is the smallest element of D. Similarly to the previous case, Proposition 3.5.1 easily follows from Proposition 3.4.1 and the following lemma.

Lemma 3.5.4. We have

(a) $\{\boldsymbol{\nu} \in D : \boldsymbol{\nu} > \boldsymbol{\mu}\} \subseteq \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$.
(b) If $k' > k$, then $\boldsymbol{\mu} \in \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$ or, equivalently, $D \subseteq \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$.
(c) If $k' \le k$, then $\boldsymbol{\mu} \notin \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$.
(d) If $p > \min \operatorname{supp}(\lambda)$, then $\left\{ \boldsymbol{\omega}^{S_{c,c+t-k',p}} \boldsymbol{\omega}^{T_{t,p}} : k' \le t \le \min\left\{ k, \sum_{l=1}^{p-1} \lambda(h_l) + k' \right\} \right\} \subseteq \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$.
(e) If $k' \le k$, then $\left\{ \boldsymbol{\omega}^{S_{c,c+t-k'-1,p}} \boldsymbol{\omega}^{T_{t,p}} : k' \le t \le \min\left\{ k, \sum_{l=1}^{p-1} \lambda(h_l) + k' \right\} \right\} \subseteq \operatorname{wt}_\ell(V_q(\boldsymbol{\mu}))$.

Proof. Observe that

$$\{\boldsymbol{\nu} \in D : \boldsymbol{\nu} \ge \boldsymbol{\mu}\} = \{\boldsymbol{\mu}_t : 0 \le t \le \min\{k', k\}\}$$

where

$$\boldsymbol{\mu}_t := \boldsymbol{\omega}^S \boldsymbol{\omega}^{T_{t,p}}.$$

Since $\boldsymbol{\mu}_0 = \boldsymbol{\lambda}$, clearly $\boldsymbol{\mu}_0 \in \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$. Define

$$\boldsymbol{\mu}'_t := \boldsymbol{\omega}^S \boldsymbol{\omega}^{T_{t,p+1}}.$$

We claim that

$$\boldsymbol{\mu}'_t \in \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda})) \quad \text{for all} \quad t = 1, \dots, \min\{k'-1, k\}.$$

Indeed, using Remark 2.6.8 with $\boldsymbol{\lambda}$ in place of $\boldsymbol{\omega}$ and $J = \{p+1, \dots, n\}$, we have $\operatorname{wt}_\ell(\chi_J(\boldsymbol{\lambda})) \subseteq \operatorname{wt}_\ell(V_q(\boldsymbol{\lambda}))$ and, hence, in order to prove the claim, we are left to show that

$$\boldsymbol{\mu}'_t \in \operatorname{wt}_\ell(\chi_J(V_q(\boldsymbol{\lambda}))) \quad \text{for all} \quad t = 1, \dots, \min\{k'-1, k\}.$$

By definition of p, applying Proposition 3.4.5 to the algebra $U_q(\tilde{\mathfrak{g}}_J)$ with $\boldsymbol{\omega}_J$ in place of $\boldsymbol{\omega}$ and $\boldsymbol{\varpi}_J$ in place of $\boldsymbol{\varpi}$ in place of $\boldsymbol{\varpi}$, we get

$$V_q(\boldsymbol{\omega}_J) \otimes V_q(\boldsymbol{\varpi}_J) \cong V_q(\boldsymbol{\omega}_J \boldsymbol{\varpi}_J).$$

Therefore, by Theorem 3.2.1, $(\boldsymbol{\omega}^{T_{t,p+1}})_J \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}_J))$. One easily checks that

$$\boldsymbol{\lambda} \cdot \iota_J \left((\boldsymbol{\omega}^{-1}\boldsymbol{\varpi}^{-1}\boldsymbol{\omega}^{T_{t,p+1}})_J\right) = \boldsymbol{\mu}'_t$$

completing the proof of the claim.

Next, we use Proposition 2.6.7 with $\boldsymbol{\lambda}$ in place of $\boldsymbol{\omega}$ and $J = \{p\}$. The previous paragraph implies that $\boldsymbol{\mu}'_t$ satisfies condition (i) of that Proposition for $t \le \min\{k'-1, k\}$. Equation (3.4.15), with $p+1$ in place of p, implies that condition (ii) is also satisfied. As for condition (iii), (3.2.11) implies that

$$\boldsymbol{\mu}'_t = \boldsymbol{\lambda} \cdot \prod_{l=1}^{t} A^{-1}_{n,p+1,r_n+2(k-l)}.$$

On the other hand, for any $\boldsymbol{\nu} \in \mathcal{P}^+_{\{p\}}$, the elements of $\mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\nu}))$ are of the form $\boldsymbol{\nu}\prod_{r\in\mathbb{Z}} A_{p,r}^{-m_r}$ with $m_r \in \mathbb{Z}_{\ge 0}$. Since $p < n$, (2.2.1) implies that condition (iii) must also be satisfied. It then follows from Proposition 2.6.7 that

$$\mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\mu}'_t)) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda})).$$

Parts (a) and (b) of the lemma now follows if we show that

$$\boldsymbol{\mu}_t \in \mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\mu}'_t)) \quad \text{for all} \quad 1 \le j \le \min\{k'-1, k\}. \tag{3.5.11}$$

Observe that (3.2.11) implies that

$$\boldsymbol{\mu}_t = \boldsymbol{\mu}'_t \left(\prod_{l=1}^{t} A^{-1}_{p,r_n+2(k-l)+n-p+1}\right). \tag{3.5.12}$$

Moreover, (3.4.15) implies that

$$\pi_p(\boldsymbol{\mu}'_t) = Y_{p,r_n+2(k-t)+n-p,t}\, Y_{p,r_p,\lambda(h_p)} \overset{(3.4.5)}{=} Y_{p,r_p+2(k'-1-t),t}\, Y_{p,r_p,\lambda(h_p)}.$$

One easily checks that, since $t \le \min\{k'-1, k\}$, the above is the q-factorization of $\pi_p(\boldsymbol{\mu}'_t)$. Thus, Corollary 2.7.2 implies that

$$V_q(\pi_p(\boldsymbol{\mu}'_t)) \cong V_q(Y_{p,r_p+2(k'-1-t),t}) \otimes V_q(Y_{p,r_p,\lambda(h_p)}) = V_q(Y_{p,r_n+2(k-t)+n-p,t}) \otimes V_q(Y_{p,r_p,\lambda(h_p)}).$$

Applying (3.2.12) to the first factor of the above tensor product and comparing with (3.5.12) one easily deduces (3.5.11).

Suppose now $k' \le k$. To prove the first statement in (c), consider the tableau S' formed by the first $\sum_{i=p+1}^{i_0} \lambda(h_i)$ columns of S, the tableau S'' formed by juxtaposing the first $\lambda(h_p) - k' + 1$ columns of length p of S and T, and finally the tableau S''' formed by the remaining columns of S (i.e., the tableau whose columns are those to the right of the b-th column of S where b is given by (3.4.10)). Then,

$$\boldsymbol{\omega}^{S'} = \prod_{j=p+1}^{i_0} Y_{j,r_j\lambda(h_j)}, \qquad \boldsymbol{\omega}^{S''} = \left(Y_{p,r_p+2(k'-1),\lambda(h_p)-k'+1}\right)\left(Y_{n,r_n,k}\right),$$

$$\boldsymbol{\omega}^{S'''} = \left(\prod_{j=1}^{p-1} Y_{j,r_j,\lambda(h_j)}\right)\left(Y_{p,r_p,k'-1}\right), \quad \text{and} \quad \boldsymbol{\omega}^{S'}\boldsymbol{\omega}^{S''}\boldsymbol{\omega}^{S'''} = \boldsymbol{\lambda}.$$

In particular, $V_q(\boldsymbol{\lambda})$ is the simple quotient of the submodule of $V_q(\boldsymbol{\omega}^{S'}) \otimes V_q(\boldsymbol{\omega}^{S''}) \otimes V_q(\boldsymbol{\omega}^{S'''})$ generated by the top weight space and part (c) follows if we show that

$$\boldsymbol{\mu} \notin \mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}^{S'}) \otimes V_q(\boldsymbol{\omega}^{S''}) \otimes V_q(\boldsymbol{\omega}^{S''})\right). \tag{3.5.13}$$

It is clear from the construction of S' and S''' that $V_q(\boldsymbol{\omega}^{S'})$ and $V_q(\boldsymbol{\omega}^{S'''})$ are increasing minimal affinizations. On the other hand, (3.4.5) implies that $V_q(\boldsymbol{\omega}^{S''})$ is a decreasing minimal affinization. In any case, the ℓ-weights of all three factors are represented by the corresponding set of semi-standard tableaux.

By (3.2.11),

$$\boldsymbol{\mu} = \boldsymbol{\lambda} \left(\prod_{l=1}^{k'} A^{-1}_{n,p,r_n+2(k-l)} \right). \tag{3.5.14}$$

This gives rise to an expression for $\boldsymbol{\lambda}^{-1}\boldsymbol{\mu}$ as a product of distinct simple ℓ-roots:

$$\boldsymbol{\lambda}^{-1}\boldsymbol{\mu} = \prod_{\xi\in\Xi} A_\xi^{-1}$$

where $\Xi \subseteq I \times \mathbb{Z}$. Equation (3.5.13) follows if we show that there is no partition $\Xi = \Xi' \cup \Xi'' \cup \Xi'''$ such that

$$\boldsymbol{\omega}^{S'} \prod_{\xi\in\Xi'} A_\xi^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{S'})), \quad \boldsymbol{\omega}^{S''} \prod_{\xi\in\Xi''} A_\xi^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{S''})), \quad \boldsymbol{\omega}^{S'''} \prod_{\xi\in\Xi'''} A_\xi^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{S'''})).$$

By contradiction, assume such a partition exists. As before, each element $\xi \in \Xi$, say $\xi = (i, r)$, corresponds to a modification of the form

$$\boxed{i}_{\,r-i} \longrightarrow \boxed{i{+}1}_{\,r-i}$$

in some of S', S'', S'''. Inspecting (3.5.14), one checks that

$$\max\{r - i : i \in I, (i, r) \in \Xi\} = s \qquad \text{and} \qquad \min\{i : (i, r) \in \Xi \text{ for some } r \in \mathbb{Z}\} = p, \tag{3.5.15}$$

where s is given by (3.4.12). This means that the boxes of S', S'', and S''' with support larger than s are not modified. Together with (3.4.13), it follows that S' and the first $\lambda(h_p) - k' + 1$ columns of S'' are left unmodified. In particular, the content of the last box of the first $\lambda(h_p) - k' + 1$ columns of S'' is p since these columns are the first $\lambda(h_p) - k' + 1$ columns of length p of S. Since $V_q(\boldsymbol{\omega}^{S''})$ is a minimal affinization, this implies, together with Theorem 3.2.1, that the first p boxes of every column of S'' are left unmodified.

On the other hand, another inspection of (3.5.14), recalling that $k' \leq k$, shows that

$$\xi_0 = (p, s - 2(k' - 1) + p) \in \Xi.$$

This corresponds to a modification of the form

$$\boxed{p}_{\,s-2(k'-1)} \longrightarrow \boxed{p{+}1}_{\,s-2(k'-1)}. \tag{3.5.16}$$

Since all the tableaux are column increasing, if the box on which this modification is being performed is the j-th box on its column, we must have $j \leq p$. Hence, it follows from the previous paragraph that $\xi_0 \in \Xi'''$. We will show that this is a contradiction.

Indeed, by construction, the last box of the last column of length p of S''' is supported at $s - 2(k' - 1) + 2$. If we had $\min \mathrm{supp}(\lambda) = p$, it would follow that S''' had no box supported at $s - 2(k' - 1)$, yielding a contradiction. Thus, we can assume that $\min \mathrm{supp}(\lambda) < p$ and, hence, the first column of S''' which has a box supported at $s - 2(k' - 1)$ has length $i < p$. This implies that, before the modification (3.5.16) can be performed, we need a modification of the form

$$\boxed{i}_{\,s-2(k'-1)} \longrightarrow \boxed{i{+}1}_{\,s-2(k'-1)}$$

which implies that $(i, s - 2(k' - 1) + i) \in \Xi$, contradicting the second statement in (3.5.15).

We now prove part (d). Fix $\boldsymbol{\nu} = \boldsymbol{\omega}^{S_{c,c+t-k',p}}\boldsymbol{\omega}^{T_{t,p}}$, $k' \le t \le \min\{k, \sum_{l=1}^{p-1} \lambda(h_l) + k'\}$. Define

$$\boldsymbol{\nu}' := \boldsymbol{\omega}^{S_{c,c+t-k',p}}\boldsymbol{\omega}^{T_{t,p+1}}.$$

Similarly to part (a), one can prove that

$$\boldsymbol{\nu}' \in \mathrm{wt}_\ell(\chi_J(V_q(\boldsymbol{\lambda}))) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda})), \tag{3.5.17}$$

where $J = I \setminus \{p\}$.

Next, we use Proposition 2.6.7 with $\boldsymbol{\lambda}$ in place of $\boldsymbol{\omega}$ and $J = \{p\}$. (3.5.17) implies that $\boldsymbol{\nu}'$ satisfies condition (i) of that Proposition. Equation (3.4.15), with $p+1$ in place of p, and (3.4.22) imply that condition (ii) is also satisfied. As for condition (iii), (3.2.11) and (3.3.9) imply that

$$\boldsymbol{\nu}' = \boldsymbol{\lambda} \cdot \left(\prod_{i=l_f+1}^{p-1} \prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,p-1,r_i+2(\lambda(h_i)-m)} \right) \left(\prod_{m=1}^{d_f} A^{-1}_{l_f,p-1,r_{l_f}+2(\lambda(h_{l_f})-m)} \right) \left(\prod_{l=1}^{t} A^{-1}_{n,p+1,r_n+2(k-l)} \right).$$

On the other hand, for any $\boldsymbol{\eta} \in \mathcal{P}^+_{\{p\}}$, the elements of $\mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\eta}))$ are of the form $\boldsymbol{\eta} \prod_{r\in\mathbb{Z}} A_{p,r}^{-m_r}$ with $m_r \in \mathbb{Z}_{\ge 0}$. Since $p < n$ and

$$t \ge k' \Rightarrow f = c + t - k' \ge c = 1 + \sum_{l=p}^{i_0} \lambda(h_l) \Rightarrow l_f \le l_c \le p-1,$$

(2.2.1) implies that condition (iii) must also be satisfied. It then follows from Proposition 2.6.7 that

$$\mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\nu}')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda})).$$

Part (d) of the lemma now follows if we show that

$$\boldsymbol{\nu} \in \mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\nu}')). \tag{3.5.18}$$

Observe that (3.2.11) implies that

$$\boldsymbol{\nu} = \boldsymbol{\nu}' \left(\prod_{l=1}^{t} A^{-1}_{p,r_n+2(k-l)+n-p+1} \right). \tag{3.5.19}$$

Moreover, (3.4.15) and (3.4.22) imply that

$$\begin{aligned}
\pi_p(\boldsymbol{\nu}') &= \prod_{l=0}^{t-k'} Y_{p,s+p-2(k'-1+m)-1} Y_{p,r_n+2(k-t)+n-p,t}\, Y_{p,r_p,\lambda(h_p)} \\
&\overset{(3.4.5)}{=} \prod_{l=0}^{t-k'} Y_{p,s+p-2(k'-1+m)-1} Y_{p,r_p+2(k'-1-t),t}\, Y_{p,r_p,\lambda(h_p)} \\
&\overset{(3.4.11)}{=} \prod_{l=0}^{t-k'} Y_{p,r_p-2(l+1)} Y_{p,r_p+2(k'-1-t),t}\, Y_{p,r_p,\lambda(h_p)} \\
&= Y_{p,r_p+2(k'-1-t),t-k'+1} Y_{p,r_p+2(k'-1-t),t}\, Y_{p,r_p,\lambda(h_p)} \\
&= Y_{p,r_p+2(k'-1-t),\lambda(h_p)+t-k'+1}\, Y_{p,r_p+2(k'-1-t),t}.
\end{aligned}$$

One easily checks that the above is the q-factorization of $\pi_p(\boldsymbol{\nu}')$. Thus, Corollary 2.7.2 implies that

$$\begin{aligned}
V_q(\pi_p(\boldsymbol{\nu}')) &\cong V_q(Y_{p,r_p+2(k'-1-t),t}) \otimes V_q(Y_{p,r_p+2(k'-1-t),\lambda(h_p)+t-k'+1}) \\
&= V_q(Y_{p,r_n+2(k-t)+n-p,t}) \otimes V_q(Y_{p,r_p+2(k'-1-t),\lambda(h_p)+t-k'+1}).
\end{aligned}$$

Applying (3.2.12) to the first factor of the above tensor product and comparing with (3.5.19) one easily deduces (3.5.18).

We now prove (e). Fix $\boldsymbol{\nu} = \boldsymbol{\omega}^{S_{c,c+t-k'-1,p}}\boldsymbol{\omega}^{T_{t,p}}$, $k' \le t \le \min\{k, \sum_{l=1}^{p-1}\lambda(h_l)+k'\}$. If $t = k'$ the result follows from part (c). Suppose $t > k'$. Define

$$\boldsymbol{\nu}' := \boldsymbol{\omega}\boldsymbol{\omega}^{T_{t,p}} \quad \text{and} \quad \boldsymbol{\nu}'' := \boldsymbol{\omega}\boldsymbol{\omega}^{T_{t,p+1}}. \tag{3.5.20}$$

Observe that

$$\boldsymbol{\nu}'' = \boldsymbol{\mu}\left(\prod_{l=k'+1}^{t} A^{-1}_{n,p+1,r_n+2(k-l)}\right). \tag{3.5.21}$$

Similarly to part (a), one can prove that

$$\boldsymbol{\nu}'' \in \mathrm{wt}_\ell(\chi_J(V_q(\boldsymbol{\mu}))) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})), \tag{3.5.22}$$

where $J = \{p+1,\dots,n\}$. Next, we use Proposition 2.6.7 with $\boldsymbol{\mu}$ in place of $\boldsymbol{\omega}$ and $J = \{p\}$. (3.5.22) implies that $\boldsymbol{\nu}''$ satisfies condition (i) of that Proposition. Equation (3.4.15), with $p+1$ in place of p, implies that condition (ii) is also satisfied. As for condition (iii), for any $\boldsymbol{\eta} \in \mathcal{P}^+_{\{p\}}$, the elements of $\mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\eta}))$ are of the form $\boldsymbol{\eta}\prod_{r\in\mathbb{Z}} A_{p,r}^{-m_r}$ with $m_r \in \mathbb{Z}_{\ge 0}$. By (3.5.21) and (2.2.1), condition (iii) must also be satisfied. It then follows from Proposition 2.6.7 that

$$\mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\nu}'')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})).$$

Observe that

$$\begin{aligned}
\pi_{\{p\}}(\boldsymbol{\nu}'') &= \pi_{\{p\}}(\boldsymbol{\mu})\ \pi_{\{p\}}\left(\prod_{l=k'+1}^{t} A^{-1}_{p,r_n+2(k-l)+n-p+1}\right)\\
&= Y_{p,r_p+2k',\lambda(h_p)-k'}\left(\prod_{l=k'+1}^{t} Y_{p,r_n+2(k-l)+n-p}\right)\\
&= Y_{p,r_p+2k',\lambda(h_p)-k'}\ Y_{p,r_n+2(k-t)+n-p,t-k'}\\
&\overset{(3.4.5)}{=} Y_{p,r_p+2k',\lambda(h_p)-k'}\ Y_{p,r_p+2(k'-t-1),t-k'}.
\end{aligned}$$

One easily checks that the above is the q-factorization of $\pi_p(\boldsymbol{\nu}'')$. Thus, Corollary 2.7.2 implies that

$$\begin{aligned}
V_q(\pi_p(\boldsymbol{\nu}'')) &\cong V_q(Y_{p,r_p+2k',\lambda(h_p)-k'}) \otimes V_q(Y_{p,r_p+2(k'-t-1),t-k'})\\
&= V_q(Y_{p,r_p+2k',\lambda(h_p)-k'}) \otimes V_q(Y_{p,r_n+2(k-t)+n-p,t-k'}).
\end{aligned}$$

Thus, applying (3.2.12) to the first factor of the above tensor product one can see that

$$\boldsymbol{\nu}' = \boldsymbol{\nu}''\left(\prod_{l=k'+1}^{t} A^{-1}_{p,r_n+2(k-l)+n-p+1}\right) \in \mathrm{wt}_\ell(\chi_{\{p\}}(\boldsymbol{\nu}'')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})). \tag{3.5.23}$$

Next, we use Proposition 2.6.7 with $\boldsymbol{\mu}$ in place of $\boldsymbol{\omega}$ and $J = \{1,\dots,p-1\}$. (3.5.23) implies that $\boldsymbol{\nu}''$ satisfies condition (i) of that Proposition. Equation (3.4.15) implies that condition (ii) is also satisfied. As for condition (iii), for any $\boldsymbol{\eta} \in \mathcal{P}^+_J$, the elements of $\mathrm{wt}_\ell(\chi_J(\boldsymbol{\eta}))$ are of the form $\boldsymbol{\eta}\prod_{i\in J,r\in\mathbb{Z}} A_{i,r}^{-m_r}$ with $m_r \in \mathbb{Z}_{\ge 0}$. Combining (3.5.21) and (3.5.23), it follows that

$$\boldsymbol{\nu}' = \boldsymbol{\mu}\left(\prod_{l=k'+1}^{t} A^{-1}_{n,p,r_n+2(k-l)}\right).$$

Hence (2.2.1) implies that condition (iii) must also be satisfied. It then follows from Proposition 2.6.7 that

$$\mathrm{wt}_\ell(\chi_{\{1,\dots,p-1\}}(\boldsymbol{\nu}')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})).$$

In particular,

$$\mathrm{wt}_\ell(\chi_{\{1,\dots,p-2\}}(\boldsymbol{\nu}')) \subseteq \mathrm{wt}_\ell(\chi_{\{1,\dots,p-1\}}(\boldsymbol{\nu}')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})).$$

Let $f = c + t - k' - 1$. Observe that (3.4.22) implies that

$$\boldsymbol{\nu} = \boldsymbol{\nu}' \left(\prod_{i=l_f+1}^{p-1} \prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,p-1,r_i+2(\lambda(h_i)-m)} \right) \left(\prod_{m=1}^{d_f} A^{-1}_{l_f,p-1,r_{l_f}+2(\lambda(h_{l_f})-m)} \right). \tag{3.5.24}$$

(3.4.15) implies that

$$\pi_{\{1,\ldots,p-2\}}(\boldsymbol{\nu}') = \prod_{i=1}^{p-2} Y_{i,r_i,\lambda(h_i)}.$$

Let $\boldsymbol{\nu}''' := \boldsymbol{\omega}^{S_{c,f,p-1}} \boldsymbol{\omega}^{T_{t,p}}$. By (3.5.20) and (3.3.9), we have $\boldsymbol{\nu}''' = \boldsymbol{\nu}'$ if $l_f = p-1$ and

$$\boldsymbol{\nu}''' = \boldsymbol{\nu}' \left(\prod_{i=l_f+1}^{p-2} \prod_{m=1}^{\lambda(h_i)} A^{-1}_{i,p-2,r_i+2(\lambda(h_i)-m)} \right) \left(\prod_{m=1}^{d_f} A^{-1}_{l_f,p-2,r_{l_f}+2(\lambda(h_{l_f})-m)} \right) \quad \text{if } l_f < p-1. \tag{3.5.25}$$

Thus, it is not difficult to see that

$$\boldsymbol{\nu}''' \in \mathrm{wt}_\ell(\chi_{\{1,\ldots,p-2\}}(\boldsymbol{\nu}')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})). \tag{3.5.26}$$

Next, we use Proposition 2.6.7 one more time with $\boldsymbol{\mu}$ in place of $\boldsymbol{\omega}$ and $J = \{p-1\}$. By (3.5.26), $\boldsymbol{\nu}'''$ satisfies condition (i) of that Proposition. Equations (3.5.25) and (3.4.15) imply that

$$\pi_{p-1}(\boldsymbol{\nu}''') = Y_{p-1,r_{p-1},\lambda(h_{p-1})} \left(\prod_{m=0}^{t} Y_{p-1,r_n+2(k-m)+n-p-1} \right), \tag{3.5.27}$$

if $l_f = p-1$, and

$$\pi_{p-1}(\boldsymbol{\nu}''') = Y_{p-1,r_{p-1},\lambda(h_{p-1})} \left(\prod_{i=l_f+1}^{p-2} \prod_{m=1}^{\lambda(h_i)} Y_{p-1,r_i+2(\lambda(h_i)-m)+p-i-1} \right)$$
$$\times \left(\prod_{m=1}^{d_f} Y_{p-1,r_{l_f}+2(\lambda(h_{l_f})-m)+p-l_f-1} \right) \left(\prod_{m=0}^{t} Y_{p-1,r_n+2(k-m)+n-p-1} \right),$$

otherwise. Hence condition (ii) is also satisfied. As for condition (iii), for any $\boldsymbol{\eta} \in \mathcal{P}^+_{\{p-1\}}$, the elements of $\mathrm{wt}_\ell(\chi_{\{p-1\}}(\boldsymbol{\eta}))$ are of the form $\boldsymbol{\eta} \prod_{r\in\mathbb{Z}} A^{-m_r}_{p-1,r}$ with $m_r \in \mathbb{Z}_{\geq 0}$. Combining (3.5.21), (3.5.23) and (3.5.25), with (2.2.1), it follows that condition (iii) must also be satisfied. It then follows from Proposition 2.6.7 that

$$\mathrm{wt}_\ell(\chi_{\{p-1\}}(\boldsymbol{\nu}''')) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu})).$$

Part (e) of the lemma now follows if we show that

$$\boldsymbol{\nu} \in \mathrm{wt}_\ell(\chi_{\{p-1\}}(\boldsymbol{\nu}''')). \tag{3.5.28}$$

Observe that

$$\boldsymbol{\nu} = \boldsymbol{\nu}''' \left(\prod_{i=l_f+1}^{p-2} \prod_{m=1}^{\lambda(h_i)} A^{-1}_{p-1,r_i+2(\lambda(h_i)-m)+p-i} \right) \left(\prod_{m=1}^{d_f} A^{-1}_{p-1,r_{l_f}+2(\lambda(h_{l_f})-m)+p-l_f} \right),$$

where $d_f = t - k' - \sum_{i=l_f+1}^{p-1} \lambda(h_i)$. Thus, using (3.3.2), one can see that

$$\boldsymbol{\nu} = \boldsymbol{\nu}''' \left(\prod_{m=0}^{t-k'-1} A^{-1}_{p-1,r_p+2(k'-t+m)} \right), \tag{3.5.29}$$

and that $\pi_{p-1}(\boldsymbol{\nu}''')$ is equal to

$$Y_{p-1,r_{p-1},\lambda(h_{p-1})}\left(\prod_{m=0}^{t-k'-1} Y_{p-1,r_p+2(k'-t+m)-1}\right)\left(\prod_{m=0}^{t} Y_{p-1,r_n+2(k-m)+n-p-1}\right), \tag{3.5.30}$$

if $l_f < p-1$.

Suppose first $l_f = p-1$, which implies $d_f = t-k' \leq \lambda(h_{p-1})$. Observe that from by (3.4.5) and (3.5.27) that

$$\begin{aligned}
\pi_{p-1}(\boldsymbol{\nu}''') &= Y_{p-1,r_{p-1},\lambda(h_{p-1})}\left(\prod_{m=0}^{t} Y_{p-1,r_p+2(k-t+m)-1}\right)\\
&= Y_{p-1,r_{p-1},\lambda(h_{p-1})}\left(\prod_{m=t-k'}^{t} Y_{p-1,r_p+2(k-t+m)-1}\right)\left(\prod_{m=0}^{t-k'-1} Y_{p-1,r_p+2(k-t+m)-1}\right)\\
&\overset{(3.3.2)}{=} Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'+1}\left(\prod_{m=0}^{t-k'-1} Y_{p-1,r_p+2(k-t+m)-1}\right)\\
&= Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'+1}Y_{p-1,r_p+2(k-t)-1,t-k'}.
\end{aligned}$$

One easily checks that the above is the q-factorization of $\pi_{p-1}(\boldsymbol{\nu}')$. Thus, Corollary 2.7.2 implies that

$$V_q(\pi_{p-1}(\boldsymbol{\nu}''')) \cong V_q(Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'+1}) \otimes V_q(Y_{p-1,r_p+2(k-t)-1,t-k'}).$$

Applying (3.2.12) to the second factor of the above tensor product and comparing with (3.5.29) one easily deduces (3.5.28).

Suppose now $l_f < p-1$. Observe that from by (3.4.5) and (3.5.30) that

$$\begin{aligned}
\pi_{p-1}(\boldsymbol{\nu}''') &= Y_{p-1,r_{p-1},\lambda(h_{p-1})}\left(\prod_{m=0}^{t-k'-1} Y_{p-1,r_p+2(k'-t+m)-1}\right)\left(\prod_{m=0}^{t} Y_{p-1,r_p+2(k-t+m)-1}\right)\\
&= Y_{p-1,r_{p-1},\lambda(h_{p-1})}\left(\prod_{m=t-k'}^{t} Y_{p-1,r_p+2(k-t+m)-1}\right)\left(\prod_{m=0}^{t-k'-1} Y^2_{p-1,r_p+2(k-t+m)-1}\right)\\
&\overset{(3.3.2)}{=} Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'+1}\left(\prod_{m=0}^{t-k'-1} Y_{p-1,r_p+2(k-t+m)-1}\right)\\
&= Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'+1}Y_{p-1,r_p+2(k-t)-1,t-k'}.
\end{aligned}$$

One easily checks that the above is the q-factorization of $\pi_{p-1}(\boldsymbol{\nu}')$. Thus, Corollary 2.7.2 implies that

$$V_q(\pi_{p-1}(\boldsymbol{\nu}''')) \cong V_q(Y_{p-1,r_{p-1},\lambda(h_{p-1})+k'+1}) \otimes V_q(Y_{p-1,r_p+2(k-t)-1,t-k'}).$$

Applying (3.2.12) to the second factor of the above tensor product and comparing with (3.5.29) one easily deduces (3.5.28). □

3.6. Ordering certain affinizations

We now use the results of the previous two sections to order certain affinizations of a given arbitrary finite-dimensional simple $U_q(\mathfrak{g})$-module. We keep the data fixed in Section 3.3. More precisely, let $\lambda \in P^+$ be such that $\lambda(h_n) = 0$, $\boldsymbol{\omega} = \boldsymbol{\omega}^S$ be such that $V_q(\boldsymbol{\omega})$ is an increasing minimal affinization of $V_q(\lambda)$ where S is given as in (3.2.2).

Given $k, r_n, \bar{r}_n \in \mathbb{Z}, k > 0$, let $T, \overline{T}$ be the tableaux such that

$$\boldsymbol{\omega}^T = Y_{n,r_n,k} \qquad \text{and} \qquad \boldsymbol{\omega}^{\overline{T}} = Y_{n,\bar{r}_n,k}$$

as in Example 3.2.2. To shorten notation, set also

$$\boldsymbol{\varpi} = \boldsymbol{\omega}^T \qquad \text{and} \qquad \overline{\boldsymbol{\varpi}} = \boldsymbol{\omega}^{\overline{T}}.$$

Our goal is to establish the ordering relation between the affinizations $V_q(\boldsymbol{\lambda})$ and $V_q(\overline{\boldsymbol{\lambda}})$ of $V_q(\lambda+k\omega_n)$ under certain conditions on r_n and $\bar{r}_n$, where

$$\boldsymbol{\lambda} = \boldsymbol{\omega}\boldsymbol{\varpi} \qquad \text{and} \qquad \overline{\boldsymbol{\lambda}} = \boldsymbol{\omega}\overline{\boldsymbol{\varpi}}.$$

We shall say that $\boldsymbol{\lambda}$ satisfy (3.5.1) if there exists $k' \leq |\lambda|$ satisfying (3.4.4). Analogously, we say that $\overline{\boldsymbol{\lambda}}$ satisfy (3.5.1) if there exists $\bar{k}' \leq |\lambda|$ satisfying (3.4.4) with $\bar{k}'$ in place of k'. Similarly, we say that $\boldsymbol{\lambda}$ (respectively $\overline{\boldsymbol{\lambda}}$) satisfy (3.5.2) if there exists a pair (p, k') (respectively $(\bar{p}, \bar{k}')$) satisfying (3.4.5) with $k' \leq k$ (respectively $\bar{k}' \leq k$).

PROPOSITION 3.6.1. Suppose one of the following conditions holds:

(i) Both $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ satisfy (3.5.1) and $k' < \bar{k}'$;
(ii) Both $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ satisfy (3.5.2), $k' < \bar{k}'$, and $p = \bar{p}$;
(iii) Both $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ satisfy (3.5.2), $k' \leq \bar{k}'$, and $p > \bar{p}$.
(iv) $\boldsymbol{\lambda}$ satisfies (3.5.1) and $\overline{\boldsymbol{\lambda}}$ satisfies (3.5.2), $\bar{p} = i_0$, $k' < \bar{k}'$.
(v) $\#(\mathrm{supp}(\lambda) \cap J) > 1$ where $J = I \setminus \{n\}$, $\boldsymbol{\lambda}$ satisfies (3.5.1) and $\overline{\boldsymbol{\lambda}}$ satisfies (3.5.2), $\bar{p} = i_0$, $k' = \bar{k}'$.

Then, $V_q(\boldsymbol{\lambda}) < V_q(\overline{\boldsymbol{\lambda}})$.

PROOF. Set $V = V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ and $\overline{V} = V_q(\boldsymbol{\omega}) \otimes V_q(\overline{\boldsymbol{\varpi}})$. Then $V_q(\boldsymbol{\lambda})$ is the simple quotient of the submodule of V generated by the top weight space and similarly for $\overline{V}$. Notice that we have isomorphisms of $U_q(\mathfrak{g})$-modules:

$$V \cong \overline{V} \cong V_q(\lambda + k\omega_n) \oplus \bigoplus_{\nu} V_q(\nu)^{\oplus t_\nu} \quad \text{and} \quad V_q(\boldsymbol{\lambda}) \cong V_q(\lambda + k\omega_n) \oplus \bigoplus_{\nu} V_q(\nu)^{\oplus m_\nu}$$

where the sums are over $\nu \in P^+$ such that $\nu < \lambda + k\omega_n$ and $m_\nu, t_\nu \in \mathbb{Z}_{\geq 0}$. Letting $\boldsymbol{\mu}$ be as in Proposition 3.5.1, it immediately follows that

$$\nu \not\leq \mathrm{wt}(\boldsymbol{\mu}) \Rightarrow t_\nu = m_\nu \quad \text{and} \quad m_{\mathrm{wt}(\boldsymbol{\mu})} = t_{\mathrm{wt}(\boldsymbol{\mu})} - 1. \tag{3.6.1}$$

Writing

$$V_q(\overline{\boldsymbol{\lambda}}) \cong V_q(\lambda + k\omega_n) \oplus \bigoplus_{\nu} V_q(\nu)^{\oplus \overline{m}_\nu}$$

and letting $\overline{\boldsymbol{\mu}}$ be given by Proposition 3.5.1 with $\overline{\boldsymbol{\varpi}}$ in place of $\boldsymbol{\varpi}$, we similarly conclude that

$$\nu \not\leq \mathrm{wt}(\overline{\boldsymbol{\mu}}) \Rightarrow t_\nu = m_\nu \quad \text{and} \quad m_{\mathrm{wt}(\overline{\boldsymbol{\mu}})} = t_{\mathrm{wt}(\overline{\boldsymbol{\mu}})} - 1. \tag{3.6.2}$$

We claim the above first four conditions imply that

$$\mathrm{wt}(\overline{\boldsymbol{\mu}}) < \mathrm{wt}(\boldsymbol{\mu}). \tag{3.6.3}$$

Assuming this, we complete the proof as follows. Let $\nu \in P^+$ be such that $\nu < \lambda + k\omega_n$. If $\nu \not\leq \mathrm{wt}(\boldsymbol{\mu})$, then we also have $\nu \not\leq \mathrm{wt}(\overline{\boldsymbol{\mu}})$ and, hence, $\overline{m}_\nu = t_\nu = m_\nu$. Otherwise, if $\nu \leq \mathrm{wt}(\boldsymbol{\mu})$, we have $m_{\mathrm{wt}(\boldsymbol{\mu})} = t_{\mathrm{wt}(\boldsymbol{\mu})} - 1 = \overline{m}_{\mathrm{wt}(\boldsymbol{\mu})} - 1$. We do case (v) separately.

In order to prove (3.6.3), assume first that condition (i) holds. It follows from (3.5.10) that

$$\mathrm{wt}(\boldsymbol{\mu}) = \lambda + k\omega_n - \left(\sum_{j=l_{k'}+1}^{i_0} \lambda(h_j) \sum_{t=j}^{n} \alpha_t \right) - d_{k'} \sum_{t=l_{k'}}^{n} \alpha_t$$

and similarly for $\mathrm{wt}(\overline{\boldsymbol{\mu}})$ with $\bar{k}'$ in place of k'. Since $k' < \bar{k}'$, we must have that

$$\text{either} \quad l_{k'} > l_{\bar{k}'} \quad \text{or} \quad l_{\bar{k}'} = l_{k'} \quad \text{with} \quad d_{k'} < d_{\bar{k}'}.$$

Either way (3.6.3) follows. If condition (ii) holds, it follows from (3.5.14) that

$$\mathrm{wt}(\boldsymbol{\mu}) = \lambda - k' \sum_{j=p}^{n} \alpha_j \qquad \text{and} \qquad \mathrm{wt}(\overline{\boldsymbol{\mu}}) = \lambda - \bar{k}' \sum_{j=p}^{n} \alpha_j$$

and (3.6.3) follows since $k' < \bar{k}'$. If (iii) holds, it follows from (3.5.14) that

$$\mathrm{wt}(\boldsymbol{\mu}) = \lambda - k' \sum_{j=p}^{n} \alpha_j \qquad \text{and} \qquad \mathrm{wt}(\overline{\boldsymbol{\mu}}) = \lambda - \bar{k}' \sum_{j=\bar{p}}^{n} \alpha_j.$$

The hypothesis $k' \leq \bar{k}'$ and $p > \bar{p}$ then clearly imply (3.6.3). Finally, if (iv) holds, since $k' \leq \bar{k}' \leq \lambda(h_{i_0})$, it follows that $l_{k'} = i_0$ and $d_{k'} = k'$. Hence, it follows from (3.5.10) and (3.5.14), respectively, that

$$\mathrm{wt}(\boldsymbol{\mu}) = \lambda - k' \sum_{j=i_0}^{n} \alpha_j \qquad \text{and} \qquad \mathrm{wt}(\overline{\boldsymbol{\mu}}) = \lambda - \bar{k}' \sum_{j=i_0}^{n} \alpha_j.$$

The hypothesis $k' < \bar{k}'$ then implies (3.6.3).

Suppose now that (v) holds. By Proposition 3.5.1, the simple factors of V are $V_q(\boldsymbol{\lambda})$ and $V_q(\boldsymbol{\mu})$, and the simple factors of $\overline{V}$ are $V_q(\overline{\boldsymbol{\lambda}})$ and $V_q(\overline{\boldsymbol{\mu}})$. Since $k' = \bar{k}' \leq \lambda(h_{i_0})$, it follows that $l_{k'} = i_0$ and $d_{k'} = k'$. Hence, by (3.5.10),

$$\boldsymbol{\mu} = \boldsymbol{\lambda} \left(\prod_{m=1}^{k'} A^{-1}_{i_0, n, r_{i_0} + 2(\lambda(h_{i_0}) - m)} \right). \tag{3.6.4}$$

Also, by (3.5.14), since $\bar{k}' = k'$ and $p = i_0$,

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\lambda}} \left(\prod_{m=1}^{k'} A^{-1}_{n, i_0, r_n + 2(k-m)} \right). \tag{3.6.5}$$

Let $s = \min\{i \in \mathrm{supp}(\lambda) : i < i_0\}$. Observe that s is well defined, since $\#(\mathrm{supp}(\lambda) \cap J) > 1$. Consider $\nu = \mathrm{wt}(\boldsymbol{\mu}) - \sum_{i=s}^{i_0-1} \alpha_i = \mathrm{wt}(\overline{\boldsymbol{\mu}}) - \sum_{i=s}^{i_0-1} \alpha_i$. We claim that

$$m_\nu(V_q(\overline{\boldsymbol{\mu}})) = 0 < m_\nu(V_q(\boldsymbol{\mu})) \quad \text{and} \quad m_\eta(V_q(\overline{\boldsymbol{\lambda}})) = m_\eta(V_q(\boldsymbol{\lambda}))$$

for all $\eta \in P^+$ such that $\lambda + k\omega_n < \eta < \nu$. In particular, $m_\eta(V_q(\overline{\boldsymbol{\mu}})) = m_\eta(V_q(\boldsymbol{\mu}))$. The second part of the claim follows from (3.6.1) and (3.6.2) using that $\mathrm{wt}(\boldsymbol{\mu}) = \mathrm{wt}(\overline{\boldsymbol{\mu}})$. For the first part, we will show that

$$\dim(V_q(\boldsymbol{\mu})_\nu) > \dim(V_q(\mathrm{wt}(\boldsymbol{\mu}))_\nu). \tag{3.6.6}$$

and

$$\dim(V_q(\overline{\boldsymbol{\mu}})_\nu) = \dim(V_q(\mathrm{wt}(\overline{\boldsymbol{\mu}}))_\nu) \tag{3.6.7}$$

Assuming this, we complete the proof of the claim as follows. Let $J' = \{s, \ldots, i_0 - 1\}$ and observe that

$$\mathrm{wt}(\overline{\boldsymbol{\mu}}_{J'}) = \mathrm{wt}(\boldsymbol{\mu}_{J'}) = \lambda(h_s)\omega_s + k'\omega_{i_0-1}.$$

One easily sees that there is no $\eta \in P^+$ such that $\mathrm{wt}(\overline{\boldsymbol{\mu}}) = \mathrm{wt}(\boldsymbol{\mu}) > \eta > \nu$. Thus, it follows from (3.6.6) that $V_q(\nu)$ is a simple factor of $V_q(\boldsymbol{\mu})$ when regarded as a $U_q(\mathfrak{g})$-module, while (3.6.7) implies that $V_q(\nu)$ is not a simple factor of $V_q(\overline{\boldsymbol{\mu}})$.

Let us prove (3.6.7). Observe from (3.6.5) that

$$\pi_{J'}(\overline{\boldsymbol{\mu}}) = Y_{s,r_s,\lambda(h_s)} \left(\prod_{m=1}^{k'} Y_{i_0-1,r_n+2(k-m)+n-i_0+1} \right).$$

By (3.3.2) we have $r_s + 2\lambda(h_s) + i_0 - s = r_{i_0}$ or, equivalently,

$$r_s + 2(\lambda(h_s) - 1) = r_{i_0} - (i_0 - s + 2). \tag{3.6.8}$$

Also, by (3.4.5),

$$r_n + 2(k - k') + n - i_0 + 1 = r_{i_0} + 2(k' - 1) - 2k' + 1 = r_{i_0} - 1.$$

Therefore,

$$r_s + 2(\lambda(h_s) - 1) + i_0 - s + 1 = r_{i_0} - 1 = r_n + 2(k - k') + n - i_0 + 1.$$

Hence $V_q(\overline{\boldsymbol{\mu}}_{J'})$ is a minimal affinization. Thus, it is irreducible as $U_q(\tilde{\mathfrak{g}}_{J'})$-module, i.e., $V_q(\overline{\boldsymbol{\mu}}_{J'}) \cong_{U_q(\tilde{\mathfrak{g}}_{J'})} V_q(\mathrm{wt}(\overline{\boldsymbol{\mu}}_{J'}))$, which implies by (2.6.2) that (3.6.7) then follows.

We now turn to (3.6.6). Observe that (3.6.4) implies

$$\pi_{J'}(\boldsymbol{\mu}) = Y_{s,r_s,\lambda(h_s)} \left(\prod_{m=1}^{k'} Y_{i_0-1,r_{i_0}+2(\lambda(h_{i_0})-m)+1} \right).$$

By (3.4.4),

$$r_{i_0} + 2(\lambda(h_{i_0}) - k') + 1 \geq r_{i_0} + 1.$$

Then,

$$r_s + 2(\lambda(h_s) - 1) + i_0 - s + 1 \overset{(3.6.8)}{=} r_{i_0} - 1 < r_{i_0} + 1 \leq r_{i_0} + 2(\lambda(h_{i_0}) - k') + 1.$$

An application of Proposition 2.7.4 to the subalgebra determined by J' implies that $m_{\nu_{J'}}(V_q(\overline{\boldsymbol{\mu}}_{J'})) > 0 = m_{\nu_{J'}}(V_q(\overline{\boldsymbol{\mu}}_{J'}))$. Thus

$$\dim(V_q(\boldsymbol{\mu}_{J'})_{\nu_{J'}}) > \dim(V_q(\overline{\boldsymbol{\mu}_{J'}})_{\nu_{J'}}) = \dim(V_q(\mathrm{wt}(\boldsymbol{\mu}_{J'}))_{\nu_{J'}})$$

and (3.6.6) follows from (2.6.2). □

3.7. Decreasing minimal affinizations

Suppose $\boldsymbol{\omega}' = \prod_{i \in I} Y_{i,r'_i,\lambda(h_i)}$ is such that $V_q(\boldsymbol{\omega}')$ is a decreasing minimal affinization of $V_q(\lambda)$. It follows from (3.2.5) that

$$r'_i = r'_{i_0} + i_0 - i + 2\ {}_{i+1}|\lambda|_{i_0} \quad \text{for all} \quad i \in \mathrm{supp}(\lambda). \tag{3.7.1}$$

For notational convenience, we define r'_i by (3.7.1) for all $1 \leq i \leq i_0$. In particular

$$r'_i = r'_j + j - i + 2\ {}_{i+1}|\lambda|_j \quad \text{for all} \quad 1 \leq i \leq j.$$

The following proposition allows us to obtain similar results for $V_q(\boldsymbol{\omega}'\boldsymbol{\varpi})$ to those we obtained for $V_q(\boldsymbol{\omega}\boldsymbol{\varpi})$ in the previous sections.

Proposition 3.7.1. If either of the following two conditions hold, we have $V_q(\boldsymbol{\omega}\boldsymbol{\varpi}) \cong_{U_q(\mathfrak{g})} V_q(\boldsymbol{\omega}'\boldsymbol{\varpi})$.

(i) There exists $1 \le k' \le \min\{|\lambda|, k\}$ such that

$$r_{i_0} + 2\lambda(h_{i_0}) + n - i_0 + 2 = r_n + 2k', \tag{3.7.2}$$

and

$$r_n + 2k + n - i_0 + 2 = r'_{i_0} + 2k'. \tag{3.7.3}$$

(ii) There exist $p \in \mathrm{supp}(\lambda)$ and $1 \le k' \le \min\{\lambda(h_p), k\}$ such that

$$r_n + 2k + n - i_0 + 2 = r_p + 2k', \tag{3.7.4}$$

and

$$r'_p + 2\lambda(h_p) + n - p + 1 = r_n + 2k'. \tag{3.7.5}$$

Proof. Writing $a_i = q^{r_i+\lambda(h_i)-1}$ and $b_j = q^{r'_i+\lambda(h_i)-1}$ for $i < n$ and $a_n = b_n = q^{r_n+k-1}$, one easily sees that condition (i) as well as condition (ii) imply that

$$\frac{a_i}{a_j} = \frac{b_j}{b_i} \qquad \text{for all} \quad i, j \in I, \tag{3.7.6}$$

and we are done by Proposition 2.1.2. $\square$

CHAPTER 4

On qcharacters and tensor products for type D

Throughout this chapter $\mathfrak{g}$ is of type D_n. The main goal of this chapter is to prove similar results to those of the previous chapter for tensor products of Kirillov-Reshetikhin modules associated to the spin nodes. The main results are

We also compare certain affinizations involving the nodes $1, n-1, n$.

4.1. Tableaux and ℓ-weights

Let $\mathbf{B} = \{1, 2, \ldots, n\} \cup \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ equipped with the partial order

$$1 < 2 < \cdots < n-1 < n, \bar{n} < \overline{n-1} < \cdots < \bar{2} < \bar{1}.$$

Following [**44**], introduce the notation

$$\boxed{i}_{\,r} = \begin{cases} Y_{i-1,r+i-1}^{-1} Y_{i,r+i-2}, & \text{if} \quad 1 \le i \le n-2; \\ Y_{n-2,r+n-2}^{-1}, & \text{if} \quad i = n-1; \\ Y_{n,r+n-1}, & \text{if} \quad i = n; \end{cases} \tag{4.1.1}$$

and

$$\boxed{\bar{i}}_{\,r} = \begin{cases} 1, & \text{if} \quad 1 \le i \le n-2; \\ Y_{n-1,r+n+1}^{-1} Y_{n,r+n+1}^{-1}, & \text{if} \quad i = n-1; \\ Y_{n-1,r+n-1}, & \text{if} \quad i = n. \end{cases} \tag{4.1.2}$$

Remark 4.1.1. The symbols were denoted by "half" boxes in [**44**] since they are related to ℓ-weights of the spin representations. The "full" boxes were reserved in [**44**] for ℓ-weights of the standard representation. Since we will only be concerned with the spin representations here, we will use use "full' boxes instead of "half" boxes.

Define column tableau, tableau, shape of a tableau, and the ℓ-weight $\boldsymbol{\omega}^T$ associated to a tableau T similarly to the type A case in Section 3.1. Let l be one of the spin nodes and, given $r \in \mathbb{Z}$, let $\mathcal{B}_l^r$ be the set of all column tableaux of the form:

$$T = \begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array}_{\,r-n+1}$$

where the contents i_j satisfy:

(1) $i_1 < \cdots < i_n$;
(2) for all $1 \le m \le n$, m is the content of a box of T if and only if $\overline{m}$ is not;
(3) if $i_j = n$ for some j, then l and j have the same parity;
(4) if $i_j = \bar{n}$ for some j, then l and j have opposite parities.

Notice that

$$Y_{l,r}=\boldsymbol{\omega}^T \quad\text{where}\quad T=\begin{array}{|c|l}\cline{1-1} 1 & \\ \cline{1-1} 2 & \\ \cline{1-1} \vdots & \\ \cline{1-1} n-1 & \\ \cline{1-1} m & \\ \cline{1-1} \end{array}_{\,r-n+1} \quad\text{with}\quad m=\begin{cases}\bar{n}, & \text{if } l=n-1,\\ n, & \text{if } l=n.\end{cases} \tag{4.1.3}$$

Moreover,

$$T\in\mathcal{B}_l^r \quad\Rightarrow\quad \boldsymbol{\omega}^T=\prod_{i\in I} Y_{i,s_i}^{p_i} \quad\text{for some}\quad s_i,p_i\in\mathbb{Z} \quad\text{with}\quad |p_i|\le 1. \tag{4.1.4}$$

The definition of gaps in a column increasing tableau is similar to that of type A. Namely, we say that a column increasing tableau T has a gap at its j-th row if the content of the j-th row is not an immediate successor of the content of the $(j-1)$-th row (in the partial order of $\mathbf{B}$). A gap in the first row means that the content of the first row is not 1.

The proof of the next lemma is similar to that of Lemma 3.1.6 and we omit the details.

LEMMA 4.1.2. Let T be a column increasing tableau. Suppose a given column of T has a gap at the j-th row. More precisely, that column has the box $\boxed{l_2}_{\,s}$ at the j-th row and the content l_1 of the previous row is not the immediate predecessor of l_2 (set $l_1=0$ if $j=1$).

(a) If $l_2\le n$, then $Y_{l_2-1,s+l_2-1}^{-1}$ appears in $\boldsymbol{\omega}^T$. Moreover, if $j>1$, then $Y_{l_1,s+l_1}$ appears in $\boldsymbol{\omega}^T$.
(b) If $l_2=\bar{n}$, then $Y_{n,s+n-1}^{-1}$ appears in $\boldsymbol{\omega}^T$. Moreover, if $j>1$, then $Y_{l_1,s+l_1}$ appears in $\boldsymbol{\omega}^T$.
(c) If $l_2>\bar{n}$, then the same column has a gap at a box whose content c is at most $\bar{n}$. $\square$

Given $T\in\mathcal{B}_l^r$, suppose $\boxed{i}_{\,s}$ and $\boxed{\overline{i+1}}_{\,s'}$ are boxes of T for some $1\le i\le n-1$. Then, $\boldsymbol{\omega}^T A_{i,s+i-1}^{-1}=\boldsymbol{\omega}^{T'}$, where $T'\in\mathcal{B}_l^r$ is obtained from T by replacing the boxes $\boxed{i}_{\,s}$ and $\boxed{\overline{i+1}}_{\,s'}$ by $\boxed{i+1}_{\,s}$ and $\boxed{\bar{i}}_{\,s'}$, respectively. In pictures:

$$\begin{array}{|c|l}\cline{1-1} \vdots & \\ \cline{1-1} i & s \\ \cline{1-1} \vdots & \\ \cline{1-1} \overline{i+1} & s' \\ \cline{1-1} \vdots & \\ \cline{1-1} \end{array} A_{i,s+i-1}^{-1} = \begin{array}{|c|l}\cline{1-1} \vdots & \\ \cline{1-1} i+1 & s \\ \cline{1-1} \vdots & \\ \cline{1-1} \bar{i} & s' \\ \cline{1-1} \vdots & \\ \cline{1-1} \end{array}. \tag{4.1.5}$$

Notice that, if $i\le n-3$, then $\boxed{\overline{i+1}}_{\,s'}=\boxed{\bar{i}}_{\,s'}=1$ and the value of s' is irrelevant. On the other hand, if $i=n-2$, conditions (1) and (2) of the definition of $\mathcal{B}_l^r$ imply that $s'=s-4$ and, if $i=n-1$, then $s'=s-2$. Similarly, if $\boxed{n-1}_{\,s+2}$ and $\boxed{n}_{\,s}$ are boxes of T, then we have $\boldsymbol{\omega}^T A_{n,s+n}^{-1}=\boldsymbol{\omega}^{T'}$, where $T'\in\mathcal{B}_l^r$ is obtained from T by replacing the boxes $\boxed{n-1}_{\,s+2}$ and $\boxed{n}_{\,s}$ by $\boxed{\bar{n}}_{\,s+2}$ and $\boxed{\overline{n-1}}_{\,s}$, respectively. In pictures:

$$\begin{array}{|c|l}\cline{1-1} \vdots & \\ \cline{1-1} n-1 & s+2 \\ \cline{1-1} n & s \\ \cline{1-1} \vdots & \\ \cline{1-1} \end{array} A_{n,s+n}^{-1} = \begin{array}{|c|l}\cline{1-1} \vdots & \\ \cline{1-1} \bar{n} & s+2 \\ \cline{1-1} \overline{n-1} & s \\ \cline{1-1} \vdots & \\ \cline{1-1} \end{array}. \tag{4.1.6}$$

LEMMA 4.1.3. Let $T \in \mathcal{B}_l^r, i \in I$, and $m \in \mathbb{Z}$. Then, $\boldsymbol{\omega}^T A_{i,m}^{-1} = \boldsymbol{\omega}^{T'}$ for some $T' \in \mathcal{B}_l^r$ if and only if the pair $(T, A_{i,m}^{-1})$ is as in the left-hand side of either (4.1.5) or (4.1.6) and T' is the column on the corresponding right-hand side.

PROOF. The "if" part is obvious. For the converse, let $\boxed{i_j}_{\,s_j}, 1 \leq j \leq n$, be the boxes of T. So $s_j = r + n + 1 - 2j$ and

$$\boldsymbol{\omega}^T = \prod_{j=1}^{n} \boxed{i_j}_{\,s_j}.$$

Suppose i is the content of a box in T but $\overline{i+1}$ is not. Hence, $(T, A_{i,m}^{-1})$ is not as in the left-hand side of (4.1.5) for any $m \in \mathbb{Z}$. Say $i = i_j$ and set $s = s_j$ to shorten notation. Conditions (1) and (2) in the definition of $\mathcal{B}_l^r$ imply that $i_{j+1} = i + 1$. Observe that

$$\begin{aligned}
&\boxed{i}_{\,s} A_{i,s+i-1}^{-1} &&= \boxed{i+1}_{\,s} \boxed{\bar{i}}_{\,s'} \left(\boxed{\overline{i+1}}_{\,s'}\right)^{-1} &&= \boxed{i+1}_{\,s} \qquad \forall\, s' \in \mathbb{Z}, \quad \text{if } i \leq n-3,\\
&\boxed{n-2}_{\,s} A_{n-2,s+n-3}^{-1} &&= \boxed{n-1}_{\,s} \left(\boxed{\overline{n-1}}_{\,s-4}\right)^{-1} \boxed{\overline{n-2}}_{\,s-4} &&= \boxed{n-1}_{\,s} \boxed{\bar{n}}_{\,s-2} \boxed{n}_{\,s-2},\\
&\boxed{n-1}_{\,s} A_{n-1,s+n-2}^{-1} &&= \left(\boxed{\bar{n}}_{\,s-2}\right)^{-1} \boxed{n}_{\,s} \boxed{\overline{n-1}}_{\,s-2} &&= \boxed{n}_{\,s} \boxed{\overline{n-1}}_{\,s-2} \boxed{n}_{\,s} \boxed{\overline{n-1}}_{\,s-2},\\
&\boxed{n}_{\,s} A_{n,s+n}^{-1} &&= \left(\boxed{n-1}_{\,s+2}\right)^{-1} \boxed{\bar{n}}_{\,s+2} \boxed{\overline{n-1}}_{\,s} &&= \boxed{\bar{n}}_{\,s+2} \boxed{\overline{n-1}}_{\,s} \left(\prod_{i=1}^{n-2} \boxed{i}_{\,s+2(n-i)}\right).
\end{aligned} \tag{4.1.7}$$

Moreover, the last expression of each line of (4.1.7) is the minimal description as a product of boxes. If $i \leq n-3$, it follows from (4.1.7) that

$$\boldsymbol{\omega}^T A_{i,m}^{-1} = \boxed{i+1}_{\,s} \boxed{i+1}_{\,s-2} \prod_{j' \neq j, j+1} \boxed{i_{j'}}_{\,s_{j'}}.$$

Since $i_{j'} \neq i$ for all $j' \neq j, j+1$, it follows that $Y_{i,s+i}^{-1}$ and $Y_{i,s+i-2}^{-1}$ appear in $\boldsymbol{\omega}^T A_{s+i-1}^{-1}$ (cf. Lemma 4.1.2). This contradicts (4.1.4) and, hence, $\boldsymbol{\omega}^T A_{i,m}^{-1}$ cannot be represented by an element of $\mathcal{B}_l^r$. The cases $i = n-2, n-1$ are similar and we omit the details. Now, suppose T has $\boxed{n}_{\,s}$ and does not have $\boxed{n-1}_{\,s+2}$. Then T must have $\boxed{\overline{n-1}}_{\,s-2}$ and (4.1.7) implies that $\boldsymbol{\omega}^T A_{n,s+n}^{-1}$ cannot be represented by an element of $\mathcal{B}_l^r$. If either i and $\overline{i+1}$ are contents of boxes in T or i is not the content of any box in T, then the proof is similar. $\square$

The following lemma is easily established.

LEMMA 4.1.4. Suppose $T \in \mathcal{B}_l^r$ has a gap. Then, there exists $(i, s) \in I \times \mathbb{Z}$ and $S \in \mathcal{B}_l^r$ such that $\boldsymbol{\omega}^T = \boldsymbol{\omega}^S A_{i,s}^{-1}$. $\square$

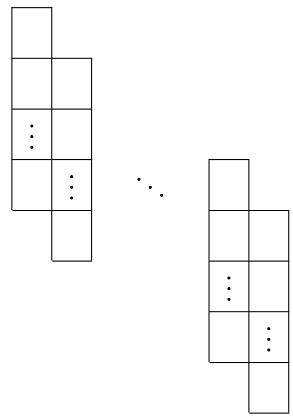
4.2. The qcharacters of spin KR modules

Fix $r \in \mathbb{Z}$, and a spin node l. It was proved in [**44**, Proposition 5.11] that

$$\mathrm{qch}(V_q(Y_{l,r})) = \sum_{T \in \mathcal{B}_l^r} \boldsymbol{\omega}^T. \tag{4.2.1}$$

The goal of this section is to prove a generalization of this formula for $\mathrm{qch}(V_q(Y_{l,r,k})), k \geq 1$. Thus, fix such k and notice that $Y_{l,r,k}$ can be represented by a tableau T made of the juxtaposition of k column-tableaux as in (4.1.3). In particular, its shape is given by the picture:

(4.2.2)



We will say that a tableau of such shape is semi-standard if the j-th column is an element of $\mathcal{B}_l^{r+2(j-1)}$ for all $1 \leq j \leq k$ and the contents in diagonals decrease from left to right and top to bottom. For any given semi-standard tableau T, we let $\mathrm{STab}(T)$ be the set of all semi-standard tableau with the same shape as T.

The proof of the next lemma is similar to that of Lemma 3.2.3 and we omit the details.

Lemma 4.2.1. Let T be a semi-standard tableau with shape as in (4.2.2), $1 \leq i \leq n, s \in \mathbb{Z}$. Suppose the box $\boxed{i}_s$ is part of the the j-th column of T. Then:

(a) The box $\boxed{i}_s$ is not in any other column of T.
(b) If $\boxed{i-1}_{s+2}$ is a box in T, it must be in the j-th column.
(c) If $i < n$ and $\boxed{i+1}_{s-2}$ is a box in T, it must be in the j-th column.
(d) If $i = n$, then the box $\boxed{\bar{n}}_s$ is not in any column of T.
(e) If the box $\boxed{\bar{n}}_s$ is part of the j-th column of T, then the box $\boxed{n}_s$ is not in any column of T.□

It is not hard to see, using Lemmas 4.1.2 and 4.2.1, that there exists a unique semi-standard tableau satisfying $\boldsymbol{\omega}^T = Y_{l,r,k}$: that with all k columns gap free. The following is the main result of this section.

Theorem 4.2.2. The module $V_q(Y_{l,r,k})$ is ℓ-minuscule, thin, and, if T is the unique semi-standard tableau satisfying $\boldsymbol{\omega}^T = Y_{l,r,k}$, then

$$\mathrm{qch}(V_q(Y_{l,r,k})) = \sum_{T' \in \mathrm{STab}(T)} \boldsymbol{\omega}^{T'}.$$

We shall need a few more lemmas about the combinatorics of semi-standard tableau. Given a tableau T, we shall denote by T^j its j-th column. The following is a generalization of Lemmas 4.1.3 and 4.1.4.

Lemma 4.2.3. Let T be a semi-standard tableau with k columns, $i \in I$, and $m \in \mathbb{Z}$.

(a) We have $\boldsymbol{\omega}^T A_{i,m}^{-1} = \boldsymbol{\omega}^S$ for some $S \in \mathrm{STab}(T)$ if and only if, there exists $1 \leq j \leq k$, such that the pair $(T^j, A_{i,m}^{-1})$ is as in the left-hand side of either (4.1.5) or (4.1.6) and S is obtained from T by replacing T^j by the column on the corresponding right-hand side.

(b) If T has a gap, there exists $(j,s) \in I \times \mathbb{Z}$ and $S \in \mathrm{STab}(T)$ such that $\boldsymbol{\omega}^T = \boldsymbol{\omega}^S A_{i,s}^{-1}$.

The next lemma is an easy consequence of Lemmas 4.1.2 and 4.2.1.

LEMMA 4.2.4. Let T be a semi-standard tableau, $i \in I$, and $s \in \mathbb{Z}$.

(a) $\boldsymbol{\omega}^T$ is i-dominant if and only if the following holds:
   (i) If $i \leq n-2$ and $i+1$ is the content of a box in a column of T, then i is the content of the box on top of it;
   (ii) If $i = n-1$ and $\overline{n-1}$ is the content of a box in a column of T, then n is the content of the box on top of it;
   (iii) If $i = n$ and $\overline{n-1}$ is the content of a box in a column of T, then $\bar{n}$ is the content of the box on top of it.

(b) $Y_{i,s}$ appears in $\boldsymbol{\omega}^T$ if and only if the following holds:
   (i) If $i \leq n-2$, then $\boxed{i}_{s-i+2}$ is a box in S and $i+1$ is not the content of any box in the same column;
   (ii) If $i = n-1$, then $\boxed{\bar{n}}_{s-n+1}$ is a box in S and $\overline{n-1}$ is not the content of any box in the same column;
   (iii) If $i = n$, then $\boxed{n}_{s-n+1}$ is a box in S and $\overline{n-1}$ is not the content of any box in the same column. $\square$

We shall also need:

LEMMA 4.2.5. Let S be a semi-standard tableau and suppose $i \in I$ and $r' < r-i+1$ are such that S has a column of the following form:

$$\begin{array}{|c|l}
\hline
\vdots & \\
\hline
i & {\scriptstyle r-i+1} \\
\hline
\vdots & \\
\hline
\overline{i+1} & {\scriptstyle r'} \\
\hline
\vdots & \\
\hline
\end{array}$$

Then, $\boxed{i+1}_{r-i+1}$ is not a box in any column of S and $\boxed{i-1}_{r-i+3}$ and $\boxed{\bar{i}}_{r''}$ with $r'' < r-i+3$ are not boxes in any same column of S. In particular, there does not exist $S' \in \mathrm{STab}(S)$ such that $\boldsymbol{\omega}^S A_{i+1,r+1}^{-1} = \boldsymbol{\omega}^{S'}$ and similarly for $\boldsymbol{\omega}^S A_{i-1,r+1}^{-1}$.

If a column of S has the boxes $\boxed{n-1}_{s+2}$ and $\boxed{n}_s$, then $\boxed{n-2}_{s+4}$ and $\boxed{\overline{n-1}}_s$ are not boxes in any same column of S. In particular, there does not exist $S' \in \mathrm{STab}(S)$ such that $\boldsymbol{\omega}^S A_{n-2,s+n+1}^{-1} = \boldsymbol{\omega}^{S'}$.

PROOF. Suppose $\boxed{i+1}_{r-i+1}$ were a box of the $(j+m)$-th column, $m \geq 1$, this column has a box supported at $r-i+1-2m$. Since S is column increasing, the content c of the box supported at $r-i+1-2m$ is at least $i+m > i$. This contradicts the assumption that S is semi-standard because the box $\boxed{i+1}_{r-i+1}$ in column j and the box $\boxed{c}_{r-i+1-2m}$ in column $j+m$ are in the same diagonal from left to right and top to bottom. Suppose now that $\boxed{i+1}_{r-i+1}$ is in the $(j-m)$-th column, $m \geq 1$. This column has a box supported at $r-i+1+2m$. Since all columns are increasing, the content c of the box supported at $r-i+1+2m$ is at most $i-m$, which implies that the content of the box supported at $r-i+1+2m-2$ is at most $i-m+1$. Since the j-th column of S has the box $\boxed{\overline{i+1}}_{r'}$ for some $r' < r-i+1$, the content of the box supported at $r-i-1$ is bigger than $i+1$. This contradicts the assumption that S is semi-standard because the box supported at $r-i+1+2m-2$ in column $j-m$ and the box supported at $r-i-1$ in column j are in the same diagonal from left to right and top to bottom. The absence of the other two kinds of boxes is proved similarly. $\square$

Henceforth, let T be as in Theorem 4.2.2 and set $\boldsymbol{\omega}=\boldsymbol{\omega}^T$. It easily follows from Lemma 4.2.3 (b) that, given $T'\in \mathrm{STab}(T)$, there exist $m\in\mathbb{Z}_{\geq 0}$, $i_j\in I$, $r_j\in\mathbb{Z}$ such that

$$\boldsymbol{\omega}^{T'}=\boldsymbol{\omega}\prod_{j=1}^{m}A_{i_j,r_j}^{-1}. \tag{4.2.3}$$

Moreover, by (2.2.1), the pairs (i_j,r_j), counted with multiplicities, are unique up to re-ordering and we can assume, by part (a) of Lemma 4.2.3, that the order is fixed in such a way that, for all $1\leq p\leq m$,

$$\boldsymbol{\omega}\prod_{j=1}^{p}A_{i_j,r_j}^{-1}=\boldsymbol{\omega}^{T_p}\quad\text{for some}\quad T_p\in\mathrm{STab}(T)$$

and there exists $1\leq t\leq k$ such that the pair $(T_p^t, A_{i_{p+1},r_{p+1}}^{-1})$ is either as in (4.1.5) or as in (4.1.6).

In light of Proposition 2.6.10, Theorem 4.2.2 follows from the following proposition.

PROPOSITION 4.2.6. The family $\mathcal{M}$ consisting of $\boldsymbol{\omega}^{T'}$ with $T'\in\mathrm{STab}(T)$ satisfies the hypothesis and conditions (i)–(iii) of Proposition 2.6.10.

PROOF OF PROPOSITION 4.2.6. It is not difficult to see that, if $T',T''\in\mathrm{STab}(T)$ are such that $T'\neq T''$, then $\boldsymbol{\omega}^{T'}\neq\boldsymbol{\omega}^{T''}$.

Condition (i): We need to show that if $T'\in\mathrm{STab}(T)\setminus\{T\}$, then $\boldsymbol{\omega}^{T'}\notin\mathcal{P}^+$. Indeed, in this case T' has a gap, which implies, since T' is semi-standard, that T' has a gap at its first column. Let $j\in\{1,\dots,n\}$ be minimal such that T' has a gap at the j-th row of its first column. It means that the j-th row of the first column contain a box $\boxed{j'}_{\,s}$ for some $j'>j$ and $s\in\mathbb{Z}$, and, if $j>1$, the $(j-1)$-th row contain a box $\boxed{j-1}_{\,s+2}$. If $j'\leq n-1$, by Lemma 4.1.2 $Y_{j'-1,s+j'-1}^{-1}$ appears in the ℓ-weight attached to this column. Supposing that $\boldsymbol{\omega}^{T'}\in\mathcal{P}^+$, this implies that $\boxed{j'-1}_{\,s+2}$ must be a box in some other column of T'. But, by Lemma 4.2.1, this box can only appear in the first column, yielding the desired contradiction. It remains the possibilities $j'=n,\bar n$, which imply that $n-1$ is not the content of any box of the first column of T and, hence, $\boxed{\overline{n-1}}_{\,s-2}$ is a box in the first column. Observe that

$$\begin{array}{|c|}\hline n\\\hline \overline{n-1}\\\hline\end{array}_{\,s-2}=Y_{n-1,s+n-1}^{-1}\qquad\text{and}\qquad \begin{array}{|c|}\hline \bar n\\\hline \overline{n-1}\\\hline\end{array}_{\,s-2}=Y_{n,s+n-1}^{-1}.$$

Supposing $\boldsymbol{\omega}^{T'}\in\mathcal{P}^+$, this implies that either $\boxed{\bar n}_{\,s}$ or $\boxed{n}_{\,s}$ must be a box in T', contradicting Lemma 4.2.1 once more.

Condition (ii): Let $\boldsymbol{\mu}\in\mathcal{M}$ and $T'\in\mathrm{STab}(T)$ such that $\boldsymbol{\omega}^{T'}=\boldsymbol{\mu}$. Suppose $(i,a)\in I\times\mathbb{C}^\times$ is such that $\boldsymbol{\mu}\boldsymbol{\alpha}_{i,a}^{-1}\notin\mathcal{M}$. We need to show that $\boldsymbol{\mu}\boldsymbol{\alpha}_{i,a}^{-1}\boldsymbol{\alpha}_{j,b}\notin\mathcal{M}$ unless $(j,b)=(i,a)$. Indeed, if $(j,b)\neq(i,a)$ is such that $\boldsymbol{\nu}:=\boldsymbol{\mu}\boldsymbol{\alpha}_{i,a}^{-1}\boldsymbol{\alpha}_{j,b}\in\mathcal{M}$, then $\boldsymbol{\omega}\boldsymbol{\nu}^{-1}\in\mathcal{Q}^+$ and (2.2.1) implies that

$$(j,b)=(i_p,q^{r_p-1})\quad\text{for some}\quad 1\leq p\leq m.$$

In other words, $\boldsymbol{\alpha}_{j,b}=A_{i_p,r_p}$. Evidently, we must have $\boldsymbol{\alpha}_{i,a}=A_{i,s}$ for some $i\in I$ and $s\in\mathbb{Z}$. In light of Lemma 4.2.3 (a), the condition $\boldsymbol{\mu}A_{i,s}^{-1}\notin\mathcal{M}$ means that, if $i<n$, then T' does not have a column affording both the boxes

$$\boxed{i}_{\,s-i+1}\quad\text{and}\quad\boxed{\overline{i+1}}_{\,s'}\quad\text{for any}\quad s'<s-i+1, \tag{4.2.4}$$

while, if $i = n$, T' does not have a column with both the boxes

$$\boxed{n-1}_{s-n+2} \quad \text{and} \quad \boxed{n}_{s-n}. \tag{4.2.5}$$

On the other hand, the condition $\boldsymbol{\nu} \in \mathcal{M}$ implies that $\boldsymbol{\nu}$ has a decomposition as in (4.2.3), which by (2.2.1), must be

$$\boldsymbol{\nu} = \boldsymbol{\omega} A_{i,s}^{-1} \prod_{t \neq p} A_{i_t,r_t}^{-1}$$

up to ordering.

From the way we chose the order of the factors in (4.2.3), we have

$$\boldsymbol{\omega} \prod_{j=1}^{p} A_{i_j,r_j}^{-1} = \boldsymbol{\omega}^{T_{p-1}} \quad \text{and} \quad \boldsymbol{\omega}^{T_{p-1}} A_{i_p,r_p}^{-1} \in \mathcal{M}.$$

Suppose $i_p < n$. Then, T_{p-1} has a column with the boxes $\boxed{i_p}_{r_p-i_p+1}$ and $\boxed{\overline{i_p+1}}_{r'}$ for some $r' < r_p - i_p + 1$. Hence, applying Lemma 4.2.5 with T_{p-1} in place of S, we get that

$$\boxed{\overline{i_p+1}}_{r_p-i_p+1} \quad \text{and} \quad \boxed{\overline{i_p-1}}_{r_p-i_p+3} \quad \text{are not a boxes in any column of} \quad T_{p-1}. \tag{4.2.6}$$

We claim that (4.2.6) together with the assumption that $\boldsymbol{\nu} \in \mathcal{M}$ implies that

$$\begin{gathered} (i_p - 1, r_p + 1) \notin \{(i_j, r_j) : j = p+1, \ldots, m\} \\ \text{and} \\ (i_p + 1, r_p + 1) \notin \{(i_j, r_j) : j = p+1, \ldots, m\}. \end{gathered} \tag{4.2.7}$$

Indeed, if either of the above did not held, since $\boldsymbol{\nu} = A_{i,s}^{-1} \boldsymbol{\omega} \prod_{t \neq p} A_{i_t,r_t}^{-1}$ has an expression as in (4.2.3) in such a way that the truncated products correspond to semi-standard tableaux, (4.2.6) would imply that $(i, s) = (i_p, r_p)$, i.e., $(i, a) = (j, b)$, yielding the desired contradiction. Similarly, if $i_p = n$ one shows that

$$(n - 2, r_p + 1) \notin \{(i_j, r_j) : j = p+1, \ldots, m\}. \tag{4.2.8}$$

We omit the details. In particular, $\boldsymbol{\omega} \prod_{t \neq p} A_{i_t,r_t}^{-1} \in \mathcal{M}$ and, in fact, we can assume that the order was chosen so that $p = m$. Let $T'' \in \mathrm{STab}(T)$ be the tableau such that

$$\boldsymbol{\omega}^{T''} = \boldsymbol{\omega} \prod_{t < m} A_{i_t,r_t}^{-1}.$$

Then,

$$\boldsymbol{\nu} = \boldsymbol{\omega}^{T''} A_{i,s}^{-1} \qquad \text{and} \qquad \boldsymbol{\mu} = \boldsymbol{\omega}^{T''} A_{i_m,r_m}^{-1}.$$

Moreover, Lemma 4.2.3 (a) implies that T'' has a column either as in (4.1.5) or as in (4.1.6). We will treat the case that (4.1.5) holds, i.e., we assume $i < n$. We omit the details for the case $i = n$ which can be dealt with similarly. In particular, T'' has columns of the form

$$\begin{array}{|c|l} \hline \vdots & \\ \hline i & {\scriptstyle s-i+1} \\ \hline \vdots & \\ \hline \overline{i+1} & {\scriptstyle s'} \\ \hline \vdots & \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{|c|l} \hline \vdots & \\ \hline i_m & {\scriptstyle r_m-i_m+1} \\ \hline \vdots & \\ \hline \overline{i_m+1} & {\scriptstyle s''} \\ \hline \vdots & \\ \hline \end{array} \tag{4.2.9}$$

and these two configurations may happen at the same column. We now proceed with a cases by case argument according to wether T' has boxes with content either i or $i + 1$.

(1) A column of T' has the box $\boxed{i}_{s-i+1}$. In this case, (4.2.4) implies that, for any $s' < s-i+1$, $\boxed{\overline{i+1}}_{s'}$ is not a box in the same column. Since T' is semi-standard, this column must have the box $\boxed{\overline{i+1}}_{s-i-1}$. In other words, T' has a column of the form

$$\begin{array}{|c|l} \hline \vdots & \\ \hline i & {\scriptstyle s-i+1} \\ \hline \overline{i+1} & {\scriptstyle s-i-1} \\ \hline \vdots & \\ \hline \end{array} \tag{4.2.10}$$

The move from T'' to T' corresponds to using (4.1.5) applied to the pair $(S, A^{-1}_{i_m,r_m})$ where S is the column containing the second configuration listed in (4.2.9). In particular, $i_m \neq i$ and, hence, T' must have a column containing the first configuration listed in (4.2.9) which contradicts the configuration (4.2.10) since, by Lemma 4.2.1, $\boxed{i}_{s-i+1}$ is not a box in any other column of T' and $i+1$ and $\overline{i+1}$ cannot be the content of boxes in the same column.

(2) T' does not have the box $\boxed{i}_{s-i+1}$. This implies that the modification corresponding to multiply $\boldsymbol{\omega}^{T''}$ by $A^{-1}_{i_m,r_m}$ changes the content of the box $\boxed{i}_{s-i+1}$ in T''. Thus $(i_m, r_m) = (i, s)$, contradiction.

Condition (iii): Let $\boldsymbol{\mu} = \boldsymbol{\omega}^{T'} \in \mathcal{M}$ and $i \in I$. We need to show that there exists i-dominant $\boldsymbol{\nu} \in \mathcal{M}$ such that

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \sum_{\boldsymbol{\eta} \in \boldsymbol{\mu}\mathcal{Q}_{\{i\}} \cap \mathcal{M}} \pi_i(\boldsymbol{\eta}). \tag{4.2.11}$$

We write down the proof for $i \leq n-2$ and omit the details for $i = n-1, n$ since they are similar. We show by induction on the number $\mathbf{C}_i = \mathbf{C}_i(T')$ of columns of T' having a box with content $i+1$ but none with content i that $\boldsymbol{\nu}$ satisfying (4.2.11) exists. Notice that Lemma 4.2.4 (a) implies that $\boldsymbol{\mu}$ is i-dominant iff $\mathbf{C}_i = 0$.

Thus, assume $\mathbf{C}_i = 0$, in which case we show that (4.2.11) holds with $\boldsymbol{\nu} = \boldsymbol{\mu}$. We start describing the left-hand side. If $i+1$ is the content of a box in every column of T' having a box with content i, then part (b) of Lemma 4.2.4 implies that $\pi_i(\boldsymbol{\nu}) = 1$ and we need to show that $\boldsymbol{\nu}$ is the unique element of $\boldsymbol{\nu}\mathcal{Q}_{\{i\}} \cap \mathcal{M}$. Indeed, notice that for such T', no change of the form (4.1.5) can be applied. Therefore, there is no $\boldsymbol{\eta} \in \boldsymbol{\nu}\mathcal{Q}_{\{i\}} \cap \mathcal{M}$ such that $\boldsymbol{\eta} < \boldsymbol{\nu}$. On the other hand, if $T'' \in \mathrm{STab}(T)$ were such that $\boldsymbol{\omega}^{T''} A^{-1}_{i,s} = \boldsymbol{\nu}$ for some s, then (4.1.5) would imply that T' had a column having a box with content i and no box with content $i+1$, contradicting our hypothesis on T'.

Suppose now that there exists a column of T' having i as content of a box while $i+1$ is not a content of any box. In that case, it is not hard to see that there exist unique $m, j_t, s_t, r_t \in \mathbb{Z}$ with $m > 0, 1 \leq j_1 < j_2 < \cdots < j_m \leq k$, satisfying

(1) $\boxed{i}_{s_t-2p}$ is a box in T'_{j_t+p} and $\boxed{i+1}_{s_t-2(p+1)}$ is not for every $0 \leq p < r_t$;
(2) Either $\boxed{i}_{s_t-2r_t}$ is not a box in $T'_{j_t+r_t}$ or $\boxed{i+1}_{s_t-2(r_t+1)}$ is a box of $T'_{j_t+r_t}$;
(3) Either $\boxed{i}_{s_t+2}$ is not a box in T'_{j_t-1} or $\boxed{i+1}_{s_t}$ is a box of T'_{j_t-1};

for every $1 \le t \le m$. Using part (b) of Lemma 4.2.4, it is not hard to see that

$$\pi_i(\boldsymbol{\nu}) = \prod_{t=1}^{m} Y_{i,s_t+i-2r_t,r_t}$$

is the q-factorization of $\pi_i(\boldsymbol{\nu})$. Hence, using the comment following (2.7.1), we get

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \pi_i(\boldsymbol{\nu}) \prod_{t=1}^{m} \left(\sum_{d=0}^{r_t} \prod_{c=0}^{d-1} A^{-1}_{i,s_t+i-1-2c} \right).$$

We now show that the right-hand side of (4.2.11) coincides with this. Notice that, for each t, the summand corresponding to a given d is equal to $\boldsymbol{\omega}^{T_{t,d}}$ where $T_{t,d}$ is the tableau obtained from T' by applying (4.1.5) to columns $j_t + p$ with $0 \le p < d$. We can rewrite the above as

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \pi_i(\boldsymbol{\nu}) \sum_{\mathbf{d}} \prod_{t=1}^{m} \prod_{c=0}^{d_t-1} A^{-1}_{i,s_t+i-1-2c}$$

where the sum runs over the set $\mathbf{D} = \{(d_1, \dots, d_m) \in \mathbb{Z}_{\ge 0} : d_t \le r_t, 1 \le t \le m\}$. Expanding the previous comment, for each $\mathbf{d} \in \mathbf{D}$, let $T_{\mathbf{d}}$ be the tableau obtained from T' by applying (4.1.5) to columns $j_t + p$ with $1 \le t \le m, 0 \le p < d_t$. Then, we can further rewrite the above as

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \sum_{\mathbf{d} \in \mathbf{D}} \boldsymbol{\omega}^{T_{\mathbf{d}}}. \tag{4.2.12}$$

Hence, we are left to show that

$$\boldsymbol{\nu}\mathcal{Q}_{\{i\}} \cap \mathcal{M} = \{\boldsymbol{\omega}^{T_{\mathbf{d}}} : \mathbf{d} \in \mathbf{D}\}. \tag{4.2.13}$$

Evidently, $\boldsymbol{\omega}^{T_{\mathbf{d}}} \in \boldsymbol{\nu}\mathcal{Q}_{\{i\}} \cap \mathcal{M}$ for all $\mathbf{d} \in \mathbf{D}$. Let $\mathbf{e}_t, 1 \le t \le m$ be the standard basis of $\mathbb{Z}_{\ge 0}$. We will show

$$\boldsymbol{\omega}^{T_{\mathbf{d}}} A^{-1}_{i,s} = \boldsymbol{\omega}^{T''} \ \Rightarrow\ T'' = T_{\mathbf{d}+\mathbf{e}_t} \quad \text{for some } t \text{ determined by } s \text{ satisfying } d_t < r_t \tag{4.2.14}$$

and

$$\boldsymbol{\omega}^{T_{\mathbf{d}}} = \boldsymbol{\omega}^{T''} A^{-1}_{i,s} \ \Rightarrow\ T'' = T_{\mathbf{d}-\mathbf{e}_t} \quad \text{for some } t \text{ determined by } s \text{ satisfying } d_t > 0, \tag{4.2.15}$$

which clearly proves (4.2.13). For showing (4.2.14), observe that (4.1.5) implies that T'' is obtained from $T_{\mathbf{d}}$ by modifying a column having a box with content i and a box with content $\overline{i+1}$. Thus, such column must be the (j_t+p)-th one for some $1 \le t \le m$ and $d_t \le p < r_t$. In particular, $d_t < r_t$ and we are left to show that $p = d_t$. One easily sees that applying (4.1.5) with $p > d_t$ would give rise to a non-semi-standard tableau. Namely, $\boxed{i}_{s-i+1}$ is a box in the $(j_t + p)$-th column of $T_{\mathbf{d}}$ and, if $p > d_t$, then $\boxed{i}_{s-i+3}$ is a box in the (j_t+p-1)-th column of $T_{\mathbf{d}}$ since this column coincides with that of T'. The change (4.1.5) would replace $\boxed{i}_{s-i+1}$ by $\boxed{i{+}1}_{s-i+1}$ and the diagonal condition for T'' be semi-standard would be violated.

For showing (4.2.15), observe that (4.1.5) implies that $T_{\mathbf{d}}$ is obtained from T'' by modifying a column having a box with content i and a box with content $\overline{i+1}$. Moreover, after the change such column would have a box with content $i+1$ and a box with content $\bar{i}$. One easily sees this column must be one of those modified for obtaining $T_{\mathbf{d}}$ from T'. Indeed, all other columns either have a box with content i or a box with content $\overline{i+1}$. Hence, the column must be the $(j_t + p)$-th one for some $1 \le t \le m$ and $0 \le p < d_t$ and we need to show that $p = d_t - 1$. For seeing this, notice that $\boxed{i}_{s-i+1}$ is a box in the $(j_t + p)$-th column of T''. Since the $(j_t + p + 1)$-th column of T'' coincides with that of $T_{\mathbf{d}}$, if $p < d_t - 1$, it would follow that $\boxed{i{+}1}_{s-i-1}$ is a box of the (j_t+p+1)-th column of

T'', violating the diagonal condition for T'' be semi-standard once more. This completes the proof for $\mathbf{C}_i = 0$.

If $\mathbf{C}_i > 0$, then T' has a column with a box having content $i+1$ and no box with content i. Suppose the last such column is the j-th one and that s is the support of the box with content $i+1$. We claim that

$$\boldsymbol{\mu} = \boldsymbol{\omega}^{T''} A_{i,s+i-1}^{-1} \quad \text{for some} \quad T'' \in \mathrm{STab}(T). \tag{4.2.16}$$

Assuming this, the inductive step is proved as follows. It follows from (4.1.5) that $\mathbf{C}_i(T'') = \mathbf{C}_i(T') - 1$ and, hence, the induction hypothesis applies to $\boldsymbol{\mu}' = \boldsymbol{\omega}^{T''}$. More precisely, there exists i-dominant $\boldsymbol{\nu} \in \mathcal{M}$ such that

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \sum_{\boldsymbol{\eta} \in \boldsymbol{\mu}' \mathcal{Q}_{\{i\}} \cap \mathcal{M}} \pi_i(\boldsymbol{\eta}). \tag{4.2.17}$$

Let $S \in \mathrm{STab}(T)$ be such that $\boldsymbol{\nu} = \boldsymbol{\omega}^S$. Then, $\mathbf{C}_i(S) = 0$ and (4.2.12) implies that

$$\mathrm{qch}(V_q(\pi_i(\boldsymbol{\nu}))) = \sum_{\mathbf{d} \in \mathbf{D}} \boldsymbol{\omega}^{S_{\mathbf{d}}}$$

where $\mathbf{D}$ and $S_{\mathbf{d}}$ are defined as before with S in place of T'. Since $\boldsymbol{\mu}' \in \boldsymbol{\mu}' \mathcal{Q}_{\{i\}} \cap \mathcal{M}$, this, together with (4.2.17), implies that $T'' = S_{\mathbf{d}}$ for some $\mathbf{d}$. Then, the same argument used for proving (4.2.14) shows that (4.2.16) implies that

$$T' = S_{\mathbf{d}+\mathbf{e}_t} \quad \text{for some } t \text{ such that } \mathbf{d} + \mathbf{e}_t \in \mathbf{D}.$$

This implies

$$\boldsymbol{\mu}' \mathcal{Q}_{\{i\}} \cap \mathcal{M} = \boldsymbol{\mu} \mathcal{Q}_{\{i\}} \cap \mathcal{M}$$

which clearly proves (4.2.11).

It remains to prove (4.2.16). In other words, we need to show that, if T'' is the tableau obtained from T' by replacing its j-th column by the one on the left-hand side of (4.1.5), then $T'' \in \mathrm{STab}(T)$. It is clear that the j-th column of T'' remains an element of $\mathcal{B}_l^{r+2(j-1)}$ and we need to check the diagonal condition for T'' be semi-standard. Let $m \in \mathbb{Z}$ be such that $1 \leq j + m \leq k$ and let i_m be the content of the box supported at $s - 2m$ of the $(j+m)$-th column of T''. In particular, $i_0 = i$. Since T' is semi-standard, we have

$$i_{1-j} \geq \cdots \geq i_{-1} \geq i + 1 \geq i_1 \geq \cdots i_{k-j}.$$

Hence, we are left to show that $i_1 \leq i$. If this was not the case, then $i_1 = i + 1$ and, by the choice of j, the $(j+1)$-th column of T' must have a box with content i whose support is necessarily s. Then, the j-th column of T' has a box supported at $s + 2$ whose content is at most $i - 1$ since i is not the content of any of its boxes. This violates the diagonal condition for T' be semi-standard, yielding the desired contradiction. $\square$

Before ending this section, we infer some information on $\mathrm{qch}(\overline{V_q(Y_{l,r,k})})$ which we shall need. We introduce the following notation. Given $i, j \in I$, let $[i, j] = \overline{\{i, j\}}$ be the minimal connected subdiagram containing i, j and

$$\alpha_{[i,j]} = \sum_{t \in [i,j]} \alpha_t.$$

Lemma 4.2.7. Let $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(Y_{l,r,k}))$ be such that $\boldsymbol{\mu} \leq Y_{l,r,k} A_{1,s}^{-1}$ for some $s \in \mathbb{Z}$ and let $s_1 = r + 2k + n - 3$. Then, $\mathrm{wt}(\boldsymbol{\mu}) \leq k\omega_l - \alpha_{[1,l]}$, $s \leq s_1$, and $\boldsymbol{\mu} \leq Y_{l,r,k} A_{1,s_1}^{-1}$.

Proof. One easily deduces using the PBW theorem that, if $j \in [1,l], j \neq 1$, then $k\omega_l-(\alpha_{1,l}-\alpha_j)$ is not an element of $P(k\omega_l)=\{w\nu : w \in \mathcal{W}, \nu \in P^+, \nu \leq k\omega_l\}$. This easily implies the first statement.

Observe that s_1 is the support of the first box of the first row of T, where T is as in Theorem 4.2.2. The hypothesis $\boldsymbol{\mu} < Y_{l,r,k}A_{1,s}^{-1}$ means that there exists $T', S \in \mathrm{STab}(T)$ such that $\boldsymbol{\omega}^S = \boldsymbol{\omega}^{T'}A_{1,s}^{-1}$ and $\boldsymbol{\mu} \leq \boldsymbol{\omega}^S$. Thus, S is obtained from T' by a modification of the type (4.1.5) with $i=1$ performed in some column of T'. But, as one can easily see, such a modification is possible only if all the columns to the left of this one have a gap at the first row. In particular, this is true for the first column and, hence, $\boldsymbol{\omega}^S \leq Y_{l,r,k}A_{1,s_1}^{-1}$. The remaining two statements are now easily deduced. $\square$

4.3. On certain partially dominant ℓ-weights of spin KR modules

We shall need the following construction associated to a tableau T with shape as in (4.2.2). Given a partition $\xi=(\xi_1,\xi_2,\ldots,\xi_k)$ with $\lfloor \frac{n-1}{2} \rfloor \geq \xi_1 \geq \cdots \geq \xi_k \geq 0$, consider the unique tableau $T_\xi^l \in \mathrm{STab}(T)$ satisfying:

(1) for all $1 \leq j \leq k$, the j-th column of T may have a gap only at the $(n-2\xi_j)$-th row;
(2) the content of the $(n-2\xi_j)$-th box is m, where $m=\bar{n}$ if $l=n-1$ and $m=n$ if $l=n$;

In pictures, the j-th column of T_ξ^l is

$$\begin{array}{|c|}\hline 1\\ \hline \vdots\\ \hline n-2\xi_j-1\\ \hline m\\ \hline \overline{n-1}\\ \hline \overline{n-2}\\ \hline \vdots\\ \hline \overline{n-2\xi_j}\\ \hline \end{array}_{\,r+2(k-j)-n+1} \qquad . \tag{4.3.1}$$

Notice that the j-th column has a gap iff $\xi_j > 0$, i.e., if the j-th column is not of the form (4.1.3). In particular,

$$Y_{l,r,k} = \boldsymbol{\omega}^{T_0^l}$$

where 0 denotes the partition with all $\xi_j = 0$.

Proposition 4.3.1. Let $J=\{1,\ldots,n-2\}\cup\{l\}$. Then, the elements of the form $\boldsymbol{\omega}^{T_\xi^l}$ are precisely the J-dominant elements of $\mathrm{wt}_\ell(V_q(Y_{l,r,k}))$.

Proof. Observe first that each column of $\boldsymbol{\omega}^{T_\xi^l}$ gives rise to a J-dominant ℓ-weight. Indeed, we have

$$Y_{i,s} = \begin{array}{|c|}\hline 1\\ \hline \vdots\\ \hline i\\ \hline \end{array}_{\,s-i+2}, \quad 1 \leq i \leq n-2, \qquad Y_{n-1,s} = \begin{array}{|c|}\hline \bar{n}\\ \hline \end{array}_{\,s-n+1}, \qquad Y_{n,s} = \begin{array}{|c|}\hline n\\ \hline \end{array}_{\,s-n+1},$$

and

$$Y_{n-1,s}^{-1} = \begin{array}{|c|}\hline n\\ \hline \overline{n-1}\\ \hline \end{array}_{\,s-n-1}, \qquad Y_{n,s}^{-1} = \begin{array}{|c|}\hline \bar{n}\\ \hline \overline{n-1}\\ \hline \end{array}_{\,s-n-1}.$$

Thus, if s is the support of the $(n-2\xi_j)$-th box of the j-th column of T^l_ξ and $\xi_j > 0$, the ℓ-weight attached to this column is

$$Y_{n-2\xi_j-1,s+n-2\xi_j-1}\, Y^{-1}_{l',n+s-1} \quad \text{with} \quad \{l,l'\} = \{n,n-1\}.$$

As we have observed before the proposition, if $\xi_j = 0$, then the ℓ-weight attached to this column is $Y_{l,s+n-1}$.

It remains to show that all J-dominant ℓ-weights of $V_q(Y_{l,r,k})$ can be represented by tableaux of the form T^l_ξ. Fix such an ℓ-weight, say $\boldsymbol{\mu}$. By Theorem 4.2.2, $\boldsymbol{\mu} = \boldsymbol{\omega}^T$ where T is a semi-standard tableaux with the given shape. We claim that it suffices to show that each column of T is of the form (4.3.1). Indeed, this determines $\xi_j \le \left\lfloor \frac{n-1}{2} \right\rfloor$ for each $1 \le j \le k$ and, since T is semi-standard, it follows that $\xi_j \ge \xi_{j+1}$ for all $1 \le j < k$.

Assume by contradiction that T has a column which is not of the form (4.3.1) and say that the j-th column is the first such column. We split the analysis in two cases:

(i) the first box of this column is either $\boxed{n}_s$ or $\boxed{\bar n}_s$, for some $s \in \mathbb{Z}$;
(ii) the content of the first box of this column is smaller than n.

In case (i), it follows that, for all $t = 2, \dots, n$, the content of the t-th box of the column is $\overline{n-t+1}$. Assume the first box is $\boxed{\bar n}_s$ (the case $\boxed{n}_s$ is similar and we omit the details). Then, the ℓ-weight attached to this column is $Y^{-1}_{n,s-n-1}$. If $l = n-1$, then n must be odd which implies that this column is of the form (4.3.1) with $\xi_j = \left\lfloor \frac{n-1}{2} \right\rfloor$, contradicting the choice of j. Therefore, $l = n$, in which case $Y^{-1}_{n,s-n-1}$ is not J-dominant. To obtain a contradiction, we have to show that $Y_{n,s-n-1}$ does not appear in the ℓ-weight attached to any other column of T. Indeed, as seen above, if $Y_{n,s-n-1}$ appears, then $\boxed{n}_s$ must be a box in another column of T. Since T has the form (4.1.3), only columns to the left of the j-th one have boxes supported at s. Suppose the $(j-m)$-th column has this box, $m \ge 1$. Then (4.1.3) implies that this column has a box supported at $s+2m$. Since all columns are increasing, the content c of the box supported at $s+2m$ is at most $n-m < \bar n$. This contradicts the assumption that T is semi-standard because the box $\boxed{c}_{s+2m}$ in column $j-m$ and the box $\boxed{\bar n}_s$ in column j are in the same diagonal from left to right and top to bottom.

In case (ii), since the j-th column is not of the form (4.3.1), it follows that there exists $1 \le i \le n-2$ such that the content of the i-th box is t for some $i < t < n$. Let i be the minimum with this property. In other words, the i-th box of the j-th column of T is $\boxed{t}_s$ for some s and, if there is a box supported at $s+2$ in this column, its content is $i-1$. Since all columns are increasing, it follows that $Y^{-1}_{t-1,s+t-1}$ appears in the ℓ-weight attached to the j-th column. Since $\boldsymbol{\mu}$ is J-dominant, $Y_{t-1,s+t-1}$ must appear in the ℓ-weight attached to some other column, i.e., $\boxed{t-1}_{s+2}$ must be a box in another column of T, contradicting Lemma 4.2.1(b). $\square$

Let us describe $\boldsymbol{\omega}^{T^l_\xi}$ more explicitly. We start with the case $l = n-1$. Assume first that $k = 1$, i.e., the tableaux have only one column and $\xi = (\xi_1)$ with $0 \le \xi_1 \le \left\lfloor \frac{n-1}{2} \right\rfloor$. If $\xi_1 > 0$, it follows from (4.1.5) and (4.1.6) that

$$\boldsymbol{\omega}^{T^{n-1}_\xi} = \boldsymbol{\omega}^{T^{n-1}_{(\xi_1-1)}} A^{-1}_{n-2\xi_1+1,n-1,r+2(\xi_1-1)} A^{-1}_{n-2\xi_1,n-2,r+2\xi_1-1} A^{-1}_{n,r+4\xi_1-1}.$$

Iterating this we get

$$\boldsymbol{\omega}^{T^{n-1}_\xi} = Y_{n-1,r} \prod_{m=1}^{\xi_1} A^{-1}_{n-2m+1,n-1,r+2m-2} A^{-1}_{n-2m,n-2,r+2m-1} A^{-1}_{n,r+4m-1}.$$

For $k > 1$, we apply this formula to each column replacing r by $r + 2(k-j)$ when working with the j-th column to get that $\boldsymbol{\omega}^{T_\xi^{n-1}}$ is equal to

$$Y_{n-1,r,k} \prod_{j=1}^{k} \prod_{m=1}^{\xi_j} A^{-1}_{n-2m+1,n-1,r+2(k-j)+2m-2} A^{-1}_{n-2m,n-2,r+2(k-j)+2m-1} A^{-1}_{n,r+2(k-j)+4m-1}. \tag{4.3.2}$$

Setting

$$|\xi| = \max\{j : \xi_j \neq 0\}$$

and using the expressions for the elements $A_{i,s}$ in terms of the elements $Y_{i,s}$, it follows that

$$\boldsymbol{\omega}^{T_\xi^{n-1}} = Y_{n-1,r,k-|\xi|} \prod_{j=1}^{|\xi|} Y_{n-2\xi_j-1,r+2(k-j)+2\xi_j} Y^{-1}_{n,r+2(k-j)+4\xi_j}. \tag{4.3.3}$$

Notice that

$$r + 2(k-j) + 4\xi_j > r + 2(k-i) + 4\xi_i \quad \text{if} \quad j < i. \tag{4.3.4}$$

Similarly, for $l = n$ we get

$$\boldsymbol{\omega}^{T_\xi^{n}} = Y_{n,r,k} \prod_{j=1}^{k} \prod_{m=1}^{\xi_j} A^{-1}_{n-2m+1,n-2,r+2(k-j)+2m-2} A^{-1}_{n-2m,n-1,r+2(k-j)+2m-1} A^{-1}_{n,r+2(k-j)+4m-3}.$$

and

$$\boldsymbol{\omega}^{T_\xi^{n}} = Y_{n,r,k-|\xi|} \prod_{j=1}^{|\xi|} Y_{n-2\xi_j-1,r+2(k-j)+2\xi_j} Y^{-1}_{n-1,r+2(k-j)+4\xi_j}.$$

Similarly to Proposition 4.3.1 one can prove the following proposition.

Proposition 4.3.2. Let $J = I \setminus \{1\}$. Then the J-dominant ℓ-weights of $\mathrm{wt}_\ell(V_q(Y_{l,r,k}))$ are the semi-standard tableaux in $\mathrm{STab}(T)$ such that each column either has no gaps or has a gap at the first row.

It follows from Proposition 4.3.2, that the $(I \setminus \{1\})$-dominant ℓ-weights of $V_q(Y_{n-1,r,k})$, other than $Y_{n-1,r,k}$, are

$$\boldsymbol{\nu}_i = Y_{n-1,r,k} \prod_{i=1}^{k} A^{-1}_{n-1,1,r+2(k-i)}, \quad i = 1, \dots, k.$$

4.4. Tensor products of spin KR modules

Fix $\boldsymbol{\omega} = Y_{n-1,r_{n-1},k_{n-1}}$ and $\boldsymbol{\varpi} = Y_{n,r_n,k_n}$, for some $r_{n-1}, r_n \in \mathbb{Z}$, $k_{n-1}, k_n \in \mathbb{Z}_{>0}$ and let S and T be semi-standard tableaux as in (4.2.2) such that

$$\boldsymbol{\omega}^S = \boldsymbol{\omega} \quad \text{and} \quad \boldsymbol{\omega}^T = \boldsymbol{\varpi}.$$

Our first goal is to describe the set

$$D := \mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})\right) \cap \mathcal{P}^+.$$

The analysis depends on whether

$$r_{n-1} + 2k_{n-1} \leq r_n + 2k_n \qquad \text{or} \qquad r_{n-1} + 2k_{n-1} \geq r_n + 2k_n.$$

Since the the latter is obtained from the former by applying the Dynkin diagram automorphism which switches the two spin nodes, we shall write down the details assuming that

$$r_{n-1} + 2k_{n-1} \le r_n + 2k_n. \tag{4.4.1}$$

We begin showing that

$$D \subseteq \{\boldsymbol{\nu}\boldsymbol{\varpi} : \boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))\}. \tag{4.4.2}$$

Let $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ and $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ be such that $\boldsymbol{\nu}\boldsymbol{\mu} \in \mathcal{P}^+$. In particular, $\boldsymbol{\nu}\boldsymbol{\mu}$ is not right negative. Suppose that $\boldsymbol{\mu} \neq \boldsymbol{\varpi}$. Then, by Proposition 2.5.5, $\boldsymbol{\mu}$ is right negative. Since the product of right negative elements is again right negative, $\boldsymbol{\nu}$ is not right negative and Proposition 2.5.5 implies that $\boldsymbol{\nu} = \boldsymbol{\omega}$. Similarly, if $\boldsymbol{\nu} \neq \boldsymbol{\omega}$, we must have $\boldsymbol{\mu} = \boldsymbol{\varpi}$. It remains to see that (4.4.1) implies that $\boldsymbol{\omega}\boldsymbol{\mu} \in \mathcal{P}^+$ only if $\boldsymbol{\mu} = \boldsymbol{\varpi}$. Indeed, if $\boldsymbol{\mu} \neq \boldsymbol{\varpi}$, then, by Proposition 2.5.5,

$$r(\boldsymbol{\mu}) > r_n + 2(k_n - 1) \ge r_{n-1} + 2(k_{n-1} - 1) = r(\boldsymbol{\omega}),$$

showing that $\boldsymbol{\omega}\boldsymbol{\mu}$ is right negative and, hence, cannot be in $\mathcal{P}^+$.

Evidently,

$$\boldsymbol{\nu}\boldsymbol{\varpi} \in D \Rightarrow \boldsymbol{\nu} \text{ is } J\text{-dominant for } J = \{1, 2, \dots, n-1\}.$$

Set $X = \{r_n + 2(m-1) : m = 1, \dots, k_n\}$ and recall the description of the J-dominant elements of $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ in Proposition 4.3.1. It then follows from (4.3.3), (4.3.4), and (4.4.2) that

$$D = \{\boldsymbol{\omega}^{S_\xi^{n-1}}\boldsymbol{\varpi} \ : \ r_{n-1} + 2(k_{n-1} - j) + 4\xi_j \in X \ \text{ for all } \ j \le |\xi|\}. \tag{4.4.3}$$

In particular, if

$$r_{n-1} + 2(k_{n-1} - 1) + 4l \neq r_n + 2(m-1) \quad \text{for all} \quad m \in \{1, \dots, k_n\},\ 1 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor, \tag{4.4.4}$$

then $D = \{\boldsymbol{\omega}\boldsymbol{\varpi}\}$ and, hence,

$$V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}) \cong V_q(\boldsymbol{\omega}\boldsymbol{\varpi}). \tag{4.4.5}$$

Indeed, (4.4.4) implies that

$$r_{n-1} + 2(k_{n-1} - 1) + 4\xi_1 \notin X$$

for all nonzero partitions ξ.

In the next lemma, we collect some partial information about

$$D \cap \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}\boldsymbol{\varpi})).$$

Lemma 4.4.1. Let $\boldsymbol{\mu} \in D$ and ξ be such that $\boldsymbol{\mu} = \boldsymbol{\omega}^{S_\xi^{n-1}}\boldsymbol{\varpi}$.

(a) If $r_{n-1} + 2(k_{n-1} - j) + 4\xi_j \in X \setminus \{r_n\}$ for all $j \le |\xi|$, then $V_q(\boldsymbol{\mu})$ is not simple a factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$.
(b) If $|\xi| = 1$ and $r_n = r_{n-1} + 2(k_{n-1} - 1) + 4\xi_1$, then $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}\boldsymbol{\varpi}))$.
(c) If $|\xi| = 2$ and $r_n = r_{n-1} + 2(k_{n-1} - 2) + 4\xi_1$, then $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}\boldsymbol{\varpi}))$.
(d) If $\xi_j = 1$ for all $j \le |\xi|$ and $r_n = r_{n-1} + 2(k_{n-1} - |\xi|) + 4$, then $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}\boldsymbol{\varpi}))$.

Proof. (a) To simplify notation, let $k = |\xi|$ and $V = V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$.

By (4.4.3), $\boldsymbol{\mu} \in D$ and there exists $1 < m \le k_n$ such that

$$r_{n-1} + 2(k_{n-1} - k) + 4\xi_k = r_n + 2(m-1). \tag{4.4.6}$$

Let $\boldsymbol{\nu} = \boldsymbol{\mu} A^{-1}_{n, r_n + 2(m-2)+1}$. We claim that it suffices to prove that:

(i) $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\mu}))$;

(ii) $\boldsymbol{\nu} \notin \mathrm{wt}_\ell(V)$.

Then it follows that

$$\mathrm{wt}_\ell(V_q(\boldsymbol{\mu})) \nsubseteq \mathrm{wt}_\ell(V),$$

which implies that $V_q(\boldsymbol{\mu})$ is not a simple factor of V.

Let us prove (i). By Remark 2.6.8, $\chi_{\{n\}}(\boldsymbol{\mu}) \subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\mu}))$ and, therefore, it suffices to prove that $\boldsymbol{\nu} \in \chi_{\{n\}}(\boldsymbol{\mu})$. It follows from (4.3.3) that

$$\boldsymbol{\mu} = Y_{n-1,r_{n-1},k_{n-1}-k} \left(\prod_{j=1}^{k} Y_{n-2\xi_j-1,r_{n-1}+2(k_{n-1}-j)+2\xi_j}\right) Y_{n,r_{n-1},m-1} \left(\prod_{r\in B} Y_{n,r}\right), \tag{4.4.7}$$

where $B = \{r_n + 2(j-1) : m < j \le k_n\} \setminus \{r_{n-1} + 2(k_{n-1}-j) + 4\xi_j : j < k\}$. In particular,

$$\pi_n(\boldsymbol{\mu}) = Y_{n,r_{n-1},m-1} \left(\prod_{r\in B} Y_{n,r}\right).$$

Since $\min(B) \ge r_n + 2(m+1)$, one easily sees that $Y_{n,r_{n-1},m-1}$ is a factor of the qfactorization of $\pi_n(\boldsymbol{\mu})$ and, hence, $V_q(Y_{n,r_{n-1},m-1})$ is a factor of the tensor product decomposition of $V_q(\pi_n(\boldsymbol{\mu}))$ as in Theorem 2.7.1. Thus, applying (3.2.12) to this factor one easily deduces $\boldsymbol{\nu} \in \chi_{\{n\}}(\boldsymbol{\mu})$ as desired.

We now prove (ii). By (4.3.2), we have

$$\boldsymbol{\mu} = \boldsymbol{\omega}\boldsymbol{\varpi} \prod_{\zeta\in\Xi} A_\zeta^{-1} \quad \text{and} \quad \boldsymbol{\nu} = \boldsymbol{\omega}\boldsymbol{\varpi}\, A_{\zeta_0}^{-1} \prod_{\zeta\in\Xi} A_\zeta^{-1}$$

where $\Xi \subseteq I \times \mathbb{Z}$ and $\zeta_0 = (n, r_n + 2(m-2) + 1)$. Therefore, setting $\Xi_0 = \Xi \cup \{\zeta_0\}$, we are left to show that there is no partition $\Xi_0 = \Xi' \cup \Xi''$ such that

$$\boldsymbol{\omega} \prod_{\zeta\in\Xi'} A_\zeta^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \qquad \text{and} \qquad \boldsymbol{\varpi} \prod_{\zeta\in\Xi''} A_\zeta^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})).$$

By contradiction, assume such a partition exists. It follows from (2.2.1), (4.1.5), and (4.1.6), that each element $A_\zeta^{-1}, \zeta \in \Xi_0$, corresponds to change the contents of two particular boxes in a column of either S or T. In what follows we show that the modifications corresponding to elements of Ξ_0 cannot be done in T, implying that $\Xi_0 \cap \Xi'' = \emptyset$. First we show that

$$(n,r) \in \Xi \Rightarrow (n,r) \notin \Xi''. \tag{4.4.8}$$

By (4.1.6), the element $(n,r) \in \Xi$ is associated to replacing the boxes $\boxed{n{-}1}_{\,r-n+2}$ and $\boxed{n}_{\,r-n}$ of a given column of either S or T by $\boxed{\bar{n}}_{\,r-n+2}$ and $\boxed{\overline{n{-}1}}_{\,r-n}$, respectively. Moreover, the modified tableau will be semi-standard after such modification only if all the previous columns have been already modified. Therefore, it suffices to show that the first column of T cannot be modified by any $(n,r) \in \Xi$.

By hypothesis,

$$r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 = r_n + 2(m'-1) \tag{4.4.9}$$

for some $1 < m' \le k_n$. Moreover, (4.4.6) and (4.3.4) implies that $m' \ge m$. Set

$$s = r_{n-1} + 2(k_{n-1}-k) - n + 1 + 4\xi_k,$$

which is the support of the $(n-2\xi_k)$-th box of the k-th column of S, and

$$s' = r_{n-1} + 2(k_{n-1}-1) - n + 1 + 4\xi_1$$

which is the support of the $(n-2\xi_1)$-th box of the first column of S. By (4.4.6) and (4.4.9),

$$s = r_n + 2(m-1) - n + 1 \quad \text{and} \quad s' = r_n + 2(m'-1) + n - 1,$$

showing that

$$s \text{ is the support of the last box of the } (k_n - m + 1)\text{-th column of } T \tag{4.4.10}$$

(or the m-th one counted from right to left) and

$$s' \text{ is the support of the last box of the } (k_n - m' + 1)\text{-th column of } T. \tag{4.4.11}$$

Inspecting (4.3.2), one checks that

$$r_0 := \max\{r : (n,r) \in \Xi\} = r_{n-1} + 2(k_{n-1} - 1) + 4\xi_1 - 1 = s' - 2 + n.$$

Thus, it follows from (4.4.11), that the modification associated to (n, r_0) cannot be done in the first $k_n - m' + 1$ columns of T. Since T has the form (4.2.2), the same holds for any other $(n,r) \in \Xi$ and (4.4.8) is proved. Now observe that, if $\zeta = (i,r) \in \Xi$ with $i < n$, then the corresponding modification cannot be done in T, since any column of T does not have the two specific contents that would be changed. Therefore,

$$\Xi \subseteq \Xi'.$$

Thus, if we show that the modification associated to ζ_0 cannot be performed neither in T nor in S_ξ^{n-1}, we obtain the desired contradiction completing the proof of (ii).

By definition of s and (4.4.6), the modification associated to ζ_0 is the

$$\text{replacement of } \boxed{n{-}1}_{\,s},\ \boxed{n}_{\,s-2} \ \text{ by } \ \boxed{\bar n}_{\,s},\ \boxed{\overline{n{-}1}}_{\,s-2}, \tag{4.4.12}$$

in a fixed column of either T or S_ξ^{n-1}. By (4.4.10), the first column of T which has a box supported at $s-2$ is the $(k_n - m + 2)$-th one. Since $k_n - m + 2 > 1$, (4.4.12) cannot be done in T because the columns to the left of the $(k_n - m + 2)$-th one have not been modified. On the other hand, since s is the support of the $(n - 2\xi_k)$-th box of the k-th column of S, it follows from (4.3.1) that the k-th column of S_ξ^{n-1} has the boxes $\boxed{\bar n}_{\,s}$ and $\boxed{\overline{n{-}1}}_{\,s-2}$. Condition (2) in the definition of the sets $\mathcal{B}_r^{n-1}$ then implies that (4.4.12) cannot be done in the k-th column of S_ξ^{n-1}. Since S_ξ^{n-1} is semi-standard, the boxes supported at s and $s-2$ in the columns to the left of the k-th one must have contents strictly greater than $\bar n$ and $\overline{n-1}$, respectively. Hence, (4.4.12) cannot be done in the columns to the left of the k-th one as well. By the definition of k, the columns of S_ξ^{n-1} to the right of the k-th one coincide with those of S and, therefore, the boxes supported at s and $s-2$ have contents equal to at most $n - 2\xi_k - 1$ and $n - 2\xi_k$, respectively. Therefore, since $\xi_k > 0$, (4.4.12) cannot be performed in such columns. This completes the proof of part (a).

(b) By hypothesis

$$r_{n-1} + 2(k_{n-1} - 1) + 4\xi_1 = r_n. \tag{4.4.13}$$

Thus, $\boldsymbol{\mu} \in D$ by (4.4.3). Let $\boldsymbol{\lambda} = \boldsymbol{\omega}\boldsymbol{\varpi}$, $\boldsymbol{\mu} = \boldsymbol{\omega}^{S_\xi^{n-1}}\boldsymbol{\varpi}$, $\boldsymbol{\nu} = \boldsymbol{\mu} A_{n, r_{n-1}+2(k_{n-1}-1)+4\xi_1-1}$ and $V = V_q(\boldsymbol{\lambda})$. We show that these data satisfy the conditions (i)–(v) of Lemma 2.6.9 with $i = n$, thus proving that $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V)$.

(i) By definition $\boldsymbol{\nu} \in \boldsymbol{\mu}\,\iota_n(\mathcal{Q}^+_{\{n\}}) \setminus \{\boldsymbol{\mu}\}$. For showing that $\boldsymbol{\nu}$ is n-dominant, observe from (4.3.3) and (4.4.13) that

$$\boldsymbol{\mu} = Y_{n-1, r_{n-1}, k_{n-1}-1} Y_{n-2\xi_1-1, r_{n-1}+2(k_{n-1}-1)+2\xi_1} Y_{n, r_n+2, k_n-1}. \tag{4.4.14}$$

Hence,

$$\begin{aligned} \boldsymbol{\nu} \;=\;& Y_{n-1, r_{n-1}, k_{n-1}-1}\, Y_{n-2\xi_1-1, r_{n-1}+2(k_{n-1}-1)+2\xi_1}\, Y^{-1}_{n-2, r_{n-1}+2(k_{n-1}-1)+4\xi_1-1} \times \\ & \times Y_{n, r_{n-1}+2(k_{n-1}-1)+4\xi_1-2}\, Y_{n, r_{n-1}+2(k_{n-1}-1)+4\xi_1}\, Y_{n, r_n+2, k_n-1} \end{aligned} \tag{4.4.15}$$

is n-dominant. Next, we show that $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V)$. We begin showing that $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}))$. Indeed, it is not difficult to see that

$$\boldsymbol{\nu} = \boldsymbol{\nu}'\boldsymbol{\varpi} \qquad \text{where} \qquad \boldsymbol{\nu}' = \boldsymbol{\omega}^{S'} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \tag{4.4.16}$$

with $S' \in \mathrm{STab}(S)$ such that all the columns of S' but the first one have no gaps and its first column has only gaps in the $(n-2\xi_1)$-th and $(n-2\xi_1+2)$-th rows, the content of the $(n-2\xi_1)$-th row is $n-1$ and the content of the $(n-2\xi_1+2)$-th row is $\overline{n-2}$. By contradiction, assume $\boldsymbol{\nu} \notin \mathrm{wt}_\ell(V)$. Since $\boldsymbol{\nu}$ is not dominant, there must exist $\boldsymbol{\zeta} \in D$ with $\boldsymbol{\lambda} > \boldsymbol{\zeta} > \boldsymbol{\nu}$ such that $V_q(\boldsymbol{\zeta})$ is a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ and $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\zeta}))$. In particular, $\boldsymbol{\zeta} > \boldsymbol{\mu}$. We claim that there is no such $\boldsymbol{\zeta}$. Indeed, if $\boldsymbol{\lambda} > \boldsymbol{\zeta} > \boldsymbol{\mu}$, it follows from (4.3.2) that $\boldsymbol{\zeta} = \boldsymbol{\omega}^{S^{n-1}_{\xi'}}\boldsymbol{\varpi}$ with $\xi' = (\xi_1', 0, \dots, 0)$ and $0 < \xi_1' < \xi_1$. Equation (4.4.13) then implies that $r_{n-1} + 2(k_{n-1}-1) + 4\xi_1' < r_n$ while (4.4.3) implies that $\boldsymbol{\zeta} \notin D$, which proves the claim.

To complete the verification of condition (i), it remains to prove the uniqueness of $\boldsymbol{\nu}$ and that $\dim(V_{\boldsymbol{\nu}}) = 1$. Let $\boldsymbol{\zeta} \in \boldsymbol{\mu}\, \iota_n(\mathcal{Q}^+_{\{n\}}) \setminus \{\boldsymbol{\mu}, \boldsymbol{\nu}\}$. We will show that $\boldsymbol{\zeta} \notin \mathrm{wt}_\ell(V)$, which settles the uniqueness. Observe from (4.3.2) that

$$\begin{aligned}\boldsymbol{\mu} \;&=\; \boldsymbol{\lambda} \prod_{m=1}^{\xi_1} A^{-1}_{n-2m+1,n-1,r_{n-1}+2(k_{n-1}-1)+2m-2} A^{-1}_{n-2m,n-2,r_{n-1}+2(k_{n-1}-1)+2m-1} \times \\ &\qquad\qquad \times A^{-1}_{n,r_{n-1}+2(k_{n-1}-1)+4m-1}.\end{aligned} \tag{4.4.17}$$

Hence, $\boldsymbol{\zeta} = \boldsymbol{\mu}\prod_{\eta\in\Xi} A_\eta$ with $\Xi \subseteq \{(n, r_{n-1} + 2(k_{n-1}-1) + 4m - 1) : m = 1, \dots, \xi_1\}$ and $\Xi \neq \{(n, r_{n-1} + 2(k_{n-1}-1) + 4\xi - 1)\}$. In particular, $(n, r_{n-1} + 2(k_{n-1}-1) + 4m - 1) \in \Xi$ for some $m = 1, \dots, \xi_1 - 1$, which implies that either $\boldsymbol{\mu}\, \iota_n(\mathcal{Q}^+_{\{n\}}) \setminus \{\boldsymbol{\mu}, \boldsymbol{\nu}\} = \emptyset$ (and uniqueness follows) or $\xi_1 > 1$. Moreover, if $\Xi_{\boldsymbol{\mu}} \subseteq I \times \mathbb{Z}$ is such that $\boldsymbol{\mu} = \boldsymbol{\lambda}\prod_{\eta\in\Xi_{\boldsymbol{\mu}}} A_\eta^{-1}$, then $\boldsymbol{\zeta} = \boldsymbol{\lambda}\prod_{\eta\in\Xi_{\boldsymbol{\mu}}\setminus\Xi} A_\eta^{-1}$. Now, we are left to show that there is no partition $\Xi_0 := \Xi_{\boldsymbol{\mu}} \setminus \Xi = \Xi' \cup \Xi''$ such that

$$\boldsymbol{\omega} \prod_{\eta\in\Xi'} A_\eta^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \qquad \text{and} \qquad \boldsymbol{\varpi} \prod_{\eta\in\Xi''} A_\eta^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})).$$

By contradiction, assume such a partition exists. It follows from (2.2.1), (4.1.5), and (4.1.6), that each element $A_\eta^{-1}, \eta \in \Xi_{\boldsymbol{\mu}}$, corresponds to change the contents of two particular boxes in a column of either S or T. In what follows we show that the modifications corresponding to elements of $\Xi_{\boldsymbol{\mu}}$ cannot be done in T, implying that $\Xi_{\boldsymbol{\mu}} \cap \Xi'' = \emptyset$. Set

$$s = r_{n-1} + 2(k_{n-1}-1) - n + 1 + 4\xi_1,$$

which is the support of the $(n-2\xi_1)$-th box of the first column of S. By (4.4.13),

$$s = r_n - n + 1,$$

showing that

$$s \text{ is the support of the last box of the } k_n\text{-th column of } T. \tag{4.4.18}$$

First we show that

$$(n, r) \in \Xi \Rightarrow (n, r) \notin \Xi''. \tag{4.4.19}$$

By (4.1.6), the element $(n, r) \in \Xi$ is associated to replacing the boxes $\boxed{n-1}_{\,r-n+2}$ and $\boxed{n}_{\,r-n}$ of a given column of either S or T by $\boxed{\bar{n}}_{\,r-n+2}$ and $\boxed{\overline{n-1}}_{\,r-n}$, respectively. Moreover, the modified tableau will be semi-standard after such modification only if all the previous columns have been already modified. Therefore, it suffices to show that the first column of T cannot be modified by any $(n,r) \in \Xi_{\boldsymbol{\mu}}$. Inspecting (4.4.17), one checks that

$$r_0 := \max\{r : (n,r) \in \Xi_{\boldsymbol{\mu}}\} = r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 - 1 = s - 2 + n.$$

Thus, it follows from (4.4.11), that the modification associated to (n, r_0) cannot be done any column of T. Since T has the form (4.2.2), the same holds for any other $(n,r) \in \Xi$ and (4.4.19) is proved. Now observe that, if $\eta = (i,r) \in \Xi_{\boldsymbol{\mu}}$ with $i < n$, then the corresponding modification cannot be done in T since no column of T has the two specific contents that would be changed. Therefore,

$$\Xi_0 \subseteq \Xi_{\boldsymbol{\mu}} \subseteq \Xi'$$

and one can check that the changes corresponding to elements of $\Xi_{\boldsymbol{\mu}}$ can only be done in the first column of S. Thus, if we show that a modification associated to an element of Ξ_0 cannot be performed in S, we obtain the desired contradiction. Let $m \in \{1, \dots, \xi_1 - 1\}$ be minimum such that $\eta_0 := (n, r_{n-1} + 2(k_{n-1}-1) + 4m - 1) \in \Xi$, which implies $\eta_0 \notin \Xi_0$. Observe that, if $i < n$ and $(i,r) \in \Xi_{\boldsymbol{\mu}}$, then $(i,r) \in \Xi_0$. Let $S'' \in \mathrm{STab}(S)$ be such that $\boldsymbol{\zeta} = \boldsymbol{\omega}^{S''}$. Now we show that the change corresponding to $\eta_1 := (n-1, r_{n-1} + 2(k_{n-1}-1) + 4(m+1) - 3) \in \Xi_0$ cannot be performed in S to obtain S''. Indeed, η_1 corresponds to changing the boxes

$$\boxed{n{-}1}_{\,s'} \quad \text{and} \quad \boxed{\bar{n}}_{\,s'-2} \qquad \text{by} \qquad \boxed{n}_{\,s'} \quad \text{and} \quad \boxed{\overline{n{-}1}}_{\,s'-2} \tag{4.4.20}$$

where $s' = r_{n-1} + 2(k_{n-1}-1) + 4m + 3 - n$. Since the modification of the boxes $\boxed{n{-}1}_{\,s'-2}$ and $\boxed{n}_{\,s'-4}$ by $\boxed{\bar{n}}_{\,s'-2}$ and $\boxed{\overline{n{-}1}}_{\,s'-4}$ corresponding to η_0 was not performed, it follows that (4.4.20) also cannot be performed, yielding the desired contradiction. This completes the proof of the uniqueness of $\boldsymbol{\nu}$.

Finally, for proving that $\dim(V_{\boldsymbol{\nu}}) = 1$, it follows from the above argument that, if $S'' \in \mathrm{STab}(S)$ and $T' \in \mathrm{STab}(T)$ are such that $\boldsymbol{\omega}^{S''}\boldsymbol{\omega}^{T'} = \boldsymbol{\nu}$, then $S'' = S'$ (see (4.4.16)) and $T' = T$. Thus, it remains to show that $\dim(V_q(\boldsymbol{\omega})_{\boldsymbol{\nu}\boldsymbol{\varpi}^{-1}}) = 1$. One can easily check this last statement by seeing, using Lemma 4.2.1 and the definition of $\boldsymbol{\omega}^{S'}$, that there is no semi-standard tableau with the shape of S other than S' such that the corresponding ℓ-weight is equal to $\boldsymbol{\nu}$.

(ii) We show that

$$\boldsymbol{\nu}' \in \mathrm{wt}_\ell(V) \cap \boldsymbol{\nu}\mathcal{Q}_{\{n\}} \Rightarrow \mathrm{ht}(\boldsymbol{\nu}'\boldsymbol{\nu}^{-1}) \leq 0, \tag{4.4.21}$$

which, together with Lemma 2.4.1, imply that condition (ii) holds. In part (i) we proved that

$$\boldsymbol{\zeta} \in \boldsymbol{\mu}\, \iota_n(\mathcal{Q}^+_{\{n\}}) \setminus \{\boldsymbol{\mu}, \boldsymbol{\nu}\} \Rightarrow \boldsymbol{\zeta} \notin \mathrm{wt}_\ell(V). \tag{4.4.22}$$

In particular, since $\boldsymbol{\nu} = \boldsymbol{\mu} A_{n, r_{n-1}+2(k_{n-1}-1)+4\xi_1-1}$, it follows that if $\boldsymbol{\zeta} \in \boldsymbol{\nu}\, \iota_n(\mathcal{Q}^+_{\{n\}}) \setminus \{\boldsymbol{\nu}\}$, then $\boldsymbol{\zeta} \notin \mathrm{wt}_\ell(V)$.

We claim that if $\boldsymbol{\nu} A_{n,r}^{-1} \in \mathrm{wt}_\ell(V)$, then $r \in \{r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 - 1, r_n + 2(k_n - 1) + 1\}$. To prove the claim, we show that for $r \neq r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 - 1, r_n + 2(k_n-1) + 1$, $\boldsymbol{\nu} A_{n,r}^{-1} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}))$. Recall from (4.4.16) that $\boldsymbol{\nu} = \boldsymbol{\nu}'\boldsymbol{\varpi}$ with $\boldsymbol{\nu}' \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. It is not hard to see that $\boldsymbol{\varpi} A_{n,r}^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ iff $r = r_n + 2(k_n - 1) + 1$. Thus, it suffices to prove that if $r \neq r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 - 1$, then $\boldsymbol{\nu}' \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. Multiplication by $A_{n,r}^{-1}$ corresponds to replacing the boxes $\boxed{n{-}1}_{\,r-n+2}$ and $\boxed{n}_{\,r-n}$ of a given column of S by $\boxed{\bar{n}}_{\,r-n+2}$ and $\boxed{\overline{n{-}1}}_{\,r-n}$. Since $s = r_{n-1} + 2(k_{n-1}-1) - n + 1 + 4\xi_1$ is the support of the $(n - 2\xi_1)$-th box of the first column of S', we can multiply $\boldsymbol{\nu}'$ by $A^{-1}_{n, r_{n-1}+2(k_{n-1}-1)+4\xi_1-1}$, which corresponds to modifications in the first column of S'. For $r \neq r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 - 1$, to multiply by $A_{n,r}^{-1}$ corresponds to modifications in other columns of S', which cannot be done since they do not have the two specific boxes, and this completes the proof of the claim.

To finish the verification of (ii), in light of (4.4.22), it suffices to observe that $\boldsymbol{\varpi} A^{-1}_{r_n+2(k_n-1)+1}A_{n,r} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ iff $r = r_n + 2(k_n-1)+1$ by seeing the corresponding modification in T, similarly to what was done previously.

(iii) Observe from (4.4.15) and (4.4.13) that $\pi_{\{n\}}(\boldsymbol{\nu}) = Y_{n,r_n-2,k_n+1}$. Thus, by (3.2.7),

$$Y_{n,r_n-2,k_n+1}A^{-1}_{n,r_{n-1}+2(k_{n-1}-1)+4\xi_1-1} = Y_{n,r_n-2,k_n+1}A^{-1}_{n,r_n-1} \notin \mathrm{wt}_\ell(V_q(\pi_{\{n\}}(\boldsymbol{\nu}))).$$

Since $\boldsymbol{\mu} = \boldsymbol{\nu}\iota_{\{n\}}(A^{-1}_{n,r_{n-1}+2(k_{n-1}-1)+4\xi_1-1})$, condition (iii) follows.

(iv) If $\boldsymbol{\nu}' \in \mathrm{wt}_\ell(U_q(\tilde{\mathfrak{g}}_n)V_{\boldsymbol{\nu}})$ is n-dominant, then

$$\mathrm{ht}(\boldsymbol{\nu}'\boldsymbol{\omega}^{-1}) = \mathrm{ht}(\boldsymbol{\nu}'\boldsymbol{\nu}^{-1}\boldsymbol{\nu}\boldsymbol{\omega}^{-1}) \overset{(4.4.21)}{\leq} \mathrm{ht}(\boldsymbol{\nu}\boldsymbol{\omega}^{-1}).$$

(v) Let $j \in I \setminus \{n\}$ and $\boldsymbol{\eta} = \boldsymbol{\mu}\prod_{\zeta\in\Xi} A_\zeta$ with $\emptyset \neq \Xi \subseteq \{j\}\times\mathbb{Z}$. We will show that $\boldsymbol{\eta} \notin \mathrm{wt}_\ell(V)$ from where (v) follows. Indeed, we will show that $\boldsymbol{\eta} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})\otimes V_q(\boldsymbol{\varpi}))$.

By (4.4.17), if $j < n-2\xi_1$, then $\Xi = \emptyset$. Otherwise, inspecting (4.4.17), one can see that:

- $\Xi \subseteq \{(n-1, r_{n-1}+2(k_{n-1}-1)+1+4k) : k = 0,\dots,\xi_1-1\}$ if $j = n-1$;
- $\Xi \subseteq \{(j, r_{n-1}+2(k_{n-1}-1)+2(m+k)) : k = 0,\dots,j-(n-2\xi_1)\}$ if $n-j = 2m$, $m \in \mathbb{Z}_{\geq 1}$;
- $\Xi \subseteq \{(j, r_{n-1}+2(k_{n-1}-1)+2(m+k)-1) : k = 0,\dots,j-(n-2\xi_1+1)\}$ if $n-j = 2m-1$, $m \in \mathbb{Z}_{\geq 2}$.

Observe that, if $\Xi_{\boldsymbol{\mu}} \subseteq I\times\mathbb{Z}$ is such that $\boldsymbol{\mu} = \boldsymbol{\lambda}\prod_{\zeta\in\Xi_{\boldsymbol{\mu}}} A^{-1}_\zeta$, then $\boldsymbol{\eta} = \boldsymbol{\lambda}\prod_{\zeta\in\Xi_{\boldsymbol{\mu}}\setminus\Xi} A^{-1}_\zeta$. Thus, it suffices to show that there is no partition $\Xi_0 := \Xi_{\boldsymbol{\mu}} \setminus \Xi = \Xi' \cup \Xi''$ such that

$$\boldsymbol{\omega}\prod_{\zeta\in\Xi'} A^{-1}_\zeta \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \qquad \text{and} \qquad \boldsymbol{\varpi}\prod_{\zeta\in\Xi''} A^{-1}_\zeta \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})).$$

Assume by contradiction that there exists such partition. In part (i) we showed that the modifications corresponding to elements of $\Xi_{\boldsymbol{\mu}}$ cannot be performed in T (implying that $\Xi_{\boldsymbol{\mu}} \cap \Xi'' = \emptyset$) and can only be performed in the first column of S. Thus, since $\Xi_0 \subseteq \Xi_{\boldsymbol{\mu}}$, if we show that some modification associated to an element of Ξ_0 cannot be performed in the first column of S, we obtain the desired contradiction.

Suppose first that $j = n-1$. Let $k \in \{0,\dots,\xi_1-1\}$ be minimum such that

$$\zeta_0 := (n-1, r_{n-1}+2(k_{n-1}-1)+4k+1) \in \Xi,$$

which implies $\zeta_0 \notin \Xi_0$. Observe that, for $i \neq n-1$ with $(i,r) \in \Xi_{\boldsymbol{\mu}}$, we have $(i,r) \in \Xi_0$. Let $S'' \in \mathrm{STab}(S)$ be such that $\boldsymbol{\zeta} = \boldsymbol{\omega}^{S''}$. Let us show that the change corresponding to $\zeta_1 := (n-2, r_{n-1}+2(k_{n-1}-1)+4k+2) \in \Xi_0$ cannot be performed in S to obtain S''. Indeed, ζ_1 corresponds to modifying the boxes

(4.4.23) $$\boxed{n\!-\!2}_{\,s'} \quad \text{and} \quad \boxed{\overline{n\!-\!1}}_{\,s'-4} \qquad \text{by} \qquad \boxed{n\!-\!1}_{\,s'} \quad \text{and} \quad \boxed{\overline{n\!-\!2}}_{\,s'-4}$$

where $s' = r_{n-1}+2(k_{n-1}-1)+4k+5-n$. Since the modification $\boxed{n\!-\!1}_{\,s'-2}$ and $\boxed{\bar{n}}_{\,s'-4}$ by $\boxed{n}_{\,s'-2}$ and $\boxed{\overline{n\!-\!1}}_{\,s'-4}$, which corresponds to ζ_0, has not been performed, (4.4.23) cannot be performed as well, yielding a contradiction. The other cases are treated similarly by choosing:

- $\zeta_0 = (j, r_{n-1}+2(k_{n-1}-1)+2(m+k)) \in \Xi$ and $\zeta_1 = (j+1, r_{n-1}+2(k_{n-1}-1)+2(m+k)+1) \in \Xi_0$ if $n-j = 2m$, $m \in \mathbb{Z}_{\geq 1}$, $j < n-2$;
- $\zeta_0 = (n-2, r_{n-1}+2(k_{n-1}-1)+2(m+k)) \in \Xi$ and $\zeta_1 = (n, r_{n-1}+2(k_{n-1}-1)+2(m+k)+1) \in \Xi_0$ if $n-j = 2m$, $m \in \mathbb{Z}_{\geq 1}$, $j = n-2$;

- $\zeta_0 = (j, r_{n-1}+2(k_{n-1}-1)+2(m+k)-1) \in \Xi$ and $\zeta_1 = (j+1, r_{n-1}+2(k_{n-1}-1)+2(m+k)) \in \Xi_0$ if $n-j = 2m-1$, $m \in \mathbb{Z}_{\geq 2}$.

(c) By hypothesis

$$r_{n-1} + 2(k_{n-1}-2) + 4\xi_1 = r_n \qquad \text{and} \qquad r_{n-1} + 2(k_{n-1}-1) + 4\xi_1 = r_n + 2.$$

Thus, by (4.4.3) $\boldsymbol{\mu} \in D$. By (4.3.2),

$$\begin{aligned}\boldsymbol{\mu} \;&=\; \boldsymbol{\lambda} \prod_{j=1}^{2}\prod_{m=1}^{\xi_1} A^{-1}_{n-2m+1,n-1,r_{n-1}+2(k_{n-1}-j)+2m-2} A^{-1}_{n-2m,n-2,r_{n-1}+2(k_{n-1}-j)+2m-1} \times \\ &\qquad \times A^{-1}_{n,r_{n-1}+2(k_{n-1}-j)+4m-1}.\end{aligned} \tag{4.4.24}$$

Let $\boldsymbol{\lambda} = \boldsymbol{\omega}\boldsymbol{\varpi}$, $\boldsymbol{\mu} = \boldsymbol{\omega}^{S_\xi^{n-1}}\boldsymbol{\varpi}$, $\boldsymbol{\nu} = \boldsymbol{\mu} A_{n,r_{n-1}+2(k_{n-1}-2)+4\xi_1-1}$ and $V = V_q(\boldsymbol{\lambda})$. As in part part (b), it suffices to show that these data satisfy the conditions (i)–(v) of Lemma 2.6.9 with $i = n$. We omit the details since they are similar to those used for proving (b).

(d) By hypothesis

$$r_{n-1} + 2(k_{n-1} - |\xi|) + 4 = r_n \quad \text{and} \quad r_{n-1} + 2(k_{n-1}-j) + 4 \in X \setminus \{r_n\} \quad \text{for all} \quad j < |\xi|. \tag{4.4.25}$$

Thus, by (4.4.3), $\boldsymbol{\mu} \in D$ and, by (4.3.2),

$$\boldsymbol{\mu} = \boldsymbol{\omega}\boldsymbol{\varpi} \prod_{j=1}^{|\xi|} A^{-1}_{n-1,r_{n-1}+2(k_{n-1}-j)+1} A^{-1}_{n-2,r_{n-1}+2(k_{n-1}-j)+2} A^{-1}_{n,r_{n-1}+2(k_{n-1}-j)+3}. \tag{4.4.26}$$

Consider $\boldsymbol{\omega}' = Y_{n-1,r_{n-1},k_{n-1}-|\xi|+1} Y_{n,r_n,k_n}$ and $\boldsymbol{\omega}'' = Y_{n-1,r_{n-1}+2(k_{n-1}-|\xi|+1),|\xi|-1}$. Observe that $\boldsymbol{\omega}'\boldsymbol{\omega}'' = \boldsymbol{\omega}\boldsymbol{\varpi}$ and $V_q(\boldsymbol{\omega}'')$ is a minimal affinization. By (4.4.25), $V_q(\boldsymbol{\omega}')$ is also a minimal affinization. We shall show that

$$\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}') \otimes V_q(\boldsymbol{\omega}'')).$$

This implies part (d) since $V_q(\boldsymbol{\omega}\boldsymbol{\varpi})$ is the simple quotient of the submodule of $V_q(\boldsymbol{\omega}') \otimes V_q(\boldsymbol{\omega}'')$ generated by the top weight space. For doing that, let $\Xi \subseteq I \times \mathbb{Z}$ be such that $\boldsymbol{\mu} = \boldsymbol{\omega}\boldsymbol{\varpi} \prod_{\zeta \in \Xi} A_\zeta^{-1}$. We have to show that there is no partition $\Xi = \Xi' \cup \Xi''$ such that

$$\boldsymbol{\omega}' \prod_{\zeta \in \Xi'} A_\zeta^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}')) \qquad \text{and} \qquad \boldsymbol{\omega}'' \prod_{\zeta \in \Xi''} A_\zeta^{-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'')).$$

By contradiction, assume such a partition exists. Let

$$\Xi_1 = \Xi \setminus \{(n-1, r_{n-1}+2(k_{n-1}-|\xi|)+1), (n-2, r_{n-1}+2(k_{n-1}-|\xi|)+2), (n, r_{n-1}+2(k_{n-1}-|\xi|)+3)\}$$

and $J = \{n-1, n-2, n\}$. We identify the subdiagram corresponding to J with the diagram of type $A_3 = J$ via the labeling correspondence

$$1 \leftrightarrow n-1, \quad 2 \leftrightarrow n-2, \quad 3 \leftrightarrow n,$$

so that $V_q(\boldsymbol{\omega}'_J)$ is an increasing minimal affinization and $\boldsymbol{\omega}'_J = \boldsymbol{\omega}^{T'}$, where T' is the semi-standard tableau which the first k_n columns are of length 3 and the last $k_{n-1} - |\xi| + 1$ columns are of length 1, the contents of each column is equal to their position in the column and T' has the form (3.2.4). Recall from (3.1.6) that each element $A_\xi^{-1}, \xi \in \Xi$, corresponds to adding 1 to the content of a particular box of T'. More precisely, if $\xi = (i, r)$, then (3.1.6) implies that the modification associated to A_ξ^{-1} is of the form

$$\boxed{i}_{\,r-i} \longrightarrow \boxed{i+1}_{\,r-i}.$$

Since $s := r_{n-1}+2(k_{n-1}-|\xi|)$ is the support of the box in the first column of length 1 of T', $s+2$ is the support of the last box of the last column of length 3 of T', and $s_n = r_n + 2k_n - 4 = s + 2k_n$ is the support of the last box of the first column of T', it is not difficult to see that

$$\zeta \in \Xi_1 \Rightarrow \boldsymbol{\omega}'_J A_\zeta^{-1} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'_J)) \overset{(2.6.2)}{\Rightarrow} \zeta \notin \Xi'.$$

Also, looking at the corresponding modifications in T', one can see that

$$\boldsymbol{\eta} := \boldsymbol{\omega}'_J A^{-1}_{n-1,r_{n-1}+2(k_{n-1}-|\xi|)+1},\ \boldsymbol{\eta}_1 := \boldsymbol{\eta} A^{-1}_{n-2,r_{n-1}+2(k_{n-1}-|\xi|)+2} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'_J)),$$

and

$$\boldsymbol{\omega}'_J A_{\zeta_0}^{-1},\ \boldsymbol{\eta} A_{\zeta_0}^{-1},\ \boldsymbol{\eta}_1 A_{\zeta_0}^{-1} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'_J)) \overset{(2.6.2)}{\Rightarrow} \zeta_0 \notin \Xi',$$

where $\zeta_0 = (n, r_{n-1} + 2(k_{n-1} - |\xi|) + 3)$.

Since $\boldsymbol{\omega}'' = \boldsymbol{\omega}^{S'}$ where S' is the semi-standard tableau formed by the first $|\xi| - 1$ columns of S ($\boldsymbol{\omega}'' = 1$ if $|\xi| = 1$), it is not difficult to see by the corresponding modification in S' that $\zeta_0 \notin \Xi''$ yielding the desired contradiction. □

We have seen that if (4.4.4) is satisfied, then $D = \{\boldsymbol{\omega\varpi}\}$. Now we suppose that there exists a pair (l, m) with $1 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor$ and $m \in \{1, \dots, k_n\}$ such that

$$r_{n-1} + 2k_{n-1} + 4l = r_n + 2m. \tag{4.4.27}$$

The pair (l, m) is not unique in general. Henceforth we fix (l, m) satisfying (4.4.27) and having the smallest possible value of m.

Lemma 4.4.2. If $m > 2$, then $l = 1$.

Proof. Suppose, by contradiction, $m > 2$ and $l > 1$. Thus:

(i) if $2l \le m + 1$, then

$$r_{n-1} + 2(k_{n-1} - 1) + 4l = r_n + 2(m - 1) \Rightarrow r_{n-1} + 2(k_{n-1} - 1) + 4 = r_n + 2(m + 2 - 2l - 1)$$

and $(1, m + 2 - 2l)$ satisfies (4.4.27) with $m + 2 - 2l < m$;

(ii) if $2l \ge m + 2$ and m is odd, then

$$r_{n-1} + 2(k_{n-1} - 1) + 4l = r_n + 2(m - 1) \Rightarrow r_{n-1} + 2(k_{n-1} - 1) + 4\left(l - \frac{1}{2}(m - 1)\right) = r_n$$

and $\left(l - \frac{1}{2}(m - 1), 1\right)$ satisfies (4.4.27) with $1 < m$;

(iii) if $2l \ge m + 2$ and m is even, then

$$r_{n-1} + 2(k_{n-1} - 1) + 4l = r_n + 2(m - 1) \Rightarrow r_{n-1} + 2(k_{n-1} - 1) + 4\left(l - \frac{1}{2}m + 1\right) = r_n + 2$$

and $\left(l - \frac{1}{2}m + 1, 2\right)$ satisfies (4.4.27) with $2 < m$;

contradicting the minimality of m. □

Proposition 4.4.3. If $m \ge 2$ and $k_{n-1} < m$, then $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ is irreducible. Otherwise, let $\xi = (\xi_1, \dots, \xi_{k_{n-1}})$ be the partition defined by

$$\xi_j = 0 \quad \text{for} \quad j > m \quad \text{and} \quad \xi_j = \xi_1 = l \quad \text{for} \quad j \le m,$$

and set $\boldsymbol{\mu} = \boldsymbol{\omega}^{S_\xi^{n-1}} \boldsymbol{\varpi}$. Then, $V_q(\boldsymbol{\mu})$ is a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ and, if $V_q(\boldsymbol{\nu})$ is a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ with $\boldsymbol{\nu} \notin \{\boldsymbol{\omega\varpi}, \boldsymbol{\mu}\}$ then $\boldsymbol{\nu} < \boldsymbol{\mu}$.

PROOF. If $m \geq 2$ and $k_{n-1} < m$, we have

$$r_{n-1} + 2(k_{n-1} - 1) + 4l = r_n + 2 \tag{4.4.28}$$

if $m = 2$, and

$$r_{n-1} + 2k_{n-1} + 4 = r_n + 2m \tag{4.4.29}$$

if $m > 2$. We show that for all $\boldsymbol{\mu} \in D \setminus \{\boldsymbol{\omega\varpi}\}$, $V_q(\boldsymbol{\mu})$ is not a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$, from where the first statement follows. To prove this, write $\boldsymbol{\mu} = \boldsymbol{\omega}^{S_\xi^{n-1}} \boldsymbol{\varpi}$ for some nonzero partition ξ. By (4.4.3),

$$r_{n-1} + 2(k_{n-1} - j) + 4\xi_j \in X = \{r_n + 2(t-1) : t = 1, \dots, k_n\} \quad \text{for all} \quad j \leq |\xi|.$$

If $m = 2$, then $k_{n-1} = 1$ implying $\xi = (\xi_1)$. Thus $r_{n-1} + 2(k_{n-1} - 1) + 4\xi_1 > r_n$ by the minimality of $m = 2$ in (4.4.28). By Lemma 4.4.1(a) the result follows. Suppose now $m > 2$. Observe that, for $j \leq |\xi|$,

$$r_{n-1} + 2(k_{n-1} - j) + 4\xi_j \geq r_{n-1} + 2(k_{n-1} - j) + 4 \overset{(4.4.29)}{=} r_n + 2(m - j) > r_n,$$

where the last inequality follows since $|\xi| \leq k_{n-1} < m$. An application of Lemma 4.4.1(a) again completes the proof of this case.

For the second statement we have $m \leq k_{n-1}$,

$$r_n = r_{n-1} + 2(k_{n-1} - m) + 4l \tag{4.4.30}$$

One easily verifies that, if $m = 1$, the hypothesis of Lemma 4.4.1(b) is satisfied, if $m = 2$, the hypothesis of Lemma 4.4.1(c) is satisfied while, for $m > 2$, Lemma 4.4.2 implies that the hypothesis of Lemma 4.4.1(d) is satisfied. In any case, it follows that $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega\varpi}))$. Hence, in order to prove that $V_q(\boldsymbol{\mu})$ is a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$, it suffices to show that

$$\text{if} \;\; \boldsymbol{\mu} < \boldsymbol{\zeta} < \boldsymbol{\omega\varpi}, \;\; \text{then} \;\; V_q(\boldsymbol{\zeta}) \;\; \text{is not a simple factor of} \;\; V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi}). \tag{4.4.31}$$

To prove this, recall that, if $V_q(\boldsymbol{\zeta})$ were a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$, then $\boldsymbol{\zeta} \in D$ is of the form $\boldsymbol{\zeta} = \boldsymbol{\omega}^{S_{\xi'}^{n-1}} \boldsymbol{\varpi}$ by (4.4.3). Moreover, the hypothesis $\boldsymbol{\zeta} > \boldsymbol{\mu}$ together with (4.3.2) implies that

$$|\xi'| \leq |\xi|, \quad \xi'_j \leq l \;\; \text{for all} \;\; j, \quad \text{and} \quad \sum_j \xi'_j < \sum_j \xi_j. \tag{4.4.32}$$

Assuming (4.4.32), we complete the proof of (4.4.31) as follows. If $m = 1$, (4.4.30) implies that $r_{n-1} + 2(k_{n-1} - 1) + 4\xi'_1 < r_n$ and, hence, $r_{n-1} + 2(k_{n-1} - 1) + 4\xi'_1 \notin X$, contradicting the characterization of D given by (4.4.3). Assume next that $m = 2$. If either $|\xi'| = 2$ or $\xi'_1 < l$, the argument is similar. Otherwise, i.e., if $|\xi'| = 1$ and $\xi'_1 = l$, an application of (4.4.30) together with Lemma 4.4.1(a) completes the proof of (4.4.31). Finally, if $m > 2$, then $l = 1, |\xi'| < |\xi| = m$, and (4.4.30) implies that $r_{n-1} + 2(k_{n-1} - j) + 4 = r_n + 2(m - j)$ for all $j \leq |\xi'|$. Since $1 < m - j \leq m - 1$, Lemma 4.4.1(a) implies that $V_q(\boldsymbol{\zeta})$ is not a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ completing the proof of (4.4.31).

It remains to show that, if $\boldsymbol{\nu} = \boldsymbol{\omega}^{S_{\xi''}^{n-1}} \boldsymbol{\varpi} \in D$ is such that $V_q(\boldsymbol{\nu})$ is a simple factor of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ with $\boldsymbol{\nu} \notin \{\boldsymbol{\omega\varpi}, \boldsymbol{\mu}\}$ then $\boldsymbol{\nu} < \boldsymbol{\mu}$. By (4.4.3), $r_{n-1} + 2(k_{n-1} - j) + 4\xi''_j \in X$ for all $j \leq |\xi''|$. Lemma 4.4.1(a) implies that

$$r_n = r_{n-1} + 2(k_{n-1} - j_0) + 4\xi''_{j_0} \tag{4.4.33}$$

for some $j_0 \leq |\xi''|$. Moreover, it follows from (4.3.4) that $j_0 = |\xi''|$. As usual, assume first that $m = 1$. If $j_0 = 1$, (4.4.30) and (4.4.33) imply that $\xi''_{j_0} = l$, i.e., $\xi'' = \xi$, contradicting $\boldsymbol{\nu} \neq \boldsymbol{\mu}$. Thus, we must have $j_0 > 1$. Combining (4.4.30) and (4.4.33) once more we see that $\xi''_{j_0} > l$. It then follows from (4.3.2) that $\boldsymbol{\nu} < \boldsymbol{\mu}$. Assume next that $m \geq 2$. The minimality of m implies that $j_0 \geq m$. If $j_0 = m$, it follows from (4.4.30) and (4.4.33) that $\xi''_{j_0} = l$. Since $\boldsymbol{\nu} \neq \boldsymbol{\mu}$, this forces $\xi''_1 > l$ and (4.3.2) implies that $\boldsymbol{\nu} < \boldsymbol{\mu}$. If $j_0 > m$, we see as before that $\xi''_{j_0} > l$ and (4.3.2) completes the proof as usual. □

LEMMA 4.4.4. Assume either one of the following conditions hold:

(i) $r_n = 2k_{n-1} + 2n - 4 + r_{n-1} + 2k$ for some $k \in \mathbb{Z}_{>0}$;
(ii) $r_{n-1} = r_n$.

Then, $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ is irreducible.

PROOF. In case (i), we show that (4.4.27) is not satisfied, from where the conclusion follows since $D = \{\boldsymbol{\omega\varpi}\}$. Suppose, by contradiction, that there exist $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$ and $m \in \{1, \ldots, k_n\}$ satisfying (4.4.27) and, as usual, that m is minimal with this property. If $m \in \{1, 2\}$, (4.4.27) together with (i) implies that

$$l = \frac{1}{2}(n - 2 + m + k) > \left\lfloor \frac{n-1}{2} \right\rfloor,$$

yielding a contradiction. If $m > 2$, then $l = 1$ by Lemma 4.4.2, and (4.4.27) together with (i) implies that

$$m = 4 - n - k < 0,$$

which is a contradiction once more.

In case (ii), if (4.4.4) is satisfied there is nothing to do. Thus, assume (4.4.27) is satisfied. We claim that we must have $m > 2$ and $k_{n-1} < m$, from where the result follows from Proposition 4.4.3. Indeed, if $m \in \{1, 2\}$, we get a contradiction since (4.4.27) implies that

$$l = \frac{1}{2}(m - k_{n-1}) < 1.$$

On the other hand, if $m > 2$, then $l = 1$ by Lemma 4.4.2, and (4.4.27) together with (ii) implies that

$$m = k_{n-1} + 2 > k_{n-1},$$

which proves the claim. □

REMARK 4.4.5. Let $\boldsymbol{\lambda} = \boldsymbol{\omega\varpi}$. Define also $a_i = q^{r_i + k_i - 1}$, $i = n-1, n$, so that $\boldsymbol{\lambda} = \boldsymbol{\omega}_{n-1, a_{n-1}, k_{n-1}} \boldsymbol{\omega}_{n, a_n, k_n}$. One easily checks that condition (4.4.27) is equivalent to

$$\frac{a_{n-1}}{a_n} = q^{-(k_{n-1} + k_n + 4s_1 - 2s_2)}$$

with $2 \leq 2s_1 \leq n-1$ and $1 \leq s_2 \leq \min\{\lambda(h_{n-1}), \lambda(h_n)\}$, by putting $l = s_1$, $m = s_2$.

4.5. Ordering affinizations supported at the spin nodes

Our next goal is to order certain affinizations of $V_q(\lambda)$ with $\mathrm{supp}(\lambda) = \{n-1, n\}$. We keep the notation fixed in Section 4.4: $\boldsymbol{\omega} = Y_{n-1, r_{n-1}, k_{n-1}} = \boldsymbol{\omega}^S$, $\boldsymbol{\varpi} = Y_{n, r_n, k_n} = \boldsymbol{\omega}^T$. We also fix $\bar{r}_n \in \mathbb{Z}$ and let $\overline{T}$ be the tableaux such that $\boldsymbol{\omega}^{\overline{T}} = Y_{n, \bar{r}_n, k_n}$. Set

$$\boldsymbol{\lambda} = \boldsymbol{\omega\varpi} \qquad \text{and} \qquad \overline{\boldsymbol{\lambda}} = \boldsymbol{\omega}\overline{\boldsymbol{\varpi}}.$$

We assume that $\boldsymbol{\lambda}$ and $\overline{\boldsymbol{\lambda}}$ satisfy (4.4.27), i.e., there exist $1 \leq l \leq \lfloor \frac{n-1}{2} \rfloor$ and $1 \leq m \leq k_n$ satisfying (4.4.27) as well as $1 \leq \bar{l} \leq \lfloor \frac{n-1}{2} \rfloor$ and $1 \leq \overline{m} \leq k_n$ satisfying (4.4.27) with $\bar{l}$ and $\overline{m}$ in place of l and m, respectively. As in Section 4.4, m and $\overline{m}$ are assumed to be minimal.

LEMMA 4.5.1. Suppose one of the following conditions hold:

(i) $m = \overline{m} = 1$ and $l < \bar{l}$;
(ii) $m = \overline{m} = 2 \le k_{n-1}$ and $l < \bar{l}$;
(iii) $m < \overline{m} \le k_{n-1}$ and $l = \bar{l}$.

Then, $V_q(\boldsymbol{\lambda}) < V_q(\overline{\boldsymbol{\lambda}})$.

PROOF. Set $V = V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})$ and $\overline{V} = V_q(\boldsymbol{\omega}) \otimes V_q(\overline{\boldsymbol{\varpi}})$. Then $V_q(\boldsymbol{\lambda})$ is the simple quotient of the submodule of V generated by the top weight space and similarly for $\overline{V}$. Notice that we have isomorphisms of $U_q(\mathfrak{g})$-modules:

$$V \cong \overline{V} \cong V_q(k_{n-1}\omega_{n-1} + k_n\omega_n) \oplus \bigoplus_\nu V_q(\nu)^{\oplus t_\nu} \quad \text{and} \quad V_q(\boldsymbol{\lambda}) \cong V_q(k_{n-1}\omega_{n-1} + k_n\omega_n) \oplus \bigoplus_\nu V_q(\nu)^{\oplus m_\nu}$$

where the sums are over $\nu \in P^+$ such that $\nu < k_{n-1}\omega_{n-1} + k_n\omega_n$ and $m_\nu, t_\nu \in \mathbb{Z}_{\ge 0}$. Letting $\boldsymbol{\mu}$ be as in Proposition 4.4.3, it immediately follows that

$$\nu \not\le \mathrm{wt}(\boldsymbol{\mu}) \Rightarrow t_\nu = m_\nu \quad \text{and} \quad m_{\mathrm{wt}(\boldsymbol{\mu})} \le t_{\mathrm{wt}(\boldsymbol{\mu})} - 1.$$

Writing

$$V_q(\overline{\boldsymbol{\lambda}}) \cong V_q(k_{n-1}\omega_{n-1} + k_n\omega_n) \oplus \bigoplus_\nu V_q(\nu)^{\oplus \overline{m}_\nu}$$

and defining $\overline{\boldsymbol{\mu}}$ similarly (with $\overline{\boldsymbol{\varpi}}$ in place of $\boldsymbol{\varpi}$), we conclude that

$$\nu \not\le \mathrm{wt}(\overline{\boldsymbol{\mu}}) \Rightarrow t_\nu = m_\nu \quad \text{and} \quad m_{\mathrm{wt}(\overline{\boldsymbol{\mu}})} \le t_{\mathrm{wt}(\overline{\boldsymbol{\mu}})} - 1.$$

We claim the above three conditions imply that

$$\mathrm{wt}(\overline{\boldsymbol{\mu}}) < \mathrm{wt}(\boldsymbol{\mu}). \tag{4.5.1}$$

Assuming this, we complete the proof as follows. Let $\nu \in P^+$ be such that $\nu < k_{n-1}\omega_{n-1} + k_n\omega_n$. If $\nu \not\le \mathrm{wt}(\boldsymbol{\mu})$, then we also have $\nu \not\le \mathrm{wt}(\overline{\boldsymbol{\mu}})$ and, hence, $\overline{m}_\nu = t_\nu = m_\nu$. Otherwise, if $\nu \le \mathrm{wt}(\boldsymbol{\mu})$, we have $m_{\mathrm{wt}(\boldsymbol{\mu})} \le t_{\mathrm{wt}(\boldsymbol{\mu})} - 1 = \overline{m}_{\mathrm{wt}(\boldsymbol{\mu})} - 1$.

In order to prove (4.5.1), assume first that condition (i) holds. It follows from (4.4.17) that

$$\mathrm{wt}(\boldsymbol{\mu}) = \lambda - \left(\sum_{k=1}^{2l-1} k\alpha_{n-2l-1+k} + l\alpha_{n-1} + l\alpha_n \right) \tag{4.5.2}$$

and similarly for $\mathrm{wt}(\overline{\boldsymbol{\mu}})$ with $\bar{l}$ in place of l. Since $l < \bar{l}$, (4.5.1) follows. If condition (ii) holds, it follows from (4.4.24) that

$$\mathrm{wt}(\overline{\boldsymbol{\mu}}) = \lambda - 2\left(\sum_{k=1}^{2\bar{l}-1} k\alpha_{n-2\bar{l}-1+k} + \bar{l}\alpha_{n-1} + \bar{l}\alpha_n \right) \tag{4.5.3}$$

and similarly for $\mathrm{wt}(\boldsymbol{\mu})$ with l in place of $\bar{l}$. Equation (4.5.1) follows as before. Finally, suppose (iii) holds. If $\overline{m} > 2$, then $l = \bar{l} = 1$ by Lemma 4.4.2 and it follows from (4.4.26) that

$$\mathrm{wt}(\boldsymbol{\mu}) = \lambda - \sum_{k=1}^{m} (\alpha_{n-1} + \alpha_{n-2} + \alpha_n) \qquad \text{and} \qquad \mathrm{wt}(\overline{\boldsymbol{\mu}}) = \lambda - \sum_{k=1}^{\overline{m}} (\alpha_{n-1} + \alpha_{n-2} + \alpha_n).$$

Since $m < \overline{m}$, (4.5.1) follows. Otherwise, we must have $m = 1$ and $\overline{m} = 2$ which imply that $\mathrm{wt}(\boldsymbol{\mu})$ is given by (4.5.2) and $\mathrm{wt}(\overline{\boldsymbol{\mu}})$ is given by (4.5.3). Since $l = \bar{l}$, (4.5.1) is proved. $\square$

4.6. Comparing certain affinizations supported at the extreme nodes

In this section we fix $\lambda = k_1\omega_1 + k_{n-1}\omega_{n-1} + k_n\omega_n \in P^+$, $k_1, k_{n-1}, k_n > 0$, and study the ordering among certain affinizations of $V_q(\lambda)$. Observe that

$$\lambda' := \lambda - \alpha_{n-2} - \alpha_{n-1} - \alpha_n = k_1\omega_1 + (k_{n-1}-1)\omega_{n-1} + (k_n - 1) + \omega_{n-3}\omega_n$$

is also dominant.

Lemma 4.6.1. If $\mu = \lambda - \sum_{i\in I}\alpha_i$, then $\dim V_q(\lambda)_\mu = 3(n-2)+1$ and $\dim V_q(\lambda')_\mu = n-3$.

Proof. This can be proved in a variety of ways (see for instance [**37**] for a proof using standard Lie theoretic techniques). Since we are using the theory of qcharacters in many of our proofs, we will sketch a proof for this lemma using qcharacters as well. More precisely, we use a monomial realization of Kashiwara's crystal $\mathcal{B}(\lambda)$ and $\mathcal{B}(\lambda')$ studied in [**32, 44**] (see also [**24**, Section 2]).

Let $\boldsymbol{\omega} = Y_{1,0}^{k_1} Y_{n-1,n}^{k_{n-1}} Y_{n,n+4}^{k_n}$. According to [**32, 44**], this ℓ-weight gives rise to a monomial realization $\mathcal{M}(\boldsymbol{\omega})$ of $\mathcal{B}(\lambda)$. The lemma can be proved by applying the algorithm that generates the monomials in this realization and then counting the number of ℓ-weights in $\mathcal{M}(\boldsymbol{\omega})$ which have classical weight μ. We simply give the list of such ℓ-weights without writing the proper justification showing that this is the correct list:

$$\begin{aligned}
\boldsymbol{\mu}_i^1 &= \boldsymbol{\omega} A_{n,n+5}^{-1} A_{n-2,n+6}^{-1} A_{n-1,n+7}^{-1} \overrightarrow{A}_{1,i,0}^{-1} \overleftarrow{A}_{n-3,i+1,n+6}^{-1},\\
\boldsymbol{\mu}_i^2 &= \boldsymbol{\omega} A_{n-1,n+1}^{-1} A_{n,n+5}^{-1} A_{n-2,n+6}^{-1} \overrightarrow{A}_{1,i,0}^{-1} \overleftarrow{A}_{n-3,i+1,n+6}^{-1},\\
\boldsymbol{\mu}_i^3 &= \boldsymbol{\omega} A_{n-1,n+1}^{-1} A_{n-2,n+2}^{-1} A_{n,n+5}^{-1} \overrightarrow{A}_{1,i,0}^{-1} \overleftarrow{A}_{n-3,i+1,n+2}^{-1},\\
\boldsymbol{\mu}^0 &= \boldsymbol{\omega} A_{1,n-2,0}^{-1} A_{n-1,n+1}^{-1} A_{n,n+5}^{-1},
\end{aligned}$$

for $i = 0, 1, \dots, n-3$, where $\overrightarrow{A}_{1,i,0}^{-1} = A_{1,i,0}^{-1}$ if $i \geq 1$ and $\overrightarrow{A}_{1,0,0}^{-1} = 1$, and, $\overleftarrow{A}_{n-3,i+1,n+6}^{-1} = A_{n-3,i+1,n+6}^{-1}$ if $i \leq n-4$ and $\overleftarrow{A}_{n-3,n-2,n+6}^{-1} = 1$. One then checks that these ℓ-weights are all distinct and, evidently, add up to $3(n-2)+1$.

For λ' one can proceed similarly, by considering $\boldsymbol{\omega}' = Y_{1,n}^{k_1} Y_{n-3,2} Y_{n-1,0}^{k_{n-1}-1} Y_{n,0}^{k_n-1}$. The list of relevant monomials in $\mathcal{M}(\boldsymbol{\omega}')$ is:

$$\boldsymbol{\mu}_i = \boldsymbol{\omega}' A_{1,i,n}^{-1} \overleftarrow{A}_{n-3,i+1,2}^{-1}, \qquad i = 1, \dots, n-3, \tag{4.6.1}$$

where $\overleftarrow{A}_{n-3,i+1,2}^{-1} = A_{n-3,i+1,2}^{-1}$ if $i \leq n-4$ and $\overleftarrow{A}_{n-3,n-2,2}^{-1} = 1$. $\square$

Lemma 4.6.2. Let $\boldsymbol{\omega} = Y_{1,r_1,k_1}$ and $\boldsymbol{\varpi} = Y_{n-1,r_{n-1},k_{n-1}}$ for some $k_1, k_{n-1} \in \mathbb{Z}_{>0}$ satisfying

$$r_1 = r_{n-1} + 2k_{n-1} + n - 2, \tag{4.6.2}$$

and set $\boldsymbol{\lambda} = \boldsymbol{\omega}\boldsymbol{\varpi}$. Then:

(a) $V_q(\boldsymbol{\lambda})$ is ℓ-minuscule.
(b) If $\boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}))$ is not right negative, then

$$\boldsymbol{\mu} \in \boldsymbol{\lambda}\mathcal{Q}^-_{I\setminus\{1\}} \qquad \text{and} \qquad r(\boldsymbol{\mu}) = r_1 + 2(k_1 - 1).$$

Proof. Set

$$\boldsymbol{\mu}_0 := \boldsymbol{\lambda} A_{n-1,1,r_{n-1}+2(k_{n-1}-1)}^{-1}.$$

We claim that

$$\mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})\right) \cap \mathcal{P}^+ = \{\boldsymbol{\lambda}, \boldsymbol{\mu}_0\}. \tag{4.6.3}$$

Assuming this, in order to prove part (a) of the lemma, it suffices to show that $\boldsymbol{\mu}_0 \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}))$. But this follows from Lemma 2.6.9 with $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, $i = 1$ and $\boldsymbol{\nu} = \boldsymbol{\mu}_0 A_{1,r_{n-1}+2(k_{n-1}-1)+n-1} = \boldsymbol{\lambda} A^{-1}_{n-1,2,r_{n-1}+2(k_{n-1}-1)}$. For condition (i) of Lemma 2.6.9 use (2.6.2) and the description of the ℓ-weights of $\mathrm{wt}_\ell(V_q(\boldsymbol{\lambda}_{\{1,\dots,n-1\}}))$ in terms of semi-standard tableaux (Proposition 3.2.1). For the other conditions of Lemma 2.6.9 the verification is similar to that in the proof of Lemma 4.4.1(b). We omit the details.

For proving (4.6.3), we start by checking that

$$\boldsymbol{\mu} \in \mathrm{wt}_\ell\left(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})\right) \cap \mathcal{P}^+ \Rightarrow \boldsymbol{\mu} = \boldsymbol{\omega}\boldsymbol{\nu} \tag{4.6.4}$$

for some $\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$. Indeed, let $\boldsymbol{\eta}\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\varpi})) \cap \mathcal{P}^+$ and suppose $\boldsymbol{\eta} \neq \boldsymbol{\omega}$. Then, by Proposition 2.5.5, $\boldsymbol{\eta}$ is right negative and

$$r(\boldsymbol{\eta}) > r(\boldsymbol{\omega}) = r_1 + 2(k_1 - 1).$$

Since the product of right negative elements is again right negative, $\boldsymbol{\nu}$ is not right negative. Proposition 2.5.5 then implies that $\boldsymbol{\nu} = \boldsymbol{\varpi}$. It follows, using (4.6.6), that

$$r(\boldsymbol{\nu}) = r_{n-1} + 2(k_{n-1} - 1) < r_1 < r(\boldsymbol{\eta})$$

which, since $\boldsymbol{\eta}$ is right negative, implies that $\boldsymbol{\eta}\boldsymbol{\nu}$ is right negative, contradicting it being dominant.

After (4.6.4), in order to complete the proof of (4.6.3), it remains to characterize the set

$$\{\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})) : \boldsymbol{\nu} \text{ is } (I \setminus \{1\})\text{-dominant and } \boldsymbol{\omega}\boldsymbol{\nu} \in \mathcal{P}^+\}.$$

As observed after Proposition 4.3.2, the $(I \setminus \{1\})$-dominant ℓ-weights of $V_q(\boldsymbol{\varpi})$, other than $\boldsymbol{\varpi}$, are

$$\boldsymbol{\nu}_i = \boldsymbol{\varpi} \prod_{i=1}^{k_{n-1}} A^{-1}_{n-1,1,r_{n-1}+2(k_{n-1}-i)}, \quad i = 1, \dots, k_{n-1}.$$

One easily checks that, for each $i = 1, \dots, k_{n-1}$,

$$\boldsymbol{\nu}_i^{\{1\}} = \prod_{i=1}^{k_{n-1}} Y^{-1}_{1,r_{n-1}+2(k_{n-1}-i)+n}.$$

A straight forward computation using (4.6.2) shows that $\boldsymbol{\omega}\boldsymbol{\nu}_i \in \mathcal{P}^+$ iff $i = 1$. This completes the proof of (4.6.3).

We now turn to part (b). By Proposition 2.5.5, all the ℓ-weights in $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \setminus \{\boldsymbol{\omega}\}$ and $\mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})) \setminus \{\boldsymbol{\varpi}\}$ are right negative. Since

$$\boldsymbol{\mu} = \boldsymbol{\eta}\boldsymbol{\nu}, \quad \text{with} \quad \boldsymbol{\eta} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})), \quad \boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi})),$$

and the product of right negative ℓ-weights is right negative, it follows that either $\boldsymbol{\eta} = \boldsymbol{\omega}$ or $\boldsymbol{\nu} = \boldsymbol{\varpi}$.

If both $\boldsymbol{\eta} = \boldsymbol{\omega}$ and $\boldsymbol{\nu} = \boldsymbol{\varpi}$, then the lemma follows.

Suppose $\boldsymbol{\eta} \neq \boldsymbol{\omega}$, which implies $\boldsymbol{\nu} = \boldsymbol{\varpi}$. Then, by Proposition 2.5.5,

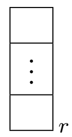
$$r(\boldsymbol{\eta}) > r_1 + 2(k_1 - 1) \overset{(4.6.2)}{=} r_{n-1} + 2k_{n-1} + n - 2 + 2(k_1 - 1) > r_{n-1} + 2(k_{n-1} - 1) = r(\boldsymbol{\varpi}).$$

Thus, since $\boldsymbol{\eta}$ is right negative, it follows that $\boldsymbol{\mu} = \boldsymbol{\eta}\boldsymbol{\varpi}$ is right negative, contradicting the hypothesis.

Finally, suppose $\boldsymbol{\eta} = \boldsymbol{\omega}$ and $\boldsymbol{\nu} \neq \boldsymbol{\varpi}$. Let T be the tableaux as in (4.2.2) such that $\boldsymbol{\omega}^T = \boldsymbol{\varpi}$. If $\boldsymbol{\nu} \in \boldsymbol{\omega}\boldsymbol{\varpi} A^{-1}_{1,s}\mathcal{Q}^-$ for some $s \in \mathbb{Z}$, then $\boldsymbol{\nu} \in \boldsymbol{\varpi} A^{-1}_{1,s}\mathcal{Q}^-$. We claim that $\boldsymbol{\nu} \in \boldsymbol{\varpi} A^{-1}_{1,r_1-1}\mathcal{Q}^-$ and the multiplication by A^{-1}_{1,r_1-1} corresponds to a change in the first column of a semi-standard with shape of T. Assuming the claim, it follows that $\boldsymbol{\nu} = \boldsymbol{\nu}' A^{-1}_{1,r_1-1} \prod_{p=1}^{t} A^{-1}_{i_p,s_p}$ for some $(i_p, r_p) \in I \times \mathbb{Z}$, with $\boldsymbol{\nu}' \in \boldsymbol{\varpi}\mathcal{Q}^-_{I\setminus\{1\}} \cap \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$, $\boldsymbol{\nu}' A^{-1}_{1,r_1-1} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$ and if $i_p = 1$ then $s_p < r_1 - 1$. Let $T' \in \mathrm{STab}(T)$ such that $\boldsymbol{\nu}' = \boldsymbol{\omega}^{T'}$. Thus the first column of T' has the boxes $\boxed{1}_{r_1-1}$ and $\boxed{\bar{2}}_{r'}$ for some $r' < r_1 - 1$. Since the first column of T' does not have a box with the content 2, it follows from Lemma 4.2.1 that Y_{1,r_1-2} appears in $\boldsymbol{\omega}^{T'}$. Observe that $\pi_1(\boldsymbol{\omega}\boldsymbol{\omega}^{T'}) = Y_{1,r_1-2,k_1+1}$, then it is not difficult to see that $\boldsymbol{\omega}\boldsymbol{\nu}' A^{-1}_{1,r_1-1} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}\boldsymbol{\varpi}))$ (using for instance the FM algorithm or Lemma 2.6.9), which implies $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}\boldsymbol{\varpi}))$, contradiction. Therefore

$$\boldsymbol{\mu} \in \boldsymbol{\omega}\boldsymbol{\varpi}\mathcal{Q}^-_{I\setminus\{1\}}. \tag{4.6.5}$$

Let us prove the claim. The only possible modification in a column of a semi-standard tableau with shape of T:

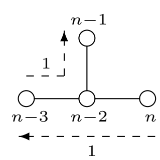


corresponding to multiply by $A^{-1}_{1,s}$ is to change the boxes $\boxed{1}_{r+2(n-1)}$ and $\boxed{\bar{2}}_{r}$ (if this column has such boxes), which implies $s = r + 2(n-1)$ (see (4.1.5)). But, if such modification is done is some column, it follows that all the columns to the left of this one have the corresponding modification (gap of size 1 in the first box), in particular the first column. Since the last box of the first column of T is supported at $r_{n-1} + 2(k_{n-1} - 1) - n + 1$ and the first box of the first column of T is supported at $r_{n-1} + 2(k_{n-1} - 1) + n - 1 = r_1 - 1$, the claim follows.

It remains to show that $r(\boldsymbol{\mu}) = r_1 + 2(k_1 - 1)$. Indeed, if $r(\boldsymbol{\mu}) > r_1 + 2(k_1 - 1)$, then $r(\boldsymbol{\mu}) = r(\boldsymbol{\nu})$, since $\boldsymbol{\eta} = \boldsymbol{\omega}$ and $r(\boldsymbol{\omega}) = r_1 + 2(k_1 - 1)$. But, $\boldsymbol{\nu}$ is right negative, hence it follows that $\boldsymbol{\mu}$ is right negative, contradiction. Therefore $r(\boldsymbol{\mu}) \le r_1 + 2(k_1 - 1)$, which implies, together with (4.6.5), that $r(\boldsymbol{\mu}) = r_1 + 2(k_1 - 1)$. $\square$

Lemma 4.6.3. Let $\boldsymbol{\omega} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$ and suppose it satisfies either of the conditions $(\mathrm{b})_l$ in Theorem 1.7.3 for some l. Then, $V_q(\boldsymbol{\omega})$ is ℓ-minuscule.

Proof. Write $\boldsymbol{\omega} = Y_{1,r_1,k_1} Y_{n-1,r_{n-1},k_{n-1}} Y_{n,r_n,k_n}$ and assume, without loss of generality, that $\boldsymbol{\omega}$ is determined by the picture



In other words, $\boldsymbol{\omega}$ satisfies the first option of conditions given by $(\mathrm{b})_{n-1}$ in Theorem 1.7.3 (recall the comments preceding and following (1.7.4)). More precisely, we have

$$r_1 = r_{n-1} + 2k_{n-1} + n - 2 \qquad \text{and} \qquad r_n = r_1 + 2k_1 + n - 2. \tag{4.6.6}$$

Then, part (a) of Lemma 4.6.2 implies that

$$V_q(\boldsymbol{\omega}^J) \text{ is } \ell\text{-minuscule for } J = [1, n-1]. \tag{4.6.7}$$

We will show that

$$D := \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^J) \otimes V_q(\boldsymbol{\omega}^{\{n\}})) \cap \mathcal{P}^+ = \{\boldsymbol{\omega}, \boldsymbol{\varpi}\} \tag{4.6.8}$$

where

$$\boldsymbol{\varpi} = \boldsymbol{\omega} A^{-1}_{1,n-2,r_1+2(k_1-1)} A^{-1}_{n,r_1+2(k_1-1)+n-1}.$$

Hence, since $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))\cap\mathcal{P}^+\subseteq D$, in order to complete the proof of the lemma, it remains to check that $\boldsymbol{\varpi}\notin\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. This can be easily done using Lemma 2.6.9 with $\boldsymbol{\lambda}=\boldsymbol{\omega}$, $\boldsymbol{\mu}=\boldsymbol{\varpi}$, $i=n$ and $\boldsymbol{\nu}=\boldsymbol{\omega}A^{-1}_{1,n-2,r_1+2(k_1-1)}=\boldsymbol{\varpi}A_{n,r_1+2(k_1-1)+n-1}$.

It remains to prove (4.6.8). We start by showing that

$$\boldsymbol{\varpi}\in D\Rightarrow\boldsymbol{\varpi}=\boldsymbol{\mu}\boldsymbol{\omega}^{\{n\}}\quad\text{with}\quad\boldsymbol{\mu}\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^J)).$$

Indeed, let $\boldsymbol{\mu}\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^J))$ and $\boldsymbol{\nu}\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{\{n\}}))$ be such that $\boldsymbol{\mu}\boldsymbol{\nu}\in D$, and suppose, by contradiction, that $\boldsymbol{\nu}\neq\boldsymbol{\omega}^{\{n\}}$. Then, by Proposition 2.5.5, $\boldsymbol{\nu}$ is right negative and

$$r(\boldsymbol{\nu})>r(\boldsymbol{\omega}^{\{n\}})=r_n+2(k_n-1).$$

Since the product of right negative elements is again right negative, $\boldsymbol{\mu}$ is not right negative. By Lemma 4.6.2

$$\boldsymbol{\mu}\in\boldsymbol{\omega}^J\mathcal{Q}^-_{I\setminus\{1\}}\qquad\text{and}\qquad r(\boldsymbol{\mu})=r_1+2(k_1-1).$$

But then, using (4.6.6), we get

$$r(\boldsymbol{\mu})=r_1+2(k_1-1)<r_n\leq r_n+2(k_n-1)<r(\boldsymbol{\nu})$$

which implies that $\boldsymbol{\mu}\boldsymbol{\nu}$ is right negative, yielding the desired contradiction.

The previous paragraph implies that, in order to describe D, we are left to finding the J-dominant elements $\boldsymbol{\mu}\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^J))$ such that $\boldsymbol{\mu}\boldsymbol{\omega}^{\{n\}}\in\mathcal{P}^+$. By (4.6.7) and Theorem 2.4.4, this can be done by applying the FM algorithm to $\boldsymbol{\omega}^J$, as follows (we use notation of [**36**, Section 4]). We assume $\boldsymbol{\mu}\neq\boldsymbol{\omega}^J$, since otherwise there is noting to do. The FM algorithm implies that $\boldsymbol{\mu}$ can be obtained from $\boldsymbol{\omega}^J$ by a sequence of i-th expansions. Let's say

$$\boldsymbol{\mu}\in\mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\nu}})\setminus\{\boldsymbol{\nu}\}$$

for some $i\in I$ with $\boldsymbol{\nu}$ either equal to $\boldsymbol{\omega}^J$ or obtained by previous expansions. In particular, $\boldsymbol{\mu}$ is not i-dominant, which implies, since we are assuming that $\boldsymbol{\mu}$ is J-dominant, that $i=n$. This in turn implies that $\boldsymbol{\nu}$ is n-dominant and $\boldsymbol{\nu}\neq\boldsymbol{\omega}^J$ since $\zeta^n_{\boldsymbol{\omega}^J}=\boldsymbol{\omega}^J$. Also, since the $(I\setminus\{n-2,n\})$-part of all elements in $\mathrm{wt}_\ell(\zeta^i_{\boldsymbol{\nu}})$ coincide, it follows that $\boldsymbol{\nu}$ must be $(I\setminus\{n-2\})$-dominant.

Set $J'=I\setminus\{n-2\}$ and let's study the J'-dominant ℓ-weights in $\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^J))\setminus\{\boldsymbol{\omega}^J\}$ whose n-th expansions give rise to J-dominant ℓ-weights $\boldsymbol{\mu}'$ such that $\boldsymbol{\mu}'\boldsymbol{\omega}^{\{n\}}\in\mathcal{P}^+$.

In the first step of the FM algorithm, we can either calculate $\zeta^1_{\boldsymbol{\omega}^J}$ or $\zeta^{n-1}_{\boldsymbol{\omega}^J}$ and after expand their ℓ-weights. Looking at $\zeta^{n-1}_{\boldsymbol{\omega}^J}$ and the j-th expansions of its ℓ-weights with $j\neq 1$, one can see that the $I\setminus\{n-2\}$-dominant ℓ-weights which appear are:

$$\boldsymbol{\nu}(t_1,t_2)=\boldsymbol{\omega}^J\left(\prod_{k=1}^{t_1}A^{-1}_{n-1,r_{n-1}+2(k_{n-1}-k)+1}\right)\left(\prod_{k=1}^{t_2}A^{-1}_{n-2,r_{n-1}+2(k_{n-1}-k)+2}\right),\quad 1\leq t_2\leq t_1\leq k_{n-1}.$$

Observe that

$$\begin{aligned}\pi_{\{n-2,n\}}(\boldsymbol{\nu}(t_1,t_2)) \quad &= \quad \left(\prod_{k=t_2+1}^{t_1}Y_{n-2,r_{n-1}+2(k_{n-1}-k)+1}\right)\left(\prod_{k=1}^{t_2}Y^{-1}_{n-2,r_{n-1}+2(k_{n-1}-k)+3}\right)\times\\ &\quad\left(\prod_{k=1}^{t_2}Y_{n,r_{n-1}+2(k_{n-1}-k)+2}\right).\end{aligned}$$

Thus, if $\boldsymbol{\mu} \in \zeta^n_{\boldsymbol{\nu}(t_1,t_2)}$, then

$$\boldsymbol{\mu} = \boldsymbol{\nu}(t_1,t_2) \prod_{k=1}^{t} A^{-1}_{n,r_{n-1}+2(k_{n-1}-k)+3}, \quad \text{for some} \quad t = 0,1,\ldots,t_2,$$

but, if $t < t_2$, then $\boldsymbol{\mu}$ is not $(n-2)$-dominant. Since $\boldsymbol{\mu}$ is J-dominant, it follows that $t = t_2 > 1$ and

$$\pi_n(\boldsymbol{\mu}) = \prod_{k=1}^{t_2} Y^{-1}_{n,r_{n-1}+2(k_{n-1}-k)+4}.$$

Since

$$\begin{aligned} r_n &= r_1 + 2k_1 + n - 2 \\ &= r_{n-1} + 2k_{n-1} + n - 2 + 2k_1 + n - 2 \\ &= r_{n-1} + 2(k_{n-1} - 1) + 2(k_1 + n - 1) \\ &> r_{n-1} + 2(k_{n-1} - 1) + 4 \end{aligned}$$

and $r_{n-1} + 2(k_{n-1} - k) + 4 \le r_{n-1} + 2(k_{n-1} - 1) + 4$ for all $1 \le k \le k_{n-1}$, it follows that $\boldsymbol{\mu}\boldsymbol{\omega}^{\{n\}}$ is not dominant. Now, observe that, the ℓ-weights obtained by j-th expansions, $j \ne 1$, of the ℓ-weights of $\zeta^{n-1}_{\boldsymbol{\omega}^J}$ which has π_1 different to 1 are the following

$$\boldsymbol{\eta}(t) = \boldsymbol{\omega}^J \prod_{k=1}^{t} A^{-1}_{n-1,2,r_{n-1}+2(k_{n-1}-k)}$$

and j-th expansions of $\boldsymbol{\eta}(t)$ with $j > 2$. We have

$$\pi_1(\boldsymbol{\eta}(t)) = Y_{1,r_1,k_1} \prod_{k=1}^{t} Y_{1,r_{n-1}+2(k_{n-1}-k)+n-2} \overset{(4.6.6)}{=} Y_{1,r_{n-1}+2(k_{n-1}-t)+n-2,k_1+t}.$$

Thus, the 1-th expansions of the ℓ-weights of $\zeta^{n-1}_{\boldsymbol{\omega}^J}$ are the 1-th expansions of the ℓ-weights of $\zeta^1_{\boldsymbol{\omega}^J}$. Looking at $\zeta^1_{\boldsymbol{\omega}^J}$ and the j-th expansions of its ℓ-weights, one can see that the $I \setminus \{n-2\}$-dominant ℓ-weights which appear, different to $\boldsymbol{\omega}^J$, are:

$$\boldsymbol{\nu}(t_1,t_2) = \boldsymbol{\omega}^J \left(\prod_{k=1}^{t_1} A^{-1}_{1,n-2,r_1+2(k_1-k)} \right) \left(\prod_{k=1}^{t_2} A^{-1}_{n-1,2,r_{n-1}+2(k_{n-1}-k)} \right),$$

with $0 \le t_2 \le k_{n-1}$, $t_1 \ge 1$ and $t_1 - k_1 \le t_2$. Observe that

$$\begin{aligned} \pi_{\{n-2,n\}}(\boldsymbol{\nu}(t_1,t_2)) = & \left(\prod_{k=1}^{t_2} Y^{-1}_{n-2,r_{n-1}+2(k_{n-1}-k)+3} Y_{n,r_{n-1}+2(k_{n-1}-k)+2} \right) \times \\ & \left(\prod_{k=1}^{t_1} Y^{-1}_{n-2,r_1+2(k_1-k)+n-1} Y_{n,r_1+2(k_1-k)+n-2} \right). \end{aligned}$$

Thus, if $\boldsymbol{\mu} \in \zeta^n_{\boldsymbol{\nu}(t_1,t_2)}$, then

$$\boldsymbol{\mu} = \boldsymbol{\nu}(t_1,t_2) \left(\prod_{k=1}^{t'_2} A^{-1}_{n,r_{n-1}+2(k_{n-1}-k)+3} \right) \left(\prod_{k=1}^{t'_1} A^{-1}_{n,r_1+2(k_1-k)+n-1} \right),$$

for some $0 \le t_2' \le t_2$ and $0 \le t_1' \le t_1$. But, if either $t_2' < t_2$ or $t_1' < t_1$, then $\boldsymbol{\mu}$ is not $(n-2)$-dominant. Since $\boldsymbol{\mu}$ is J-dominant, it follows that $t_2' = t_2$ and $t_1' = t_1 > 1$ and

$$\pi_n(\boldsymbol{\mu}) = \left(\prod_{k=1}^{t_2} Y^{-1}_{n,r_{n-1}+2(k_{n-1}-k)+4}\right)\left(\prod_{k=1}^{t_1} Y^{-1}_{n,r_1+2(k_1-k)+n}\right).$$

Observe from (4.6.6) that this factorization (set by the parenthesis) is the q-factorization of $\pi_n(\boldsymbol{\mu})$. Since

$$r_n = r_1 + 2k_1 + n - 2,$$

it follows that $\boldsymbol{\mu}\boldsymbol{\omega}^{\{n\}}$ is dominant if and only if $t_2 = 0$ and $t_1 = 1$. $\square$

Proposition 4.6.4. Let $\boldsymbol{\omega} = Y_{1,r_1,k_1}Y_{n-1,r_{n-1},k_{n-1}}Y_{n,r_n,k_n}$ and $\boldsymbol{\omega}' = Y_{1,r_1',k_1}Y_{n-1,r_{n-1}',k_{n-1}}Y_{n,r_n',k_n}$ for some $r_i, r_i' \in \mathbb{Z}$. Suppose $r_1 = r_{n-1}+2k_{n-1}+n-2$ and that either one of the following conditions holds:

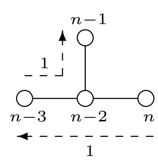
(i) $r_n = r_1 + 2k_1 + n - 2$ and $r_{n-1}' = r_n' = r_1' + 2k_1 + n - 2$;

(ii) $r_1' = r_{n-1}' + 2k_{n-1} + n - 2$, $r_{n-1} = r_n + 2k_n + 2$, and $r_n' = r_{n-1}' + 2k_{n-1} + 2$.

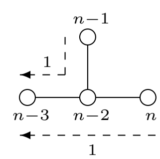
Then, $V_q(\boldsymbol{\omega}) < V_q(\boldsymbol{\omega}')$. Moreover, if $\nu = \lambda - \sum_{i\in I}\alpha_i$, then

$$m_\nu(V_q(\boldsymbol{\omega})) = 0 < m_\nu(V_q(\boldsymbol{\omega}')). \tag{4.6.9}$$

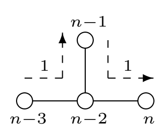
Proof. Notice that, in the sense of Section 1.7, the conditions in (i) mean that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\omega}')$ correspond, respectively, to the following pictures:



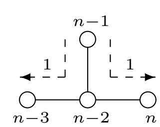
and



Similarly, the ones in (ii) mean that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\omega}')$ correspond, respectively, to the following pictures:



and



In particular (cf. Theorem 1.6.1), setting $I_j = I \setminus \{j\}$ for $j = n-1, n$, (i) implies that

$$V_q(\boldsymbol{\omega}_{I_j}) \quad \text{and} \quad V_q(\boldsymbol{\omega}'_{I_j}) \quad \text{are minimal affinizations for} \quad j = n-1, n. \tag{4.6.10}$$

while (ii) implies that

$$V_q(\boldsymbol{\omega}_{I_n}), \quad V_q(\boldsymbol{\omega}_{\{n-1,n-2,n\}}), \quad V_q(\boldsymbol{\omega}'_{I_n}), \quad V_q(\boldsymbol{\omega}'_{\{n-1,n-2,n\}}) \quad \text{are minimal affinizations.}$$

We write down the proof in case (i) is satisfied. The proof for (ii) is similar.

Thus, we have to show that for all $\mu \in P^+$, $\mu < \lambda$, either $m_\mu(V_q(\boldsymbol{\omega})) \le m_\mu(V_q(\boldsymbol{\omega}'))$ or there exists $\mu' > \mu$ such that $m_{\mu'}(V_q(\boldsymbol{\omega})) < m_{\mu'}(V_q(\boldsymbol{\omega}'))$. If $m_\mu(V_q(\boldsymbol{\omega})) = 0$, there is nothing to do. Thus, assume $m_\mu(V_q(\boldsymbol{\omega})) > 0$ and write

$$\mu = \lambda - \sum_{i\in I} s_i\alpha_i.$$

If $j \in \{n-1, n\}$ is such that $s_j = 0$, then, by Lemma 2.6.1,

$$m_\mu(V_q(\boldsymbol{\omega})) = m_{\mu_{I_j}}(V_q(\boldsymbol{\omega}_{I_j})) \quad \text{and} \quad m_\mu(V_q(\boldsymbol{\omega})) = m_{\mu_{I_j}}(V_q(\boldsymbol{\omega}_{I_j})).$$

On the other hand, (4.6.10) implies that

$$m_{\mu_{I_j}}(V_q(\boldsymbol{\omega}_{I_j})) = m_{\mu_{I_j}}(V_q(\boldsymbol{\omega}'_{I_j})) = 0.$$

In particular, it follows that $m_\mu(V_q(\boldsymbol{\omega})) = 0$, contradicting the choice of μ. Hence, we must have $s_j > 0$ for $j = n-1, n$. An application of Lemma 2.6.2 with $J_1 = \{n\}$, $i_0 = n-2$, $J_2 = \{1, \dots, n-3, n-1\}$, shows that $s_{n-2} > 0$ as well. This proves that, for $\boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}'$, we have

$$m_\nu(V_q(\boldsymbol{\varpi})) > 0 \quad \Rightarrow \quad \nu \le \lambda - \vartheta \quad \text{where} \quad \vartheta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n. \tag{4.6.11}$$

Another application of Lemma 2.6.2, this time with $J_1 = \{1, \dots, j-1\}$, $i_0 = j$, $J_2 = \{j+1, \dots, n\}$, gives that

$$\text{if } j < n-2 \text{ is such that } s_j = 0, \text{ then } s_i = 0 \text{ for all } i \le j. \tag{4.6.12}$$

Suppose $s_{n-3} = 0$, from where it follows that $s_1 = \cdots = s_{n-3} = 0$, and let $J = \{n-2, n-1, n\}$. Since

$$\frac{a_{n-1}}{a_n} = q^{-k_{n-1}-k_n-2k_1-2n+4} \ne q^{\pm(k_{n-1}+k_n+4-2t)} \quad \text{for all} \quad 1 \le t \le \min\{k_{n-1}, k_n\},$$

and

$$\frac{a'_{n-1}}{a'_n} = q^{k_{n-1}-k_n} \ne q^{\pm(k_{n-1}+k_n+4-2t)} \quad \text{for all} \quad 1 \le t \le \min\{k_{n-1}, k_n\},$$

Proposition 2.7.5 and Lemma 2.6.1 implies that

$$m_\mu(V_q(\boldsymbol{\varpi})) = m_\mu(V_q(\boldsymbol{\varpi}_J)) \quad \text{for} \quad \boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}'.$$

Moreover, the latter can be computed using Proposition 2.7.5 and we get, for $\boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}'$, that

$$m_\mu(V_q(\boldsymbol{\varpi})) \quad = \quad \begin{cases} 1, & \text{if } \mu = \lambda - t\vartheta \text{ for some } 1 \le t \le \min\{k_{n-1}, k_n\}; \\ 0, & \text{otherwise.} \end{cases} \tag{4.6.13}$$

Suppose next that there exists

$$j = \max\{i < n-3 : s_i = 0\} \quad \text{and that} \quad s_{j+1} \ne 0.$$

In particular, $s_i > 0$ for all $i > j$ and $s_i = 0$ for all $i \le j$. Letting $J = \{j+1, \dots, n\}$ and using Lemma 2.6.1 we get

$$m_\mu(V_q(\boldsymbol{\varpi})) = m_{\mu_J}(V_q(\boldsymbol{\varpi}_J)) \quad \text{for} \quad \boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}'.$$

On the other hand, condition (i) together with Lemma 4.4.4 applied to the subalgebra $U_q(\tilde{\mathfrak{g}}_J)$ imply that

$$V_q(\boldsymbol{\varpi}_J) \cong V_q(\boldsymbol{\varpi}_J^{\{n-1\}}) \otimes V_q(\boldsymbol{\varpi}_J^{\{n\}}) \quad \text{for} \quad \boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}'.$$

Hence,

$$m_\mu(V_q(\boldsymbol{\omega})) = m_\mu(V_q(\boldsymbol{\omega}')).$$

So far we have proved that

$$s_i = 0 \quad \text{for some} \quad i \in I \quad \Rightarrow \quad m_\mu(V_q(\boldsymbol{\omega})) = m_\mu(V_q(\boldsymbol{\omega}')).$$

Thus, henceforth we suppose $s_i > 0$ for all $i \in I$. Recall that we have set $\nu = \lambda - \sum_{i \in I} \alpha_i \in P^+$. Since $s_i > 0$ for all $i \in I$, we have $\mu \le \nu$. Thus, the proposition will be proved once (4.6.9) is proved.

For proving (4.6.9), we begin by checking that, if $\eta \in P^+$ is such that $\nu < \eta < \lambda$ and $\boldsymbol{\varpi} \in \{\boldsymbol{\omega}, \boldsymbol{\omega}'\}$, then

$$m_\eta(V_q(\boldsymbol{\varpi})) > 0 \quad \Leftrightarrow \quad \eta = \lambda - \vartheta. \tag{4.6.14}$$

The implication $\Leftarrow$ follows as a particular case of (4.6.13). For the converse, notice that (4.6.11), (4.6.12), and the condition $\eta > \nu$ imply that

$$\eta = \lambda - \sum_{i=j}^{n} \alpha_i \quad \text{for some} \quad j \in \{2, \ldots, n-2\}.$$

One easily checks that $\eta \in P^+$ iff $j = n-2$, thus completing the proof of (4.6.14).

Notice that (4.6.14) implies that, for $\boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}'$,

$$\dim V_q(\boldsymbol{\varpi})_\nu \geq \dim V_q(\lambda)_\nu + \dim V_q(\lambda - \vartheta)_\nu = 4(n-2),$$

where we used Lemma 4.6.1 in the last equality. Since

$$m_\lambda(V_q(\boldsymbol{\varpi})) = m_{\lambda-\vartheta}(V_q(\boldsymbol{\varpi})) = 1 \quad \text{for} \quad \boldsymbol{\varpi} = \boldsymbol{\omega}, \boldsymbol{\omega}',$$

(4.6.9) follows if we show that

$$\dim V_q(\boldsymbol{\omega})_\nu = 4(n-2) \quad \text{and} \quad \dim V_q(\boldsymbol{\omega}')_\nu > 4(n-2). \tag{4.6.15}$$

For proving the second statement in (4.6.15), it suffices to show that

$$\#\{\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}')) : \mathrm{wt}(\boldsymbol{\nu}) = \nu\} > 4(n-2).$$

This can be done by repeatedly using Proposition 2.6.7 to check that the following are elements of $\mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$:

$$\boldsymbol{\mu}_i^1 = \boldsymbol{\omega}' A^{-1}_{n,r_n+1} A^{-1}_{n-2,r_n+2} A^{-1}_{n-1,r_n+3} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_n+2}, \quad i = 0,1,\ldots,n-3;$$
$$\boldsymbol{\mu}_i^2 = \boldsymbol{\omega}' A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{n,r_n+1} A^{-1}_{n-2,r_n+2} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_n+2}, \quad i = 0,1,\ldots,n-3;$$
$$\boldsymbol{\mu}_i^3 = \boldsymbol{\omega}' A^{-1}_{n,r_n+1} A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{n-2,r_{n-1}+2} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_{n-1}+2}, \quad i = 0,1,\ldots,n-3;$$
$$\boldsymbol{\mu}_i^4 = \boldsymbol{\omega}' A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{n-2,r_{n-1}+2} A^{-1}_{n,r_{n-1}+3} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_{n-1}+2}, \quad i = 0,1,\ldots,n-3;$$
$$\boldsymbol{\mu} = \boldsymbol{\omega}' A^{-1}_{n,r_n+1} A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{1,n-2,r_1}.$$

Here $\overrightarrow{A}_{i,j,r} = A_{i,j,r}$ if $j \geq i$ and $\overrightarrow{A}_{i,j,r} = 1$ if $j < i$, and, $\overleftarrow{A}_{i,j,r} = A_{i,j,r}$ if $j \leq i$ and $\overleftarrow{A}_{i,j,r} = 1$ if $j > i$.

As for the first statement in (4.6.15), it follows from Lemma 4.6.3 and Theorem 2.4.4 that we can use the FM algorithm for computing $\dim V_q(\boldsymbol{\omega})_\nu$. By using the algorithm, we see that following are the elements of $\mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}))$:

$$\boldsymbol{\mu}_i^1 = \boldsymbol{\omega} A^{-1}_{n,n-2,r_n} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_n+2}, \quad i = 0,1,\ldots,n-3;$$
$$\boldsymbol{\mu}_i^2 = \boldsymbol{\omega} A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{n,r_n+1} A^{-1}_{n-2,r_n+2} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_n+2}, \quad i = 0,1,\ldots,n-3;$$
$$\boldsymbol{\mu}_i^3 = \boldsymbol{\omega} A^{-1}_{n,r_n+1} A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{n-2,r_{n-1}+2} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_{n-1}+2}, \quad i = 1,\ldots,n-3;$$
$$\boldsymbol{\mu}_i^4 = \boldsymbol{\omega} A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{n-2,r_{n-1}+2} A^{-1}_{n,r_{n-1}+3} \overrightarrow{A}^{-1}_{1,i,r_1} \overleftarrow{A}^{-1}_{n-3,i+1,r_{n-1}+2}, \quad i = 1,\ldots,n-3;$$
$$\boldsymbol{\mu}^5 = \boldsymbol{\omega} A^{-1}_{n,r_n+1} A^{-1}_{n-1,r_{n-1}+1} A^{-1}_{1,n-2,r_1};$$
$$\boldsymbol{\mu}^6 = \boldsymbol{\omega} A^{-1}_{n,r_n+1} A^{-1}_{1,n-1,r_1}.$$

Hence,

$$\#\{\boldsymbol{\nu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) : \mathrm{wt}(\boldsymbol{\nu}) = \nu\} = 4(n-2),$$

Again by the FM algorithm, one can see that the corresponding ℓ-weight spaces are one-dimensional, which completes the proof. $\square$

Remark 4.6.5. Proposition 4.6.4 together with [**12**, Theorem 2.2] proves Theorem 1.7.1 (recall also the comment on Remark 1.7.2).

We also record the next lemma which is easily checked using Lemma 2.1.2.

LEMMA 4.6.6. Let $\boldsymbol{\omega} = Y_{1,r_1,k_1}Y_{n-1,r_{n-1},k_{n-1}}Y_{n,r_n,k_n}$ and $\boldsymbol{\omega}' = Y_{1,r'_1,k_1}Y_{n-1,r'_{n-1},k_{n-1}}Y_{n,r'_n,k_n}$ for some $r_i, r'_i \in \mathbb{Z}$ and suppose either one of the following conditions hold:

(i) $r_1 = r_{n-1}+2k_{n-1}+n-2, r'_1 = r'_n+2k_n+n-2, r_n = r_1+2k_1+n-2$, and $r'_{n-1} = r'_1+2k_1+n-2$;
(ii) $r_{n-1} = r_n = r_1 + k_1 + n - 2, r'_1 = r'_{n-1} + 2k_{n-1} + n - 2$, and $r'_n = r'_{n-1} + 2(k_{n-1} - k_n)$;
(iii) $r_1 = r_{n-1}+2k_{n-1}+n-2, r'_{n-1} = r'_1+2k_1+n-2, r_{n-1} = r_n+2k_n+2$, and $r'_n = r'_{n-1}+2k_{n-1}+2$;
(iv) $r_1 = r_{n-1} + 2k_{n-1} + n - 2, r_n = r_{n-1} + 2k_{n-1} + 2$, $r'_1 = r'_n + 2(k_n - k_1 + 2) - n$, and $r'_{n-1} = r'_n + 2k_n + 2$.

Then, $V_q(\boldsymbol{\omega}) \cong V_q(\boldsymbol{\omega}')$.

CHAPTER 5

Minimal affinizations for D_n

We now prove the main theorems stated in Section 1.7. The reader should recall the notation fixed there. In particular, we fix $\lambda \in P^+$ such that $\lambda(h_{n-2}) = 0$ and $\mathrm{supp}(\lambda)$ bounds a subdiagram of type D_4. Recall also the definitions of i_λ and f_λ and that $A = \{i \in I : i < n-2\}$.

5.1. Rephrasing the main theorem

Let $a_i \in \mathbb{C}^\times, i \in I$, and assume $\boldsymbol{\omega} = \prod_{i\in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)}$ is such that $V_q(\boldsymbol{\omega}_A)$ is a minimal affinization. Assume also that $r_i \in \mathbb{Z}, i \in I$, are such

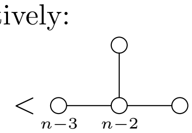
$$\boldsymbol{\omega} = \prod_{i\in I} Y_{i,r_i,\lambda(h_i)}.$$

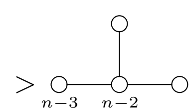
The conditions in the main theorems were described in terms the parameters a_i. We now rephrase them in terms of the parameters r_i which will be more convenient for the proof.

Set

$$I_n = I \setminus \{n\}, \quad I_{n-1} = I \setminus \{n-1\}, \quad \text{and} \quad I_\lambda = \{f_\lambda + 1, f_\lambda + 2, \ldots, n\}.$$

The condition that $V_q(\boldsymbol{\omega}_A)$ is increasing or decreasing will be graphically denoted by the following pictures, respectively:





If $\#(\mathrm{supp}(\lambda) \cap A) = 1$, then the terminology increasing or decreasing for $V_q(\boldsymbol{\omega}_A)$ is vacuous and we use either of the pictures.

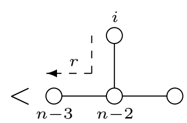
Let $\{i, j\} = \{n-1, n\}$. Assume first that $V_q(\boldsymbol{\omega}_A)$ is increasing. Then, condition (3.5.1) applied to $\boldsymbol{\omega}_{I_j}$ means that there exists $1 \le r \le \min\{|\lambda|_{n-2}, \lambda(h_i)\}$ such that

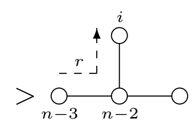
$$r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r_i + 2r. \tag{5.1.1}$$

Similarly, if $V_q(\boldsymbol{\omega}_A)$ is decreasing, condition (3.7.3) applied to $\boldsymbol{\omega}_{I_j}$ means that there exists $1 \le r \le \min\{|\lambda|_{n-2}, \lambda(h_i)\}$ such that

$$r_i + 2\lambda(h_i) + n - f_\lambda + 1 = r_{f_\lambda} + 2r. \tag{5.1.2}$$

Conditions (5.1.1) and (5.1.2) will be indicated graphically by the following pictures





Observe that, if $r = 1$, then $V_q(\boldsymbol{\omega}_{I_j})$ is a minimal affinization. One easily checks using Remark 3.5.2 that (5.1.1) and (5.1.2) are equivalent to (1.7.2), justifying the use of the above pictures.

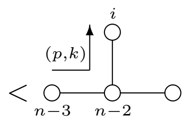
We now turn to the rephrasing of (1.7.3). If $V_q(\boldsymbol{\omega}_A)$ is increasing, condition (3.5.2) restricted to the subdiagram I_j means that there exist $p \in \operatorname{supp}(\lambda) \cap A$ and $1 \leq k \leq \min\{\lambda(h_p), \lambda(h_i)\}$ such that

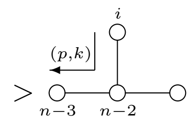
$$r_i + 2\lambda(h_i) + n - p + 1 = r_p + 2k. \tag{5.1.3}$$

If $V_q(\boldsymbol{\omega}_A)$ is decreasing, condition (3.7.5) restricted to the subdiagram I_j is: there exist $p \in \operatorname{supp}(\lambda) \cap A$ and $1 \leq k \leq \min\{\lambda(h_p), \lambda(h_i)\}$ such that

$$r_p + 2\lambda(h_p) + n - p + 1 = r_i + 2k. \tag{5.1.4}$$

Conditions (5.1.3) and (5.1.4) will be indicated graphically by the following pictures



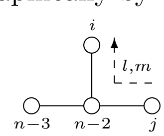


Observe that, if $(p, k) = (f_\lambda, 1)$, then $V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n-2,i\}})$ is a minimal affinization. Notice also that, if $\#(\operatorname{supp}(\lambda) \cap J) = 1$, then (5.1.1) is equivalent to (5.1.4) and (5.1.2) is equivalent to (5.1.3), with $p = f_\lambda$ and $k = r$. One easily checks using Remark 3.5.2 that (5.1.3) and (5.1.4) are equivalent to (1.7.3) with $r = |\lambda|_{p-1} + p - i_\lambda + k$.

We also consider condition (4.4.27) restricted to the subdiagram I_λ. If $r_i + 2\lambda(h_i) \leq r_j + 2\lambda(h_j)$, it means that there exist $1 \leq l \leq \left\lfloor \frac{n-f_\lambda-1}{2} \right\rfloor$ and $1 \leq m \leq \min\{\lambda(h_i), \lambda(h_j)\}$ such that

$$r_i + 2\lambda(h_i) + 4l = r_j + 2m. \tag{5.1.5}$$

Recall that we consider such pair with m minimum. Thus, by Lemma 4.4.2, if $m > 2$ we have $l = 1$. Condition (5.1.5) will be indicated graphically by the following picture



Observe that, if $(l, m) = (1, 1)$, then $V_q(\boldsymbol{\omega}_{I_\lambda})$ is a minimal affinization. We shall often omit l from the above pictures in the case it is 1. One easily checks using Remark 4.4.5 that (5.1.5) is equivalent to (1.7.5) with $l = s_1$ and $m = s_2$.

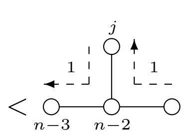
For the reader's convenience, we restate Theorem 1.7.5 using the above rephrasing of the conditions. In fact, part of the theorem has already been proved. Namely, let $\boldsymbol{\omega} \in \mathcal{P}^+$ be such that $V_q(\boldsymbol{\omega})$ is a minimal affinization of $V_q(\lambda)$. Then, Proposition 2.6.4 implies that $V_q(\boldsymbol{\omega}_A)$ and $V_q(\boldsymbol{\omega}_{\{i\}})$ are minimal affinizations for all $i \in I$. Moreover, Corollary 2.5.2 implies that we can assume that

$$\boldsymbol{\omega} = \prod_{i \in I} Y_{i, r_i, \lambda(h_i)}$$

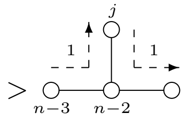
for some $r_i \in \mathbb{Z}$. The remainder of the statement of Theorem 1.7.5 can be phrased as follows.

Theorem 5.1.1. Suppose $n \geq 5$ and $\#(\operatorname{supp}(\lambda) \cap A) > 1$. Then, the parameters r_i's are related by one of the following conditions.

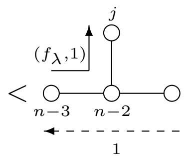
(a)$_j$ The parameters are related by either of the following pictures, where j is a spin node:



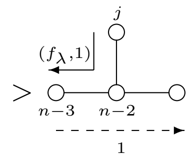
or



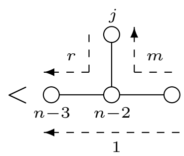
(b)$_j$ The parameters are related by either of the following pictures, where j is a spin node:



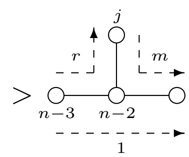
or



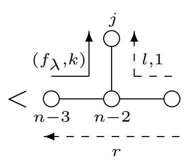
(c)$_j^r$ The parameters are related by either of the following pictures, where j is a spin node and $m = \lambda(h_j) + 3 - r$:



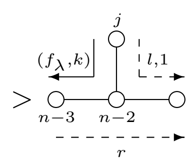
or



(d)$_j^{k,l}$ The parameters are related by either of the following pictures



or



where j is a spin node and $r = \lambda(h_{f_\lambda}) + n - f_\lambda + 2 - k - 2l$ and, setting $\{j, j'\} = \{n-1, n\}$ the parameters satisfy the following conditions:

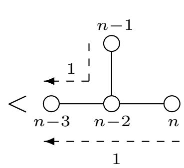
(i) If $r \leq k$, then $\lambda(h_j) + 3 - r > \min\{l, \lambda(h_{j'})\}$.

(ii) If $p \in \operatorname{supp}(\lambda) \cap A \setminus \{f_\lambda\}$, $0 \leq k' < k$ and $1 \leq l' \leq l$ are such that

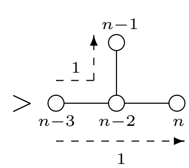
$$1 \leq k'' := {}_p|\lambda|_{f_\lambda} + n - p + 2 - r - 2l - 2l' - k' - \lambda(h_j) \leq \min\{\lambda(h_p), \lambda(h_{j'})\},$$

then $r - {}_p|\lambda|_{f_\lambda} \leq k''$.

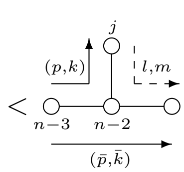
(e) The parameters are related by either of the following pictures, where j is a spin node:



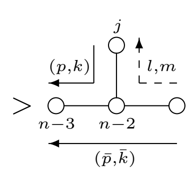
or



(f)$_{j,l}^{(\bar{p},\bar{k}),(p,k)}$ The parameters are related by either of the following pictures



or



where j is a spin node, $m = \lambda(h_j) + 2l + \bar{k} - ({}_{\bar{p}}|\lambda|_{p-1} + p - \bar{p} + k)$ and, setting $\{j, j'\} = \{n-1, n\}$, the parameters satisfy one of the following three conditions:

(i) $\bar{k} = 1$, $l > 1$, $m \leq 2$, $k \leq 2$, with $k = 1$ if $m = 2$. Besides, if $\bar{p} < f_\lambda$, given $i, \bar{i} \in \text{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < \max\{k, m\}$ such that $\bar{p} < \bar{i} \leq i \leq f_\lambda$, $p \leq i$ and $0 \leq \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k'$ is even, then

$$l < \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k') + 1.$$

Moreover, if $p < f_\lambda$, given $i, \bar{i} \in \text{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < \max\{k, m\}$ such that $p \leq i < \bar{i} \leq f_\lambda$ and $\bar{i} - i + {}_i|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_{j'})$ is even then

$$\text{either} \quad \frac{1}{2}(\bar{i}-i+{}_i|\lambda|_{\bar{i}-1}+2l-k+k'+2-\lambda(h_n)) < l \quad \text{or} \quad \frac{1}{2}(\bar{i}-i+{}_i|\lambda|_{\bar{i}-1}+2l-k+k'+2-\lambda(h_{j'})) > 2l.$$

(ii) $\bar{k} = l = 1$, $p > \bar{p}$ and

$$k < \lambda(h_{f_\lambda}) + p - \bar{p} + n - f_\lambda - 1.$$

Besides, if $p < f_\lambda$, given $i, \bar{i} \in \text{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < k$ such that $\bar{p} \leq \bar{i} < i \leq f_\lambda$; $p < i$ and $\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} \leq i - p + {}_p|\lambda|_{i-1} - k'$, then

$$m < i - p + {}_p|\lambda|_{i-1} - k' - \bar{i} + \bar{p} - {}_{\bar{p}}|\lambda|_{\bar{i}-1} + 1.$$

Moreover, if $p < f_\lambda$, given $i, \bar{i} \in \text{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < k$ such that $p \leq i < \bar{i} \leq f_\lambda$ then

$$\text{either} \quad \lambda(h_{j'}) + 1 + k - k' - \bar{i} + i - {}_i|\lambda|_{\bar{i}-1} < 1 \quad \text{or} \quad \lambda(h_{j'}) + 1 + k - k' - \bar{i} + i - {}_i|\lambda|_{\bar{i}-1} > m.$$

(iii) $\bar{k} > 1$, $k = m = 1$. Besides, if $\bar{p} < f_\lambda$, given $i, \bar{i} \in \text{supp}(\lambda)$ and $0 \leq l' < l$ such that $\bar{p} < \bar{i} \leq i \leq f_\lambda$; $p \leq i$ with either $l' > 0$ or $p < i$, and $i-p+{}_p|\lambda|_{i-1}+2l' \leq \bar{i}-\bar{p}+{}_{\bar{i}}|\lambda|_{\bar{p}-1}$, then

$$\text{either} \quad \bar{k} < \bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-i+p-{}_p|\lambda|_{i-1}-2l'+1 \quad \text{or} \quad \bar{k} > \bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-i+p-{}_p|\lambda|_{i-1}-2l'+\lambda(h_{\bar{i}}).$$

Moreover, if $\bar{p} < f_\lambda$, given $i, \bar{i} \in \text{supp}(\lambda)$ and $\max\{0, \bar{k} - \lambda(h_{\bar{i}})\} \leq \bar{k}' < \bar{k}$ such that $p \leq i \leq \bar{i} \leq f_\lambda$, $\bar{i} > \bar{p}$ and $-\lambda(h_{j'}) + \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k'$ is even, then

$$\text{either} \quad l > \frac{1}{2}(-\lambda(h_{j'}) + \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k')$$

$$\text{or} \quad \frac{1}{2}(-\lambda(h_{j'}) + \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k') > 2l - 1.$$

Furthermore, these conditions parameterize the distinct equivalence classes of minimal affinizations. In particular, the two pictures listed for each condition give rise to equivalent affinizations.

The very last statement of this theorem is an easy consequence of Proposition 2.1.2. We shall organize the proof of the other statements as follows. Notice that each condition (a)–(f) defines a family of affinizations. Henceforth, given $\boldsymbol{\omega} \in \mathcal{P}^+$, we shall say that $V_q(\boldsymbol{\omega})$ is an affinization of type (x), or simply that $\boldsymbol{\omega}$ is of type (x), if $\boldsymbol{\omega}$ satisfies the conditions $(\text{x})_!^?$ for some choice of indices ? and !. The proof of Theorem 5.1.1 will be split in the following steps.

Step 1. If $V_q(\boldsymbol{\omega})$ is a minimal affinization, then $\boldsymbol{\omega}$ must be of one of the types (a)–(f).

Step 2. Two affinizations of the same type are comparable iff they are associated to the same indices.

Step 3. An affinization of type (x) is not comparable to an affinization of type (y) if x≠y.

These steps will be carried out in Sections 5.3, 5.4, and 5.5, respectively.

5.2. One more comparison of affinizations

We will need the following lemma.

LEMMA 5.2.1. Let $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\omega}) = \mathrm{wt}(\boldsymbol{\omega}') = \lambda$ and $V_q(\boldsymbol{\omega}_A), V_q(\boldsymbol{\omega}'_A)$ are increasing minimal affinizations. Fix $j \in \{n-1, n\}$.

(a) If $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ satisfy, respectively, the conditions determined by the pictures in either one of the options (i)–(vi) below, then $V_q(\boldsymbol{\omega}) < V_q(\boldsymbol{\omega}')$.

(i)

with

$$k \le k', \quad p \ge p'; \qquad \bar{k} \le \bar{k}', \quad \bar{p} \ge \bar{p}'; \qquad l \le l', \quad m \le m';$$

and at least one of these inequalities being strict.

(ii)

with

$$k \le k', \quad p \ge p'; \qquad r \le r'; \qquad l \le l', \quad m \le m';$$

and at least one of these inequalities being strict.

(iii)

with

$$r \le k \quad \text{and} \quad m \le m'.$$

(iv)

with

$$r \le k \quad \text{and} \quad m \le l.$$

(v)

with

$$k \le k', \qquad r - {}_{\bar{p}+1}|\lambda|_{f_\lambda} > \bar{k}, \qquad l \le l'.$$

(vi)

with

$$r \le k \quad \text{and} \quad m > 1.$$

(b) If $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ satisfy, respectively, the conditions determined by the pictures in either one of the options (i) or (ii) below, then $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\omega}')$ are not comparable affinizations.

(i)

with $k < r$.

(ii)

with $\bar{k} < r$ and $r \le {}_{\bar{p}+1}|\lambda|_{f_\lambda}$.

Proof. Write

$$\boldsymbol{\omega} = \prod_{i \in I} Y_{i, r_i, \lambda(h_i)} \quad \text{and} \quad \boldsymbol{\omega}' = \prod_{i \in I} Y_{i, r'_i, \lambda(h_i)}, \quad \text{with} \quad r_i, r'_i \in \mathbb{Z}.$$

Without loss of generality, suppose $j = n-1$ and $r_i = r'_i$ for all $i \in A$.

Let us prove part (a). Assume condition (i) holds, consider the diagram subalgebra determined by I_n, and let $\boldsymbol{\lambda} = \boldsymbol{\omega}_J$. Consider also the element $\boldsymbol{\mu}$ from Lemma 3.5.4 which is of the form $\boldsymbol{\lambda\eta}$ for some $\boldsymbol{\eta} \in \mathcal{Q}_J$. Set

$$\boldsymbol{\mu}_1 = \boldsymbol{\omega}\ \iota_{I_n}(\boldsymbol{\eta}).$$

Define $\boldsymbol{\mu}_2$ in the same manner by considering the diagram subalgebra determined by I_{n-1}. Part (c) of Lemma 3.5.4 implies that

$$(\boldsymbol{\mu}_1)_{I_n} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{I_n})) \qquad \text{and} \qquad (\boldsymbol{\mu}_2)_{I_{n-1}} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{I_{n-1}})).$$

Similarly, we define $\boldsymbol{\mu}_3$ by considering the diagram subalgebra determined by I_λ and letting $\boldsymbol{\mu}$ be as in Proposition 4.4.3. In particular, $(\boldsymbol{\mu}_3)_{I_\lambda} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{I_\lambda}))$. Equation (2.6.2) implies that

$$\boldsymbol{\mu}_i \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})) \qquad \text{for} \quad i = 1, 2, 3.$$

Analogous constructions and conclusions apply to $\boldsymbol{\omega}'$ and we denote the corresponding elements by $\boldsymbol{\mu}'_1, \boldsymbol{\mu}'_2, \boldsymbol{\mu}'_3$.

Set $V := V_q(\boldsymbol{\omega}^{\{1,\dots,f_\lambda\}}) \otimes V_q(\boldsymbol{\omega}^{\{n-1\}}) \otimes V_q(\boldsymbol{\omega}^{\{n\}})$. We will prove that

$$\boldsymbol{\mu} \in \mathrm{wt}_\ell(V) \cap \mathcal{P}^+ \quad \text{and} \quad \boldsymbol{\mu} \not\geq \boldsymbol{\mu}_i \quad \text{for} \quad i = 1, 2, 3 \quad \Rightarrow \quad \boldsymbol{\mu} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})). \tag{5.2.1}$$

Fix such $\boldsymbol{\mu}$ and, for each $i=1,2,3$, let $\Xi_i \subseteq I\times\mathbb{Z}$ be such that

$$\boldsymbol{\mu}_i = \boldsymbol{\omega}\prod_{\zeta\in\Xi_i} A_\zeta^{-1}.$$

We claim that

$$\boldsymbol{\mu} = \boldsymbol{\omega}\prod_{\zeta\in\Xi} A_\zeta^{-1} \qquad \text{for some} \quad \Xi\subseteq \Xi_1\cup\Xi_2\cup\Xi_3. \tag{5.2.2}$$

Assuming (5.2.2), we prove (5.2.1) as follows. Let $\Xi_4=\Xi\cap\Xi_1$, $\Xi_5=\Xi\cap\Xi_2$, and $\Xi_6=\Xi\cap\Xi_3$. Observe that $\Xi_4\subsetneqq\Xi_1$, $\Xi_5\subsetneqq\Xi_2$, and $\Xi_6\subsetneqq\Xi_3$, since otherwise, if $\Xi_{i+3}=\Xi_i$ for some $i=1,2,3$ we would have $\boldsymbol{\mu}\ge\boldsymbol{\mu}_i$, contradicting the choice of $\boldsymbol{\mu}$. Since $\boldsymbol{\mu}$ is dominant and $\boldsymbol{\omega}\prod_{\zeta\in\Xi_{i+3}} A_\zeta^{-1}<\boldsymbol{\mu}_i$, by (3.4.6) and (4.4.3),

$$\boldsymbol{\nu}_1 := \boldsymbol{\omega}\prod_{\zeta\in\Xi_4} A_\zeta^{-1} = \boldsymbol{\omega}\left(\prod_{t=1}^{k_1} A^{-1}_{n-1,p_1,r_{n-1}+2(\lambda(h_{n-1})-t)}\right), \quad \text{with} \quad p_1\ge p, \quad k_1\le k,$$

and at least one of the inequalities above is strict,

$$\boldsymbol{\nu}_2 := \boldsymbol{\omega}\prod_{\zeta\in\Xi_5} A_\zeta^{-1} = \boldsymbol{\omega}\left(\prod_{t=1}^{\bar k_1} A^{-1}_{n,r_n+2(\lambda(h_n)-t)+1}A^{-1}_{n-2,\bar p_1,r_n+2(\lambda(h_n)-t)+1}\right), \quad \text{with} \quad \bar p_1\ge \bar p, \quad \bar k_1\le \bar k,$$

and at least one of the inequalities above is strict, and

$$\boldsymbol{\nu}_3 := \boldsymbol{\omega}\prod_{\zeta\in\Xi_6} A_\zeta^{-1} = \boldsymbol{\omega}\prod_{t=1}^{m_1}\prod_{s=1}^{l_1} A^{-1}_{n-2s+1,n-1,r_{n-1}+2(\lambda(h_{n-1})-t)+2s-2}A^{-1}_{n-2s,n-2,r_{n-1}+2(\lambda(h_{n-1})-t)+2s-1}\cdot$$
$$\cdot A^{-1}_{n,r_{n-1}+2(\lambda(h_{n-1})-t)+4s-1}, \qquad \text{with} \quad m_1\le m, \quad l_1\le l.$$

Suppose first $m_1\ge k_1$. By Lemma 4.4.1(a), $\boldsymbol{\nu}_3\in \mathrm{wt}_\ell(\chi_{I_\lambda}(V_q(\boldsymbol{\omega})))\subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. Let

$$\boldsymbol{\nu}_2' := \boldsymbol{\nu}_3\left(\prod_{t=1}^{\bar k_1} A^{-1}_{n,r_n+2(\lambda(h_n)-t)+1}A^{-1}_{n-2,f_\lambda+1,r_n+2(\lambda(h_n)-t)+1}\right)\left(\prod_{t=1}^{k_1} A^{-1}_{n-2l_1-1,f_\lambda+1,r_{n-1}+2(\lambda(h_{n-1})-t)+2l_1+1}\right).$$

Then, it is not difficult to see that

$$\boldsymbol{\nu}_2'\in \mathrm{wt}_\ell(\chi_{I_\lambda}(V_q(\boldsymbol{\omega})))\subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\omega})),$$

and one can check that $\boldsymbol{\nu}_2'$ satisfies conditions (ii) and (iii) of Proposition 2.6.7 with $J=I\setminus I_\lambda$. Hence $\mathrm{wt}_\ell(\chi_J(V_q(\boldsymbol{\nu}_2')))\subseteq \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$. Observe that

$$\boldsymbol{\mu} = \boldsymbol{\nu}_2'\left(\prod_{t=1}^{k_1} A^{-1}_{f_\lambda,p_1,r_{n-1}+2(\lambda(h_{n-1})-t)+n-f_\lambda-1}\right)\left(\prod_{t=1}^{\bar k_1} A^{-1}_{f_\lambda,\bar p_1,r_n+2(\lambda(h_n)-t)+n-f_\lambda-1}\right)$$

To finish the proof of (5.2.1), it is not hard to see that $\boldsymbol{\mu}\in \mathrm{wt}_\ell(\chi_J(V_q(\boldsymbol{\nu}_2')))$. If $k_1>m_1$ the proof is similar, the unique difference is to consider

$$\boldsymbol{\nu}_2' := \boldsymbol{\nu}_3\left(\prod_{t=1}^{\bar k_1} A^{-1}_{n,r_n+2(\lambda(h_n)-t)+1}A^{-1}_{n-2,f_\lambda+1,r_n+2(\lambda(h_n)-t)+1}\right)\times$$
$$\times\left(\prod_{t=1}^{m_1} A^{-1}_{n-2l_1-1,f_\lambda+1,r_{n-1}+2(\lambda(h_{n-1})-t)+2l_1+1}\right)\left(\prod_{t=m_1+1}^{k_1} A^{-1}_{n-1,f_\lambda+1,r_{n-1}+2(\lambda(h_{n-1})-t)}\right).$$

Let us prove (5.2.2). We show that, if $\boldsymbol{\mu}=\boldsymbol{\omega}\prod_{\zeta\in\Xi}A_\zeta^{-1}\in\mathrm{wt}_\ell(V)\cap\mathcal{P}^+$ and $\Xi\not\subseteq\Xi_1\cup\Xi_2\cup\Xi_3$, then $\boldsymbol{\mu}>\boldsymbol{\mu}_i$ for some $i=1,2,3$. Indeed, write $\boldsymbol{\mu}=\boldsymbol{\nu}_1\boldsymbol{\nu}_2\boldsymbol{\nu}_3$ where $\boldsymbol{\nu}_1\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{\{1,\dots,f_\lambda\}}))$, $\boldsymbol{\nu}_2\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{\{n-1\}}))$ and $\boldsymbol{\nu}_3\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}^{\{n\}}))$. Since $\boldsymbol{\mu}$ is dominant, $\boldsymbol{\mu}$ is not right negative, and since the product of right negative ℓ-weights is again right negative, it follows that either of $\boldsymbol{\nu}_i$ is not right negative. If $\boldsymbol{\nu}_2$ is right negative, then $\boldsymbol{\nu}_2\neq\boldsymbol{\omega}^{n-1}$. By (4.4.2), $\boldsymbol{\nu}_2\boldsymbol{\nu}_3$ is not dominant and the negative power that appears in $\boldsymbol{\nu}_2$ must be canceled with $\boldsymbol{\nu}_1$. Since $\Xi\not\subseteq\Xi_1\cup\Xi_2\cup\Xi_3$, it follows that $\boldsymbol{\nu}>\boldsymbol{\mu}_1$. If $\boldsymbol{\nu}_2$ is not right negative, by Proposition 2.5.5, $\boldsymbol{\nu}_2=\boldsymbol{\omega}^{\{n-1\}}$. Since $\Xi\not\subseteq\Xi_1\cup\Xi_2\cup\Xi_3$, either $\boldsymbol{\nu}>\boldsymbol{\mu}_2$ or $\boldsymbol{\nu}>\boldsymbol{\mu}_3$, completing the proof of (5.2.2).

Similarly, setting $V':=V_q((\boldsymbol{\omega}')^{\{1,\dots,f_\lambda\}})\otimes V_q((\boldsymbol{\omega}')^{\{n-1\}})\otimes V_q((\boldsymbol{\omega}')^{\{n\}})$, if $\boldsymbol{\mu}'\in\mathrm{wt}_\ell(V)$ is dominant and $\boldsymbol{\mu}'\not\geq\boldsymbol{\mu}'_i$ for all $i=1,2,3$, then $\boldsymbol{\mu}'\in\mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'))$.

To finish the proof, observe that $V_q(\boldsymbol{\omega})$ is the simple quotient of the submodule of V generated by the top weight space and similarly for V'. Notice that we have isomorphisms of $U_q(\mathfrak{g})$-modules:

$$V\cong V'\cong V_q(\lambda)\oplus\bigoplus_\nu V_q(\nu)^{\oplus t_\nu},\quad V_q(\boldsymbol{\omega})\cong V_q(\lambda)\oplus\bigoplus_\nu V_q(\nu)^{\oplus m_\nu}\quad\text{and}\quad V_q(\boldsymbol{\omega}')\cong V_q(\lambda)\oplus\bigoplus_\nu V_q(\nu)^{\oplus m'_\nu}$$

where the sums are over $\nu\in P^+$ such that $\nu<\lambda$ and $m_\nu,m'_\nu,t_\nu\in\mathbb{Z}_{\geq0}$. It follows from the claim that

$$\nu\not\leq\mathrm{wt}(\boldsymbol{\mu}_i)\Rightarrow t_\nu=m_\nu\quad\text{and}\quad m_{\mathrm{wt}(\boldsymbol{\mu}_i)}\leq t_{\mathrm{wt}(\boldsymbol{\mu}_i)}-1,$$

and similarly

$$\nu\not\leq\mathrm{wt}(\boldsymbol{\mu}'_i)\Rightarrow t_\nu=m_\nu\quad\text{and}\quad m'_{\mathrm{wt}(\boldsymbol{\mu}'_i)}\leq t_{\mathrm{wt}(\boldsymbol{\mu}'_i)}-1.$$

Condition (i) implies that

$$\mathrm{wt}(\boldsymbol{\mu}'_i)\leq\mathrm{wt}(\boldsymbol{\mu}_i)\quad\text{for all}\quad i=1,2,3,\tag{5.2.3}$$

with at lest one of the inequalities above is strict. Let $\nu\in P^+$ be such that $\nu<\lambda$. If $\nu\not\leq\mathrm{wt}(\boldsymbol{\mu})$, then we also have $\nu\not\leq\mathrm{wt}(\boldsymbol{\mu}')$ and, hence, $m'_\nu=t_\nu=m_\nu$. Otherwise, if $\nu\leq\mathrm{wt}(\boldsymbol{\mu})$, we have $m_{\mathrm{wt}(\boldsymbol{\mu})}\leq t_{\mathrm{wt}(\boldsymbol{\mu})}-1=m'_{\mathrm{wt}(\boldsymbol{\mu})}-1$, which concludes the proof of part (a).

The proof of part (a) for the other conditions are similar and we omit the details.

We now prove (b). We do case (i), the other is similar. Write

$$\boldsymbol{\omega}=\prod_{i\in I}Y_{i,r_i,\lambda(h_i)}\quad\text{and}\quad\boldsymbol{\omega}'=\prod_{i\in I}Y_{i,r'_i,\lambda(h_i)},\quad\text{with}\quad r_i,r'_i\in\mathbb{Z}.$$

Without loss of generality suppose $j=n-1$ and $r_i=r'_i$ for all $i\in A$. Let $s\in\{1,\dots,f_\lambda\}$ be such that ${}_{s+1}|\lambda|_{f_\lambda}<r\leq{}_s|\lambda|_{f_\lambda}$. Consider

$$\boldsymbol{\mu}=\boldsymbol{\omega}\left(\prod_{i=s+1}^{f_\lambda}\prod_{t=1}^{\lambda(h_i)}A^{-1}_{i,n-1,r_i+2(\lambda(h_i)-t)}\right)\left(\prod_{t=1}^{r-{}_{s+1}|\lambda|_{f_\lambda}}A^{-1}_{s,n-1,r_s+2(\lambda(h_s)-t)}\right),$$
$$\boldsymbol{\mu}'=\boldsymbol{\omega}\left(\prod_{t=1}^{k}A^{-1}_{n-1,p,r'_{n-1}+2(\lambda(h_{n-1})-t)}\right).$$

Let $V:=V_q(\boldsymbol{\omega}^{\{1,\dots,f_\lambda\}})\otimes V_q(\boldsymbol{\omega}^{\{n-1\}})\otimes V_q(\boldsymbol{\omega}^{\{n\}})$ and $V':=V_q((\boldsymbol{\omega}')^{\{1,\dots,f_\lambda\}})\otimes V_q((\boldsymbol{\omega}')^{\{n-1\}})\otimes V_q((\boldsymbol{\omega}')^{\{n\}})$. Observe that $V_q(\boldsymbol{\omega})$ is the simple quotient of the submodule of V generated by the top weight space and similarly for V'. Notice that we have isomorphisms of $U_q(\mathfrak{g})$-modules:

$$V\cong V'\cong V_q(\lambda)\oplus\bigoplus_\nu V_q(\nu)^{\oplus t_\nu},\quad V_q(\boldsymbol{\omega})\cong V_q(\lambda)\oplus\bigoplus_\nu V_q(\nu)^{\oplus m_\nu}\quad\text{and}\quad V_q(\boldsymbol{\omega}')\cong V_q(\lambda)\oplus\bigoplus_\nu V_q(\nu)^{\oplus m'_\nu}$$

where the sums are over $\nu \in P^+$ such that $\nu < \lambda$ and $m_\nu, m'_\nu, t_\nu \in \mathbb{Z}_{\geq 0}$. By Proposition 3.5.1, $(\boldsymbol{\mu})_{I_n} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{I_n}))$, and $(\boldsymbol{\mu}')_{I_n} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'_{I_n}))$. Thus, by definition of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ and (2.6.2), we have $\boldsymbol{\mu} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}))$ and $\boldsymbol{\mu}' \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}'))$. Thus, it follows that

$$\nu \not\leq \mathrm{wt}(\boldsymbol{\mu}) \Rightarrow t_\nu = m_\nu \quad \text{and} \quad m_{\mathrm{wt}(\boldsymbol{\mu})} \leq t_{\mathrm{wt}(\boldsymbol{\mu})} - 1,$$

and similarly

$$\nu \not\leq \mathrm{wt}(\boldsymbol{\mu}') \Rightarrow t_\nu = m_\nu \quad \text{and} \quad m'_{\mathrm{wt}(\boldsymbol{\mu}')} \leq t_{\mathrm{wt}(\boldsymbol{\mu}')} - 1.$$

If either $s < p$ or $s = p$ with $p < f_\lambda$ and $r - {}_{p+1}|\lambda|_{f_\lambda} < k$, since $k < r$, it follows that

$$\mathrm{wt}(\boldsymbol{\mu}') \not\leq \mathrm{wt}(\boldsymbol{\mu}) \quad \text{and} \quad \mathrm{wt}(\boldsymbol{\mu}) \not\leq \mathrm{wt}(\boldsymbol{\mu}').$$

Let $\nu \in P^+$ be such that $\nu < \lambda$. If $\nu \not\leq \mathrm{wt}(\boldsymbol{\mu})$, then we also have $\nu \not\leq \mathrm{wt}(\boldsymbol{\mu}')$, then $m'_\nu = t_\nu = m_\nu$. Otherwise, if $\nu \leq \mathrm{wt}(\boldsymbol{\mu}')$, we have $m_{\mathrm{wt}(\boldsymbol{\mu}')} \leq t_{\mathrm{wt}(\boldsymbol{\mu}')} - 1 = m'_{\mathrm{wt}(\boldsymbol{\mu}')} - 1$ and if $\nu \leq \mathrm{wt}(\boldsymbol{\mu})$, we have $m'_{\mathrm{wt}(\boldsymbol{\mu})} \leq t_{\mathrm{wt}(\boldsymbol{\mu})} - 1 = m_{\mathrm{wt}(\boldsymbol{\mu})} - 1$. Hence $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\omega}')$ cannot be related.

If either $s > p$ or $s = p = f_\lambda$, since $k < r$ it follows

$$\mathrm{wt}(\boldsymbol{\mu}') > \mathrm{wt}(\boldsymbol{\mu}).$$

Let $\nu \in P^+$ be such that $\nu < \lambda$. If $\nu \not\leq \mathrm{wt}(\boldsymbol{\mu}')$, then we also have $\nu \not\leq \mathrm{wt}(\boldsymbol{\mu})$ and, hence, $m'_\nu = t_\nu = m_\nu$. Otherwise, if $\nu \leq \mathrm{wt}(\boldsymbol{\mu}')$, we have $m_{\mathrm{wt}(\boldsymbol{\mu}')} \leq t_{\mathrm{wt}(\boldsymbol{\mu}')} - 1 = m'_{\mathrm{wt}(\boldsymbol{\mu}')} - 1$. Hence $V_q(\boldsymbol{\omega}) \not\leq V_q(\boldsymbol{\omega}')$. The same occurs if $s = p < f_\lambda$ and $r - {}_{p+1}|\lambda|_{f_\lambda} \geq k$.

Let $\nu = \lambda - \sum_{i=f_\lambda}^{n-2} \alpha_i - \alpha_n$. By Proposition 2.7.4, $m_\nu(V_q(\boldsymbol{\omega}'_{\{f_\lambda,\dots,n-2,n\}})) > 0 = m_\nu(V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n-2,n\}}))$. Thus, by Lemma 2.6.1, $m_\nu(V_q(\boldsymbol{\omega}')) > 0 = m_\nu(V_q(\boldsymbol{\omega}))$. Now, it is not difficult to see, looking at $V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n-2,n\}})$ and $V_q(\boldsymbol{\omega}'_{\{f_\lambda,\dots,n-2,n\}})$, that if $\lambda \geq \eta > \nu$, then $m_\eta(V_q(\boldsymbol{\omega}')) = m_\eta(V_q(\boldsymbol{\omega}))$. Hence $V_q(\boldsymbol{\omega}') \not\leq V_q(\boldsymbol{\omega})$, completing the proof of part (b). □

5.3. Proof of Step 1

Without loss of generality, assume $V_q(\boldsymbol{\omega}_A)$ is an increasing minimal affinization. Given $J \subseteq I$, denote $U_J := U_q(\mathfrak{g}_J)$. The proof will be done by treating the following cases separately:

1. (5.1.5) is not satisfied for neither choices of the ordered pair (i, j);
2. (5.1.5) is satisfied for some choice of the ordered pair (i, j) and there exists $i \in \{n-1, n\}$ such that neither (5.1.1) nor (5.1.3) is satisfied;
3. (5.1.5) is satisfied for some choice of the ordered pair (i, j) and, given $i \in I$, either (5.1.1) or (5.1.3) is satisfied.

This is clearly a complete and non intersecting list of possibilities. We will show that $\boldsymbol{\omega}$ is of type (b) or (e) in case 1, of type (a) in case 2, and of type (c), (d) or (f) in case 3.

5.3.1. Case 1. Given $i \in \{n-1, n\}$, let $J_i = \{f_\lambda\} \cup I_\lambda \setminus \{i\}$. These define subdiagrams of type A and we regard f_λ as being the first node. Consider the following possibilities:

(i) There exists $i \in \{n-1, n\}$ such that $V_q(\boldsymbol{\omega}_{J_i})$ is not a minimal affinization;
(ii) $V_q(\boldsymbol{\omega}_{J_i})$ is a decreasing minimal affinization for $i = n-1, n$;
(iii) $V_q(\boldsymbol{\omega}_{J_i})$ is an increasing minimal affinization for $i = n-1, n$;
(iv) There exists a choice of ordered pair (i, j) such that $\{i, j\} = \{n-1, n\}$, $V_q(\boldsymbol{\omega}_{J_i})$ is an increasing minimal affinization and $V_q(\boldsymbol{\omega}_{J_j})$ is a decreasing minimal affinization.

Notice that, since we assuming $V_q(\boldsymbol{\omega}_A)$ is an increasing minimal affinization, possibility (iii) means $\boldsymbol{\omega}$ is of type (e) while (iv) means $\boldsymbol{\omega}$ is of type $(\mathrm{b})_j$. Moreover, if $\#(\mathrm{supp}(\lambda)\cap A) = 1$, Proposition 4.6.4 implies that (iii) does not determine a minimal affinization. It remains to check that (i) and (ii) contradict the minimality of $V_q(\boldsymbol{\omega})$. For (i), without loss of generality we assume that $V_q(\boldsymbol{\omega}_{J_n})$ is not a minimal affinization. Observe that for (ii), if $\#(\mathrm{supp}(\lambda)\cap A) > 1$, then $V_q(\boldsymbol{\omega}_{I_i})$ is not a minimal affinization for $i = n-1, n$. Under this assumption, we treat case (ii) together with (i) as follows. Comments for the case (ii) with $\#(\mathrm{supp}(\lambda)\cap A) = 1$ will be made afterwards.

Choose $r'_{n-1}, r'_n \in \mathbb{Z}$ such that, letting $\boldsymbol{\varpi}$ be defined by

$$\boldsymbol{\varpi}_A = \boldsymbol{\omega}_A \qquad \text{and} \quad \boldsymbol{\varpi}_i = Y_{i,r'_i,\lambda(h_i)} \quad \text{for} \quad i = n-1, n,$$

we have

(5.3.1) $\boldsymbol{\varpi}_{I_{n-1}}$ satisfies (5.1.3) with $(p,k) = (f_\lambda, 1)$ and $\boldsymbol{\varpi}_{I_n}$ satisfies (5.1.1) with $r = 1$.

Observe that $V_q(\boldsymbol{\varpi})$ is of type $(\mathrm{b})_n$. We will show that, both for (i) and (ii), $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

We have to show that for all $\mu \in P^+$, $\mu \le \lambda$, we have either $m_\mu(V_q(\boldsymbol{\varpi})) \le m_\mu(V_q(\boldsymbol{\omega}))$ or there exists $\mu' > \mu$ such that $m_{\mu'}(V_q(\boldsymbol{\varpi})) < m_{\mu'}(V_q(\boldsymbol{\omega}))$. Let

$$\nu = \lambda - \sum_{i=s}^{n-1} \alpha_i \quad \text{where} \quad s = \begin{cases} f_\lambda, & \text{for (i);} \\ \max\{i \in \mathrm{supp}(\lambda) : i < f_\lambda\}, & \text{for (ii).} \end{cases}$$

Let $J = \{i \in I : s \le i < n\}$. Then, Lemma 2.6.1 implies that

$$m_\nu(V_q(\boldsymbol{\varpi})) = m_{\nu_J}(V_q(\boldsymbol{\varpi}_J)) \quad \text{and} \quad m_{\nu_J}(V_q(\boldsymbol{\omega}_J)) = m_\nu(V_q(\boldsymbol{\omega})).$$

In case (i), $V_q(\boldsymbol{\varpi}_J)$ is a minimal affinization and hence, $m_{\nu_J}(V_q(\boldsymbol{\varpi}_J)) = 0$. On the other hand, an application of Proposition 2.7.4 to the subdiagram of type A determined by J gives

$$m_{\nu_J}(V_q(\boldsymbol{\omega}_J)) > 0.$$

Hence,

$$m_\nu(V_q(\boldsymbol{\varpi})) = 0 < m_\nu(V_q(\boldsymbol{\omega})). \tag{5.3.2}$$

In particular, $V_q(\boldsymbol{\varpi}) \not\cong_{U_q(\mathfrak{g})} V_q(\boldsymbol{\omega})$.

Clearly $m_\lambda(V_q(\boldsymbol{\omega})) = m_\lambda(V_q(\boldsymbol{\varpi})) = 1$. Let $\mu < \lambda$ and set $\eta = \lambda - \mu = \sum_{i\in I} s_i\alpha_i$, $s_i \in \mathbb{Z}_{\ge 0}$. If $m_\mu(V_q(\boldsymbol{\varpi})) = 0$ obviously $m_\mu(V_q(\boldsymbol{\varpi})) \le m_\mu(V_q(\boldsymbol{\omega}))$. Then, suppose μ is such that $m_\mu(V_q(\boldsymbol{\varpi})) > 0$. In the following, we show that it must occur $s_j \ne 0$ for $j = n-2, n$. If $s_n = 0$, then $\mu \in \lambda - Q^+_{I_n}$ and, since $V_q(\boldsymbol{\varpi}_{I_n})$ is a minimal affinization, by Lemma 2.6.1,

$$m_\mu(V_q(\boldsymbol{\varpi})) = m_{\mu_{I_n}}(V_q(\boldsymbol{\varpi}_{I_n})) = 0,$$

contradicting the choice of μ. Thus $s_n > 0$. Applying Lemma 2.6.2 with $J_1 = \{n\}$, $i_0 = n-2$, $J_2 = \{1, \dots, n-3, n-1\}$, we obtain $s_{n-2} > 0$. Similarly, for $j \in \{2, \dots, n-3\}$, considering $J_1 = \{1, \dots, j-1\}$, $i_0 = j$, $J_2 = \{j+1, \dots, n\}$, it follows that if $s_j = 0$, then $s_i = 0$ for all $i \in \{1, \dots, j-1\}$. Consider $j \in \{1, \dots, n-3\}$ the biggest index such that $s_j = 0$. Thus $s_{j+1}, \dots, s_{n-2}, s_n > 0$. Suppose $j < s$. In this case take $\mu' = \lambda - \sum_{i=s}^{n-1}\alpha_i$, then $\mu' \ge \mu$ and $m_{\mu'}(V_q(\boldsymbol{\varpi})) < m_{\mu'}(V_q(\boldsymbol{\omega}))$ by (5.3.2).

Observe that (5.3.1) implies that

$$r'_{n-1} - r'_n = 2(\lambda(h_n) + \lambda(h_{f_\lambda}) + n - f_\lambda - 1).$$

Since

$$\lambda(h_n)+\lambda(h_{f_\lambda})+n-f_\lambda-1=\lambda(h_n)+2l-m \Leftrightarrow m=2l-(n-f_\lambda-1)-\lambda(h_{f_\lambda}),$$

and

$$\lambda(h_n)+\lambda(h_{f_\lambda})+n-f_\lambda-1=m-2l-\lambda(h_{n-1}) \Leftrightarrow m=\lambda(h_n)+\lambda(h_{n-1})+\lambda(h_{f_\lambda})+n-f_\lambda-1+2l,$$

it follows that

$$\boldsymbol{\varpi}_{I_\lambda} \quad \text{does not satisfy} \quad (5.1.5). \tag{5.3.3}$$

Thus, if $j \geq f_\lambda$, by Lemma 2.6.1, $m_\mu(V_q(\boldsymbol{\varpi}))=m_{\mu_{\{j+1,\dots,n\}}}(V_q(\boldsymbol{\varpi}_{\{j+1,\dots,n\}}))$, and the same for $\boldsymbol{\omega}$. By assumption (1), (5.3.3) and (4.4.5), one has

$$V_q((\boldsymbol{\varpi})_{\{j+1,\dots,n\}}) \cong_{U_{\{j+1,\dots,n\}}} V_q(\lambda(h_{n-1})\omega_{n-1}) \otimes V_q(\lambda(h_n)\omega_n) \cong_{U_{\{j+1,\dots,n\}}} V_q(\boldsymbol{\omega}_{\{j+1,\dots,n\}}). \tag{5.3.4}$$

Then $m_\mu(V_q(\boldsymbol{\varpi}))=m_\mu(V_q(\boldsymbol{\omega}))$.

Suppose now that $\boldsymbol{\omega}$ satisfies (ii) and $s \leq j < f_\lambda$. If $s_{n-1}>0$, take $\mu'=\lambda-\sum_{i=f_\lambda}^n \alpha_i$. By Proposition 4.6.4 together with Lemma 2.6.1, $m_{\mu'}(V_q(\boldsymbol{\varpi}))=0<m_{\mu'}(V_q(\boldsymbol{\omega}))$. If $s_{n-1}=0$, then, by Lemma 2.6.1, $m_\mu(V_q(\boldsymbol{\varpi}))=m_{\mu_{\{j+1,\dots,n-1\}}}(V_q(\boldsymbol{\varpi}_{\{j+1,\dots,n-1\}}))=0$, contradicting the choice of μ.

If $\#(\mathrm{supp}(\lambda)\cap A)=1$, then it follows from Proposition 2.1.2 that case (ii) is equivalent to case (iii), which does not define a minimal affinization as aforementioned.

5.3.2. Case 2. Without loss of generality, assume that (5.1.1) and (5.1.3) are not satisfied with $i=n$. Consider the following subcases:

(i) $r_{f_\lambda}+2(\lambda(h_{f_\lambda})-1)+n-f_\lambda+1 \neq r_{n-1}$;

(ii) $r_{f_\lambda}+2(\lambda(h_{f_\lambda})-1)+n-f_\lambda+1 = r_{n-1}$ and $r_{n-1}+2(\lambda(h_{n-1})-1)+4 \neq r_n$;

(iii) $r_{f_\lambda}+2(\lambda(h_{f_\lambda})-1)+n-f_\lambda+1 = r_{n-1}$ and $r_{n-1}+2(\lambda(h_{n-1})-1)+4 = r_n$.

The conditions in (iii) imply that $\boldsymbol{\omega}$ is of type $(\mathrm{a})_{n-1}$. We will show that (i) and (ii) contradict the minimality of $V_q(\boldsymbol{\omega})$.

Choose $r'_{n-1}, r'_n \in \mathbb{Z}$ such that, letting $\boldsymbol{\varpi}$ be defined by

$$\boldsymbol{\varpi}_A=\boldsymbol{\omega}_A \qquad \text{and} \qquad \boldsymbol{\varpi}_i=Y_{i,r'_i,\lambda(h_i)} \quad \text{for} \quad i=n-1,n,$$

we have

$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.1) with } r=1 \quad \text{and} \quad \boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } j=n,\ (l,m)=(1,1). \tag{5.3.5}$$

Observe that $V_q(\boldsymbol{\varpi})$ is of type $(\mathrm{a})_{n-1}$ and $V_q(\boldsymbol{\varpi}_{I_n})$ is a minimal affinization. We will show that $V_q(\boldsymbol{\varpi})<V_q(\boldsymbol{\omega})$ with similar ideas to those used in Case 1.

Suppose $m_\mu(V_q(\boldsymbol{\varpi}))>0$, where $\mu=\lambda-\eta$ with $\eta \in \sum_{i\in I} s_i\alpha_i$, $s_i \in \mathbb{Z}_{\geq 0}$, and $\eta \neq 0$. We begin showing that $s_{n-2}, s_n>0$. If $s_n=0$, by Lemma 2.6.1, $m_\mu(V_q(\boldsymbol{\varpi}))=m_{\mu_{I_n}}(V_q(\boldsymbol{\varpi}_{I_n}))=0$, where the second equality, which contradicts the choice of μ, follows from the minimality of $V_q(\boldsymbol{\omega}_{I_n})$. Applying the Lemma 2.6.2 with $J_1=\{n\}$, $i_0=n-2$, and $J_2=\{1,\dots,n-3,n-1\}$, we get $s_{n-2}>0$. Similarly, for $j \in \{2,\dots,n-3\}$, considering $J_1=\{1,\dots,j-1\}$, $i_0=j$, $J_2=\{j+1,\dots,n\}$, it follows that

$$s_j=0 \quad \Rightarrow \quad s_i=0 \quad \text{for all} \quad i<j.$$

Next, we show that

$$s_{n-1}=0 \quad \Rightarrow \quad m_\mu(V_q(\boldsymbol{\varpi}))=m_\mu(V_q(\boldsymbol{\omega})).$$

For proving this, we begin observing that $\boldsymbol{\varpi}_{I_{n-1}}$ does not satisfy neither (5.1.1) nor (5.1.3). Indeed, (5.3.5) implies that $r_{f_\lambda} + 2(\lambda(h_{f_\lambda}) - 1) < r'_n + 2(\lambda(h_n) - 1)$. Hence $\boldsymbol{\varpi}_{I_{n-1}}$ does not satisfy (5.1.3). Also

$$r'_n - r_{f_\lambda} = 2(\lambda(h_{n-1}) + \lambda(h_{f_\lambda})) + n - f_\lambda + 1.$$

Since

$$2(\lambda(h_{n-1}) + \lambda(h_{f_\lambda})) + n - f_\lambda + 1 = 2(\lambda(h_{f_\lambda}) - r) + n - f_\lambda + 1 \Leftrightarrow r = -\lambda(h_{n-1}) < 0,$$

it follows that $\boldsymbol{\varpi}_{I_{n-1}}$ does not satisfy (5.1.1). Then, Proposition 3.5.1 applied to the subdiagram of type A determined by I_{n-1}, implies that we have isomorphisms of $U_q(\mathfrak{g}_{I_{n-1}})$-modules:

$$V_q(\boldsymbol{\varpi}_{I_{n-1}}) \cong V_q(\lambda^A_{I_{n-1}}) \otimes V_q(\lambda^{\{n\}}_{I_{n-1}}) \cong V_q(\boldsymbol{\omega}_{I_{n-1}}). \tag{5.3.6}$$

Therefore, $m_\mu(V_q(\boldsymbol{\varpi})) = m_\mu(V_q(\boldsymbol{\omega}))$, as claimed.

Henceforth we assume $s_{n-1} > 0$. Observe that $V_q(\boldsymbol{\varpi}_J)$ is a minimal affinization for $J = \{n-2, n-1, n\}$. Then, by Lemma 2.6.1,

$$m_{\mu'}(V_q(\boldsymbol{\varpi})) = m_{\mu'_J}(V_q(\boldsymbol{\varpi}_J)) = 0 \quad \text{for all dominant} \quad \mu' \in \lambda - Q_J,\ \mu' \neq \lambda.$$

Suppose first that $V_q(\boldsymbol{\omega}_J)$ is not a minimal affinization and let $\mu' = \lambda - (\alpha_{n-1} + \alpha_{n-2} + \alpha_n) > \mu$. Similarly, Lemma 2.6.1 implies that $m_{\mu'}(V_q(\boldsymbol{\omega})) = m_{\mu'_J}(V_q(\boldsymbol{\omega}_J))$, but Proposition 2.7.4 applied to the subalgebra $U_q(\tilde{\mathfrak{g}}_J)$ implies that $m_{\mu'}(V_q(\boldsymbol{\omega})) > 0$. Thus, from now on we assume $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization, which implies

$$V_q(\boldsymbol{\omega}_{I_\lambda}) \quad \text{is a minimal affinization.} \tag{5.3.7}$$

Let

$$j = \max\{i \in I : s_i = 0\},$$

if $\{i \in I : s_i = 0\} \neq \emptyset$ and $j = 0$ otherwise. If $j \geq f_\lambda$, (5.3.7) implies that there exists an isomorphism of $U_q(\mathfrak{g}_{I_\lambda})$-modules: $V_q(\boldsymbol{\varpi}_{I_\lambda}) \cong V_q(\boldsymbol{\omega}_{I_\lambda})$. Thus, by Lemma 2.6.1 with I_λ in place of J, $m_\mu(V_q(\boldsymbol{\varpi})) = m_\mu(V_q(\boldsymbol{\omega}))$.

Next, assume $\#(A \cap \mathrm{supp}(\lambda)) > 1$ and set $s = \max\{i \in \mathrm{supp}(\lambda) \cap A : i < f_\lambda\} \neq \emptyset$. Comments for the case $\#(\mathrm{supp}(\lambda) \cap A) = 1$ will be made afterwards. Suppose first that $j < s$. For subcase (i) there are two possibilities:

(a) $V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n-1\}})$ is a minimal affinization (and $V_q(\boldsymbol{\omega}_{\{s,\dots,n-1\}})$ is not);
(b) $V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n-1\}})$ is not a minimal affinization.

Set

$$\nu = \lambda - \sum_{i=s}^{n-1} \alpha_i \quad \text{for (a) and} \quad \nu = \lambda - \sum_{i=f_\lambda}^{n-1} \alpha_i \quad \text{for (b).}$$

Then, $\nu > \mu$ and Lemma 2.6.1 together with Proposition 2.7.4 gives

$$m_\nu(V_q(\boldsymbol{\varpi})) = 0 < m_\nu(V_q(\boldsymbol{\omega})), \tag{5.3.8}$$

contradicting the minimality of $V_q(\boldsymbol{\omega})$. In subcase (ii), since $V_q(\boldsymbol{\omega}_{\{n-1,n-2,n\}})$ is a minimal affinization, we have

$$r_n + 2(\lambda(h_n) - 1) + 4 = r_{n-1}. \tag{5.3.9}$$

Set ν as in case (b) above. This time (5.3.8) follows from Lemma 2.6.1 together with Proposition 4.6.4. In particular, $V_q(\boldsymbol{\varpi}) \not\cong_{U_q(\mathfrak{g})} V_q(\boldsymbol{\omega})$, implying that subcases (i) and (ii) does not define minimal affinizations.

Finally, suppose $s \le j < f_\lambda$. For subcase (i), we again consider the possibilities (a) and (b) above. Possibility (b) is dealt with exactly as before. In possibility (a) we must have

$$r_{n-1} + 2(\lambda(h_{n-1}) - 1) + n - f_\lambda + 1 = r_{f_\lambda}.$$

If $r_{n-1} + 2(\lambda(h_{n-1}) - 1) + 4 = r_n$, set ν as in (b) and observe that (5.3.8) follows from Lemma 2.6.1 together with Proposition 4.6.4. Otherwise, by the hypothesis of Case 2, (5.3.9) must hold. It then follows from Proposition 4.6.4 that, setting $J = \{f_\lambda, \dots, n\}$, we have an isomorphism of $U_q(\mathfrak{g}_J)$-modules $V_q(\boldsymbol{\varpi}_J) \cong V_q(\boldsymbol{\omega}_J)$. Thus, by Lemma 2.6.1, $m_\mu(V_q(\boldsymbol{\varpi})) = m_\mu(V_q(\boldsymbol{\omega}))$. In subcase (ii) the result follows similarly.

If $\#(\mathrm{supp}(\lambda) \cap A) = 1$, then in the case (a) of subcase (i):

- is a minimal affinization of type $(\mathrm{a})_{n-1}$ (second diagram) if
$$r_n + 2\lambda(h_n) + 4 = r_{n-1} + 2;$$
- by Proposition 2.1.2 is not a minimal affinization if
$$r_{n-1} + 2\lambda(h_{n-1}) + 4 = r_n + 2;$$
- is not a minimal affinization otherwise (is bigger than class $(\mathrm{a})_{n-1}$).

5.3.3. Case 3. Fix the ordered pair (i, j) and assume (5.1.5) is satisfied with this choice. We have the following possibilities:

(i) (5.1.3) is also satisfied for the same choice of (i, j) while (5.1.1) is satisfied with the opposite choice;
(ii) (5.1.1) is satisfied for both choices;
(iii) (5.1.3) is satisfied for both choices;
(iv) (5.1.1) is also satisfied for the same choice of (i, j) while (5.1.3) is satisfied with the opposite choice;

We begin showing that possibility (iv) cannot happen. Indeed, isolating r_i in (5.1.1), r_j in (5.1.3) and plugging their values in (5.1.1) we get

$$2(\lambda(h_i) + \lambda(h_j) + \lambda(h_{f_\lambda}) + 1 - r - m) + r_{f_\lambda} + n - f_\lambda + 4l + n - p = r_p + 2k. \tag{5.3.10}$$

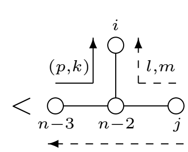
Using (3.3.2) to compute r_p in in terms of r_{f_λ} and plugging this back in (5.3.10) gives

$$k + m + r = n - p + 1 + 2l + \lambda(h_i) + \lambda(h_j) + \sum_{t=p}^{f_\lambda} \lambda(h_t),$$

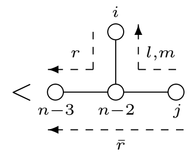
which is a contradiction since $n - p + 1 + 2l > 0$, $1 \le k \le \lambda(h_p)$, $1 \le r \le \lambda(h_i)$, and $1 \le m \le \lambda(h_j)$.

Performing similar computations for the other three possibilities we get the following conditions on the parameters:

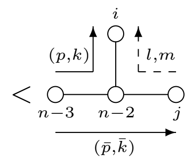
(i) $k + r + 2l - m = {}_p|\lambda|_{f_\lambda} + n - p + 1$



(ii) $r+m-\bar{r}=\lambda(h_i)+2l$.



(iii) ${}_{\bar{p}}|\lambda|_{p-1}+p-\bar{p}+m+k-\bar{k}=\lambda(h_i)+2l$.



We will show that we must have:

(i) $m=1$ and $p=f_\lambda$, with the indices satisfying the conditions stated in Theorem 5.1.1 for type $(\mathrm{d})_i^{k,l}$;
(ii) $\bar{r}=l=1$, showing that $\boldsymbol{\omega}$ is of type $(\mathrm{c})_i^r$;
(iii) $\boldsymbol{\omega}$ is of type $(\mathrm{f})_{i,l}^{(\bar{p},\bar{k}),(p,k)}$ with the indices satisfying the conditions stated in Theorem 5.1.1.

Henceforth, we assume, without loss of generality, that $i=n-1$ in (i), (ii), and (iii).

Possibility (i). We begin observing that the condition $k\le\lambda(h_p)$ implies that $r\ge 3$. Indeed, if $r\le 2$, we would have

$$k>k-m={}_p|\lambda|_{f_\lambda}+n-p+1-r-2l\ge{}_p|\lambda|_{f_\lambda}+n-p-1-2l\ge\lambda(h_p)$$

where the last inequality follows because $2l\le n-f_\lambda-1$ implies that $n-p-1-2l\ge 0$.

Next, we show that $m=1$. Suppose, by contradiction, that $m>1$ and choose $r'_n\in\mathbb{Z}$ such that, letting $\boldsymbol{\varpi}$ be defined by

$$\boldsymbol{\varpi}_{I_n}=\boldsymbol{\omega}_{I_n}\qquad\text{and}\qquad\boldsymbol{\varpi}_n=Y_{n,r'_n,\lambda(h_n)},$$

we have

$$\boldsymbol{\varpi}_{I_\lambda}\ \text{ satisfies }\ (5.1.5)\ \text{ with }\ i=n-1\ \text{ and }\ (l,1). \tag{5.3.11}$$

Hence, combining (5.1.3) with $i=n-1$ and (5.3.11), one gets

$$r_{f_\lambda}+2(\lambda(h_{f_\lambda})-1)+n-f_\lambda+1=r_n+2({}_p|\lambda|_{f_\lambda}+n-p+1-k-2l).$$

In other words, $\boldsymbol{\varpi}_{I_{n-1}}$ satisfies (5.1.1) with

$$r':={}_p|\lambda|_{f_\lambda}+n-p+1-k-2l+1$$

in place of r. Thus, if $m>1$, we have

$$r'<{}_p|\lambda|_{f_\lambda}+n-p+1-k-2l+m=r.$$

Lemma 5.2.1(a) then implies that $V_q(\boldsymbol{\varpi})<V_q(\boldsymbol{\omega})$, yielding a contradiction.

It remains to show that $p=f_\lambda$. Let $p'=\min\{i\in\operatorname{supp}(\lambda):i>p\}$. Suppose, by contradiction, that $p<f_\lambda$. Assume first that

$$l\ge\frac{1}{2}(p'-p+\lambda(h_p)-k+2)+1 \tag{5.3.12}$$

and

$$p'-p+\lambda(h_p)-k+2\quad\text{is odd.} \tag{5.3.13}$$

Let
$$r'_{n-1} = r_{n-1} - 2(p' - p + \lambda(h_p) - k + 1)$$
and let $\boldsymbol{\varpi}$ be defined by
$$\boldsymbol{\varpi}_{I_{n-1}} = \boldsymbol{\omega}_{I_{n-1}} \qquad \text{and} \quad \boldsymbol{\varpi}_{n-1} = Y_{n-1,r'_{n-1},\lambda(h_{n-1})}.$$
Using (5.1.3) and (3.3.2) with p and p', we have
$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - p' + 1 = r_{p'} + 2,$$
By (5.1.5),
$$r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2 \Rightarrow r_n + 2\lambda(h_n) + 4\left(l - \frac{1}{2}(p' - p + \lambda(h_p) - k + 1)\right) = r'_{n-1} + 2$$
Let $l' = \frac{1}{2}(p' - p + \lambda(h_p) - k + 1)$. Observe that
$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (p', 1) \text{ and } \boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l - l', 1).$$
Since $\boldsymbol{\varpi}_{I_{n-1}} = \boldsymbol{\omega}_{I_{n-1}}$, $p' > p$ and $l - l' < l$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. If $l \geq \frac{1}{2}(p' - p + \lambda(h_p) - k + 2) + 1$ and $p' - p + \lambda(h_p) - k + 2$ is even, let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and $\boldsymbol{\varpi}_{n-1} = Y_{i,r'_i,\lambda(h_i)}$ for $i = n-1, n$, where
$$r'_{n-1} = r_{n-1} - 2(p' - p + \lambda(h_p) - k + 1) \qquad \text{and} \qquad r'_n = r_n + 2.$$
Hence, using (3.3.2) with p and p',
$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - p' + 1 = r_{p'} + 2.$$
Also
$$r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2 \Rightarrow r'_n + 2\lambda(h_n) + 4\left(l - \frac{1}{2}(p' - p + \lambda(h_p) - k + 2)\right) = r'_{n-1} + 2,$$
and
$$r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r_n + 2r \Rightarrow r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r'_n + 2(r-1).$$
Let $l' = \frac{1}{2}(p' - p + \lambda(h_p) - k + 2)$. Since $l - l' \geq 1$ and $r > 1$, it follows that
$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.1) with } r - 1,$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (p', 1),$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l - l', 1).$$
Since $l - l' \leq l$, $p' > p$ and $r - 1 < r$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. Finally, suppose $l < \frac{1}{2}(p' - p + \lambda(h_p) - k + 2) + 1$. Observe that
$$k + r + 2l = {}_p|\lambda|_{f_\lambda} + n - p + 2 \Rightarrow n - p + \lambda(h_p) - k - 2l - r + 2 = -{}_{p+1}|\lambda|_{f_\lambda} < 0.$$
Hence,
$$r > n - p + \lambda(h_p) - k - 2l + 2 \geq p' - p + \lambda(h_p) - k - 2l + 4. \tag{5.3.14}$$
Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and $\boldsymbol{\varpi}_{n-1} = Y_{i,r'_i,\lambda(h_i)}$ for $i = n-1, n$, where
$$r'_{n-1} = r_{n-1} - 2(p' - p + \lambda(h_p) - k + 1) \qquad \text{and} \qquad r'_n = r_n + 2(p' - p + \lambda(h_p) - k - 2l + 3).$$
Hence, using (3.3.2) with p and p',
$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - p' + 1 = r_{p'} + 2.$$

Also

$$r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2 \Rightarrow r'_n + 2\lambda(h_n) + 4 = r'_{n-1} + 2,$$

and condition (5.1.1) with $i = n-1$ and r implies that

$$r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r'_{n-1} + 2(r - (p' - p + \lambda(h_p) - k - 2l + 3)).$$

Let $r' = p' - p + \lambda(h_p) - k - 2l + 3$. By (5.3.14), $r - r' \geq 1$. Thus it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.1) with } r - r',$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (p', 1),$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (1,1).$$

Since $r - r' < r$, $p' > p$ and $1 \leq l$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Moreover, we show that the following conditions hold:

(1) If $r \leq k$, then $\lambda(h_{n-1}) + 3 - r > \min\{l, \lambda(h_n)\}$.
(2) If $p \in \operatorname{supp}(\lambda) \cap A \setminus \{f_\lambda\}$, $0 \leq k' < k$ and $1 \leq l' \leq l$ are such that

$$1 \leq k := {}_p|\lambda|_{f_\lambda} + n - p + 2 - r - 2l - 2l' - k' - \lambda(h_{n-1}) \leq \min\{\lambda(h_p), \lambda(h_n)\},$$

then $r - {}_p|\lambda|_{f_\lambda} \leq k$.

Suppose that (1) fails. Then $r \leq k$ and $\lambda(h_{n-1}) + 3 - r \leq \min\{l, \lambda(h_n)\}$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\operatorname{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and for $i = n-1, n$, $\boldsymbol{\varpi}_i = Y_{i, r'_i, \lambda(h_i)}$ where

$$r'_{n-1} = r_{n-1} + 2(\lambda(h_{n-1}) + 2l - 1) \quad \text{and} \quad r'_n = r_n + 2(r-1).$$

Hence,

$$r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r_n + 2r \Rightarrow r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r'_n + 2,$$
$$r_{n-1} + 2\lambda(h_{n-1}) + 4l = r_n + 2 \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + 4 = r'_n + 2(\lambda(h_{n-1}) + 3 - r),$$

and

$$r_{n-1} + 2\lambda(h_{n-1}) + n - f_\lambda + 1 = r_{f_\lambda} + 2k \Rightarrow r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r'_{n-1} + 2(\lambda(h_{f_\lambda}) + n - f_\lambda + 2 - k - 2l).$$

Let $m = \lambda(h_{n-1}) + 3 - r$. Since $r = \lambda(h_{f_\lambda}) + n - f_\lambda + 2 - k - 2l$ and $r \leq k$, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.1) with } r,$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.1) with } 1$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n-1 \text{ and } (1, m).$$

Since $m \leq l$ and $r \leq k$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Now suppose that (2) fails. Then there are $p \in \operatorname{supp}(\lambda) \cap A \setminus \{f_\lambda\}$, $0 \leq k' < k$ and $1 \leq l' \leq l$ such that

$$1 \leq k'' := {}_p|\lambda|_{f_\lambda} + n - p + 2 - r - 2l - 2l' - k' - \lambda(h_{n-1}) \leq \min\{\lambda(h_p), \lambda(h_n)\},$$

and $r - {}_p|\lambda|_{f_\lambda} > k''$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\operatorname{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and for $i = n-1, n$, $\boldsymbol{\varpi}_i = Y_{i, r'_i, \lambda(h_i)}$ where

$$r'_{n-1} = r_{n-1} - 2k' \quad \text{and} \quad r'_n = r_n + 2(2 - 2l - 2l' - \lambda(h_{n-1}) - \lambda(h_n) - k').$$

Hence,

$$r_{n-1} + 2\lambda(h_{n-1}) + n - f_\lambda + 1 = r_{f_\lambda} + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - f_\lambda + 1 = r_{f_\lambda} + 2(k - k'),$$
$$r_{n-1} + 2\lambda(h_{n-1}) + 4l = r_n + 2 \Rightarrow r'_n + 2\lambda(h_n) + 4 = r'_n + 2,$$

and

$$r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r_n + 2r \Rightarrow r'_n + 2\lambda(h_n) + n - p + 1 = r_p + 2k''.$$

Thus it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (f_\lambda, k - k'),$$

$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (p, k'')$$

$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l', 1).$$

Since $k - k' \leq k$, $l' \leq l$ and $r - {}_p|\lambda|_{f_\lambda} > k''$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Possibility (ii). Observe that $r \geq 3$ since, otherwise, $m > m - \bar{r} = \lambda(h_{n-1}) + 2l - r \geq \lambda(h_{n-1})$. Also, $m \geq 3$ since, otherwise, $r > r - \bar{r} = \lambda(h_{n-1}) + 2l - m \geq \lambda(h_{n-1})$, implying by Lemma 4.4.2 that $l = 1$. Similarly one easily verifies that $m - \bar{r} \geq 2$, implying in particular that $m > \bar{r}$. Now we show that $\bar{r} = 1$. Suppose, by contradiction, $\bar{r} > 1$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in I_n$, $\boldsymbol{\varpi}_n = Y_{n,r'_n,\lambda(h_n)}$ and

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.1) with } r' = 1. \tag{5.3.15}$$

Hence, combining (5.1.5) with $i = n - 1$ and (5.3.5), one gets

$$r_{n-1} + 2\lambda(h_{n-1}) + 4 = r'_n + 2(m - \bar{r} + 1).$$

In other words, $\boldsymbol{\varpi}_{I_\lambda}$ satisfies (5.1.5) with $i = n - 1$, $l = 1$ and

$$m' := m - \bar{r} + 1 < m,$$

since $\bar{r} > 1$.

Observe that $V_q(\boldsymbol{\varpi})$ is of type $(\mathrm{c})^r_{n-1}$. We show that $[V_q(\boldsymbol{\varpi})] < [V_q(\boldsymbol{\omega})]$, which gives a contradiction. Evidently, $m_\mu(V_q(\boldsymbol{\varpi})) = m_\mu(V_q(\boldsymbol{\omega})) = 1$. Suppose $m_\mu(V_q(\boldsymbol{\varpi})) > 0$ where $\mu = \lambda - \eta$, $\eta = \sum_{i \in I} n_i \alpha_i$, $n_i \in \mathbb{Z}_{\geq 0}$, $\eta \neq 0$. If $n_n = 0$, since $\boldsymbol{\varpi}_{I_n} = \boldsymbol{\omega}_{I_n}$, it follows from Lemma 2.6.1 that

$$m_\mu(V_q(\boldsymbol{\varpi})) = m_{\mu_{I_n}}(V_q((\boldsymbol{\varpi})_{I_n})) = m_{\mu_{I_n}}(V_q(\boldsymbol{\omega}_{I_n})) = m_\mu(V_q(\boldsymbol{\omega})).$$

Suppose $n_n > 0$. Applying Lemma 2.6.2 with $J_1 = \{n\}$, $i_0 = n-2$ and $J_2 = \{1, \ldots, n-3, n-1\}$, it follows $n_{n-2} > 0$. Similarly, for $j \in \{2, \ldots, n-3\}$, considering $J_1 = \{1, \ldots, j-1\}$, $i_0 = j$, $J_2 = \{j+1, \ldots, n\}$, it follows that if $n_j = 0$, then $n_i = 0$ for all $i \in \{1, \ldots, j-1\}$. Consider $j \in \{1, \ldots, n-3\}$ the biggest index such that $n_j = 0$. Thus $n_{j+1}, \ldots, n_{n-2}, n_{n-1}, n_n > 0$. If $j \geq f_\lambda$, then it follows from Lemma 4.5.1(iii) that

$$[V_q(\boldsymbol{\varpi}_{\{j+1,\ldots,n\}})] < [V_q(\boldsymbol{\omega}_{\{j+1,\ldots,n\}})].$$

Suppose $j < f_\lambda$. It follows from Proposition 3.5.1 that if

$$\boldsymbol{\mu} = \boldsymbol{\varpi} A^{-1}_{f_\lambda, n-2, r_{f_\lambda}+2(\lambda(h_{f_\lambda})-1)} A^{-1}_{n, r_{f_\lambda}+2(\lambda(h_{f_\lambda})-1)+n-f_\lambda},$$

then $\boldsymbol{\mu}_{I_{n-1}} \notin \mathrm{wt}_\ell(V_q(\boldsymbol{\varpi}_{I_{n-1}}))$ and $\boldsymbol{\mu}_{I_{n-1}} \in \mathrm{wt}_\ell(V_q(\boldsymbol{\omega}_{I_{n-1}}))$. Also, it follows from the proof of Proposition 3.5.1 that, if $\nu = \mathrm{wt}(\boldsymbol{\mu}) = \lambda - \sum_{i=f_\lambda}^{n-2} \alpha_i - \alpha_n$, then $m_\nu(V_q(\boldsymbol{\varpi}_{I_{n-1}})) < m_\nu(V_q(\boldsymbol{\omega}_{I_{n-1}}))$. Therefore, by Lemma 2.6.1,

$$m_\nu(V_q(\boldsymbol{\varpi})) < m_\nu(V_q(\boldsymbol{\omega})).$$

Since $\nu \geq \mu$, the proof of case (ii) follows.

Possibility (iii). First observe, combining hypotheses (5.1.1) with $i = n - 1$, (5.1.3) with $i = n$ and (5.1.5) with $i = n$, that

$$p \geq \bar{p}.$$

Now we show that either k or $\bar{k}$ is equal to 1. Suppose, by contradiction $k, \bar{k} > 1$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, $\boldsymbol{\varpi}_i = Y_{i,r'_i,\lambda(h_i)}$ for $i = n-1, n$ where

$$r'_{n-1} = r_{n-1} - 2 \quad \text{ and } \quad r'_n = r_n - 2.$$

Thus,

$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2(k-1),$$

and

$$r_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_p + 2\bar{k} \Rightarrow r'_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_{\bar{p}} + 2(\bar{k}-1).$$

Since $k, \bar{k} > 1$, it follows that

$$\boldsymbol{\varpi}_{I_n} \text{ and } \boldsymbol{\varpi}_{I_{n-1}} \text{ satisfy (5.1.3) with } (p, k-1) \text{ and } (\bar{p}, \bar{k}-1), \text{ respectively.}$$

Also, one can easily checks that

$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l, m).$$

Therefore, by Lemma 5.2.1(a), $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Suppose $\bar{k} = 1$. If $l > 1$, which implies by Lemma 4.4.2 that $m = 1, 2$, we show that the following conditions hold.

(1) $k \le 2$. Besides, if $m = 2$, then $k = 1$.
(2) If $\bar{p} < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \le k' < \max\{k, m\}$ such that $\bar{p} < \bar{i} \le i \le f_\lambda$; $p \le i$ and $0 \le \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k'$ is even, then
$$l < \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k') + 1.$$
(3) If $p < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \le k' < \max\{k, m\}$ such that $p \le i < \bar{i} \le f_\lambda$ and $\bar{i} - i + {}_i|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_n)$ is even then

$$\text{either} \quad \frac{1}{2}(\bar{i} - i + {}_i|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_n)) < l \quad \text{or} \quad \frac{1}{2}(\bar{i} - i + {}_i|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_n)) > 2l.$$

Suppose that (1) fails. Thus $k > 2$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in I_{n-1}$, and $\boldsymbol{\varpi}_{n-1} = Y_{n-1, r'_{n-1}, \lambda(h_{n-1})}$ where $r'_{n-1} = r_{n-1} - 4$. Hence,

$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2(k-2),$$

and

$$r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2m \Rightarrow r_n + 2\lambda(h_n) + 4(l-1) = r'_{n-1} + 2m.$$

Since $l > 1$ and $k > 2$, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (p, k-2) \text{ and } \boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l-1, m).$$

Since $\boldsymbol{\varpi}_{I_{n-1}} = \boldsymbol{\omega}_{I_{n-1}}$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. Suppose now $m = 2$. If $k > 1$, letting $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in I_{n-1}$, and $\boldsymbol{\varpi}_{n-1} = Y_{n-1, r'_{n-1}, \lambda(h_{n-1})}$ where $r'_{n-1} = r_{n-1} - 2$, one can similarly checks that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (p, \bar{k}-1) \text{ and } \boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l-1, 1),$$

which implies $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$.

Suppose now that (2) fails. Then, there exist $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \le k' < \max\{k, m\}$ such that $\bar{p} < \bar{i} \le i \le f_\lambda$; $p \le i$, $0 \le \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k'$ is even and

$$l \ge \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k') + 1.$$

First consider $m = 1$, which implies $\max\{k, m\} = k = 1, 2$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and $\boldsymbol{\varpi}_i = Y_{i,r'_i,\lambda(h_i)}$ for $i = n-1, n$, where

$$r'_n = r_n + 2(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1}) \quad \text{and} \quad r'_{n-1} = r_{n-1} + 2(i - p + {}_p|\lambda|_{i-1} - k'). \tag{5.3.16}$$

Hence, using (3.3.2) with $\bar{p}$ and $\bar{i}$,

$$r_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_{\bar{p}} + 2 \Rightarrow r'_n + 2\lambda(h_n) + n - \bar{i} + 1 = r_{\bar{i}} + 2. \tag{5.3.17}$$

Also, (3.3.2) with p and i,

$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - i + 1 = r_i + 2(k - k'). \tag{5.3.18}$$

Finally, condition $r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2$ implies that

$$r'_n + 2\lambda(h_n) + 4\left(l - \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k')\right) = r'_{n-1} + 2.$$

Let $l' = l - \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k')$. By the conditions on the parameters, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (\bar{i}, 1),$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (i, k - k'),$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l', 1).$$

Since $\bar{i} > \bar{p}$, $i \geq p$, $k - k' \leq k$ and $l' \leq l$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. Consider now $m = 2$, which implies $k = 1$, $\max\{0, k - \lambda(h_i)\} = 0$ and $\max\{k, m\} = 2$. If $k' = 0$, letting $\boldsymbol{\varpi}$ be defined as above one gets similarly a contradiction. If $k' = 1$, let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and $\boldsymbol{\varpi}_i = Y_{i,r'_i,\lambda(h_i)}$ for $i = n-1, n$, where

$$r'_n = r_n + 2(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1}) \quad \text{and} \quad r'_{n-1} = r_{n-1} + 2(i - p + {}_p|\lambda|_{i-1}). \tag{5.3.19}$$

Hence, using (3.3.2) with $\bar{p}$ and $\bar{i}$,

$$r_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_{\bar{p}} + 2 \Rightarrow r'_n + 2\lambda(h_n) + n - \bar{i} + 1 = r_{\bar{i}} + 2. \tag{5.3.20}$$

Also, (3.3.2) with p and i,

$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2 \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - i + 1 = r_i + 2. \tag{5.3.21}$$

Finally, condition $r_n + 2\lambda(h_n) + 4l = r_{n-1} + 4$ implies that

$$r'_n + 2\lambda(h_n) + 4\left(l - \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + 1)\right) = r'_{n-1} + 2.$$

Let $l' = l - \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + 1)$. By the conditions on the parameters, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (\bar{i}, 1),$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (i, 1),$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l', 1).$$

Since $\bar{i} > \bar{p}$, $i \geq p$, $l' \leq l$ and $1 < 2 = m$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Suppose now that (3) fails. Then, there exist $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < \max\{k, m\}$ such that $p \leq i < \bar{i} \leq f_\lambda$, $\bar{i} - i + {}_i|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_n)$ is even and

$$0 \leq -l + \frac{1}{2}(\bar{i} - i + {}_i|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_n)) \leq l.$$

First consider $m = 1$, which implies $\max\{k, m\} = k = 1, 2$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be defined as in (5.3.16). Then, (5.3.17) and (5.3.18) holds. Finally, condition $r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2$ implies that

$$r'_{n-1} + 2\lambda(h_{n-1}) + 4\left(-l + \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} + 2 - i + p - {}_{p}|\lambda|_{i-1} + k' - \lambda(h_{n-1}) - \lambda(h_n))\right) = r'_n + 2.$$

Let $l' = -l + \frac{1}{2}(\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} + 2 - i + p - {}_{p}|\lambda|_{i-1} + k' - \lambda(h_{n-1}) - \lambda(h_n))$. Observe that, condition

$$(5.3.22) \qquad {}_{\bar{p}}|\lambda|_{p-1} + p - \bar{p} + m + k = \lambda(h_{n-1}) + 2l + 1$$

implies that

$$l' = -l + \frac{1}{2}(\bar{i} - i + {}_{i}|\lambda|_{\bar{i}-1} + 2l - k + k' + 2 - \lambda(h_n)).$$

By the conditions on the parameters, it follows that

$\boldsymbol{\varpi}_{I_{n-1}}$ satisfies (5.1.3) with $(\bar{i}, 1)$,

$\boldsymbol{\varpi}_{I_n}$ satisfies (5.1.3) with $(i, k - k')$,

$\boldsymbol{\varpi}_{I_\lambda}$ satisfies (5.1.5) with $i = n-1$ and $(l', 1)$.

Since $\bar{i} > p \geq \bar{p}$, $i \geq p$, $l' \leq l$ and $k - k' \leq k$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. If $m = 2$ similarly one can get a contradiction.

If $l = 1$, we first show that the following condition holds.

(1) $k < \lambda(h_{f_\lambda}) + p - \bar{p} + n - f_\lambda - 1$.
(2) If $p < f_\lambda$, given $i, \bar{i} \in \operatorname{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < k$ such that $\bar{p} \leq \bar{i} \leq i \leq f_\lambda$; $p < i$ and $\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} \leq i - p + {}_{p}|\lambda|_{i-1} - k'$, then
$$m < i - p + {}_{p}|\lambda|_{i-1} - k' - \bar{i} + \bar{p} - {}_{\bar{p}}|\lambda|_{\bar{i}-1} + 1.$$
(3) If $p < f_\lambda$, given $i, \bar{i} \in \operatorname{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \leq k' < k$ such that $p \leq i < \bar{i} \leq f_\lambda$ then
$$\text{either} \quad \lambda(h_n) + 1 + k - k' - \bar{i} + i - {}_{i}|\lambda|_{\bar{i}-1} < 1 \quad \text{or} \quad \lambda(h_n) + 1 + k - k' - \bar{i} + i - {}_{i}|\lambda|_{\bar{i}-1} > m.$$

Suppose that (1) fails. Thus,

$$k \geq \lambda(h_{f_\lambda}) + p - \bar{p} + n - f_\lambda - 1.$$

Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, $\boldsymbol{\varpi}_i = Y_{i, r'_i, \lambda(h_i)}$ for $i = n-1, n$ where

$$r'_{n-1} = r_{n-1} + 2({}_{\bar{p}}|\lambda|_{f_\lambda - 1} + f_\lambda - p - 1 + m) \quad \text{and} \quad r'_n = r_n + 2({}_{\bar{p}}|\lambda|_{f_\lambda - 1} + f_\lambda - p).$$

Hence, using (3.3.2) with $\bar{p}$ and f_λ,

$$r_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_{\bar{p}} + 2 \Rightarrow r'_n + 2\lambda(h_n) + n - f_\lambda + 1 = r_{f_\lambda} + 2.$$

Also, (3.3.2) with p and f_λ, condition $r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2k$ implies that

$$r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 = r'_n + 2({}_{p}|\lambda|_{f_\lambda} + n - f_\lambda + 2 + \lambda(h_{n-1}) - k - {}_{\bar{p}}|\lambda|_{f_\lambda - 1} - m).$$

Let $r := {}_{p}|\lambda|_{f_\lambda} + n - f_\lambda + 2 + \lambda(h_{n-1}) - k - {}_{\bar{p}}|\lambda|_{f_\lambda - 1} - m$. Condition described in (iii) (with $\bar{k} = l = 1$) implies that

$$r = \lambda(f_\lambda) + n - f_\lambda - 1.$$

Finally, condition $r_n + 2\lambda(h_n) + 4 = r_{n-1} + 2$ implies that

$$r'_n + 2\lambda(h_n) + 4 = r'_{n-1} + 2.$$

Since $1 \leq r \leq k$, it follows that

$\boldsymbol{\varpi}_{I_{n-1}}$ satisfies (5.1.1) with r,

$\boldsymbol{\varpi}_{I_n}$ satisfies (5.1.3) with $(f_\lambda, 1)$,

$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (1,1).$$

Since $f_\lambda > \bar{p}$, $r \le k$ and $m \ge 1$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Suppose that (2) fails. Thus, there exist $i, \bar{i} \in \operatorname{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \le k' < k$ such that

$$m \ge i - p + {}_p|\lambda|_{i-1} - k' - \bar{i} + \bar{p} - {}_{\bar{p}}|\lambda|_{\bar{i}-1} + 1.$$

Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be defined as in (5.3.16). Thus, one can see that (5.3.17) and (5.3.18) hold. Condition $r_n + 2\lambda(h_n) + 4 = r_{n-1} + 2m$ implies that

$$r'_n + 2\lambda(h_n) + 4 = r'_{n-1} + 2(m - (i - p + {}_p|\lambda|_{i-1} - k' - \bar{i} + \bar{p} - {}_{\bar{p}}|\lambda|_{\bar{i}-1})).$$

Let $m' = m - (i - p + {}_p|\lambda|_{i-1} - k' - \bar{i} + \bar{p} - {}_{\bar{p}}|\lambda|_{\bar{i}-1})$. By the conditions on the parameters, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (\bar{i}, 1),$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (i, k - k'),$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (1, m').$$

Since $i > p$, $\bar{i} \ge \bar{p}$, $k - k' \le k$ and $m' \le m$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$. Now, suppose $p = \bar{p}$. Thus, from the condition given in (iii), we have

$$k + m = \lambda(h_{n-1}) + 3. \tag{5.3.23}$$

Hence $k \ge 3$, since otherwise $m = \lambda(h_{n-1}) + 3 - k > \lambda(h_{n-1})$. Similarly, $m \ge 3$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\operatorname{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and $\boldsymbol{\varpi}_i = Y_{i, r'_i, \lambda(h_i)}$ for $i = n-1, n$ and

$$r'_{n-1} = r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda + 1 - 2k \quad \text{and} \quad r'_n = r_{f_\lambda} + 2\lambda(h_{f_\lambda}) + n - f_\lambda - 1. \tag{5.3.24}$$

If follows from (5.3.24) that

$$r'_{n-1} + 2\lambda(h_{n-1}) + 4 = r'_n + 2(\lambda(h_{n-1}) - k + 3) \overset{(5.3.23)}{=} r'_n + 2m. \tag{5.3.25}$$

Since $1 \le k \le \min\{\lambda(h_p), \lambda(h_{n-1})\} \le \min\{|\lambda|_{n-2}, \lambda(h_{n-1})\}$, it follows from (5.3.24) and (5.3.25) that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.1) with } k,$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.1) with } 1,$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n-1 \text{ and } (1, m).$$

In other words $\boldsymbol{\varpi}$ is of type $(c)^k_{n-1}$. By Lemma 5.2.1(a), $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, yielding a contradiction.

Suppose now that (3) fails. Thus, there exist $i, \bar{i} \in \operatorname{supp}(\lambda)$ and $\max\{0, k - \lambda(h_i)\} \le k' < k$ such that

$$1 \le \lambda(h_n) + 1 + k - k' - \bar{i} + i - {}_i|\lambda|_{\bar{i}-1} \le m.$$

Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be defined as in (5.3.16). Thus, one can see that (5.3.17) and (5.3.18) hold. Condition $r_n + 2\lambda(h_n) + 4 = r_{n-1} + 2m$ implies that

$$r'_{n-1} + 2\lambda(h_{n-1}) + 4 = r'_n + 2(\lambda(h_n) + \lambda(h_{n-1}) + 4 - m + i - p + {}_p|\lambda|_{i-1} - k' - (\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1})).$$

Let $m' = \lambda(h_n) + \lambda(h_{n-1}) + 4 - m + i - p + {}_p|\lambda|_{i-1} - k' - (\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1})$. It follows from (5.3.22) that

$$m' = \lambda(h_n) + 1 + k - k' - \bar{i} + i - {}_i|\lambda|_{\bar{i}-1}.$$

By the conditions on the parameters, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (\bar{i}, 1),$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (i, k - k'),$$

$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i=n-1 \text{ and } (1,m').$$

Since $i \geq p \geq$, $\bar{i} > p \geq \bar{p}$, $k-k' \leq k$ and $m' \leq m$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Suppose now $\bar{k} > 1$, which implies $k=1$. We show that $m=1$. Indeed, suppose by contradiction $m>1$. Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in I_n$, and $\boldsymbol{\varpi}_n = Y_{n,r'_n,\lambda(h_n)}$ where $r'_n = r_n - 2$. Thus,

$$r_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_{\bar{p}} + 2\bar{k} \Rightarrow r'_n + 2\lambda(h_n) + n - \bar{p} + 1 = r_{\bar{p}} + 2(\bar{k}-1),$$

and

$$r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2m \Rightarrow r'_n + 2\lambda(h_n) + 4l = r_{n-1} + 2(m-1).$$

Since $m, \bar{k} > 1$, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (\bar{p}, \bar{k}-1) \text{ and } \boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i=n \text{ and } (l, m-1).$$

Since $\boldsymbol{\varpi}_{I_{n-1}} = \boldsymbol{\omega}_{I_{n-1}}$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Besides, the following condition hold. If $\bar{p} < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $0 \leq l' < l$ such that $\bar{p} < \bar{i} \leq i \leq f_\lambda$; $p \leq i$ with either $l' > 0$ or $p < i$, and $i - p + {}_p|\lambda|_{i-1} + 2l' \leq \bar{i} - \bar{p} + {}_{\bar{i}}|\lambda|_{\bar{p}-1}$, then

$$\text{either} \quad \bar{k} < \bar{i} - \bar{p} + {}_{\bar{i}}|\lambda|_{\bar{p}-1} - i + p - {}_p|\lambda|_{i-1} - 2l' + 1 \quad \text{or} \quad \bar{k} > \bar{i} - \bar{p} + {}_{\bar{i}}|\lambda|_{\bar{p}-1} - i + p - {}_p|\lambda|_{i-1} - 2l' + \lambda(h_{\bar{i}}).$$

Suppose, by contradiction, that this is not the case. Thus, there exist $i, \bar{i}, l'$, under the above conditions such that

$$\bar{i} - \bar{p} + {}_{\bar{i}}|\lambda|_{\bar{p}-1} - i + p - {}_p|\lambda|_{i-1} - 2l' + 1 \leq \bar{k} \leq \bar{i} - \bar{p} + {}_{\bar{i}}|\lambda|_{\bar{p}-1} - i + p - {}_p|\lambda|_{i-1} - 2l' + \lambda(h_{\bar{i}}).$$

Let $\boldsymbol{\varpi} \in \mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi}) = \lambda$, $\boldsymbol{\varpi}_i = \boldsymbol{\omega}_i$ for all $i \in A$, and $\boldsymbol{\varpi}_i = Y_{i,r'_i,\lambda(h_i)}$ for $i = n-1, n$, where

$$r'_{n-1} = r_{n-1} + 2(i - p + {}_p|\lambda|_{i-1}) \quad \text{and} \quad r'_n = r_n + 2(i - p + {}_p|\lambda|_{i-1} + 2l'). \tag{5.3.26}$$

Hence, using (3.3.2) with p and i,

$$r_{n-1} + 2\lambda(h_{n-1}) + n - p + 1 = r_p + 2 \Rightarrow r'_{n-1} + 2\lambda(h_{n-1}) + n - i + 1 = r_i + 2. \tag{5.3.27}$$

Also,

$$r_n + 2\lambda(h_n) + 4l = r_{n-1} + 2 \Rightarrow r'_n + 2\lambda(h_n) + 4(l - l') = r'_{n-1} + 2.$$

Finally, using (3.3.2) with $\bar{p}$ and $\bar{i}$, condition (5.1.3) (with $i = n$ and $(\bar{p}, \bar{k})$) implies that

$$r'_n + 2\lambda(h_n) + n - \bar{i} + 1 = r_{\bar{i}} + 2(\bar{k} - (\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} - 2l')). \tag{5.3.28}$$

Let $\bar{k}' = \bar{k} - (\bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} - 2l')$. By the conditions on the parameters, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}} \text{ satisfies (5.1.3) with } (\bar{i}, \bar{k}'),$$
$$\boldsymbol{\varpi}_{I_n} \text{ satisfies (5.1.3) with } (i, 1),$$
$$\boldsymbol{\varpi}_{I_\lambda} \text{ satisfies (5.1.5) with } i = n \text{ and } (l - l', 1).$$

Since $\bar{i} > \bar{p}$, $i \geq p$, $l - l' \leq l$ and $\bar{k}' \leq \bar{k}$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

Moreover we show that the following condition holds. If $\bar{p} < f_\lambda$, given $i, \bar{i} \in \mathrm{supp}(\lambda)$ and $\max\{0, \bar{k} - \lambda(h_{\bar{i}})\} \leq \bar{k}' < \bar{k}$ such that $p \leq i \leq \bar{i} \leq f_\lambda$, $\bar{i} > \bar{p}$ and $-\lambda(h_n) + \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k'$ is even, then

$$\text{either} \quad l > \frac{1}{2}(-\lambda(h_n) + \bar{i} - \bar{p} + {}_{\bar{p}}|\lambda|_{\bar{i}-1} - i + p - {}_p|\lambda|_{i-1} + k')$$

$$\text{or}\quad \frac{1}{2}(-\lambda(h_n)+\bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-i+p-{}_{p}|\lambda|_{i-1}+k')>2l-1.$$

Suppose, by contradiction, that this is not the case. Thus, there exist $i,\bar{i},\bar{k}'$, under the above conditions such that

$$1\le 1-l+\frac{1}{2}(-\lambda(h_n)+\bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-i+p-{}_{p}|\lambda|_{i-1}+k')\le l.$$

Let $\boldsymbol{\varpi}\in\mathcal{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\varpi})=\lambda$, $\boldsymbol{\varpi}_i=\boldsymbol{\omega}_i$ for all $i\in A$, and $\boldsymbol{\varpi}_i=Y_{i,r_i',\lambda(h_i)}$ for $i=n-1,n$, where

$$r_n'=r_n+2(\bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-\bar{k}')\quad\text{and}\quad r_{n-1}'=r_{n-1}+2(i-p+{}_{p}|\lambda|_{i-1}).$$

Hence, using (3.3.2) with $\bar{p}$ and $\bar{i}$,

$$r_n+2\lambda(h_n)+n-\bar{p}+1=r_{\bar{p}}+2\Rightarrow r_n'+2\lambda(h_n)+n-\bar{i}+1=r_{\bar{i}}+2(\bar{k}-\bar{k}'). \tag{5.3.29}$$

Also, (3.3.2) with p and i,

$$r_{n-1}+2\lambda(h_{n-1})+n-p+1=r_p+2k\Rightarrow r_{n-1}'+2\lambda(h_{n-1})+n-i+1=r_i+2. \tag{5.3.30}$$

Finally, condition $r_n+2\lambda(h_n)+4l=r_{n-1}+2$ implies that

$$r_{n-1}'+2\lambda(h_{n-1})+4\left(1-l+\frac{1}{2}(-\lambda(h_n)+\bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-i+p-{}_{p}|\lambda|_{i-1}+k')\right)=r_n'+2.$$

Let $l'=1-l+\frac{1}{2}(-\lambda(h_n)+\bar{i}-\bar{p}+{}_{\bar{p}}|\lambda|_{\bar{i}-1}-i+p-{}_{p}|\lambda|_{i-1}+k')$. By the conditions on the parameters, it follows that

$$\boldsymbol{\varpi}_{I_{n-1}}\text{ satisfies (5.1.3) with }(\bar{i},\bar{k}-\bar{k}'),$$
$$\boldsymbol{\varpi}_{I_n}\text{ satisfies (5.1.3) with }(i,1),$$
$$\boldsymbol{\varpi}_{I_\lambda}\text{ satisfies (5.1.5) with }i=n-1\text{ and }(l',1).$$

Since $\bar{i}>\bar{p}$, $i\ge p$, $l'\le l$ and $\bar{k}-\bar{k}'\le\bar{k}$, it follows from Lemma 5.2.1(a) that $V_q(\boldsymbol{\varpi})<V_q(\boldsymbol{\omega})$, contradicting the minimality of $V_q(\boldsymbol{\omega})$.

5.4. Proof of Step 2

We show that a minimal affinization $V_q(\boldsymbol{\omega})$ of type (a)-(f) cannot be related to any minimal affinization $V_q(\boldsymbol{\varpi})$ of the same type and some different parameters. Let $\{j,j'\}=\{n-1,n\}$.

◇ Let $\boldsymbol{\omega}$ be of type $(\mathrm{a})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{a})_{j'}$. Then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_{j'}})<V_q(\boldsymbol{\varpi}_{I_{j'}})$ and $V_q(\boldsymbol{\omega}_{I_j})>V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

◇ Let $\boldsymbol{\omega}$ be of type $(\mathrm{b})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{b})_{j'}$. Then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j})<V_q(\boldsymbol{\varpi}_{I_j})$ and $V_q(\boldsymbol{\omega}_{I_{j'}})>V_q(\boldsymbol{\varpi}_{I_{j'}})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

◇ Let $\boldsymbol{\omega}$ be of type $(\mathrm{c})_j^r$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{c})_{j'}^{r'}$. In Case 3 we showed that $r\ge 3$ in type $(\mathrm{c})_j^r$, then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j})<V_q(\boldsymbol{\varpi}_{I_j})$ and $V_q(\boldsymbol{\omega}_{I_{j'}})>V_q(\boldsymbol{\varpi}_{I_{j'}})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related. If $\boldsymbol{\omega}$ be of type $(\mathrm{c})_j^r$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{c})_j^{r'}$, $r\ne r'$, suppose $r<r'$. Since

$$r+m=\lambda(h_j)+3=r'+m',$$

it follows that $m>m'$. Then, $V_q(\boldsymbol{\omega}_{I_{j'}})<V_q(\boldsymbol{\varpi}_{I_{j'}})$ and $V_q(\boldsymbol{\omega}_{I_\lambda})>V_q(\boldsymbol{\varpi}_{I_\lambda})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ also cannot be related.

◇ Let $\boldsymbol{\omega}$ be of type $(\mathrm{d})_j^{k,l}$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{d})_{j'}^{k',l'}$. By condition described in this type:

$$r+k+2l=\lambda(h_{f_\lambda})+n-f_\lambda+2=r'+k'+2l'.$$

Suppose, without loss of generality, $2l' \leq 2l$. Thus $r'-k \geq r-k'$. If $r \leq k'$, by Proposition 3.6.1(iv) $V_q(\boldsymbol{\omega}_{I_j}) < V_q(\boldsymbol{\varpi}_{I_j})$. If $2l' < 2l$, then $V_q(\boldsymbol{\omega}_{I_\lambda}) > V_q(\boldsymbol{\varpi}_{I_\lambda})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related. If $2l' = 2l$, then $r' \leq k$, and, by Proposition 3.6.1(iv), $V_q(\boldsymbol{\omega}_{I_{j'}}) > V_q(\boldsymbol{\varpi}_{I_{j'}})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ also cannot be related. Finally, suppose $k' < r$. Then $r'-k \geq r-k' > 0$. Hence $r' > k$ and again by Proposition 3.6.1(iv), $V_q(\boldsymbol{\omega}_{I_{j'}}) < V_q(\boldsymbol{\varpi}_{I_{j'}})$, and studying cases $2l' < 2l$ and $2l' = 2l$ one gets that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related. If $\boldsymbol{\omega}$ be of type $(\mathrm{d})_j^{k,l}$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{d})_j^{k',l'}$ it is another application of Proposition 3.6.1(iv) to verify that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{f})_{j,l}^{(\bar{p},\bar{k}),(p,k)}$. One can see in the proof of Case 3 that the conditions described in this type are exactly that ones for not to exist $\boldsymbol{\varpi}$ of type $(\mathrm{f})_{j,l'}^{(\bar{p}',\bar{k}'),(p',k')}$ or $(\mathrm{f})_{j',l'}^{(\bar{p}',\bar{k}'),(p',k')}$ such that $V_q(\boldsymbol{\varpi}) < V_q(\boldsymbol{\omega})$.

5.5. Proof of Step 3

We show that a minimal affinization $V_q(\boldsymbol{\omega})$ of type (a)-(f) cannot be related to any minimal affinization $V_q(\boldsymbol{\varpi})$ of other type (a)-(f). Let $\{j, j'\} = \{n-1, n\}$.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{a})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{b})_j$. Then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_\lambda}) < V_q(\boldsymbol{\varpi}_{I_\lambda})$ and $V_q(\boldsymbol{\omega}_{I_j}) > V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{a})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{c})_j^r$. In Case 3 we showed that $r \geq 3$ in type $(\mathrm{c})_j^r$, then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_{j'}}) < V_q(\boldsymbol{\varpi}_{I_{j'}})$ and $V_q(\boldsymbol{\omega}_{I_j}) > V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{a})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{d})_j^{k,l}$. Then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_{j'}}) < V_q(\boldsymbol{\varpi}_{I_{j'}})$ and $V_q(\boldsymbol{\omega}_{I_j}) > V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{a})_j$ and $\boldsymbol{\varpi}$ be of type (e). Then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_\lambda}) < V_q(\boldsymbol{\varpi}_{I_\lambda})$ and $V_q(\boldsymbol{\omega}_{I_j}) > V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{a})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{f})_{j,l}^{(\bar{p},\bar{k}),(p,k)}$. Then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_{j'}}) < V_q(\boldsymbol{\varpi}_{I_{j'}})$ and $V_q(\boldsymbol{\omega}_{I_j}) > V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{b})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{c})_j^r$. In Case 3 we showed that $r \geq 3$ in type $(\mathrm{c})_j^r$, then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n-2,l\}}) < V_q(\boldsymbol{\varpi}_{\{f_\lambda,\dots,n-2,l\}})$ and $V_q(\boldsymbol{\omega}_{I_\lambda}) > V_q(\boldsymbol{\varpi}_{I_\lambda})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{b})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{d})_j^{k,l}$. In Case 3 we showed that $r \geq 3$ in type $(\mathrm{d})_j^{k,l}$, then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j}) < V_q(\boldsymbol{\varpi}_{I_j})$ and $V_q(\boldsymbol{\omega}_{I_\lambda}) > V_q(\boldsymbol{\varpi}_{I_\lambda})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{b})_j$ and $\boldsymbol{\varpi}$ be of type (e). By Proposition 4.6.4(a), $V_q(\boldsymbol{\omega}_{\{f_\lambda,\dots,n\}}) < V_q(\boldsymbol{\varpi}_{\{f_\lambda,\dots,n\}})$ and, since $\#(\mathrm{supp}(\lambda) \cap A) > 1$, $V_q(\boldsymbol{\omega}_{I_{j'}}) > V_q(\boldsymbol{\varpi}_{I_{j'}})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{b})_j$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{f})_{j,l}^{(\bar{p},\bar{k}),(p,k)}$. Then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j}) < V_q(\boldsymbol{\varpi}_{I_j})$ and $V_q(\boldsymbol{\omega}_{I_\lambda}) > V_q(\boldsymbol{\varpi}_{I_\lambda})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(\mathrm{c})_j^r$ and $\boldsymbol{\varpi}$ be of type $(\mathrm{d})_j^{k,l}$. In Case 3 we showed that $r \geq 3$ in type $(\mathrm{d})_j^{k,l}$, then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j}) < V_q(\boldsymbol{\varpi}_{I_j})$. Now one can see that, if $V_q(\boldsymbol{\omega}) < V_q(\boldsymbol{\varpi})$, then condition (1) described in the possibility (i) of Case 3 is not satisfied.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(c)_j^r$ and $\boldsymbol{\varpi}$ be of type (e). Then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_\lambda}) < V_q(\boldsymbol{\varpi}_{I_\lambda})$. In Case 3 we showed that $r \geq 3$ in type $(c)_j^r$, then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_{j'}}) > V_q(\boldsymbol{\varpi}_{I_{j'}})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(c)_j^r$ and $\boldsymbol{\varpi}$ be of type $(f)_{j,l}^{(\bar{p},\bar{k}),(p,k)}$. If $\boldsymbol{\varpi}$ satisfies (i) or (ii) of $(f)_{j,l}^{(\bar{p},\bar{k}),(p,k)}$, then $k \leq 2$. In Case 3 we showed that $r \geq 3$ in type $(c)_j^r$, thus $k < r$, and by Lemma 5.2.1(b), $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ are not related. Suppose now that $\boldsymbol{\varpi}$ satisfies (iii) of $(f)_{j,l}^{(\bar{p},\bar{k}),(p,k)}$. It is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j}) < V_q(\boldsymbol{\varpi}_{I_j})$. Thus, either $V_q(\boldsymbol{\omega}) < V_q(\boldsymbol{\varpi})$ or they are not related. Suppose, by contradiction, $V_q(\boldsymbol{\omega}) < V_q(\boldsymbol{\varpi})$. Then $V_q(\boldsymbol{\omega}_J) \leq V_q(\boldsymbol{\varpi}_J)$ for all $J \subseteq I$ connected. In particular, for $J = I_\lambda$ we get

$$\lambda(h_j) + 3 - r \leq \lambda(h_j) + 3 - k - ({}_{\bar{p}}|\lambda|_{p-1} + p - \bar{p}).$$

Hence

$$r \geq k + {}_{\bar{p}}|\lambda|_{p-1} + p - \bar{p} > k.$$

Then, by Lemma 5.2.1(b), $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ are not related, yielding a contradiction.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(d)_j^{k,l}$ and $\boldsymbol{\varpi}$ be of type (e). Then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_\lambda}) < V_q(\boldsymbol{\varpi}_{I_\lambda})$. In Case 3 we showed that $r \geq 3$ in type $(d)_j^{k,l}$, then it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j}) > V_q(\boldsymbol{\varpi}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related.

$\diamond$ Let $\boldsymbol{\omega}$ be of type $(d)_j^{k',l'}$ and $\boldsymbol{\varpi}$ be of type $(f)_{j,l}^{(\bar{p},\bar{k}),(p,k)}$ or with j' in place of j. One can see that, if $V_q(\boldsymbol{\omega}) > V_q(\boldsymbol{\varpi})$, then condition (2) described in the possibility (i) of Case 3 is not satisfied. Now, if $\bar{k}, k \leq 2$, since $r \geq 3$, either $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ are not comparable affinizations (Lemma 5.2.1(b)) or $V_q(\boldsymbol{\varpi}_{I_j}) < V_q(\boldsymbol{\omega}_{I_j})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ are not related. If $k \geq 3$, then one can see that if $V_q(\boldsymbol{\omega}) \leq V_q(\boldsymbol{\varpi})$, then condition (1) of the possibility (iii) of Case 3 is not satisfied. If $\bar{k} > 3$, then $k = m = 1$ and one can verify that $V_q(\boldsymbol{\omega}) \not\leq V_q(\boldsymbol{\varpi})$.

$\diamond$ Let $\boldsymbol{\omega}$ be of type (e) and $\boldsymbol{\varpi}$ be of type $(f)_{j,l}^{(\bar{p},\bar{k}),(p,k)}$. Then, it is not difficult to see that $V_q(\boldsymbol{\omega}_{I_j}) < V_q(\boldsymbol{\varpi}_{I_j})$ and $V_q(\boldsymbol{\omega}_{I_\lambda}) > V_q(\boldsymbol{\varpi}_{I_\lambda})$, implying that $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\varpi})$ cannot be related. $\square$

REMARK 5.5.1. If $\#(\mathrm{supp}(\lambda) \cap A) = 1$, then class (f) degenerates to a class of type (c) or (d).

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