Guido Gerson Espiritu Ledesma

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Orientadora: Ketty Aboroa de Rezende

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Abstract

In this work Lyapunov graphs are used as a combinatorial tool in order to obtain a complete classification of Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ and Morse-Novikov flows on orientable and non-orientable surfaces. This classification consists in determining necessary and sufficient conditions that must be satisfied by an abstract Lyapunov graph so that it is associated to a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ or to a Morse-Novikov flow on a surface respectively.
In summary in this doctoral thesis we obtain the following results:

1. The local conditions that must be satisfied by each vertex on a Lyapunov graph is determinated as well as the global conditions on the graph in order for it to be associated to a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ or a Morse-Novikov flow on a surface.

2. The realization of these graphs subject to the conditions found above as Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ or as Morse-Novikov flows on surfaces respectively is obtained.

Keywords: Circle-valued Morse functions, Conley index theory, Lyapunov graphs, Smale flows.

Resumo

Neste trabalho, usamos os grafos de Lyapunov como uma ferramenta combinatória para obter classificações completas de fluxos Smale sobre $\mathbb{S}^2 \times \mathbb{S}^1$ e fluxos Morse-Novikov sobre superfícies orientáveis e não orientáveis. Esta classificação consiste em obter condições necessárias e suficientes que devem ser satisfeitas por um grafo de Lyapunov abstrato de forma a ser associado a um fluxo Smale sobre $\mathbb{S}^2 \times \mathbb{S}^1$ ou um fluxo Morse-Novikov sobre uma superfície respectivamente. Assim nesta tese de doutorado obtemos os seguintes resultados:

1. As condições locais que devem ser satisfeitas por cada vértice do grafo de Lyapunov, assim como as condições globais que devem ser satisfeitas pelos grafos para estarem associados a um fluxo Smale sobre $\mathbb{S}^2 \times \mathbb{S}^1$ ou a um fluxo Morse-Novikov sobre uma superfície são determinadas.

2. A realização destes grafos abstratos sujeita às condições determinadas acima, como fluxos Smale sobre $\mathbb{S}^2 \times \mathbb{S}^1$ ou fluxos Morse-Novikov sobre superfícies respectivamente, são obtidas.

Palavras-chave: Funções de Morse circulares, teoria do índice de Conley, grafos de Lyapunov, fluxos de Smale.

Contents

To my mother Antonia and my wife Meylin.

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Introduction

The qualitative study of a class of dynamical systems consists first in describing geometrically the behavior of the flow on an invariant set (basic set) where the dynamics of the system is concentrated. Second, in describing the disposition of the basic sets within the manifold.

In this direction, in 1980, Franks [15],[16] worked with non-singular Smale flows. A non-singular Smale flow is a structurally stable flow with one dimensional basic sets and without singularities. In a Smale flow the saddle set may contain infinitely many closed orbits, while attractors and repellers must still be a collection of finitely many orbits. A theorem of Bowen [6] asserts that a one dimensional hyperbolic basic set (saddle set) must be equivalent to a suspension of a subshift of finite type.

In 1985, Franks introduced the concept of Lyapunov graphs to give an affirmative answer to a question raised by himself, Pugh and Shub, years earlier *"Given any subshift of finite type $\sigma_A : \Sigma_A \to \Sigma_A$ is there a non-singular Smale flow on $\mathbb{S}^3$ with the suspension of σ_A as a basic set?"*. Moreover, Franks [16] gives an abstract classification for non-singular Smale flows on $\mathbb{S}^3$. This is an algorithm which is used to decide if a given Lyapunov graph can be realized by a non-singular Smale flow on $\mathbb{S}^3$.

In this thesis, we investigate the relationship between the topology and dynamics of Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ and Morse-Novikov flows on surfaces. Our approach will be qualitative in nature and make extensive use of the methods introduced by Franks in [16] and further developed by de Rezende in [13] and by Bin Yu in [20]. A strategy that has been very successful in analyzing smooth flows on manifolds is to consider first the local dynamics described by the chain recurrent behavior on the basic sets and secondly to examine how these basic sets fit together.

The latter issue is in fact an embedding problem, which analyzes how the collection of basic sets form $\mathbb{S}^2 \times \mathbb{S}^1$. In this work, as in [13] and [20] we make use of a Lyapunov graph as a combinatorial tool which gives a global picture of the disposition of the basic sets within the manifold.

The main results, Propositions 4.1.1, 4.2.1, 4.2.2 and Lemma 4.2.2 are existence theorems which determine under what conditions an abstract Lyapunov graph is associated to a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$. It is also shown in Theorems 4.1.1, 4.2.1 and 4.2.2 that these conditions are necessary in order for a Smale flow to exist on $\mathbb{S}^2 \times \mathbb{S}^1$.

Similar results were obtained by Franks [16] for non-singular Smale flows on $\mathbb{S}^3$. In [20] Bin Yu obtains results for non-singular Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$. Our work completes this analysis by considering Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ which admit singularities. The difficulty in working with $\mathbb{S}^2 \times \mathbb{S}^1$ is precisely the embedding problem of the basic sets, due to the complexity of its topology as can be verified in Proposition 4.2.4.

Finally, this thesis considers, in the Conley index theory context, flows arising from circle valued Lyapunov functions both on orientable and non-orientable surfaces. In order to achieve this, our methods led us to consider circular Morse functions as a starting point and to build from there, the theory in the general case. In Section 1.4 an extension of Franks' definition of Lyapunov graph [16] is presented for circle valued Morse functions and referred to as *circular Morse digraphs*. This turns out to be an extremely useful combinatorial tool to explore the local topology of basic blocks containing singularities as well as the global topology of the surface. In Chapter 5, Theorem 5.2.1 characterizes the Morse circular digraphs associated with gradients of circle valued Morse functions. This is achieved by establishing local conditions on the degree of the vertices in relation to the type of singularity, as well as, the global condition on the cycle rank of the graph in relation to the genus of the underlying surface. Furthermore, this result is exploited in three different directions. Lastly, Theorem 5.2.1 sets the foundation to establish Theorem 5.3.1 which characterizes Lyapunov circular digraphs emerging from more general smooth flows connected to circle valued Lyapunov functions on surfaces. We direct a final word with a view towards further development of the methods and tools presented herein which can be adapted for the higher dimensional case. These methods, in conjunction with the theory of continuation and realizability of Lyapunov graphs (see [2], [3] and references therein) should produce extremely interesting results on the interplay between dynamics and topology.

This thesis is divided as follows: In Chapter 1 we introduce the necessary background material. In Chapter 2 we provide some topological properties of embedded surfaces in $\mathbb{S}^2 \times \mathbb{S}^1$. We introduce the notion of manifolds of handlebody type. In Chapter 3 we study the relationship between the Lyapunov graph and manifolds of handlebody type. In Chapter 4 we state and prove the main results. In Chater 5 we extend the definition of Lyapunov graph and give a characterization of Morse digraph associated with Morse-Novikov flows. Throughout this thesis, we consider homology with $F_2 = \mathbb{Z}_2$ coefficients.

Chapter 1

Background

The chain-recurrent set is a relevant object in Dynamical Systems and other areas of mathematics. In this chapter, we introduce Smale flows, which are structurally stable flows with invariant sets that are hyperbolic chain-recurrent sets with dimension less than or equal to one. Also, we give the definition of abstract Lyapunov graphs and this definition in the context of Morse-Novikov flows. Moreover, the definition of the genus of 3-manifolds and some results that relate the genus and the cycle rank of a Lyapunov graph are given. Finally, a result that shows the relationship between smooth flows and homology of the manifolds is given. The background material used in this chapter can be found in Bowen [6], Bowen and Franks [7], Conley [8], Cruz and de Rezende [10], Franks [15], [16] and Pajitnov [18].

1.1 Chain-recurrent set

We restrict our attention to smooth flows on compact n-manifolds.

Definition 1.1.1. If $\phi_t : M \to M$ is a flow and $\epsilon > 0$, we say there is an ϵ-chain from x to y provided that there exist points $x_1 = x, x_2, \ldots, x_n = y$ and real numbers $t(i) > 1$ such that

$$d(\phi_{t(i)}(x_i), x_{i+1}) < \epsilon$$

for all $1 \leq i < n$. A point x is called **chain-recurrent** if for any $\epsilon > 0$ there is an ϵ-chain from x to x. The set of chain recurrent points $\mathfrak{R}$ is called the **chain recurrent set**.

If M is a compact, it is easy to see that the chain recurrent set is compact and invariant under the flow.

Definition 1.1.2. If $\phi_t : M \to M$ is a smooth flow, then a smooth function $f : M \to \mathbb{R}$ will be called a **Lyapunov function** provided

1. $\frac{d}{dt}(f(\phi_t x)) < 0$ if it is not in the chain recurrent set $\mathfrak{R}$.
2. If $x, y \in \mathfrak{R}$ then $f(x) = f(y)$ if and only if for each $\epsilon > 0$ there are ϵ-chains from x to y and from y to x.

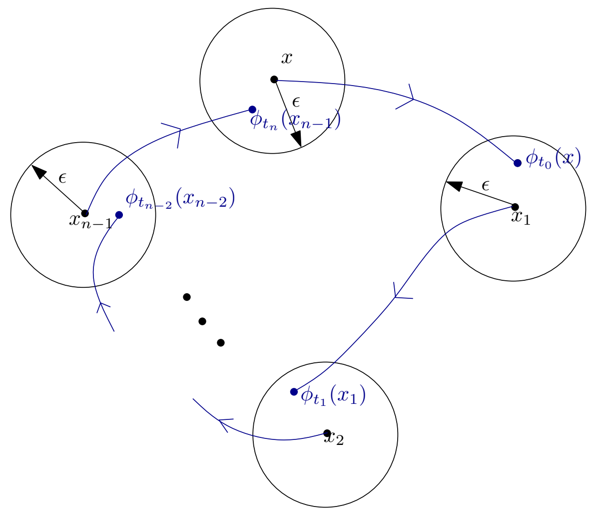


Figure 1.1: x is a chain recurrent point.

A compact invariant set Λ for a smooth flow $\phi_t : M \to M$ is said to have a **hyperbolic structure** provided that the tangent bundle of M restricted to Λ can be written as the Whitney sum of three sub-bundles $E^u + E^s + E^c$, each being invariant under $D\phi_t$ for all t, in such a way that E^c is spanned by the vector field tangent to the flow and there are constants C, $\alpha > 0$ satisfying

$$\|D\phi_t(v)\| \leq Ce^{-\alpha t} \|v\| \quad \text{for } v \in E^s,\ t \geq 0$$

and

$$\|D\phi_t(v)\| \geq C^{-1}e^{\alpha t} \|v\| \quad \text{for } v \in E^u,\ t \geq 0.$$

An important consequence of the hyperbolicity is that $\mathfrak{R}$ is decomposed into finitely many irreducible pieces [19].

Theorem 1.1.1. Suppose that the chain recurrent set $\mathfrak{R}$ of a flow on a compact manifold $\phi_t : M \to M$ has a hyperbolic structure. Then $\mathfrak{R}$ is a finite disjoint union of compact invariant sets $\Lambda_1, \ldots, \Lambda_n$ and each Λ_i contains a point whose forward orbit is dense in Λ_i.

The sets Λ_i are called **basic sets** of the flow and are precisely the chain transitive pieces of $\mathfrak{R}$, which are not separated by Lyapunov function.

If Λ is a compact invariant hyperbolic set for a flow then each orbit in Λ has a **stable** and **unstable manifold**. These are defined as follows for $x \in \Lambda$,

$$W^s(x) = \{y \in M \,|\, \text{for some}\, r \in \mathbb{R},\ d(\phi_t(y), \phi_{t+r}(x)) \to 0 \,\text{as}\, t \to \infty\}$$

and

$$W^u(x) = \{y \in M \,|\, \text{for some}\, r \in \mathbb{R},\ d(\phi_t(y), \phi_{t+r}(x)) \to 0 \,\text{as}\, t \to -\infty\}$$

Definition 1.1.3. A flow $\phi_t : M \to M$ with hyperbolic chain recurrent set $\mathfrak{R}$ is said to satisfy the **transversality condition** provided that for each $x, y \in \mathfrak{R}$, the manifolds $W^s(x)$ and $W^u(y)$ intersect transversally.

1.2 Smale flow

Definition 1.2.1. A smooth flow $\phi_t : M \to M$ on a compact manifold is called a **Smale flow** provided:

1. its chain recurrent set $\mathfrak{R}$ has a hyperbolic structure and $\dim \mathfrak{R} \leq 1$;

2. it satisfies the transversality condition.

Bowen [6] gives a complete dynamical description of the basic sets of Smale flows.

Theorem 1.2.1. Let ϕ_t be a flow with hyperbolic chain recurrent set and Λ a basic set of dimension 1, then ϕ_t restricted to Λ is topologically conjugate to the suspension of a subshift of finite type associated to an irreducible matrix.

Hence, the chain recurrent set of a Smale flow is made up of singularities, periodic orbits and suspensions of subshifts of finite type.

Definition 1.2.2. If A and B are non-negative integer matrices they are **flow equivalent** provided the suspension of the subshifts of finite type $\sigma(A)$ and $\sigma(B)$ are topologically equivalent.

1.3 Lyapunov graph

Given a Lyapunov function $f : M \to \mathbb{R}$ define the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. We call $M/ \sim_f$ a **Lyapunov graph** and denote it by L. The **cycle rank** of the graph is the maximum number of edges that can be removed without disconnecting the graph. We will denote it by $\beta(L)$. The **indegree** of a vertex u in L is the number of incoming edges of u and the **outdegree** of u is the number of outgoing edges of u. See Figure 4.21.

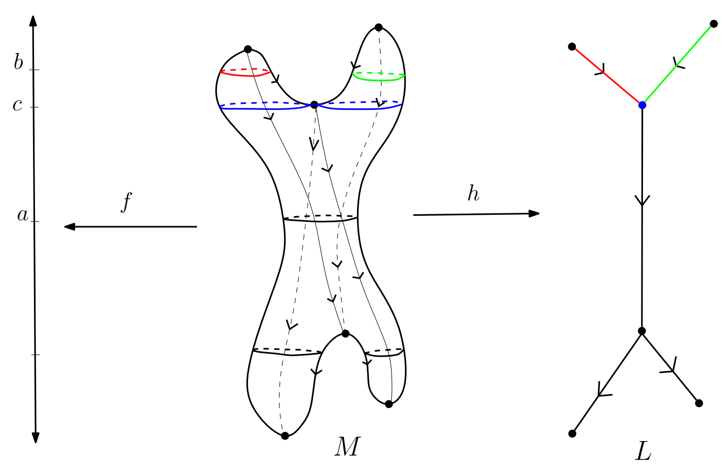


Figure 1.2: L is a Lyapunov graph.

Definition 1.3.1. An **abstract Lapunov graph** is a finite connected oriented graph L which possesses no oriented cycles and with each vertex labelled with a chain recurrent flow on a compact space. Each edge will be labelled with a non-negative integer g, which we refer to as the weight on the edge.

1.4 Circular Morse functions

In this section we try to extend the definition of real Morse functions. A nice reference for this topic is Pajinov [18].
Let $f : M \to \mathbb{S}^1$ be a circle-valued function. Let $\text{Exp} : \mathbb{R} \to \mathbb{S}^1$ denote the exponential function $x \to e^{2\pi ix}$. Finally, let $E : \overline{M} \to M$ be the covering[1] of M induced by f and Exp. By definition of the induced covering, we have a map $F : \overline{M} \to \mathbb{R}$, which makes the following diagram commute:

$$\begin{tikzcd} \overline{M} \arrow[r, "F"] \arrow[d, "E"'] & \mathbb{R} \arrow[d, "Exp"] \\ M \arrow[r, "f"] & \mathbb{S}^1 \end{tikzcd} \tag{1.4.1}$$

Note that if f is homotopic to a constant map, then $\overline{M}$ consists of infinitely many disjoint copies of M. In particular, this situation occurs when the first Betti number of the original manifold M, computed with respect to $\mathbb{Z}$, is null. For example, this is the case of $\mathbb{S}^2$ and $\mathbb{RP}^2$.

Definition 1.4.1. With the above notation, f is said to be a **circular Morse** function on M if only if F is a (classical) Morse function on $\overline{M}$.

A value x of $\mathbb{S}^1$ is said to be critical for f if it is of the form $\text{Exp}(c)$, where c is a critical value of F. The singularities of f on M are naturally induced by the singularities of F on $\overline{M}$ via E.

The facts that E is a local homeomorphism and that the diagram commutes make the definitions of singularity and critical value consistent with the corresponding definitions in the differential setting, since these definitions are local.

Definition 1.4.2. Let $a \in \mathbb{R}$ be a regular value of F. The set $W = F^{-1}([a-1, a])$ is a compact cobordism W with

$$\partial_1 W = F^{-1}(a) \quad \text{and} \quad \partial_0 W = F^{-1}(a-1)$$

which is called a **fundamental cobordism**.

Example 1.4.1. In Figure 4.6, b is a regular value of a circular value Morse function f. Then $W = F^{-1}([b-1, b])$ is a fundamental cobordism, see Figure 4.6, right side.

---

[1] Let us underline that we choose here to deal with any induced covering E of M and not only with the infinite cyclic covering for two reasons. On the one hand we wish to emphasize the interplay between the classical and the new situation. On the other hand, this choice allows us to include in our further discussion the cases of the sphere $\mathbb{S}^2$ and of the projective plane $\mathbb{RP}^2$.

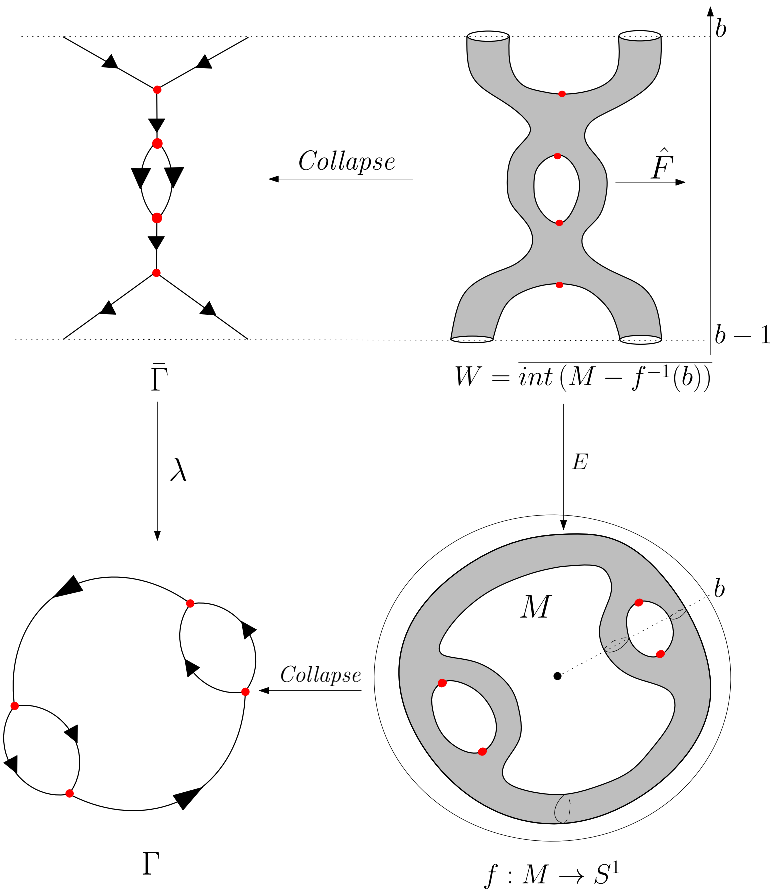


Figure 1.3: W is a fundamental cobordism

1.4.1 Morse circular digraphs

Let us recall that a classical Morse function $f : M \to \mathbb{R}$, induces a Lyapunov graph Γ, whose points are the connected components of the level hypersurfaces $f^{-1}(c)$, for any real c. One makes it into a graph Γ by considering that vertices exactly correspond to the connected components containing a singularity. Moreover, the orientation of the flow associated with the Lyapunov function f induces an orientation on the edges of Γ, so that it can be considered also as a directed graph (or, shortly, digraph). See Figure 4.6.

Note that such a graph is finite for compact manifolds. This is not the case when we consider a (classical) Morse function F on $\overline{M}$ as in the diagram above. For our purpose we need to consider not only infinite graphs (with infinitely many vertices of finite degree) but also the "graph" homeomorphic to $\mathbb{R}$, with empty vertex set and edge set made of this copy of $\mathbb{R}$ constituting the graph itself. This last graph is associated with the trivial vertical flow on an infinite cylinder. Apart from this exception, the graph $\overline{\Gamma}$ associated with F on $\overline{M}$ is infinite because the covering (and hence the singularities) are infinite.

In the same way, in order to state the following theorem, we shall need to extend a little more the definition of a graph by considering as a "graph" also a homeomorphic copy of $\mathbb{S}^1$. In this case, the vertex set is the empty set while the edge set contains this copy of $\mathbb{S}^1$, which is the graph itself. Such a graph will be associated with a trivial flow on the torus $\mathbb{T}^2$ possessing no singularities, induced by the circular Morse function equal to the projection of $\mathbb{S}^1 \times \mathbb{S}^1$ on the first factor.

Proposition 1.4.1. With the above notation, let $\overline{\Gamma}$ be the Morse digraph associated with F via the collapsing procedure defined above. Let λ denote the map induced on $\overline{\Gamma}$ by the covering E. Then λ is a projection of $\overline{\Gamma}$ onto a digraph Γ, preserving vertices, which is a local isomorphism of digraphs, as well as a local homeomorphism of complexes. Moreover, the digraph Γ can also be obtained from M and f by the same collapsing procedure as above, and the following diagram commutes:

$$\begin{array}{ccccc} \overline{\Gamma} & \xleftarrow{\textit{collapse}} & \overline{M} & \xrightarrow{F} & \mathbb{R} \\ \downarrow\ \lambda & & \downarrow\ E & & \downarrow\ \mathrm{Exp} \\ \Gamma & \xleftarrow{\textit{collapse}} & M & \xrightarrow{f} & \mathbb{R} \end{array}$$

Figure 4.6 illustrates Proposition 1.4.1.

Proof. Straightforward from the constructions and definitions. □

1.5 Genus of three-manifolds

Definition 1.5.1. Let M be a smooth, compact, connected 3-manifold with boundary. The **genus of manifold** M, $g(M)$ is the maximal number of mutually disjoint, smooth, compact, connected, two-side codimension one submanifolds that do not disconnect M.

The following result gives the relation between the cycle rank of the Lyapunov graph L associated with a smooth flow on M and the genus of the fundamental group of M.

Proposition 1.5.1. Let M be a connected, closed smooth 3-manifold. Let ϕ_t be a smooth flow on M with associated Lyapunov function f. Let L be a Lyapunov graph associated to f. Then

$$\beta(L) \leq g(M)$$

See [10] for the proof.

In particular, let ϕ_t be a smooth flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with L its associated Lyapunov graph. Since, $g(\mathbb{S}^2 \times \mathbb{S}^1) = 1$ by Proposition 1.5.1, one has

$$\beta(L) \leq 1$$

Thus, Lyapunov graphs associated with smooth flows, in particular Smale flows, on $\mathbb{S}^2 \times \mathbb{S}^1$ have at most one cycle.

1.6 Morse-Smale flows

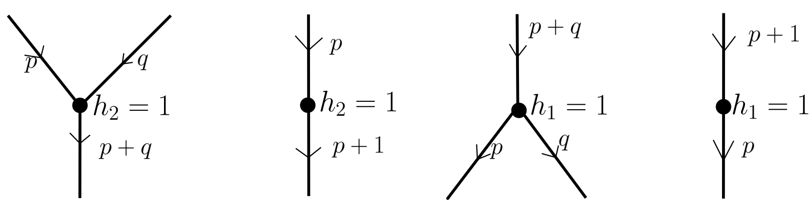
A flow ϕ_t on a compact manifold M is **Morse-Smale** if the chain recurrent set $\mathfrak{R}$ consists of hyperbolic singularity or closed orbits and the unstable manifold of any singularity or closed orbit has transverse intersection with the stable manifold of any singularity or closed orbit. Let L be the Lyapunov graph associated to this flow. Each vertex will be labelled with $(h_0, h_1, h_2) \in \mathbb{N}^3$ which means that for $j = 0, 1, 2$, the rank of the j-th Conley homology index of the isolating block associated with the vertex is equal to h_j. Then one has the following result due to de Rezende [12].

Proposition 1.6.1. Let ϕ_t be a Morse-Smale flow on a closed orientable 3-manifold with Lyapunov function f. Let L be a Lyapunov graph with respect to f. The bounds on the degree of a vertex v of L and the weights on the incident edges of v are as follows:

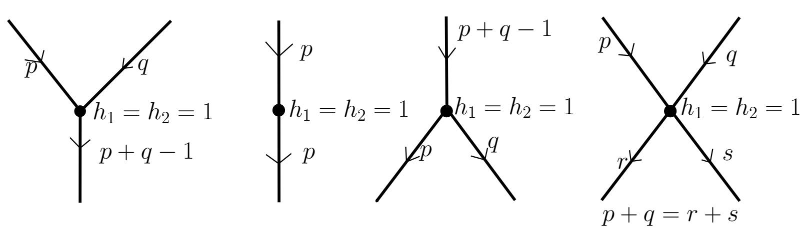
1. If v is a vertex labelled with a source, $h_3 = 1$ (sink, $h_0 = 1$), then $e^- = 1$ and $G^- = 0$ ($e^+ = 1$ and $G^+ = 0$).

2. If v is a vertex labelled with a repelling periodic orbit, $h_2 = h_3 = 1$, (attracting periodic orbit, $h_0 = h_1 = 1$) then $e^- = 1$ and $G^- = 1$ ($e^+ = 1$ and $G^+ = 1$).

3. If v is a vertex labelled with a saddle of index 2, $h_2 = 1$ (saddle of index 1, $h_1 = 1$) then $e^- = 1$ and $e^+ \leq 2$ ($e^+ = 1$ and $e^- \leq 2$). Furthermore, the weights on the incoming and outgoing edges of v must satisfy:



4. If v is a vertex labelled with a saddle type periodic orbit, $h_1 = h_2 = 1$, then $e^+ \leq 2$ and $e^- \leq 2$ and the weights on the incoming and outgoing edges of v must satisfy:



1.7 Filtration and homology

The existence of a smooth Lyapunov function for a flow ϕ_t implies the existence of a filtration associated to it.

Definition 1.7.1. If $\phi_t : M \to M$ is a flow with hyperbolic chain recurrent set with basic sets $\{\Lambda_i\}$ $(i = 1, \ldots, n)$, a **filtration** associated with ϕ_t is a collection of submanifolds $M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that

1. $\phi_t(M_i) \subset \operatorname{int} M_i$, for each $t > 0$;

2. $\Lambda_i = \bigcap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1})$.

The following is a result due to Bowen and Franks [7].

Theorem 1.7.1. Let ϕ_t be a Smale flow and let M_i, $i = 1, \ldots, n$ be a filtration associated to ϕ_t. Suppose that $\Lambda_i = \bigcap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1})$ is a basic set of index k labelled with a structure matrix $A_{n\times n}$. Then

$$H_s(M_i, M_{i-1}, F_2) \cong 0, \text{ se } \quad s \neq k, k+1;$$
$$H_k(M_i, M_{i-1}, F_2) \cong F_2^n/(I-B)F_2^n;$$
$$H_{k+1}(M_i, M_{i-1}, F_2) \cong \ker\left((I-B) \text{ on } F_2^n\right).$$

where $B = A \bmod 2$ and $F_2 = \mathbb{Z}_2$

See [15] for the proof.

The following theorem in [13] for Smale flows on S^3 completely classifies Lyapunov graphs for flows on S^3.

Theorem 1.7.2. Let L be an abstract Lyapunov graph. L is associated with a Smale flow ϕ_t on $\mathbb{S}^3$ if only if the following conditions hold:

1. The underlying graph L is a tree with exactly one edge attached to each sink or source vertex. Moreover, the sink (source) vertex are each labelled with an index 0 (index 3) singularity or an attracting (repelling) periodic orbit.

2. If a vertex is labelled with a singularity of index 2 (index 1) then $1 \le e^+ \le 2$ and $e^- = 1$ ($e^+ = 1$ and $1 \le e^- \le 2$). If a vertex is labelled with a suspension of a subshift of finite type and $A_{n\times n}$ is the non-negative integer matrix representing this subshift, then

$$e^+,\, e^- > 0,$$
$$k+1-G^- \le e^+ \le k+1, \text{ with } \quad G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \quad \text{and}$$
$$k+1-G^+ \le e^- \le k+1 \text{ with } \quad G^+ = \textstyle\sum_{j=1}^{e^+} g_j^+$$

where $k = \dim \ker\left((I-B) : F_2^m \to F_2^m\right)$, $F_2 = \mathbb{Z}_2$ and $b_{ij} = a_{ij} \bmod 2$, $e^+(e^-)$ is the number of incoming (outgoing) edges of the vertex and $g_j^+(g_i^-)$ is the weight on an incoming (outgoing) edge of the vertex.

3. All vertices satisfy the Poincaré Hopf condition, i.e., for a vertex labelled with a singularity of index r, the condition is

$$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^- ,$$

and for a vertex labelled with a suspension of a subshift of finite type or a periodic orbit, the condition is

$$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^- .$$

Chapter 2

Surfaces Embedded in $\mathbb{S}^2 \times \mathbb{S}^1$

In this chapter we establish a topological result on embedding of surfaces in $\mathbb{S}^2 \times \mathbb{S}^1$ which is important in this work. Due to the topology of $\mathbb{S}^2 \times \mathbb{S}^1$ these surfaces can be separable or non-separable and thus determine interesting results on the embedding. Also, we introduce the concept of manifolds of handlebody type and a result which establishes a relationship between embedded separable surfaces and manifolds of handlebody type.

2.1 Some topological facts in $\mathbb{S}^2 \times \mathbb{S}^1$

For our work it is important to have a characterization of the generators of the two dimensional homology of $\mathbb{S}^2 \times \mathbb{S}^1$.

Lemma 2.1.1. Let S be an orientable closed connected surface in $\mathbb{S}^2 \times \mathbb{S}^1$ such that S is non-separable and $i : S \to \mathbb{S}^2 \times \mathbb{S}^1$ is the inclusion map. Then the induced homomorphism

$$i_* : H_2(S) \to H_2(\mathbb{S}^2 \times \mathbb{S}^1)$$

is an isomorphism.

Proof. Since S is non-separable, there are two points A and B on a tubular neighborhood of S which can be connected by a path that intersects S in one single point C. This path can be extended to a loop which intersects S in C. It follows that the one dimensional homology class $[\gamma]$ and the two dimensional homology class $[S]$ have a non-zero intersection number. Therefore, both $[\gamma]$ and $[S]$ are non-trivial elements in the homology of $\mathbb{S}^2 \times \mathbb{S}^1$. Since, $H_2(\mathbb{S}^2 \times \mathbb{S}^1, F_2) \cong F_2$, one has that i_* is an isomorphism. $\square$

Definition 2.1.1. Let S be a closed surface in $\mathbb{S}^2 \times \mathbb{S}^1$ and let $i : S \to \mathbb{S}^2 \times \mathbb{S}^1$ be the inclusion embedding. Let

$$i_* : \pi_1(S) \to \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$$

be the homomorphism induced by i. We say that S is π_1-**trivial** in $\mathbb{S}^2 \times \mathbb{S}^1$ if i_* is trivial and π_1-**non-trivial** otherwise.

The study of embedded surfaces in $\mathbb{S}^2 \times \mathbb{S}^1$ is essential to our work. Some of these embeddings have been previously studied, such as, the embedding of the sphere $\mathbb{S}^2$ [1] and the embedding of the torus in [20]. In this section, we present other embeddings that are relevant for our work. The following result of the embedding of $\mathbb{S}^2$ in $\mathbb{S}^2 \times \mathbb{S}^1$ is a well known result with a nice proof in [1].

Proposition 2.1.1. Let S be a differential embedding of the 2-sphere in $\mathbb{S}^2 \times \mathbb{S}^1$, then

1. either S bounds a 3-ball,
2. or S is homotopic to a fiber $S^2 \times \{t\}$, with $t \in S^1$.

Now let ϕ_t be a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov function $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$ and let L be the Lyapunov graph associated to f. Let $c \in \mathbb{R}$ be a regular value, then $f^{-1}(c)$ is the disjoint collection of orientable and closed surfaces. Let $\mathcal{T}_g \subset f^{-1}(c)$ be a connected and closed surface of genus g, then by the topology of $\mathbb{S}^2 \times \mathbb{S}^1$, $\mathcal{T}_g$ can be a separable or a non-separable surface in $\mathbb{S}^2 \times \mathbb{S}^1$.

If $\mathcal{T}_g$ is non-separable, there exists another connected component of the regular level set $\mathcal{T}_{\widehat{g}}$ non-parallel to $\mathcal{T}_g$, such that $\mathbb{S}^2 \times \mathbb{S}^1 - (\mathcal{T}_g \sqcup \mathcal{T}_{\widehat{g}})$ has two components. We denote their closures by M_1 and M_2. Therefore, M_1 and M_2 satisfy

$$M_1 \cup M_2 = \mathbb{S}^2 \times \mathbb{S}^1 \quad \text{and} \quad M_1 \cap M_2 = \mathcal{T}_g \sqcup \mathcal{T}_{\widehat{g}}$$

Also, $\partial M_1 = \partial M_2 = \mathcal{T}_g \sqcup \mathcal{T}_{\widehat{g}}$. The following lemma gives us homological information on M_1 and M_2.

Lemma 2.1.2. M_1 and M_2 as defined above satisfy:

1. $H_2(M_i) \cong F_2$,
2. $\dim H_1(M_i) = \widehat{g} + g$.

Proof. We consider the Mayer-Vietoris exact sequence of the pair (M_1, M_2),

$$0 \to H_3(M_1) \oplus H_3(M_2) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(M_1 \cap M_2) \to H_2(M_1) \oplus H_2(M_2) \xrightarrow{\alpha} H_2(\mathbb{S}^2 \times \mathbb{S}^1). \tag{2.1.1}$$

Since, M_1 and M_2 are compact manifolds with boundary, then $H_3(M_1) \cong H_3(M_2) \cong 0$. On the other hand, $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2$ and as $M_1 \cap M_2$ has two components, then $H_2(M_1 \cap M_2) \cong F_2^2$. Therefore, one has

$$\dim H_2(M_1) + \dim H_2(M_2) \leq 2. \tag{2.1.2}$$

On the other hand, we consider the exact sequence of the pair $(M_i, \partial M_i)$,

$$0 \to H_3(M_i, \partial M_i) \to H_2(\partial M_i) \to H_2(M_i) \to H_2(M_i, \partial M_i). \tag{2.1.3}$$

Since, M_i is a 3-manifold with boundary, then $H_3(M_i, \partial M_i) \cong F_2$. Also ∂M_i has two components, and hence $H_2(\partial M_i) \cong F_2^2$. Therefore,

$$\dim H_2(M_i) \geq 1. \tag{2.1.4}$$

By (2.1.2) and (2.1.4), one concludes that

$$H_2(M_i) \cong F_2.$$

So, one writes the exact sequence (2.1.3) as

$$0 \to H_2(M_i, \partial M_i) \to H_1(\partial M_i) \to H_1(M_i) \to H_1(M_i, \partial M_i) \to \widetilde{H}_0(\partial M_i) \to 0.$$

Therefore,

$$\dim H_1(M_i) = \dim H_1(\partial M_i) - \dim H_2(M_i, \partial M_i) + \dim H_1(M_i, \partial M_i) - \dim \widetilde{H}_0(\partial M_i). \tag{2.1.5}$$

At this point, one needs to consider the following proposition.

Proposition 2.1.2. Let (X, A) be a pair of topological spaces, and let F_2 be a field. There is a natural isomorphism

$$\beta : H^n(X, A; F_2) \to Hom_{F_2}(H_n(X, A; F), F_2) \cong H_n(X, A; F_2)^*$$

Since, $\partial M_i \cong \mathcal{T}_g \sqcup \mathcal{T}_{\hat{g}}$, then $\dim H_1(\partial M_i) = 2g + 2\hat{g}$ and $\widetilde{H}_0(\partial M_i) \cong F_2$. On the other hand, by Proposition 2.1.2 and by Poincaré duality, one has that

$$H_1(M_i, \partial M_i) \cong (H_2(M_i))^* \quad \text{and} \quad H_2(M_i, \partial M_i) \cong (H_1(M_i))^*.$$

In (2.1.5) one has

$$\dim H_1(M_i) = g + \hat{g}. \tag{2.1.6}$$

□

If $\mathcal{T}_g$ is separable, this embedding splits $\mathbb{S}^2 \times \mathbb{S}^1$ in two submanifolds M_1 and M_2 such that:

$$\mathbb{S}^2 \times \mathbb{S}^1 = M_1 \cup M_2 \quad \text{and} \quad M_1 \cap M_2 = \mathcal{T}_g.$$

Lemma 2.1.3. Let M_1 and M_2 be defined as above. Then

1. $\dim H_2(M_1) + \dim H_2(M_2) \leq 1$,
2. If $H_2(M_1) \cong 0$, then $\dim H_1(M_1) = g$,
3. If $H_2(M_1) \cong F_2$, then $\dim H_1(M_1) = g + 1$.

Analogously for M_2.

Proof. Consider the Mayer-Vietoris exact sequence of the pair (M_1, M_2).

$$\to H_3(M_1) \oplus H_3(M_2) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(\mathcal{T}_g) \to H_2(M_1) \oplus H_2(M_2) \to H_2(\mathbb{S}^2 \times \mathbb{S}^1). \tag{2.1.7}$$

Since, M_1 and M_2 are compact manifolds with boundary, then $H_3(M_1) \cong H_3(M_2) \cong 0$. Also, $M_1 \cap M_2$ has one component, hence $H_2(M_1 \cap M_2) \cong F_2$. Therefore, by the sequence (2.1.7) one has

$$\dim H_2(M_1) + \dim H_2(M_2) \leq 1. \tag{2.1.8}$$

Hence, there are the following possibilities:

(a) $H_2(M_1) \cong 0$ and $H_2(M_2) \cong 0$

(b) $H_2(M_1) \cong F_2$ and $H_2(M_2) \cong 0$

(c) $H_2(M_1) \cong 0$ and $H_2(M_2) \cong F_2$.

Now consider the exact sequence of the pair $(M_i, \partial M_i)$,

$$0 \to H_3(M_i, \partial M_i) \to H_2(\partial M_i) \to H_2(M_i) \to H_2(M_i, \partial M_i). \tag{2.1.9}$$

Since, M_i is a 3-manifold with boundary $H_3(M_i, \partial M_i) \cong F_2$. Also, ∂M_i has one component, hence $H_2(\partial M_i) \cong F_2$. Therefore, $H_3(M_i, \partial M_i)$ is isomorphic to $H_2(\partial M_i)$. Thus, we can write the exact sequence (2.1.9) as

$$0 \to H_2(M_i) \to H_2(M_i, \partial M_i) \to H_1(\partial M_i) \to H_1(M_i) \to H_1(M_i, \partial M_i) \to 0$$

which implies that

$$2g = \dim H_2(M_i, \partial M_i) - \dim H_2(M_i) + \dim H_1(M_i) - \dim H_1(M_i, \partial M_i). \tag{2.1.10}$$

By Proposition 2.1.2 and by Poincaré duality, one has that

$$H_1(M_i, \partial M_i) \cong F_2 \quad \text{and} \quad H_2(M_i, \partial M_i) \cong (H_1(M_i))^* .$$

Hence, equation (2.1.10) can be written as

$$g = \dim H_1(M_i) - \dim H_2(M_i). \tag{2.1.11}$$

Now one concludes the proof with a case analysis.

Case (a)**:** Consider $H_2(M_1) \cong 0$ and $H_2(M_2) \cong 0$.

Hence, by equation (2.1.11) one has

$$g = \dim H_1(M_i).$$

Case (b)**:** Consider $H_2(M_1) \cong F_2$ and $H_2(M_2) \cong 0$.

By the prior case, if $H_2(M_2) \cong 0$ then $H_1(M_2) \cong F_2^g$. In the case $H_2(M_1) \cong F_2$, equation (2.1.11) is written as

$$g = \dim H_1(M_1) - 1,$$

which implies that $\dim H_1(M_1) = g + 1$.

Case (c)**:** Consider $H_2(M_1) \cong 0$ and $H_2(M_2) \cong F_2$.

In a similar fashion to the previous case one has that

(i) if $H_2(M_1) \cong 0$, then $\dim H_1(M_1) = g$,

(ii) if $H_2(M_2) \cong F_2$, then $\dim H_1(M_2) = g + 1$.

$\square$

2.2 Manifolds of handlebody type

Definition 2.2.1. A connected manifold M^3 is of **handlebody type** whenever ∂M^3 is homeomorphic to a closed and orientable surface of genus g, for some $g \geq 0$ and verifies

$$H_2(M^3) \cong 0 \quad \text{and} \quad \dim H_1(M^3) = g.$$

The positive number g is defined as the genus of the manifold of handlebody type.

Remark 2.2.1. By Lemma 2.1.3, at least one of M_1 or M_2 is of handlebody type.

Lemma 2.2.1. With the above notation, M_2 is not a manifold of handlebody type if and only if

$$H_2(M_2) \cong F_2.$$

Proof. Necessity. By inequality (2.1.8) one has that $\dim H_2(M_2) \leq 1$, hence we have two possibilities. If $H_2(M_1) \cong 0$, by equality (2.1.11), it follows that $\dim H_1(M_2) = g$. Hence, M_2 is of handlebody type contradicting the hypothesis. Therefore, $H_2(M_1) \cong F_2$.

Sufficiency. It follows from equality (2.1.11), in the proof of Lemma 2.1.3. □

Lemma 2.2.2. Let N be a 3-submanifold of $\mathbb{S}^2 \times \mathbb{S}^1$, such that N is of handlebody type and ∂N is a π_1-trivial surface in $\mathbb{S}^2 \times \mathbb{S}^1$. Then the inclusions $j : N \to \mathbb{S}^2 \times \mathbb{S}^1$ and $i : \partial N \to N$ induces homomorphisms

$$j_* : H_1(N) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1) \quad \text{and} \quad i_* : H_1(\partial N) \to H_1(N)$$

which j_* is trivial and i_* is surjective.

Proof. Consider the following diagram

$$\begin{tikzcd} \partial N \arrow[d, "i"] \arrow[dr, "k"] & \\ N \arrow[r, "j"] & \mathbb{S}^2 \times \mathbb{S}^1 \end{tikzcd}$$

where i, j and k are inclusions. These functions induce the following diagram.

$$\begin{tikzcd} H_1(\partial N) \arrow[d, "i_*"] \arrow[dr, "k_*"] & \\ H_1(N) \arrow[r, "j_*"] & H_1(\mathbb{S}^2 \times \mathbb{S}^1) \end{tikzcd}$$

Since, ∂N is π_1-trivial, then k_* is trivial. Thus, in order to prove that j_* is trivial, it is only necessary to prove that i_* is surjective. Now, consider the exact homology sequence of the pair $(N, \partial N)$,

$$0 \to H_2(N, \partial N) \to H_1(\partial N) \xrightarrow{i_*} H_1(N) \to H_1(N, \partial N).$$

By Proposition 2.1.2 and the Poincaré duality, one has

$$H_1(N, \partial N) \cong (H_2(N))^*.$$

On the other hand, one has that $H_2(N) \cong 0$. Hence the lemma follows.

$\square$

Lemma 2.2.3. Let Y and Z be two disjoint 3-submanifolds of handlebody type in $\mathbb{S}^2 \times \mathbb{S}^1$, such that ∂Y and ∂Z are π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$. Let X be the closure of the complement of Y in $\mathbb{S}^2 \times \mathbb{S}^1$. Then the inclusion $j : Z \to X$ induces a homomorphism

$$j_* : H_1(Z) \to H_1(X)$$

which is trivial.

Proof. Consider the following diagram

$$\begin{tikzcd}
\partial Z \arrow[d, "k"'] \arrow[dr, "i"] & \\
Z \arrow[r, "j"] & X
\end{tikzcd}$$

where i, j and k are inclusions. These functions induce the following diagram

$$\begin{tikzcd}
H_1(\partial Z) \arrow[d, "k_*"'] \arrow[dr, "i_*"] & \\
H_1(Z) \arrow[r, "j_*"] & H_1(X)
\end{tikzcd} \tag{2.2.1}$$

Since, Y and Z are disjoint, one has that ∂Z is π_1-trivial in X. This follows by considering the Mayer-Vietoris exact homology sequence.

$$\to H_1(X \cap Y) \to H_1(Y) \oplus H_1(X) \xrightarrow{\beta} H_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \widetilde{H}_0(X \cap Y).$$

Since, $\widetilde{H}_0(X \cap Y) \cong 0$, one has that β is surjective. On the other hand, since Y is π_1-trivial, then the restricted homomorphism

$$H_1(X) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1) \tag{2.2.2}$$

is surjective. Now consider the following diagram.

$$\begin{tikzcd}
\pi_1(\partial Z) \arrow[r, "\alpha"] \arrow[dr, "\gamma"'] & \pi_1(\mathbb{S}^2 \times \mathbb{S}^1) \\
& \pi_1(X) \arrow[u, "\beta_1"']
\end{tikzcd}$$

By (2.2.2), one has that β_1 is surjective and by the hypothesis, α is trivial and hence γ is trivial. Therefore, ∂Z is π_1-trivial in X. Hence, by (2.2.1), one has i_* is trivial. Thus, by Lemma 2.2.2 one concludes that k_* is surjective. Therefore, j_* is trivial. □

Lemma 2.2.4. Let $\mathcal{T}_g$ be a separable connected component of a regular level set associated to some Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$. Let M_1 and M_2 be the closure of the two connected components of $\mathbb{S}^2 \times \mathbb{S}^1 - \mathcal{T}_g$. Hence,

1. if $\mathcal{T}_g$ is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, then either M_1 or M_2 is of handlebody type;
2. if $\mathcal{T}_g$ is π_1-non-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, then M_1 and M_2 are of handlebody type.

Proof. 1. It follows by Remark 2.2.1, that at least one of the submanifolds M_1 or M_2 is of handlebody type. Suppose that, M_1 is a manifold of handlebody type. By Lefschetz Duality, one has that

$$H_1(\mathbb{S}^2 \times \mathbb{S}^1, M_1) \cong H_2(M_2). \tag{2.2.3}$$

Now, consider the homology exact sequence of the pair $(\mathbb{S}^2 \times \mathbb{S}^1, M_1)$,

$$\to H_2(\mathbb{S}^2 \times \mathbb{S}^1, M_1) \to H_1(M_1) \xrightarrow{\eta} H_1(\mathbb{S}^2 \times \mathbb{S}^1) \to H_1(\mathbb{S}^2 \times \mathbb{S}^1, M_1) \to 0. \tag{2.2.4}$$

Since, $\mathcal{T}_g$ is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, by Lemma 2.2.2, one has that η is equal to zero. Therefore, $H_1(\mathbb{S}^2 \times \mathbb{S}^1) \cong H_1(\mathbb{S}^2 \times \mathbb{S}^1, M_1)$ and by (2.2.3) one has that

$$H_2(M_2) \cong F_2.$$

The result now follows by Lemma 2.2.1.

2. If $\mathcal{T}_g$ is π_1-non-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$ there exists a simple closed curve β in $\mathcal{T}_g$ such that $[\beta] \in \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$ is non-trivial. Furthermore, β intersects every non-separable orientable surface S in $\mathbb{S}^2 \times \mathbb{S}^1$. Since $M_1 \cap M_2 = \mathcal{T}_g$ it is possible to find two simple closed curves β_1 and β_2 homotopic to β and such that $\beta_1 \subset int(M_1)$ and $\beta_2 \subset int(M_2)$. Suppose that M_1 is not of handlebody type hence, $H_2(M_1) \cong F_2$. Therefore, there exists a non-separable surface S in the interior of M_1. However, since β_2 is homotopic to β one has that $[\beta_1] \in \pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$ is non-trivial and hence must intersect every non-separable orientable surface in $\mathbb{S}^2 \times \mathbb{S}^1$. In particular, β_1 intersects S which is a contradiction.

□

Chapter 3

Manifolds of Handlebody Type and Lyapunov Graphs

In the previous chapter we obtained a relationship between an embedded separable surface S in $\mathbb{S}^2 \times \mathbb{S}^1$ and a manifold of handlebody type. More specifically, if S is π_1-non-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, then S bounds a manifold of handlebody type on both sides. If S is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, then S bounds a manifold of handlebody type only on one side. In this chapter we study the relationship between manifolds of handlebody type and Lyapunov graph L with $\beta(L) = 0$ associated with a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$.

3.1 Manifolds of handlebody type and Lyapunov graphs

In this section, we study the relation between Lyapunov graphs and manifolds of handlebody type. Let ϕ_t be a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov function f and let L be the associated Lyapunov graph. Suppose L is a tree. Choose a regular level set which is a separable surface $\mathcal{T}_g$ which divides $\mathbb{S}^2 \times \mathbb{S}^1$ in two 3-submanifolds, $\mathcal{M}^-_{\mathcal{T}_g}$ and $\mathcal{M}^+_{\mathcal{T}_g}$, and such that the flow is transversal and outward going on $\mathcal{M}^-_{\mathcal{T}_g}$ and inward going on $\mathcal{M}^+_{\mathcal{T}_g}$. See Figure 3.1.

Remark 3.1.1. By Lemma 2.2.4 one has that:

- if $\mathcal{T}_g$ is π_1-trivial, then either $\mathcal{M}^+_{\mathcal{T}_g}$ or $\mathcal{M}^-_{\mathcal{T}_g}$ is of handlebody type;
- if $\mathcal{T}_g$ is π_1-non-trivial, then $\mathcal{M}^+_{\mathcal{T}_g}$ and $\mathcal{M}^-_{\mathcal{T}_g}$ are of handlebody type;
- if we cut $\mathbb{S}^2 \times \mathbb{S}^1$ along a separable regular level set, the graph L is disconnected into two subgraphs with dangling edges that represent each one of the manifolds $\mathcal{M}^+_{\mathcal{T}_g}$ and $\mathcal{M}^-_{\mathcal{T}_g}$.

Proposition 3.1.1. Let u be a vertex in L and let e be an incoming edge of u with weight g. Suppose that there exists a vertex v labelled with a basic set Λ such that Λ is contained in $\mathcal{M}^-_{\mathcal{T}_g}$. Then there exists an incoming (outgoing) edge with weight $\widehat{g}$ of v such that $\mathcal{M}^+_{\mathcal{T}_{\widehat{g}}}$ ($\mathcal{M}^-_{\mathcal{T}_{\widehat{g}}}$) is contained in $\mathcal{M}^-_{\mathcal{T}_g}$.

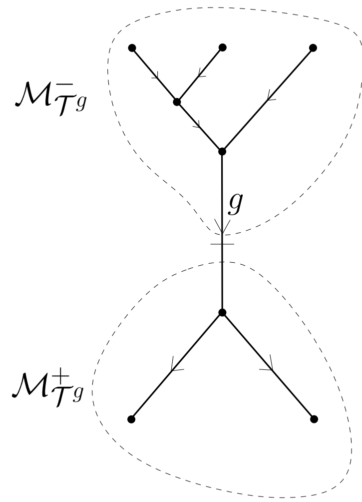


Figure 3.1: g is the genus of the regular level set.

Proof. Since L is a tree, there exists an incoming or outcoming edge e_1 (with weight $\widehat{g}$) of v such that cutting along this edge separates u and v in distinct subgraphs. First suppose that e_1 is an incoming edge of v. Hence, if we cut $\mathbb{S}^2 \times \mathbb{S}^1$ along $\mathcal{T}_{\widehat{g}}$ one has $\mathcal{M}^+_{\mathcal{T}_{\widehat{g}}} \subset \mathcal{M}^-_{\mathcal{T}_g}$. On the other hand, if e_1 is an outgoing edge of v, $\mathcal{M}^-_{\mathcal{T}_{\widehat{g}}} \subset \mathcal{M}^-_{\mathcal{T}_g}$. □

We have a similar result if the edge e is outgoing of u. In what follows other interesting properties of L are determined. Given a vertex v of the graph L labelled with a basic set Λ_v, each edge point represents a regular level set which is a surface of genus g_i^+ (g_j^-) embedded in $\mathbb{S}^2 \times \mathbb{S}^1$ denoted by $\mathcal{T}_{g_i^+}$ ($\mathcal{T}_{g_j^-}$). The incoming (outgoing) edges are labelled with weights corresponding to g_i^+'s(g_j^-'s).

Corollary 3.1.1. Let v be a vertex on L and with the above notation.

1. For $\alpha \in \{1, \ldots, e_v^-\}$ fixed, one has

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}$$

   with $k \in \{1, \ldots, e^+\}$ and $j \in \{1, \ldots, e^-\} - \{\alpha\}$.

2. For $\beta \in \{1, \ldots, e^+\}$ fixed, one has

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \subset \mathcal{M}^+_{\mathcal{T}_{g_\beta^+}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}} \subset \mathcal{M}^+_{\mathcal{T}_{g_\beta^+}}$$

   with $k \in \{1, \ldots, e^+\} - \{\beta\}$ and $j \in \{1, \ldots, e^-\}$.

Proof. 1. Since $\mathcal{T}_{g_\alpha^-}$ corresponds to an outgoing edge of v, let u be the vertex on which the edge is incoming. Note that each $\mathcal{T}_{g_k^+}$ corresponds to an outgoing edge of vertices v_k that are incoming of v. Hence, by the Lemma 2.2.4 it follows that

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}.$$

Analogously, each $\mathcal{T}_{g_j^-}$ corresponds to an outgoing edge of vertices v_j and by Lemma 2.2.4 it follows that

$$\mathcal{M}^+_{\mathcal{T}_{g_j^-}} \subset \mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}.$$

2. To prove 2, it suffices to reverse the orientation on the graph.

$\square$

Corollary 3.1.2. Let v be a vertex on L and with the notation above, one has

1. if $\mathcal{M}^-_{\mathcal{T}_{g_\alpha^-}}$ is of handlebody type for $\alpha \in \{1, \dots, e^-\}$, then

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}}$$

are of handlebody type for $k \in \{1, \dots, e^+\}$ and $j \in \{1, \dots, e^-\} - \{\alpha\}$;

2. if $\mathcal{M}^+_{\mathcal{T}_{g_\beta^+}}$ is of handlebody type for $\beta \in \{1, \dots, e^+\}$, then

$$\mathcal{M}^-_{\mathcal{T}_{g_k^+}} \quad \text{and} \quad \mathcal{M}^+_{\mathcal{T}_{g_j^-}}$$

are of handlebody type for $k \in \{1, \dots, e^+\} - \{\beta\}$ and $j \in \{1, \dots, e^-\}$.

Proof. This follows directly from Remark 3.1.1, of Corollary 3.1.1 and from the fact that a submanifold with a surface boundary of a manifold of handlebody type is of handlebody type. $\square$

Chapter 4

Smale Flows on $\mathbb{S}^2 \times \mathbb{S}^1$

In this chapter we used all the theory developed in the previous chapters to obtain the main result on Lyapunov graphs associated with Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$. This is done as follows: first, Lyapunov graphs with cycle rank equal to one are characterized. Secondly, Lyapunov graphs without cycles are characterized.

Let ϕ_t be a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov function $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$. Let Λ be a basic set of ϕ_t and $f(\Lambda) = c$. Choose $\epsilon > 0$ sufficiently small so that c is the only critical value in $[c - \epsilon, c + \epsilon]$. Let $X_0 = f^{-1}(-\infty, c + \epsilon]$ and $Z_0 = f^{-1}(-\infty, c - \epsilon]$.

4.1 Lyapunov graph with a cycle

Let U be the closure of the component of $X_0 - Z_0$ which contains Λ. Then U is a neighborhood of Λ whose boundary consists of closed and orientable surfaces to which the flow is transverse. The flow enters on e^+ of these surfaces and exits on the remaining e^-.

Proposition 4.1.1. Suppose ϕ_t is a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$, with Lyapunov graph L, such that $\beta(L) = 1$. Let v be a vertex of L on the cycle, labelled with a suspension of a subshift of finite type and $A_{n\times n}$ is the non-negative integer matrix representing this subshift. Let $k = \dim\ker \overline{I - A} : F_2^n \to F_2^n$ where $F_2 = \mathbb{Z}_2$ and $\overline{I - A}$ is the $\bmod 2$ reduction of $I - A$. Then, if e^+ and e^- are respectively the indegree and outdegree of v, one has:

$$\begin{gathered} e^+, e^- > 0, \\ e^+ \leq k+1, \\ e^- \leq k+1, \\ k+1-G^- \leq e^+, \text{ with } \quad G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \quad \text{and} \\ k+1-G^+ \leq e^-, \text{ with } \quad G^+ = \textstyle\sum_{j=1}^{e^+} g_j^+. \end{gathered}$$

Where g_i^+'s (g_j^-)'s are the weights on the incoming (outgoing) edges of the vertex v.

Proof. Suppose Λ is the basic set of ϕ_t corresponding to v. First of all, suppose that both edges of the cycle are incoming edges of the vertex v, as shown in Figure 4.1. Let Z be the union of the components of $\mathbb{S}^2 \times \mathbb{S}^1 - (\partial U \cap Z_0)$ which do not contain Λ. By Lemma 2.2.4, each component

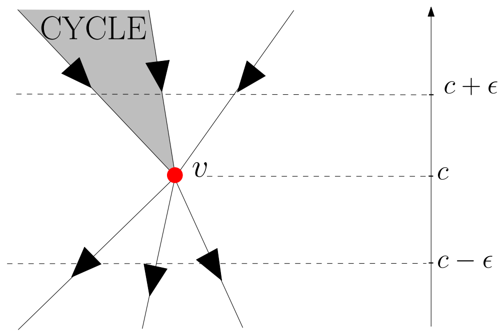


Figure 4.1: v is a vertex on the cycle.

of Z is a manifold of handlebody type. Now we call $X = Z \cup U$ and $Y = \mathbb{S}^2 \times \mathbb{S}^1 - X$. Suppose Y_1 is a component of Y such that it corresponds to the cycle on the graph. Therefore, by the Lemma 2.1.2 one has

$$\dim H_1(Y_1) = g_1^+ + g_2^+ \quad \text{and} \quad \dim H_2(Y_1) = 1. \tag{4.1.1}$$

and any other component of Y is of handlebody type.

Assertion 4.1.1. Let X_0, Z_0, X and Z be defined as above, then

$$H_2(X_0, Z_0) \cong H_2(X, Z).$$

This Assertion follows by excising $U = X \cap X_0^c$, where X_0^c is the complement of X_0, one has that,

$$H_2(X, Z) \cong H_2(X - U, Z - U) \cong H_2(X_0, Z - U).$$

Since Z_0 is a strong deformation retract of $Z - U$ this implies that

$$H_2(X, Z) \cong H_2(X_0, Z_0).$$

This conclude to proof of this assertion. By Assertion 4.1.1 and Theorem 1.7.1, we conclude that $\dim H_2(X, Z) = \dim H_2(X_0, Z_0) = k$. We consider the exact homology sequence of the pair (X, Z),

$$H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X). \tag{4.1.2}$$

Since, $H_3(X, Z) \cong 0$ and $H_2(Z) \cong 0$, this implies

$$\dim H_2(X) \leq \dim H_2(X, Z) = k. \tag{4.1.3}$$

On other hand, we consider the Mayer-Vietoris exact homology sequence,

$$H_3(X) \oplus H_3(Y) \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(X \cap Y) \to H_2(X) \oplus H_2(Y) \xrightarrow{\beta} H_2(\mathbb{S}^2 \times \mathbb{S}^1) \to \tag{4.1.4}$$

Both X and Y are compact 3$-$manifolds with boundary, so $H_3(X) \cong H_3(Y) \cong 0$. Also, $X \cap Y$ is composed of e^+ closed orientable surfaces, so $H_2(X \cap Y) \cong F_2^{e^+}$. On the other hand, we know that

$H_3(\mathbb{S}^2\times\mathbb{S}^1)\cong H_2(\mathbb{S}^2\times\mathbb{S}^1)\cong F_2$. Since, $H_2(Y)\cong F_2$, this submanifold contains a non-separable regular level set. By Lemma 2.1.1, it follows that β is surjective. Therefore,

$$\dim H_2(X) = e^+ - 1$$

By inequality (4.1.3), one has

$$e^+ \leq k+1.$$

If $a = \dim \operatorname{Im}\alpha$, from the exact sequence (4.1.2), one has

$$k = e^+ - 1 + G^- - a.$$

Hence,

$$k+1-G^- \leq e^+. \tag{4.1.5}$$

On the other hand, U satisfies the Poincaré equality,

$$e^+ + G^- = e^- + G^+.$$

By inequality (4.1.5), one has

$$k+1-G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transform A to A^t so k is unchanged.

We call $W = U\cup Y$ and in this case the flow enters in U through Y. We consider the exact homology sequence of the pair (W,Y),

$$H_3(W,Y)\to H_2(Y)\to H_2(W)\to H_2(W,Y). \tag{4.1.6}$$

By Theorem 1.7.1, one has $H_3(W,Y)\cong 0$ and $H_2(Y)\cong F_2$. Therefore, by sequence (4.1.6)

$$\dim H_2(W)\leq k+1. \tag{4.1.7}$$

Now, consider the Mayer-Vietoris exact homology sequence,

$$H_3(W)\oplus H_3(Z)\to H_3(\mathbb{S}^2\times\mathbb{S}^1)\to H_2(W\cap Z)\to H_2(W)\oplus H_2(Z)\xrightarrow{\gamma} H_2(\mathbb{S}^2\times\mathbb{S}^1). \tag{4.1.8}$$

Both W and Z are compact 3-manifolds with boundary, so $H_3(W)\cong H_3(Z)\cong 0$. $W\cap Z$ is composed of e^- closed orientable surfaces, hence $H_2(X\cap Y)\cong F_2^{e^-}$. Since the submanifold W contains a non-separable regular level set by Lemma 2.1.1, γ is surjective. Therefore, $\dim H_2(W) = e^-$ and by inequality (4.1.7) one has,

$$e^-\leq k+1.$$

The proof is complete in this case.

Now, suppose that one edge enters v and the other exists v, as shown in Figure 4.2. Note that, $X_0\cap Y_0$ and $W_0\cap Z_0$ are composed by $e^+ + 1$ and $e^- + 1$ components respectively. One can suppose the component Y_1 of Y_0 and the component Z_1 of Z_0 correspond to an incoming and outgoing edge of the cycle respectively. Let Z be the union of Z_1 with the components of

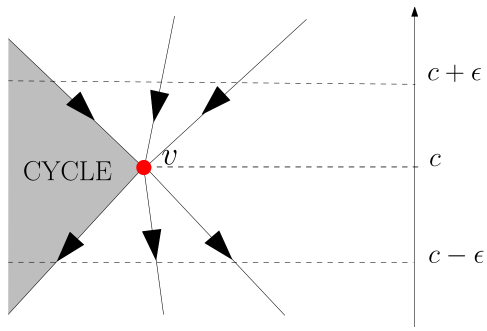


Figure 4.2: v is a vertex in the cycle.

$\mathbb{S}^2 \times \mathbb{S}^1 - (\partial U \cap (Z_0 - Z_1))$ which do not contain Λ. By Lemma 2.2.4, each component of $Z - Z_1$ is a manifold of handlebody type. Now we define $X = Z \cup U$ and $Y = \mathbb{S}^2 \times \mathbb{S}^1 - X$. By Lemma 2.2.4 one has that each component of $(Y - Y_1)$ is a manifold of handlebody type. Note that, $Z \cap Y$ is a closed and orientable surface of genus g. Moreover, by Lemma 2.1.1 one has $H_2(Z) \cong F_2$ and $\dim H_1(Z) = \sum g_i^- + g$. Now let $G^- = \dim H_1(Z)$. With this notation, one has the following assertion:

Assertion 4.1.2. Let X_0, Z_0, X and Z be as defined above, then

$$H_2(X_0, Z_0) \cong H_2(X, Z).$$

Assertion 4.1.2 follows in the same way as Assertion 4.1.1. By Assertion 4.1.2 and Theorem 1.7.1, we conclude that $\dim H_2(X, Z) = \dim H_2(X_0, Z_0) = k$. Now, consider the exact homology sequence of the pair (X, Z),

$$H_3(X, Z) \to H_2(Z) \to H_2(X) \to H_2(X, Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X). \tag{4.1.9}$$

Since, $H_3(X, Y) \cong 0$ and $H_2(Z) \cong 0$, this implies

$$\dim H_2(X) \leq \dim H_2(X, Z) = k + 1. \tag{4.1.10}$$

Consider the Mayer-Vietoris exact homology sequence,

$$0 \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(X \cap Y) \to H_2(X) \oplus H_2(Y) \xrightarrow{\beta} H_2(\mathbb{S}^2 \times \mathbb{S}^1). \tag{4.1.11}$$

Since, $X \cap Y$ is composed of e^+ closed orientable surfaces, hence $H_2(X \cap Y) \cong F_2^{e^+ + 1}$. In addition, we know that $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2$. Furthermore, there exists a non-separable regular level set contained in Y. By Lemma 2.1.1, it implies that β is surjective. Therefore,

$$\dim H_2(X) = e^+.$$

By inequality (4.1.3), one has

$$e^+ \leq k + 1.$$

Also, if $a = \dim \operatorname{Im} \alpha$, from the exact sequence (4.1.2), one has

$$k = e^+ - 1 + \sum g_i^- + g - a.$$

Hence,

$$k + 1 - G^- \leq e^+. \tag{4.1.12}$$

Moreover, U satisfies the Poincaré equality,

$$e^+ + G^- = e^- + G^+.$$

By inequality (4.1.12), one has

$$k + 1 - G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transform A to A^t so that k remains unchanged. $\square$

Theorem 4.1.1. Let L be an abstract Lyapunov graph. L is associated with a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ such that there exists a non-separable regular level set if and only if the following conditions hold:

1. The underlying graph L is an oriented graph and $\beta(L) = 1$ with exactly one edge attached to each sink or source vertex. Moreover, the sink (source) vertex are each labelled with an index 0 (index 3) singularity or an attracting (repelling) periodic orbit.

2. If a vertex is labelled with a singularity of index 2 (index 1), then $1 \leq e^+ \leq 2$ and $e^- = 1$ ($1 \leq e^- \leq 2$ and $e^+ = 1$).

3. If a vertex v is labelled with a suspension of a subshift of finite type and $A_{m\times m}$ is the non-negative integer matrix that is associated to this subshift, then

$$\begin{gathered} e^-, e^+ > 0, \\ k + 1 - G^- \leq e^+ \leq k + 1, \quad \text{with } G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \text{ and} \\ k + 1 - G^+ \leq e^- \leq k + 1, \quad \text{with } G^+ = \textstyle\sum_{i=1}^{e^+} g_i^+, \end{gathered}$$

   where $k = \dim \ker \left((I - B) : F_2^m \to F_2^m\right)$, $F_2 = \mathbb{Z}/2$, $b_{ij} = a_{ij} \bmod 2$ and $g_i^+ (g_j^-)$ are the weights on the incoming (outgoing) edges of the vertex v.

4. All vertices must satisfy the Poincaré-Hopf condition, i.e., for a vertex labelled with a singularity of index r, the condition is

$$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

   and for a vertex labelled with a suspension of a subshift of finite type or a periodic orbit, the condition is

$$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

Proof. Necessity. By Proposition 4.2.2 one has that $\beta(L) \leq 1$. On the other hand, since L has a non-separable regular level set, then the cycle rank of L verifies $\beta(L) \geq 1$. Therefore, $\beta(L) = 1$. Items (1) and (2) follow from Proposition 1.6.1 and item (3) follows from Proposition 4.1.1. *Sufficiency.* Let L be a Lyapunov graph that satisfies conditions (1), (2) and (3) of this theorem. Suppose a is an edge in the cycle of L with weight g and consider the graphs L_1 and L_2 each with a dangling edge. Now cut L along a and glue the graphs L_1 and L_2 by the dangling edges, as

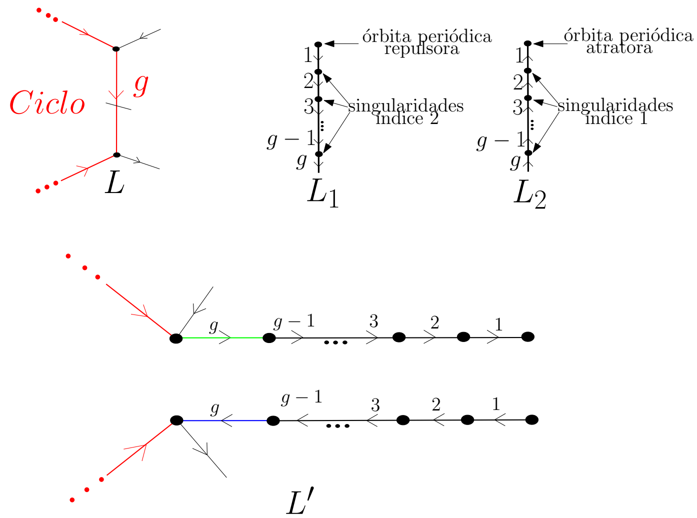


Figure 4.3: The graphs L_1 and L_2 have dangling edge.

shown in Figure 4.3. Then a new Lyapunov graph L' is obtained such that, L' is a tree and each vertex satisfies the conditions in Theorem 1.7.2. Therefore, there exists a Smale flow ψ_t on S^3 with Lyapunov graph L' and such that L_1 and L_2 are associated to two handlebodies H_1^g and H_2^g, whose boundaries are unlinked in $\mathbb{S}^3$, as shown in Figure 4.4. Now we cut the two neighborhoods

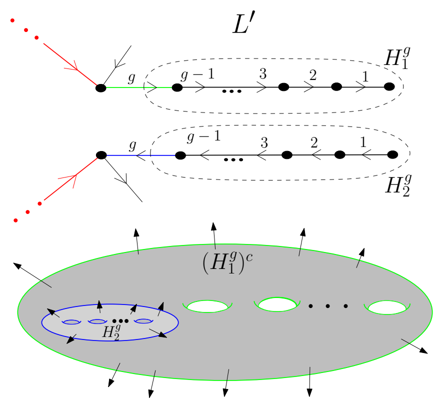


Figure 4.4: L' corresponds with a Smale flow on $\mathbb{S}^3$

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$N(H_1^g)$ and $N(H_2^g)$ of H_1^g and H_2^g respectively. Then gluing $S^3 - (N(H_1^g) \sqcup N(H_2^g))$ along $\partial N(H_1^g)$ and $\partial N(H_2^g)$ suitably, we obtain a Smale flow φ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov graph L.

□

4.2 Lyapunov graph without cycles

We continue to use the notation established in the beginning of this section. Let U be the closure of the component of $X_0 - Z_0$ which contains Λ. Then U is a neighborhood of Λ whose boundary consists of closed and orientable surfaces. Also, the flow is transverse to the boundary of U. The flow enters on e^+ of these surfaces and exits on the remaining e^- surfaces. Let Z be the union of the components of $\mathbb{S}^2 \times \mathbb{S}^1 - (\partial U \cap Z_0)$ which do not contain Λ. Now define $X = Z \cup U$ and $Y = \mathbb{S}^2 \times \mathbb{S}^1 - X$.

Lemma 4.2.1. Let X_0, Z_0, X and Z be defined as above, then

$$H_2(X_0, Z_0) \cong H_2(X, Z).$$

Proof. By excising $U = X \cap X_0^c$, where X_0^c is the complement of X_0, we have that

$$H_2(X, Z) \cong H_2(X - U, Z - U) \cong H_2(X_0, Z - U).$$

Since, Y_0 is a strong deformation retract of $Y - U$ this implies that

$$H_2(X, Y) \cong H_2(X_0, Y_0).$$

□

First of all, suppose that some component of ∂U^- is a π_1-non-trivial regular level set. Furthermore, we can suppose that ∂Z_1 is also π_1-non-trivial regular level set. By Lemma 2.2.4 and Corollary 3.1.2, one has that each component of Z and each component of Y are manifolds of handlebody type. As shown in Figure 4.5.

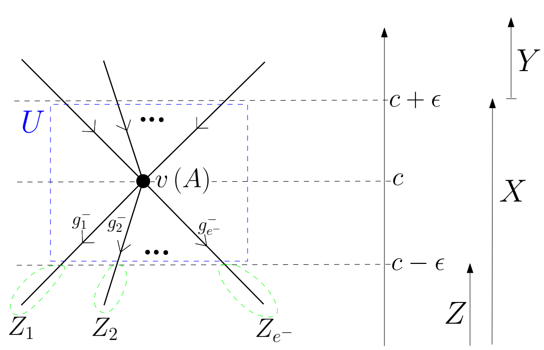


Figure 4.5: ∂Z_1 is π_1-non-trivial

.

Proposition 4.2.1. Suppose ϕ_t is a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$, with Lyapunov graph L, such that L is a tree. Furthermore, suppose there exists a π_1-non-trivial regular level set. Let v be a vertex of L labelled with a suspension of a subshift of finite type and $A_{n\times n}$ is the non-negative integer matrix representing this subshift. Let $k = \dim \ker \overline{I-A} : F_2^n \to F_2^n$ where $F_2 = \mathbb{Z}_2$ and $\overline{I-A}$ is the $\bmod 2$ reduction of $I - A$. Then, if e^+ and e^- are respectively the indegree and outdegree of v, one has:

$$\begin{gathered} e^+, e^- > 0, \\ e^+ \leq k+1, \\ e^- \leq k+1, \\ k+1-G^- \leq e^+, \text{ with } \quad G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \quad \text{and} \\ k+1-G^+ \leq e^-, \text{ with } \quad G^+ = \textstyle\sum_{j=1}^{e^+} g_j^+, \end{gathered}$$

where g_i^+'s (g_j^-)'s are the weights on the incoming (outgoing) edges of the vertex v.

Proof. Suppose Λ is the basic set of ϕ_t corresponding to v. By Lemma 4.2.1 and Theorem 1.7.1, we conclude that $\dim H_2(X,Z) = \dim H_2(X_0, Z_0) = k$. We consider the exact homology sequence of the pair (X, Z),

$$H_3(X,Z) \to H_2(Z) \to H_2(X) \to H_2(X,Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X). \tag{4.2.1}$$

Since, $H_3(X,Y) \cong 0$ and $H_2(Z) \cong 0$, this implies that

$$\dim H_2(X) \leq \dim H_2(X,Z) = k. \tag{4.2.2}$$

Now consider the exact homology reduced sequence of the pair (M, Y),

$$\widetilde{H}_1(Y) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \to \widetilde{H}_0(Y) \to \widetilde{H}_0(\mathbb{S}^2 \times \mathbb{S}^1). \tag{4.2.3}$$

Since, $\widetilde{H}_0(\mathbb{S}^2 \times \mathbb{S}^1) \cong 0$ and $\dim(\widetilde{H}_0(Y)) = e^+ - 1$, we conclude that

$$\dim \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \geq e^+ - 1.$$

Moreover, by Lefschetz duality $H_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1 - Y, \mathbb{S}^2 \times \mathbb{S}^1 - \mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(X)$. Therefore, $\dim H_2(X) \geq e^+ - 1$ and from inequality (4.2.2), one has

$$e^+ \leq k+1$$

Also, from sequence (4.2.3), we have that

$$\dim H_2(X) \leq e^+ - 1 + \dim H_1(\mathbb{S}^2 \times \mathbb{S}^1) = e^+. \tag{4.2.4}$$

If $a = \dim \operatorname{Im} \alpha$, from the exact sequence (4.2.1), one has

$$k = \dim H_2(X) + G^- - a. \tag{4.2.5}$$

The hypothesis that there exits a π_1-non-trivial regular level set in Z, implies that $\operatorname{Im} \alpha \neq 0$. Therefore, from inequality (4.2.5) we can conclude that

$$\dim H_2(X,Z) < \dim H_2(X) + G^-. \tag{4.2.6}$$

Now by inequality (4.2.4), one has that

$$k + 1 - G^- \leq e^+.$$

On the other hand, U satisfies the Poincaré equality,

$$e^+ + G^- = e^- + G^+.$$

Hence, one has

$$k + 1 - G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transform A to A^t so that k is unchanged.

For the general case, we can suppose that a π_1-non-trivial regular level set is contained in some component of Z. Then it suffices to repeat the arguments in the above proof. $\square$

Remark 4.2.1. By Proposition 2.1.1 any sphere in $\mathbb{S}^2 \times \mathbb{S}^1$, separable or non-separable, is π_1-trivial. For this reason, if there exists a π_1-non-trivial surface in $\mathbb{S}^2 \times \mathbb{S}^1$, then its genus must be greater than zero.

Theorem 4.2.1. Let L be an abstract Lyapunov graph, with $\beta(L) = 0$. L is associated with a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ such that there exists a π_1-non-trivial separable regular level set if and only if the following conditions hold:

1. The underlying graph L is an oriented graph and $\beta(L) = 0$ with exactly one edge attached to each vertex labelled with a sink or a source. Moreover, the sink (source) vertex are each labelled with an index 0 (index 3) singularity or an attracting (repelling) periodic orbit.

2. If a vertex is labelled with a singularity of index 2 (index 1), then $1 \leq e^+ \leq 2$ and $e^- = 1$ ($1 \leq e^- \leq 2$ and $e^+ = 1$).

3. If a vertex v is labelled with a suspension of a subshift of finite type and $A_{m\times m}$ is the non-negative integer matrix that is associated to this subshift, then

$$\begin{gathered} e^-, e^+ > 0, \\ k + 1 - G^- \leq e^+ \leq k + 1, \quad \text{with } G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \text{ and} \\ k + 1 - G^+ \leq e^- \leq k + 1, \quad \text{with } G^+ = \textstyle\sum_{i=1}^{e^+} g_i^+, \end{gathered}$$

   where $k = \dim\ker\left((I - B) : F_2^m \to F_2^m\right)$, $F_2 = \mathbb{Z}/2$, $b_{ij} = a_{ij} \bmod 2$ and $g_i^+ (g_j^-)$ are the weights on the incoming (outgoing) edges of the vertex v.

4. All vertices must satisfy the Poincaré-Hopf condition, i.e., for a vertex labelled with a singularity of index r, the condition is

$$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

   and for a vertex labelled with a suspension of a subshift of finite type or a periodic orbit, the condition is

$$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

Proof. *Necessity.* It follows by Proposition 1.6.1 and Proposition 4.2.1.
Sufficiency. Let L be a Lyapunov graph that satisfies conditions (1), (2) and (3) of this theorem. By Remark 4.2.1, there exists an edge a of L, such that the weight of a is $g > 0$. Now we cut L along a and glue the graphs L_1 and L_2 by dangling edges, as shown Figure 4.6. Then new

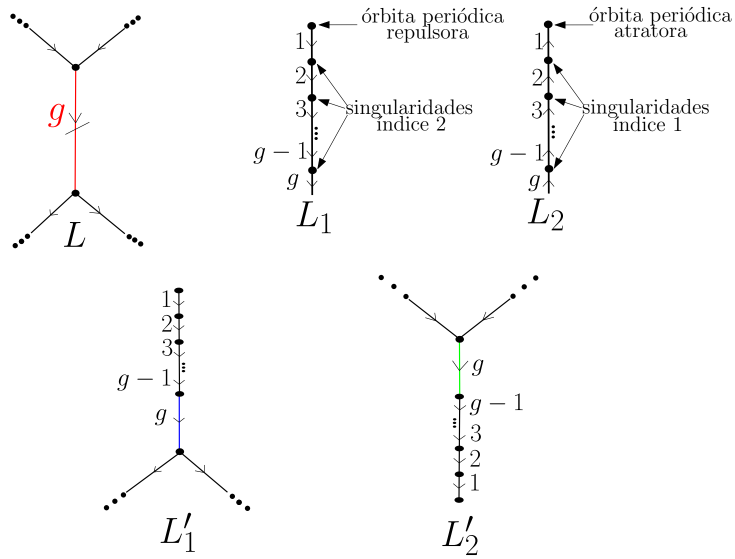


Figure 4.6: The graphs L_1 and L_2 have dangling edge.

Lyapunov graphs L_1' and L_2' is obtained. Since, L_1' and L_2' satisfy the conditions of Theorem 1.7.2, there exist Smale flows ψ_t and φ_t on $\mathbb{S}^3$, such that the graph L_1 corresponds to a handlebody manifold H_1^g and the graph L_2 corresponds to a handlebody manifold H_2^g, as shown in Figure 4.7.

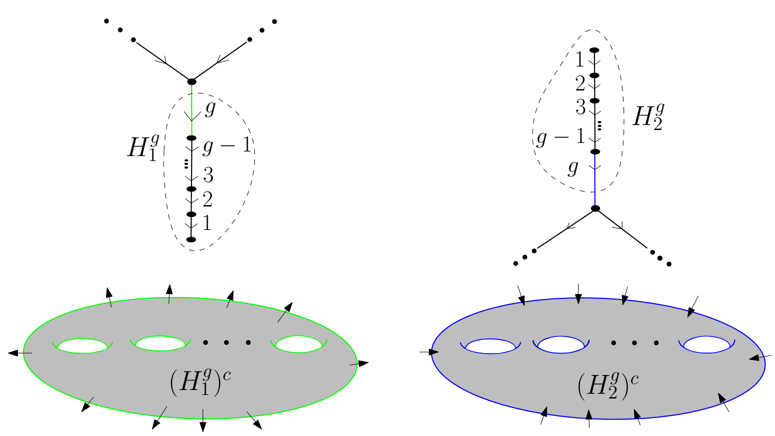


Figure 4.7: The graphs L_1' and L_2' correspond with Smale flows on $\mathbb{S}^3$.

Then we glue $((H_1^g)^c, \psi_t|_{(H_1^g)^c})$ and $((H_2^g)^c, \psi_t|_{(H_2^g)^c})$ suitably along their boundaries. We obtain a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov graph L. $\square$

Now we suppose that each connected component of a regular level set $\mathcal{T}_g$ be π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$.

Proposition 4.2.2. Suppose ϕ_t is a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$, with Lyapunov graph L. Let v be a vertex of L labelled with a suspension of a subshift of finite type and $A_{n\times n}$ is the non-negative integer matrix representing this subshift and $k = \dim \ker \overline{I - A} : F_2^n \to F_2^n$ where $F_2 = \mathbb{Z}_2$ and $\overline{I-A}$ is the $\bmod 2$ reduction of $I - A$. Let e^+ and e^- be respectively the indegree and outdegree of v. If v represents a basic set, which is contained in some manifold of handlebody type M in $\mathbb{S}^2 \times \mathbb{S}^1$, one has:

$$\begin{gathered} e^+, e^- > 0, \\ e^+ \leq k+1, \\ e^- \leq k+1, \\ k+1-G^- \leq e^+, \text{ with } \quad G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \quad \text{ and} \\ k+1-G^+ \leq e^-, \text{ with } \quad G^+ = \textstyle\sum_{j=1}^{e^+} g_j^+, \end{gathered}$$

where g_i^+'s (g_j^-)'s are the weights on the incoming (outgoing) edges of the vertex v.

Proof. By Remark 2.2.1, we can suppose that the component Y_1 is not a manifold of handlebody type. Then by Corollary 3.1.2 the other components of Y and all components of Z are manifolds of handlebody type. In other words $H_2(Y) \cong F_2$, $H_2(Z) \cong 0$ and $\dim H_1(Z) = G^-$.

Consider the exact homology sequence of the pair (X, Z),

$$H_3(X,Z) \to H_2(Z) \to H_2(X) \to H_2(X,Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X). \tag{4.2.7}$$

Now we can use similar arguments as in the proof of Proposition 4.2.1 to prove:

$$e^+ \leq k+1 \quad \text{and} \quad \dim H_2(X) \geq e^+ - 1. \tag{4.2.8}$$

Now we consider the Mayer-Vietoris exact homology sequence,

$$0 \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(X \cap Y) \to H_2(X) \oplus H_2(Y) \xrightarrow{\beta} H_2(\mathbb{S}^2 \times \mathbb{S}^1). \tag{4.2.9}$$

Since, $X \cap Y$ is composed of e^+ closed orientable surfaces, one has that $H_2(X \cap Y) \cong F_2^{e^+}$. Moreover, we know that $H_3(\mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1) \cong F_2$. Therefore,

$$\dim H_2(X) \leq e^+ - 1. \tag{4.2.10}$$

By inequalities (4.2.8) and (4.2.10), one concludes that

$$\dim H_2(X) = e^+ - 1.$$

If $a = \dim \operatorname{Im} \alpha$, from exact sequence (4.2.7), one has that

$$k = \dim H_2(X) + G^- - a.$$

Hence,

$$k + 1 - G^- \leq e^+. \tag{4.2.11}$$

On the other hand, U satisfies the Poincaré equality,

$$e^+ + G^- = e^- + G^+.$$

Thus,

$$k + 1 - G^+ \leq e^-.$$

Now for the last inequality, we need to consider the reverse flow which switches the roles of e^+ and e^- and transform A to A^t so that k is unchanged. In this case, the flow enters U through Z and exits through Y. We call $W = Y \cup U$ and we consider the exact homology sequence of the pair (W, Y),

$$\to H_3(W, Y) \to H_2(Y) \to H_2(W) \to H_2(W, Y). \tag{4.2.12}$$

By Theorem 1.7.1, one has $H_3(W, Y) \cong 0$ and $H_2(Y) \cong F_2$. Therefore, by sequence (4.2.12)

$$\dim H_2(W) \leq k + 1 \quad \text{and} \quad \dim H_2(W) \geq 1. \tag{4.2.13}$$

Now consider the Mayer-Vietoris exact homology sequence,

$$0 \to H_3(\mathbb{S}^2 \times \mathbb{S}^1) \to H_2(W \cap Z) \to H_2(W) \oplus H_2(Z) \xrightarrow{\gamma} H_2(\mathbb{S}^2 \times \mathbb{S}^1). \tag{4.2.14}$$

As long as, $W \cap Z$ is composed of e^- closed orientable surfaces, one has that $H_2(X \cap Y) \cong F_2^{e^-}$. Since, $H_2(W) \cong F_2$, this submanifold contains a non-separable surface, then by Lemma 2.1.1, this implies that γ is surjective. Therefore, $\dim H_2(W) = e^-$ and by inequality (4.2.13) one has,

$$e^- \leq k + 1.$$

□

Let $k = \dim \ker \overline{I - A} : F_2^m \to F_2^m$ where $F_2 = \mathbb{Z}_2$ and $\overline{I - A}$ is the $\bmod 2$ reduction of $I - A$. If e^+ and e^- are respectively the indegree and outdegree of v. With the definition of X, Y and Z in Proposition 4.2.2 one has the following lemma.

Lemma 4.2.2. Let v be a vertex of L labelled with a suspension of a subshift of finite type and $A_{n\times n}$ is the non-negative integer matrix representing this subshift. Suppose each component of Y and Z are manifolds of handlebody type, one has:

$$\begin{gathered} e^+, e^- > 0, \\ e^+ \leq k + 1, \\ e^- \leq k + 1, \\ k - G^- = e^+, \text{ with } \quad G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \quad \text{and} \\ k - G^+ = e^-, \text{ with } \quad G^+ = \textstyle\sum_{j=1}^{e^+} g_,^+. \end{gathered}$$

where g_i^+'s (g_j^-)'s are the weights on the incoming (outgoing) edges at v.

Proof. Since, Z and Y are composed by manifolds of handlebody type, one has that $H_2(Z) \cong H_2(Y) \cong 0$, $\dim H_1(Z) = G^-$ and $\dim H_2(Y) = G^+$. Now consider the exact homology sequence of the pair (X, Z),

$$H_3(X,Z) \to H_2(Z) \to H_2(X) \to H_2(X,Z) \to H_1(Z) \xrightarrow{\alpha} H_1(X). \tag{4.2.15}$$

Now by a similar argument as in the proof of Proposition 4.2.1 to prove,

$$e^+ \leq k+1 \quad \text{and} \quad e^- \leq k+1. \tag{4.2.16}$$

By hypothesis, each component of ∂Z is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$ and $Z \cap Y = \emptyset$ by Lemma 2.2.3, which implies that $\alpha = 0$. Now from the exact sequence (4.2.15), one has that

$$k = \dim H_2(X) + G^-. \tag{4.2.17}$$

Consider the reduced exact homology sequence for the pair (M, Y),

$$\widetilde{H}_1(Y) \xrightarrow{\beta} \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1) \to \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \to \widetilde{H}_0(Y) \to 0. \tag{4.2.18}$$

Since, each component of ∂Y is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$, by Lemma 2.2.2, one concludes that $\beta = 0$. Now by sequence (4.2.18), one has

$$\dim \widetilde{H}_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) = e^+.$$

On the other hand, by the Lefschetz duality $H_1(\mathbb{S}^2 \times \mathbb{S}^1, Y) \cong H_2(\mathbb{S}^2 \times \mathbb{S}^1 - Y, \mathbb{S}^2 \times \mathbb{S}^1 - \mathbb{S}^2 \times \mathbb{S}^1) \cong H_2(X)$. Therefore, $\dim H_1(X) = e^+$ and from equality (4.2.17), one has

$$k - G^- = e^+.$$

Furthermore, U satisfies Poincare's equality,

$$e^+ + G^- = e^- + G^+.$$

Hence, one has

$$k - G^+ = e^-.$$

□

4.2.1 Construction of the basic blocks

In this section the construction of basic block for singularities, periodic orbits and subshifts of finite type that verify the conditions in Theorem 1.7.2 are presented. In what follows we present a construction of a basic block for some vertex v that satisfies the condition of Lemma 4.2.2. For that, we need some definitions.

Definition 4.2.1. Let C_1 and C_2 be solid concentric cylinders with $C_2 \subset C_1$. A **round handle** R is a $3-$manifold homeomorphic to $C_1 - C_2$ containing a saddle type periodic orbit of period equal to one and with a flow defined on R as follows: the flow enters on two disjoint boundary components of R homeomorphic to annuli and exits on two other disjoint boundary components also homeomorphic to annuli. See Figure 4.8

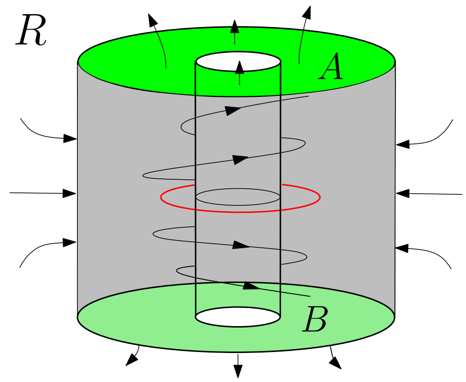


Figure 4.8: Round handle.

Definition 4.2.2. A one-handle H_i is a strip $D^1 \times D^1$. A one-handle H_s in a round handle R_s will be chosen so that $H_s \times \mathbb{S}^1 = R_s$ and H_s is transversal to the flow. A one-handle H_t in a nilpotent handle N_t will be chosen so that $H_t \times D^1 = N_t$.

Definition 4.2.3. A **basic block** for a one-dimensional set Λ of a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ and Lyapunov function $f : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{R}$ with $f(\Lambda) = c$ is the component of $f^{-1}([c-\epsilon, c-\epsilon])$, X, which contain Λ, where $\epsilon > 0$ is chosen so that X contains no other basic set and

1. there exist (not necessarily connected) codimension one submanifolds with boundary U and V in X with $U \subset V$ is everywhere transversal to the flow.

2. The first return map $\mu : U \to \operatorname{int} V$ is a well defined smooth map and there is a hyperbolic handle set $H \subset U$ with every orbit of Λ intersecting H and every $h_i \subset H$ intersecting Λ.

3. if $x \in H$ but $\mu(x) \notin H$ then $\phi_t(x) \cap H = \emptyset$ $\forall t > 0$ and $f(\phi_{t_0}(x)) = c - \epsilon$ for some $t_0 > 0$. Likewise, if $x \in H$ but $\mu^{-1}(x) \notin H$ then $\phi_t(x) \cap H = \emptyset$ $\forall t < 0$ and $f(\phi_{t_0}(x)) = c + \epsilon$ for some $t_0 < 0$.

4.2.2 A special Smale flow on handlebodies

In this subsection, we build specific Smale flows on handlebodies which are transverse to its boundary. This construction will be very important for this subsection. There exists a Smale flow on a handlebody H_g of genus g, with g attracting periodic orbits λ_{a_i}, $g-1$ saddle periodic orbits λ_{s_i} and $g-1$ repelling singularities p_i. H_g is obtained by gluing $g-1$ round handles to g solid tori as shown in Figure 4.9. The flow on H_g is induced by the periodic orbits and singularities and will be denoted by ρ_t. Since the flow is transversal and points inward on ∂H_g, one has

$$\epsilon = d(\partial H_g, \rho_1(\partial H_g)) > 0.$$

Proposition 4.2.3. Let S be a surface homeomorphic to $D^1 \times D^1$ embedded in a tubular $\epsilon/2$-neighborhood of the boundary of handlebody, such that the flow described above is transversal to S. Then there exists a neighborhood V of S such that the first return map, $\mu : S \to V$, is smooth.

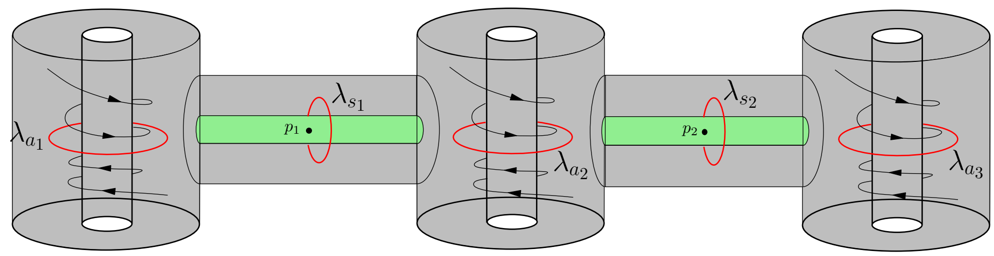


Figure 4.9: H_3 is a handlebody of genus 3.

Proof. Suppose that S is embedded in some round handle R_j with periodic orbit λ_{s_j}. Hence, one can extend S to a surface V_S, which is homeomorphic to $D^1 \times D^1$ and has transversal intersection with λ_{s_j}, as shown in Figure 4.10.

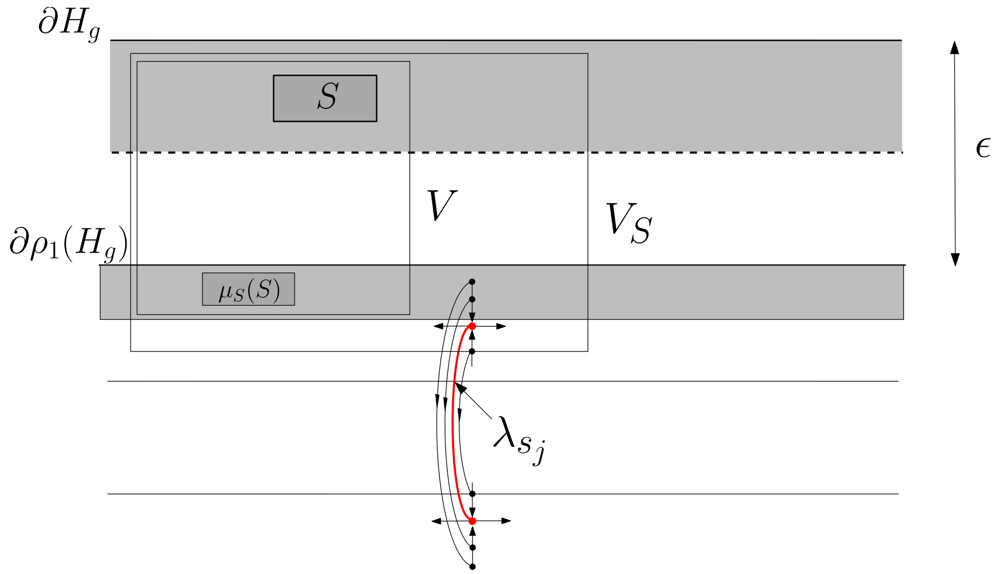
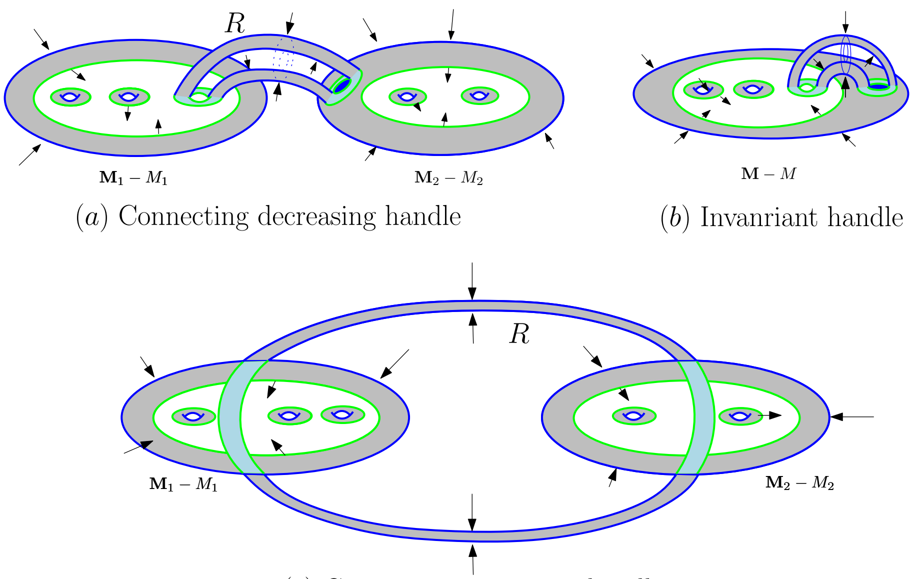


Figure 4.10: It is the interior of H_j.

Therefore, V_S is a cross section of λ_{s_j}, so the first return map μ_S is defined for V_S and is smooth. Now if it is necessary one can extend V_S, such that V_S contains $\mu_S(S)$. On the other hand, since λ_{s_j} has an index 1 periodic orbit, one has that $\mu_S(S)$ does not intersect λ_{s_j}. Hence, one can find a surface $V \subset V_S$ such that V contains S and $\mu_S(S)$ and does not intersect with λ_{s_j}. See Figure 4.10. Now, suppose that S is contained in some neighborhood of a attracting periodic orbit λ_{a_i}, one proceeds in a similar way. For the general case, one can choose a smaller neighborhood contained in S and argue as above. $\square$

Given a handlebody M, we consider a flow on M as described above, and the restriction of this flow to the tubular neighborhood of ∂M. Note that, the tubular neighborhood of ∂M is homeomorphic to $\partial M \times I$, the collaring of the boundary. For this reason we refer to the tubular neighborhood of ∂M as the collaring of ∂M.

In [13] three types of gluing of round handles, to a collaring of ∂M are presented. See Figure 4.11.



(a) Connecting decreasing handle

(b) Invanriant handle

(c) Connecting invanriant handle

Figure 4.11: Distinct gluings of round handles.

Due to the topology of $\mathbb{S}^2 \times \mathbb{S}^1$ there is another case to consider. In order to describe it, we build a round handle R_1 in $\mathbb{S}^2 \times \mathbb{S}^1$. Start with a non-separable sphere $\mathbb{S}^2$ and remove two disjoint disk D_1 and D_2. Now consider a product of $\mathbb{S}^2 - (D_1 \sqcup D_2)$ with an interval I. This manifold is a 3-manifold, R_1, such that the boundary ∂R_1 is composed by four annuli. One can put a saddle periodic orbit inside of R_1, as shown in Figure 4.12.

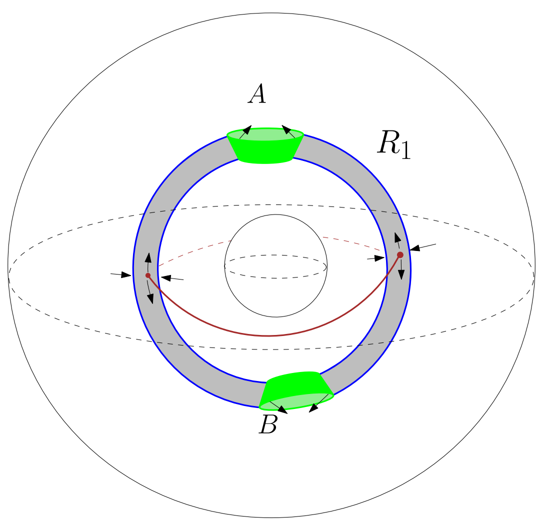


Figure 4.12: Special type of round handle in $\mathbb{S}^2 \times \mathbb{S}^1$.

Let N be a compact, connected 3-manifold with nonempty and connected boundary which is inside a 3-ball in $\mathbb{S}^2\times\mathbb{S}^1$. In the boundary of N, ∂N consider two disjoint disks D_1 and D_2 and inside each one of them other smaller disks $\overline{D}_1 \subset D_1$ and $\overline{D}_2 \subset D_2$ such that the annuli $A_1 = \overline{D_1 - \overline{D}_1}$ and $B_1 = \overline{D_2 - \overline{D}_2}$ have boundaries $\alpha_1' = \partial D_1$, $\alpha_2' = \partial \overline{D}_1$, $\beta_1' = \overline{D}_2$ and $\beta_2' = \partial \overline{D}_2$. A round handle R is homeomorphic to $\mathbb{S}^1 \times [a,b] \times [c,d]$. Consider the following circles $\beta_1 = \mathbb{S}^1 \times \{a\} \times \{c\}$, $\beta_2 = \mathbb{S}^1 \times \{b\} \times \{c\}$, $\alpha_1 = \mathbb{S}^1 \times \{a\} \times \{d\}$ and $\alpha_2 = \mathbb{S}^1 \times \{b\} \times \{d\}$. Also consider the annuli $A = \mathbb{S}^1 \times [a,b] \times \{d\}$, $B = \mathbb{S}^1 \times [a,b] \times \{c\}$, $E = \mathbb{S}^1 \times \{a\} \times [c,d]$ and $F = \mathbb{S}^1 \times \{b\} \times [c,d]$. With this decomposition we now describe the special gluing of R_1 to the 3-manifold N. The annulus A will be glued to A_1 and B to B_1 in such a way that α_1 is identified to α_2' and α_2 is identified to α_1'. Also, β_1 is identified to β_1' and β_2 is identified to β_2'. See Figure 4.13.

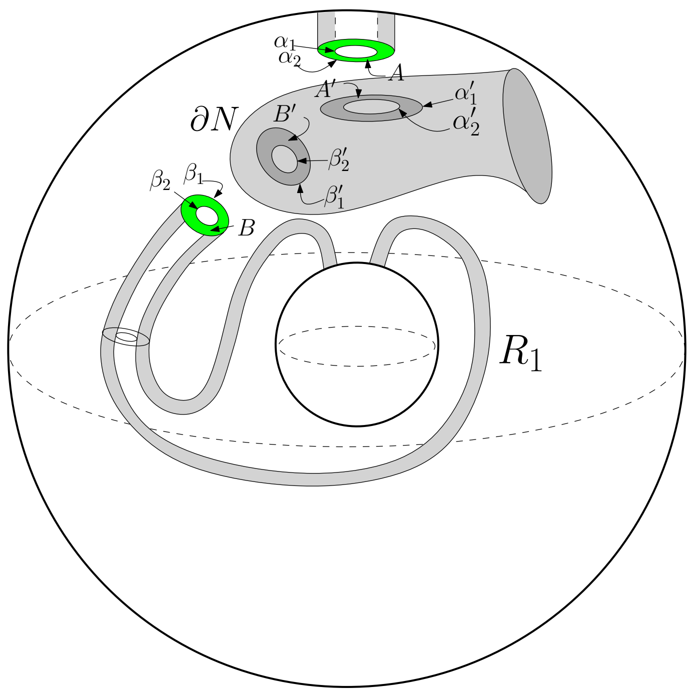


Figure 4.13: Special gluing of a round handle in $\mathbb{S}^2 \times \mathbb{S}^1$.

Lemma 4.2.3. With the notation above, $\partial(N \cup R_1)$ is homeomorphic to ∂N.

Proof. We have that

$$\partial N = \overline{D}_2 \cup_{\beta_2'} B_1 \cup_{\beta_1'} C \cup_{\alpha_1'} A_1 \cup_{\alpha_2'} \overline{D}_1$$

where $C = \overline{\partial N - (D_1 \cup D_2)}$. On the other hand, one has

$$\partial(N \cup R_1) = \overline{D}_1 \cup_{\alpha_1=\alpha_1'} E \cup_{\beta_1=\beta_1'} C \cup_{\alpha_2=\alpha_2'} F \cup_{\beta_2=\beta_2'} \overline{D}_2$$

Note that both $\overline{D}_1 \cup E$ and $F \cup \overline{D}_2$ are disks. Hence, ∂N is homeomorphic to $\partial(N \cup R_1)$. $\square$

Corollary 4.2.1. Let N be a handlebody of genus g embedded in a 3-ball in $\mathbb{S}^2 \times \mathbb{S}^1$. Let

$$X = N \cup R_1$$

as described above, then $\mathbb{S}^2 \times \mathbb{S}^1 - X$ is homeomorphic to a handlebody of genus g.

Proof. Recalling the construction of R_1, we have that

$$R_1 = \overline{\mathbb{S}^2 - (D_1^2 \cup D_2^2)} \times I$$

where $\mathbb{S}^2$ is a non-separable sphere. The annuli $\partial D_1^2 \times I$ and $\partial D_2^2 \times I$ are the gluing regions. For the case of a genus zero handlebody, a 3-disk should be glued to the above annuli using two 2-disks on the boundary of the 3-disk, as shown in the Figure 4.14. Note that the final manifold is isotopic

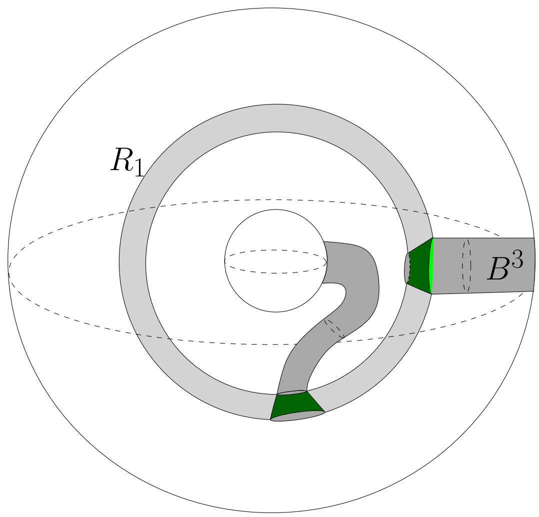


Figure 4.14: R_1 is glued to B^3.

to a tubular neighborhood of $\mathbb{S}_a^2 \vee \mathbb{S}_b^1$, where $\mathbb{S}_a^2$ is a fiber of $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^1$ and $\mathbb{S}_b^1$ is a fiber of $\mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2$. Thus, the complement in $\mathbb{S}^2 \times \mathbb{S}^1$ is homeomorphic to B^3.
In the general case of a handlebody of any genus, it is enough to observe that when we glue a handle in B^3, we increase the genus of the complement. This completes the proof. $\square$

Given $A_{m\times m}$, a non-negative irreducible integer matrix which is not a permutation matrix, our aim is to construct a block for the suspension of $\sigma(A)$ so that it verifies the conditions in Lemma 4.2.2.

Now suppose that $A_{m\times m}$ is a matrix with k ones on the diagonal and all other entries equal to zero. Then one has the following proposition.

Proposition 4.2.4. Let v be a vertex of L labelled with the suspension of a subshift of finite type $\sigma(A)$, where $A_{m\times m}$ is a matrix with k ones on the diagonal and other entries equal to zero. Let e^+ and e^- be the indegree and outdegree of v and $\left\{g_i^+\right\}$ and $\left\{g_j^-\right\}$ are the weights on the incoming and outgoing edges of v respectively. Suppose that

$$\begin{gathered} k - \sum g_i^- = e^+ \quad \text{and} \\ k - \sum g_i^+ = e^- \end{gathered}$$

Then, there exists a basic block X in $\mathbb{S}^2 \times \mathbb{S}^1$ for the suspension of $\sigma(A)$ with e^+ (e^-) entering (exiting) boundary components each being a 2-manifold of genus g_i^+ (g_j^-). Furthermore, $\mathbb{S}^2 \times \mathbb{S}^1 - X$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$.

Proof. The dynamics associated to the suspension of $\sigma(A)$ is simple. It consists of a basic set with k periodic orbits. One can rewrite the hypothesis

$$\begin{array}{c} (k-1) + 1 - \sum g_i^- = e^+, \\ (k-1) + 1 - \sum g_i^+ = e^-. \end{array}$$

By Proposition 8.2 in there exists a way of gluing $k-1$ round handles to e^- collarings of $\partial M_1 \ldots, \partial M_{e^-}$, where each handlebody M_i has genus g_i^-, such that the resulting 3−manifold X_1 is a basic block which contains $k-1$ periodic orbits. Also, $\partial X_1^+(\partial X_1^-)$ is composed by $e^+(e^-)$ closed surfaces of genus $g_i^+(g_j^-)$ respectively and the complement in S^3 is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$. Recall that, since X_1 is a compact 3-manifold in $\mathbb{S}^3$, then one can embed X_1 in $\mathbb{S}^2 \times \mathbb{S}^1$. For simplicity we continue to denote this embedding by X_1. Now, glue X_1 to R_1 in order to construct a new basic block $X_2 = X_1 \cup R_1$ with k periodic orbits. By Lemma 4.2.3 and Corollary 4.2.1, X_2 verifies $\partial X_2 \approx \partial X_1^+ \cup \partial X_1^-$ and $\mathbb{S}^2 \times \mathbb{S}^1 - X_2$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$. At this point, a basic set with k periodic orbits corresponding to the k ones on the diagonal of A has been constructed. Also, $m-k$ nilpotent handles must be glued to X_2. Recall that these handles are homeomorphic to $D^1 \times D^1 \times D^1$ and contain no recurrent points. These handles will correspond to the $m-k$ zeros on the diagonal of A. Finally, denote X as X_2 glued with $m-k$ nilpotent handles. Recall that these nilpotent handles do not modify ∂X_2. □

For the general case, we can construct a basic block for the suspension of $\sigma(N)$ where N is a matrix that is flow equivalent to A. By definition, this implies that the suspensions of $\sigma(A)$ and $\sigma(N)$ are topologically equivalent. An essential proposition at this point is due to Franks [16].

Proposition 4.2.5. If A is a non-negative irreducible integer matrix which is not a permutation matrix then given an integer $M > 0$ there is a matrix which is flow equivalent to N and which has every entry greater than M and non-diagonal entries even. The size of N depends only on A and not on M.

By Proposition 4.2.5, the matrix $A_{m\times m}$ is flow equivalent to some matrix $N_{n\times n}$ which has all non-diagonal entries even,

$$\dim \ker \left((I - \overline{N} : F_2^n \to F_2^n) \right) = \dim \ker \left((I - \overline{A} : F_2^n \to F_2^n) \right) = k,$$

where $\overline{N}$ denotes the $\bmod 2$ reduction of N. By Proposition 4.2.4 we are able to construct a basic block X for the suspension of $\sigma(\overline{N})$. Our goal is to construct a basic block for the suspension of $\sigma(N)$. In order to achieve this, we wish to maintain the block X while modifying the flow within it. We will use the notion of one-handles H_i within round handles R_i and nilpotent handles N_t.

Let x be a point of $D^1 \times p \subset H_i$ where $p \in D^1$. The interval $D^1 \times p$ will be denoted by $W_i^u(x)$. Similarly, let x be a point of $q \times D^1$. The interval $q \times D^1$ will be denoted by $W_i^s(x)$. The reason for introducing one-handles is that if H_i intersects m times $\mu(H_j)$ then the ijth entry of $\overline{N}$ increases by m, where $\mu(x)$ is the first return map for the one-handle of the round handle.

This is precisely what we want to do, namely, increase the entries of $\overline{N}$ to obtain N. For this reason we will construct a connected surface U, such that, U contains all one handles and the flow is transversal to U.

For each one handle H_i, we consider two disjoint subsets E_1^i and E_2^i contained in H_i as shown in Figure 4.15, homeomorphic to $D^1 \times D^1$ such that

$$\mu(E_1^i \sqcup E_2^i) \cap H_i = \emptyset,$$

we will call E_1^i and E_2^i the ***set of ends*** of H_i. See Figure 4.15.

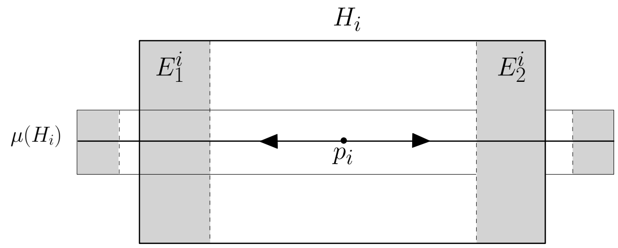


Figure 4.15: p_i is the center of the one handle H_i

Thus, our aim is the construction of a surface $U \subset X$ such that U contains all one handles H_i and all nilpotent handles N_j.

On the other hand, in order to connect the one handles we consider disjoint rectangles embedded in the union of collaring of ∂M_i for $i \in \{1, \ldots, e^-\}$ and consider the flow transversal to the collaring as described in Proposition 4.2.3. We will use these rectangles to connect the ends of the one handles associated to the collaring.

First fix a collaring of a handlebody $\partial M_i \times I$ for $i \in \{1, \ldots, e^-\}$. Now consider the round handles $R_{j_1}, \ldots, R_{j_l}, \ldots, R_{j_{n(i)}}$ which glue to $\partial M_i \times I$. Without loss of generality, we can suppose that $R_{j_1}, \ldots, R_{j_l}$ are round handles, that have only one annulus which glues to $\partial M_i \times I$ and the remaining ones are invariant round handles with two annuli which glue to $\partial M_i \times I$. The special round handle R_1 shall be considered as if R_1 has one annulus glued to $\partial M_1 \times I$. Now, we connect the ends of ones handles $H_{j_1}, \ldots, H_{j_l}$ with a rectangle embedded in the collaring of ∂M_i as described above. Thus, we can find a connected surface U_1^i such that, U_1^i contains all one handles $H_{j_1}, \ldots, H_{j_l}$ of the round handles $R_{j_1}, \ldots, R_{j_l}$ glued to $\partial M_i \times I$. For the invariant round handles, we glue a rectangle to $E_1^{j_{l+1}}$ and another rectangle to $E_2^{j_{n(i)}}$. Now, we connect $H_{j_{l+1}}$ with $H_{j_{l+2}}$ by gluing a rectangle joining $E_2^{j_{l+1}}$ and $E_1^{j_{l+2}}$. We do this successively until the last pair $H_{j_{n(i)-1}}$ and $H_{j_{n(i)}}$ have been joined. Thus, we can find a connected surface U_2^i, such that U_2^i contains all one handles $H_{j_{l+1}}, \ldots, H_{j_{n(i)}}$. Now we connect U_1^i and U_2^i with a rectangle to obtain a connected surface U^i. Note that the flow is transversal to U^i.

In this fashion, we construct a collection of $\mathcal{U} = \{U_i \mid 1 \leq i \leq e^-\}$. Note that, $U^i \cap U^j$ is composed by the one handles that connect the collaring of ∂M_i with the collaring of ∂M_j. Define

$$U_1 = \bigcup_{i=1}^{e^-} U^i.$$

This surface U_1 contains all one handles associated to $R_1, \ldots, R_k$. Extend U_1 by a rectangle $\hat{R}$, which contains all nilpotent one handles. See Figure 4.16.

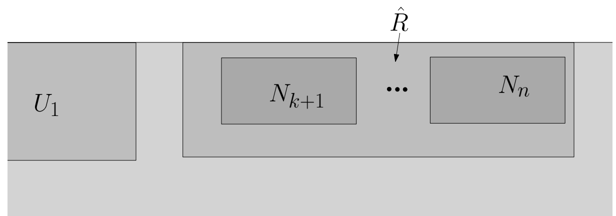


Figure 4.16: $\hat{R}$ contains all nilpotent one handles.

Thus, by Proposition 4.2.3 one can find a surface U that contains U_1 and $\hat{R}$. Actually, one finds a surface V, such that $U \subset V$ and the first return map $\mu : U \to V$ is smooth. By construction, one has that

$$\overline{U} = (U - \mu(U)) \cup \bigcup_{i=1}^{e^-} \mu(E_1^i \sqcup E_2^i)$$

is connected.

Lemma 4.2.4. Let U_0 be a small neighborhood of U in V as defined above. There exists an isotopy $\Psi_t : V \to V$ supported on the interior of U_0 such that Ψ_0 is the identity. Also, $\mu_1 = \Psi_1 \circ \mu : U \to V$ satisfies that its suspension flow with induced flow φ_t has a chain recurrent set which is topologically equivalent to the suspension of a subshift of finite type with matrix $A_{n\times n}$.

Proof. We have that $\overline{U}$ is connected. Hence, for $p \in \operatorname{int}(\mu(W_i^u(p_i) \cap (\mathbf{M}_i - M_i))$ and $q \in \operatorname{int} H_j$ such that $W_j^u(q)$ and $\mu(H_\alpha)$ are disjoint $\forall \alpha \in \{1, \ldots, e^-\}$, there exists a curve $\gamma : [0,3] \to \operatorname{int} \overline{U}$ such that $\gamma(1) = p$ and $\gamma(2) = q$. Also, the intersection of $\mu(H_i)$ and $\gamma([0,3])$ is connected and equal to $\mu(W_i^s(\mu^{-1}(p)))$ and its intersection with H_j lies in $W_j^u(q)$. For the other H_l's with $l \in \{1, \ldots, n\} - \{i, j\}$ and $\gamma([0,3]) \cap H_l \neq \emptyset$, γ is transversal to $W_l^u(p_l)$.

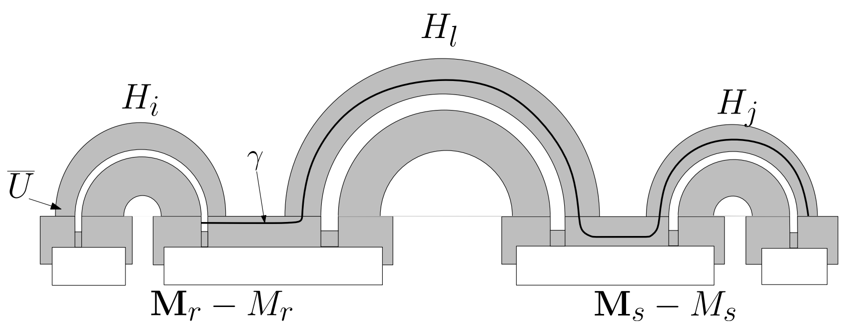


Figure 4.17: H_l is a connecting handle.

Thus, there is an isotopy supported in a tubular neighborhood of $\gamma([0,1])$ which pushes a small interval of $\mu(W_i^u(p_i))$ along the curve γ until it intersects $W_j^s(p_j)$ in two points. It is possible that this process causes intersection with other $W_l^s(p_l)$ but always an even number of them. See Figure 4.18.

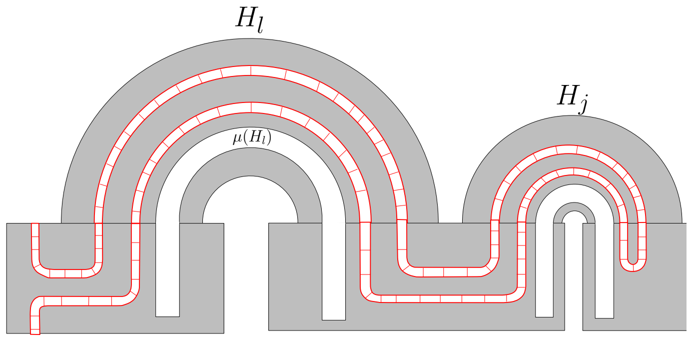


Figure 4.18: Deformation of $\phi_1(H_i)$.

Note that this isotopy preserves the hyperbolicity of the one handles with respect to the first return map ϕ_1. In others words, if $\mu(W_i^u(x)) \cap W_j^u(y) \neq \emptyset$ then $W_j^u(y) \subset \mu(W_i^u(x))$ and similarly if $\mu(W_i^s(x)) \cap W_j^s(y) \neq \emptyset$ then $\mu(W_i^s(x)) \subset W_j^s(y)\ \forall x \in H_i$ and $y \in H_j$.
Hence, to achieve the geometric intersection matrix N, which is flow equivalent to A, we argue as in [16]. First one needs to find a pairwise disjoint family of embedded curves $\gamma_{ij} : [0,3] \to \overline{U}$ with the properties described above for $i, j \in \{1, \ldots, n\}$. The existence of this family $\{\gamma_{ij}\}$ with these properties is easy to see except for the fact that they are disjoint. Now, by using an isotopy supported on a neighborhood of γ_{11} we push off all other curves. We do this for γ'_{12} and in the same way for all the remaining curves.
By Proposition 4.2.5 we can, if necessary, replace the matrix N for another matrix which is flow equivalent to it, congruent $\mod 2$ and has every entry as large as we want. In particular larger than the corresponding entry of A. We can suppose this has been done and for simplicity of notation continue to call this matrix N. Now for each γ_{ij} we can increase the intersection points of $W_j^s(p_j)$ with the image of $W_i^u(p_i)$ if we push back the curve through H_j again. See Figure 4.19.

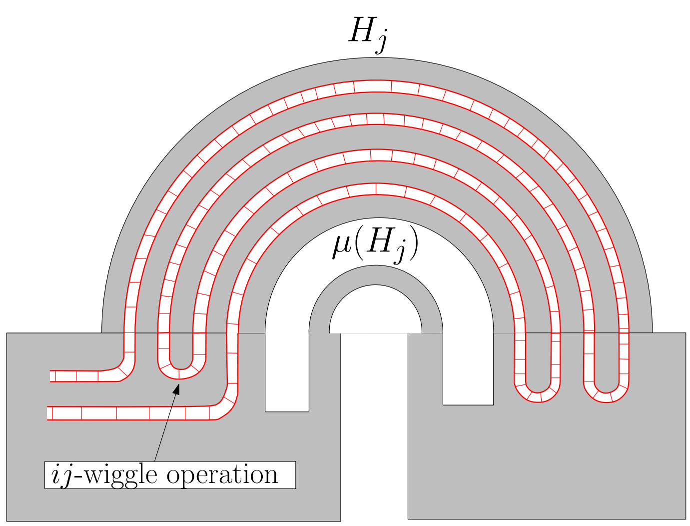


Figure 4.19: $ij-$wiggle operation.

This is referred to as the $ij-$wiggle operation. If this operation is repeated r times, $2r + 1$

intersection points are added. Do this with $r = \frac{1}{2}[(A)_{ij} - (N)_{ij})]$ so that the ij entry of the matrix $\overline{N}$ agrees with $(N)_{ij}$. Repeat this process in order to get to the matrix N. Now isotope μ so that the final map has the desired property. $\square$

Proposition 4.2.6. Let $A_{n\times n}$ be a non-negative integer matrix with $k = \dim \ker : (\overline{(I-A)}F_2^m \to F_2^m$, where $F_2 = \mathbb{Z}_2$ and $\overline{A}$ is the $\bmod 2$ reduction of A. Suppose also that $e^+, e^-, g_j^+, g_i^- \in \mathbb{N}$, with $i = 1, \ldots, e^+$ and $j = 1 \ldots, e^-$, are positive integers satisfying

1. $e^- - e^+ = \sum g_i^- - \sum g_j^+$,
2. $k - \sum g_i^- = e^+ \leq k+1$ and
3. $k - \sum g_i^- = e^+ \leq k+1$.

Then there exists a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ with a basic block X such that

1. The flow ϕ_t restricted to the basic set $\Lambda \subset X$ is topologically equivalent to the suspension of $\sigma(A)$, and
2. $\partial X^+(\partial X^-)$ has $e^+(e^-)$ components composed by surfaces of genus $g_i^+(g_j^-)$.
3. $(\mathbb{S}^2 \times \mathbb{S}^1) - X$ is homoemorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$

Proof. By Proposition 4.2.5, one knows that $A_{n\times n}$ is flow equivalent to $N_{m\times m}$. By Proposition 4.2.4, we can build a basic block X_1 for $\overline{N}$, which satisfies assertions 2 and 3 of this proposition. For the first assertion, by Lemma 4.2.4, one has that, for U_0 chosen sufficiently small, $\mu : U_0 \to V$ is smooth and $\tau : U_0 \to \mathbb{R}$ is the smallest $t > 0$ such that $\phi_t(x) = \mu(x)$. Then the partial flow on

$$Z = \{\phi_t(x) | x \in U_0,\ 0 \leq t \leq \tau(x)\}.$$

is the suspension flow for μ. Now by the construction, the suspension flow for μ_1 is the same as for μ since μ and μ_1 are isotopic. On the other hand, since the isotopy is supported on the interior of U_0, one has that μ and μ_1 agree near the boundary of U_0. Then near the boundary of Z the suspension flow of μ_1, φ_t and the suspension flow of μ, ϕ_t agree. Now, let η_t be a flow on $\mathbb{S}^2 \times \mathbb{S}^1$ which is generated by the vector field which is tangent to the suspension flow of μ_1 on Z and tangent to ϕ_t elsewhere. Hence η_t is a Smale flow which satisfies this lemma. $\square$

Theorem 4.2.2. Let L be an abstract Lyapunov graph with $\beta(L) = 0$. L is associated with a Smale flow ϕ_t on $\mathbb{S}^2 \times \mathbb{S}^1$ such that each separable regular level set is π_1-trivial in $\mathbb{S}^2 \times \mathbb{S}^1$ if and only if the following conditions hold:

1. The underlying graph L is an oriented graph and $\beta(L) = 0$ with exactly one edge attached to each vertex labelled with a sink or a source. Moreover, the sink (source) vertex is labelled with an index 0 (index 3) singularity or an attracting (repelling) periodic orbit.
2. If a vertex is labelled with a singularity of index 2 (index 1), then $1 \leq e^+ \leq 2$ and $e^- = 1$ ($1 \leq e^- \leq 2$ and $e^+ = 1$).

3. There exists a unique vertex v labelled with a suspension of a subshift of finite type and $A_{m\times m}$ is the non-negative integer matrix that is associated to this subshift, such that,

$$\begin{gathered} e^-, e^+ > 0, \\ k - G^- = e^+ \leq k+1, \quad \text{with } G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \text{ and} \\ k - G^+ = e^- \leq k+1, \quad \text{with } G^+ = \textstyle\sum_{i=1}^{e^+} g_i^+, \end{gathered}$$

where $k =$ dim ker $((I-B): F_2^m \to F_2^m)$, $F_2 = \mathbb{Z}/2$, $b_{ij} = a_{ij} \bmod 2$ and g_i^+'s (g_j^-'s) are the weights on an incoming (outgoing) edges of v.

4. Any other vertex labelled with a suspension of a subshift of finite type, where $C_{l\times l}$ is the non-negative integer matrix that is associated to this subshift, must satisfy

$$\begin{gathered} e^-, e^+ > 0, \\ k+1 - G^- \leq e^+ \leq k+1, \quad \text{with } G^- = \textstyle\sum_{i=1}^{e^-} g_i^- \text{ and} \\ k+1 - G^+ \leq e^- \leq k+1, \quad \text{with } G^+ = \textstyle\sum_{i=1}^{e^+} g_i^+, \end{gathered}$$

where $k =$ dim ker $((I-L): F_2^m \to F_2^m)$, $F_2 = \mathbb{Z}/2$, $l_{ij} = c_{ij} \bmod 2$ and g_i^+'s (g_j^-'s) are the weights on the incoming (outgoing) edges of this vertex.

5. All vertices must satisfy the Poincaré-Hopf condition, i.e., for a vertex labelled with a singularity of index r, the condition is

$$(-1)^r = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

and for a vertex labelled with a suspension of a subshift of finite type or a periodic orbit, the condition is

$$0 = e^+ - e^- - \sum g_j^+ + \sum g_i^-$$

Proof. Necessity: It follows by Proposition 1.6.1, Proposition 4.2.2 and Lemma 4.2.2.

Sufficiency. Let L be a Lyapunov graph which satisfies (1), (2), (3) and (4). Let v be a vertex that verifies condition (2). By Proposition 4.2.6, there exists a Smale flow ϕ_t in $\mathbb{S}^2 \times \mathbb{S}^1$ and a basic block X embedded in $\mathbb{S}^2 \times \mathbb{S}^1$, associated to the vertex v. Also, $\mathbb{S}^2 \times \mathbb{S}^1 - X$ is homeomorphic to $H_1^{g_1^+} \sqcup \cdots \sqcup H_{e^+}^{g_{e^+}^+} \sqcup H_1^{g_1^-} \sqcup \cdots \sqcup H_{e^-}^{g_{e^-}^-}$. Now, if we cut L along all incoming or outgoing edges of v, we obtain $e^+ + e^-$ subgraphs with dangling edges which can be denoted by $L_1^+, \ldots, L_{e^+}^+$ and $L_1^- \ldots, L_{e^-}^-$, as shown in Figure 4.20. We use the graphs L_i' and L_j' with dangling edges to create the new graphs $\hat{L}_1, \ldots, \hat{L}_{e^+}$ and $\widetilde{L}_1 \ldots, \widetilde{L}_{e^-}$. Then by Proposition 4.2.2, there exists Smale flows ϕ_t^i and φ_t^j for $\hat{L}_i$ and $\widetilde{L}_j$ such that L_i^+ and L_j^- correspond to handlebodies H_i^+ and H_j^- respectively, as shown in Figure 4.21. Thus we have flows defined on $(X, \phi_t|_X)$, $(H_i^+, \phi_t^i|_{H_i^+})$ and $(H_j^-, \varphi_t^i|_{H_j^-})$ for $i \in \{1, \ldots, e^+\}$ and $j \in \{1, \ldots, e^-\}$. Finally we glue these manifolds suitably, in order to obtain a Smale flow on $\mathbb{S}^2 \times \mathbb{S}^1$ with Lyapunov graph L. $\square$

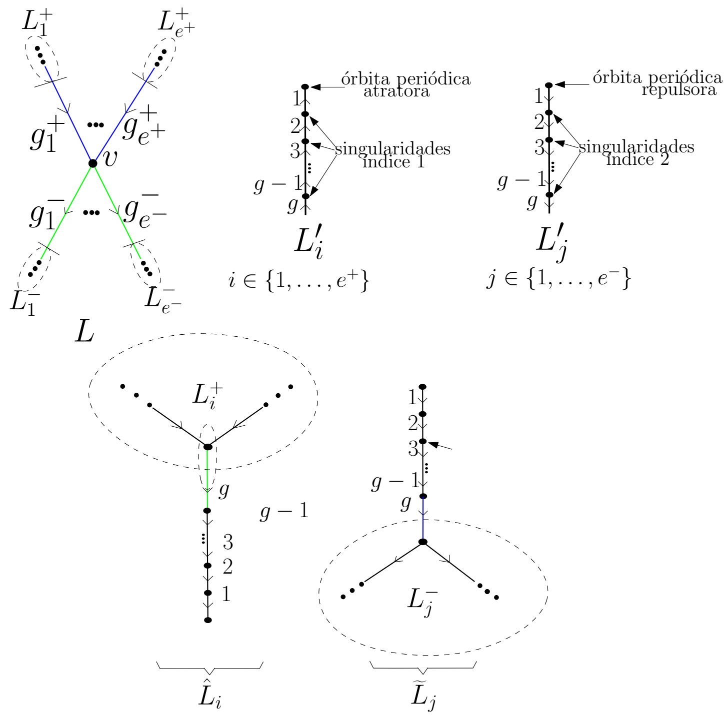


Figure 4.20: The graphs L'_i and L'_j have dangling edge.

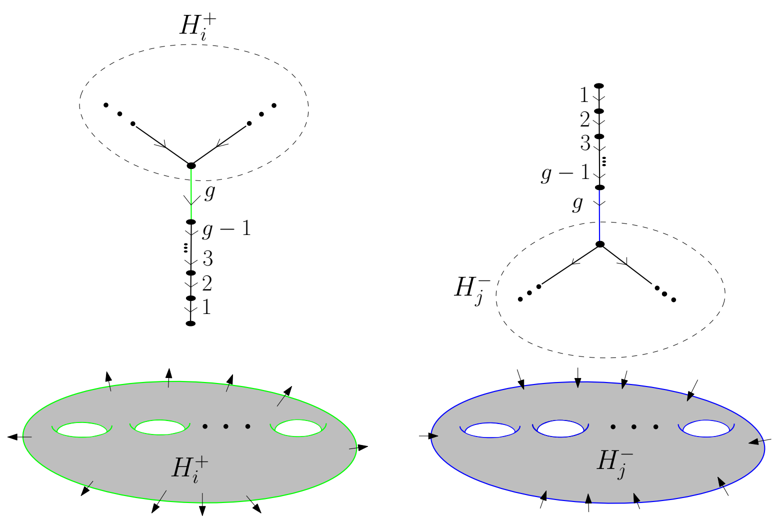


Figure 4.21: The graphs $\hat{L}_i$ and $\widetilde{L}_j$ correspond with Smale flows on $\mathbb{S}^3$.

Chapter 5

Morse-Novikov Flows

This chapter is in the spirit of the ideas developed in [14] which characterized Lyapunov graphs for smooth flows on surfaces. In this sense we obtain a characterization of the Morse digraph associated with Morse-Novikov flows that arise from Morse circular functions on orientable and non-orientable surfaces. Also, we generalize the concept of Lyapunov function in the context of Novikov theory in order to obtain the characterization of the Lypunov digraphs associated with smooth flows on surfaces.

5.1 Gradient-like flows on surfaces

We focus on surfaces, both orientable and nonorientable, endowed with circular Morse functions.

5.1.1 Relations with the classical Morse setting

Before classifying the global and local combinatorics of the circular Morse digraphs, let us discuss some operations one can perform in the circular Morse case in order to be able to refer to the classical results.

Let N be a fundamental cobordism, that is,

$$N = \text{closure}(M - f^{-1}(\text{Exp}(a))) = \overline{M} \setminus (F^{-1}(]-\infty, a-1[) \cup F^{-1}(]a, \infty[)$$

where a is a regular value of F and, by definition, $\text{Exp}(a)$ is a regular value of f.

If f is onto, the surface N has nonempty boundary $\partial N = \partial_- N \sqcup \partial_+ N$ with the same number $\mid \partial_- N \mid$ of entering and exiting components.

The surface N is not necessarily connected, and we shall denote by c the number of its connected components. Up to reindexing, for i from 1 to $\mid \partial_- N \mid$, let us denote by $C_i^+ \subseteq \partial_+ N$ and $C_i^- \subseteq \partial_- N$ the components such that $E(C_i^+) = E(C_i^-)$. In particular, the original surface M is obtained from N by identifying C_i^+ and C_i^- via E.

Now, let $\hat{N}$ denote the closed surface obtained from N by gluing to C_i^+ (C_i^-) a disk D_i^+ (D_i^-) along its boundary. If moreover each D_i^+ (D_i^-) contains a source (a sink), we can assume that the gluing extends the classical Morse function F restricted to N to a classical Morse function $\hat{F} : \hat{N} \to \mathbb{R}$, Note that, by construction, M is obtained from $\hat{N}$ by removing from it the interiors of the disks D_i^+ and D_i^- and by identifying $\partial D_i^+ = C_i^+$ with $\partial D_i^- = C_i^-$.

5.1.2 Comparison between the surfaces M and $\hat{N}$

Let us recall that the genus of the orientable surface $M = n\mathbb{T}^2$ corresponding to the connected sum of n tori (and by convention to the sphere $\mathbb{S}^2$ if $n = 0$)) is equal to n. If the surface is nonorientable, then $M = n\mathbb{RP}^2$ is the connected sum of n projective planes and its genus is n.

Lemma 5.1.1. With the above notation, for all i from 1 to $c \geq 1$, let $\hat{N}_i$ denote a compact connected surface such that $\hat{N} = \sqcup_{i=1}^{c} \hat{N}_i$. If M is orientable, then we have

$$M = \left(\hat{N}_1 \# \hat{N}_2 \# \dots \# \hat{N}_c\right) \#(\mid \partial_- N \mid -c+1)\mathbb{T}^2$$

and

$$\text{genus}(M) = \sum_{i=1}^{c} \text{genus}(\hat{N}_i) + \mid \partial_- N \mid -c+1$$

Proof. Since M is orientable, so is each component $\hat{N}_i$ of $\hat{N}$.

Let us first assume $\hat{N}$ connected, that is, $c = 1$. Each time one removes from $\hat{N}$ the interiors of two disks D_i^+ and D_i^- and performs the identification of the corresponding boundaries, this results in the connected sum of $\hat{N}$ with a torus. Therefore

$$M = \hat{N} \# \mid \partial_- N \mid \mathbb{T}^2 = \left(\text{genus}(\hat{N}) + \mid \partial_- N \mid\right) \mathbb{T}^2$$

and we are done, as for the connected case.

Let us now assume $c \gneq 1$. Since M is connected, there must be $c-1$ indices $j_1, \dots, j_{c-1}$ such that the removal of the interiors of the $2(c-1)$ disks D_j^+ and D_j^-, $j \in \{j_1, \dots, j_{c-1}\}$, followed by the identification of the associated boundaries, transforms $\hat{N}$ into a connected compact manifold given, by construction, by the connected sum of the c components $\hat{N}_i$ of $\hat{N}$. The remaining $\mid \partial_- N \mid -c+1$ identifications to be performed are of the same type as those made in the connected case. Combining these procedures yields

$$M = \left(\hat{N}_1 \# \hat{N}_2 \# \dots \# \hat{N}_c\right) \#(\mid \partial_- N \mid -c+1)\mathbb{T}^2$$

and we can conclude by observing that each $\hat{N}_i$ is the connected sum of genus$(\hat{N}_i)$ tori. $\square$

Lemma 5.1.2. With the above notation, for all i from 1 to $c \geq 1$, let $\hat{N}_i$ denote a compact connected surface such that $\hat{N} = \sqcup_{i=1}^{c} \hat{N}_i$. Moreover, let $b \in \mathbb{N}$, $0 \leq b \leq c$, such that for all i from 1 to b, the surface $\hat{N}_i$ is orientable, while for all i from $b+1$ to c, the surface $\hat{N}_i$ is nonorientable. If M is nonorientable, then we have

$$M = \left(\underbrace{\hat{N}_1 \# \hat{N}_2 \# \dots \# \hat{N}_b}_{\textit{orientable}} \# \underbrace{\hat{N}_{b+1} \# \dots \# \hat{N}_c}_{\textit{nonorientable}}\right) \# 2(\mid \partial_- N \mid -c+1)\mathbb{RP}^2$$

and

$$\text{genus}(M) = 2\sum_{i=1}^{b} \text{genus}(\hat{N}_i) + \sum_{i=b+1}^{c} \text{genus}(\hat{N}_i) + 2(\mid \partial_- N \mid -c+1)$$

Proof. We follow the line of the proof of Lemma 5.1.2. First, let $\hat{N}$ be connected, that is, $c = 1$. If $\hat{N}$ is also nonorientable, then $b = 0$. In this case, the removal from $\hat{N}$ of the interiors of two disks D_i^+ and D_i^- followed by the identification of the corresponding boundaries can be interpreted both as making the connected sum of $\hat{N}$ with a torus and equivalently, because of the non-orientability of $\hat{N}$, as making the connected sum of $\hat{N}$ with a Klein bottle $\mathbb{K} = 2\mathbb{RP}^2$. After repeating this procedure for each couple of self-associated components, we have

$$M = \hat{N}\#2 \mid \partial_- N \mid \mathbb{RP}^2$$

and, since $\hat{N}$ is made of genus$(\hat{N})$ connected copies of $\mathbb{RP}^2$, we also have

$$\text{genus}(M) = \text{genus}(\hat{N}) + 2 \mid \partial_- N \mid$$

If $\hat{N}$ is (still connected) but orientable, then $b = c = 1$. Since M is nonorientable, there must exist a couple of disks D_i^+ and D_i^- whose boundaries are identified by reversing the orientation. Let us assume this identification is the first we perform. Then this identification corresponds to making the connected sum of $\hat{N}$ with a Klein bottle. The following identifications will then be made on a nonorientable manifold and, as in the previous case, we can conclude

$$M = \hat{N}\#2 \mid \partial_- N \mid \mathbb{RP}^2$$

Moreover, since in this situation we must have $\mid \partial_- N \mid \geq 1$ we can deduce

$$M = \text{genus}(\hat{N})\mathbb{T}^2\#2 \mid \partial_- N \mid \mathbb{RP}^2 = 2\,\text{genus}(\hat{N})\mathbb{RP}^2\#2 \mid \partial_- N \mid \mathbb{RP}^2$$

and therefore

$$\text{genus}(M) = 2\,\text{genus}(\hat{N}) + 2 \mid \partial_- N \mid$$

which ends the proof in the case $\hat{N}$ connected.

Let us now assume $\hat{N}$ not connected, that is, $c \gneq 1$. The connectivity of M implies that there must be $2(c-1)$ components of ∂N whose identification yields a connected surface. This corresponds on $\hat{N}$ to removing $2(c-1)$ disks D_i^+ and D_i^- so that, after the corresponding $(c-1)$ identifications of their boundaries, we obtain a manifold which is the connected sum of all of the surfaces $\hat{N}_i$.
Now, the same arguments of the connected case, applied to $\hat{N}_1\#\hat{N}_2\#\ldots\#\hat{N}_c$, insure us that

$$M = \left(\underbrace{\hat{N}_1\#\hat{N}_2\#\ldots\#\hat{N}_b}_{\textit{orientable}}\#\underbrace{\hat{N}_{b+1}\#\ldots\#\hat{N}_c}_{\textit{nonorientable}}\right)\#2(\mid \partial_- N \mid -c+1)\mathbb{RP}^2$$

Moreover, since at least one of the factors of the above connected sum is nonorientable, the formula for the resulting genus is straightforward. $\square$

5.1.3 Comparison between the Morse digraphs of (M, f), (N, F) and $(\hat{N}, \hat{F})$

Finally, let us compare the (classical) Morse digraph $\hat{\Gamma}(\hat{N}, \hat{F})$ underlying $\hat{N}$ and $\hat{F}$, with the circular Morse digraph $\Gamma(M, f)$ associated with M and f.

Let $\Gamma(N, F)$ denote the Morse digraph defined for N and F by the usual collapsing procedure. Properly speaking, $\Gamma(N, F)$ is a generalized digraph since whenever ∂N is nonempty, dangling edges[1] appear. When oriented by the flow, we shall denote by $(\cdot, v)$ and by $(w, \cdot)$ the dangling edge entering v and exiting w, respectively. We say that the two dangling edges $(\cdot, v)$ and $(w, \cdot)$ are merged whenever they are both removed and replaced by the oriented edge (w, v). (Note that the merging of degenerate dangling edges can also be defined, but produces no effect).

Let us recall that the cycle rank (or **cyclomatic number**) of a connected graph Γ is its first Betti number. Denoted by $\beta(\Gamma)$, it corresponds to the number of edges that can be removed from Γ without disconnecting it. It is given by the formula $\beta = 1-$ number of vertices $+$ number of edges.

Lemma 5.1.3. Following the above notation, for all i from 1 to $c \geq 1$, let $\hat{N}_i$ denote a compact connected surface such that $\hat{N} = \sqcup_{i=1}^{c} \hat{N}_i$. Let $\hat{\Gamma}_i(\hat{N}_i, \hat{F})$ denote the Morse digraphs associated with $\hat{N}_i$ and $\hat{F} \mid_{\hat{N}_i}$. Then the cycle rank of $\Gamma(M, f)$ is given by

$$\beta(\Gamma(M, f)) = \sum_{i=1}^{c} \beta(\hat{\Gamma}_i(\hat{N}_i, \hat{F})) + \mid \partial_{-} N \mid - c + 1$$

Proof. Let a_i and r_i respectively denote the sink and source contained in D_i^- and D_i^+ used in the construction of $\hat{N}$ from N. Let u_{a_i} and u_{r_i} denote the vertices of $\hat{\Gamma}(\hat{N}, \hat{F})$ associated with a_i and r_i. Removing D_i^- (D_i^+) from $\hat{N}$ corresponds to replacing the oriented edge (w_i, u_{a_i}) $((u_{r_i}, v_i))$ by the dangling edge $(w_i, \cdot)$ $((\cdot, v_i))$. Note that the obtained graph is nothing but the Morse digraph $\Gamma(N, F)$. Moreover this operation leaves the cycle rank of each component of $\hat{\Gamma}(\hat{N}, \hat{F})$ unchanged.
Now, the graph $\Gamma(M, f)$ is obtained from $\hat{\Gamma}(\hat{N}, \hat{F}) = \sqcup_{i=1}^{c} \hat{\Gamma}_i(\hat{N}_i, \hat{F})$ by merging two by two all the dangling edges. First, by the connectivity of M, the graph $\Gamma(M, f)$ is also connected, therefore there are $(c-1)$ mergings of dangling edges whose role is to connect the c graphs $\hat{\Gamma}_i(\hat{N}_i, \hat{F})$. The resulting graph is thus connected and, by construction, its cycle rank is equal to the sum of the cycles ranks of the $\hat{\Gamma}_i(\hat{N}_i, \hat{F})$'s. Since we are left with $\mid \partial_{-} N \mid - c + 1$ mergings to perform, each of which increases the cycle rank by 1 (by the connectivity of the underlying graph), the desired formula follows. $\square$

5.2 General description of a circular Morse digraph

In Therorem 5.2.1 we classify the Morse digraphs associated with circular Morse functions both from the local and from the global point of view, and we prove their realizability.

[1] A *nondegenerate dangling edge* is an edge with only one extremal vertex. It is therefore homeomorphic to a half-open interval $]x, y]$, where y corresponds to the unique extremal vertex. A *degenerate dangling edge* is an edge with only no extremal vertices, homeomorphic to an open interval.

Theorem 5.2.1. Let M be a closed connected surface. The connected digraph Γ is the circular Morse digraph associated with a circular Morse function on M if and only if the following conditions are satisfied.

1. (*Local conditions*)
   (a) The vertex v corresponds to a source (sink), and the number of exiting (entering) edges e_v^- (e_v^+) is equal to 1.
   (b) The vertex v corresponds to a saddle inside an orientable surface with boundary, and we have that either $e_v^- = 2$ and $e_v^+ = 1$, or $e_v^- = 1$ and $e_v^+ = 2$.
   (c) The vertex v corresponds to a saddle inside a nonorientable surface with boundary, and $e_v^- = e_v^+ = 1$.
2. (*Global conditions*)
   (a) if M is orientable, the cycle rank $\beta(\Gamma)$ of Γ must be equal to the genus of M.
   (b) If M is nonorientable, twice the cycle rank of Γ plus the number of nonorientable vertices must be equal to the genus of M.

Proof. Let us prove the necessity of the local conditions (Item 1) by using the construction and results of Sec. 5.1.1. Each vertex of a circular Morse digraph $\Gamma(M, f)$ has the same incoming and outgoing edges of the corresponding vertex of the classical Morse digraph $\hat{\Gamma}(\hat{N}, \hat{F})$. Hence we are done by applying Theorem 1-(1) of [14] to $\hat{\Gamma}(\hat{N}, \hat{F})$.

The necessity of the global conditions for the orientable case (Item 2-(a)) can be obtained by combining the formulae of Lemmas 5.1.1 and 5.1.3, together with Theorem 1-(2)-(a) of [14] applied to the $\hat{\Gamma}_i(\hat{N}_i, \hat{F})$'s. As for the nonorientable case, the global conditions of Item 2-(b) follow from the formulae of Lemmas 5.1.1, 5.1.2 and 5.1.3, together with Theorem 1-(2) ((a) and (b)) of [14] applied to the $\hat{\Gamma}_i(\hat{N}_i, \hat{F})$'s.

We show the sufficiency of the conditions of our theorem, that is, digraphs Γ satisfying them can be seen as the Morse digraphs associated with a surface M and a circular Morse function on it. If Γ contains no oriented cycles, we are in the classical case, and by Theorem 1 of [14] there is nothing to prove.

If Γ contains an oriented cycle, by suitably cutting as many edges as needed, say, k, one gets a connected graph $\tilde{\Gamma}$ with $2k$ dangling edges (k entering and k exiting), possessing no oriented cycles. Following the procedure given in [14], $\tilde{\Gamma}$ can be realized as the (classical) Morse digraph of a connected surface $\tilde{N}$ whose entering and exiting boundary both consist of k components.

If Γ contains no nonorientable vertices, then so does $\tilde{\Gamma}$, and we can assume by construction that the realizing surface $\tilde{N}$ is orientable. In order to realize Γ, we just need to identify the boundary components of $\tilde{N}$, whenever they correspond to the same initial edge of Γ that has been cut. Up to a perturbation, we can assume that the gluing is compatible with the flow and that no saddle connections occur. In this case, the orientability of the resulting surface depends only on whether all the gluings of the boundary components of $\tilde{N}$ preserve the orientation or not.

In the case Γ contains a nonorientable vertex, then so does $\tilde{\Gamma}$, and $\tilde{N}$ is necessarily nonorientable.

After gluing the associated boundary components of $\tilde{N}$, the resulting surface is nonorientable and, as before, together with the induced flow, it gives a realization of Γ. $\square$

5.3 Smooth flows on surfaces

In this last section, we briefly adress the combinatorics of Lyapunov digraphs, which generalize Morse digraphs in the same way as Lyapunov functions generalize Morse functions, and the Conley index theory generalizes Morse theory.

Let $f : M \to \mathbb{S}^1$ be a circle-valued function. Let $\text{Exp} : \mathbb{R} \to \mathbb{S}^1$ denote the exponential function $x \to e^{2\pi i x}$. Finally, let $E : \overline{M} \to M$ be the covering of M induced by f and Exp. By definition of the induced covering, we have a map $F : \overline{M} \to \mathbb{R}$, which makes the following diagram commute:

$$\begin{tikzcd} \overline{M} \arrow[r, "F"] \arrow[d, "E"'] & \mathbb{R} \arrow[d, "Exp"] \\ M \arrow[r, "f"] & \mathbb{S}^1 \end{tikzcd} \tag{5.3.1}$$

Definition 5.3.1. The function f is said to be a **circular Lyapunov** function on M if only if F is a (classical) Lyapunov function on $\overline{M}$.

Let M be a smooth manifold and let $\Phi_t : M \to M$ be a smooth flow. A (classical) Lyapunov function $F : M \to \mathbb{R}$ for Φ_t is a function which strictly decreases along orbits of Φ_t outside its chain recurrent set and is constant on components of the chain recurrent set. The values of F on the chain recurrent set are called the critical values of F, while the complementary ones are called regular.

A relevant difference of this new setting concerns the labels of the vertices of a Lyapunov digraph, now related to the dimension of the homology Conley indices. In the case of Lyapunov functions on surfaces, a vertex label $(h_0, h_1, h_2) \in \mathbb{N}^3$ means that for $j = 0, 1, 2$, the rank of the j-th Conley homology index of the isolating block associated with the vertex is equal to h_j. Moreover, the *genus of a vertex* v of a Lyapunov digraph associated with (M, f) and labelled with (h_0, h_1, h_2), is denoted by g_v and is given by

$$g_v = \begin{cases} \dfrac{(-h_2 + h_1 - h_0 - e_v + 2)}{2} & \text{if } M \text{ is orientable} \\ (-h_2 + h_1 - h_0 - e_v + 2) & \text{if } M \text{ is nonorientable} \end{cases}$$

An example of a smooth flow of this type is discussed at the end of this section, Example 5.3.1.

Here is a complete characterization of labelled realizable digraphs as circular Lyapunov digraphs.

Theorem 5.3.1. Let M be a compact connected surface and Λ be a connected digraph whose vertices are labelled with $(h_0, h_1, h_2) \in \mathbb{N}^3$. Then Λ is the circular Lyapunov graph associated with a smooth flow on M with a finite-component chain recurrent set and a circular Lyapunov function f if and only if the following conditions are satisfied.

1. (*Local conditions*) The labels of all vertices v satisfy the Poincaré-Hopf inequalities:

$$e_v^+ \leq h_1 + 1, \quad e_v^- \leq h_1 + 1, \quad e_v^+ + e_v^- \leq h_1 - h_2 - h_0 + 2$$

   where the latter inequality is strict if v corresponds to a chain recurrent component inside a nonorientable surface with boundary. Furthermore, if $e_v^+ = 0$ ($e_v^- = 0$), then $h_2 = 1$ ($h_0 = 1$), otherwise $h_2 = 0$ ($h_0 = 0$).

2. (*Global condition*) The genus of M is given by

$$\text{genus}(M) = \begin{cases} \beta(\Lambda) + \sum_{i=1}^{V} g_{v_i} & \text{if } M \text{ is orientable} \\ 2\beta(\Lambda) + \sum_{i=1}^{V} g_{v_i} & \text{if } M \text{ is nonorientable} \end{cases}$$

   where the sums are taken over all of the V vertices of Λ.

Proof. A digraph Λ is realizable on a surface if and only if it is locally realizable (that is, each vertex v is realizable by a surface with boundary, of genus g_v, with e_v^+ entering components and e_v^- exiting components which are copies of $\mathbb{S}^1$). The local conditions in our context are the same as the ones in the classical Lyapunov situation, because circular Lyapunov functions are locally classical. Hence the inequalities of Item (1) are necessary and sufficient for the local realizability, by Theorem 2-(1) of [14].

The absence of dangling edges for Λ are equivalent to the fact that the boundary components of the isolating blocks realizing the vertices can all be glued two by two, respecting the flow orientation. The same arguments as in Lemmas 5.1.1, 5.1.2 and 5.1.3, lead us to the formulae of Item (2) for the genus of any surface M' resulting from a gluing. □

Example 5.3.1. In Figure 5.1, we have a smooth flow defined on $2\mathbb{T}^2$. It has two singularities, a monkey saddle, for which and $(h_0, h_1, h_2) = (0, 2, 0)$, and a repelling periodic orbit, for which $(h_0, h_1, h_2) = (0, 1, 1)$. Therefore the Lyapunov digraph Λ associated with the given circular Lyapunov function has two vertices v_1 and v_2, corresponding to the monkey saddle and the repelling periodic orbit, respectively, and labelled by the corresponding triple (h_0, h_1, h_2). Both v_1 and v_2 have genus 0. The graph Λ has cycle rank $\beta(\Lambda) = 2 = \text{genus}(M)$.

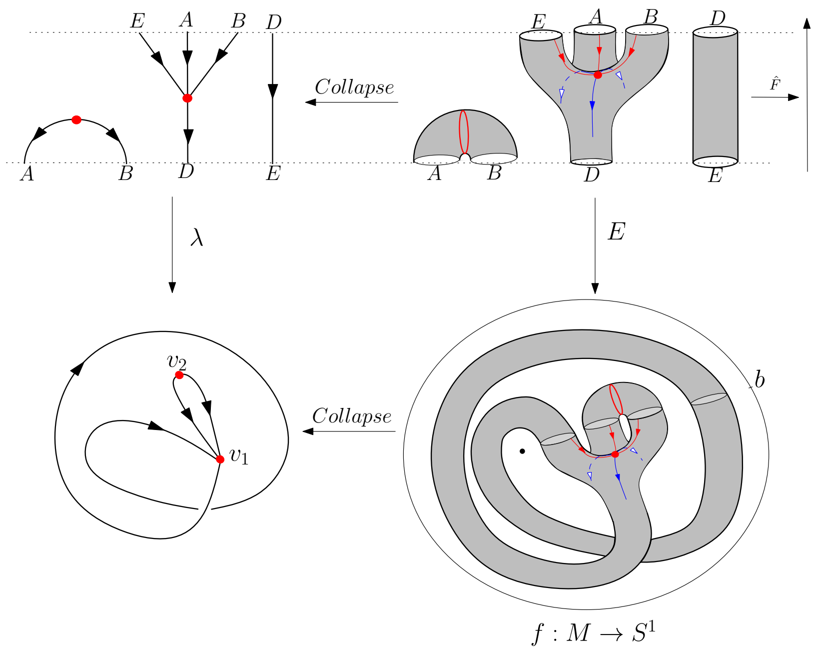


Figure 5.1: Smooth flow, circular Lyapunov function, fundamental cobordism and associated digraphs

Conclusion

This thesis has as an underlying theme the study of Smale and Morse-Novikov flows in the spirit of [15], [13] e [20].

Lyapunov graphs are used in order to understand the local topology of the basic blocks that contain each invariant set and how they fit together to constitute the manifold. This is an extremely useful tool for this purpose, however, an even deeper dynamical question remains unanswered. In rough terms what is the dynamics of the stable and unstable manifolds of the invariant sets? How much knowledge can be obtained by the use of topological methods to gain further insight on intersection properties?

In this thesis, Lyapunov graph theory provides us with a global vision of the distribution of basic blocks within the manifold and information on their topology by establishing a characterization of these graphs for Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ and Morse-Novikov flows on both orientable and non-orientable surfaces. As a by-product some obvious intersection properties of stable and unstable manifolds are obtained in such a classification. However, a deeper study will probably involve the use of connection matrices as in Cornea [9] and Franzosa [17].

In Chapter 2 and 3 the study of embeddings of surfaces in $\mathbb{S}^2 \times \mathbb{S}^1$ as done in Lemmas 2.1.2 and 2.1.3 give homological information on the manifolds bounded by these surfaces. This is shown in Lemma 2.2.4, using manifolds of handlebody type. In Corollary 3.1.2 this is established by relating properties of the Lyapunov graphs and manifolds of handlebody type.

In Chapter 4, Section 4.1, Lyapunov graphs associated to Smale flows on $\mathbb{S}^2 \times \mathbb{S}^1$ which possess a non-separable surface as a level set are considered. So, in Proposition 4.1.1 the necessary local conditions on the degree of the vertices labelled with suspensions of subshifts of finite type when this vertex is on a cycle are determined and in Theorem 4.1.1 are shown to be sufficient.

In Section 4.2 Lyapunov graphs with cycle rank equal to zero, associated to Smale flows, are studied. Firstly, one supposes that there exists a π_1-non-trivial surface as a level set, hence in Proposition 4.2.1 local necessary conditions on the degree of the vertices labelled with suspensions of subshifts of finite type are determined and shown to be sufficient in Theorem 4.2.1. Secondly, one supposes that all level sets are π_1-trivial surfaces, thus in Proposition 4.2.2 local necessary conditions on the degree of the vertices labelled with suspensions of subshifts of finite type are determined and later proven to be sufficient. This case brings about a novelty not found previously in [13]. More specifically we show in Lemma 4.2.2 the necessary conditions the degree of vertices associated to a subshift of finite type must satisfy are enlarged in comparison to the S^3 case. Hence in Section 4.2.1 and 4.2.2 the construction of the special type of basic blocks that respects these special degree conditions are realized, thus proving in Theorem 4.2.2 that these special bounds on the degree of such vertices are sufficient.

This same type of problem can be formulated for connected sums of $\mathbb{S}^2 \times \mathbb{S}^1$'s and constitutes a probable line of future research.

In Chapter 5 we extend Lyapunov graph theory to circular Morse functions and to circular Lyapunov functions on orientable and non-orientable surfaces. Hence in Theorems 5.2.1 and 5.3.1 these graphs are characterized determining necessary and sufficient conditions in order to be associated to Morse-Novikov flows. This extension of Lyapunov graph theory for Morse-Novikov flows opens a wide range of new questions in Novikov theory. Recall that circular Lyapunov graphs in conjunction with Conley index theory are a useful combinatorial tools used to explore the local topology of basic blocks containing singularities, periodic orbits, subshifts of finite type or more generally invariant sets which possess an isolating neighborhood as well as the global topology of the manifold.

Of course, the question of obtaining Lyapunov graph characterizations for circular Morse functions on generalized n-tori is fascinating. The choice of these manifolds stems from the fact that this classification was obtained in [11] and subsequently continuation properties studied in [3], [4] and [5]. Finally in [2], basic blocks were constructed realizing these graphs on generalized n-tori. Hence, relying on this foundation, one can explore the case for Morse-Novikov flows.

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