DÉBORA APARECIDA FRANCISCO ALBANEZ

Continuous Data Assimilation for Navier-Stokes-alpha model

Assimilação contínua de dados para o modelo Navier-Stokes-alpha

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CONTINUOUS DATA ASSIMILATION FOR NAVIER-STOKES-ALPHA MODEL

ASSIMILAÇÃO CONTÍNUA DE DADOS PARA O MODELO NAVIER-STOKES-ALPHA

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in mathematics.

Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em matemática.

Orientador: Milton da Costa Lopes Filho

Coorientadora: Helena Judith Nussenzveig Lopes

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Abstract

Motivated by the presence of the finite number of determining parameters (degrees of freedom) such as modes, nodes and local spatial averages for dissipative dynamical systems, specially Navier-Stokes equations, we present in this thesis a new continuous data assimilation algorithm for the three-dimensional Navier-Stokes-alpha model, which consists of introducing a general type of approximation interpolation operator, (that is constructed from observational measurements), into the Navier-Stokes-alpha equations.

The main result provides conditions on the finite-dimensional spatial resolution of the collected data, sufficient to guarantee that the approximating solution, that is obtained from these collected data, converges to the unknown reference solution (physical reality) over time. These conditions are given in terms of some physical parameters, such as kinematic viscosity, the size of the domain and the forcing term.

Keywords: Determining Modes, Volume Elements, Nodes, Continuous Data Assimilation, Three-dimensional Navier-Stokes-alpha Equations.

Resumo

Motivados pela existência de um número finito de parâmetros determinantes (graus de liberdade), tais como modos, nós e médias espaciais locais para sistemas dinâmicos dissipativos, principalmente as equações de Navier-Stokes, apresentamos nesta tese um novo algoritmo de assimilação contínua de dados para o modelo tridimensional das equações Navier-Stokes-alpha, o qual consiste na introdução de um tipo geral de operador interpolante de aproximação (construído a partir de medições observacionais) dentro das equações de Navier-Stokes-alpha.

O principal resultado garante condições sob a resolução espacial de dimensão finita dos dados coletados, suficientes para que a solução aproximada, construída a partir desses dados coletados, convirja para a referente solução que não conhecemos (realidade física) no tempo. Essas condições são dadas em termos de alguns parâmetros físicos, tais como a viscosidade cinemática, o tamanho do domínio e o termo de força.

Palavras-chave: Modos determinantes, Elementos de Volume, Nós, Assimilação Contínua de Dados, Equações de Navier-Stokes-alpha Tridimensionais.

Contents

À minha família

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Chapter 1

Introduction

In Section 1, we present and discuss the relevance of Navier-Stokes-α equations in fluid dynamics theory. In the second section, we discuss *data assimilation* and its importance in approximation of solutions of certain PDE. We also treat determining parameters in Section 3.

In Section 4, we present a method of data assimilation studied by A. Azouani, E. Olson and E.S. Titi, in [1], in 2013, on which this thesis was based.

In Section 5, we present our new data assimilation model for Navier-Stokes-α equations.

1.1 The dynamical model: The Navier-Stokes-α Equations

Turbulent flows are conventionally visualized as a cascade of large eddies (large-scale components of the flow) breaking up successively into ever smaller sized eddies (fine-scale components of the flow). In this process, the energy cascades toward ever smaller scales until it reaches the dissipation scale. This cascade is a characteristic feature of turbulence.

This phenomenon is also the main difficulty in simulating turbulence numerically, because all the numerical simulations will have finite resolution and it will not be able to keep up with the cascade all the way to the dissipation scale, specially for flows near walls.

From the physical point of view, the effects of subgrid-scale fluid motions (small eddies, swirls, vortices) occurring below the grid resolution should be modeled. We present here a modeling scheme - called **Navier-Stokes-α** (NS-α) model in three dimensions with periodic boundary conditions $\Omega = [0,L]^3 \equiv \left(\frac{\mathbb{R}}{L\mathbb{Z}}\right)^3 \equiv \mathbb{T}^3$ (the three-dimensional torus), where $L > 0$ is the size of periodic domain:

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\Delta v - u \times (\nabla \times v) = -\nabla p + f, \\ \text{div } u = \text{div } v = 0, \end{cases} \tag{1.1.1}$$

where:

(i) $u = (u_1, u_2, u_3)$ is the velocity of the fluid (called filtered velocity);

(ii) $p = \widetilde{p} - \frac{1}{2}\nabla(|u|^2 + \alpha^2|\nabla u|^2) + v \cdot u$ is the modified pressure;

(iii) $\widetilde{p}$ is the pressure;

(iv) $\nu > 0$ is the kinematic viscosity;

(v) The function f is a given body forcing;

(vi) $\alpha > 0$ is taken as a constant with dimension of length and the filtering relation $u = G_\alpha * v$ for the advection velocity in the NS-α model is specified as

$$v \equiv u - \alpha^2\Delta u,$$

that is, the filtering kernel G_α for the NS-α model turns out to be the Green's function for the Helmholtz operator, $(I - \alpha^2\Delta)$. The velocity v is also known as unfiltered velocity.

The parameter α specifies the smallest scale that actively participates in the dynamics, scales larger than α are resolved explicitly, while motion on scales below α is "swept" by the large scales, a consequence of Taylor's Hypothesis.

The notation $\nabla \times u$ for some vector-value field $u = (u_1, u_2, u_3)$ represents the curl of u, which is well known in three dimensions by

$$\nabla \times u = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right).$$

The non-linear term in (1.1.1) satisfies the following identity:

$$-u \times (\nabla \times v) = -\sum_{i=1}^{3}(u_i\partial_i v - u_i\nabla v_i) = (u \cdot \nabla)v + \sum_{i=1}^{3} u_i\nabla v_i - \nabla(v \cdot u), \tag{1.1.2}$$

with $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$ provided (for instance) $u, v \in C^1(\Omega)$ (see [9]). For $u = v$, we have

$$-v \times (\nabla \times v) = (v \cdot \nabla)v - \frac{1}{2}\nabla\left(\sum_{i=1}^{3} v_i v_i\right).$$

The Navier-Stokes-α model is also refered in the literature as the viscous **Camassa-Holm** equations or **Lagrangian-averaged Navier-Stokes-α** (LANS-α) model. An equivalent alternative formulation is to rewrite (1.1.1) as

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu\Delta v + u \cdot \nabla v + (\nabla u)^T \cdot v = -\nabla P + f, \\ \text{div } u = \text{div } v = 0, \end{cases} \tag{1.1.3}$$

where $P = \widetilde{p} - \frac{1}{2}\nabla(|u|^2 + \alpha^2|\nabla u|^2)$.

The LANS-α motion equation satisfies the Kelvin circulation Theorem:

$$\begin{aligned} \frac{d}{dt}\oint_{\Gamma(u)} v \cdot dx &= \oint_{\Gamma(u)} \left(\frac{\partial v}{\partial t} + u \cdot \nabla v + (\nabla u)^T \cdot v\right) \cdot dx \\ &= \oint_{\Gamma(u)} (\nu\Delta v + f) \cdot dx. \end{aligned}$$

For $\alpha = 0$, the NS-α model reduces to the exact 3D Navier-Stokes system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u \cdot \nabla)u = -\nabla p + f, \\ \text{div } u = 0, \end{cases} \tag{1.1.4}$$

and for $\nu = \alpha = 0$, NS-α reduces to 3D Euler equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p + f, \\ \text{div } u = 0. \end{cases} \tag{1.1.5}$$

It is known (see [19]) that the L^2-energy (kinetic energy) is conserved quantity in Euler equations, in the absence of external forces. For Navier-Stokes equations, the kinetic energy $E(T)$ satisfies the differential equation

$$\frac{d}{dt}E(t) = -\nu \int_\Omega |\nabla u|^2 dx,$$

where u is a smooth solution to the Navier-Stokes equations. In the case of NS-α equations, the corresponding kinetic energy is given by

$$E_\alpha(t) = \int_\Omega |u(x,t)|^2 + \alpha^2|\nabla u(t,x)|^2 dx,$$

is globally bounded in time.

Numerical results for the NS-α model are given in [5], where the authors compare the structures of velocity and vorticity fields in a direct numerical simulation (DNS) of the viscous NS-α model with the corresponding results for the Navier-Stokes equations. For that, they computed the Navier–Stokes alpha equations for several values of the α parameter, including the limiting case $\alpha \to 0$, in which the Navier–Stokes equations are recovered.

The Navier-Stokes-α model can be seen as a regularized approximation of the 3D Navier-Stokes system, depending on the small positive parameter α in some terms of which the unknown velocity function v is replaced by a smoother vector-valued function u related to v by means of the elliptic system $v = u - \alpha^2 \Delta u$.

1.2 Data Assimilation Theory

For a mathematical model of some physical system, as weather forecasting or a flow (viscous or not), it becomes necessary to combine the real-world observations in a physically consistent way with this model, given by partial differential equations.

The process of establishing a connection between the theoretical models of physical conservations laws and these real-world measurements (such observational data sometimes are done on a rough way), in order to extract a better information of the physical system is called *Data Assimilation*.

Data assimilation arises in a vast array of different topics: traditionally in meteorological and hydrology modelling, wind tunnel or water tunnel experiments and recently from biomedical engineering.

A summary of the use of data assimilation in pratical weather forecasting is described in [8], which succeded the idea of obtaining improved estimates of current atmospheric state using the equations of the atmosphere themselves, proposed by Charney, Harlem and Jastrow in [4]. Ocean state estimation is another example of a physical system which essentially resorts to data assimilation as the technique to integrate measurements into a dynamical model.

The measured data usually contains inaccuracies and is given with low spatial and/or temporal resolution. In general, the prodution of an accurate information of the true state of the atmosphere or a fluid in a given time is not possible, so alternatively the question is: how does one find a good approximation of the true state? Or, in other words, is it possible to find a good asymptotic approximation of the solution to some model of the physical reality at time t?

The most ordinary types of data assimilation are: *Discrete Data Assimilation* and *Continuous Data Assimilation*. In *Discrete Data Assimilation*, the approximating solution is coupled to the reference solution at a discrete sequence of points in time. One application of this to Lorenz and 2D Navier-Stokes equations can be found in [13].

The main focus of this thesis is on *Continuos Data Assimilation*. In this mode of assimilation, a feasible state trajectory is found that best fits the observed data over a time interval, and the estimated states at the end of the interval are used to produce the next forecasts.

In this work, data assimilation for time dependent fluid flow, specifically Navier-Stokes-α system, is considered; that is, the flow is assumed to satisfy the given partial differential equation (1.1.1), representing the mathematical model.

First, we discuss a manner of doing this model of assimilation, developed by Olson and Titi (found in [20]), by introducing an observation-dependent forcing term in the 2D Navier-Stokes equations: they have considered $u_1(t)$ the real state at time t of the dynamical system, i.e., the exact solution of the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \nu \Delta u_1 + (u_1 \cdot \nabla) u_1 = -\nabla \pi_1 + f, \\ \text{div } \ u_1 = 0, \end{cases} \tag{1.2.1}$$

where π_1 is the pressure, with initial conditions $u_1(0) = u_0$, on the L-periodic torus $\Omega = [0, L]^2$, where π_1 is the pressure and f is the forcing term. The observational measurements corresponding to $u_1(t)$ at time t we represent by $P_\lambda u_1(t)$, where P_λ is defined using the Fourier space representation for a such that $\|a\|_{L^2(\Omega)} < \infty$:

$$P_\lambda a = \sum_{|k|^2 \le \lambda} \widehat{a}_k \phi_k \quad \text{and} \quad Q_\lambda = I - P_\lambda,$$

where $\phi_k(x) = e^{2\pi i \frac{k \cdot x}{L}}$. Here λ represents a parameter, namely the resolution of the measuring equipment.

In this model of assimilation, the main difficulty is that it is not possible to obtain u_0 exactly by measurement, because if we had u_0 exactly, that is, the detailed reality at time $t = 0$, it would be enough to integrate the Navier-Stokes equations and therefore get $u_1(t)$ exactly for any $t > 0$.

To deal with this problem, the authors have considered $u_2(t)$, an approximation to $u_1(t)$ obtained from the observational measurements and then, they found conditions on λ in terms of other physical parameters of the system (as viscosity and forcing term) to ensure the convergence of $u_2(t) - u_1(t) \to 0$ in L^2 and H^1-norms.

The idea of the construction of $u_2(t)$ from the observational measurements $P_\lambda u_1(t)$ was to rewrite the Navier-Stokes equations (1.2.1) as a system of two coupled differential equations, where $u_1 = p_1 + q_1$ with $p_1 = P_\lambda u_1$ (the observational measurements) and $q_1 = Q_\lambda u_1$ (the unknown modes of Fourier). Then, applying the orthogonal projectors P_λ and Q_λ in (1.2.1), we get

$$\left\{\begin{array}{l} \dfrac{\partial p_1}{\partial t} + P_\lambda[(p_1+q_1)\cdot\nabla(p_1+q_1)] - \nu\Delta p_1 = -\nabla P_\lambda \pi_1 + P_\lambda f, \nabla\cdot p_1 = 0, \\ \\ \dfrac{\partial q_1}{\partial t} + Q_\lambda[(p_1+q_1)\cdot\nabla(p_1+q_1)] - \nu\Delta q_1 = -\nabla Q_\lambda \pi_1 + Q_\lambda f, \nabla\cdot q_1 = 0. \end{array}\right. \tag{1.2.2}$$

Since $q_1(0)$ is not known, it is impossible to integrate the second equation. Then the authors computed an approximation $q_2(t)$ of $q_1(t)$ by integrating

$$\frac{\partial q_2}{\partial t} + Q_\lambda[(p_1+q_2)\cdot\nabla(p_1+q_2)] - \nu\Delta q_2 = -\nabla Q_\lambda \pi_2 + Q_\lambda f, \quad \nabla\cdot q_2 = 0, \tag{1.2.3}$$

with π_2 the new pressure and $q_2(0) = \eta$, where $\eta = Q_\lambda \eta$ represents a guess of initial data of the new high modes $q_2(t)$ of the exact solution. Therefore, from here on, we consider (1.2.3) instead of the second equation of (1.2.2). Finally, adding (1.2.3) and the first equation of (1.2.2), we obtain the approximating solution $u_2(t)$ of the solution $u_1(t)$ by the following system:

$$\left\{\begin{array}{l} \dfrac{\partial u_2}{\partial t} - \nu\Delta u_2 + (u_2\cdot\nabla)u_2 = -\nabla\pi_2 + f_2, \\ \mathrm{div}\;\; u_2 = 0, \end{array}\right. \tag{1.2.4}$$

where $u_2(t) = p_1(t) + q_2(t) = P_\lambda u_1(t) + q_2(t)$, with initial conditions $u_2(0) = P_\lambda u(0) + \eta$ where $P_\lambda u(0)$ is the initial observational measurement and $\eta = Q_\lambda \eta$. Note that we have now a known initial data $u_2(0)$ to deal, since $P_\lambda u(0)$ and η are known. Moreover,

$$f_2 = f + P_\lambda[(u_2\cdot\nabla)u_2 - (u_1\cdot\nabla)u_1],$$

since $(u_2 \cdot \nabla)u_2 = P_\lambda[(u_2 \cdot \nabla)u_2] + Q_\lambda[(u_2 \cdot \nabla)u_2]$.

The global existence and uniqueness for the system (1.2.4) is found in Theorem 3.1 in [20], as well as the convergence of u_2 to u_1 when time goes to infinity in Lemma 3.2 of [20].

A few years later, Peter Korn focused on the same technique of [20], but for 3D Navier-Stokes-α equations, in [17]. The true evolution of NS-α equation was denoted in [17] by $u_1(t)$, as well as $v_1 = u_1 - \alpha^2 \Delta u_1$. Therefore, applying the projectors P_λ and Q_λ, we have

$$\begin{cases} \dfrac{\partial P_\lambda v_1}{\partial t} - P_\lambda[u_1 \times (\nabla \times v_1)] - \nu \Delta P_\lambda v_1 = -\nabla P_\lambda \pi_1 + P_\lambda f, \;\; \nabla \cdot P_\lambda u_1 = 0 \\ \\ \dfrac{\partial Q_\lambda v_1}{\partial t} - Q_\lambda[u_1 \times (\nabla \times v_1)] - \nu \Delta Q_\lambda v_1 = -\nabla Q_\lambda \pi_1 + Q_\lambda f, \;\; \nabla \cdot Q_\lambda u_1 = 0, \end{cases} \tag{1.2.5}$$

Denote $P_\lambda u_1(t) := \overline{u}_1(t)$ the observational data on time t (that are known by measurements) and $Q_\lambda u_1(t) = \widetilde{u}_1(t)$, and since $u_1 = P_\lambda u_1 + Q_\lambda u_1$, we write $u_1 = \overline{u}_1 + \widetilde{u}_1$, as well as $v_1 = \overline{v}_1 + \widetilde{v}_1$, with $P_\lambda v_1(t) = \overline{v}_1(t)$ and $Q_\lambda v_1(t) = \widetilde{v}_1(t)$. With these notations, we get

$$\begin{cases} \dfrac{\partial \overline{v}_1}{\partial t} - P_\lambda[u_1 \times (\nabla \times v_1)] - \nu \Delta \overline{v}_1 = -\nabla P_\lambda \pi_1 + P_\lambda f, \;\; \nabla \cdot \overline{u}_1 = 0 \\ \\ \dfrac{\partial \widetilde{v}_1}{\partial t} - Q_\lambda[u_1 \times (\nabla \times v_1)] - \nu \Delta \widetilde{v}_1 = -\nabla Q_\lambda \pi_1 + Q_\lambda f, \;\; \nabla \cdot \widetilde{u}_1 = 0, \end{cases} \tag{1.2.6}$$

In the same way as [20], to solve the problem that the high-frequency component of initial data $Q_\lambda u_1(0) = \widetilde{u}_1(0)$ (and therefore $\widetilde{v}_1(0) = \widetilde{u}_1(0) - \alpha^2 A \widetilde{u}_1(0)$) are unkown, the second equation in (1.2.6) is replaced by

$$\frac{\partial \widetilde{v}_2}{\partial t} - Q_\lambda\{(\overline{u}_1 + \widetilde{u}_2) \times [\nabla \times (\overline{v}_1 + \widetilde{v}_2)]\} - \nu \Delta \widetilde{v}_2 = -\nabla Q_\lambda \widetilde{\pi}_2 + Q_\lambda f, \tag{1.2.7}$$

with a guess of the initial conditions $\widetilde{u}_2(0) = \eta = Q_\lambda \eta$ and $\widetilde{v}_2 = \widetilde{u}_2 - \alpha^2 \Delta \widetilde{u}_2$. Then (1.2.7) is added to the first equation of (1.2.6) to obtain

$$\begin{cases} \dfrac{\partial v_2}{\partial t} - u_2 \times (\nabla \times v_2) - \nu \Delta v_2 = -\nabla \pi_2 + f_2, \\ \text{div } v_2 = \text{div } u_2 = 0, \end{cases} \tag{1.2.8}$$

i.e., $v_2(t) = \overline{v}_1(t) + \widetilde{v}_2(t)$ or equivalently, $u_2(t) = \overline{u}_1(t) + \widetilde{u}_2(t)$, with initial data $u_2(0) = \overline{u}_1(0) + \eta$ and

$$f_2 = f + P_\lambda[(u_1 \times (\nabla \times v_1)] - P_\lambda[u_2 \times (\nabla \times v_2)].$$

The global well-posedness of the system (1.2.8) and the covergence of this approximation u_2 to the original solution u_1 in L^2 and H^1-norms is found in Theorem 10 and Lemma 11 in [17], respectively.

1.3 Determination of the Solutions by Determining Parameters

The standard theory of turbulence asserts that turbulent flows are determinated by a finite number of *degrees of freedom*, that is, the number of independent "pieces" of data which are used to make an exact calculation is finite.

The first mathematically rigorous indication that the large time behavior of the solutions to the 2D Navier-Stokes equations has a finite number of degrees of freedom was given in [11]. After that, there have been many studies to estimate the number of degrees of freedom of the solutions for the Navier-Stokes equations in terms of the Grashoff number G, a nondimensional quantity proportional to the forcing term f.

The Grashoff number is defined in terms of f, the viscosity ν and some other parameter with the dimension of length, usually taken as the first eigenvalue of the Stokes operator (that will be defined later) λ_1, which has the dimension of length l^{-2}. For dimensional reasons, the definition of G depends on the spatial dimension, so it is defined for two-dimensional case as

$$G = \frac{F}{\lambda_1 \nu^2},$$

and for three dimensional case as

$$G = \frac{F}{\lambda_1^{3/4} \nu^2}, \tag{1.3.1}$$

where

$$F = \limsup_{t \to \infty} \left(\int_\Omega |f(t,x)|^2 dx \right)^{\frac{1}{2}}.$$

Returning to degrees of freedom, a natural question is: if we know the behavior of the velocity vectors $u(x,t)$ of a fluid for all time (or for large times), on a set of finite points

$$\mathbb{E} = \{x^1, x^2, \dots, x^N\}, \tag{1.3.2}$$

what information can we deduce for the large time behavior of the flow?

The answer given in [12] for 2D Navier-Stokes equations is: if the set of points $\mathbb{E}$ is sufficiently dense (but still finite), then the large time behavior of the flow is uniquely determined by the knowledge of $u(x,t)$ for all $x \in \mathbb{E}$ and for all time (or for all t sufficiently large). For instance, if for all $x \in \mathbb{E}$, $u(x,t)$ tends to some time-periodic function $\widetilde{u}(x,t)$:

$$\widetilde{u}(x,t) = \widetilde{u}(x, t+T)$$

as $t \to \infty$, then $u(\cdot, t)$ tends as well to a time-periodic solution $\overline{u}(x,t) = \overline{u}(x, t+T)$ for all x and

$$\overline{u}(x,t) = \widetilde{u}(x,t), \quad \text{for all} \quad x \in \mathbb{E}.$$

We present next the notions of determining modes, determining nodes and determining volume elements for the regular theory of turbulence. These notions are rigorous attempts to identifly those parameters that determine turbulent flows. Most of the research on estimating these parameters has been concentrated on 2D Navier-Stokes equations, since in the 3D case, the question of global existence and uniqueness of strong solutions are still open.

We consider u and v two solutions of the 2D Navier-Stokes equations, respectively:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = -\nabla p + f, \;\; u(0) = u_0, \tag{1.3.3}$$

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v = -\nabla \widetilde{p} + g, \;\; v(0) = v_0, \tag{1.3.4}$$

where f, g are given forces in a suitable space. We denote P_m the orthogonal projection onto the liner space spanned by $\{w_1, w_2, ..., w_m\}$, the first m eigenfunctions of the Stokes operator, with periodic boundary conditions.

Definition 1.3.1. A set of modes $\{w_j\}_{j=1}^m$ is called determining if we have

$$\lim_{t \to \infty} \| u(t) - v(t) \|_{L^2(\Omega)} = 0,$$

for u and v solving (1.3.3) and (1.3.4) respectively, whenever

$$\lim_{t \to \infty} \|f(t) - g(t)\|_{L^2(\Omega)} = 0,$$

and

$$\lim_{t \to \infty} \|P_m u(t) - P_m v(t)\|_{L^2(\Omega)} = 0.$$

In other words, the modes $\{w_j\}_{j=1}^m$ is called determining if they determine completely the behavior of the solution in the limit $t \to \infty$.

A first estimate of the number of determining modes for 2D Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^2$, provided boundary condition $u = 0$ was given in [11], where the authors proved that the modes are determining if

$$\lambda_{m+1} > \frac{2}{\nu^2} \limsup \int_\Omega |\nabla \times u(t,x)|^2 dx,$$

where λ_i is the i-th eigenvalue of the Stokes Operator (it will be defined in next chapter).

An improved estimate on the number of determining modes for the 2D Navier-Stokes equations under the same conditions was obtained in [10], where the authors proved that if

$$\frac{\lambda_{m+1}}{\lambda_1} \geq cG(1 + \log G)^{1/2}, \tag{1.3.5}$$

where G is the Grashoff number and $c > 0$ is a dimensionless constant coming from the properties of the nonlinear term of 2D Navier-Stokes equations, then the number of determining modes is not larger than m. However, it is argued heuristically in that paper that the number of determining modes should be of the order m, where m satisfies $\dfrac{\lambda_{m+1}}{\lambda_1} \geq cG$. In 1993, the authors of [16] showed how to eliminate the logarithmic term in (1.3.5) (for the case of periodic boundary conditions) and consequently that if m satisfies

$$\frac{\lambda_{m+1}}{\lambda_1} \geq \sqrt{3}cG,$$

then the number of determining modes is not larger than m.

Another way to characterize the degrees of freedom of a physical flow model is the *determining finite volume elements*. It consists, for 2D Navier-Stokes equations with periodic boundary conditions $\Omega = [0, L]^2$, to divide Ω into N equal squares Q_j (with j=1,. . . ,N) of side $l = L/\sqrt{N}$. We set a volume element (also called local spatial average) as

$$\langle u\rangle_{Q_j} = \frac{N}{L^2}\int_{Q_j} u(x)dx$$

for every $1 \leq j \leq N$. This leads us to following definition:

Definition 1.3.2. A set of volume elements is said to be determining if for two solutions u and v solving (1.3.3) and (1.3.4), respectively, and satisfying $\lim_{t\to\infty}\|f(t) - g(t)\|_{L^2(\Omega)} = 0$ and

$$\lim_{t\to\infty}(\langle u\rangle_{Q_j} - \langle v\rangle_{Q_j}) = 0,$$

for all $j = 1, \ldots, N$, we have

$$\lim_{t\to\infty}\|u(t) - v(t)\|_{L^2(\Omega)} = 0.$$

The existence of determining volume elements for the 2D Navier-Stokes equations was shown in [14]: the authors proved that the volume elements are determining provided

$$N \geq 4(10 + 4\sqrt{2})^2 G^2,$$

where N is the number of squares that the periodic domain is divided. One year later, the same authors of [14] presented in [16] an improved upper bound on the number of determining volume elements: $N > cG$, where G is the Grashoff number and c is a constant coming from nonlinearity properties of 2D Navier-Stokes equations.

Definition 1.3.3. Let $\mathbb{E}$ be a collection of points in the domain Ω (also called nodal values in Finite Elements Method Theory), as in (1.3.2). This set is called a set of determining nodes if for two solutions u and v solving equations (1.3.3) and (1.3.4), respectively, and satisfying for all $j = 1, \ldots, N$,

$$\lim_{t\to\infty}(u(x^j, t) - v(x^j, t)) = 0,$$

and $\lim_{t\to\infty}\|f(t) - g(t)\|_{L^2(\Omega)} = 0$, we have

$$\lim_{t\to\infty}\|u(t) - v(t)\|_{L^2(\Omega)} = 0.$$

The existence of a set of determining nodes for 2D Navier-Stokes equations in an open bounded set of $\mathbb{R}^2$ (with sufficiently smooth boundary) and for periodic boundary conditions, it was first proven in [12]. Later in [15], in 1992, an upper bound for the number of determining nodes for periodic case was found to be proportional to $G^2(1+\log G)$. In 1993, the authors of [16] presented an improved upper bound: if $\mathbb{E} = \{x_1, x_2, ..., x_N\}$ is a set of nodes, it is determining provided $N \geq 4\sqrt{2}\pi^2 cG$.

1.4 The new algorithm to insert observational measurements

In 2013, Titi and Azouani (see [2],[1]) introduced a finite-dimensional feedback control scheme for stabilizing solutions of infinite-dimensional dissipative equations, such as the Navier-Stokes equations, Kuramoto-Sivashinsky equation and reaction-diffusion equations.

This new idea was originated from the fact that such systems possess a finite number of determining parameters (degrees of freedom), such as determining Fourier modes and determining nodes, in accordance with that cited in the previous section.

The classical method of continuous data assimilation requires special care concerning how the observations are inserted into a model in pratice. For example, as we saw earlier, it is necessary in general to separate the slow $P_N u$ parts and the fast $Q_N u$ parts of a solution before inserting the observations into the model. The method proposed in [1] does not require such a decomposition. Rather than inserting the measurements directly into the model, i.e., into the nonlinear term, they introduced a feedback control term that forces the model toward the reference solution that is corresponding to the observations.

In [1], this sort of continuos data assimilation was considered in the 2D Navier-Stokes system for $\Omega \subset \mathbb{R}^2$ an open, bounded and connected set with $u \mid_{\partial\Omega} = 0$ (no-slip Dirichlet boundary conditions) and also for periodic boundary conditions.

We present next the construction of this new continuous data assimilation, which was done in [1].

Firstly, the authors have considered $u(t)$ representing the real physical state at time t of the 2D Navier-Stokes dynamical system and $I_h(u(t))$ representing the observations of the system at a rough spatial resolution of size h.

With $I_h(u(t))$, the observational measurements in hand for all $t \in [0, T]$, the next step is to construct an increasingly accurrate initial condition from which predictions of $u(t)$, for $t > T$ can be made. And this is done by constructing an approximate solution $v(t)$ that converges to $u(t)$ over time.

The algorithm developed by the authors for constructing $v(t)$ from the observational measurements $I_h(u(t))$ for all $t \in [0, T]$ is given by

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v + \nabla p = f - \mu (I_h(v) - I_h(u)), \\ v(0) = v_0, \ \operatorname{div} \, v = 0. \end{cases} \tag{1.4.1}$$

on the interval $[0, T]$. Here, v_0 is taken to be arbitrary, $\mu > 0$ is a parameter inverse-time dimensional and h is a parameter with dimension of length that should be related to μ in terms of $\nu, f, G, \lambda_1 = (2\pi/L)^2$ (the first eigenvalue of Stokes operator, with periodic boundary conditions $\Omega = [0, L]^3$), and other constants, for instance, related to nonlinearity properties, in order to (1.4.1) make sense and also to ensure the convergence of the approximating solution to the real solution.

The method of constructing v given by (1.4.1), allows the use of general interpolant observables, given by linear interpolation operators

$$I_h : H^1(\Omega) \to L^2(\Omega),$$

which is an approximate interpolant of order h of the inclusion map $i : H^1(\Omega) \hookrightarrow L^2(\Omega)$ that satisfies the following estimate, for some $c_0 > 0$,

$$\|\varphi - I_h \varphi\|_{L^2(\Omega)} \leq c_0 h \|\varphi\|_{H^1(\Omega)} \tag{1.4.2}$$

This inequality is a version of the well-known Bramble-Hilbert inequality, that appears in the context of finite elements method (see [6]).

In addition, it was considered interpolant observables given by linear interpolants

$$I_h : H^2(\Omega) \to L^2(\Omega)$$

that satisfy the following approximation property:

$$\|\varphi - I_h\varphi\|^2_{L^2(\Omega)} \leq c_1 h^2 \|\varphi\|^2_{H^1(\Omega)} + c_2 h^4 \|\varphi\|^2_{H^2(\Omega)}, \tag{1.4.3}$$

for every $\varphi \in H^2(\Omega)$. In [1], there are examples of such interpolant.

The great advantage of this approach is that it works for a general class of interpolant observables without modification, only respecting (1.4.2) or (1.4.3).

The expected results of this sort of continuos data assimilation is to yield conditions, on the finite-dimensional spatial resolution of the collected data, sufficient to ensure that the approximating solution, which is obtained by this algorithm from the measurement data, converges to the unkwown form solution $u(t)$ over time.

Since u_0 is not known, it would make sense to take $v_0 = I_h(u(0))$, which is the initial observation of the solution u. However, v_0 chosen this way might not be in the suitable space for solving the equation. The main point of this new method given in (1.4.1) is to avoid the difficulties which come from the direct insertion of observational measurements into the approximate solution, specially initial observational measurements. The results obtained in [1] holds when v_0 is chosen to be any element of the suitable space.

The proof that the data assimilation equations (1.4.1) are globally well-posed when I_h satisfies (1.4.2), for both no-slip Dirichlet and periodic boundary conditions, is found in Theorem 5 (see [1]), since $\mu c_0 h^2 \leq \nu$.

The existence and uniqueness of strong solutions for data assimilation equations which I_h satisfies (1.4.3) is found in Theorem 6 (see [1]) for periodic boundary conditions, since $\mu\gamma h^2 \leq \nu$, where γ is a constant depending on c_1 and c_2.

The main result of [1] can be stated as follows:

Theorem 1.4.1. Let Ω be an open, bounded and connected set in $\mathbb{R}^2$ with C^2 boundary, and let u be a solution to 2D Navier-Stokes equations with no-slip Dirichlet boundary conditions, i.e., $u\mid_{\partial\Omega} = 0$. Assume that I_h satisfies (1.4.2), with h small enough such that

$$h^2 \leq (k_1 \lambda_1 G^2)^{-1}$$

where k_1 depends on c_0 and the nonlinearity properties, G is the Grashoff number and λ_1 is the first eigenvalue of Stokes operator under boundary condition $u\mid_{\partial\Omega}= 0$. Then there exists $\mu > 0$ (given explicitly), such that $\|v - u\|_{L^2(\Omega)} \to 0$ exponentially, as $t \to \infty$.

A similar result to Theorem 1.4.1 was proven also in [1], when the interpolant I_h satisfies (1.4.3) for periodic boundary conditions, and sharper estimates may be obtained:

Theorem 1.4.2. Let $\Omega = [0, L]^2$ and let u be a solution to 2D Navier-Stokes equations with periodic boundary conditions. Let I_h satisfy either (1.4.2) or (1.4.3), with h small enough such that

$$1/h^2 \geq c_3\lambda_1 G(1 + \log(1 + G)),$$

where c_3 depends on c_1, c_2 and nonlinearity properties, G is the Grashoff number and $\lambda_1 = (2\pi/L)^2$ is the first eigenvalue of Stokes operator under periodic boundary conditions $\Omega = [0, L]^2$. Then there exists $\mu > 0$ (given explicitly), such that $\|v - u\|_{H^1(\Omega)} \to 0$ exponentially, as $t \to 0$.

To finish this section, in [2], the authors used the Chafee-Infante reaction-diffusion equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta u_{xx} - \alpha u + u^3 = 0, \\ u_x(0) = u_x(L) = 0, \end{cases} \tag{1.4.4}$$

with $\alpha > 0$, to consider the following general feedback system of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta u_{xx} - \alpha u + u^3 = -\mu I_h u, \\ u_x(0) = u_x(L) = 0, \end{cases} \tag{1.4.5}$$

where $I_h : H^1([0, L]) \to L^2([0, L])$ is an interpolant can be thought as a controller that is used to stabilize the system. This interpolant must satisfy

$$\|\varphi - I_h\varphi\|_{L^2([0,L])} \leq c_4 h\|\varphi\|_{H^1([0,L])}, \tag{1.4.6}$$

for all $\phi \in H^1([0, L])$. Some examples of such approximate interpolant are: the finite volumes elements, the approximate interpolant based on nodal values and the interpolant given as projections onto Fourier modes.

The proof that such interpolants satisfies (1.4.6) is found in [2]. If $\nu > \mu c^2 h^2$, then Theorem 4.1 (see [2]) guarantees the global existence and uniqueness for the system (1.4.5), with an arbitrary initial condition $u_0 \in H^1([0, L])$.

The main result of [2] is:

Theorem 1.4.3. Let $I_h : H^1([0, L]) \to L^2([0, L])$ be a linear map, which is an approximate interpolant of order h of the inclusion map $i : H^1 \hookrightarrow L^2$, that satisfies the approximation inequality (1.4.6). Moreover, assume that μ is large enough that

$$\mu > 2\alpha + \frac{\beta}{L^2}$$

and h is small enough such that

$$\mu c^2 h^2 < \beta$$

Then, for every $u_0 \in H^1([0, L])$, the global unique solution of (1.4.5) decays exponentially to zero.

Next, we discuss the results of this thesis, that is, how this new technique can be applied in Navier-Stokes-α equations.

1.5 The new continuous data assimilation model applied to Navier-Stokes-α model

Consider $u(t)$ representing the true evolution at time t of the incompressible three-dimensional Navier-Stokes-α equations in the periodic box $\Omega = [0, L]^3$:

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \alpha^2 \Delta u) - \nu \Delta (u - \alpha^2 \Delta u) - u \times (\nabla \times v) + \nabla p = f, \\ \text{div } u = \text{div } v = 0, \end{cases} \tag{1.5.1}$$

where $v = u - \alpha^2 \Delta u$ and the initial data, u_0, is unknown. Consider also a linear interpolant

$$I_h : H^1(\Omega) \longrightarrow L^2(\Omega),$$

that satisfies, for all $\varphi \in H^1(\Omega)$, the following approximation property:

$$\|\varphi - I_h \varphi\|^2_{L^2(\Omega)} \leq c_1^2 h^2 \|\nabla \varphi\|^2_{L^2(\Omega)}. \tag{1.5.2}$$

In addition, we will also consider interpolant observables given by linear interpolants I_h : $H^2(\Omega) \longrightarrow L^2(\Omega)$, that satisfy the following approximation property:

$$\|\varphi - I_h\varphi\|^2_{L^2(\Omega)} \leq c_2^2 h^2 \|\nabla\varphi\|^2_{L^2(\Omega)} + c_2^2 h^4 \|\varphi\|^2_{H^2(\Omega)} \tag{1.5.3}$$

We now write the continuous data assimilation for the system (1.5.1). Let I_h be the interpolation operator satisfying (1.5.2) (or (1.5.3)). Suppose that u must be recovered from the observational measurements $I_h(u(t))$, that have been continuously recorded for times t in $[0, T]$. Then, the approximaing solution w or, equivalently, $z = w - \alpha^2\Delta w$, with initial condition w_0 chosen arbitrarily in an appropriate space (that will be stated in the next chapter) , shall be given by

$$\begin{aligned} &\frac{\partial}{\partial t}(w - \alpha^2\Delta w) - \nu\Delta(w - \alpha^2\Delta w) - w \times (\nabla \times z) + \nabla p \\ &\qquad\qquad = f - \mu(I_h w - I_h u) + \mu\alpha^2\Delta(I_h w - I_h u), \end{aligned} \tag{1.5.4}$$

on the interval $[0, T]$, and div $w =$ div $z = 0$.

This thesis is divided as follow:

In Chapter 2, we present the functional setting of the 3D Navier-Stokes-α, and the spaces and norms required to ensure the global existence and uniqueness of (1.5.4). We also present some compactness theorems, as well as some properties of the nonlinear term of (1.5.1).

In Chapter 3, we show the global existence in time and uniqueness of the solution to the system (1.5.4).

In Chapter 4, we show the conditions of μ and h in terms of physical parameters to guarantee the convergence, in time, of w (given in (1.5.4)), or equivalently, z, to real state u, or equivalently, v, the solution of (1.5.1). To finish, we exhibit three examples of interpolants I_h, and prove that two of them satisfies (1.5.2), and one satisfies (1.5.3), which are obtained from observable measurements.

Chapter 2

Preliminaries

Our purpose in this chapter is to review some of the standard facts and functional setting on Navier-Stokes-α equations, to present some of the properties of the nonlinear term and the Stokes operator. Additionally, we enunciate an important compacteness theorem, known as Aubin-Lion theorem. We also fix notation and terminology.

2.1 Basic Concepts and the Stokes Operator

In this work, we are considering the autonomous Cauchy problem for the three-dimensional Navier-Stokes-α, as already presented in Chapter 1:

$$\begin{cases} \dfrac{\partial}{\partial t}(u - \alpha^2 \Delta u) - \nu \Delta (u - \alpha^2 \Delta u) - u \times (\nabla \times (u - \alpha^2 \Delta u)) + \nabla p = f, \\ \text{div } \ u = 0, \end{cases} \tag{2.1.1}$$

with $u(x, 0) = u_0(x)$ and f time-independent; we consider this system under periodic boundary conditions, i.e., on the periodic domain $\Omega = [0, L]^3$:

$$\begin{aligned} u(x_1, x_2, x_3, t) &= u(x_1 + L, x_2, x_3, t), \\ u(x_1, x_2, x_3, t) &= u(x_1, x_2 + L, x_3, t), \\ u(x_1, x_2, x_3, t) &= u(x_1, x_2, x_3 + L, t). \end{aligned}$$

Thanks to (2.1.1), using integration by parts we have:

$$\frac{d}{dt} \int_\Omega [u(x) - \alpha^2 \Delta u(x)] dx = \int_\Omega f(x) dx.$$

Furthermore, because of the spatial periodicity of the solution, we have $\int_\Omega \Delta u(x)dx = 0$. As a result, we have

$$\frac{d}{dt}\int_\Omega u(x)dx = \int_\Omega f(x)dx.$$

Note that if the average of the force vanishes, then the average velocity is conserved. In this work, we will consider forcing terms and initial values with spatial averages are zero, i.e., we will assume

$$\int_\Omega u_0(x)dx = \int_\Omega f(x)dx = 0,$$

and therefore $\int_\Omega u(x)dx = 0$.

Let us denote by $\mathcal{V}$ the set

$$\begin{aligned} \mathcal{V} \ &= \{\varphi;\ \ \varphi \text{ is a vector valued trigonometric polynomial defined on } \Omega, \\ &\qquad \text{such that div } \varphi = 0 \text{ and } \int_\Omega \varphi(x)dx = 0\}, \end{aligned}$$

where $\Omega = [0, L]^3$ is the periodic domain. If $Z \subset L^1(\Omega)$, we will denote by

$$\dot{Z} = \{\varphi \in Z, \text{ such that } \int_\Omega \varphi(x)dx = 0\}.$$

Denote by H and V:

$$\begin{aligned} H &= \text{closure of } \ \mathcal{V} \text{ in } (L^2_{per}(\Omega))^3, \\ V &= \text{closure of } \ \mathcal{V} \text{ in } (H^1_{per}(\Omega))^3. \end{aligned} \tag{2.1.2}$$

For characterizing (2.1.2), we have the following proposition, whose proof can be found in [7] or also in [22].

Proposition 2.1.1. Let $\Omega = [0, L]^3 \subset \mathbb{R}^3$ the periodic box. Then

$$H^\perp = \{u \in (L^2_{per}(\Omega))^3;\ \ u = \nabla p,\ \ p \in (H^1_{per}(\Omega))^3\} \tag{2.1.3}$$

$$H = \{u \in (L^2_{per}(\Omega))^3;\ \text{div } u = 0 \text{ and } \int_\Omega u(x)dx = 0\} \tag{2.1.4}$$

$$V = \{u \in (H^1_{per}(\Omega))^3;\ \text{div } u = 0 \text{ and } \int_\Omega u(x)dx = 0\}, \tag{2.1.5}$$

The most useful for this work are (2.1.4) and (2.1.5) and, since the proof of (2.1.5) requires more results that we are not interested, we prove only (2.1.4) for sake of completeness.

Proof: Assume (2.1.3). To prove (2.1.4), denote $\widetilde{H}$ the space on the right-hand side of (2.1.4). Cleary $H \subset \widetilde{H}$ by definition. To prove that $\widetilde{H} \subset H$, suppose that H is not the whole space $\widetilde{H}$ and let $\mathcal{H}^{\perp}$ be the orthogonal complement of H in $\widetilde{H}$. By (2.1.3), every $u \in \mathcal{H}^{\perp}$ is the gradient of some $p \in (H^1_{per}(\Omega))^3$. As a result of $u = \nabla p$, we have

$$\Delta p - \text{div } u = 0,$$

and since we have periodic boundary conditions, this implies that p is a constant and $u = 0$; therefore $\mathcal{H}^{\perp} = \{0\}$ and $H = \widetilde{H}$. $\square$

To simplify the notation, from here on we will denote $(L^2_{per}(\Omega))^3$ and $(H^1_{per}(\Omega))^3$ by $L^2(\Omega)$ and $H^1(\Omega)$, respectively; i.e., we will omit the index *per* of the periodic spaces. Additionally, we will denote by $|\cdot|$ the L^2-norm:

$$|u| := \|u\|_{L^2(\Omega)} = \left(\int_{\Omega} |u(x)|^2 dx\right)^{\frac{1}{2}}$$

and by $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$:

$$(u, v) = \int_{\Omega} u(x)\overline{v(x)} dx.$$

Taking into account the equality (2.1.3) of Proposition 2.1.1, we state the following theorem, known as the Hodge decomposition Theorem, that can be found in Proposition 1.18 of [19]:

Proposition 2.1.2. Every vector field $v \in C^{\infty}(\Omega)$, with $\Omega = [0, L]^3$ the periodic box, has the unique orthogonal decomposition

$$v = v_1 + v_2 + \nabla q, \quad v_1 = \int_{\Omega} v(x) dx, \quad \text{div}\, v_2 = 0,$$

where

$$\nabla q(x) = \sum_{|k| \neq 0} \frac{k \otimes k}{|k|^2} \widehat{v}(k)$$

and $k \otimes k = (k_i k_j)$. This decomposition has the following properties

(i) $v_2, \nabla q \in C^{\infty}(\Omega)$,

(ii) $v_2 \perp \nabla q$ in $L^2(\Omega)$, i.e., $\displaystyle\int_{\Omega} v_2 \cdot \nabla q \, dx = 0$,

(iii) $\|v-v_1\|^2_{L^2(\Omega)}=\|v_2\|^2_{L^2(\Omega)}+\|\nabla q\|^2_{L^2(\Omega)}$, and in general, for any multi-index of the derivative D^β,

$$\|D^\beta v\|^2_{L^2(\Omega)}=\|D^\beta v_2\|^2_{L^2(\Omega)}+\|\nabla D^\beta q\|^2_{L^2(\Omega)}.$$

Note that, by arguments of density, we can extend the result above for any function $v\in L^2(\Omega)$, with the same periodic boundary condition $\Omega=[0,L]^3$. Therefore the projection operator $\mathcal{P}:L^2(\Omega)\to H$ given by

$$\mathcal{P}v=v_1$$

projects the vector field on the divergence-free space.

Definition 2.1.3. The decomposition of $v\in L^2(\Omega)$ into $v=v_1++v_2+\nabla q$ is called the *Hodge Decomposition*, and the projection operador $\mathcal{P}$ is called *Leray Projector*.

Actually, if $v\in H^m(\Omega)$, then we have the following lemma, which is found in Lemma 3.6 of [19] for domain $\mathbb{R}^n$, and can be modified for the periodic box $\Omega=[0,L]^3$:

Lemma 2.1.4. Every vector field $v\in H^m(\Omega)$, with $m\in\mathbb{N}\cup\{0\}$ has the unique orthogonal decomposition $v=v_1+v_2+\nabla q$, such that the Leray projector $\mathcal{P}v=v_2\in H^m(\Omega)$ commutes with the distribution derivatives: for all $v\in H^m(\Omega)$ and $|\alpha|\leq m$,

$$\mathcal{P}D^\alpha v=D^\alpha\mathcal{P}v.$$

and also we have that $\mathcal{P}$ is symmetric, i.e.,

$$(\mathcal{P}u,v)_{H^m(\Omega)}=(u,\mathcal{P}v)_{H^m(\Omega)}.$$

Definition 2.1.5. The decomposition of $v\in L^2(\Omega)$ into $v=v_1++v_2+\nabla q$ is called the *Hodge Decomposition*, and the projection operador $\mathcal{P}$ is called *Leray Projector*.

We introduce next the Stokes operator, with the standard notation in the literature of Navier-Stokes equations (see [7] and [22]).

Definition 2.1.6. Consider the space $D(A)=H^2(\Omega)\cap V.$, where $\Omega=[0,L]^3$. The Stokes Operator $A:D(A)\subset H\to H$ is defined by $A=-\mathcal{P}\Delta$, with $-\Delta$ under periodic boundary conditions.

The Stokes operator preserves the self-adjointness property of the operator $-\Delta$:

Proposition 2.1.7. The Stokes operator is symmetric, i.e.,

$$(Au, v)_{L^2(\Omega)} = (u, Av)_{L^2(\Omega)}, \tag{2.1.6}$$

for all $u, v \in D(A)$.

Notice that in the case of periodic boundary conditions, $A = -\Delta|_{D(A)}$. Additionally, we state one more result about the Stokes operator, that can be found in Proposition 4.2 of [7]:

Theorem 2.1.8. The Stokes operator is selfadjoint and positive operator, and its inverse, A^{-1}, is a compact operator in H.

Since A is a self-adjoint positive operator with compact inverse, the space H has an orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$ of eigenfunctions of A, i.e.,

$$A\phi_k = \lambda_k \phi_k,$$

with $\phi_k(x) = \dfrac{1}{L^{3/2}} e^{2\pi i \frac{k \cdot x}{L}}$, with the eigenvalues of the form

$$\left(\frac{2\pi}{L}\right)^2 |k|^2, \text{ where } k \in \mathbb{Z}^3 \backslash \{0\}.$$

We denote these eigenvalues by

$$0 < \lambda_1 = (2\pi/L)^2 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

Consequently, we can express any element in H as a Fourier Series

$$u(x) = \sum_{k \in \mathbb{Z}^3} \widehat{u}_k \phi_k(x) = \sum_{k \in \mathbb{Z}^3} \widehat{u}_k \frac{e^{2\pi i \frac{k \cdot x}{L}}}{L^{3/2}}, \tag{2.1.7}$$

where

$$\widehat{u}_k = (u, \phi_k) = \frac{1}{L^{3/2}} \int_{\Omega} u(x) e^{-2\pi i \frac{k \cdot x}{L}} dx,$$

with $\widehat{u}_0 = 0, \widehat{u}_k = \overline{\widehat{u}_{-k}}$ and due to incompressibility of elements of H, $k \cdot \widehat{u}_k = 0$.

In the next definition, we regard a generalization of the operator A.

Definition 2.1.9. Let β be a real number. The fractional powers of the operator A are defined by linearity from their action on eigenfunctions:

$$A^{\beta}\phi_j = \lambda_j^{\beta}\phi_j, \quad \text{for} \quad j = 0, 1, 2, \ldots$$

with domain

$$D(A^{\beta}) = \{u \in H; \sum_{j=0}^{\infty} \lambda_j^{2\beta}(u, \phi_j)^2 < \infty\}.$$

Hence

$$A^{\beta}u = \sum_{j=1}^{\infty} \lambda_j^{\beta}(u, \phi_j)\phi_j, \quad \text{for} \quad u = \sum_{j=1}^{\infty}(u, \phi_j)\phi_j, \quad u \in D(A^{\beta}).$$

If $\beta > 0$, the definition above for $A^{-\beta}$ is equivalent to the dual space of A^{β}, i.e., $A^{-\beta} \equiv (A^{\beta})'$.

The spaces $D(A^{\beta})$ are endowed with the norm

$$\|u\|^2_{D(A^{\beta})} = L^3 \sum_{k\in\mathbb{Z}^3} |k|^{2\beta}|\widehat{u}_k|^2.$$

We can make $D(A^{\beta})$ into a Hilbert space by using the inner product

$$((u, v))_{D(A^{\beta})} = (A^{\beta}u, A^{\beta}v),$$

i.e.,

$$((u, v))_{D(A^{\beta})} = \sum_{j=1}^{\infty} \lambda_j^{2\beta}\widehat{u}_j\widehat{v}_{-j},$$

thus,

$$((u, v))_{D(A^{\beta})} = \sum_{k\in\mathbb{Z}^3} |k|^{2\beta}\widehat{u}_k\widehat{v}_k,$$

where $u = \sum_{j=1}^{\infty} \widehat{u}_j\phi_j$ and $v = \sum_{j=1}^{\infty} \widehat{v}_j\phi_j$. This inner product gives rise to a corresponding norm

$$\|u\|_{D(A^{\beta})} = |A^{\beta}u|.$$

According to [7] (see also [22]), in the case of $\beta = \frac{1}{2}$, we have $D(A^{1/2}) = V$ and, since $\int_{\Omega} u(x)dx = 0$, Poincaré's Inequality guarantees the equivalence of norms:

$$c|A^{\frac{1}{2}}u| \leq \|u\|_{H^1(\Omega)} \leq \tilde{c}|A^{\frac{1}{2}}u| \text{ for every } u \in V.$$

We use the following notation for the norm in V, which is equivalent to the H^1-norm:

$$\|u\|:= |A^{1/2}u|= \left(\int_\Omega \sum_{j=1}^{3} \frac{\partial u}{\partial x_j}(x) \cdot \frac{\partial u}{\partial x_j}(x)dx\right)^{\frac{1}{2}} = \left(\sum_{j=0}^{\infty} \lambda_j (u, \phi_j)^2\right)^{\frac{1}{2}}.$$

We also denote the inner product in V by

$$((u, v)) := (A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v),$$

which is equivalent to the H^1-inner product, when restricted to V.

The Poincaré inequality also ensures that there exist positive constants $c, \widetilde{c}$ such that

$$c|Au|\leq \|u\|_{H^2(\Omega)}\leq \widetilde{c}|Au|,$$

hence we adopt, for future calculations, $\|\cdot\|_{H^2(\Omega)}= |A\cdot|$. The advantage of using this norm lies in the fact that there exist many properties of the operator A to be used on calculations.

More generally, we can use the regularity theory to characterize the Sobolev spaces under periodic boundary conditions $\Omega = [0, L]^3$:

$$H^s(\Omega) = \{u;\, u = \sum_{k\in\mathbb{Z}^3} c_k e^{2\pi i \frac{k\cdot x}{L}},\ k \cdot c_k = 0,\ \overline{c}_k = c_{-k},\ \sum_{k\in\mathbb{Z}^3} |k|^{2s}|c_k|^2< \infty\},$$

for $s \in \mathbb{R}$, in terms of the fractional powers of the operator A so that

$$H^s(\Omega) = D(A^{\frac{s}{2}}),$$

endowed with the norm

$$\|u\|_{H^s(\Omega)}= |A^{\frac{s}{2}}u|.$$

We shall need the following version of Poincaré's inequality, whose proof can be found in [20] and we include here the proof for the sake of completeness.

Proposition 2.1.10 (Poincaré's Inequality)**.** For all $u \in V$ and $v \in D(A)$,

$$|u|^2\leq \lambda_1^{-1}\|u\|^2 \text{ and } \|v\|^2\leq \lambda_1^{-1}|Av|^2. \tag{2.1.8}$$

Proof: Let $u \in V$ and write u in the form $u(x) = \sum_{k\in\mathbb{Z}^3} \widehat{u}\phi_k(x)$. Consider the projections

$$P_\lambda u(x) = \sum_{|k|^2\leq\lambda} \widehat{u}_k\phi_k(x) \text{ and } Q_\lambda = I - P_\lambda.$$

Note that, for $\alpha < \beta$, we have

$$\|Q_\lambda u\|_\alpha^2 = L^3 \sum_{|k|^2>\lambda} |k|^{2\alpha}|\widehat{u}_k|^2 \leq L^3\lambda^{\alpha-\beta} \sum_{|k|^2>\lambda} |k|^{2\beta}|\widehat{u}_k|^2 = \lambda^{\alpha-\beta}\|Q_\lambda u\|_\beta^2. \tag{2.1.9}$$

Since $Q_{\lambda_1}u = u$, where $\lambda_1 = (2\pi/L)^2$, then

$$\|u\|_\alpha^2 \leq \lambda_1^{\alpha-\beta}\|u\|_\beta^2 \text{ for } \alpha > \beta. \tag{2.1.10}$$

Taking $\alpha = 0, \beta = 1$, we conclude the first inequality of (2.1.8). For the second inequality, it is sufficient to take $\alpha = 1$ and $\beta = 2$. □

The continuous extension of operator A is established in the next theorem.

Theorem 2.1.11. (i) The operator A can be extended continuously to be defined on $V = D(A^{1/2})$ with values in $V' = D(A^{-1/2})$ so that

$$\langle Au, v\rangle_{V',V} = (A^{1/2}u, A^{1/2}v) = \int_\Omega (\nabla u : \nabla v)dx,$$

for every $u, v \in V$, where $(A : B) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}$, with $A = (a_{ij})$ and $B = (b_{i,j})$ matrices of order $m \times n$.

(ii) Similarly, the operator A^2 can be extended continuously to be defined on $D(A)$ with values in $D(A)'$, the dual space of the Hilbert space $D(A)$, so that

$$\langle A^2u, v\rangle_{D(A)',D(A)} = (Au, Av),$$

for every $u, v \in D(A)$.

Proof: The proof of (i) can be found in [7] and also in [22]. The proof of (ii) is a straightforward extension of that of (i).

□

Since we have the following sequence of continuous and dense embeddings:

$$D(A) \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow D(A)', \tag{2.1.11}$$

and A is a self-adjoint, the operator A can also be extended continuously to be defined on H with values in $D(A)'$ such that

$$\begin{array}{rclcccl} A : H & \longrightarrow & D(A)' & & & & \\ u & \mapsto & Au : & D(A) & \longrightarrow & \mathbb{R} & \\ & & & v & \mapsto & \langle Au, v\rangle_{D(A)',D(A)} = (u, Av). & \end{array} \tag{2.1.12}$$

Next, we present a result that will be useful in the theorem of existence for the system (2.1.1), to ensure the continuity of the solution. A proof of this fact can be found in Theorem 7.2 of [21]:

Theorem 2.1.12. Suppose that $u \in L^2([0,T];V)$ and $\dfrac{du}{dt} \in L^2([0,T];V')$. Then $u \in C([0,T];H)$, with

$$\sup_{t\in[0,T]} |u(t)| \leq C(T)(\|u\|_{L^2([0,T];V)} + \|du/dt\|_{L^2([0,T];V')}) \tag{2.1.13}$$

As a generalization of the previous lemma, for the operators A^β we have the following lemma, which is due to Lions-Magenes:

Lemma 2.1.13. For some $k \geq 0$, suppose that

$$u \in L^2([0,T];D(A^{(k+1)/2})) \ \text{ and } \ \frac{du}{dt} \in L^2([0,T];D(A^{(k-1)/2})),$$

Then u is continuous from $[0,T]$ into $D(A^{\frac{k}{2}})$. Furthermore,

$$\frac{d}{dt}|A^{\frac{k}{2}}u|^2 = \left\langle A^{\frac{k}{2}}\frac{du}{dt}, A^{\frac{k}{2}}u \right\rangle_{D(A^{k/2})',D(A^{k/2})}.$$

Proof: It can be justified in Theorem 7.2 and Corollary 7.3 of [21].

□

Since we have periodic boundary conditions and therefore $A = -\Delta$, which is a maximal monotone operator, we have the following result of Functional Analysis:

Proposition 2.1.14. If A is a maximal monotone operator, then

(i) A is closed;

(ii) For every $\eta > 0, (I + \eta A)$ is a bijection from $D(A)$ to H.

(iii) $(I + \eta A)^{-1}$ is a bounded operator and $\|(I + \eta A)^{-1}\|_{\mathcal{L}(H,H)} \leq 1$.

Proof: It can be found in Proposition 5.95 of [3]. $\square$

2.2 The NS-α nonlinearity properties

In this section we present some of the relevant properties of the non-linear term of NS-α: $u \times (\nabla \times v)$. We start regarding the nonlinear term of Navier-Stokes equations, and following the notation of classical Navier-Stokes equations theory, we denote the Leray projector of the nonlinear term as

$$B(u, v) = \mathcal{P}[(u \cdot \nabla)v] = \mathcal{P} \sum_{j=1}^{3} u_j \frac{\partial v}{\partial x_j}.$$

If div $u = 0$, then

$$B(u, v) = \mathcal{P} \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(u_j v).$$

Using the Leray projector into the autonomous Navier-Stokes system with periodic boundary conditions, we have the non-linear functional differential equation

$$\frac{\partial v}{\partial t} + \nu A v + \mathcal{P} \sum_{i=1}^{3} v_i \frac{\partial v}{\partial x_i} = \mathcal{P} f, \tag{2.2.1}$$

with div $v = 0$ and $v(0) = v_0$ in a suitable space. If we consider $\mathcal{P} f = f$, we get

$$\begin{cases} \dfrac{\partial v}{\partial t} + \nu A v + B(v, v) = f(x), \\ \text{div } \ v = 0, \ \ v(0) = v_0. \end{cases} \tag{2.2.2}$$

We set $B(v)u = B(u, v)$ for every $u, v \in V$. For every fixed $v \in V$, $B(v)$ is a linear operator acting on u. Notice that

$$(B(u, v), w) = -(B(u, w), v) \text{ for every } \ u, v, \in V.$$

Furthermore, for all $w \in D(A)$ and $u, v \in V$ we have the estimate (see [7]):

$$|\langle B(u,v), w\rangle_{D(A)',D(A)}| \leq c|u|\,\|v\|\,\|w\|_{L^\infty(\Omega)} \leq \lambda_1^{-\frac{1}{4}}|u|\,\|v\|\,|Aw|,$$

and therefore

$$\|B(u,v)\|_{D(A)'} \leq c\lambda_1^{-\frac{1}{4}}|u|\,\|v\|. \tag{2.2.3}$$

Now we are ready to consider the Navier-Stokes-α nonlinearity. In order to deal with that term, let us denote for every $u, v \in \mathcal{V}$,

$$\widetilde{B}(u,v) = -\mathcal{P}(u \times (\nabla \times v)). \tag{2.2.4}$$

Since $\mathcal{V}$ is dense in V, (2.2.4) holds for every $u, v \in V$.

Furthermore, using (1.1.2), we have for every $u, v, w \in V$,

$$(\widetilde{B}(u,v), w) = (B(u,v), w) - (B(w,v), u) = (B(v)u - B^*(v)u, w), \tag{2.2.5}$$

where $B^*(v)$ denotes the adjoint operator of the linear operator $B(v)$. As a result, we have

$$\widetilde{B}(u,v) = (B(v) - B^*(v))u \text{ for every } u, v \in V.$$

As presented in Chapter 1, the nonlinear term in (2.1.1) satisfies the following identity:

$$u \times (\nabla \times v) = \sum_{i=1}^{3}(u_i\partial_i v - u_i\nabla v_i) = (u \cdot \nabla)v - \sum_{i=1}^{3} u_i \nabla v_i, \tag{2.2.6}$$

and for $u = v$ we have

$$v \times (\nabla \times v) = (v \cdot \nabla)v - \frac{1}{2}\nabla \sum_{i=1}^{3} u_i v_i. \tag{2.2.7}$$

Since $\mathcal{P}$ projects any gradient function onto zero, i.e.,

$$\mathcal{P}\nabla\left(\sum_{i=1}^{3} u_i v_i\right) = 0,$$

we conclude by (2.2.7) that

$$\widetilde{B}(v,v) = B(v,v). \tag{2.2.8}$$

In the next lemma we state some properties and estimates about the bilinear operator $\widetilde{B}$, that are similar to properties of the operator B. The proof can be found in [9] and we will reproduce here with more details.

Lemma 2.2.1. The operator $\widetilde{B}$ can be extended continuously from $V \times V$ with values in V', and in particular it satisfies for all $u, v, w \in V$

$$|\langle \widetilde{B}(u,v), w\rangle_{V',V}| \leq c|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\,\|w\|, \tag{2.2.9}$$

and

$$|\langle \widetilde{B}(u,v), w\rangle_{V',V}| \leq c\|u\|\,\|v\|\,|w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}. \tag{2.2.10}$$

Furthermore, for every $u, v, w \in V$,

$$\langle \widetilde{B}(u,v), w\rangle_{V',V} = -\langle \widetilde{B}(w,v), u\rangle_{V',V} \tag{2.2.11}$$

and in particular, for every $u, v \in V$,

$$\langle \widetilde{B}(u,v), u\rangle_{V',V} = 0. \tag{2.2.12}$$

Moreover, we have for every $u \in H, v \in V$ and $w \in D(A)$,

$$|\langle \widetilde{B}(u,v), w\rangle_{D(A)',D(A)}| \leq c|u|\,\|v\|\,\|w\|^{\frac{1}{2}}|Aw|^{\frac{1}{2}}, \tag{2.2.13}$$

and by symmetry we have for every $u \in D(A), v \in V$ and $w \in H$,

$$|(\widetilde{B}(u,v), w)| \leq \|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}\|v\|\,|w|. \tag{2.2.14}$$

Also, for every $u \in V, v \in H$ and $w \in D(A)$,

$$|\langle \widetilde{B}(u,v), w\rangle_{D(A)',D(A)}| \leq c(|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v|\,|Aw| + |v|\,\|u\|\,\|w\|^{\frac{1}{2}}|Aw|^{\frac{1}{2}}). \tag{2.2.15}$$

In addition, for every $u \in D(A), v \in H$ and $w \in V$,

$$|\langle \widetilde{B}(u,v), w\rangle_{V',V}| \leq c(\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|v|\,\|w\| + |Au|\,|v|\,|w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}). \tag{2.2.16}$$

Proof: To prove (2.2.9), let us first consider the case when $u, v, w \in \mathcal{V}$. Using (2.2.5) and the fact that

$$(B(u,v), w) = \sum_{i,j=1}^{3} \int_\Omega u_i \left(\frac{\partial v_j}{\partial x_i}\right) w_j dx,$$

by generalized Hölder's inequality,

$$\left|\int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx\right| \leq \|u_i\|_{L^3(\Omega)} \left|\frac{\partial v_j}{\partial x_i}\right| \|w_j\|_{L^6(\Omega)},$$

since $\frac{1}{3}+\frac{1}{2}+\frac{1}{6}=1$. Therefore

$$|\langle \widetilde{B}(u,v), w\rangle_{V',V}| \leq c\|u\|_{L^3(\Omega)} |\nabla v| \, \|w\|_{L^6(\Omega)}, \tag{2.2.17}$$

Recall the following Sobolev inequalities in three dimensions:

$$\|\varphi\|_{L^3(\Omega)} \leq c\|\varphi\|^{\frac{1}{2}}_{L^2(\Omega)} \|\varphi\|^{\frac{1}{2}}_{H^1(\Omega)}, \tag{2.2.18}$$

$$\|\varphi\|_{L^6(\Omega)} \leq c\|\varphi\|_{H^1(\Omega)} \text{ for all } \varphi \in H^1(\Omega). \tag{2.2.19}$$

Using (2.2.18) and (2.2.19) into (2.2.17), we get (2.2.9) for $u, v, w \in \mathcal{V}$. Since $\mathcal{V}$ is dense in V, the result follows.

The estimate (2.2.10) follows the same steps of (2.2.9), but considering $\|u\|_{L^6(\Omega)}$ and $\|w\|_{L^3(\Omega)}$ instead of $\|u\|_{L^3(\Omega)}$ and $\|w\|_{L^6(\Omega)}$.

The identity (2.2.11) follows from the vector calculus formula:

$$(a \times b) \cdot c = \det[a,b,c] = -\det[c,b,a] = -(c \times b) \cdot a$$

for all $a, b, c \in \mathbb{R}^3$. Taking $a = u, b = \nabla \times v$ and $c = w$, we get (2.2.11). The result (2.2.12) is a particular case of (2.2.11) with $w = u$.

Let us now prove (2.2.13). Again consider firstly the case where $u, v, w \in \mathcal{V}$. Then

$$\begin{aligned} |\langle \widetilde{B}(u,v), w\rangle_{D(A)',D(A)}| &= \left|\int_\Omega [u \times (\nabla \times v)] \cdot w dx\right| \\ &\leq |u| \, |\nabla v| \, \|w\|_{L^\infty(\Omega)}. \end{aligned} \tag{2.2.20}$$

Using the three-dimensional Agmon's inequality:

$$\|\varphi\|_{L^\infty(\Omega)} \leq c\|\varphi\|^{\frac{1}{2}}_{H^1(\Omega)} \|\varphi\|^{\frac{1}{2}}_{H^2(\Omega)}, \tag{2.2.21}$$

by (2.2.20) and (2.2.21), we conclude (2.2.13):

$$|\langle \widetilde{B}(u,v), w\rangle_{D(A)',D(A)}| \leq c|u| \, \|v\| \, \|w\|^{\frac{1}{2}} |Aw|^{\frac{1}{2}}. \tag{2.2.22}$$

And again by arguments of density, we have (2.2.22) for every $u \in H, v \in V$ and $w \in D(A)$.

The proof of (2.2.14) is analogue to (2.2.13), provided that we have symetry by (2.2.11). Therefore

$$|(\widetilde{B}(u,v),w)| \leq \|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}\|v\|\,|w|,$$

for every $u \in D(A), v \in V$ and $w \in H$.

Let us prove (2.2.15). Consider again $u, v, w \in \mathcal{V}$. Using (2.2.5), we get

$$\begin{aligned} |\langle \widetilde{B}(u,v),w\rangle_{D(A)',D(A)}| &\leq \left|\int_\Omega ((u\cdot\nabla)v)\cdot w dx\right| + \left|\int_\Omega ((w\cdot\nabla)u)\cdot v dx\right| \\ &\leq \left|\int_\Omega ((u\cdot\nabla)w\cdot v)dx\right| + |v|\,|\nabla u|\,\|w\|_{L^\infty(\Omega)} \\ &\leq c\|u\|_{L^3(\Omega)}\|\nabla w\|_{L^6(\Omega)}|v| + c|v|\,\|u\|\,\|w\|_{L^\infty(\Omega)}, \end{aligned}$$

Applying (2.2.18) in $\|u\|_{L^3(\Omega)}$, (2.2.19) in $\|\nabla w\|_{L^6(\Omega)}$ and (2.2.21) in $\|w\|_{L^\infty(\Omega)}$, we have

$$|\langle \widetilde{B}(u,v),w\rangle_{D(A)',D(A)}| \leq c(|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v|\,|Aw| + |v|\,\|u\|\,\|w\|^{\frac{1}{2}}|Aw|^{\frac{1}{2}}).$$

Finally, the proof for (2.2.16) is similar (2.2.15). Note that

$$\begin{aligned} |\langle \widetilde{B}(u,v),w\rangle_{V',V}| &\leq \left|\int_\Omega ((u\cdot\nabla)v)\cdot w dx\right| + \left|\int_\Omega ((w\cdot\nabla)u)\cdot v dx\right| \\ &\leq \left|\int_\Omega ((u\cdot\nabla)w\cdot v)dx\right| + |w|_{L^3(\Omega)}\,\|\nabla u\|_{L^6(\Omega)}|v| \\ &\leq c\|u\|_{L^\infty(\Omega)}\|w\|\,|v| + c\|w\|_{L^3(\Omega)}\|\nabla u\|_{L^6(\Omega)}|v| \\ &\leq \|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}\|w\|\,|v| + c|Au|\,|v|\,|w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}, \end{aligned}$$

which completes the proof.

□

Note that from (2.2.15) follows

$$|\langle \widetilde{B}(u,v),w\rangle_{D(A)',D(A)}| \leq \|u\|\,|v|\,|Aw|$$

for all $u, v \in V$, and $w \in D(A)$. this means that $\widetilde{B}$ maps $V \times H$ into $D(A)'$ and

$$\|\widetilde{B}(u,v)\|_{D(A)'} \leq c\|u\|\,|v|.$$

cf. (2.2.3), the estimate of $B(u,v)$.

2.3 The equivalent form of NS-α system

To obtain an equivalent form of the system of equations (2.1.1), we apply the operator $\mathcal{P}$ to (2.1.1) and use the definition of the operators A and $\widetilde{B}$. So we get:

$$\begin{cases} \dfrac{d}{dt}(u+\alpha^2 Au)+\nu A(u+\alpha^2 Au)+\widetilde{B}(u,u+\alpha^2 Au)=\mathcal{P}f, \\ \text{div } \ u=0,\ u(0)=u_0, \end{cases} \tag{2.3.1}$$

where again we assume periodic boundary conditions $\Omega=[0,L]^3$ and f time-independent forcing term. For convenience, we will assume that $\mathcal{P}f=f$; otherwise, we add the gradient part of f to the modified pressure and rename $\mathcal{P}f$ by f, provided that $f=\mathcal{P}f+\nabla\phi$ (see Proposition 2.1.2).

Alternatively, if we denote

$$v=u+\alpha^2 Au,$$

the system (2.3.1) can be write in the short form

$$\begin{cases} \dfrac{dv}{dt}+\nu Av+\widetilde{B}(u,v)=f, \\ u(0)=u_0, \end{cases} \tag{2.3.2}$$

with $\operatorname{div} v=\operatorname{div} u=0$.

The solution to the system (2.3.1) is defined as follows, and can be found in Definition 2 of [9]:

Definition 2.3.1. Let $f\in H$. A function $u\in C([0,\infty);V)\cap L^2([0,\infty);D(A))$ with $\dfrac{du}{dt}\in L^2([0,\infty);H)$ is a regular solution to (2.3.1) in the interval $[0,T)$, for any $T>0$, if it satisfies, for every $g\in D(A)$:

$$\left\langle \frac{d}{dt}(u+\alpha^2 Au),g\right\rangle_{D(A)',D(A)}+\nu\langle A(u+\alpha^2 Au),g\rangle_{D(A)',D(A)} \\ +\langle \widetilde{B}(u,u+\alpha^2 Au),g\rangle_{D(A)',D(A)}=(f,g)$$

for almost every $t\in[0,T)$, and $u(0)=u_0\in V$. The above equation assumes the following sense: For every $t_0,t\in[0,T)$,

$$(u(t)+\alpha^2 Au(t),g)-(u(t_0)+\alpha^2\, u\,(t_0),g)+\nu\int_{t_0}^{t}(u(s)+\alpha^2 Au(s),Ag)ds \\ +\int_{t_0}^{t}\langle \widetilde{B}(u(s),v(s)),g\rangle_{D(A)',D(A)}ds=(f,g)(t-t_0)$$

The well posedness of the system (2.3.1) according to definition 2.3.1 was proved by E.S. Titi, C. Foias and D.D. Holm in Theorem 3 of[9]:

Theorem 2.3.2 (Global existence and uniqueness)**.** Let $f \in H$ and $u_0 \in V$. Then for any $T > 0$, the system (2.3.1) has a unique regular solution u on $[0, T)$. Moreover, this solution satisfies:

(i) $u \in L^\infty_{loc}((0, T]; H^3(\Omega))$.

(ii) There are constants R_k, for $k = 0, 1, 2, 3$, which depend only on ν, α and f, but not on u_0, such that

$$\limsup_{t\to\infty}(|A^{\frac{k}{2}}u|^2+\alpha^2|A^{\frac{k+1}{2}}u|^2) = R_k^2.$$

In particular, we have

$$R_0^2 = \frac{1}{\lambda_1}\min\left\{\frac{|A^{-\frac{1}{2}}f|^2}{\nu}, \frac{|A^{-1}f|^2}{\nu\alpha^2}\right\} \leq \min\left\{\frac{|f|^2}{\nu^2\lambda_1^2}, \frac{|f|^2}{\nu^2\lambda_1^3\alpha^2}\right\},$$

i.e.,

$$R_0^2 \leq \frac{G^2\nu^2}{\lambda_1^{1/2}}\min\left\{1, \frac{1}{\alpha^2\lambda_1}\right\} = \frac{G^2\nu^2}{\gamma\lambda_1^{1/2}},$$

where $G = \frac{|f|}{\nu^2\lambda_1^{3/4}}$ is the Grashoff number, and $\frac{1}{\gamma} = \min\{1, \frac{1}{\alpha^2\lambda_1}\}$. Furthermore, for all $t \geq 0$,

$$\limsup_{T\to\infty}\frac{\nu}{T}\int_t^{t+T}(\|u(s)\|^2+\alpha^2|Au(s)|^2)ds \leq \nu\lambda_1 R_0^2 \leq \frac{G^2\nu\lambda_1^{\frac{1}{2}}}{\gamma}.$$

Since in chapter 4, we will analyze the behavior of $w(t) - u(t)$ when time goes to infinity, where $w(t)$ is the solution of the system (1.5.4), it is necessary to know the behavior of the solutions of NS-α equations when $t \to \infty$. In other words, we need to know the global attractor of the semigroup $S(t)$ of the solution operator to system (2.3.1), i.e., $u(t) = S(t)u_0$, in terms of physical parameters of the equation (2.1.1).

Proposition 2.3.3. Fix $T > 0$. Let G the Grashoff number $G = \dfrac{|f|}{\nu^2\lambda_1^{3/4}}$ and suppose that u is the solution given by Theorem 2.3.2. Then there exists a time t_0, which depends on u_0, such that for $t \geq t_0 > 0$ we have

$$\|u(t)\|^2 \leq \frac{2G^2\nu^2}{\lambda_1^{1/2}\alpha^2} \tag{2.3.3}$$

Moreover,

$$\int_t^{t+T} \|u(s)\|^2 + \alpha^2 |Au(s)|^2 ds \le (2 + \nu\lambda_1 T)\frac{\nu G^2}{\lambda_1^{1/2}} \tag{2.3.4}$$

Proof: The proofs of (2.3.3) and (2.3.4) follows from some of the estimates obtained in Theorem 3 of [9]. For completeness of the thesis, we reproduce here: taking the L^2-inner product of (2.3.1) with u and using (2.2.12), we have

$$\frac{1}{2}\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) = (f, u) \tag{2.3.5}$$

Note that

$$\begin{aligned} |(f,u)| = |(A^{-\frac{1}{2}}f, A^{\frac{1}{2}}u)| &\le |A^{-\frac{1}{2}}f|\,\|u\| \le \frac{\nu}{2}\|u\|^2 + \frac{1}{2\nu}\|f\|_{V'}^2 \\ &\le \frac{\nu}{2}(\|u\|^2 + \alpha^2|Au|^2) + \frac{1}{2\nu\lambda_1}|f|^2 \end{aligned}$$

Therefore

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \nu(\|u\|^2 + \alpha^2|Au|^2) \le \frac{|f|^2}{\nu\lambda_1}. \tag{2.3.6}$$

Using Poincaré's Inequality, we obtain

$$\frac{d}{dt}(|u|^2 + \alpha^2\|u\|^2) + \nu\lambda_1(|u|^2 + \alpha^2\|u\|^2) \le \frac{1}{\nu\lambda_1}|f|^2, \tag{2.3.7}$$

and by Gronwall's Inequality,

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \le (|u_0|^2 + \alpha^2\|u_0\|^2)e^{-\nu\lambda_1 t} + \frac{1}{\nu^2\lambda_1^2}|f|^2. \tag{2.3.8}$$

Thus, there exists $t_0 > 0$ depending on $|u_0|$ and $\|u_0\|$ such that, if $t \ge t_0$,

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \le \frac{2|f|^2}{\nu^2\lambda_1^2} = \frac{2G^2\nu^2}{\lambda_1^{1/2}}, \tag{2.3.9}$$

and consequently,

$$\|u(t)\|^2 \le \frac{2|f|^2}{(\nu\lambda_1\alpha)^2} = \frac{2G^2\nu^2}{\alpha^2\lambda_1^{1/2}}.$$

To obtain (2.3.4), we integrate (2.3.6) over the interval $(t, t+T)$:

$$|u(t+T)|^2 + \alpha^2\|u(t+T)\|^2 - (|u(t)|^2 + \alpha^2\|u(t)\|^2) + \nu\int_t^{t+T} \|u(s)\|^2 + \alpha^2|Au(s)|^2 ds \le \frac{|f|^2}{\nu\lambda_1}T$$

Using (2.3.9), it follows that, for all $t \geq t_0$,

$$\nu \int_t^{t+T} \|u(s)\|^2 + \alpha^2 |Au(s)|^2 ds \leq \frac{|f|^2}{\nu\lambda_1} T + \frac{2|f|^2}{\nu^2\lambda_1^2} = G^2\nu^3\lambda_1^{\frac{1}{2}} T + 2G^2\nu^2\lambda_1^{-\frac{1}{2}}$$

and we conclude that

$$\int_t^{t+T} \|u(s)\|^2 + \alpha^2 |Au(s)|^2 ds \leq (2 + \nu\lambda_1 T)\nu\lambda_1^{-\frac{1}{2}} G^2.$$

$\square$

The next three theorems are essential to prove the results of existence for the system (1.5.4). The first one is the well-known Picard Theorem for ODE's:

Theorem 2.3.4 (Picard)**.** Let $O \subseteq B$ be an open subset of a Banach space B, and let $F(X)$ a nonlinear operator satisfying the following criteria:

(i) F maps O to B.

(ii) $F(X)$ is locally Lipschitz continuous, i.e., for any $X \in O$, there exists $L > 0$ and an open neighborhood $U_X \subset O$ do X such that

$$\|F(X_1) - F(X_2)\|_B \leq L\|X_1 - X_2\|_B \quad \text{for all} \quad X_1, X_2 \in U_X$$

Then for any $X_0 \in O$, there exists a time T such that the ODE

$$\frac{dX}{dt} = F(X), \quad X(0) = X_0 \in O$$

has a unique local solution $X \in C^1([0,T]; O)$.

The other two theorems are compacteness theorems. The next one is the well-known Banach-Alaoglu Theorem:

Theorem 2.3.5 (Weak Compactness)**.** Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is bounded. Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and $u \in X$ such that $u_{k_j} \rightharpoonup u$.

The third theorem is the Aubin-Lion Theorem, whose proof can be found in [18], but for completeness of the thesis, we will prove it. Before, we prove an auxiliary lemma:

Lemma 2.3.6. Let B_0, B and B_1 be three Banach spaces such that $B_0 \hookrightarrow B \hookrightarrow B_1$. and the embedding $B_0 \hookrightarrow B$ is compact. Then for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$\|v\|_B \leq \eta \|v\|_{B_0} + c_\eta \|v\|_{B_1}. \tag{2.3.10}$$

Proof: Suppose that the statement is false. So for some $\eta > 0$, there exists $v_m \in B_0$ and $c_m \to \infty$ such that

$$\|v_m\|_B > \eta \|v_m\|_{B_0} + c_m \|v_m\|_{B_1}.$$

Considering $w_m = v_m / \|v_m\|_{B_0}$, we have

$$\|w_m\|_B > \eta + c_m \|w_m\|_{B_1} > \eta, \tag{2.3.11}$$

and $\|w_m\|_B \leq k_1$, $\|w_m\|_{B_0} = 1$. From (2.3.11), we obtain

$$\|w_m\|_{B_1} \to 0$$

. But $\|w_m\|_{B_0} = 1$ and since $B_0 \hookrightarrow B$ is compact, one can extract a subsequence w_{m_k} that converges strongly in B and necessarily it converges to 0; i.e., $\|w_{m_k}\|_B \to 0$, which contradicts (2.3.11).

$\square$

Theorem 2.3.7 (Aubin-Lion)**.** Let B_0, B and B_1 be three Banach spaces such that

$$B_0 \hookrightarrow B \hookrightarrow B_1$$

where the embedding are continuous, B_0 and B_1 are reflexive and the embedding $B_0 \hookrightarrow B$ is compact.

Let $T > 0$ be a fixed finite number, and let $1 < p_0, p_1 < \infty$ and regard the space

$$W = \{u \in L^{p_0}([0,T]; B_0), \quad u' = \frac{du}{dt} \in L^{p_1}([0,T]; B_1)\}$$

which is provided with the norm

$$\|u\|_W = \|u\|_{L^{p_0}([0,T];B_0)} + \|u'\|_{L^{p_1}([0,T];B_1)}$$

Then the embedding $W \hookrightarrow L^{p_0}([0,T]; B)$ is compact.

Proof: Let $\{v_m\}_{m=1}^{\infty}$ be a bounded sequence in W, and we will denote for simplicity $v_m \in W$. The aim is to prove that there exists a subsequence $\{v_{m_k}\}_{k=1}^{\infty}$ of v_m such that $v_{m_k} \to v$ strongly in $L^{p_0}([0,T];B)$. Since $L^{p_0}([0,T];B_0)$ is reflexive, one can extract a subsequence $v_{m_k} \to v$ weakly in W, i.e.,

$$v_{m_k} - v \to 0 \text{ weakly in } W.$$

Changing the notations, the problem turns into as follows: let v_m a sequence in W such that $v_m \to 0$ weakly in W. Then

$$v_m \to 0 \text{ strongly in } L^{p_0}([0,T];B). \tag{2.3.12}$$

Indeed, for all $\eta > 0$, there exists c_η by Lemma 2.3.6 such that

$$\|v_m\|_B \leq \eta \|v_m\|_{B_0} + c_\eta \|v_m\|_{B_1},$$

and therefore, for all $\eta > 0$, there exists d_η such that

$$\|v_m\|_{L^{p_0}(0,T;B)} \leq \eta \|v_m\|_{L^{p_0}(0,T;B_0)} + d_\eta \|v_m\|_{L^{p_0}(0,T;B_1)}. \tag{2.3.13}$$

Let $\varepsilon > 0$. Since

$$\|v_m\|_{L^{p_0}(0,T;B)} \leq c, \tag{2.3.14}$$

we can choose $\eta > 0$ such that $\eta < \dfrac{\varepsilon}{2c}$. Hence from (2.3.13) and (2.3.14),

$$\|v_m\|_{L^{p_0}(0,T;B)} \leq \frac{\varepsilon}{2} + d_\eta \|v_m\|_{L^{p_0}(0,T;B_1)}.$$

Consequently, to prove (2.3.12) is sufficient to show that

$$v_m \to 0 \text{ strongly in } L^{p_0}([0,T];B_1). \tag{2.3.15}$$

Indeed, we have

$$\|v_m(t)\|_{B_1} \leq C, \tag{2.3.16}$$

provided $W \subset C([0,T];B_1)$. According to Lebesgue Dominated Convergence Theorem, we have (2.3.15) if we prove that, for all $s \in [0,T]$,

$$v_m(s) \to 0 \text{ strongly in } B_1 \tag{2.3.17}$$

Without loss of generality, one can suppose $s = 0$; i.e., prove that

$$v_m(0) \to 0 \text{ strongly in } B_1. \tag{2.3.18}$$

We define

$$w_m(t) = v_m(\lambda t), \ \ \lambda > 0 \text{ fixed.}$$

Then $v_m(0) = w_m(0)$ and

$$\begin{aligned} \|w_m\|^{p_0}_{L^{p_0}([0,T];B_0)} &= \int_0^T \|w_m(s)\|^{p_0}_{B_0} ds = \int_0^T \|v_m(\lambda s)\|^{p_0}_{B_0} ds \\ &= \int_0^T \|v_m(t)\|^{p_0}_{B_0} \frac{1}{\lambda} dt \leq \frac{1}{\lambda}\overline{c}, \end{aligned}$$

and therefore

$$\|w_m\|_{L^{p_0}([0,T];B_0)} \leq \widetilde{c}\lambda^{-\frac{1}{p_0}} \tag{2.3.19}$$

Similarly, we get

$$\|w'_m\|_{L^{p_1}(0,T;B_1)} \leq \widetilde{c}\lambda^{1-\frac{1}{p_1}}. \tag{2.3.20}$$

Moreover, if φ is a function in $[0,T]$ with $\varphi(0) = -1$ and $\varphi(T) = 0$, then

$$w_m(0) = \int_0^T [\varphi(t)w_m(t)]' dt = \int_0^T \varphi'(t)w_m(t)dt + \int_0^T \varphi(t)w'_m(t)dt.$$

Denoting $\beta_m = \int_0^T \varphi(t)w'_m(t)dt$ and $\gamma_m = \int_0^T \varphi'(t)w_m(t)dt$, we have from (2.3.20):

$$\|v_m(0)\|_{B_1} \leq \|\beta_m\|_{B_1} + \|\gamma_m\|_{B_1} \leq c\lambda^{1-\frac{1}{p_1}} + \|\gamma_m\|_{B_1}.$$

If $\varepsilon > 0$, we choose $\lambda > 0$ such that

$$\widetilde{c}\lambda^{1-\frac{1}{p_1}} \leq \frac{\varepsilon}{2},$$

and prove (2.3.18) is therefore to prove that

$$\gamma_m \to 0 \text{ strongly in } B_1.$$

Provided that we can assume $\lambda \leq 1$ and $w_m(t) = v_m(\lambda t)$, we have $w_m \to 0$ weakly in $L^{p_0}(0,T;B_0)$ and thus $\gamma_m \to 0$ weakly in B_0. By assumption, $B_0 \hookrightarrow B_1$ is compact and as a result $\gamma_m \to 0$ strongly in B_1, which completes de proof.

$\square$

Chapter 3

Global well-posedness and Uniqueness

In section 1, we present the definition of a regular solution to the problem (1.5.4) and prove the existence of a solution for two cases of I_h (see (3.1.2) and (3.1.3) below.)

In section 2, we prove the uniqueness of the solutions for both cases.

3.1 Existence

Consider the continuous data assimilation equations for the incompressible Navier-Stokes-α equations, as presented in Chapter 1, $\Omega = [0, L]^3$, under periodic boundary conditions:

$$\begin{cases} \dfrac{\partial}{\partial t}(w - \alpha^2 \Delta w) - \nu \Delta (w - \alpha^2 \Delta w) - w \times (\nabla \times z) + \nabla p \\ \qquad\qquad\qquad\qquad = f - \mu (I_h w - I_h u) + \mu \alpha^2 \Delta (I_h w - I_h u), \\ \operatorname{div} \ w = 0, \end{cases} \tag{3.1.1}$$

on the interval $[0, T]$, $z = w - \alpha^2 \Delta w$ with initial condition $w(0) = w_0 \in V$ chosen arbitrarily and $I_h(u(t))$ representing our observations of the Navier-Stokes-α system. We will deal with this problem in two cases: when the interpolant $I_h : \dot{H}^1(\Omega) \to \dot{L}^2(\Omega)$ satisfies

$$|\varphi - I_h \varphi|^2 \leq c_1^2 h^2 |\nabla \varphi|^2 \quad \text{for every} \quad \varphi \in \dot{H}^1(\Omega), \tag{3.1.2}$$

and when the interpolant $I_h : \dot{H}^2(\Omega) \to \dot{L}^2(\Omega)$ satisfies

$$|\varphi - I_h \varphi|^2 \leq c_2^2 h^2 |\nabla \varphi|^2 + c_2^2 h^4 \|\varphi\|_{H^2(\Omega)}^2 \quad \text{for every} \quad \varphi \in \dot{H}^2(\Omega). \tag{3.1.3}$$

Applying the Leray Projector and using the functional setting presented in Chapter 2, the above system is equivalent to

$$\begin{cases} \dfrac{d}{dt}(w+\alpha^2 Aw)+\nu A(w+\alpha^2 Aw)+ \ \widetilde{B}(w,w+\alpha^2 Aw) \\ \qquad\qquad\qquad\qquad\qquad\qquad = f-\mu\mathcal{P}(I-\alpha^2\Delta)(I_h(w)-I_h(u)), \\ \text{div } \ w=0, \end{cases} \tag{3.1.4}$$

In order to rewrite the system (3.1.4) in a simpler way, we have the following lemma:

Lemma 3.1.1. Suppose $\varphi \in \dot{H}^2(\Omega)$. Then $\mathcal{P}\varphi \in \dot{H}^2(\Omega)$ and $-\mathcal{P}\Delta\varphi = -\mathcal{P}\Delta\mathcal{P}\varphi = A\mathcal{P}\varphi$.

Proof. If $\varphi \in \dot{H}^2(\Omega)$, by the Helmholtz decomposition, there exists a unique $\psi \in V$ and $p \in \dot{H}^1(\Omega)$ such that $\varphi = \psi + \nabla p$, with div $\psi = 0$ and $\mathcal{P}\varphi = \psi$. Moreover, we also have $\Delta p = \operatorname{div}\varphi \in \dot{H}^1(\Omega)$, and it follows that $p \in \dot{H}^3(\Omega)$. Since $\varphi \in \dot{H^2}(\Omega)$, we conclude that $\psi \in H^2(\Omega)$. On the other hand,

$$-\Delta\varphi = -\Delta\psi - \nabla(\Delta p),$$

and consequently,

$$-\mathcal{P}\Delta\varphi = -\Delta\psi = -\Delta\mathcal{P}\varphi. \tag{3.1.5}$$

This also implies that

$$A\varphi = -\mathcal{P}(\Delta\varphi) = -\mathcal{P}^2(\Delta\varphi) = -\mathcal{P}\Delta(\mathcal{P}\varphi) = A\mathcal{P}\varphi.$$

$\square$

Using Lemma 3.1.5, the system (3.1.4) is equivalent to

$$\begin{cases} \dfrac{d}{dt}(w+\alpha^2 Aw)+\nu A(w+\alpha^2 Aw)+\widetilde{B}(w,w+\alpha^2 Aw) \\ \qquad\qquad\qquad\qquad\qquad\qquad = f-\mu(I+\alpha^2 A)\mathcal{P}(I_h(w)-I_h(u)), \\ \text{div } \ w=0, \end{cases} \tag{3.1.6}$$

on the interval $[0,T]$, with $w(0)=w_0 \in V, z = w+\alpha^2 Aw$. Furthermore, inequalities (3.1.2) and (3.1.3) become

$$|\mathcal{P}(\varphi - I_h\varphi)|^2 \leq c_1^2 h^2 \|\varphi\|^2, \text{ for every } \varphi \in V, \tag{3.1.7}$$

and

$$|\mathcal{P}(\varphi - I_h\varphi)|^2 \le c_2^2 h^2 \|\varphi\|^2 + c_2^2 h^4 |A\varphi|^2, \text{ for every } \varphi \in D(A). \tag{3.1.8}$$

We present next the definition of a regular solution to the system (3.1.6).

Definition 3.1.2. Let $f \in H$ and $T > 0$. A function $w \in C([0,T);V) \cap L^2([0,T);D(A))$ with $\dfrac{du}{dt} \in L^2([0,T);H)$ is a regular solution to (3.1.6) on the interval $[0,T)$ if it satisfies, for every $g \in D(A)$:

$$\begin{aligned} \left\langle \frac{d}{dt}(w + \alpha^2 Aw), g \right\rangle_{D(A)',D(A)} &+ \nu \langle A(w + \alpha^2 Aw), g\rangle_{D(A)',D(A)} \\ &+ \langle \widetilde{B}(w, w + \alpha^2 Aw), g\rangle_{D(A)',D(A)} = (f,g) \\ -\mu\langle \mathcal{P}(I_h w - I_h u), g\rangle_{D(A)',D(A)} &- \mu\alpha^2 \langle A\mathcal{P}(I_h w - I_h u), g\rangle_{D(A)',D(A)} \end{aligned} \tag{3.1.9}$$

for almost every $t \in [0,T)$, and $w(0) = w_0 \in V$. The above equation assumes the following sense: for every $t_0, t \in [0,T)$,

$$\begin{aligned} (w(t) + \alpha^2 Aw(t), g) - (w(t_0) + \alpha^2 w(t_0), g) + \nu \int_{t_0}^{t} (w(s) + \alpha^2 Aw(s), Ag) ds \\ + \int_{t_0}^{t} \langle \widetilde{B}(w(s), z(s)), g\rangle_{D(A)',D(A)} ds = (f,g)(t - t_0) \\ -\mu \int_{t_0}^{t} (\mathcal{P}(I_h w(s) - I_h u(s)), g) ds - \mu\alpha^2 \int_{t_0}^{t} (I_h w(s) - I_h u(s), Ag) ds \end{aligned} \tag{3.1.10}$$

Theorem 3.1.3. Let $f \in H, w_0 \in V$ and $\mu > 0$ given. Suppose that I_h satisfies (3.1.2) (and hence (3.1.7)) and $\mu c_1^2 h^2 < \dfrac{\nu}{2}$, where $c_1 > 0$ is the constant given in (3.1.7). Let u be the solution of NS-α equations with initial data $u(0) = u_0 \in V$, ensured by Theorem 2.3.2. Then the continuous data assimilation equations (3.1.6) have a regular solution w on $[0,T)$ for any $T > 0$ in the sense of definition 3.1.2.

Proof: Firstly, we apply the bounded operator $(I + \alpha^2 A)^{-1} \in \mathcal{L}(H,H)$ in the equation (3.1.6) and using that

$$(I + \alpha^2 A)^{-1} \frac{d}{dt}(I + \alpha^2 A)w = \frac{d}{dt}(I + \alpha^2 A)^{-1}(I + \alpha^2 A)w = \frac{dw}{dt}$$

and

$$(I + \alpha^2 A)^{-1} A (I + \alpha^2 A)w = A(I + \alpha^2 A)^{-1}(I + \alpha^2 A)w = Aw,$$

we obtain

$$\frac{dw}{dt} + \nu Aw + (I + \alpha^2 A)^{-1}\widetilde{B}(w, z) = (I + \alpha^2 A)^{-1} f - \mu\mathcal{P}(I_h w - I_h u) \tag{3.1.11}$$

Note that to prove the existence of the solution to the equation (3.1.11) is equivalent to prove the existence of solution to (3.1.6). Define

$$\overline{f}(s) = (I + \alpha^2 A)^{-1} f + \mu\mathcal{P} I_h u(s).$$

Note that for all $s \in [0, T]$,

$$|\mathcal{P} I_h u(s)| \leq |\mathcal{P}(u(s) - I_h u(s))| + |u(s)| \leq c_1 h \|u(s)\| + |u(s)|. \tag{3.1.12}$$

Since the Navier-Stokes-α solution u satisfies $u \in C([0, T]; V)$, we conclude that $I_h u \in C([0, T]; H)$. Moreover, we have

$$\begin{aligned} |\overline{f}| &\leq |(I + \alpha^2 A)^{-1} f| + \mu|\mathcal{P} I_h u| \\ &\leq |f| + \mu|\mathcal{P} I_h u| \\ &\leq |f| + \mu c_1 h\|u\| + \mu|u|, \end{aligned}$$

and therefore $\overline{f} \in C([0, T]; H)$, i.e., there exists a constant M such that $|\overline{f}| < M$ for every $t \in [0, T]$.

The purpose now is to stabilish the global existence of solutions to (3.1.6). For that, we use the Faedo-Galerkin method. Let $H_m = \text{spam}\{\phi_1, \ldots, \phi_m\}$, where $A\phi_j = \lambda_j \phi_j$. We denote by P_m the orthogonal projection from H onto H_m. Let $w_m \in H_m$ satisfy the finite-dimensional Faedo-Galerkin system of ordinary differential equations:

$$\begin{cases} \dfrac{dw_m}{dt} + \nu A w_m + P_m (I + \alpha^2 A)^{-1} \widetilde{B}(w_m, z_m) = P_m \overline{f} - \mu P_m \mathcal{P} I_h w_m \\ u_m(0) = P_m u_0, \end{cases} \tag{3.1.13}$$

Since system (3.1.13) has a quadratic non-linearity, therefore it is locally Lipschitz and as a result by theorem 2.3.4, it has a unique short time solution. The next step is to prove that the solution is uniformly bounded in time and m; and thereby we shall ensure the global existence in time of w_m for all m.

Denote by $[0, T_m^{\max})$ the maximal interval of existence for (3.1.13). Our goal is to show that $T_m^{\max} = T$. Focusing on $[0, T_m^{\max})$, we take the dual spaces action $D(A)', D(A)$ on w_m in (3.1.13), we have

$$\frac{1}{2}\frac{d}{dt}|w_m|^2+\nu\|w_m\|^2+\langle P_m(I+\alpha^2 A)^{-1}\widetilde{B}(w_m,z_m),w_m\rangle_{D(A)',D(A)} = (P_m\overline{f},w_m)-\mu(\mathcal{P}I_h w_m,w_m), \tag{3.1.14}$$

where $\langle \frac{dw_m}{dt}, w_m\rangle_{D(A)',D(A)} = \frac{1}{2}\frac{d}{dt}|w_m|^2$ is due to Lemma 2.1.13.

Taking L^2-inner product of (3.1.13) with Aw_m and then multiplying the equation by α^2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\alpha^2\|w_m\|^2+\nu\alpha^2|Aw_m|^2+ & (P_m(I+\alpha^2 A)^{-1}\widetilde{B}(w_m,z_m),\alpha^2 Aw_m) \\ = & (P_m\overline{f},\alpha^2 Aw_m)-\mu(\mathcal{P}I_h w_m,\alpha^2 Aw_m).\end{aligned} \tag{3.1.15}$$

Adding (3.1.14) and (3.1.15), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+ & \nu(\|w_m\|^2+\alpha^2|Aw_m|^2) \\ + & (P_m(I+\alpha^2 A)^{-1}\widetilde{B}(w_m,z_m),w_m+\alpha^2 Aw_m) \\ = & (P_m\overline{f},w_m+\alpha^2 Aw_m)-\mu(\mathcal{P}I_h w_m,w_m+\alpha^2 Aw_m).\end{aligned} \tag{3.1.16}$$

Taking into account that A is self-adjoint, we have that $(I+\alpha^2 A)$ is self-adjoint and therefore

$$\begin{aligned}((I+\alpha^2 A)^{-1}\widetilde{B}(w_m,z_m),(I+\alpha^2 A)w_m) &= ((I+\alpha^2 A)(I+\alpha^2 A)^{-1}\widetilde{B}(w_m,z_m),w_m) \\ &= (\widetilde{B}(w_m,z_m),w_m)=0.\end{aligned} \tag{3.1.17}$$

Using (3.1.17) and the symmetry of $\mathcal{P}$, we have from (3.1.16),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\nu(\|w_m\|^2+ \alpha^2|Aw_m|^2) &= (P_m\overline{f},w_m)+\alpha^2(P_m\overline{f},Aw_m) \\ &- \mu(I_h w_m,w_m)-\mu\alpha^2(I_h w_m,Aw_m).\end{aligned}$$

By Young's inequality we get

$$(P_m\overline{f},w_m)\le|\overline{f}|\,|w_m|\le\frac{\mu}{4}|w_m|^2+\frac{1}{\mu}|\overline{f}|^2,$$

$$\alpha^2(P_m\overline{f}, Aw_m) \le \alpha^2|\overline{f}|\,|Aw_m| \le \frac{\nu}{4}\alpha^2|Aw_m|^2 + \frac{\alpha^2}{\nu}|\overline{f}|^2,$$

and we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2) \ &+\nu(\|w_m\|^2+\alpha^2|Aw_m|^2)\\ &\le \ \frac{1}{\mu}|\overline{f}|^2+\frac{\mu}{4}|w_m|^2+\frac{\alpha^2}{\nu}|\overline{f}|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2\\ &- \ \mu(I_hw_m, w_m) - \mu\alpha^2(I_hw_m, Aw_m).\end{aligned}$$

Including $\pm\mu(w_m, w_m)$ and $\pm\mu\alpha^2(w_m, Aw_m)$ on the right-hand side of the inequality above,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+ \ \nu(\|w_m\|^2+\alpha^2|Aw_m|^2) &\le \left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)|\overline{f}|^2\\ &+ \ \frac{\mu}{4}|w_m|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2\\ &+ \ \mu(w_m - I_hw_m, w_m) - \mu|w_m|^2\\ &+ \ \mu\alpha^2(w_m - I_hw_m, Aw_m) - \mu\alpha^2\|w_m\|^2.\end{aligned}$$

Note that, using Cauchy-Schwarz inequality and the condition (3.1.7), we have

$$(w_m - I_hw_m, w_m) = (w_m - I_hw_m, \mathcal{P}w_m) \le |\mathcal{P}(w_m - I_hw_m)|\,|w_m| \le c_1h\|w_m\|\,|w_m|,$$

and

$$(w_m - I_hw_m, Aw_m) = (w_m - I_hw_m, \mathcal{P}Aw_m) \le |\mathcal{P}(w_m - I_hw_m)|\,|Aw_m| \le c_1h\|w_m\|\,|Aw_m|,$$

and therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+ \ \nu(\|w_m\|^2+\alpha^2|Aw_m|^2) &\le \left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)|\overline{f}|^2\\ &+ \ \frac{\mu}{4}|w_m|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2\\ &+ \ \mu c_1h\|w_m\|\,|w_m| - \mu|w_m|^2\\ &+ \ \mu\alpha^2c_1h\|w_m\|\,|Aw_m| - \mu\alpha^2\|w_m\|^2.\end{aligned} \tag{3.1.18}$$

Note that by Young's inequality,

$$\mu c_1h\|w_m\|\,|w_m| = \mu^{\frac{1}{2}}|w_m|\mu^{\frac{1}{2}}c_1h\|w_m\| \le \frac{\mu}{4}|w_m|^2+\mu c_1^2h^2\|w_m\|^2 \tag{3.1.19}$$

and

$$\mu\alpha^2 c_1 h\|w_m\|\,|Aw_m|= \mu^{\frac{1}{2}}\alpha\|w_m\|\mu^{\frac{1}{2}}c_1 h\alpha|Aw_m|\leq \frac{\mu}{2}\alpha^2\|w_m\|^2+\frac{\mu c_1^2h^2}{2}\alpha^2|Aw_m|^2. \tag{3.1.20}$$

Replacing (3.1.19) and (3.1.20) into (3.1.18), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+ \ \alpha^2\|w_m\|^2) + \nu(\|w_m\|^2+\alpha^2|Aw_m|^2)&\\ \leq& \left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)|\overline{f}|^2+\frac{\mu}{4}|w_m|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2\\ +& \frac{\mu}{4}|w_m|^2+\mu c_1^2h^2\|w_m\|^2-\mu|w_m|^2\\ +& \frac{\mu}{2}\alpha^2\|w_m\|^2+\frac{\mu c_1^2h^2}{2}\alpha^2|Aw_m|^2-\mu\alpha^2\|w_m\|^2.\end{aligned}$$

Under the hyphotesis that h is sufficiently small so that $\mu c_1^2h^2 < \dfrac{\nu}{2}$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+ \ \alpha^2\|w_m\|^2) + \nu(\|w_m\|^2+\alpha^2|Aw_m|^2)&\\ \leq& \left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)|\overline{f}|^2+\frac{\mu}{4}|w_m|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2\\ +& \frac{\mu}{4}|w_m|^2+\frac{\nu}{2}\|w_m\|^2-\mu|w_m|^2\\ +& \frac{\mu}{2}\alpha^2\|w_m\|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2-\mu\alpha^2\|w_m\|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+ \ \alpha^2\|w_m\|^2) + \frac{\nu}{2}(\|w_m\|^2+\alpha^2|Aw_m|^2)&\\ \leq& \left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)|\overline{f}|^2-\frac{\mu}{2}(|w_m|^2+\alpha^2\|w_m\|^2).\end{aligned} \tag{3.1.21}$$

Using the Poincaré's inequality,

$$\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2) + (\nu\lambda_1+\mu)(|w_m|^2+\alpha^2\|w_m\|^2) \leq 2\left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)|\overline{f}|^2, \tag{3.1.22}$$

for all $t\in[0,T_m^{\max})$. By Gronwall's inequality we conclude that for all $t\in[0,T_m^{\max})$,

$$|w_m(t)|^2+\alpha^2\|w_m(t)\|^2\leq (|w_0|^2+\alpha^2\|w_0\|^2)e^{-(\nu\lambda_1+\mu)t}+2\left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)\frac{M}{\nu\lambda_1+\mu}(1-e^{-(\nu\lambda_1+\mu)t}).$$

Since $e^{-(\nu\lambda_1+\mu)t}\leq 1$ and $1-e^{-(\nu\lambda_1+\mu)t}\leq 1$ for all $t\in[0,T_m^{\max})$, we reach

$$|w_m(t)|^2+\alpha^2\|w_m(t)\|^2\leq (|w_0|^2+\alpha^2\|w_0\|^2)+2\left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)\frac{M}{\nu\lambda_1+\mu} := M_1. \tag{3.1.23}$$

Since the right-hand side of (3.1.23) is bounded, then $T_m^{\max} = T$, otherwise we can extend the solution beyond $T_m^{\max}$, which contradicts the definition of $T_m^{\max}$.

The estimate (3.1.23) is uniform in m and t, and therefore we have the global existence of w_m in time and also

$$\|w_m\|^2_{L^\infty([0,T];V)} \leq \frac{M_1}{\alpha^2} \quad \text{and} \quad \|z_m\|^2_{L^\infty([0,T];V')} \leq M_1. \tag{3.1.24}$$

Additionally, we shall find H^2-estimates for w_m. From (3.1.21) we get

$$\frac{d}{dt}(|w_m(t)|^2+\alpha^2\|w_m(t)\|^2) + \nu(\|w_m(t)\|^2+|Aw_m(t)|^2) \leq 2\left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)M. \tag{3.1.25}$$

With (3.1.25) in hand, we integrate over the interval $(0,t)$:

$$\begin{aligned} |w_m(t)|^2+\alpha^2\|w_m(t)\|^2+\nu\int_0^t \|w_m(s)\|^2 +\alpha^2|Aw_m(s)|^2 ds \\ \leq |w_m(0)|^2+\alpha^2\|w_m(0)\|^2+2\left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)Mt, \end{aligned}$$

and it follows that

$$\int_0^t \|w_m(s)\|^2+\alpha^2|Aw_m(s)|^2 ds \leq \frac{1}{\nu}(|w_0|^2+\alpha^2\|w_0\|^2) + 2\left(\frac{1}{\mu}+\frac{\alpha^2}{\nu}\right)\frac{MT}{\nu} := M_2(T). \tag{3.1.26}$$

Hence,

$$\|w_m\|^2_{L^2([0,T];D(A))} \leq \frac{M_2(T)}{\alpha^2} \quad \text{and} \quad \|z_m\|^2_{L^2([0,T];H)} \leq M_2(T). \tag{3.1.27}$$

Note that from (3.1.26) we also obtain

$$\|w_m\|^2_{L^2([0,T];V)} \leq M_2(T). \tag{3.1.28}$$

Now we establish uniform estimates in m for derivates $\dfrac{dw_m(t)}{dt}$ and $\dfrac{dz_m(t)}{dt}$. Returning to equation

$$\begin{aligned} \frac{dz_m(t)}{dt} + \nu Az_m(t) + P_m\widetilde{B}(w_m,z_m) = P_mf + \mathcal{P}I_h(u(t)) + P_mA\mathcal{P}I_h(u(s)) \\ - \mu P_m\mathcal{P}I_h(w_m(t)) - \mu\alpha^2 P_mA\mathcal{P}I_h(w_m(t)), \end{aligned} \tag{3.1.29}$$

we shall estimate $\dfrac{dz_m(t)}{dt}$ in $L^2([0,T];D(A)')$. Note that by (2.2.15),

$$\begin{aligned} |\widetilde{B}(w_m(t),z_m(t))| &\leq k_2(|w_m(t)|^{1/2}\|w_m(t)\|^{1/2}|z_m(t)|+\lambda_1^{-1/4}|z_m(t)|\|w_m(t)\|) \\ &\leq 2k_2\lambda_1^{-1/4}|z_m(t)|\|w_m(t)\|. \end{aligned}$$

Consequently, and thanks to (3.1.24) and (3.1.27), we have

$$\begin{aligned}\|\widetilde{B}(w_m, z_m)\|^2_{L^2([0,T];H)} &= \int_0^T |\widetilde{B}(w_m(s), z_m(s))|^2 ds \\ &\leq \frac{4k_2^2}{\lambda_1^{1/2}} \int_0^T |z_m(s)|^2 \|w_m(s)\|^2 ds \\ &\leq \frac{4k_2^2 M_1}{\lambda_1^{1/2}} \int_0^T |z_m(s)|^2 ds \\ &= \frac{4k_2^2 M_1}{\lambda_1^{1/2}} \|z_m\|^2_{L^2([0,T];H)} \leq \frac{4k_2^2 M_1}{\lambda_1^{1/2}} M_2(T).\end{aligned}$$

To estimate the right-hand side of (3.1.29), we use the fact that $I_h(u) \in C([0,T];H)$ and so $A\mathcal{P}I_h(u) \in C([0,T];D(A)')$. Moreover, we have the two following estimates:

$$|I_h(w_m)| \leq |I_h(w_m) - w_m| + |w_m| \leq c_1 h \|w_m\| + |w_m|,$$

$$\|AI_h(w_m)\|_{D(A)'} = |A^{-1}AI_h(w_m)| \leq |I_h w_m - w_m| + |w_m| \leq c_1 h \|w_m\| + |w_m|$$

Therefore, we conclude that

$$\left\|\frac{dz_m}{dt}\right\|^2_{L^2([0,T];H)} \leq M_3(\nu, \lambda_1, \overline{f}, \alpha, T) \text{ and } \left\|\frac{dw_m}{dt}\right\|^2_{L^2([0,T];D(A)')} \leq M_4(\nu, \lambda_1, \overline{f}, \alpha, T),$$

for some M_3 and M_4.

The next step is to extract subsequences which are convergent in some related spaces. For that, we will make use of the Compactness Theorems 2.3.5 and 2.3.7 .

First, consider the space

$$Y = \{z_m \in L^2([0,T];H), \quad \frac{dz_m}{dt} \in L^2([0,T];D(A)')\}.$$

Since $D(A) \subset V \subset H$ and the injection $V \subset H$ is compact, we have

$$H \equiv H' \subset V' \subset D(A)',$$

with $H \subset V'$ a compact injection. Thus by Theorem 2.3.7, the injection $Y \subset L^2([0,T];V')$ is compact, i.e., there exists a subsequence $\{z_{m_j}\}_{j=1}^\infty$ of $\{z_m\}_{m=1}^\infty$, which from this point, we denote with the same label $\{z_m\}_{m=1}^\infty$ such that

$$z_m \to z \quad \text{strongly in} \quad L^2([0,T];V'), \tag{3.1.30}$$

or equivalently,

$$w_m \to w \quad \text{strongly in} \quad L^2([0,T];V). \tag{3.1.31}$$

Second, by (3.1.27), we have that $\{w_m\}$ is bounded in $L^2([0,T];D(A))$, which is a Hilbert space and in particular, a reflexive space. Using Theorem 2.3.5 we conclude that there exists a subsequence of $\{w_m\}$ such that

$$w_m \to w \quad \text{weakly in} \quad L^2([0,T];D(A)), \tag{3.1.32}$$

or equivalently,

$$z_m \to z \quad \text{weakly in} \quad L^2([0,T];H). \tag{3.1.33}$$

With these convergences in hand, we need to pass the limit when m goes to infinity in the following equation, coming from (3.1.13):

$$\begin{aligned}(z_m(t),g) - (z_m(t_0),g) + \nu\int_{t_0}^{t}(z_m(s),Ag)ds + \int_{t_0}^{t}\langle \widetilde{B}(w_m(s),z_m(s)),P_m g\rangle_{D(A)',D(A)}ds \\ = (f,P_m g)(t-t_0) - \mu\int_{t_0}^{t}(I_h w_m(s) - I_h u(s),g)ds - \mu\alpha^2\int_{t_0}^{t}(I_h w_m(s) - I_h u(s),Ag)ds,\end{aligned}$$

for every $g \in D(A)$ and for all $t_0, t \in [0,T]$. We shall analyze each one of these terms in details.

1. Terms $(z_m(t),g)$ and $(z_m(t_0),g)$:

Since $z_m \to z$ weakly in $L^2([0,T];H)$, we have that $z_m(s) \to z(s)$ weakly in H for all $t \in [0,T]\backslash G$, where med(G)=0. Then $F(z_m(s)) \to F(z(s))$ as $m \to \infty$ strongly for all $F \in H' \equiv H$. For $g \in D(A)$ fixed, but arbitrary, regard F the mapping $F = (\cdot,g) : h \in H \mapsto (h,g)$. Then $F \in H'$ and we conclude, as $m \to \infty$,

$$(z_m(t),g) \to (z(t),g) \quad \text{and} \quad (z_m(t_0),g) \to (z(t_0),g).$$

(2) Term $\int_{t_0}^{t}(z_m(s),Ag)ds$:

Using that $z_m \to z$ weakly in $L^2([0,T];H)$, we construct a linear functional $F \in [L^2([0,T];H)]' \equiv L^2([0,T];H)$ and then we apply the property $F(z_m) \to F(z)$ as $m \to \infty$.

$$\begin{aligned} F : L^2(0,T;H) &\longrightarrow \mathbb{R} \\ z &\longmapsto \int_{t_0}^{t}(z(s),Ag)ds. \end{aligned}$$

F is well-defined and continuous:

$$\begin{aligned}|F(z)| &\leq \int_{t_0}^{t}|(z(s),Ag)|ds \leq \int_{t_0}^{t}|z(s)||Ag|ds \\ &\leq \left(\int_{t_0}^{t}|z(s)|^2 ds\right)^{\frac{1}{2}}\left(\int_{t_0}^{t}|Ag|^2 ds\right)^{\frac{1}{2}} \\ &\leq T^{\frac{1}{2}}|Ag|\|z\|_{L^2([0,T];H)}.\end{aligned}$$

Therefore as $m \to \infty$,

$$\int_{t_0}^{t}(z_m(s),Ag)ds \to \int_{t_0}^{t}(z(s),Ag)ds.$$

(3) Term $\int_{t_0}^{t}\langle \widetilde{B}(w_m(s),z_m(s)),P_m g\rangle_{D(A)',D(A)}ds$:

We have to prove that, as $m \to \infty$,

$$\left|\int_{t_0}^{t}(\widetilde{B}(w_m(s),z_m(s)),P_m g) - \langle \widetilde{B}(w(s),z(s)),g\rangle_{D(A)',D(A)}ds\right| \to 0.$$

Note that

$$\begin{aligned}&\left|\int_{t_0}^{t}(\widetilde{B}(w_m(s),z_m(s)),P_m g) - \langle \widetilde{B}(w(s),z(s)),g\rangle_{D(A)',D(A)}ds\right| \\ &\leq \left|\int_{t_0}^{t}(\widetilde{B}(w_m(s),z_m(s)),P_m g) - \langle \widetilde{B}(w_m(s),z_m(s)),g\rangle_{D(A)',D(A)}ds\right| \\ &+\left|\int_{t_0}^{t}\langle \widetilde{B}(w_m(s),z_m(s)),g\rangle_{D(A)',D(A)} - \langle \widetilde{B}(w(s),z_m(s)),g\rangle_{D(A)',D(A)}ds\right| \\ &+\left|\int_{t_0}^{t}\langle \widetilde{B}(w(s),z_m(s)),g\rangle_{D(A)',D(A)} - \langle \widetilde{B}(w(s),z(s)),g\rangle_{D(A)',D(A)}ds\right|.\end{aligned}$$

The first term on the right-hand side of the inequality can be estimated as follows:

$$\begin{aligned}&|\int_{t_0}^{t}\langle \widetilde{B}(w_m(s),z_m(s)),\ P_m g - g\rangle_{D(A)',D(A)}ds| \\ &\leq \int_{t_0}^{t}\frac{c}{\lambda_1^{1/4}}\|w_m(s)\||z_m(s)||P_m Ag - Ag|ds \\ &= \frac{c}{\lambda_1^{1/4}}|P_m Ag - Ag|\int_{t_0}^{t}\|w_m(s)\||z_m(s)|ds \\ &= \frac{c}{\lambda_1^{1/4}}|P_m Ag - Ag|\left(\int_0^T\|w_m(s)\|^2\right)^{\frac{1}{2}}\left(\int_0^T|z_m(s)|^2 ds\right)^{\frac{1}{2}} \\ &\leq \frac{c}{\lambda_1^{1/4}}|P_m Ag - Ag|\|w_m\|_{L^2([0,T];V)}\|z_m\|_{L^2([0,T];H)} \\ &\leq \frac{c}{\lambda_1^{1/4}}|P_m Ag - Ag|M_2(T),\end{aligned}$$

where we used Hölder's Inequality, (3.1.27), (3.1.28) and the following inequality for every $u \in V, v \in H$ and $w \in D(A)$ (see Lemma 2.2.1, item (2.2.15)):

$$|\langle \widetilde{B}(u,v), w\rangle_{D(A)',D(A)}| \leq c(|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v||Aw| + |v|\|u\|\|w\|^{\frac{1}{2}}|Aw|^{\frac{1}{2}}). \tag{3.1.34}$$

Since $|P_m Ag - Ag| \to 0$ as $m \to \infty$, it follows that

$$\left|\int_{t_0}^{t} \langle \widetilde{B}(w(s), z(s)), P_m g - g\rangle_{D(A)',D(A)} ds\right| \to 0. \tag{3.1.35}$$

Similarly, the second term,

$$\begin{aligned}
&|\int_{t_0}^{t} \langle \widetilde{B}(w_m(s) - w(s), z_m(s)), g\rangle_{D(A)',D(A)} ds| \\
&\qquad \leq \int_{t_0}^{t} \frac{c}{\lambda_1^{1/4}} \|w_m(s) - w(s)\||z_m(s)||Ag| ds \\
&\qquad = \frac{c}{\lambda_1^{1/4}} |Ag| \int_{t_0}^{t} \|w_m(s) - w(s)\||z_m(s)| ds \\
&\qquad = \frac{c}{\lambda_1^{1/4}} |Ag| \left(\int_0^T \|w_m(s) - w(s)\|^2\right)^{\frac{1}{2}} \left(\int_0^T |z_m(s)|^2 ds\right)^{\frac{1}{2}} \\
&\qquad \leq \frac{c}{\lambda_1^{1/4}} |Ag| \|w_m - w\|_{L^2([0,T];V)} \|z_m\|_{L^2([0,T];H)} \\
&\qquad \leq \frac{c}{\lambda_1^{1/4}} |Ag| M_2(T)^{1/2} \|w_m - w\|_{L^2([0,T];V)}.
\end{aligned}$$

Furthermore, from (3.1.31),

$$\left|\int_{t_0}^{t} \langle \widetilde{B}(w_m(s) - w(s), z_m(s)), g\rangle_{D(A)',D(A)} ds\right| \to 0. \tag{3.1.36}$$

To estimate the third term

$$\left|\int_{t_0}^{t} \langle \widetilde{B}(w(s), z_m(s) - z(s)), g\rangle_{D(A)',D(A)} ds\right|,$$

we note that, since $z_m \to z$ weakly in $L^2([0,T];H)$, we have for any $F \in [L^2([0,T];H)]' \equiv L^2([0,T];H)$ the convergence $F(z_m) \to F(z)$, as $m \to \infty$. For $w \in L^2([0,T];V)$ and $g \in D(A)$ fixed, but arbitrary, consider the linear function

$$\begin{aligned}
F : L^2(0,T;H) &\longrightarrow \mathbb{R} \\
z &\longmapsto \int_{t_0}^{t} \langle \widetilde{B}(w(s), z(s)), g\rangle_{D(A)',D(A)} ds.
\end{aligned}$$

F is well-defined and continuous:

$$
\begin{aligned}
|F(z)| = |\int_{t_0}^{t} \langle \widetilde{B}(w(s), z(s)\ ), g\rangle_{D(A)', D(A)} ds| & \\
&\leq \frac{c}{\lambda_1^{1/4}} |Ag| \int_{t_0}^{t} \|w(s)\| |z(s)| ds \\
&\leq \frac{c}{\lambda_1^{1/4}} |Ag| \left(\int_0^T \|w(s)\|^2 \right)^{\frac{1}{2}} \left(\int_0^T |z(s)|^2 ds \right)^{\frac{1}{2}} \\
&= \frac{c}{\lambda_1^{1/4}} |Ag| \|w\|_{L^2([0,T];V)} \|z\|_{L^2([0,T];H)}.
\end{aligned}
$$

Then $F \in [L^2([0,T];H)]'$ and $F(z_m) \to F(z)$ as $m \to \infty$, which implies

$$
\int_0^T \langle \widetilde{B}(w(s), z_m(s)), g\rangle_{D(A)', D(A)} ds \to \int_0^T \langle \widetilde{B}(w(s), z(s)), g\rangle_{D(A)', D(A)} ds,
$$

i.e.,

$$
\left| \int_{t_0}^{t} \langle \widetilde{B}(w(s), z_m(s) - z(s)), g\rangle_{D(A)', D(A)} ds \right| \to 0. \tag{3.1.37}
$$

From (3.1.35), (3.1.36) and (3.1.37) we conclude that

$$
\int_{t_0}^{t} (\widetilde{B}(w_m(s), z_m(s)), P_m g) \to \int_{t_0}^{t} \langle \widetilde{B}(w(s), z(s)), g\rangle_{D(A)', D(A)} ds,
$$

as $m \to \infty$.

(4) Term $(f, P_m g)(t - t_0)$:

Note that

$$
(f, P_m g) = (f, P_m g - g) + (f, g) \leq |f| |P_m g - g| + (f, g) \to (f, g),
$$

as $m \to \infty$, since $|P_m g - g| \to 0$. Hence

$$
(f, P_m g)(t - t_0) \to (f, g)(t - t_0), \quad \text{as} \quad m \to \infty.
$$

(5) Term $\int_{t_0}^{t} (I_h w_m(s) - I_h u(s), g) ds$:

To prove that $\int_{t_0}^{t} (I_h w_m(s) - I_h u(s), g) ds \to \int_{t_0}^{t} (I_h w(s) - I_h u(s), g) ds$, as $m \to \infty$, it is sufficient to prove that

$$
\int_{t_0}^{t} (I_h w_m(s), g) ds \to \int_{t_0}^{t} (I_h w(s), g) ds.
$$

First of all, note that $I_h|_{D(A)}\colon \dot{H}^2(\Omega) \to \dot{L}^2(\Omega)$ can be seen as a continuous linear operator. Indeed, if $w \in D(A)$,

$$|I_h w|\leq |I_h w - w|+|w|\leq \frac{c_1 h}{\lambda_1^{1/2}}|Aw|+\frac{1}{\lambda_1}|Aw|= \left(\frac{c_1 h}{\lambda_1^{1/2}} + \frac{1}{\lambda_1}\right)\|w\|_{D(A)}. \tag{3.1.38}$$

Now, provided that $w_m \to w$ weakly in $L^2([0,T];D(A))$, we have that $F(w_m) \to F(w)$ for every $F \in [L^2([0,T];D(A))]' \equiv L^2([0,T];D(A)')$. For $g \in D(A)$, consider the linear function:

$$\begin{array}{rcl} F : L^2(0,T;D(A)) & \longrightarrow & \mathbb{R} \\ w & \longmapsto & \displaystyle\int_{t_0}^{t}(I_h w(s), g)ds. \end{array}$$

F is well-defined and continuous: If $w \in L^2([0,T];D(A))$, from Hölder's Inequality and (3.1.38),

$$\begin{aligned} |F(w)|= |&\int_{t_0}^{t} (I_h w(s), g)ds| \\ \leq &\int_{t_0}^{t}|I_h w(s)||g|ds = |g|\int_0^T |I_h w(s)|ds \\ \leq &|g|\left(\frac{ch}{\lambda_1^{1/2}} + \frac{1}{\lambda_1}\right)\int_0^T \|w(s)\|_{D(A)}ds \\ \leq &|g|\left(\frac{ch}{\lambda_1^{1/2}} + \frac{1}{\lambda_1}\right)T^{\frac{1}{2}}\left(\int_0^T \|w(s)\|^2_{D(A)}ds\right)^{\frac{1}{2}} \\ = &|g|\left(\frac{ch}{\lambda_1^{1/2}} + \frac{1}{\lambda_1}\right)T^{\frac{1}{2}}\|w\|_{L^2([0,T];D(A))} \end{aligned}$$

Therefore $F \in L^2([0,T];D(A)')$ and $F(w_m) \to F(w)$, as $m \to \infty$, i.e.,

$$\int_{t_0}^{t}(I_h w_m(s), g)ds \to \int_{t_0}^{t}(I_h w(s), g)ds,$$

which implies

$$\int_{t_0}^{t}(I_h w_m(s) - I_h u(s), g)ds \to \int_{t_0}^{t}(I_h w(s) - I_h u(s), g)ds.$$

(6) Term $\int_{t_0}^{t}(I_h w_m(s) - I_h u(s), Ag)ds$:

For this term, we proceed as the preceding term, only changing the funcional F. If $g \in D(A)$, define F as

$$\begin{array}{rcl} F : L^2(0,T;D(A)) & \longrightarrow & \mathbb{R} \\ w & \longmapsto & \displaystyle\int_{t_0}^{t}(I_h w(s), Ag)ds. \end{array}$$

Likewise $F \in L^2([0,T]; D(A)')$ and equally, $F(w_m) \to F(w)$ as $m \to \infty$, i.e.,

$$\int_{t_0}^{t} (I_h w_m(s), Ag) ds \to \int_{t_0}^{t} (I_h w(s), Ag) ds,$$

then

$$\int_{t_0}^{t} (I_h w_m(s) - I_h u(s), Ag) ds \to \int_{t_0}^{t} (I_h w(s) - I_h u(s), Ag) ds,$$

and therefore we have completed the estimates. Finally, passing the limit in (3.1.34) as $m \to \infty$, we have

$$\begin{aligned}&(z(t), g) - (z(t_0), g) + \nu \int_{t_0}^{t} (z(s), Ag) ds + \int_{t_0}^{t} \langle \widetilde{B}(w(s), z(s)), Pg \rangle_{D(A)', D(A)} ds = \\ &(f, Pg)(t - t_0) - \mu \int_{t_0}^{t} (I_h w(s) - I_h u(s), g) ds - \mu\alpha^2 \int_{t_0}^{t} (I_h w(s) - I_h u(s), Ag) ds,\end{aligned} \tag{3.1.39}$$

for every $g \in D(A)$ and $t_0, t \in [0,T] \backslash G$.

To conclude the theorem and ensure the solution in the sense of definition 3.1.2, we claim that $w \in C([0,T], V)$ (and equivalently, $v \in C([0,T]; V')$). Indeed, from (**??**), we can extract a subsequence such

$$\frac{dw_k}{dt} \to \dot{w} \text{ weakly in } L^2([0,T]; H),$$

and we have written $\dot{w}$ because it is not immediately obvious that in fact $\dot{w} = dw/dt$. However, if we use the definiton of weak convergence of dw_k/dt to $\dot{w}$ we have

$$\int_0^T \frac{dw_k}{dt}(t)\psi(t) dt \to \int_0^T \dot{w}(t)\psi(t) dt,$$

for all $\psi \in L^2([0,T]; H)$. Now, if $\psi \in C_c^\infty([0,T]; H)$ then we can integrate the left-hand side by parts to give

$$\begin{aligned}\int_0^T \frac{dw_k}{dt}(t)\psi(t) dt &= -\int_0^T w_k(t) \frac{d\psi}{dt}(t) dt \\ &\to -\int_0^T w_k(t) \frac{d\psi}{dt}(t) dt,\end{aligned}$$

using the weak convergence of w_k to w, since $d\psi/dt \in C_c^\infty([0,T]; H) \subset L^2([0,T]; H)$. Therefore we have

$$\int_0^T \dot{w}(t)\psi(t) dt = -\int_0^T w(t) \frac{d\psi}{dt}(t) dt \text{ for all } \psi \in C_c^\infty([0,T]; H),$$

and so $\dot{w} = dw/dt$ as required.

Thus we have that $w \in L^2([0,T]; D(A))$ and $dw/dt \in L^2([0,T]; H)$, and we conclude by Lemma 2.1.13 (taking $k = 1$) that $u \in C([0,T]; V)$. Hence

$$\begin{aligned} &\frac{d}{dt}(w + \alpha^2 Aw) + \nu A(w + \alpha^2 Aw) + \widetilde{B}(w, w + \alpha^2 Aw) \\ &\qquad\qquad = f - \mu\mathcal{P}(I_h(w) - I_h(u)) - \mu\alpha^2 A\mathcal{P}(I_h(w) - I_h(u)), \end{aligned} \tag{3.1.40}$$

and we have the existence of a regular solution for (4.1.3).

□

Consider now the same system (3.1.6), but now with the interpolant I_h satisfying

$$|\mathcal{P}(\varphi - I_h\varphi)|^2 \leq c_2^2 h^2 \|\varphi\|^2 + c_2^2 h^4 |A\varphi|^2 \tag{3.1.41}$$

instead of (3.1.7). So we have the following result of existence:

Theorem 3.1.4. Let $f \in H, w_0 \in V$ and $\mu > 0$ given. Suppose that I_h satisfies (3.1.41) (and hence (3.1.8)) and the two conditions below are valid:

$$\mu\bar{c}_2 h^2 < \frac{\nu}{2} \tag{3.1.42}$$

and

$$\mu c_2^2 h^4 < \frac{\nu\alpha^2}{2}, \tag{3.1.43}$$

where $c_2 > 0$ is the constant given in (3.1.41) and $\bar{c}_2 = \max\{c_2, c_2^2\}$. Let u be the solution of NS-α equations with initial data $u(0) = u_0 \in V$, ensured by Theorem 2.3.2. Then the continuous data assimilation equations (4.1.3) for Navier-Stokes-α have a regular solution w on $[0,T)$ for any $T > 0$ in the sense of definition (3.1.2).

Proof: The proof is similar to that of Theorem 3.1.3; so we will exhibit here only the main steps. Firstly, the proof follows the same method as up to inequality (3.1.18):

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(|w_m|^2 + \alpha^2\|w_m\|^2) + \nu(\|w_m\|^2 + \alpha^2|Aw_m|^2) \leq{}& \left(\frac{1}{\mu} + \frac{2\alpha^2}{\nu}\right)|\overline{f}|^2 \\ &+ \frac{\mu}{4}|w_m|^2 + \frac{\nu}{8}\alpha^2|Aw_m|^2 \\ &+ \mu(w_m - I_h w_m, w_m) - \mu|w_m|^2 \\ &+ \mu\alpha^2(w_m - I_h w_m, Aw_m) - \mu\alpha^2\|w_m\|^2. \end{aligned}$$

Using Cauchy-Schwarz and Young's inequality,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\ \nu(\|w_m\|^2+\alpha^2|Aw_m|^2) \le& \left(\frac{1}{\mu}+\frac{2\alpha^2}{\nu}\right)|\overline{f}|^2\\ +&\ \frac{\mu}{4}|w_m|^2+\frac{\nu}{8}\alpha^2|Aw_m|^2\\ +&\ \frac{\mu}{2}|\mathcal{P}(w_m-I_hw_m)|^2+\frac{\mu}{2}|w_m|^2-\mu|w_m|^2\\ +&\ \frac{\mu^2\alpha^2}{\nu}|\mathcal{P}(w_m-I_hw_m)|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2-\mu\alpha^2\|w_m\|^2.\end{aligned}$$

Applying (3.1.41), we proceed in a similar way as Theorem 3.1.3:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\ \nu(\|w_m\|^2+\alpha^2|Aw_m|^2) \le& \left(\frac{1}{\mu}+\frac{2\alpha^2}{\nu}\right)|\overline{f}|^2\\ +&\ \frac{\mu}{4}|w_m|^2+\frac{3\nu}{8}\alpha^2|Aw_m|^2\\ +&\ \frac{\mu c_2^2h^2}{2}\|w_m\|^2+\frac{\mu c_2^2h^4}{2}|Aw_m|^2-\frac{\mu}{2}|w_m|^2\\ +&\ \frac{\mu^2c_2^2h^2}{\nu}\alpha^2\|w_m\|^2+\frac{\mu^2c_2^2h^4}{\nu}\alpha^2|Aw_m|^2-\mu\alpha^2\|w_m\|^2.\end{aligned}$$

By assumption, h satisfies

$$\mu\bar{c}_2h^2<\frac{\nu}{2},\ \text{ and }\ \mu c_2^2h^4<\frac{\nu\alpha^2}{2}.$$

Therefore

$$\mu c_2^2h^2\le\mu\bar{c}_2h^2<\frac{\nu}{2},$$

and

$$\mu^2c_2^2h^4=\mu c_2h^2\cdot\mu c_2h^2<\mu\bar{c}_2h^2\cdot\mu\bar{c}_2h^2<\frac{\nu}{2}\cdot\frac{\nu}{2}=\frac{\nu^2}{4},$$

so we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\ \nu(\|w_m\|^2+\alpha^2|Aw_m|^2) \le& \left(\frac{1}{\mu}+\frac{2\alpha^2}{\nu}\right)|\overline{f}|^2\\ +&\ \frac{\mu}{4}|w_m|^2+\frac{3\nu}{8}\alpha^2|Aw_m|^2\\ +&\ \frac{\nu}{4}\|w_m\|^2+\frac{\nu\alpha^2}{4}|Aw_m|^2-\frac{\mu}{2}|w_m|^2\\ +&\ \frac{\mu}{2}\alpha^2\|w_m\|^2+\frac{\nu}{4}\alpha^2|Aw_m|^2-\mu\alpha^2\|w_m\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\nu(\|w_m\|^2+\alpha^2|Aw_m|^2)&\\ \leq \left(\frac{1}{\mu}+\frac{2\alpha^2}{\nu}\right)|\overline{f}|^2+\frac{\nu}{4}\|w_m\|^2+\frac{7\nu}{8}\alpha^2|Aw_m|^2-\frac{\mu}{4}|w_m|^2-\frac{\mu}{2}\alpha^2\|w_m\|^2,&\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2) &+ \frac{\nu}{8}(\|w_m\|^2+\alpha^2|Aw_m|^2)\\ &\leq \left(\frac{1}{\mu}+\frac{2\alpha^2}{\nu}\right)|\overline{f}|^2-\frac{\mu}{4}(|w_m|^2+\alpha^2\|w_m\|^2),\end{aligned}$$

and using the Poincaré Inequality,

$$\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\left(\frac{\mu}{2}+\frac{\lambda_1\nu}{4}\right)(|w_m|^2+\alpha^2\|w_m\|^2)\leq\left(\frac{2}{\mu}+\frac{4\alpha^2}{\nu}\right)|\overline{f}|^2. \tag{3.1.44}$$

By Gronwall's Inequality, it follows that

$$\begin{aligned}|w_m(t)|^2+\alpha^2\|w_m(t)\|^2 &\leq (|w_0|^2+\alpha^2\|w_0\|^2)e^{-(\frac{\mu}{2}+\frac{\lambda_1\nu}{4})t}\\ &+ \left(\frac{2}{\mu}+\frac{4\alpha^2}{\nu}\right)\left(\frac{\mu}{2}+\frac{\lambda_1\nu}{4}\right)^{-1}\cdot M(1-e^{-(\frac{\mu}{2}+\frac{\lambda_1\nu}{4})t}),\end{aligned}$$

where $M>0$ is the same constant of Theorem 3.1.3. Taking into account that $e^{-(\frac{\mu}{2}+\frac{\lambda_1\nu}{4})t}\leq 1$ and $(1-e^{-(\frac{\mu}{2}+\frac{\lambda_1\nu}{4})t})\leq 1$, we have

$$|w_m(t)|^2+\alpha^2\|w_m(t)\|^2\leq(|w_0|^2+\alpha^2\|w_0\|^2)+(\frac{2}{\mu}+\frac{4\alpha^2}{\nu})(\frac{\mu}{2}+\frac{\nu\lambda_1}{4})^{-1}M:=\widetilde{M}_1. \tag{3.1.45}$$

Thus

$$\|w_m\|^2_{L^\infty([0,T];V)}\leq\frac{\widetilde{M}_1}{\alpha^2}\ \text{ or }\ \|w_m\|^2_{L^\infty([0,T];V')}\leq\widetilde{M}_1.$$

For H^2-estimates, we have from (3.1.44):

$$\frac{d}{dt}(|w_m|^2+\alpha^2\|w_m\|^2)+\frac{\nu}{4}(\|w_m\|^2+\alpha^2|Aw_m|^2)\leq\left(\frac{2}{\mu}+\frac{4\alpha^2}{\nu}\right)M, \tag{3.1.46}$$

and integrating (3.1.46) from 0 to t, we reach

$$\int_0^t\|w_m(s)\|^2+\alpha^2|Aw_m(s)|^2ds\leq\frac{4}{\nu}(|w_0|^2+\alpha^2\|w_o\|^2)+\frac{4}{\nu}\left(\frac{2}{\mu}+\frac{4\alpha^2}{\nu}\right)MT:=\widetilde{M}_2(T).$$

Therefore it follows that

$$\|w_m\|^2_{L^2([0,T];D(A))} \leq \frac{\widetilde{M}_2(T)}{\alpha^2},$$
$$\|z_m\|^2_{L^2([0,T];D(A))} \leq \widetilde{M}_2(T)$$

and

$$\|w_m\|^2_{L^2([0,T];V)} \leq \widetilde{M}_2(T).$$

To estimate $\dfrac{dw_m}{dt}$ in $L^2([0,T];H)$ and $\dfrac{dz_m}{dt}$ in $L^2([0,T];D(A)')$, we use the same technique presented in Theorem 3.1.3, only substituing (**??**) for the following estimate:

$$\begin{aligned}|P_m\mathcal{P}I_hw_m| \leq |\mathcal{P}(I_hw_m - w_m)| + |w_m| &\leq c_2h\|w_m\| + c_2h^2|Aw_m| + |w_m| \\ &\leq (c_2h\lambda_1^{-1/2} + c_2h^2 + \lambda_1^{-1})|Aw_m|,\end{aligned}$$

where we used that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. Therefore

$$\begin{aligned}\|P_m\mathcal{P}I_hw_m\|^2_{L^2([0,T];H)} &= (c_2h\lambda_1^{-1/2} + c_2h^2 + \lambda_1^{-1})^2\int_0^T |Aw_m(s)|^2ds \\ &\leq (c_2h\lambda_1^{-1/2} + c_2h^2 + \lambda_1^{-1})^2\frac{\widetilde{M}_2(T)}{\alpha^2}.\end{aligned}$$

The estimate $\dfrac{dz_m}{dt}$ in $L^2([0,T];D(A)')$ follows as estimate (**??**).

Following the proof steps of Theorem 3.1.3, we can extract all subsequences desired with no changes, using the compacteness theorems cited in that theorem.

Now, we are interested in passing the limit, when $t \to \infty$, in

$$\begin{aligned}&(z_m(t), g) - (z_m(t_0), g) + \nu\int_{t_0}^t (z_m(s), Ag)ds + \int_{t_0}^t \langle \widetilde{B}(w_m(s), z_m(s)), P_mg\rangle_{D(A)',D(A)}ds \\ &= (f, P_mg)(t - t_0) - \mu\int_{t_0}^t (I_hw_m(s) - I_hu(s), g)ds - \mu\alpha^2\int_{t_0}^t (I_hw_m(s) - I_hu(s), Ag)ds.\end{aligned}$$

We are concerned only with proving that

$$\int_{t_0}^t (I_hw_m(s) - I_hu(s), g)ds \to \int_{t_0}^t (I_hw(s) - I_hu(s), g)ds. \tag{3.1.47}$$

Indeed, note that $I_h|_{D(A)}\colon H^2(\Omega) \to L^2(\Omega)$ is a continuous linear operator: the proof is similar to inequality (3.1.38):

$$
\begin{aligned}
|I_h w|^2 &\leq (|I_h w - w| + |w|)^2 \leq 2|I_h w - w|^2 + 2|w|^2 \\
&\leq 2c_2^2 h^2 \|w\|^2 + 2c_2^2 h^4 |Aw|^2 + \frac{2}{\lambda_1^2}|Aw|^2 \\
&\leq \left(\frac{2c_2^2 h^2}{\lambda_1} + 2c_2^2 h^4 + \frac{2}{\lambda_1^2}\right)|Aw|^2.
\end{aligned}
$$

We continue in the same way as in Theorem 3.1.3 to obtain (3.1.47), and we conclude the proof.

□

3.2 Uniqueness of solutions

The strategy for showing the uniqueness to (3.1.6) is via continuous dependence of regular solutions on the initial data.

We divide in two cases:

(i) the interpolant I_h satisfying (3.1.7);

(ii) the interpolant I_h satisfying (3.1.8).

Theorem 3.2.1. The solution to the problem (3.1.6) given by Theorem 3.1.3 is unique.

Proof: Let w and $\overline{w}$ be two solutions of (4.1.3) on the interval $[0, T]$, with initial data $w(0) = w_0$ and $\overline{w}(0) = \overline{w}_0$, respectively. Denote $z = w + \alpha^2 Aw$ and $\overline{z} = \overline{w} + \alpha^2 A\overline{w}$. Then taking the difference of

$$
\begin{aligned}
&\frac{d}{dt}(w + \alpha^2 Aw) + \nu A(w + \alpha^2 Aw) + \widetilde{B}(w, w + \alpha^2 Aw) \\
&\qquad = f - \mu\mathcal{P}(I_h(w) - I_h(u)) - \mu\alpha^2 A(I_h(w) - I_h(u)),
\end{aligned}
\tag{3.2.1}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}(\overline{w} + \alpha^2 A\overline{w}) + \nu A(\overline{w} + \alpha^2 A\overline{w}) + \widetilde{B}(\overline{w}, \overline{w} + \alpha^2 A\overline{w}) \\
&\qquad = f - \mu\mathcal{P}(I_h(\overline{w}) - I_h(u)) - \mu\alpha^2 A(I_h(\overline{w}) - I_h(u)),
\end{aligned}
\tag{3.2.2}
$$

and denoting $\theta = \overline{w} - w$, we have

$$\frac{d}{dt}(\theta + \alpha^2 A\theta) + \nu A(\theta + \alpha^2 A\theta) + \widetilde{B}(\overline{w}, \overline{z}) - \widetilde{B}(w, z) = -\mu \mathcal{P} I_h \theta - \mu\alpha^2 A\theta. \tag{3.2.3}$$

Notice that

$$\begin{aligned} \widetilde{B}(\overline{w}, \overline{z}) - \widetilde{B}(w, z) &= \widetilde{B}(\overline{w}, \overline{z}) - \widetilde{B}(w, \overline{z}) + \widetilde{B}(w, \overline{z}) - \widetilde{B}(w, z) \\ &= \widetilde{B}(\overline{w} - w, \overline{z}) + \widetilde{B}(w, \overline{z} - z) \\ &= \widetilde{B}(\theta, \overline{z}) + \widetilde{B}(w, \theta + \alpha^2 A\theta). \end{aligned} \tag{3.2.4}$$

Replacing (3.2.4) into (3.2.3) and taking the dual action $D(A)', D(A)$ on θ,

$$\begin{aligned} \left\langle \frac{d}{dt}(\theta + \alpha^2 A\theta), \theta \right\rangle_{D(A)', D(A)} &+ \nu \langle A(\theta + \alpha^2 A\theta), \theta \rangle_{D(A)', D(A)} + \langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta \rangle_{D(A)', D(A)} \\ &= -\mu(I_h\theta, \theta) - \mu\alpha^2 \langle A I_h \theta, \theta \rangle_{D(A)', D(A)}. \end{aligned}$$

Consequently, since $\langle \widetilde{B}(\theta, \overline{z}), \theta \rangle_{D(A)', D(A)} = 0$, using (2.1.12) and Lemma 2.1.13 , we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(|\theta|^2 + \alpha^2 \|\theta\|^2) + \nu(\|\theta\|^2 + \alpha^2 |A\theta|^2) &+ \langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta \rangle_{D(A)', D(A)} \\ &= -\mu(I_h\theta, \theta) - \mu\alpha^2 (I_h\theta, A\theta). \end{aligned}$$

In the same way done in Theorem 3.1.3, we estimate the following terms using Young's inequality:

$$\begin{aligned} -\mu(I_h\theta, \theta) &= \mu(\theta - I_h\theta, \theta) - \mu|\theta|^2 \\ &\leq \mu|\mathcal{P}(\theta - I_h\theta)|\,|\theta| - \mu|\theta|^2 \\ &\leq \mu c_1 h \|\theta\|\,|\theta| - \mu|\theta|^2 \\ &\leq \frac{\mu c_1^2 h^2}{2}\|\theta\|^2 + \frac{\mu}{2}|\theta|^2 - \mu|\theta|^2 \\ &\leq \frac{\nu}{4}\|\theta\|^2 - \frac{\mu}{2}|\theta|^2, \end{aligned} \tag{3.2.5}$$

and similarly,

$$\begin{aligned} -\mu\alpha^2(I_h\theta, A\theta) &= \mu\alpha^2(\theta - I_h\theta, A\theta) - \mu\alpha^2\|\theta\|^2 \\ &\leq \mu\alpha^2|\mathcal{P}(\theta - I_h\theta)|\,|A\theta| - \mu\alpha^2\|\theta\|^2 \\ &\leq \mu\alpha^2 c_1^2 h\|\theta\|\,|A\theta| - \mu\alpha^2\|\theta\|^2 \\ &\leq \alpha^2\mu c_1^2 h^2 |A\theta|^2 + \frac{\mu\alpha^2}{2}\|\theta\|^2 - \mu\alpha^2\|\theta\|^2 \\ &\leq \frac{\nu}{4}\alpha^2|A\theta|^2 - \frac{\mu}{2}|\theta|^2, \end{aligned} \tag{3.2.6}$$

provided that $\mu c_1^2 h^2 \leq \dfrac{\nu}{2}$. Moreover, using that $-\dfrac{\mu}{2}|\theta|^2, -\dfrac{\mu}{2}\alpha^2\|\theta\|^2 \leq 0$, it follows that

$$\frac{1}{2}\frac{d}{dt}(|\theta|^2+\alpha^2\|\theta\|^2) + \nu(\|\theta\|^2+\alpha^2|A\theta|)^2 + \langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta\rangle \leq \frac{\nu}{4}(\|\theta\|^2+\alpha^2|A\theta|^2).$$

Thus

$$\frac{1}{2}\frac{d}{dt}(|\theta|^2+\alpha^2\|\theta\|^2) + \frac{3\nu}{4}(\|\theta\|^2+\alpha^2|A\theta|)^2 \leq |\langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta\rangle|. \tag{3.2.7}$$

To estimate the right-hand side above, we use the property

$$|\langle \widetilde{B}(u, v), z\rangle_{V',V}| \leq c(\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|v|\|z\| + |Au||v||z|^{\frac{1}{2}}\|z\|^{\frac{1}{2}}) \tag{3.2.8}$$

for every $u \in D(A), v \in H, z \in V$, as well as Young's inequality:

$$\begin{aligned}
|\langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta\rangle| &\leq c(\|w\|^{1/2}|Aw|^{1/2}|\theta + \alpha^2 A\theta|\,\|\theta\| + |Aw|\,|\theta + \alpha^2 A\theta|\,|\theta|^{1/2}\|\theta\|^{1/2})\\
&\leq c\|w\|^{1/2}|Aw|^{1/2}|\theta|\,\|\theta\| + c\alpha^2\|w\|^{1/2}|Aw|^{1/2}|A\theta|\,\|\theta\|\\
&+ c|Aw|\,|\theta|\,|\theta|^{1/2}\|\theta\|^{1/2} + c\alpha^2|Aw|\,|A\theta|\,|\theta|^{1/2}\|\theta\|^{1/2}\\
&\leq \frac{c^2}{\nu}\|w\|\,|Aw|\,|\theta|^2 + \frac{\nu}{4}\|\theta\|^2 + \frac{c^2}{\nu}\alpha^2\|w\|\,|Aw|\,\|\theta\|^2 + \frac{\nu}{4}\alpha^2|A\theta|^2\\
&+ \frac{c^2}{\lambda_1^{1/2}\nu}|Aw|^2|\theta|^2 + \frac{\lambda_1^{1/2}\nu}{4}|\theta|\,\|\theta\| + \frac{c^2}{\nu}\alpha^2|Aw|^2|\theta|\,\|\theta\| + \frac{\nu}{4}\alpha^2|A\theta|^2.
\end{aligned}$$

Using Young and Poincaré's inequality again, it follows that

$$\begin{aligned}
|\langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta\rangle| &\leq \frac{c^2}{\lambda_1^{1/2}\nu}|Aw|^2|\theta|^2 + \frac{c^2}{\lambda_1^{1/2}\nu}\alpha^2|Aw|^2\|\theta\|^2\\
&+ \frac{c^2}{\lambda_1^{1/2}\nu}|Aw|^2|\theta|^2 + \frac{c^2}{\lambda_1^{1/2}\nu}\alpha^2|Aw|^2\|\theta\|^2\\
&+ \frac{\nu}{2}(\|\theta\|^2+\alpha^2|A\theta|^2),
\end{aligned}$$

so we conclude the following estimate to non-linear term:

$$|\langle \widetilde{B}(w, \theta + \alpha^2 A\theta), \theta\rangle| \leq \frac{\nu}{2}(\|\theta\|^2+\alpha^2|A\theta|^2) + \frac{2c^2}{\lambda_1^{1/2}\nu}|Aw|^2(|\theta|^2+\alpha^2\|\theta\|^2). \tag{3.2.9}$$

From (3.2.7) and (3.2.9) we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(|\theta|^2+\alpha^2\|\theta\|^2) + &\frac{\nu}{4}(\|\theta\|^2+\alpha^2|A\theta|^2)\\
&\leq \frac{2c^2}{\lambda_1^{1/2}\nu}|Aw|^2(|\theta|^2+\alpha^2\|\theta\|^2),
\end{aligned}$$

which implies

$$\frac{d}{dt}(|\theta(t)|^2+\alpha^2\|\theta(t)\|^2) \leq \frac{4c^2}{\lambda_1^{1/2}\nu}|Aw(t)|^2(|\theta(t)|^2+\alpha^2\|\theta(t)\|^2).$$

By Gronwall's Inequality on the interval $[0,t]$,

$$|\theta(t)|^2+\alpha^2\|\theta(t)\|^2\leq (|\theta(0)|^2+\alpha^2\|\theta(0)\|^2)e^{\frac{4c^2}{\lambda_1^{1/2}\nu}\int_0^t|Aw(s)|^2ds}$$

and since $w\in L^2([0,T];D(A))$, we have the continuous dependence of the regular solution on the initial data and in particular the uniqueness of regular solution.

□

We prove next the uniqueness of the solution for the case (ii), i.e., when I_h satisfies (3.1.8).

Theorem 3.2.2. The solution to the problem (3.1.6) given by Theorem 3.1.4 is unique.

Proof: Following the same notation as in first case, the proof is the same up to estimates (3.2.5) and (3.2.6), which (3.2.5) will be replaced by

$$\begin{aligned} -\mu(I_h\theta,\theta) &= \mu(\theta - I_h\theta,\theta) - \mu|\theta|^2 \\ &\leq \mu|\mathcal{P}(\theta - I_h\theta)|\,|\theta|-\mu|\theta|^2 \\ &\leq \frac{\mu}{2}|\mathcal{P}(\theta - I_h\theta)|^2+\frac{\mu}{2}|\theta|^2-\mu|\theta|^2 \\ &\leq \frac{\mu c_2^2h^2}{2}\|\theta\|^2+\frac{\mu}{2}c_2^2h^4|A\theta|^2-\frac{\mu}{2}|\theta|^2. \end{aligned} \tag{3.2.10}$$

Since $\mu c_2^2h^2\leq\mu\bar{c}_2h^2<\frac{\nu}{2}$ and $\mu c_2^2h^4<\frac{\nu\alpha^2}{2}$, we have

$$-\mu(I_h\theta,\theta)\leq\frac{\nu}{4}\|\theta\|^2+\frac{\nu}{4}\alpha^2|A\theta|^2-\frac{\mu}{2}|\theta|^2=\frac{\nu}{4}(\|\theta\|^2+\alpha^2|A\theta|^2)-\frac{\mu}{2}|\theta|^2. \tag{3.2.11}$$

The estimate (3.2.6) must be replaced by

$$\begin{aligned} -\mu\alpha^2(I_h\theta,A\theta) &= \mu\alpha^2(\theta - I_h\theta,A\theta) - \mu\alpha^2\|\theta\|^2 \\ &\leq \mu\alpha^2|\mathcal{P}(\theta - I_h\theta)|\,|A\theta|-\mu\alpha^2\|\theta\|^2 \\ &\leq \frac{\mu^2\alpha^2}{\nu}|\mathcal{P}(\theta - I_h\theta)|^2+\frac{\nu}{4}\alpha^2|A\theta|^2-\mu\alpha^2\|\theta\|^2 \\ &\leq \frac{\mu^2\alpha^2c_2^2h^2}{\nu}\|\theta\|^2+\frac{\mu^2\alpha^2c_2^2h^4}{\nu}|A\theta|^2+\frac{\nu}{4}\alpha^2|A\theta|^2-\mu\alpha^2\|\theta\|^2, \end{aligned}$$

and using that $\mu c_2^2 h^2 < \dfrac{\nu}{2}$, we have

$$\mu^2 c_2^2 h^2 = \mu \cdot \mu c_2^2 h^2 \leq \frac{\mu\nu}{2},$$

and also

$$\mu^2 c_2^2 h^4 = \mu c_2 h^2 \cdot \mu c_2 h^2 \leq \mu \bar{c}_2 h^2 \cdot \mu \bar{c}_2 h^2 < \frac{\nu}{2} \cdot \frac{\nu}{2} = \frac{\nu^2}{4}.$$

Therefore

$$\begin{aligned} -\mu\alpha^2(I_h\theta, A\theta) &\leq \frac{\mu\nu\alpha^2}{2\nu}\|\theta\|^2 + \frac{\nu^2\alpha^2}{4\nu}|A\theta|^2 + \frac{\nu}{4}\alpha^2|A\theta|^2 - \mu\alpha^2\|\theta\|^2 \\ &= \frac{\mu}{2}\alpha^2\|\theta\|^2 + \frac{\nu}{2}\alpha^2|A\theta|^2 - \mu\alpha^2\|\theta\|^2 \\ &= \frac{\nu}{2}\alpha^2|A\theta|^2 - \frac{\mu}{2}\alpha^2\|\theta\|^2, \end{aligned} \tag{3.2.12}$$

and thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(|\theta|^2 + \alpha^2\|\theta\|^2) &+ \nu(\|\theta\|^2 + \alpha^2|A\theta|^2) \leq |\langle \widetilde{B}(\overline{w}, \theta + \alpha^2 A\theta), \theta\rangle| \\ &+ \frac{\nu}{4}(\|\theta\|^2 + \alpha^2|A\theta|^2) - \frac{\mu}{2}|\theta|^2 + \frac{\nu}{2}\alpha^2|A\theta|^2 - \frac{\mu}{2}\alpha^2\|\theta\|^2. \end{aligned}$$

From the above inequality, we obtain

$$\frac{d}{dt}(|\theta|^2 + \alpha^2\|\theta\|^2) + \frac{\nu}{2}(\|\theta\|^2 + \alpha^2|A\theta|^2) \leq 2|\langle \widetilde{B}(\overline{w}, \theta + \alpha^2 A\theta), \theta\rangle|, \tag{3.2.13}$$

and the rest of the proof is exactly the same as for the first case (Theorem 3.2.1), following the same steps from inequality (3.2.7), with (3.2.13) in the place of (3.2.7)and the continuous dependence of initial data is reached and consequently, we have the desired uniqueness.

□

Chapter 4

The Convergence Theorem and Examples of Interpolants

4.1 Stabilization using I_h as a feedback control

Let $I_h : \dot{H}^1(\Omega) \to \dot{L}^2(\Omega)$, where Ω is the periodic domain $[0, L]^3$, a linear map which satisfies for all $\varphi \in \dot{H}^1(\Omega)$,

$$|\mathcal{P}(\varphi - I_h\varphi)|^2 \leq c_1^2 h^2 \|\varphi\|^2, \tag{4.1.1}$$

where $c_1 > 0$ is a constant. This bounded linear operator, when restricted to V, can be seen as an approximate interpolant of order h of the inclusion V into $\dot{L}^2(\Omega)$.

We shall suppose that an evolution u is governed by the incompressible three-dimensional Navier-Stokes-α system, under periodic boundary conditions $\Omega = [0, L]^3$:

$$\begin{cases} \dfrac{d}{dt}(u + \alpha^2 Au) + \nu A(u + \alpha^2 Au) + \widetilde{B}(u, u + \alpha^2 Au) = f \\ \text{div } u = 0, \end{cases} \tag{4.1.2}$$

From this point forward we denote $v := u + \alpha^2 Au$.

Suppose now that $u(\cdot)$ (and consequently, $v(\cdot)$) has to be recovered as $t \to \infty$ from the observational measurements $I_h(u)$, that have been continuously recorded for times $t \in [0, T]$. Then, the approximating solution w with initial data $w_0 \in V$ (which we can choose arbitrarily), shall be given by

$$\frac{d}{dt}(w+\alpha^2 Aw)+\nu A(w+\alpha^2 Aw) \ + \ \widetilde{B}(w, w+\alpha^2 Aw) \\ = f-\mu(I+\alpha^2 A)\mathcal{P}(I_h(w)-I_h(u)), \tag{4.1.3}$$

with div $w=0$ and $z=w+\alpha^2 Aw$.

Before proving the Convergence Theorem, we enunciate the generalized Gronwall inequality, proved in Lemma 4 of [14].

Lemma 4.1.1 (Uniform Gronwall's Inequality)**.** Let $T>0$ be fixed, β be locally integrable real valued function on $(0,\infty)$, satisfying the following conditions:

$$\liminf_{t\to\infty}\int_t^{t+T}\beta(s)ds=\gamma>0 \ \text{ and } \ \limsup_{t\to\infty}\int_t^{t+T}\beta^-(s)ds=\Gamma<\infty,$$

where $\beta^-=\max\{-\beta,0\}$. Furthermore, let ψ be a real valued locally integrable function defined on $(0,\infty)$ such that

$$\lim_{t\to\infty}\int_t^{t+T}\psi^+(s)ds=0,$$

where $\psi^+=\max\{\psi,0\}$. Suppose that ξ is an absolutely continuous non-negative function on $(0,\infty)$ such that

$$\frac{d\xi(t)}{dt}+\beta(t)\xi(t)\le\psi(t),$$

almost everywhere on $(0,\infty)$. Then $\xi(t)\to 0$ exponentially as $t\to\infty$.

We state and prove next the main result.

Theorem 4.1.2. Let u be a solution of the incompressible three-dimensional Navier-Stokes-α equations (4.1.2) and let $I_h:\dot{H}^1(\Omega)\to\dot{L}^2(\Omega)$ a linear map satisfying (3.1.7). Assume that $\mu>0$ is large enough satisfying

$$\mu>24c^2\lambda_1 G^2+\frac{15c^2\nu G^2}{\alpha^2}, \tag{4.1.4}$$

where c is the constant deriving of the non-linearity property (2.2.16), λ_1 is the first eigenvalue of Stokes operator under periodic boundary conditions, i.e., $\lambda_1=(2\pi/L)^2$ and $G=\dfrac{|f|}{\lambda_1^{3/4}\nu^2}$ is the Grashoff number (in our case, for time-independent forcing term) defined in (1.3.1). Moreover, assume that h small enough such that

$$h^2<\frac{\nu}{2\mu c_1^2} \tag{4.1.5}$$

where c_1 is the constant given on (4.1.1). Then, the global unique solution $w \in L^2([0,\infty); D(A)) \cap C([0,\infty); V)$ of (4.1.3), given by Theorem 3.1.3 satisfies $(w(t) - u(t)) \to 0$, as $t \to \infty$, in $|\cdot|$ and $\|\cdot\|$-norms.

Proof: Considering u the solution of the Navier-Stokes-α equations, we have

$$\frac{d}{dt}(u + \alpha^2 Au) + \nu A(u + \alpha^2 Au) + \widetilde{B}(u, v) = f. \tag{4.1.6}$$

Let $\delta = w - u$, where $w(\cdot)$ satisfies (4.1.3). Taking the difference of (4.1.3) and (4.1.6) we have

$$\frac{d}{dt}(\delta + \alpha^2 A\delta) + \nu A(\delta + \alpha^2 A\delta) + \widetilde{B}(w, z) - \widetilde{B}(u, v) = -\mu \mathcal{P} I_h \delta - \mu\alpha^2 A I_h \delta. \tag{4.1.7}$$

Note that

$$\begin{aligned}
\widetilde{B}(w,z) - \widetilde{B}(u,v) &= \widetilde{B}(w,z) + \widetilde{B}(w,v) - \widetilde{B}(w,v) - \widetilde{B}(u,z-v) + \widetilde{B}(u,z-v) - \widetilde{B}(u,v) \\
&= \widetilde{B}(w,z-v) + \widetilde{B}(w-u,v) - \widetilde{B}(u,z-v) + \widetilde{B}(u,z-v) \\
&= \widetilde{B}(w-u,z-v) + \widetilde{B}(w-u,v) + \widetilde{B}(u,\delta+z-v) \\
&= \widetilde{B}(\delta,\delta+\alpha^2 A\delta) + \widetilde{B}(\delta,v) + \widetilde{B}(u,\delta+\alpha^2 A\delta).
\end{aligned}$$

Replacing (4.1.8) into (4.1.7), we have

$$\frac{d}{dt}(\delta + \alpha^2 A\delta) + \nu A(\delta + \alpha^2 A\delta) + \widetilde{B}(\delta, \delta + \alpha^2 A\delta) + \widetilde{B}(\delta, v) + \widetilde{B}(u, \delta + \alpha^2 A\delta) = -\mu \mathcal{P} I_h \delta - \mu\alpha^2 A I_h \delta.$$

Taking the dual spaces action on δ and using the fact that we have from (2.2.12):

$$\langle \widetilde{B}(\delta, \delta + \alpha^2 A\delta), \delta \rangle_{D(A)', D(A)} = \langle \widetilde{B}(\delta, v), \delta \rangle_{D(A)', D(A)} = 0,$$

we obtain

$$\begin{aligned}
&\left\langle \frac{d}{dt}\delta, \delta \right\rangle_{D(A)',D(A)} + \alpha^2 \left\langle \frac{d}{dt}A\delta, \delta \right\rangle_{D(A)',D(A)} + \nu\langle A\delta, \delta\rangle_{D(A)',D(A)} + \nu\alpha^2 \langle A^2\delta, \delta\rangle_{D(A)',D(A)} \\
&\quad + \langle \widetilde{B}(u, \delta + \alpha^2 A\delta), \delta \rangle_{D(A)',D(A)} = -\mu\langle \mathcal{P} I_h \delta, \delta\rangle_{D(A)',D(A)} - \mu\alpha^2 \langle A I_h \delta, \delta \rangle_{D(A)',D(A)}.
\end{aligned}$$

From Theorem 2.1.11 and Lemma 2.1.13,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(|\delta|^2 + \alpha^2\|\delta\|^2) + \nu(\|\delta\|^2 + \alpha^2|A\delta|^2) \; + \; &\langle \widetilde{B}(u, \delta + \alpha^2 A\delta), \delta \rangle_{D(A)',D(A)} \\
&= -\mu(I_h\delta, \delta) - \mu\alpha^2 \langle A I_h \delta, \delta \rangle_{D(A)',D(A)}.
\end{aligned} \tag{4.1.8}$$

Also, we have $\langle AI_h\delta, \delta\rangle_{D(A)',D(A)} = (I_h\delta, A\delta)$ by (2.1.12). Estimating the right-hand side terms of (4.1.8) using Young's inequality,

$$\begin{aligned} -\mu(I_h\delta,\delta) &= \mu(\delta - I_h\delta, \delta) - \mu|\delta|^2 \\ &\leq \mu|\mathcal{P}(\delta - I_h\delta)|\,|\delta| - \mu|\delta|^2 \\ &\leq \mu c_1 h\|\delta\|\,|\delta| - \mu|\delta|^2 \\ &\leq \frac{\mu}{2}|\delta|^2 + \frac{\mu c_1^2 h^2}{2}\|\delta\|^2 - \mu|\delta|^2, \end{aligned} \tag{4.1.9}$$

and similarly,

$$\begin{aligned} -\mu\alpha^2(I_h\delta, A\delta) &= \mu\alpha^2(\delta - I_h\delta, A\delta) - \mu\alpha^2\|\delta\|^2 \\ &\leq \mu\alpha^2|\mathcal{P}(\delta - I_h\delta)|\,|A\delta| - \mu\alpha^2\|\delta\|^2 \\ &\leq \mu\alpha^2 c_1 h\|\delta\|\,|A\delta| - \mu\alpha^2\|\delta\|^2 \\ &= \frac{\mu}{2}\alpha^2\|\delta\|^2 + \frac{\mu c_1^2 h^2}{2}\alpha^2\|\delta\|^2 - \mu\alpha^2\|\delta\|^2. \end{aligned} \tag{4.1.10}$$

Therefore using (4.1.9) and (4.1.10), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(|\delta|^2 + \alpha^2\|\delta\|^2) &+ \nu(\|\delta\|^2 + \alpha^2|A\delta|^2) + \langle \widetilde{B}(u, \delta + \alpha^2 A\delta), \delta\rangle_{D(A)',D(A)} \\ &\leq \frac{\mu}{2}|\delta|^2 + \frac{\mu c_1^2 h^2}{2}\|\delta\|^2 - \mu|\delta|^2 \\ &+ \frac{\mu}{2}\alpha^2\|\delta\|^2 + \frac{\mu c_1^2 h^2}{2}\alpha^2\|\delta\|^2 - \mu\alpha^2\|\delta\|^2. \end{aligned} \tag{4.1.11}$$

The next step is to estimate the non-linear term. Using (2.2.16), we have

$$\begin{aligned} |\langle \widetilde{B}(u, \delta + \alpha^2 A\delta), \delta\rangle| &\leq c(\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|\delta + \alpha^2 A\delta|\,\|\delta\| + |Au|\,|\delta + \alpha^2 A\delta|\,|\delta|^{\frac{1}{2}}\|\delta\|^{\frac{1}{2}}) \\ &\leq c\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|\delta|\,\|\delta\| + c\alpha^2\|u\|^{\frac{1}{2}}|Au|^{\frac{1}{2}}|A\delta|\,\|\delta\| \\ &+ c|Au|\,|\delta|\,|\delta|^{\frac{1}{2}}\|\delta\|^{\frac{1}{2}} + c\alpha^2|Au|\,|A\delta|\,|\delta|^{\frac{1}{2}}\|\delta\|^{\frac{1}{2}}. \end{aligned}$$

For the terms above, we use the Young inequality again, and it is necessary to be careful with the dimensional analysis of each term of right-hand side below, because these terms need to have the same units, in order to add them:

$$
\begin{aligned}
|\langle \widetilde{B}(u,\delta+\alpha^2A\delta),\delta\rangle| &\leq \frac{2}{\nu}c^2\|u\|\,|Au|\,|\delta|^2+\frac{\nu}{8}\|\delta\|^2\\
&+ \frac{2}{\nu}c^2\alpha^2\|u\|\,|Au|\,\|\delta\|^2+\frac{\nu}{8}\alpha^2|A\delta|^2\\
&+ \frac{2}{\nu\lambda_1^{1/2}}c^2|Au|^2|\delta|^2+\frac{\nu\lambda_1^{1/2}}{8}|\delta|\,\|\delta\|\\
&+ \frac{2}{\nu}c^2\alpha^2|Au|^2|\delta|\,\|\delta\|+\frac{\nu}{8}|A\delta|^2.
\end{aligned}
$$

Using that

$$\frac{2}{\nu}c^2\|u\|\,|Au|\,|\delta|^2\leq \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}\|u\|^2|\delta|^2+\frac{c^2}{2\nu\lambda_1^{1/2}}|Au|^2|\delta|^2,$$

$$\frac{2}{\nu}c^2\alpha^2\|u\|\,|Au|\,\|\delta\|^2\leq \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}\alpha^2\|\delta\|^2\|u\|^2+\frac{c^2}{2\nu\lambda_1^{1/2}}\alpha^2|Au|^2\|\delta\|^2,$$

and

$$\frac{2}{\nu}c^2\alpha^2|Au|^2|\delta|\,\|\delta\|\leq \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}\alpha^2|Au|^2|\delta|^2+\frac{c^2}{2\nu\lambda_1^{1/2}}\alpha^2|Au|^2\|\delta\|^2,$$

we obtain

$$
\begin{aligned}
|\langle \widetilde{B}(u,\delta+\alpha^2A\delta),\delta\rangle| &\leq \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2|\delta|^2+\alpha^2|Au|^2|\delta|^2)\\
&+ \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}\alpha^2\|\delta\|^2\|u\|^2+\frac{\nu}{4}(\|\delta\|^2+\alpha^2|A\delta|^2)\\
&+ \frac{5c^2}{2\nu\lambda_1^{1/2}}|Au|^2(|\delta|^2+\alpha^2\|\delta\|^2),
\end{aligned}
$$

which implies

$$
\begin{aligned}
|\langle \widetilde{B}(u,\delta+\alpha^2A\delta),\delta\rangle| &\leq \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2+\alpha^2|Au|^2)(|\delta|^2+\alpha^2\|\delta\|^2)\\
&+ \frac{2c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2+\alpha^2|Au|^2)(|\delta|^2+\alpha^2\|\delta\|^2)\\
&+ \frac{5c^2}{2\nu\lambda_1^{1/2}}|Au|^2(|\delta|^2+\alpha^2\|\delta\|^2)+\frac{\nu}{4}(\|\delta\|^2+\alpha^2|A\delta|^2).
\end{aligned}
\tag{4.1.12}
$$

Thus from (4.1.11) and (4.1.12),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\delta|^2+\alpha^2\|\delta\|^2) \;+\; \nu(\|\delta\|^2+\alpha^2|A\delta|^2)&\\ &\le \frac{4c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2+\alpha^2|Au|^2)(|\delta|^2+\alpha^2\|\delta\|^2)\\ &+\; \frac{5c^2}{2\nu\lambda_1^{1/2}}|Au|^2(|\delta|^2+\alpha^2\|\delta\|^2)+\frac{\nu}{4}(\|\delta\|^2+\alpha^2|A\delta|^2)\\ &+\; \frac{\mu}{2}|\delta|^2+\frac{\mu c_1^2h^2}{2}\|\delta\|^2-\mu|\delta|^2+\frac{\mu}{2}\alpha^2\|\delta\|^2+\frac{\mu c_1^2h^2}{2}\alpha^2\|\delta\|^2-\mu\alpha^2\|\delta\|^2.\end{aligned}$$

Since by assumption $\mu c_1^2h^2<\frac{\nu}{2}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\delta|^2+\alpha^2\|\delta\|^2) \;+\; \frac{\nu}{2}(\|\delta\|^2+\alpha^2|A\delta|^2)&\\ &\le \frac{4c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2+\alpha^2|Au|^2)(|\delta|^2+\alpha^2\|\delta\|^2)\\ &+\; \frac{5c^2}{2\nu\lambda_1^{1/2}}|Au|^2(|\delta|^2+\alpha^2\|\delta\|^2)-\frac{\mu}{2}(|\delta|^2+\alpha^2\|\delta\|^2).\end{aligned}\tag{4.1.13}$$

Therefore we conclude by (4.1.13) that

$$\frac{d}{dt}(|\delta(t)|^2+\alpha^2\|\delta(t)\|^2)+\beta(t)(|\delta(t)|^2+\alpha^2\|\delta(t)\|^2)\le 0,\tag{4.1.14}$$

where

$$\beta(t)=\mu-\frac{8c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u(t)\|^2+\alpha^2|Au(t)|^2)-\frac{5c^2}{\nu\lambda_1^{1/2}}|Au(t)|^2.$$

To make use of Lemma 4.1.1, note that for $T>0$,

$$\int_t^{t+T}\beta(s)ds=\mu T-\frac{8c^2\lambda_1^{\frac{1}{2}}}{\nu}\int_t^{t+T}\|u(s)\|^2+\alpha^2|Au(s)|^2ds-\frac{5c^2}{\nu\lambda_1^{1/2}}\cdot\frac{1}{\alpha^2}\int_t^{t+T}\alpha^2|Au(s)|^2ds.\tag{4.1.15}$$

Taking $T=\frac{1}{\nu\lambda_1}$ in Proposition 2.3.3, we have for $t\ge t_0$,

$$\int_t^{t+T}\|u(s)\|^2+\alpha^2|Au(s)|^2ds\le\frac{3\nu G^2}{\lambda_1^{1/2}},$$

thus

$$-\frac{8c^2\lambda_1^{\frac{1}{2}}}{\nu}\int_t^{t+T}\|u(s)\|^2+\alpha^2|Au(s)|^2ds\ge -24c^2G^2\tag{4.1.16}$$

and

$$-\frac{5c^2}{\nu\lambda_1^{1/2}}\cdot\frac{1}{\alpha^2}\int_t^{t+T}\alpha^2|Au(s)|^2ds\geq-\frac{15c^2G^2}{\lambda_1\alpha^2}. \tag{4.1.17}$$

Therefore, if we want $\liminf_{t\to\infty}\int_t^{t+T}\beta(s)ds=\gamma>0$ (using $T=(\nu\lambda_1)^{-1}$), it is sufficient from (4.1.15),(4.1.16) and (4.1.17) to have

$$\frac{\mu}{\nu\lambda_1}-24c^2G^2-\frac{15c^2G^2}{\lambda_1\alpha^2}>0,$$

i.e.,

$$\mu>24c^2\nu\lambda_1G^2+\frac{15c^2\nu G^2}{\alpha^2}, \tag{4.1.18}$$

which is given by assumption (4.1.4). Finally, taking $\psi\equiv 0$ in Lemma 4.1.1, we conclude that

$$|\delta(t)|^2+\alpha^2\|\delta(t)\|^2\longrightarrow 0,\ \text{ as }\ t\to\infty,$$

i.e., $(w(t)-u(t))\to 0$ in L^2 and H^1-norm, exponentially in time, and the proof is complete.

$\square$

Now, we consider the second case of interpolant observables $I_h:\dot{H}^2(\Omega)\to\dot{L}^2(\Omega)$, with $\Omega=[0,L]^3$ the periodic domain, that satisfies for some constant $c_2>0$ and all $\varphi\in D(A)$,

$$|\mathcal{P}(\varphi-I_h\varphi)|^2\leq c_2^2h^2\|\varphi\|^2+c_2^2h^4|A\varphi|^2. \tag{4.1.19}$$

Theorem 4.1.3. Let u be the solution of Navier-Stokes-α equations (4.1.2) $I_h:H^2(\Omega)\to L^2(\Omega)$ a linear map satisfying (4.1.19). Suppose that $\mu>0$ is large enough satisfying

$$\mu>24c^2\nu\lambda_1G^2+\frac{15c^2\nu G^2}{\alpha^2}, \tag{4.1.20}$$

where c is the constant deriving of the non-linearity property (2.2.16), λ_1 is the first eigenvalue of Stokes operator under periodic boundary conditions, i.e., $\lambda_1=(2\pi/L)^2$ and $G=\dfrac{|f|}{\lambda_1^{3/4}\nu^2}$ is the Grashoff number (in our case, for time-independent forcing term) defined in (1.3.1). Suppose also that h small enough such that

$$\mu\bar{c}_2h^2\leq\frac{\nu}{2}\ \text{ and }\ \mu c_2^2h^4\leq\frac{\nu\alpha^2}{2}, \tag{4.1.21}$$

where $\bar{c}_2=\max\{c_2,c_2^2\}$. Then, the global unique solution $w\in L^2([0,\infty);D(A))\cap C([0,\infty);V)$ of (4.1.3), ensured by Theorem 3.1.4, satisfies $(w(t)-u(t))\to 0$, as $t\to\infty$, in L^2 and H^1-norms.

Proof: The idea of proof is the same as Theorem 4.1.2, except from the fact that we need to estimate $-\mu(I_h\delta,\delta)$ and $-\mu\alpha^2(I_h\delta,A\delta)$ as follows:

$$\begin{aligned}
-\mu(I_h\delta,\delta) &= \mu(\delta-I_h\delta,\delta)-\mu|\delta|^2\\
&\le \mu|\mathcal{P}(\delta-I_h\delta)||\delta|-\mu|\delta|^2\\
&\le \frac{\mu}{2}|\mathcal{P}(\delta-I_h\delta)|^2+\frac{\mu}{2}|\delta|^2-\mu|\delta|^2\\
&\le \frac{\mu c_2^2h^2}{2}\|\delta\|^2+\frac{\mu}{2}c_2^2h^4|A\delta|^2-\frac{\mu}{2}|\delta|^2.
\end{aligned}\tag{4.1.22}$$

Since by assumption $\mu c_2^2h^2\le\mu\bar{c}_2h^2<\frac{\nu}{2}$ and $\mu c_2^2h^4<\frac{\nu\alpha^2}{2}$, we have

$$-\mu(I_h\delta,\delta)\le\frac{\nu}{4}\|\delta\|^2+\frac{\nu}{4}\alpha^2|A\delta|^2-\frac{\mu}{2}|\delta|^2=\frac{\nu}{4}(\|\delta\|^2+\alpha^2|A\delta|^2)-\frac{\mu}{2}|\delta|^2,\tag{4.1.23}$$

as well as

$$\begin{aligned}
-\mu\alpha^2(I_h\delta,A\delta) &= \mu\alpha^2(\delta-I_h\delta,A\delta)-\mu\alpha^2\|\delta\|^2\\
&\le \mu\alpha^2|\mathcal{P}(\delta-I_h\delta)||A\delta|-\mu\alpha^2\|\delta\|^2\\
&\le \frac{\mu^2\alpha^2}{\nu}|\mathcal{P}(\delta-I_h\delta)|^2+\frac{\nu}{4}\alpha^2|A\delta|^2-\mu\alpha^2\|\delta\|^2\\
&\le \frac{\mu^2\alpha^2c_2^2h^2}{\nu}\|\delta\|^2+\frac{\mu^2\alpha^2c_2^2h^4}{\nu}|A\delta|^2+\frac{\nu}{4}\alpha^2|A\delta|^2-\mu\alpha^2\|\delta\|^2.
\end{aligned}$$

Using that $\mu c_2^2h^2<\frac{\nu}{2}$, we have

$$\mu^2c_2^2h^2=\mu\cdot\mu c_2^2h^2\le\frac{\mu\nu}{2},$$

and also

$$\mu^2c_2^2h^4=\mu c_2h^2\cdot\mu c_2h^2\le\mu\bar{c}_2h^2\cdot\mu\bar{c}_2h^2<\frac{\nu}{2}\cdot\frac{\nu}{2}=\frac{\nu^2}{4}.$$

Therefore

$$\begin{aligned}
-\mu\alpha^2(I_h\delta,A\delta) &\le \frac{\mu\nu\alpha^2}{2\nu}\|\delta\|^2+\frac{\nu^2\alpha^2}{4\nu}|A\delta|^2+\frac{\nu}{4}\alpha^2|A\delta|^2-\mu\alpha^2\|\delta\|^2\\
&= \frac{\mu}{2}\alpha^2\|\delta\|^2+\frac{\nu}{2}\alpha^2|A\delta|^2-\mu\alpha^2\|\delta\|^2\\
&= \frac{\nu}{2}\alpha^2|A\delta|^2-\frac{\mu}{2}\alpha^2\|\delta\|^2.
\end{aligned}\tag{4.1.24}$$

The estimate for the non-linear term is the same estimate (4.1.12) (from Theorem 4.1.2). Thus using (4.1.23) and (4.1.24) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(|\delta|^2+\alpha^2\|\delta\|^2) \quad &+ \nu(\|\delta\|^2+\alpha^2|A\delta|^2)\\ &\leq \frac{4c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2+\alpha^2|Au|^2)(|\delta|^2+\alpha^2\|\delta\|^2)\\ &+ \frac{5c^2}{2\nu\lambda_1^{1/2}}|Au|^2(|\delta|^2+\alpha^2\|\delta\|^2)+\frac{\nu}{4}(\|\delta\|^2+\alpha^2|A\delta|^2)\\ &+ \frac{\nu}{4}(\|\delta\|^2+\alpha^2|A\delta|^2)-\frac{\mu}{2}|\delta|^2+\frac{\nu}{2}\alpha^2|A\delta|^2-\frac{\mu}{2}\alpha^2\|\delta\|^2,\end{aligned}$$

and it follows that

$$\begin{aligned}\frac{d}{dt}(|\delta|^2+\alpha^2\|\delta\|^2) \quad &\leq \frac{8c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u\|^2+\alpha^2|Au|^2)(|\delta|^2+\alpha^2\|\delta\|^2)\\ &+ \frac{5c^2}{2\nu\lambda_1^{1/2}}|Au|^2(|\delta|^2+\alpha^2\|\delta\|^2)-\mu(|\delta|^2+\alpha^2\|\delta\|^2).\end{aligned}$$

In the same way as Theorem 4.1.2, we have

$$\frac{d}{dt}(|\delta(t)|^2+\alpha^2\|\delta(t)\|^2)+\beta(t)(|\delta(t)|^2+\alpha^2\|\delta(t)\|^2)\leq 0, \tag{4.1.25}$$

where

$$\beta(t)=\mu-\frac{8c^2\lambda_1^{\frac{1}{2}}}{\nu}(\|u(t)\|^2+\alpha^2|Au(t)|^2)-\frac{5c^2}{\nu\lambda_1^{1/2}}|Au(t)|^2.$$

and with exactly the same calculus of Theorem 4.1.2, we make use of Lemma 4.1.1 to conclude that if

$$\mu > 24c^2\nu\lambda_1 G^2+\frac{15c^2\nu G^2}{\alpha^2}, \tag{4.1.26}$$

and (4.1.21) holds, then

$$|\delta(t)|^2+\alpha^2\|\delta(t)\|^2\longrightarrow 0, \ \text{ as } \ t\to\infty,$$

exponentially in time, which is the desired conclusion.

$\square$

Actually, from Theorems 4.1.2 and 4.1.3, we estabilish that the algorithm used for constructing $w(t)$ from the observational measures $I_h u(t)$ given by

$$\begin{aligned}\frac{d}{dt}(w+\alpha^2Aw) \quad &+ \quad \nu A(w+\alpha^2Aw)+\widetilde{B}(w,w+\alpha^2Aw)\\ &= \quad f-\mu\mathcal{P}(I_h(w)-I_h(u))-\mu\alpha^2A\mathcal{P}(I_h(w)-I_h(u)),\end{aligned}$$

yields a good aproximation for $u(t)$ in the sense that

$$|w(t)-u(t)|^2+\alpha^2\|w(t)-u(t)\|^2\longrightarrow 0$$

exponentially, as time goes to infinity, provided that these observational data have fine enough spatial resolution (and it is imposed in hyphotesis (4.1.5) and (4.1.21)).

These results assert that if we want to accurately predict the physical reality u for a time $\bar{t}$ into the future, it is sufficient to obtain the observational data $I_h u(t)$ accumulated over an interval of time proportional to $\bar{t}$ in the immediate past.

To exemplify, suppose that we want to predict $u(t)$ with accuracy $\varepsilon > 0$ on the interval $[t_1, t_1+\bar{t}\,]$, where t_1 is the present time and $\bar{t} > 0$ is how far into the future we want to predict. Consider then h small enough and μ large enough such that Theorem 4.1.2 (or Theorem 4.1.3, depending on if I_h satisfies (3.1.2) or (3.1.3)) is satisfied. Thus, there exists $\gamma > 0$ and $C > 0$ such that

$$|w(t)-u(t)|^2+\alpha^2\|w(t)-u(t)\|^2\leq Ce^{-\gamma t}, \text{ for all } t\geq 0.$$

All we need to do now is to use $w(t_1)$ as initial condition to make a future prediction: let $\overline{u}$ the solution of (2.3.1), with initial data $\overline{u}(t_1) = w(t_1)$. The existence and uniqueness Theorem found in [9] gives us a result about continuous dependence on initial conditions, which implies there exists $\xi > 0$ such that

$$|\overline{u}(t)-u(t)|^2+\alpha^2\|\overline{u}(t)-u(t)\|^2\leq (|\overline{u}(t_1)-u(t_1)|^2+\alpha^2|\overline{u}(t_1)-u(t_1)|^2)e^{\xi(t-t_1)}$$

for all $t \geq t_1$. Therefore for $t \in [t_1, t_1+\bar{t}\,]$,

$$|\overline{u}(t)-u(t)|^2+\alpha^2\|\overline{u}(t)-u(t)\|^2\leq Ce^{-\gamma t_1+\xi\bar{t}} < \varepsilon^2,$$

provided that $\gamma t_1 \geq \xi\bar{t}+\ln(C/\varepsilon^2)$. Thus $\overline{u}(t)$ (that is known) predicts $u(t)$ with accuracy ε on the interval $[t_1, t_1+\bar{t}]$.

4.2 Examples of interpolant I_h

In this section, we present three examples of linear interpolant observables: the first two examples satisfying for all $\varphi \in H^1(\Omega)$,

$$|\varphi - I_h\varphi|^2 \leq c_1^2 h^2 |\nabla\varphi|^2, \tag{4.2.1}$$

and the third one satisfying for all $\varphi \in H^2(\Omega)$,

$$|\varphi - I_h\varphi|^2 \leq c_2^2 h^2 |\nabla\varphi|^2 + c_2^2 h^4 \|\varphi\|_{H^2(\Omega)}^2.$$

4.2.1 Projection Fourier Modes

Initially, consider the L^2-orthonormal and H^1-orthogonal basis $\phi_k = \frac{1}{L^{3/2}} e^{2\pi i \frac{k\cdot x}{L}}, k \in \mathbb{Z}^3 \backslash \{0\}$, as presented in Chapter 2, given by the eigenfunctions of the operator $A = -\mathcal{P}\Delta$. Remember that $\varphi \in D(A)$ can be written as

$$\varphi(x) = \sum_{k\in\mathbb{Z}^3\backslash\{0\}} \widehat{\varphi}_k \phi_k(x),$$

with

$$\widehat{\varphi}_k = (\varphi, \phi_k) = \frac{1}{L^{3/2}} \int_\Omega \varphi(x) e^{-2\pi i \frac{k\cdot x}{L}} dx.$$

Remembering the Fourier L^2-orthogonal projection as the truncated series defined as

$$P_N\varphi = \sum_{|k|\leq N} \widehat{\varphi}_k \phi_k,$$

we can construct an interpolant observable I_h, considering $N = \frac{1}{h}$. Define I_h as

$$I_h\varphi = P_{\frac{1}{h}}\varphi = \sum_{|k|\leq \frac{1}{h}} \widehat{\varphi}_k \phi_k,$$

what represents the low Fourier modes with wave numbers k such that $|k| \leq 1/h$.

Next we prove that this interpolant I_h given as Fourier projection satisfies (4.2.1).

Lemma 4.2.1. For all $\varphi \in D(A)$,

$$|\varphi - I_h\varphi| = |\varphi - P_N\varphi| \leq h\|\varphi\|,$$

where $N = \dfrac{1}{h}$.

Proof: Using the representation of high modes $Q_N = I_d - P_N$, we have for $N = \dfrac{1}{h}$:

$$\begin{aligned} |\varphi - I_h\varphi|^2 &= |\varphi - P_N\varphi|^2 = |\sum_{|k|>N} \widehat{\varphi}_k \phi(x)|^2 \\ &= \sum_{|k|>N} |\widehat{\varphi}_k|^2 \leq \sum_{|k|>N} |\widehat{\varphi}_k|^2 |k|^2 h^2 \\ &\leq h^2 \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |\widehat{\varphi}_k|^2 |k|^2 = h^2 \|\varphi\|^2, \end{aligned}$$

because if we have $|k| > N = \dfrac{1}{h}$, then $|k|^2 h^2 > 1$. Therefore

$$|\varphi - I_h\varphi|^2 \leq h^2 \|\varphi\|^2.$$

□

Note that h and N are dependent parameters and we use them interchangeably with the understanding that as $h \to 0$, $N \to \infty$ and conversely.

4.2.2 Volume Elements - Local Averages

First of all, for $N \in \mathbb{N}$, consider the periodic domain $\Omega = [0, L]^3$ divided in Ω_k, $k = 1, ..., N$, where Ω_k is the cube with edge $\dfrac{L}{\sqrt[3]{N}}$, and so $|\Omega_k| = \dfrac{L^3}{N}$.

As cited in Chapter 1, the local average of u in Ω_k is defined as

$$\langle u \rangle_{\Omega_k} = \frac{1}{|\Omega_k|} \int_{\Omega_k} u(x)dx = \frac{N}{L^3} \int_{\Omega_k} u(x)dx.$$

We will denote the characteristic function on Ω_k by χ_{Ω_k}:

$$\chi_{\Omega_k}(x) = \begin{cases} 1 \ \text{if} \ x \in \Omega_k, \\ 0 \ \text{if} \ x \notin \Omega_k. \end{cases}$$

To construct the linear operator I_h, we suppose that the average values of φ on each of the Ω_k's is known. Then, consider $I_h : L^2(\Omega) \to L^2(\Omega)$ defined as

$$I_h(\varphi(x)) = \sum_{k=1}^{N} \langle \varphi \rangle_{\Omega_k} \chi_{\Omega_k}(x), \tag{4.2.2}$$

where $h = \dfrac{L}{\sqrt[3]{N}}$.

To prove that (4.2.2) satisfies (4.2.1), we will prove next a Poincaré's Inequality version for periodic boundary condition in three-dimensions: $\Omega = [0, L]^3$.

Lemma 4.2.2. Let $u \in V$ and denote $\Omega = [0, L]^3$ the three-dimensional torus. For N a positive integer, divide this domain into N cubes of edge $l = \dfrac{L}{\sqrt[3]{N}}$ and denote Ω_j the j-th cube, with $|\Omega_j|= l^3$, $j = 1, ..., N$. Then

$$\|u\|^2_{L^2(\Omega_j)} \leq l^3 \langle u\rangle^2_{\Omega_j} + \frac{l^2}{3}\|\nabla u\|^2_{L^2(\Omega_j)},$$

for all $j = 1, ..., N$ and in particular,

$$\|u\|^2_{L^2(\Omega)} \leq l^3 \gamma^2(u) + \frac{l^2}{3}\|u\|^2,$$

where $\gamma(u) = \max\limits_{1\leq j\leq N}|\langle u\rangle_{\Omega_j}|$.

Proof: It was showed in [14], for the one-dimensional case, the following inequality:

$$\|u\|^2_{L^2(I_j)} \leq l\langle u\rangle^2_{I_j} + \frac{l^2}{3}\|\nabla u\|^2_{L^2(I_j)}.$$

for every $u \in H^1([0, L])$, for all $j = 1, ..., N$, where I_j is the interval $[\frac{(j-1)L}{N}, \frac{jL}{N}] \subset [0, L]$ of length $l = \frac{L}{N}$.

In the case of two dimensions, it was proved in the same work that

$$\|u\|^2_{L^2(K_j)} \leq l^2\langle u\rangle^2_{K_j} + \frac{l^2}{3}\|\nabla u\|^2_{L^2(K_j)}.$$

for all $u \in H^1([0, L]^2)$, where the domain $[0, L]^2$ has been divided into N squares K_j with side $l = \frac{L}{\sqrt[2]{N}}$, $j = 1, ..., N$, with $|K_j|= \frac{L^2}{N}$.

To obtain the three-dimensional version, let initially $u \in C_0^\infty(\mathbb{R}^3)$. We fix the first coordinate x_1 and apply the two-dimensional version to $v(x_1) = v(x_1)(x_2, x_3) = u(x_1, x_2, x_3) \in C_0^\infty(\mathbb{R}^2)$:

$$\|v(x_1)\|^2_{L^2(K_j)} \leq l^2\langle v(x_1)\rangle^2_{K_j} + \frac{l^2}{3}\|\nabla v(x_1)\|^2_{L^2(K_j)},$$

i.e.,

$$\begin{aligned}\int_0^l\int_0^l|u(x_1, x_2, x_3)|^2 dx_2 dx_3 &\leq l^2\left(\frac{1}{l^2}\int_0^l\int_0^l u(x_1, x_2, x_3)dx_2dx_3\right)^2 \\ &+ \frac{l^2}{3}\int_0^l\int_0^l\left|\frac{\partial u}{\partial x_2}(x_1, x_2, x_3)\right|^2 dx_2dx_3 + \frac{l^2}{3}\int_0^l\int_0^l\left|\frac{\partial u}{\partial x_3}(x_1, x_2, x_3)\right|^2 dx_2dx_3.\end{aligned}$$

Integrating over x_1, we obtain

$$\begin{aligned}\int_0^l\int_0^l\int_0^l|u(x_1,x_2,x_3)|^2dx_2dx_3dx_1 &\leq l^2\int_0^l\left(\frac{1}{l^2}\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3\right)^2dx_1\\ &+ \frac{l^2}{3}\int_0^l\int_0^l\int_0^l\left|\frac{\partial u}{\partial x_2}(x_1,x_2,x_3)\right|^2dx_2dx_3dx_1\\ &+ \frac{l^2}{3}\int_0^l\int_0^l\int_0^l\left|\frac{\partial u}{\partial x_3}(x_1,x_2,x_3)\right|^2dx_2dx_3dx_1.\end{aligned}$$

The next step is to apply the one dimensional case to

$$z(x_1)=\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3,$$

and so

$$\|z(x_1)\|^2_{L^2([0,l])}\leq l\langle z(x_1)\rangle^2_{[0,l]}+\frac{l^2}{3}\|\nabla z(x_1)\|^2_{L^2([0,l])},$$

i.e.,

$$\begin{aligned}\int_0^l\left(\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3\right)^2dx_1 &\leq \frac{1}{l}\left(\int_0^l\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3dx_1\right)^2\\ &+ \frac{l^2}{3}\int_0^l\left|\frac{\partial}{\partial x_1}\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3\right|^2dx_1.\end{aligned}$$

Multiplying (4.2.3) by $\frac{1}{l^2}$ and using Hölder's Inequality, we have

$$\begin{aligned}&l^2\int_0^l\left(\frac{1}{l^2}\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3\right)^2dx_1\leq\frac{1}{l^3}\left(\int_0^l\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3dx_1\right)^2\\ &\quad+\frac{1}{3}\int_0^l\left|\frac{\partial}{\partial x_1}\int_0^l\int_0^l u(x_1,x_2,x_3)dx_2dx_3\right|^2dx_1\\ &\quad\leq l^3\langle u\rangle^2_{[0,l]^3}+\frac{1}{3}\int_0^l\left[\int_0^l\left(\int_0^l 1dx_2\right)^{\frac{1}{2}}\left(\int_0^l\left|\frac{\partial u}{\partial x_1}(x_1,x_2,x_3)\right|^2dx_2\right)^{\frac{1}{2}}dx_3\right]^2dx_1\\ &\quad= l^3\langle u\rangle^2_{[0,l]^3}+\frac{l}{3}\int_0^l\left[\int_0^l\left(\int_0^l\left|\frac{\partial u}{\partial x_1}(x_1,x_2,x_3)\right|^2dx_2\right)^{\frac{1}{2}}dx_3\right]^2dx_1.\end{aligned}$$

By Hölder's Inequality again,

$$
\begin{aligned}
&l^2 \int_0^l \left(\frac{1}{l^2} \int_0^l \int_0^l u(x_1, x_2, x_3) dx_2 dx_3 \right)^2 dx_1 \\
&\quad \leq \; l^3 \langle u \rangle^2_{[0,l]^3} + \frac{l}{3} \int_0^l \left[\left(\int_0^l 1 dx_3 \right) \int_0^l \left(\int_0^l \left| \frac{\partial u}{\partial x_1}(x_1, x_2, x_3) \right|^2 dx_2 \right) dx_3 \right] dx_1 \\
&\quad = \; l^3 \langle u \rangle^2_{[0,l]^3} + \frac{l^2}{3} \int_0^l \int_0^l \int_0^l \left| \frac{\partial u}{\partial x_1}(x_1, x_2, x_3) \right|^2 dx_2 dx_3 dx_1.
\end{aligned}
$$

Finally, replacing (4.2.3) into (4.2.3) we conclude that

$$
\begin{aligned}
\int_0^l \int_0^l \int_0^l |u(x_1, x_2, x_3)|^2 dx_2 dx_3 dx_1 &\leq l^3 \langle u \rangle^2_{[0,l]^3} + \frac{l^2}{3} \int_0^l \int_0^l \int_0^l \left| \frac{\partial u}{\partial x_1}(x_1, x_2, x_3) \right|^2 dx_2 dx_3 dx_1 \\
+ \; \frac{l^2}{3} \int_0^l \int_0^l \int_0^l \left| \frac{\partial u}{\partial x_2}(x_1, x_2, x_3) \right|^2 dx_2 dx_3 dx_1 &+ \frac{l^3}{3} \int_0^l \int_0^l \int_0^l \left| \frac{\partial u}{\partial x_3}(x_1, x_2, x_3) \right|^2 dx_2 dx_3 dx_1.
\end{aligned}
$$

Therefore for all $u \in C_0^\infty(\mathbb{R}^3)$,

$$
\|u\|^2_{L^2(\Omega_j)} \leq l^3 \langle u \rangle^2_{\Omega_j} + \frac{l^2}{3} \|\nabla u\|^2_{L^2(\Omega_j)},
$$

for all $j = 1, ..., N$ and in particular,

$$
\|u\|^2_{L^2(\Omega)} \leq l^3 \gamma^2(u) + \frac{l^2}{3} \|u\|^2,
$$

where $\gamma(u) = \max_{1 \leq j \leq N} |\langle u \rangle_{\Omega_j}|$. Provided that $C_0^\infty(\mathbb{R}^3)|_{\Omega_j}$ is dense in $H^2(\Omega_j)$, the inequality follows for all $u \in H^2(\Omega_j)$.

□

We present next the proof that (4.2.2) satisfies (4.2.1).

Lemma 4.2.3. For all $\varphi \in D(A)$,

$$
|I_h \varphi - \varphi|^2 = |\varphi - \sum_{k=1}^N \langle \varphi \rangle_{\Omega_k} \chi_{\Omega_k}|^2 \leq \frac{1}{3} h^2 \|\varphi\|^2,
$$

where $h = \dfrac{L}{\sqrt[3]{N}}$.

Proof: If $\varphi \in D(A)$,

$$\begin{aligned}
|\varphi - \sum_{k=1}^{N}\langle\varphi\rangle_{\Omega_k}\chi_{\Omega_k}|^2 &= \int_\Omega |\varphi(x) - \sum_{k=1}^{N}\langle\varphi\rangle_{\Omega_k}\chi_{\Omega_k}(x)|^2 dx \\
&= \int_\Omega |\varphi(x)\sum_{k=1}^{N}\chi_{\Omega_k}(x) - \sum_{k=1}^{N}\langle\varphi\rangle_{\Omega_k}\chi_{\Omega_k}(x)|^2 dx \\
&= \int_\Omega \left[\sum_{k=1}^{N}\varphi(x)\chi_{\Omega_k}(x) - \sum_{k=1}^{N}\langle\varphi\rangle_{\Omega_k}\chi_{\Omega_k}(x)\right]\left[\sum_{j=1}^{N}\varphi(x)\chi_{\Omega_j}(x) - \sum_{j=1}^{N}\langle\varphi\rangle_{\Omega_j}\chi_{\Omega_j}(x)\right] dx \\
&= \int_\Omega \left[\sum_{k=1}^{N}(\varphi(x) - \langle\varphi\rangle_{\Omega_k})\chi_{\Omega_k}(x)\right]\left[\sum_{j=1}^{N}(\varphi(x) - \langle\varphi\rangle_{\Omega_j})\chi_{\Omega_j}(x)\right] dx \\
&= \int_\Omega \sum_{k=1}^{N}\sum_{j=1}^{N}(\varphi(x) - \langle\varphi\rangle_{\Omega_k})(\varphi(x) - \langle\varphi\rangle_{\Omega_j})\chi_{\Omega_k}(x)\chi_{\Omega_j}(x)dx \\
&= \sum_{k=1}^{N}\int_\Omega (\varphi(x) - \langle\varphi\rangle_{\Omega_k})^2\chi_{\Omega_k}(x)dx = \sum_{k=1}^{N}\int_{\Omega_k}|\varphi(x) - \langle\varphi\rangle_{\Omega_k}|^2 dx,
\end{aligned}$$

since $\chi_{\Omega_k}(x)\chi_{\Omega_j}(x) = \delta_{kj}\Omega_k(x)$. For every $1 \leq k \leq N$, we have from Lemma 4.2.2,

$$\begin{aligned}
\|\varphi - \langle\varphi\rangle_{\Omega_k}\|^2_{L^2(\Omega_k)} &\leq \left(\frac{L}{\sqrt[3]{N}}\right)^3\left(\frac{1}{|\Omega_k|}\int_{\Omega_k}\varphi(x) - \langle\varphi\rangle_{\Omega_k}dx\right)^2 + \frac{1}{3}\left(\frac{L}{\sqrt[3]{N}}\right)^2\|\nabla(\varphi - \langle\varphi\rangle_{\Omega_k})\|^2_{L^2(\Omega_k)} \\
&= \left(\frac{L}{\sqrt[3]{N}}\right)^3\left(\frac{1}{|\Omega_k|}\int_{\Omega_k}\varphi(x)dx - \frac{1}{|\Omega_k|}\int_{\Omega_k}\left(\frac{1}{|\Omega_k|}\int_{\Omega_k}\varphi(y)dy\right)dx\right)^2 \\
&+ \frac{1}{3}\left(\frac{L}{\sqrt[3]{N}}\right)^2\|\nabla\varphi\|^2_{L^2(\Omega_k)} \\
&= \left(\frac{L}{\sqrt[3]{N}}\right)^3\left(\frac{1}{|\Omega_k|}\int_{\Omega_k}\varphi(x)dx - \frac{1}{|\Omega_k|}\int_{\Omega_k}\varphi(y)dy\right)^2 + \frac{1}{3}\left(\frac{L}{\sqrt[3]{N}}\right)^2\|\nabla\varphi\|^2_{L^2(\Omega_k)} \\
&= \frac{1}{3}\left(\frac{L}{\sqrt[3]{N}}\right)^2\|\nabla\varphi\|^2_{L^2(\Omega_k)} = \frac{1}{3}h^2\|\nabla\varphi\|^2_{L^2(\Omega_k)},
\end{aligned}$$

where $h = \frac{L}{\sqrt[3]{N}}$. As a result of summing over $k = 1, ..., N$,

$$\sum_{k=1}^{N}\int_{\Omega_k}|\varphi(x) - \langle\varphi\rangle_{\Omega_k}|^2 dx \leq \frac{1}{3}\sum_{k=1}^{N}h^2\|\nabla\varphi\|^2_{L^2(\Omega_k)} = \frac{1}{3}h^2|\nabla\varphi|^2 = \frac{1}{3}h^2\|\varphi\|^2.$$

Therefore

$$|\varphi - I_h\varphi|^2 \leq \frac{1}{3}h^2\|\varphi\|^2.$$

□

4.2.3 Nodal Values

The most physically interesting example of an interpolant I_h which satisfies

$$|\varphi - I_h\varphi|^2 \leq c_2^2 h^2 \|\varphi\|^2 + c_2^2 h^4 |A\varphi|^2, \tag{4.2.3}$$

can be constructed using measurements at a discrete set of nodal points in $\Omega = [0, L]^3$.

Indeed, similarly as previous example of volume elements, to construct such interpolant using nodal values, we divide the domain Ω in N cubes of edge $\dfrac{L}{\sqrt[3]{N}}$, for $N \in \mathbb{N}$ and thus $|\Omega_j| = \dfrac{L^3}{N}$, $j = 1, ..., N$, where Ω_j denote the $j-$th cube and $\Omega = \cup_{j=1}^N \Omega_j$. Then we consider arbitraty points $x_j \in \Omega_j$ that represent the points where observational measurements of the velocity of the flow are done.

Define this interpolant as

$$I_h\varphi(x) = \sum_{k=1}^{N} \varphi(x_k)\chi_{\Omega_k}(x). \tag{4.2.4}$$

To prove that the interpolant (4.2.4) satisfies (4.2.3), it is necessary to make use of the following two lemmas:

Lemma 4.2.4. Let $\overline{Q} = [0, \Lambda] \times [0, d]$ and $u \in H^1(\overline{Q})$. Then

$$\int_0^\Lambda |u(x, 0)|^2 dx \leq \frac{2}{d} \|u\|^2_{L^2(\overline{Q})} + d \left\| \frac{\partial u}{\partial y} \right\|^2_{L^2(\overline{Q})}, \tag{4.2.5}$$

and simmetrically,

$$\int_0^d |u(0, y)|^2 dy \leq \frac{2}{\Lambda} \|u\|^2_{L^2(\overline{Q})} + \Lambda \left\| \frac{\partial u}{\partial x} \right\|^2_{L^2(\overline{Q})}. \tag{4.2.6}$$

Proof: This can be found in Lemma 6.1 of [16]. In this thesis, we prove the following lemma, that will be also useful for prove the interpolant (4.2.4) satisfies (4.2.3):

Lemma 4.2.5. Let $\overline{\Omega} = [0, l] \times [0, l] \times [0, l]$ and x and z be two points of $\overline{\Omega}$, where the third coordinates of x and z are the same, i.e., $x = (x_1, y_1, z_1)$ and $z = (x_2, y_2, z_1)$. Then for every $\varphi \in H^2(\overline{\Omega})$, we have

$$|\varphi(x) - \varphi(z)| \leq \frac{2}{l^{1/2}} \left(4\|\nabla\varphi\|^2_{L^2(\overline{\Omega})} + l^2 \left\| \frac{\partial^2 \varphi}{\partial x \partial y} \right\|^2_{L^2(\overline{\Omega})} \right)^{\frac{1}{2}}. \tag{4.2.7}$$

Simmetrically, if y and z are two points in $\overline{\Omega}$ such that the second coordinate of y and z are the same, i.e., $y = (x_2, y_2, z_2)$ and $z = (x_3, y_2, z_3)$, then

$$|\varphi(y) - \varphi(z)| \leq \frac{2}{l^{1/2}} \left(4\|\nabla\varphi\|^2_{L^2(\overline{\Omega})} + l^2 \left\| \frac{\partial^2 \varphi}{\partial x \partial z} \right\|^2_{L^2(\overline{\Omega})} \right)^{\frac{1}{2}}, \tag{4.2.8}$$

for every $\varphi \in H^2(\overline{\Omega})$.

Proof: We will show only the first estimate, and the second one is analogue. We begin by considering the square $Q = [0, l] \times [0, l]$. For any two points in $\overline{\Omega}$ of the form (x_1, y, z_1) and (x_2, y, z_1), with $y \in [0, l]$, we have

$$\begin{aligned} |\varphi(x_1, y, z_1) \ - \ \varphi(x_2, y, z_1)|^2 &= \left| \int_{x_1}^{x_2} \frac{\partial \varphi}{\partial x}(s, y, z_1) ds \right|^2 \\ &\leq \ \left\| \frac{\partial \varphi}{\partial x}(\cdot, y, z_1) \right\|^2_{L^2([0,l])} \cdot \int_{x_1}^{x_2} 1 dx \leq l \left\| \frac{\partial \varphi}{\partial x}(\cdot, y, z_1) \right\|^2_{L^2([0,l])}. \end{aligned}$$

Since the third coordinate z_1 is fixed and the points (x_1, y, z_1) and $\varphi(x_2, y, z_1)$ are in a plane parallel to xy plan, we can apply Lemma 4.2.4 for $\dfrac{\partial \varphi}{\partial x}(\cdot, y, z_1)$, with d replaced with the maximal distance of the y-coordinate of the points $(x_1, y, z_1), (x_2, y, z_1)$ from the horizontal walls; i.e.,

$$l \geq d = \max\{y, l - y\} \geq \frac{l}{2}$$

and therefore

$$\int_0^l \left| \frac{\partial \varphi}{\partial x}(x, y, z_1) \right|^2 dx \leq \frac{4}{l} \left\| \frac{\partial \varphi}{\partial x} \right\|^2_{L^2(Q)} + l \left\| \frac{\partial^2 u}{\partial y \partial x} \right\|^2_{L^2(Q)},$$

since $\dfrac{1}{d} \leq \dfrac{2}{l}$. Then we have

$$l \left\| \frac{\partial \varphi}{\partial x}(\cdot, y, z_1) \right\|^2_{L^2([0,l])} \leq 4 \left\| \frac{\partial \varphi}{\partial x} \right\|^2_{L^2(Q)} + l^2 \left\| \frac{\partial^2 \varphi}{\partial y \partial x} \right\|^2_{L^2(Q)}. \tag{4.2.9}$$

Replacing (4.2.9) into (4.2.9), we have that

$$|\varphi(x_1, y, z_1) - \varphi(x_2, y, z_1)|^2 \leq 4 \left\| \frac{\partial \varphi}{\partial x} \right\|^2_{L^2(Q)} + l^2 \left\| \frac{\partial^2 \varphi}{\partial y \partial x} \right\|^2_{L^2(Q)}. \tag{4.2.10}$$

By symmetry, we have the similar inequality for points of the form (x, y_1, z_1) and (x, y_2, z_1), where $x \in (0, l)$:

$$|\varphi(x, y_1, z_1) - \varphi(x, y_2, z_1)|^2 \leq 4 \left\| \frac{\partial \varphi}{\partial y} \right\|^2_{L^2(Q)} + l^2 \left\| \frac{\partial^2 \varphi}{\partial x \partial y} \right\|^2_{L^2(Q)}. \tag{4.2.11}$$

Thus

$$\begin{aligned}
|\varphi(x)-\varphi(z)|^2 &= |\varphi(x_1,y_1,z_1)-\varphi(x_2,y_2,z_1)|^2 \\
&\leq (|\varphi(x_1,y_1,z_1)-\varphi(x_2,y_1,z_1)|+|\varphi(x_2,y_1,z_1)-\varphi(x_2,y_2,z_1)|)^2 \\
&\leq 2|\varphi(x_1,y_1,z_1)-\varphi(x_2,y_1,z_1)|^2+2|\varphi(x_2,y_1,z_1)-\varphi(x_2,y_2,z_1)|^2 \\
&\leq 2\left(4\left\|\frac{\partial\varphi}{\partial x}\right\|^2_{L^2(Q)}+l^2\left\|\frac{\partial^2\varphi}{\partial y\partial x}\right\|^2_{L^2(Q)}\right)+2\left(4\left\|\frac{\partial\varphi}{\partial y}\right\|^2_{L^2(Q)}+l^2\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(Q)}\right),
\end{aligned}$$

and it follows that

$$|\varphi(x)-\varphi(z)|^2\leq 4\left(4\|\nabla\varphi\|^2_{L^2(Q)}+l^2\left\|\frac{\partial^2\varphi}{\partial y\partial x}\right\|^2_{L^2(Q)}\right). \tag{4.2.12}$$

Our aim is to obtain estimates in $L^2(\overline{\Omega})$, where $\overline{\Omega}=[0,l]\times[0,l]\times[0,l]$ instead of $L^2(Q)$. For this, we integrate (4.2.12) from 0 to l in z-coordinate:

$$\begin{aligned}
\int_0^l|\varphi(x_1,y_1,z_1)-\varphi(x_2,y_2,z_1)|dz &\leq 2\int_0^l\left(4\|\nabla\varphi(\cdot,\cdot,z)\|^2_{L^2(Q)}+l^2\left\|\frac{\partial^2\varphi}{\partial y\partial x}(\cdot,\cdot,z)\right\|^2_{L^2(Q)}\right)^{\frac{1}{2}}dz \\
&\leq 2\int_0^l\left(4\|\nabla\varphi(\cdot,\cdot,z)\|^2_{L^2(Q)}+l^2\left\|\frac{\partial^2\varphi}{\partial y\partial x}(\cdot,\cdot,z)\right\|^2_{L^2(Q)}dz\right)^{\frac{1}{2}}l^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$l|\varphi(x_1,y_1,z_1)-\varphi(x_2,y_2,z_1)|\leq 2l^{\frac{1}{2}}\left(4\|\nabla\varphi\|^2_{L^2(\overline{\Omega})}+l^2\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\overline{\Omega})}\right)^{\frac{1}{2}},$$

i.e.,

$$|\varphi(x_1,y_1,z_1)-\varphi(x_2,y_2,z_1)|^2\leq \frac{4}{l}\left(4\|\nabla\varphi\|^2_{L^2(\overline{\Omega})}+l^2\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\overline{\Omega})}\right). \tag{4.2.13}$$

and we have the desired conclusion.

□

We are ready to prove that the interpolant I_h constructed using measurements at nodal points satisfies (4.2.3):

Lemma 4.2.6. For all $\varphi \in D(A)$, the interpolant I_h defined in (4.2.4) satisfies

$$|\varphi - I_h\varphi|^2 \leq 32h^2\|\varphi\|^2 + 4h^4|A\varphi|^2 \tag{4.2.14}$$

where $h = L/\sqrt[3]{N}$.

Proof: Note that

$$\begin{aligned}
|\varphi - \sum_{k=1}^{N}\varphi(x_k)\chi_{\Omega_k}|^2 &= \int_\Omega |\varphi(x) - \varphi(x_k)\chi_{\Omega_k}(x)|^2 dx \\
&= \int_\Omega |\varphi(x)\sum_{k=1}^{N}\chi_{\Omega_k}(x) - \sum_{k=1}^{N}\varphi(x_k)\chi_{\Omega_k}(x)|^2 dx \\
&= \int_\Omega \left[\sum_{k=1}^{N}\varphi(x)\chi_{\Omega_k}(x) - \sum_{k=1}^{N}\varphi(x_k)\chi_{\Omega_k}(x)\right]\left[\sum_{j=1}^{N}\varphi(x)\chi_{\Omega_j}(x) - \sum_{j=1}^{N}\varphi(x_k)\chi_{\Omega_k}(x)\right] dx.
\end{aligned}$$

Since $\chi_{\Omega_k}(x)\chi_{\Omega_j}(x) = \chi_{\Omega_k}(x)\delta_{kj}$, we have

$$\begin{aligned}
|\varphi - \sum_{k=1}^{N}\varphi(x_k)\chi_{\Omega_k}|^2 &\leq \int_\Omega \left[\sum_{k=1}^{N}(\varphi(x) - \varphi(x_k))\chi_{\Omega_k}(x)\right]\left[\sum_{j=1}^{N}(\varphi(x) - \varphi(x_k))\chi_{\Omega_j}(x)\right] dx \\
&= \int_\Omega \sum_{k=1}^{N}\sum_{j=1}^{N}(\varphi(x) - \varphi(x_k))(\varphi(x) - \varphi(x_k))\chi_{\Omega_k}(x)\chi_{\Omega_j}(x)dx \qquad (4.2.15) \\
&= \sum_{k=1}^{N}\int_\Omega (\varphi(x) - \varphi(x_k))^2\chi_{\Omega_k}(x)dx.
\end{aligned}$$

Next, we find an estimate for

$$|\varphi(x) - \varphi(x_k)|^2.$$

Consider Ω_k for k fixed, but arbitrary. Choose $z \in \Omega_k$ such that z is in the line of the intersection of two plans: the plane which contains the point x and is parallel to xy-plane and the plane which contais the point x_k and is parallel to the xz-plane in three-dimensions.

In other words, if x and x_k are such that $x = (\xi_1, \xi_2, \xi_3)$ and $x_k = (\eta_1, \eta_2, \eta_3)$, then $z = (\tau_1, \eta_2, \xi_3)$. Therefore

$$\begin{aligned}
|\varphi(x) - \varphi(x_k)| &\leq |\varphi(x) - \varphi(z)| + |\varphi(z) - \varphi(x_k)| \\
&\leq |\varphi(\xi_1, \xi_2, \xi_3) - \varphi(\tau_1, \eta_2, \xi_3)| + |\varphi(\tau_1, \eta_2, \xi_3) - \varphi(\eta_1, \eta_2, \eta_3)|.
\end{aligned}$$

Now me make use of Lemma 4.2.5, applying (4.2.7) for the difference $|\varphi(\xi_1,\xi_2,\xi_3)-\varphi(\tau_1,\eta_2,\xi_3)|$ and (4.2.8) for the difference $|\varphi(\tau_1,\eta_2,\xi_3)-\varphi(\eta_1,\eta_2,\eta_3)|$:

$$\begin{aligned}
|\varphi(x)-\varphi(x_k)|^2 &\leq (|\varphi(\xi_1,\xi_2,\xi_3)-\varphi(\tau_1,\eta_2,\xi_3)|+|\varphi(\tau_1,\eta_2,\xi_3)-\varphi(\eta_1,\eta_2,\eta_3)|)^2 \\
&\leq 2|\varphi(\xi_1,\xi_2,\xi_3)-\varphi(\tau_1,\eta_2,\xi_3)|^2+2|\varphi(\tau_1,\eta_2,\xi_3)-\varphi(\eta_1,\eta_2,\eta_3)|^2 \\
&\leq \frac{4}{h}\left(4\|\nabla\varphi\|^2_{L^2(\Omega_k)}+h^2\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\Omega_k)}\right)+\frac{4}{h}\left(4\|\nabla\varphi\|^2_{L^2(\Omega_k)}+h^2\left\|\frac{\partial^2\varphi}{\partial x\partial z}\right\|^2_{L^2(\Omega_k)}\right),
\end{aligned}$$

where h is the edge of de cubes Ω_k, i.e., $h=L/\sqrt[3]{N}$. Then we conclude that

$$|\varphi(x)-\varphi(x_k)|^2\leq\frac{4}{h}\left(8\|\nabla\varphi\|^2_{L^2(\Omega_k)}+h^2\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\Omega_k)}+h^2\left\|\frac{\partial^2\varphi}{\partial x\partial z}\right\|^2_{L^2(\Omega_k)}\right). \tag{4.2.16}$$

Therefore from (4.2.15) and (4.2.16), it follows that

$$\begin{aligned}
|\varphi-\sum_{k=1}^N\varphi(x_k)\chi_{\Omega_k}|^2 &\leq \sum_{k=1}^N\int_\Omega(\varphi(x)-\varphi(x_k))^2\chi_{Q_k}(x)dx \\
&\leq \sum_{k=1}^N\int_\Omega\frac{4}{h}\left(8\|\nabla\varphi\|^2_{L^2(\Omega_k)}+h^2\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\Omega_k)}+h^2\left\|\frac{\partial^2\varphi}{\partial x\partial z}\right\|^2_{L^2(\Omega_k)}\right)\chi_{\Omega_k}dx \\
&= \sum_{k=1}^N\left(\frac{32}{h}\|\nabla\varphi\|^2_{L^2(\Omega_k)}\int_\Omega\chi_{\Omega_k}dx+4h\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\Omega_k)}\int_\Omega\chi_{\Omega_k}dx+4h\left\|\frac{\partial^2\varphi}{\partial x\partial z}\right\|^2_{L^2(\Omega_k)}\int_\Omega\chi_{\Omega_k}dx\right).
\end{aligned}$$

Since $|\Omega_k|=h^3$ for all $k=1,...,N$, we obtain

$$|\varphi-\sum_{k=1}^N\varphi(x_k)\chi_{Q_k}|^2\leq 32h^2\|\nabla\varphi\|^2_{L^2(\Omega)}+4h^4\left\|\frac{\partial^2\varphi}{\partial x\partial y}\right\|^2_{L^2(\Omega)}+4h^4\left\|\frac{\partial^2\varphi}{\partial x\partial z}\right\|^2_{L^2(\Omega)},$$

and thus

$$|\varphi-\sum_{k=1}^N\varphi(x_k)\chi_{Q_k}|^2\leq 32h^2\|\varphi\|^2+8h^4|A\varphi|^2. \tag{4.2.17}$$

□

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