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TEORIA DE CALIBRE EM VARIEDADES DE HOLONOMIA ESPECIAL

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GAUGE THEORY ON SPECIAL HOLONOMY MANIFOLDS

Orientador: Prof. Dr. Marcos Benevenuto Jardim

TEORIA DE CALIBRE EM VARIEDADES DE HOLONOMIA ESPECIAL

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Resumo

Neste trabalho estudamos teorias de calibre em variedades de dimensão alta, com ênfase em variedades Calabi-Yau, G_2 e $Spin(7)$. Começamos desenvolvendo a teoria de conexões em fibrados e seus grupos de holonomia, culminando com o teorema de Berger que classifica as possíveis holonomias de variedades Riemannianas e o teorema de Wang relacionando a holonomia à existência de espinores paralelos. A seguir, descrevemos em mais detalhes as estruturas geométricas resultantes da redução da holonomia, incluindo aspectos topológicos (homologia e grupo fundamental) e geométricos (curvatura). No último capítulo desenvolvemos o formalismo de teoria de calibre em dimensão quatro: introduzimos o espaço de moduli de instantons e realizamos as reduções dimensionais das equações de anti-autodualidade. Com esta motivação procedemos a estudar teorias de calibre em variedades de holonomia especial e também algumas de suas reduções dimensionais.

Palavras-chave: Grupos de Holonomia, Teoria de Calibre, Variedades Calabi-Yau, Variedades G_2, Variedades $Spin(7)$.

Abstract

In this work we study gauge theory on high dimensional manifolds with emphasis on Calabi-Yau, G_2 and $Spin(7)$ manifolds. We start by developing the theory of connections on fiber bundles and their associated holonomy groups, culminating with Berger's theorem classifying the holonomies of RIemannian manifolds and Wang's theorem relating the holonomy groups to the existence of parallel spinors. We proceed to describe in more detail the geometric structures resulting from holonomy reduction, including topological (homology and fundamental group) and geometric (curvature) aspects. In the last chapter we develop the formalism of gauge theory in dimension four: we introduce the moduli space of instantons and the dimensional reductions of the anti-selfduality equations. With this motivation in mind, we proceed to study gauge theories on manifolds of special holonomy and also some of their dimensional reductions.

Keywords: Holonomy Groups, Gauge Theory, Calabi-Yau manifolds, G_2-manifolds, $Spin(7)$-manifolds.

Contents

Introdução

Teoria de calibre é o estudo de conexões em fibrados principais. O seu sucesso na descrição de invariantes refinados de variedades de dimenão baixa - por exemplo, os polinômios de Donaldson de 4-variedades, e os grupos de homologia de Floer de 3-variedades - levou os matemáticos a inquirir que tipo de estrutura é necessária em dimensões mais altas para que uma teoria similar seja viável. O trabalho de diversos matemáticos como Donaldson, Thomas, Tian, Atiyah, Hitchin e Witten levou a uma resposta satisfatória - as variedades precisam possuir holonomia especial, no sentido da classificação de Berger de grupos de holonomia. O caso de variedades Calabi-Yau é particularmente bem comportado, e o trabalho de Donaldson e Thomas mostra que praticamente todas as ideias e construções da teoria em baixas dimensões podem ser transportadas para o contexto de variedades Calabi-Yau.

O objetivo deste trabalho é descrever os fundamentos das teorias de calibre em altas dimensões, e mostrar a interrelação entre as teorias definidas em variedades de holonomia especial através de reduções dimensionais das equações de anti-autodualidade generalizadas.

No capítulo 1 descrevemos em detalhes a teoria de conexões em fibrados (principais e vetorias) e dos seus grupos de holonomia. O objetivo é enunciar o teorema de Berger sobre a classificação dos grupos de holonomia da conexão de Levi-Civita de uma variedade Riemanniana e o teorema de Wang sobre as geometrias que admitem espinores paralelos. No caminho demonstramos diversos teoremas que elucidam a natureza geométrica dos grupos de holonomia: o teorema de Ambrose-Singer e outros resultados relacionados, que estabelecem que a holonomia de uma conexão é completamente determinada pela curvatura e que a curvatura é uma medida infinitesimal da holonomia; o teorema da redução, que determina a equivalência entre o problema de classificação de holonomias e a classificação de G-estruturas; provamos também a correspondência entre tensores fixos pela holonomia e campos tensoriais paralelos; e finalmente mostramos que a classificação de holonomias de espaços simétricos pode ser determinada através da teoria de representação de álgebras de Lie reais. Ao fim do capítulo mostramos como os grupos de holonomia podem fornecer informação topológica a respeito de variedades Riemannianas compactas, em particular no caso em que a métrica é Ricci-plana.

No capítulo 2 descrevemos em maiores detalhes algumas das geometrias descritas pelo teorema de Berger. Nossa intenção final é focar nas variedades de holonomia especial - Calabi-Yau (holonomia $SU(n)$), G_2 e $Spin(7)$. Para isso, começamos descrevendo a geometria de variedades complexas, com ênfase no estudo de variedades Kähler. Mostramos como a estrutura complexa determina uma decomposição do espaço tangente complexificado, e consequentemente das álgebras tensorial e exterior. Descrevemos brevemente algumas propriedades do tensor de curvatura de variedades Kähler, e num apêndice desenvolvemos a teoria de Hodge para estas variedades. A seguir, explicamos o teorema de Calabi-Yau que garante a existência de variedades Kähler compactas com métricas Ricci-planas. Em seguida explicamos brevemente a teoria de subvariedades lagrangianas especiais e o teorema de McLean sobre deformações destas estruturas, e mencionamos algumas propriedades das superfícies $K3$, que constituem a família mais simples de variedades Calabi-Yau não-triviais (compactas e com holonomia $SU(2)$). Terminamos o capítulo explicando um pouco sobre a geometria de variedades G_2 e $Spin(7)$, em particular como certos “operadores de Hodge generalizados” definidos pelas respectivas formas de calibração fornecem uma decomposição específica da álgebra exterior.

Começamos o capítulo 3 descrevendo teorias de calibre em dimensão 4. Apresentamos a equação de anti-autodualidade e mostramos que o espaço de moduli de instantons admite uma descrição local em termos da cohomologia de um certo complexo. Esta é a base para a teoria de Donaldson, que define invariantes da estrutura suave da 4-variedade em termos de invariantes topológicos do espaço de moduli. A seguir, efetuamos as reduções dimensionais das equações de anti-autodualidade para dimensão 3 (equações de Bogomol'nyi), dimensão 2 (equações de Hitchin) e dimensão 1 (equações de Nahm). A discussão de teorias de calibre em baixa dimensão motiva a questão de qual tipo de estrutura deve-se exigir de uma variedade de dimensão *complexa* 4. Essa questão foi abordada e resolvida por Donaldson e Thomas [7], e a resposta é que a variedade deve ser Calabi-Yau (i.e., ter holonomia contida em $SU(4)$). Isso nos motiva a introduzir uma teoria de calibre em variedades Calabi-Yau como sendo a generalização correta das ideias previamente discutidas para o contexto complexo. Em geral, pode-se definir uma equação de anti-autodualidade para cada uma das três geometrias especiais usando suas respectivas formas de calibração, e neste sentido pode-se estudar teorias de calibre em variedades de holonomia especial. Por fim, realizamos algumas reduções dimensionais destas equações de instantons generalizadas, e mostramos que os resultados por vezes se conectam com outros aspectos de teorias de calibre, como o estudo de fibrados holomorfos estáveis sobre variedades Kähler [2] (através da correspondência de Hitchin-Kobayashi [33]) e uma certa versão não-abeliana das equações de Seiberg-Witten [34].

Deve-se mencionar que, até onde sabemos, este é o primeiro trabalho a derivar as equações de Kähler-instantons perturbadas (3.5.11) a partir da redução dimensional de um G_2-instanton. Este resultado mostra claramente a conexão existente entre holonomia excepcional e geometria complexa no contexto de teorias de calibre em dimensões altas, um fato que é muitas vezes mencionado mas não explicitamente mostrado na literatura. As reduções dimensionais de um $Spin(7)$-instanton a um G_2-instanton e de um $Spin(7)$-instanton a uma variedade $K3$ foram primeiramente apresentadas no trabalho de B. Jurke [18], e a redução dimensional de um Calabi-Yau instanton em dimensão complexa 4 a um Calabi-Yau instanton em dimensão complexa 3 apareceu originalmente no artigo de Singer et al. [2]. Deve-se notar, contudo, que vários destes resultados já eram conhecidos na literatura ao menos na forma de “folclore”, como pode ser visto por exemplo no artigo de Donaldson e Thomas [7].

Chapter 1

Differential Geometry

1.1 Principal Bundles and Connections

In what follows, M will denote a n-dimensional smooth manifold.

Definition 1.1.1. A (smooth) principal G-bundle is a triple (P, M, G) where P and M are smooth manifolds and G a Lie group, satisfying the following:

1. There is a smooth, free (right) action of G on P.
2. There is a smooth projection map $\pi : P \to M$ satisfying $\forall p \in P$, $g \in G$, $\pi(pg) = \pi(p)$. Therefore, π is a fibration whose fibers $\pi^{-1}(m)$ are the orbits of the action, so that each of them is diffeomorphic to G.
3. There is an open covering $\{U_\alpha\}$ of M that trivializes P; this means that we have diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that $p_1 \circ \phi_\alpha = \pi$, where p_1 is projection in the first factor. The functions ϕ_α are called *local trivializations*.

It follows from 2. that $M \cong P/G$ as smooth manifolds. Also, we can write $\phi_\alpha = (\pi, g_\alpha)$ and for each pair of neighborhoods U_α, U_β with non-trivial intersection we can define *transition functions* $g_{\alpha\beta} = g_\alpha \circ g_\beta^{-1} : U_\alpha \cap U_\beta \to G$. They satisfy the *cocycle condition*:

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha}(x) = e \tag{1.1.1}$$

These functions measure how the bundle "twists" between neighborhoods; this can be seen from the fact that, if $g_{\alpha\beta}(x) = e \; \forall x \in U_\alpha \cap U_\beta$, then $g_\alpha = g_\beta$ in $U_\alpha \cap U_\beta$ and in fact $U_\alpha \cup U_\beta$ also trivializes the bundle. Thus, the transition functions "glue together" the twists of the bundle.

From this intuitive picture, it should come as no surprise that, given a set of neighborhoods U_α of M and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, there is a unique (up to isomorphism) G-bundle $P \to M$ with these functions as transition functions. This is constructed in the obvious way: one takes the union of all $U_\alpha \times G$ and quotient by a relation which identifies $(x, g) \sim (x, g_{\alpha\beta}(x)g)$ whenever $x \in U_\alpha \cap U_\beta$. The fact that this is an equivalence relation follows from the cocycle condition. Also, the projection is the obvious one, and one also gets a free right action from right multiplication in the G factors.

We should also mention that if $\tilde{g}_{\alpha\beta}$ is a new set of transition functions related to the old ones by

$$\tilde{g}_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1} \tag{1.1.2}$$

(for functions $h_\alpha : U_\alpha \to G$, $h_\beta : U_\beta \to G$), then they give rise to a principal bundle isomorphic to P. In this case, we say the transition functions differ by a *coboundary*.

The reason for this terminology is that, as we explain in appendix A.1, the space of cocycles modulo coboundaries gives rise to the first Čech cohomology group[1] of M associated to the covering $\{U_\alpha\}$, with coefficients in G. What we have shown is that isomorphism classes of principal G-bundles over M with the given trivializing neighborhoods are in bijection with this group. For an illustration on how Čech cohomology can give information on the geometry of principal bundles, see appendix A.1.

Definition 1.1.2. A (rank k) vector bundle over M is a manifold E together with a projection $\pi : E \to M$ such that:

1. For every $x \in M$, $\pi^{-1}(x) \cong \mathbb{R}^k$
2. There is a covering $\{U_\alpha\}$ by open sets and isomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$. The functions ϕ_α are called *local trivializations.*

In this definition, M is called the *base* of the bundle, $E_x = \pi^{-1}(x)$ the *fiber over* x and E the *total space* of the bundle. We also usually identify $E_x \cong V$ for a k-dimensional vector space V, usually called the *typical fiber.*

As a trivial example, given a vector space V we can form the *trivial vector bundle* $M \times V$. The projection in this case is just the usual projection on M. Non-trivial examples are given by the tangent and cotangent bundle of M, as well as all the other tensor bundles and bundles of differential k-forms; we denote this last one by $\Omega^k(M)$.

Another important example is the following: Let $f : M \to N$ be a smooth map between smooth manifolds M and N, and let $E \xrightarrow{\pi} N$ be a vector bundle. The *pullback bundle* f^*E over M is defined to be the set $\{(m, e) \in M \times E; \pi(e) = f(m)\}$. One endows this with the induced smooth structure, and with the projection $\pi'(m, e) = m$ it has a natural vector bundle structure: if the local trivialzations of E with respect to a covering U_α of N are given by $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, then with respect to the covering $f^{-1}(U_\alpha)$ of M the local trivializations of f^*E are defined by:

$$\phi'_\alpha(m, e) = (m, \pi_2(\phi_\alpha(e)))$$

where $\pi_2 : U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection onto the second factor. Notice that the typical fiber of f^*E is $\mathbb{R}^k$, the same as E. Also, we have an induced map $\tilde{f} : f^*E \to E$ given by $\pi \circ \tilde{f} = f \circ \pi'$.

Given a rank k vector bundle, one can form the frame bundle which is a principal bundle with structure group $GL(V)$: the fiber over a point x is simply the set of all basis of the vector space V_x at that point, or equivalently, the set of isomorphisms between V_x and $\mathbb{R}^k$. Conversely, given a principal G-bundle $P \xrightarrow{\pi} M$ and a representation $\rho : G \to GL(k)$ we can form a vector bundle $\rho(P)$ with typical fiber V isomorphic to $\mathbb{R}^k$ by the following procedure: let G act on $P \times V$ in the following manner: $(p, v).g = (pg, \rho(g^{-1})v)$. This action is clearly free since $(pg, \rho(g)^{-1}v) = (p, v) \Rightarrow pg = p \Rightarrow g = e$, since the action of G on P is free. The quotient manifold $\rho(P) := P \times V/G$ inherits a natural vector bundle structure: the projection map defined by $\bar{\pi}([p, v]) = \pi(p)$ is well-defined, because $\bar{\pi}([pg, \rho(g)^{-1}v]) = \pi(pg) = \pi(p) = \bar{\pi}([p, v])$. Also, if we fix a $p \in P_x$, the map $v \mapsto [p, v]$ defines an isomorphism between V and the fiber $\rho(P)_x$. An example that will appear often in this work is the adjoint bundle (often denoted by $Ad(P)$ or simply by $\mathfrak{g}$) associated to the adjoint representation $Ad : G \to GL(\mathfrak{g})$ of G on its Lie algebra $\mathfrak{g}$.

Given a vector bundle $E \to M$ with typical fiber V, we can also form other vector bundles over the same base by performing standard operations on V: E^* is the bundle with typical fiber V^*, and

[1] Actually, this space is not really a group, but only a pointed set.

$End(E)$ the bundle with fiber $End(V)$. Given another vector bundle $F \to M$ we can also form $E \oplus F$ and $E \otimes F$. In particular, if $E = \Omega^k(M)$, then we will denote its tensor product with a trivial bundle with fiber V by $\Omega^k(M, V)$.

Definition 1.1.3. A *connection* on a principal G-bundle $P \to M$ is a smooth distribution $\mathcal{H} \subset TP$ of "‘horizontal spaces"’ such that:

1. $\forall p \in P$, $T_pP = \mathcal{H}_p \oplus V_p$, where $V_p = Ker(d\pi_p)$ is the "‘vertical space"’ at p.
2. The distribution is equivariant with respect to the G-action: $\mathcal{H}_{pg} = R_g^*\mathcal{H}_p$.

In fact, the connection $\mathcal{H}$ is a vector subbundle of TP, and it is often useful to think of it that way. Also, the the differential of the projection $d\pi_p$ induces an isomorphism $\mathcal{H}_p \cong T_{\pi(p)}M$, which justifies the terminology "horizontal space" for $\mathcal{H}_p$.

An important observation is that $V_p \cong \mathfrak{g}$ as vector spaces ($\mathfrak{g}$ is the Lie algebra of G). The isomorphism is given by taking the differential of the map $p : G \to P$, $p(g) = pg$ at the identity of G, and restricting the codomain:

$$dp_e|_{V_p} : \mathfrak{g} \to V_p \tag{1.1.3}$$

Notice that $V_p = Im(dp_e)$, since the action sends fibers to fibers.

Let us show that this is indeed an isomorphism. First of all, it is clearly a linear mapping. Now suppose that $dp_e(A) = 0$. This is the same as $\frac{d}{dt}p.exp(tA) = 0$, which implies $p.exp(tA) = p$, and since the action is free, we have that $exp(tA) = e \Rightarrow A = 0$. This proves injectivity, and surjectivity follows from the fact that the dimensions coincide (recall the typical fiber is isomorphic to G). It remains to show that dp_e commutes with the bracket. We write $x_t = exp(tX)$, and we will use the following formula for the bracket of vector fields X and Y:

$$[X, Y] = \lim_{t\to 0} \frac{1}{t}(Y - R_{x_t}Y) \tag{1.1.4}$$

Now it is easy to see that $(R_{x_t}dp_eB)_p = R_{x_t}d(p(x_t)^{-1})_e(Y_e) = dp_e(ad(x_t^{-1})Y_e)$. Thus:

$$\begin{aligned} [dp_eA, dp_eB] &= \lim_{t\to 0} \frac{1}{t}(dp_eB_e - dp_e(ad(x_t^{-1})B_e) \\ &= dp_e([A, B]_e) = (dp[A, B])_p \end{aligned} \tag{1.1.5}$$

Therefore, we can define the *connection 1-form* $\omega : TP \to \mathfrak{g}$ to be the element of $\Omega^1(P, \mathfrak{g})$ that restricts fiberwise to $\omega_p = dp_e^{-1} : T_pP \to \mathfrak{g}$. Notice that $\omega(v) = 0$ if v is horizontal (i.e., lies in $\mathcal{H}$).

Now, define the *Maurer-Cartan form* to be the element $\omega^{MC} \in \Omega^1(G, \mathfrak{g})$ obtained from the identity map in $T_eG \cong \mathfrak{g}$ by left translation.

The connection 1-form ω has the following properties:

1. $p^*\omega = \omega^{MC}$
2. $g^*\omega_{pg} = Ad(g^{-1})\omega_p$

Proof. The first formula follows directly from from the definitions: $(p^*\omega)_e(X) = \omega_p(dp_e(X)) = X = \omega_e^{MC}(X)$. In general:

$$\begin{aligned}
(p^*\omega)_g(X) &= \omega_{pg}(dp_g X) \\
&= \omega_{pg}(d(p \circ L_g \circ L_g^{-1})_g X) \\
&= ((p \circ L_g)^*\omega)_e(dL_g^{-1} X) \\
&= (L_g^{-1})^*((pg)^*\omega)_e(X) \\
&= (L_g^{-1})^*\omega_e^{MC}(X) \\
&= \omega_g^{MC}(X)
\end{aligned}$$

In the second formula, both sides vanish when applied to horizontal vectors. So it suffices to check on vertical vectors; any such vector in V_p is of the form $dp_e(X)$ for some $X \in \mathfrak{g}$. The left hand side is then

$$\begin{aligned}
\omega_{pg}(dg_p \circ dp_e(X)) &= d(pg)_e^{-1}(dg_p \circ dp_e(X)) \\
&= d(p \circ L_g)_e^{-1}(dg_p \circ dp_e(X)) \\
&= dL_{g^{-1}} \circ dp_e^{-1} \circ dg_p \circ dp_e(X) \\
&= Ad(g^{-1})(X)
\end{aligned}$$

since $L_{g^{-1}} \circ p^{-1} \circ g \circ p(h) = C_{g^{-1}}(h)$. On the other hand:

$$Ad(g^{-1}\omega_p(dp_e(X)) = Ad(g^{-1})(dp)_e^{-1} dp_e(X) = Ad(g^{-1})(X)$$

□

Conversely, every form satisfying these two properties defines a connection in P (one simply defines $\mathcal{H}_p = Ker(\omega_p)$, where we see ω_p as a map from T_pP to $\mathfrak{g}$). So we will use both definitions of a connection interchangeably.

A fact that is often important is that we can also see the connection as a 1-form on M with values in the adjoint bundle $Ad(P)$. This is due to the fact that the connection 1-form is Ad-invariant, so that it descends to the adjoint bundle via the general construction of an associated principal bundle.

Every principal bundle can be given a connection. We sketch a proof of this fact. The first thing to notice is that a trivial bundle has a natural connection - namely, the distribution given by the tangent spaces of the base. Then we can define connections in each trivializing chart of a general bundle. Now, given two connections ω_1 and ω_2 in P, their difference $\omega_1 - \omega_2 = \phi$ is also an Ad-equivariant element of $\Omega^1(P, \mathfrak{g})$; also, ϕ vanishes on the vertical spaces (which are of course the same for both ω_1 and ω_2). The space of forms with this property can be identified with $\Omega^1(M) \otimes Ad(P)$ via π^*. The space of connections is then an *affine space* modelled in $\Omega^1(M) \otimes Ad(P)$.

So, in each trivializing chart U_α one can define a connection A_α, and by gluing these together using a partition of unity $\{\xi_\alpha\}$ subordinate to the trivializing cover, one gets a connection $A = \sum \xi_\alpha A_\alpha$ defined in the whole bundle.

If $P \to M$ is a principal bundle and $f : N \to M$ is a continuous map, we can define the *pull-back bundle* f^*P over N in the same way we did for vector bundles: it is just the subbundle of $N \times P$ defined by $f^*P = \{(n, p) \in N \times P; f(n) = \pi(p)\}$. The projection is the obvious one, and the action is given by

$(n,p)g = (n,pg)$, $\forall g \in G$. We also get an induced mapping $\tilde{f} : f^*P \to P$ in the same way as for vector bundles, which allows us to identify the fibers of f^*P with the fibers of P.

A connection ω in P induces a connection $f^*\omega$ in the pull-back bundle f^*P. This implies, of course, that a horizontal distribution on P pulls-back to a horizontal distribution on f^*P.

We have mentioned before that the connection induces an isomorphism between the horizontal subspace $\mathcal{H}_p$ and $T_{\pi(p)}M$. In fact, we have an isomorphism of bundles $\mathcal{H} \cong \pi^*(TM)$ over P.

A useful consequence of these facts is that we can lift paths on M to horizontal paths on P:

Proposition 1.1.4. Given a principal bundle $P \to M$ with a connection $\mathcal{H}$, a point $x \in M$ and a curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$, then for every $p \in \pi^{-1}(x)$, there is a unique curve $\tilde{\gamma} : [0,1] \to P$ with $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}'(t) \in \mathcal{H}_{\tilde{\gamma}(t)}$.

Proof. The pullback of P by γ is a principal bundle γ^*P over $[0,1]$. The connection determines a unique horizontal vector field on γ^*P that projects to d/dt on $[0,1]$. This field integrates to a unique horizontal curve on γ^*P with the given initial condition, and upon identification between the fibers of P and γ^*P, we obtain the required curve on P. on γ^*P. $\square$

This gives a geometric interpretation of the connection: first, notice that the lifting of paths is equivariant with respect to the action: the lift of $\gamma.g$ is just $\tilde{\gamma}.g$. Therefore, given a smooth curve γ in M between x and y the connection determines a G-equivariant isomorphism between the fibers $\pi^{-1}(x)$ and $\pi^{-1}(y)$.

Let us now turn to connections on vector bundles:

Definition 1.1.5. A *connection* (or a *covariant derivative*) on a vector bundle $E \to M$ is a $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(M)$ satisfying $\nabla(fs) = df \otimes s + f\nabla s$ for every $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

We can see the connection also as a map $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ that is $\mathbb{R}$-linear in sections and $C^\infty(M)$-linear in the vector entry. So the connection is a generalization of the usual covariant derivative in Riemannian geometry to arbitrary vector bundles, not just the tangent bundle.

The same argument used in Riemannian geometry to show that the Christoffel symbols are anti-symmetric shows that locally a connection can be written as $\nabla = d + A$, where $A \in \Omega^1(M; \mathfrak{g})$. This can also be seen from a fact we have mentioned before - that the space of connections is an a affine space modelled on $\Omega^1(M; Ad(E))$, and we are considering d to be (locally) a flat connection.

The relation between connections in principal and vector bundles goes as follows: given a connection one-form ω in a principal bundle P and a representation $\rho : G \to GL(V)$, we can form the associated bundle $P \times_G V$ whose sections are of the form $\sigma(x) = [p(x), v(x)]$. Given a vector field $T \in TM$ we define:

$$(\nabla\sigma)_x(T_x) = [p(x), d\rho_e(\omega_x(\tilde{T}_x))(v(x)) + dv_x(T_x)] \tag{1.1.6}$$

where $\tilde{T}_x$ is the horizontal lift of T_x.

Intuitively, this means that a horizontal distribution on a principal bundle $H \subset TP$ induces horizontal distributions in every associated vector bundle in a natural way. On the other hand, a covariant derivative in a vector bundle E gives rise to a natural notion of horizontal distribution on its frame bundle $F(E)$, namely, the distribution generated by parallel sections.

1.1.1 Curvature

We will start by giving a geometric definition of the curvature of a connection. Let $\mathcal{H} \subset TP$ be a connection on the principal bundle $P \to M$. Now, we have seen that we can always lift a smooth curve in M to a (unique) horizontal curve in P. The curvature of the connection will measure the obstruction to such a lift for higher-dimensional submanifolds.

To see how this goes, choose a point $x \in M$ and vectors v, w in T_xM, and consider a coordinate system $(x_1, \ldots, x_n)$ around x such that $\frac{\partial}{\partial x_i}|_x = v_i$. We take a rectangle $[0, \epsilon] \times [0, \epsilon]$ in the (x_1, x_2)-space and lift its image to P in a counterclockwise fashion, in such a way that the final point of one segment is the initial point for the lifting of the next. If p is the initial point of the first segment lifted, it is clear that the endpoint is in the same fiber, but is not necessarily equal to p. Thus, it is of the form $pg(\epsilon)$ for some $g(\epsilon) \in G$. It is also clear that, if ϵ is sufficiently small, then $g(\epsilon)$ is close enough to the identity so that it is inside the neighborhood of e where $exp : \mathfrak{g} \to G$ is a diffeomorphism. Therefore, it makes sense to define:

$$F_p(v, w) = -lim_{\epsilon \to 0} \frac{log(g(\epsilon))}{\epsilon^2} \tag{1.1.7}$$

Theorem 1.1.6. The element $[p, F_p(v, w)]$ in $Ad(P)$ is linear and skew-symmetric with respect to v and w. We can then identify F as an element of $\Omega^2(M) \otimes Ad(P)$, called the curvature of the connection $\mathcal{H}$.

So we can understand F as an infinitesimal measure of the *holonomy* of the connection (see section 1.2).

Another way of understanding this is by means of the Lie bracket of vector fields; in fact, one can prove that F is the obstruction for the horizontal distribution $\mathcal{H}$ to be integrable. Let us sketch how this definition works, as it is more useful in practice[2] and connects well with the theory for vector bundles.

Start with a vector field X on TM. From the isomorphism $\pi^*(TM) \cong D$ we find that there is a unique section $h(X) \in \Gamma(\mathcal{H})$ that projects down to X (this is, of course, just the horizontal lift of X). It is easy to see that the quantity

$$\tilde{F}(X, Y) = [h(X), h(Y)] - h([X, Y]) \tag{1.1.8}$$

defined for vector fields X and Y, is linear and anti-symmetric in X and Y. Also, since $\Gamma(TM)$ and $\mathcal{H}$ are isomorphic as Lie algebras (remember that $\mathcal{H}$ is involutive), we have that $d\pi([h(X), h(Y)]) = [X, Y] = d\pi(h([X, Y]))$, so that $\tilde{F}(X, Y) \in ker(d\pi) \cong \mathfrak{g}$. So we can see $\tilde{F}(X, Y)$ as a (G-invariant) section of $P \times \mathfrak{g} \to P$, which of course corresponds to a section F of $\Lambda^2 T^*M \otimes Ad(P)$, i.e., an element of $\Omega^2(M; Ad(P))$. This is the curvature of the connection $\mathcal{H}$. It satisfies:

$$\bar{\pi}^*(F(X, Y)) = \tilde{F}(X, Y) \tag{1.1.9}$$

where $\bar{\pi} : P \times \mathfrak{g} \to Ad(P)$ is the projection into the orbits of the G-action.

For vector bundles, it is more useful to work with the curvature in terms of the covariant derivative $\nabla : \Gamma(E) \to \Gamma(E) \otimes \Omega^1(M)$. Locally we can write $\nabla = d + A$ for A a $\mathfrak{g}$-valued 1-form. We define a $\mathfrak{g}$-valued 2-form by $F(v, w) = [\nabla_v, \nabla_w] - \nabla_{[v,w]}$. Notice that this agrees with the statement above that the curvature should measure the non-integrability of the horizontal distribution. In local coordinates

[2] for the proof of equivalence between the definitions, see [9].

on M, we can write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Indeed, choosing a coordinate basis $\{\partial_\mu\}$ for TM and a local frame $\{e_i\}$ of E, we have:

$$\begin{aligned} F_{\mu\nu} e_i &= (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) e_i \\ &= \nabla_\mu (A^j_{\nu i} e_j) - \nabla_\nu (A^j_{\mu i} e_j) \\ &= \partial_\mu A^j_{\nu i} e_j + A^k_{\mu j} A^j_{\nu i} e_k - \partial_\mu A^j_{\nu i} e_j - A^k_{\nu j} A^j_{\mu i} e_k \\ &= (\partial_\mu A^j_{\nu i} - A^j_{\nu k} A^k_{\mu i} + A^j_{\mu k} A^k_{\nu i} - \partial_\mu A^j_{\nu i}) e_j \end{aligned} \tag{1.1.10}$$

which is the previous formula in matrix notation.

The *covariant exterior differential* is defined to be the extension d_nabla of ∇ to all E-valued forms using linearity and the Leibniz rule (i.e., $d_\nabla(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d_\nabla \eta$, where $\omega \in \Omega^p(M)$ and $\eta \in \Omega(M; E)$).

This operator is especially useful in the case when $E = Ad(P)$ since it allows us to express the global 2-form F in terms of the covariant derivative: just notice that unlike the ordinary exterior differential, in general $d^2_\nabla \neq 0$. In fact, we have the following:

Proposition 1.1.7. If $\eta \in \Gamma(E \otimes \Lambda^k T^* M)$, then $d^2_\nabla \eta = F \wedge \eta$.

Proof. We work in local coordinates $\{x^\mu\}$ on M. Write $\eta = s_I \otimes dx^I$ for I an index set, and s_I section on E. Then:

$$\begin{aligned} d^2_\nabla \eta &= d_\nabla (\nabla_\mu s_I \otimes dx^\mu \wedge dx^I) \\ &= \nabla_\nu \nabla_\mu s_I \otimes dx^\nu \wedge dx^\mu \wedge dx^I \\ &= \frac{1}{2} [\nabla_\nu, \nabla_\mu] s_I \otimes dx^\nu \wedge dx^\mu \wedge dx^I \\ &= F \wedge \eta \end{aligned} \tag{1.1.11}$$

where we used the fact that, since $\{\partial_\mu\}$ is a coordinate basis for the tangent spaces, $[\partial_\mu, \partial_\nu] = 0$. $\square$

In differential forms notation, we can write $F = dA + A \wedge A$, where the wedge acts as the commutator in the Lie algebra part and as the usual wedge product in the 1-form part. This is just a compressed version of (1.1.10), but it can also be proved directly from the local formula $d_\nabla = d + A$:

$$\begin{aligned} d^2_\nabla \eta &= d_\nabla (d\eta + A \wedge \eta) \\ &= A \wedge d\eta + dA \wedge \eta - A \wedge d\eta + A \wedge A \wedge \eta \\ &= (dA + A \wedge A) \wedge \eta \end{aligned}$$

for every E-valued form η.

To finish this section, we prove the *Bianchi identity*:

$$d_\nabla F = 0 \tag{1.1.12}$$

Proof. Let η be any E-valued form. Then $d^3_\nabla \eta = d_\nabla (d^2_\nabla \eta) = d_\nabla (F \wedge \eta) = d_\nabla F \wedge \eta + F \wedge d_\nabla \eta$. On the other hand, $d^3_\nabla \eta = d^2_\nabla (d_\nabla \eta) = F \wedge d_\nabla \eta$. Because this is true for any η, we must have $d_\nabla F = 0$. $\square$

1.2 Holonomy Groups

Let $E \to M$ be a rank k vector bundle endowed with a connection ∇, and fix points x, $y \in M$. For each curve γ such that $\gamma(0) = x$, $\gamma(1) = y$, and each $v \in E_x M$ we can consider the parallel transport equation:

$$(\gamma^*\nabla)_{\gamma'(t)} s \circ \gamma(t) = 0 \tag{1.2.1}$$

with $s : U \to E$ a section of E defined in an open set U containing $Im(\gamma)$, with $s(0) = v$. What this equation means is that at each point $\gamma(t)$, $s(\gamma(t))$ is the element of $E_{\gamma(t)}$ obtained from v by requiring the entire section to be constant with respect to ∇ over γ. More precisely, we need to work with the pullback connection $\gamma^*\nabla$ over the pullback bundle γ^*E, and we are requiring the resulting section over $[0, 1]$ to be parallel. This is an O.D.E. for the section $s \circ \gamma$, and so there is a unique solution; the element $s(\gamma(1)) \in E_y$ is called the *parallel transport* of v along γ. Now, the parallel transport of a basis of E_x defines a linear map $P_{x,\gamma} : E_x \to E_y$.

Definition 1.2.1. The *holonomy group* of ∇ at x is:

$$Hol_x(\nabla) = \{P_{x,\gamma} \in GL(k); \gamma \text{ loop based at } x\}$$

This is in fact a group: $P^{-1}_{x,\gamma} = P_{x,\gamma^{-1}}$, where $\gamma^{-1}(t) = \gamma(1 - t)$ is just the loop with the orientation reversed; also, if α and β are loops based at x, then $P_{x,\alpha} \circ P_{x,\beta} = P_{x,\alpha*\beta}$, where $\alpha*\beta$ is just the loop defined by going first around β and then around α. Both of these equalities follow from the uniqueness of solutions to the parallel transport equation.

Now, fix attention to the null-homotopic loops based at x. A homotopy between two such loops γ_1 and γ_2 that preserves the base point can be seen as a path in loop space, and therefore defines a path in $H_x := Hol_x(\nabla)$ between P_{x,γ_1} and P_{x,γ_2}. This means that the component $H^0_x := Hol^0_x(\nabla)$ of H_x consisting of parallel transport operators through null-homotopic loops is path-connected, and therefore it is a (connected) Lie subgroup of $GL(k)$, the connected component of the identity in $GL(k)$. By this argument, one can see that in general there is a map between $\pi_1(M, x)$ and connected components of H_x. In fact, since H^0_x is normal in H_x[3] it is in fact a *Lie subgroup* of $GL(k)$, and we can define the *monodromy group* $Mon_x(\nabla) := Hol_x(\nabla)/Hol^0_x(\nabla)$; what we get then is an epimorphism $\pi_1(M, x) \to Mon_x(\nabla)$ given by $[\alpha] \mapsto P_\alpha H^0_x$. Also, since $\pi_1(M)$ is enumerable, H_x has at most countably many connected components. Of course, it follows that H_x is connected if M is simply-connected.

Furthermore, assuming M is path-connected, the groups $Hol^0_x(\nabla)$ and $Hol^0_y(\nabla)$ are conjugate to each other: indeed, if γ is a path joining x and y, then we get a parallel transport operator $P_\gamma : T_x M \to T_y M$. If we fix an element $h \in Hol^0_y(\nabla)$, there must be a loop α at y such that $h = P_\alpha$. Then $\gamma^{-1} \circ \alpha \circ \gamma$ is a loop at x, so it defines an element $P_{\gamma^{-1}} P_\alpha P_\gamma$ of $Hol^0_x(\nabla)$. That every element of $Hol^0_x(\nabla)$ is of this form follows from the same argument applied to y instead of x.

We can see the holonomy groups H^0_x as Lie subgroups of $GL(k)$, and since they are all conjugate to each other it follows that they are isomorphic as Lie groups. Therefore, it doesn't matter which base point we choose, and from now on we will simply refer to the holonomy group $H^0 := Hol^0(M)$.

Definition 1.2.2. The *holonomy algebra* $\mathfrak{h} := \mathfrak{hol}(\nabla)$ is the Lie algebra of $H^0(\nabla)$.

Of course, since H^0_x is only defined up to conjugation, $\mathfrak{h}$ is also only defined up to the adjoint action of $GL(k)$. Whenever needed, we will refer to the holonomy algebra at a point x by $\mathfrak{hol}_x(\nabla)$.

[3] This follows from the fact that if α is null-homotopic, then for any loop β, $\beta * \alpha * \beta^{-1}$ is also null-homotopic

We can also define the holonomy group of connections on principal bundles. Given a principal G-bundle $P \to M$ and a connection $\mathcal{H} \subset TP$, then given a loop γ based at $x \in M$ and an element $p \in P_x$, there is a unique lift of γ to a horizontal cuve $\tilde{\gamma}$ in P such that $\tilde{\gamma}(0) = p$. Since $q := \tilde{\gamma}(1) \in P_x$, there is an element $g \in G$ such that $q = pg$. We will write $p \sim q$ to mean that there exists a horizontal curve connecting p to q. This can be easily seen to define an equivalence relation among the points in the fiber P_p.

Definition 1.2.3. The *holonomy group* of $\mathcal{H}$ at p is

$$H_p := Hol_p(\mathcal{H}) = \{g \in G; p \sim pg\}$$

We can also define the holonomy algebra of a connection on a principal bundle $\mathfrak{hol}_p(\mathcal{H})$ as the Lie algebra of $Hol_p(\mathcal{H})$. Notice that for principal bundles we have a copy of the holonomy group and algebra attached to each point of P, and not only of M as in the case of vector bundles.

All the properties of holonomy groups of vector bundles also hold for principal bundles. In fact, there is a close relationship between the two: the holonomies of a connection in a vector bundle and the induced connection in the frame bundle are the same, and given a principal G-bundle P with connection $\mathcal{H}$ and an associated vector bundle given by a representation $\rho : G \to GL(k)$, then the holonomy of the associated bundle is just $\rho(Hol(\mathcal{H}))$.

Theorem 1.2.4. Let E be a vector bundle with a connection ∇, and $S_0 \in \mathcal{T}^k(E_x)$ a k-tensor over E_x that is fixed by the holonomy action, $T(S_0) = S_0$, $\forall T \in Hol(\nabla)$. We will also denote by ∇ the induced connection on $\mathcal{T}^k(E)$. Then there is a unique section S of $\mathcal{T}^k(E)$ such that $S(x) = S_0$ and $\nabla S = 0$. Conversely, given a parallel section S of $\mathcal{T}^k(E)$, then at every fiber S is fixed by the holonomy action.

Proof. Suppose first that S_0 is fixed by the holonomy action $\rho : H \to GL(E_x)$. The required section is defined as follows. For every point y we choose a path α connecting x to y, and define S_y to be the parallel transport $P_\alpha S_0$ of S_0 along α with respect to ∇. If β is another such path, then $\alpha^{-1} \circ \beta$ is a loop at x, so that $P_{\alpha^{-1} \circ \beta} = P_{\alpha^{-1}} P_\beta = \rho(h)$ for some $h \in H$. But $\rho(h) S_0 = S_0$, so that $P_\alpha S_0 = P_\beta S_0$, which means that S_y doesn't depend on the choice of curve. This gives a well-defined tensor S which satisfies $\nabla S = 0$ by construction.

Conversely, if we start with a parallel section S, then for every loop γ at x we have $P_\gamma S_{\gamma(0)} = S_{\gamma(1)}$, i.e., $P_\gamma S_x = S_x$. This shows that S is fixed at every fiber by the holonomy action. □

An easy corollary of this result is the following: if $G \subset GL(T_x M)$ is a subgroup fixing all parallel tensors S_x, then $Hol(\nabla) \subset G$. This result is very important, because it establishes a relation between geometric structures (i.e., reductions of the holonomy) and parallel tensors. We show now that, in fact, a principal bundle P with a connection admits a principal subbundle whose structure group is the holonomy group of P. This result is known as the *reduction theorem*:

Theorem 1.2.5 (Reduction theorem)**.** Let $P \xrightarrow{\pi} M$ be a principal bundle with structure group G, and $\mathcal{H}$ a connection on P. Assume that $H := Hol_p(\mathcal{H})$ is a Lie subgroup of G. Fix a point $p \in P$ and consider the subset $P(p) = \{q \in P; p \sim q\}$ of all points that can be reached from p by a horizontal curve. Then $P(p)$ is a subbundle of P with structure group $Hol_p(\mathcal{H})$, and the connection D reduces to a connection D' in $P(p)$.

Proof. It is not difficult to see that $P(p)$ is a principal subbundle with fiber H: suppose that $p \overset{\gamma}{\sim} q$, i.e., γ is a horizontal curve connecting p to q. If $h \in H$, then of course $p \sim ph$. But $ph \overset{\gamma h}{\sim} qh$, so that $p \sim qh$. Thus, $qh \in P(p)$ and therefore H acts on $P(p)$. This actios is free because it is te restriction of the

G-action on P. Also, the fibers of $\pi|_{P(p)} : P(p) \to M$ are clearly the orbis of H, so that $M \cong P(p)/H$, and a local trivialization can be constructed by parallel translation of coordinate curves.

The fact that the connection reduces to a connection on $P(p)$ follows from the horizontal subspaces being tangent to $P(p)$; then we simply define the horizontal subspaces of the new connection to be the projection of the former horizontal spaces onto $TP(p)$. $\square$

There is also a similar result which is really important since it establishes a relation between the holonomy of a connection on the tangent bundle and the existence of certain subbundles of the frame bundle. For that, we need a definition: a connection ∇ on the tangent bundle is said to be *torsion-free* if the torsion tensor $\mathcal{T}_\nabla$ defined by:

$$\mathcal{T}_\nabla(v, w) = \nabla_v w - \nabla_w v - [v, w]$$

vanishes. The most important example of such a connection is the *Levi-Civita connection* on a (semi)Riemannian manifold (M, g), that is, the unique connection ∇ on TM such that $\mathcal{T}_\nabla = 0$ and $\nabla g = 0$.

The result we refered to is the following:

Theorem 1.2.6. Let $F \to M^n$ be the frame bundle of M and $G \subset GL(n)$ a Lie subgroup. Then M admits a torsion-free connection with holonomy $H \subset G$ if and only if it admits a reduction to a torsion-free principal G-bundle over M.

Proof. The proof is similar to the proof of the reduction theorem, so we will skip it. $\square$

Definition 1.2.7. Let $F \to M$ be the frame bundle of M, and $G \subset GL(n)$ a Lie subgroup. We call a principal G-subbundle of F a *G-structure* on M.

Intuitively, a G-structure on M should be regarded as a special class of frames on F determined by the group G. Given a G-structure, one says that the structure group has been reduced to G, since the transition functions take values in G. Also, a connection that reduces to a G-structure is said to be *compatible* with it, and we say the G-structure is torsion-free whenever the connection is. We will see that there is an intimate relationship between G-structures and additional geometric structures on M.

Example 1.2.8. A $O(n)$-structure $O \to M$ is equivalent to a Riemannian metric on M. In fact, given the standard inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$, for each $x \in M$, any choice of $p \in O_x$ determines a metric g in M given by $g(X, Y) = \langle p^{-1}(X), p^{-1}(Y)\rangle$. Conversely, a choice of metric on M determines a special class of elements in F_x, namely the isometries between T_xM and $\mathbb{R}^n$, and this clearly defines a principal $O(n)$-subbundle of F.

Example 1.2.9. A pseudo-symplectic structure on M is defined to be a $Sp(n)$-structure, which is in turn the same as a non-degenerate 2-form ω on M. If ω is also closed M is called a *symplectic manifold.* These manifolds are always even-dimensional, and the condition $d\omega = 0$ is an integrability condition that makes the structure locally trivial, a result known as *Darboux's theorem.*

Example 1.2.10. A $GL^+(n)$-structure on M is equivalent to a choice of orientation, since the associated transition functions have positive determinant. Combining with the previous example, we see that a $SO(n)$-structure is equivalent to a choice of oriented orthonormal frames on M.

The last example shows that it might not be possible to find a G-structure on M, since there are topological obstructions. For example, we need the first and second Stiefel-Whitney classes of TM to vanish for M to admit an orientation or a spin structure, respectively[4].

[4] More on this on the appendix - including the definition of a spin structure.

Many of the results already proved can be interpreted in the language of G-structures; for instance, what theorem 1.2.6 means in this language is that a torsion-free G-structure on M is equivalent to a torsion-free connection with holonomy $H \subset G$. This will be useful after we give the classification of holonomy groups at the end of this section.

The holonomy group of a principal bundle $P \to M$ with connection $\mathcal{H}$ is a global invariant that contains information about the curvature F of $\mathcal{H}$, as we will now explain.

First of all, we define the holonomy subbundle of $Ad(P)$. Consider the projection $\bar{\pi} : P \times \mathfrak{g} \to Ad(P)$. For $x \in M$, we define $\mathfrak{hol}_x(\mathcal{H}) = \bar{\pi}(x, \mathfrak{hol}_p(\mathcal{H}))$, where $\pi(p) = x$. Since $\mathfrak{hol}_{pg}(\mathcal{H}) = Ad(g^{-1})\mathfrak{hol}_p(\mathcal{H})$, this definition doesn't depend on the choice of p. We define $\mathfrak{hol}(\mathcal{H})$ to be the subbundle of $Ad(P)$ with fiber $\mathfrak{hol}_m(\mathcal{H})$ over m. Then we have the following:

Proposition 1.2.11. The curvature F_m of $(P, \mathcal{H})$ at $m \in M$ is an element of $\Omega^2(\mathfrak{hol}_m(\mathcal{H}))$.

Proof. This is a simple consequence of the definition of $\tilde{F}$ as a section of $P \times \mathfrak{g}$ (equation (1.1.8)) and the fact that the restriction of $V_p = Ker(d\pi)_p$ to $P(p)$ is exactly $\mathfrak{hol}_p(\mathcal{H})$. Thus, if X and Y are vector fields on M, $\tilde{F}(X,Y) = \bar{\pi}^* F(X,Y) \in \mathfrak{hol}_p(\mathcal{H})$, so that $F(X,Y) \in \mathfrak{hol}_m(\mathcal{H})$. □

This agrees well with formula 1.1.7 defining the curvature as an infinitesimal measure of the holonomy: the holonomy around an infinitesimal square of side ϵ is of the form $Id - \epsilon^2 F_{\mu\nu} + \mathcal{O}(\epsilon^3)$. Also, compare with theorem 1.1.6 - what we have just shown is that the holonomy group restricts the curvature of the connection.

There is an important kind of converse to this result - this is the celebrated *Ambrose-Singer theorem*:

Theorem 1.2.12 (Ambrose-Singer). $\mathfrak{hol}_p(\mathcal{H}) = span\,\{\bar{\pi}^*(F_q(X,Y)); X, Y \in \Gamma(TM), q \in P(p)\}$.

Proof. Since we are dealing with the holonomy algebra, we may assume that P is the holonomy bundle $P(p)$, i.e., that $G = Hol(\mathcal{H})$. Let $\mathfrak{h}'_p$ be the subspace generated by elements of the form $\pi^* F_q(X,Y)$ for $q \in P(p)$ and X, Y horizontal vectors at q. We have to show that this generates $\mathfrak{h}_p = \mathfrak{hol}_p(\mathcal{H})$.

First of all, $\mathfrak{h}'_p$ is an ideal of $\mathfrak{h}_p$; indeed, from $R_g^* F = Ad(g^{-1})F$, it follows that $\mathfrak{h}'_p$ is Ad-invariant and thus also ad-invariant, since this is the infinitesimal representation associated to Ad. Therefore, it is invariant by $ad(G)$.

We will show that $\mathfrak{h}'_p = \mathfrak{h}_p$ by proving the dimensions are the same. Consider the distribution $\mathcal{D}_p = \mathcal{H}_p + \mathfrak{h}'_p$, where we are seeing the last space through the identification $\mathfrak{h}_p \cong Ker(d\pi_p)$. One can prove this distribution is smooth and involutive, and if $dim(M) = n$ and $dim(\mathfrak{h}'_p) = r$ then clearly $dim\mathcal{D} = n + r$.

Let P_0 be the maximal integral manifold associated to $\mathcal{D}$. Since we are working in the holonomy bundle, any point $q \in P$ can be joined to p by a horizontal curve, so that all tangent vectors of such curves are contained in $\mathcal{D}$. Therefore, the whole curve is contained in P_0, and thus $P \subset P_0$ i.e., $P_0 = P$.

Thus, $dim(\mathfrak{h}_p) = dim(P) - n = dim(P_0) - n = r = dim(\mathfrak{h}'_p)$.

□

From this result one can prove the converse to the reduction theorem:

Corollary 1.2.13. Let H be a connected Lie subgroup of G. If P restricts to a subbundle with structure group H, then there is a connection $\mathcal{H}$ on P such that $Hol(\mathcal{H}) = H$.

Therefore, the problem of constructing connections with prescribed holonomies is equivalent to the problem of constructing G-structures, and is highly dependent on the topological features of the bundle in question. However, the classification of *Riemannian* holonomy groups is tractable with purely geometric methods, as we will explain next.

1.2.1 Tangent Bundle and Riemannian Holonomy

We recall that we have proved earlier some results specific to tangent bundles and their tensor powers, theorems 1.2.4 and 1.2.6. From now on, we will restrict attention to connections on the tangent bundle, especially the Levi-Civita connection of a Riemannian manifold (M, g). From a connection ∇ on TM we get a connection on the frame bundle FM, which in turn induces connections on all tensor bundles on M through the various representations of $GL(n)$. In the case of the Levi-Civita connection this grup is reduced to $O(n)$, since the metric is a parallel tensor for this connection.

The curvature tensor of the Levi-Civita connection on a Riemannian manifold is often denoted by R^a_{bcd}, so we will stick to this index notation[5]. Here the last two indices correspond to the 2-form part of the curvature, and the first two to the Lie algebra $\mathfrak{so}(n)$. We make no distinction between them since $\mathfrak{so}(n)$ has a natural action on tangent spaces by anti-symmetric traceless matrices. We can also define $R_{abcd} = g_{ae}R^e_{bcd}$. From this one can prove the formulas:

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} \tag{1.2.2}$$

and also the Bianchi identities:

$$R_{abcd} + R_{adbc} + R_{acdb} = 0 \tag{1.2.3}$$

$$\nabla_a R_{bcde} + \nabla_d R_{bcea} + \nabla_e R_{bcad} = 0 \tag{1.2.4}$$

This last one is just the usual Bianchi identity in different notation.

The *Ricci curvature* of (M, g) is defined by $R_{ab} = R^c_{acb}$, and the *scalar curvature* is $r = g^{ab}R_{ab}$. We notice that the Ricci curvature is a symmetric tensor, $R_{\mu\nu} = R_{\nu\mu}$. We say that (M, g) is *Ricci-flat* if $R_{ab} = 0$, and it is *Einstein* if there is a constant α such that $R_{ab} = \alpha g_{ab}$.

Theorem 1.2.14. $R_{abcd} \in Sym^2\mathfrak{hol}(g) \subset \Lambda^2T^*M \otimes \Lambda^2T^*M$, the second symmetric power of $hol(g)$.

Proof. This is a simple consequence of 1.2.11 and 1.2.2. □

1.2.2 Holonomy of Symmetric Spaces

There is a special class of Riemannian manifolds which are constructed from Lie groups, and the classification of the possible holonomy groups for these manifolds follows from the classification of (representations of) Lie groups. These are the so-called *symmetric spaces*, which we now briefly introduce (for a complete survey, see [14]).

Definition 1.2.15. A Riemannian manifold (M, g) is called a *symmetric space* if for each $x \in M$ there is an involutive isometry $s_x : M \to M$ such that x is an isolated fixed point for s_x. The map s_x is called the *symmetry* at x.

It is easy to see that the symmetry reflects geodesics: $s_x(exp(v)) = exp_x(-v)$. Indeed, since $s_x^2 = Id$, it follows that $(ds_x)^2_x = Id$, and thus $(ds_x)_x = -Id$.

Every symmetric space is also complete as a Riemannian manifold: if exp is only defined on $(0, \epsilon)$ in a geodesic α, then by choosing a point in $(\frac{\epsilon}{2}, \epsilon)$ and using the reflection symmetry we can extend the geodesic beyond ϵ.

[5] We use greek indices to refer to components of tensors, and latin indices when we want to refer to the tensor itself. Also, whenever we use the old notation, we will use the letter R for the curvature of a Riemannian manifold, instead of F.

Furthermore, symmetric spaces are always homogeneous spaces. This is a consequence of the fact that the group generated by composition of symmetries[6] acts transitively on M, so that if we fix a point $p \in M$, we have $M = G/G_p$, where G_p is the isotropy subgroup at p.

Theorem 1.2.16. Suppose (M, g) is a symmetric space of the form G/G_p with $p \in M$. Then $\nabla R = 0$ and $Hol_p^0(g) = G_p$.

Note: Intuitively, this means that in a symmetric space the curvature does not vary.

Proof. If we fix $p \in M$, the isometry s_p preserves R (and therefore preserves ∇R) and is such that ds_p acts as $-Id$ in T_pM. Therefore, $\nabla R|_p = 0$, $\forall p \in M$, and then $\nabla R = 0$.

The second part follows from two basic facts concerning symmetric spaces (see [14]):

- (*Cartan decomposition*): We can decompose the Lie algebra of G as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with $\mathfrak{h}$ the Lie algebra of G_p and $\mathfrak{m} \cong T_pM$, and such that $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$.

- R acts in tangent vectors through the Lie bracket: $R(u, v, w) = [u, [v, w]]$.

We can then identify $\mathcal{R} := span\,\{R(v, w); w, w \in T_pM\}$ with $ad(\mathfrak{h})$. By the Ambrose-Singer theorem, $\mathcal{R} = \mathfrak{hol}_p(g)$. The groups G_p and $Hol_p^0(g)$ are both connected, so $Hol_p(g) = Ad(G_p)$, and since the adjoint representation is faithful, $Hol_p^0(g) \cong G_p$[7]. $\square$

Symmetric spaces can be classified using the Cartan decomposition of the Lie algebra of G and the representation theory of real Lie algebras, which is well-understood. Therefore, we can classify all isotropy representations, and thus the holonomy groups and representations themselves.

1.2.3 The Berger Classification

Now we would like to classify the holonomy groups of manifolds that are not symmetric spaces. A first fact to notice is that, if (M, g) is a *reducible* manifold (i.e., if it is isometric to a product manifold of the form $(M_1 \times M_2, g_1 \times g_2)$), then $Hol(M, g) \cong Hol(M_1, g_1) \times Hol(M_2, g_2)$. This means that reducible manifolds are such that the holonomy representation is reducible. There is a kind of converse to this result, known as *de Rham's decomposition theorem*:

Theorem 1.2.17 (de Rham)**.** Suppose (M, g) is a complete simply connected Riemannian manifold, and that the holonomy representation is reducible. Then there are manifolds (M_i, g_i), $i = 1 \dots k$ which are also complete and simply connected, such that $Hol(g_i)$ acts irreducibly on the tangent spaces, and $Hol(M, g) = Hol(M_1, g_1) \times \dots \times Hol(M_k, g_k)$. Moreover, (M, g) is isometric to $(M_1 \times \dots \times M_k, g_1 \times \dots \times g_k)$.

Proof. We will prove the statement about the holonomy group, but we will not prove that (M, g) is reducible - rather, we will argue why the metric should be *locally reducible*. For a complete proof, the reader is refered to section 6 of chapter IV in [20].

Suppose then that the holonomy representation is reducible, fix $p \in M$ and let $V_p \subset T_pM$ be a non-trivial subspace invariant by $Hol_p(g)$. We define a distribution on TM by parallel transport of V_p: for $q \in M$, define $V_q = P_\gamma(V_p)$, where γ is a curve from p to q. This clearly doesn't depend on the

[6] This is a Lie group, since it is a path-connected subgroup of the group of isometries of M, which is itself a Lie group due to the Myers-Steenrod theorem.

[7] Here it is understood that the adjoint action of G_p is the restriction of the adjoint action of G to $\mathfrak{m}$, since it comes from the identification $\mathcal{R} \cong \mathfrak{hol}_p(g)$ and $R(v, w)$ acts in elements of $T_pM \cong \mathfrak{m}$.

choice of curve (if α is another such curve, then $P_{\gamma^{-1}*\alpha}(V_p) = V_p \Rightarrow P_\alpha(V_p) = P_\gamma(V_p)$). Thus, we get a distribution $V \subset TM$.

Actually, we get *two* distributions, since the orthogonal complement $V_p^\perp$ is also invariant by $Hol_p(g)$; indeed, since parallel translation is an isometry, for $v \in V_p$ and $w \in V_p^\perp$ we have $0 = g_p(v,w) = g_p(P_\alpha(v), P_\alpha(w))$ for every loop α at p. Since V_p is invariant, $P_\alpha(v) \in V_p$ and thus $P_\alpha(w) \in V_p^\perp$. Therefore, we get two distributions V and $V^\perp$ such that $TM = V \oplus V^\perp$.

The important fact is that these distributions are *integrable*; in fact, if X, $Y \in \Gamma V$, then $\nabla_X Y \in \Gamma(V)$; this follows from the pointwise formula:

$$\nabla_X Y(x_0) = lim_{t\to 0}(P_{x(t)}Y(x(t)) - Y(x_0)) \tag{1.2.5}$$

for $\nabla_X Y$ in terms of parallel translation of Y along the integral curve $x(t)$ of X passing through x_0. Since at each time t the term $P_{x(t)}Y(x(t)) - Y(x_0)$ defines a field in V_{x_0} (rememeber that V_{x_0} is closed by parallel translation) it follows that in the limit we also get an element of $\Gamma(V_{x_0})$, since V_{x_0}, being a vector space, is closed.

The same is true of course for $\nabla_X Y$. It follows that $[X,Y] = \nabla_X Y - \nabla_Y X \in \Gamma(V)$ and hence V is integrable. The same argument with the obvious changes shows that $V^\perp$ is also integrable.

The crucial point now is that the maximal integral manifolds of these distributions define a local product structure on M: if $p \in M$, there is a neighborhood of p of the form $U \times U'$ such that $T(U\times U') \cong TU \times TU'$ and $TU = V|_{U\times U'}$, $TU' = V^\perp|_{U\times U'}$. This fact can be proved by using an appropriate coordinate system. One can also show that $g|_{U\times U'} \cong g_U \times g_{U'}$; the idea is to use the fact that the Levi-Civita connection is torsion-free and $\nabla g = 0$ to show that $g_V|_{U\times U'}$ is independent of the U' coordinates, and therefore is the pullback to $U \times U'$ of a metric g_U in U (and the same for $g_{V^\perp}$). For more details, see [20].

This argument shows that reducibility of the holonomy action implies local reducibility of the metric. We want to show now that $Hol_p^0(g) = H_V \times H_{V^\perp}$, with $H_V \subset SO(V_p)$ acting trivially in $V_p^\perp$, and $H_{V^\perp}$ acting trivially in V_p.

We will only sketch the argument: the idea is to use the symmetries of the Riemann tensor to give a decomposition of the holonomy algebra. In fact, one uses the Bianchi identities to see that R_{abcd} is a section of the bundle $Sym^2(\Lambda^2 V^*) \oplus Sym^2(\Lambda^2(v^\perp)^*) \subset Sym^2(\Lambda^2 T^*M)$, from which one can show that $span\,\{R_q(u,v); u,v \in T_qM\}$ breaks as $A_q \oplus B_q$ with $A_q \subset V_q \otimes V_q^*$ and $B_q \subset V_q^\perp \otimes (V_q^\perp)^*$. These subspaces are closed by parallel transport, since V_q and its duals/complements are. Thus:

$$\begin{aligned}\mathfrak{hol}_p(g) &= span\,\{P_{\gamma^{-1}}R_q(u,v)P_\gamma; u,v \in T_qM, q \in M, \gamma \text{ curve from } p \text{ to } q\} \\ &= \mathfrak{h}_V \otimes \mathfrak{h}_{V^\perp}\end{aligned}$$

with $\mathfrak{h}_V \subset V_p \otimes V_p^*$ and $\mathfrak{h}_{V^\perp} \subset V_p^\perp \otimes (V_p^\perp)^*$.

Therefore, we conclude that there are Lie groups H_V and $H_{V^\perp}$ with $Lie(H_V) = \mathfrak{h}_V$ and $Lie(H_{V^\perp}) = \mathfrak{h}_{V^\perp}$ such that $Hol_p^0(g) = H_V \times H_{V^\perp}$. In general, $Hol(g_U) \subset H_V$ and $Hol(g_{U'}) \subset H_{V^\perp}$, H_V acts trivially on $V^\perp$ and $H_{V^\perp}$ acts trivially on V.

In this way, by decomposing T_pM into irreducible pieces, we finally obtain $Hol_p^0(g) = H_1 \times \ldots \times H_k$, with each H_i being a connected Lie subgroup acting irreducibly.

□

Corollary 1.2.18. If (M,g) is an irreducible Riemannian manifold with $dim(M) = n$, then the holonomy representations on $\mathbb{R}^n$ are irreducible.

These results can be used, for instance, to show that if M is simply connected, then $Hol(g)$ is a closed connected Lie subgroup of $SO(n)$, and therefore compact (see [16]).

An important point is that the H_i's in the proof are not necessarily holonomy groups of smaller manifolds. In fact, they are constructed as groups generated by parallel transport along small laces, and the topological space constructed by gluing together the neighborhoods in which they are defined is not necessarily Hausdorff. Nevertheless, this is not a big problem for the purposes of classification, since the next theorem (the Berger classification) also works for the H_i's.

Theorem 1.2.19 (Berger Classification). Suppose (M^n, g) is an irreducible simply connected Riemannian manifold which is not a symmetric space. Then $Hol(g)$ must be one of the following:

- $SO(n)$
- $U(m)$ with $n = 2m$ (Kähler manifolds)
- $SU(m)$ with $n = 2m$ (Calabi-Yau manifolds)
- $Sp(k)$ with $n = 4k$ (Hyperkähler manifolds)
- $Sp(1)Sp(k)$ with $n = 4k$ (Quaternionic-Kähler manifolds)
- G_2 with $n = 7$ (G_2-manifolds)
- $Spin(7)$ with $n = 8$ ($Spin(7)$-manifolds)

Each of these geometries has its own special features, many of which will be addressed in the next chapter - although we will not have much to say about Hyperkähler and Quaternionic-Kähler manifolds in this work.

The Berger classification was proved by Berger through a case by case analysis. Roughly speaking, what happens is that the Ambrose-Singer theorem places a strong restriction on the holonomy group (the holonomy algebra must be big enough to accommodate the whole image of the curvature map) and at the same time, the space of such tensors are required to satisfy the Bianchi identities, so it shouldn't be too big. The groups on Berger's list are exactly the intermediate cases that can accommodate both requirements. Later, Simons proved that a holonomy group satisfying Berger's hypothesis must act transitively in S^{n-1}, explaining Berger's list - the list of Lie groups acting transitively on spheres was known at the time, and it contains Berger's list. A geometric proof of this result was recently given by Olmos [29] - his idea was to show that the normal bundle to an orbit of the holonomy action is mapped by the exponential map to a locally symmetric manifold. Then, if the action is not transitive on a sphere, he shows that one can generate a sufficient number of these exponentiated manifolds so that M itself is a locally symmetric space.

Suppose (M, g) is a spin manifold, as explained in appendix A.1. One of the many applications of Berger's theorem is to describe the space of parallel spinors of M. Indeed, as explained in the appendix, the Levi-Civita connection induces a natural connection ∇^S on the spinor bundle, and using theorem 1.2.4 one can use $Hol(\nabla^S)$ to study the space of parallel spinors, and in this way classify all possible holonomies of spin manifolds admitting parallel spinors. This is the content of *Wang's theorem*:

Theorem 1.2.20 (Wang). Suppose (M^n, g) is an orientable and irreducible simply-connected spin manifold, and let D denote the number of linearly independent parallel spinor fields on M. If n is even, we write $D = D_+ + D_-$ with $D_\pm$ being the dimension of the space of parallel spinors in $\Gamma(S^\pm)$. If $D \geq 1$, then one of the following holds[8]:

[8] Notice that this depends on a choice of orientation; reverting it would exchange the values of D_+ and D_-.

1. $Hol(g) = SU(2m)$ with $n = 4m$, and $D_+ = 2$, $D_- = 0$
2. $Hol(g) = SU(2m+1)$ with $n = 4m+2$, and $D_+ = 1$, $D_- = 1$
3. $Hol(g) = Sp(m)$ with $n = 4m$ and $D_+ = m+1$, $D_- = 0$
4. $Hol(g) = G_2$ with $D = 1$
5. $Hol(g) = Spin(7)$ with $D_+ = 1$, $D_- = 0$

It is interesting to notice that the allowed geometries are exactly those with Ricci-flat metrics. Although we won't prove Wang's theorem, we give a simple argument why a spin Riemannian manifold admitting a constant spinor should be Ricci-flat with holonomy strictly contained in $SO(n)$. Suppose $\psi \in \Gamma(S)$ is a parallel spinor, that is $\nabla^S \psi = 0$, and let the spin representation of the Clifford algebra $Cl(n)$ be given by the *gamma matrices* γ^a (these satisfy $\gamma^a\gamma^b + \gamma^b\gamma^a = -2Id$ for Riemannian manifolds). Ricci-flatness follows from the way the Riemann tensor acts on spinors:

$$\gamma^a[\nabla^S_a, \nabla^S_b]\psi = \gamma^a R_{ab}\psi \tag{1.2.6}$$

Since the first term vanishes and ψ is non-zero, $R_{ab} = 0$.[9]

Also, the holonomy representation on TM induces a representation on spin space, and a parallel spinor is invariant under this action (its isotropy is the entire holonomy group), so it defines a one-dimensional (trivial) representation. But the spin representation is not trivial, and it is irreducible if the holonomy is $SO(n)$. Thus, $Hol(g) \subset SO(n)$.

In Wang's theorem we assumed (M, g) to be spin and to possess at least one non-zero parallel spinor. The following result is a converse to Wang's theorem:

Theorem 1.2.21. Assume $n > 2$ and that the holonomy group H of (M, g) is simply connected (in particular, H can be any of the Ricci-flat holonomy groups). Then M is spin and the space of parallel spinors is non-empty.

Proof. Since H is a subgroup of $SO(n)$ (we are assuming M oriented), the immersion $i : H \hookrightarrow SO(n)$ extends to a homomorphism $\tilde{i} : \tilde{H} \to Spin(n)$ between the universal covers. Since H is simply-connected, $\tilde{H} = H$, and thus $p \circ \tilde{i} = i$, where $p : Spin(n) \to SO(n)$ is the two-fold covering map.

Now, the holonomy bundle $H(M)$ is a H-structure on M, in fact a reduction of the $SO(n)$-structure on the frame bundle $F(M)$. We can define another $SO(n)$-structure P on M by acting with $SO(n)$ on $H(M)$, $P = SO(n)H(M)$ (i.e., P is the restriction of $F(M)$ to $H(M)$). But this $SO(n)$-structure has a natural lift to a $Spin(n)$-structure $\tilde{P}$, given by the associated bundle construction $\tilde{P} = P \times_H Spin(n)$, where we are using the immersion $\tilde{i} : H \to Spin(n)$ to define an action of H on $Spin(n)$. □

1.2.4 The Holonomy action on Cohomology

This section depends on material from appendix A.3.

Suppose (M, g) is a Riemannian manifold and G a Lie group acting on the tangent spaces. This action induces a representation of G on the tensor algebra of M. In particular, we get representations

[9] Friedrich [10] (pag. 67) proves this in a different way: he first shows that a spin structure is equivalent to a trivialization of the $U(1)$ part of the associated $Spin^c$ structure, and then shows that in the presence of a parallel spinor the norm of the Ricci tensor squared equals the norm of the curvature of the $U(1)$ part of the $Spin^c$ bundle.

ρ^k of G in $\Omega^k(M)$. We can then break these representations into irreducible pieces to get $\Omega^k = \bigoplus_{i_k} \Omega^k_{i_k}$. More generally, if we have a G-structure P on M we get subbundles $\rho(P)$ of the frame bundle FM for every representation ρ of G. We are mostly interested in representations in the exterior algebra; now, it may happen that even if the representation of G on the tangent spaces is irreducible, the representations ρ^k might not be. Then there are irreducible representations ρ^k_i, $i \in I^k$, for I^k an index set, such that:

$$\Omega^k(M) = \bigoplus_{i \in I^k} \rho^k_i(P)$$

In this way, the bundle of k-forms breaks up as a direct sum of irreducible subbundles $\Omega^k_i := \rho^k_i(P)$. Of course, this is not a feature particular to differential forms - the same arguent makes it true for all the tensor bundles and for the spin bundles when M is spin.

Example 1.2.22. An example of this is the *Hodge star operator*, defined in appendix A.3. This is an isomorphism of vector bundles $* : \Omega^k(M) \to \Omega^{n-k}(M)$ for M a n-dimensional oriented Riemannian manifold. As is explained in the appendix, in the case $n = 4$ and $k = 2$, $*$ is an automorphism of $\Omega^2(M)$ that squares to $-Id$, so that its eigenvalues are ± 1. Therefore, the bundle of 2-forms splits as a direct sum of two eigenspaces which are in fact irreducible representations of $SO(n)$, each of which has dimension 6.

Now, the important point is the following:

Proposition 1.2.23. Let (M, g) be a compact Riemannian manifold, P a G-structure on M with $Hol(g) \subset G$, and $\Delta : \Omega^k(M) \to \Omega^k(M)$ the Laplace-Beltrami operator, as defined in A.3. Suppose that $\xi \in \Omega^k_i$. Then we also have $\Delta\xi \in \Omega^k_i$.

Proof. The proof is based on the following *Weitzenböck formula* for Δ:

$$\Delta\xi = \nabla^*\nabla\xi - 2S(\xi) \tag{1.2.7}$$

which is proved in proposition 4.10 of [30]. Here ∇^* is the adjoint of ∇ with respect to g and S is a certain tensor constructed solely with the curvature tensor R^a_{bcd} and the metric. From the fact that $R_{abcd} \in Sym^2(\mathfrak{hol}(g))$ (theorem 1.2.14) one can show that S actually preserves the splitting $\Omega^k(M) = \bigoplus \Omega^k_i$, since G itself does and $Hol(g) \subset G$. From this last fact it also follows that ∇ is compatible with P, so that it preserves the splitting. Thus, $\nabla^*\nabla\xi \in \Omega^k_i$. Therefore, $\Delta\xi \in \Omega^k_i$, i.e., Δ also preserves the splitting. □

The reason this is important is that as a consequence, the spaces of harmonic forms $\mathfrak{H}^k$ also breaks into irreducible representations of G, $\mathfrak{H}^k = \bigoplus \mathfrak{H}^k_i$. By the hodge theorem A.3.3, it follows that the de Rham cohomology groups break as:

$$H^k(M) = \bigoplus_{i \in I^k} H^k_i$$

i.e., into irreducible representations of G. In particular, when $G = Hol(g)$, we have the result that the de Rham cohomology of M breaks into irreducible representations of the holonomy group. We then get *refined Betti numbers*:

$$b^k_i = dim(H^k_i)$$

for $i \in I^k$ and $k = 1, \ldots, n$, which can carry information about the topology of M and the geometry of the holonomy bundle. This link with topology is important since one can then study the geometry of G-structures with the aid of algebraic topology, more specifically, by studying the cohomology of the manifold. For example, a reduction of the frame bundle for a group G might impose conditions on the refined Betti numbers; if these conditions are not met, then one knows that the manifold cannot admit such a G-structure.

We have proved in theorem 1.2.4 that there is a bijection between parallel tensor fields and tensors invariant by the holonomy action. Now, an invariant tensor is the same as a trivial representation. Therefore, the number of factors of trivial representations in the decomposition of the exterior algebra determines the number of independent parallel forms. Since $\nabla\xi = 0$ implies $d\xi = d^*\xi = 0$, it follows that such a parallel form is also harmonic. Therefore, to each trivial representation Ω_i^k the associated $\mathfrak{H}_i^k$ is a space of parallel forms, and each parallel form $\xi \in \mathfrak{H}_i^k$ defines a cohomology class $[\xi] \in H_i^k(M)$. Thus, each G-structure comes equipped with cohomology classes corresponding to the parallel tensors associated to the reduction to G.

The importance of this is that one often is interested in studying the *moduli space* $\mathcal{M}_G$ of G-structures on M, defined as the space of G-structures quotiented by the diffeomorphisms of M isotopic to the identity. For example, one might be interested in studying the moduli space of G_2 structures on a 7-manifold; then the cohomology classes of parallel tensors furnish maps $\mathcal{M}_G \to H_i^k(M)$ that can be interpreted as coordinate systems for submanifolds of $\mathcal{M}_G$, which can be used to exploit the local geometry of the moduli space.

1.2.5 Topology of compact Ricci-flat manifolds

We will show now how Ricci-flatness combined with reduced holonomy places strong restrictions on the topology of a *compact* Riemannian manifold M. Specifically, we aim to prove the following:

Theorem 1.2.24. If (M, g) is a compact Riemannian manifold with holonomy either $SU(n)$, $Sp(n)$, G_2 or $Spin(7)$, then $H^1(M) = \{0\}$ and $\pi_1(M)$ is finite.

For this, we will need the following Weitzenböck formula for a 1-form ω (which is a particular case of formula (1.2.7)):

$$\Delta\omega = \nabla^*\nabla\omega + R_{ab}g^{bc}\omega_c \tag{1.2.8}$$

Proof. We will only prove that $H^1(M) = \{0\}$, and will comment about the fundamental group at the end of this section. First of all, from the Weitzenbock formula for 1-forms 1.2.8 we have that if ω is harmonic, then:

$$\nabla^*\nabla\omega_a + R_{ab}g^{bc}\omega_c = 0 \tag{1.2.9}$$

We can then take the inner product with ω and integrate on M with respect to the Riemannian volume element to get:

$$||\nabla\omega||_{L^2}^2 + \int R_{ab}g^{bc}g^{ad}\omega_c\omega_d = 0 \tag{1.2.10}$$

It follows that, if the metric is Ricci-flat, then $\nabla\omega = 0$, i.e., *harmonic* 1*-forms are constant.* However, the constant forms are exactly those fixed by the holonomy group, by theorem 1.2.4. Since $SU(m)$, $Sp(k)$, G_2 or $Spin(7)$ do not fix any point in $\mathbb{R}^n \setminus \{0\}$ (where n is either $2m$, $4k$, 7 or 8, respectively), it follows that they also cannot fix 1-forms. Thus, there are no constant 1-forms, and therefore $\mathfrak{H}^1 = \{0\}$, i.e., there are no harmonic 1-forms.

Now, the Hodge theorem asserts that $H^k(M) \cong \mathfrak{H}^k$, and thus $H^1(M) = \{0\}$. □

Notice that the result doesn't hold for k-forms in general, due to the fact that the Weitzenbock formula for these forms has more terms involving the Ricci tensor.

Also, if we drop the assumption on the holonomy, we can only conclude that if $dim\mathfrak{H}^1 = l$, then we have l independent constant 1-forms, so that the holonomy representation on $\Lambda^1 T^*M$ is reducible and splits as $\mathbb{R}^k \oplus \mathbb{R}^{(n-k)}$, acting trivially on the first factor. Now, compactness of M assures that its universal cover $\tilde{M}$ carries a complete metric, and by theorem 1.2.17 it follows that $\tilde{M}$ is isometric to $\mathbb{R}^l \times N$. This result is known as the *Bochner theorem.*

The proof that $\pi_1(M)$ is finite relies on the following theorem of Riemannian goemetry (which we won't prove):

Theorem 1.2.25. If (M, g) is compact and Ricci-flat, then it admits a finite cover of the form $\mathbb{T}^k \times N$, where $\mathbb{T}^k$ is the flat torus and N is compact and simply-connected. If the Ricci tensor of M is positive definite, then $\pi_1(M)$ is finite.

1.3 Calibrated Geometry

In this section we develop the theory of *calibrated submanifolds* of Riemannian manifolds [13], which will be important in later chapters.

Let (M, g) be a Riemannian manifold and α a closed k-form on M. We will call α a *calibration* on M if $\forall x \in M$ and for every oriented k-plane $W \subset T_xM$ we have $\alpha_x|_V = c.dV_W$ for some $c \leq 1$. An immersed k-dimensional oriented submanifold N of M is said to be *calibrated* by α if $\alpha|_{T_xN} = dV_{T_xN}$ for every $x \in N$. We emphasize that the notion of a calibrated submanifold depends on a choice of orientation for the submanifold.

The main result about calibrated submanifolds is the following:

Proposition 1.3.1. If (M, g) is a Riemannian manifold and N is a compact submanifold calibrated by a k-form α, then N minimizes volume in its homology class.

Proof. The proof is quite simple. First of all, since α is closed, it defines a cohomology class $[\alpha] \in H^k(M)$. Also, N defines an element $[N] \in H_k(M)$. Then:

$$vol(N) = \int_N \alpha|_N = [\alpha].[N] \tag{1.3.1}$$

If $\tilde{N}$ is another k-dimensional compact submanifold in the same homology class, $[\tilde{N}] = [N]$, then:

$$vol(\tilde{N}) \geq \int_{\tilde{N}} \alpha|_{\tilde{N}} = [\alpha].[\tilde{N}] = vol(N) \tag{1.3.2}$$

where the inequality follows from the fact that α is a calibration.

□

We are mainly interested in calibrated submanifolds due to the following: we showed in theorem 1.2.4 that a reduction in the holonomy of a Riemannian manifold is essentially given by a parallel tensor. In some of the geometries in the Berger classification these tensors are closed k-forms, which are in fact calibrations. The reason is that these k-forms are defined in a very specific way: one takes an alternate

k-tensor ϕ_0 that is fixed by the standard action of the holonomy group on tangent spaces, and then extended it to a parallel, and hence closed k-form ϕ using theorem 1.2.4. However, the ϕ_0 can be rescaled if necessary so that for every oriented k-plane $V \subset \mathbb{R}^n$, $\phi_0|_V \leq vol_V$ and so that there is at least one k-plane U such that $\phi_U = vol_U$. Therefore, the associated k-form ϕ will be a calibration on the manifold.

In what follows we will come across some important examples of calibrations; these include the real and imaginary parts of the holomorphic volume form of a Calabi-Yau manifold, the G_2-structure and its Hodge dual on a G_2-manifold and the $Spin(7)$-structure on a $Spin(7)$-manifold (c.f. sections 2.3.1, 2.4 and 2.5).

Chapter 2

Special Geometries

2.1 Complex Geometry

Definition 2.1.1. Let M be a $2n$-dimensional smooth manifold with coordinate maps $\phi_\alpha : M \to \mathbb{R}^{2n} \cong \mathbb{C}^n$. We say that M is a *complex* manifold if the transition maps $\phi_\alpha \circ \phi_\beta^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ are holomorphic.

Definition 2.1.2. Let M be an even dimensional real manifold. An *almost complex structure* on M is a smooth tensor field J^a_b such that $J^2 = -Id$ (acting on vector fields)

Now, define the *Nijenhuis tensor* associated to J by:

$$N_J(v,w) = [v,w] + J([Jv,w] + [v,Jw]) - [Jv,Jw] \tag{2.1.1}$$

(one can easily verify this is indeed a tensor by checking it is C^∞-linear).

We call J a *complex structure* if $N_J = 0$, and in this case (M, J) is called a *complex manifold.* The reason for this terminology is the following theorem:

Theorem 2.1.3 (Newlander-Nirenberg)**.** A necessary and sufficient condition for a real manifold M with an almost complex structure J to admit a holomorphic atlas is that $N_J = 0$

This theorem is very difficult to prove. We refer the reader to page 14 of [28] for an exposition of a proof due to Malgrange.

One can also formulate an equivalent definition of a complex manifold in terms of G-structures: a $2k$-dimensional manifold M is a complex manifold if it admits a torsion-free $GL(k,\mathbb{C})$-structure.

Example 2.1.4. The n-dimensional complex projective space $\mathbb{CP}^n$ is defined as the set of all lines through the origin of $\mathbb{C}^{n+1}$. We write $[z_0,\ldots,z_n]$ for the line passing through the point $(z_0,\ldots,z_n)$. This space is a complex manifold; indeed, we can define charts $U_j = \{[z_0,\ldots,z_n]; z_j \neq 0\}$ (these are clearly open in the quotient topology). Now, every element in $\mathbb{CP}^n$ can be written in the form $[z_0,\ldots,z_{j-1},1,\ldots,z_n]$. For every $j = 0,\ldots,n$ we define maps:

$$\psi_j([z_0,\ldots,z_{j-1},1,\ldots,z_n]) = (z_0,\ldots,z_{j-1},z_{j+1},\ldots,z_n) \tag{2.1.2}$$

These maps are diffeomorphisms and the transition maps are holomorphic. Thus, $\mathbb{CP}^n$ is complex.

We say that a map $f : M \to N$ between complex manifolds is *holomorphic* if $J_N(d_pf(v)) = d_pf(J_M(v))$ holds for every $p \in M$ and $v \in T_pM$. The map is called a *biholomorphism* if it is invertible and the inverse map is also holomorphic. Also, a submanifold $L \subset M$ is a *complex submanifold* if $J_M(T_pL) = T_pL$ for every $p \in L$. It is clear that the restriction of a complex structure to a complex submanifold endows it with a complex structure.

Now, a complex structure J gives a way of decomposing tensors into irreducible representations of $GL(n, \mathbb{C})$. Let us describe how this works: for every $p \in M$ the map $J_p : T_pM \to T_pM$ satisfies $J_p^2 = -Id_{T_pM}$. We can extend this map to the complexified tangent space $T_pM \otimes_{\mathbb{R}} \mathbb{C}$ by requiring linearity in the $\mathbb{C}$ factor. The eigenvalues of this maps are $\pm i$, and we define $T_p^{(1,0)}M$ to be the eigenspace with eigenvalue i, and $T_p^{(0,1)}M$ to be the eigenspace with eigenvalue $-i$. Then $T_pM \otimes_{\mathbb{R}} \mathbb{C} = T_p^{(1,0)}M \oplus T_p^{(0,1)}M$.

The same trick holds for all the tensor bundles on M, by taking the induced actions of J in the relevant spaces. In this way, any tensor can be written in the form $T = T_1 + T_2$ with T_1 in the i-eigenspace and T_2 in the $-i$-eigenspace.

We can also write the k-forms as

$$\Lambda^k T^* M = \bigoplus_{j=0}^{k} \Lambda^j T^{*(1,0)} M \otimes \Lambda^{k-j} T^{*(0,1)} M$$

Elements of $\Lambda^p T^{*(1,0)} M \otimes \Lambda^q T^{*(0,1)} M$ are called (p, q)-forms. Of particular interest to us are the $(p, 0)$-forms: these are p-forms living in the eigenspace $\Lambda^q T^{*(1,0)} M$ associated to the eigenvalue i of J.

A nice consequence of this decomposition is that it allows us to write the exterior derivative as $d = \partial + \overline{\partial}$, where ∂ maps (p, q)-forms to $(p + 1, q)$-forms, and $\overline{\partial}$ maps (p, q)-forms to $(p, q + 1)$-forms. These operators satisfy $\partial^2 = 0$ and $\overline{\partial}^2 = 0$, so that we can define cohomology groups for them; in particular, the groups $H_{\overline{\partial}}^{(p,q)}(M)$ are known as the *Dolbeault cohomology groups* of M; they depend on the complex structure J on M.

Given a vector bundle $E \to M$, we will usually write $\Omega^{(p,q)}(M; E)$ as short-hand notation for the space of sections of the bundle $\Lambda^{(p,q)} T^* M \otimes E$.

We now proceed to study vector bundles on complex manifolds.

Definition 2.1.5. Let (M, J) be a complex manifold. We say that $E \xrightarrow{\pi} M$ is a *holomorphic vector bundle* of rank k if the following conditions are satisfied:

1. E is a complex manifold, and π is a holomorphic map.

2. There is an open covering $\{U_\alpha\}$ of M and biholomorphic maps $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ such that for each $p \in U_\alpha$, $\pi^{-1}(p, \mathbb{C}^k)$ is isomorphic to $\mathbb{C}^k$. This space is called the fiber over p.

Example 2.1.6. There is an important example of a holomorphic vector bundle over complex projective space $\mathbb{CP}^n$, the *tautological line bundle*, denoted by $\mathcal{O}(-1)$ and defined as follows: it is the subset of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ given by

$$\mathcal{O}(-1) = \{([w], z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1}; z \in [w]\}$$

The projection $\pi : \mathcal{O}(-1) \to \mathbb{CP}^n$ is simply projection into the first factor. The trivializing cover for this bundle is taken to be the standard open cover of $\mathbb{CP}^n$ given by the sets $U_j = \{[z_0, \ldots, z_n]; z_j \neq 0\}$. The local trvializations $\psi_j : \pi{-}1(U_j) \to U_j \times \mathbb{C}$ are defined by $\psi_j([w], z) = ([w], z_j)$, where $z = (z_0, \ldots, z_n)$. Their inverses are given by $\psi_j^{-1}([w], z) = ([w], \frac{z}{w_i} w)$, where w_i is the unique complex number such that w satisfies $[w] = [\frac{w_0}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, 1, \ldots, \frac{w_n}{w_i}]$. These maps are both holomorphic, so they define a structure of holomorphic vector bundle on $\mathcal{O}(-1)$.

Let $E \xrightarrow{\pi} M$ be a holomorphic vector bundle. A section $s : M \to E$ is called *holomorphic* if it is holomorphic as a map between complex manifolds.

In the same spirit as for real manifolds, one can define many operations on holomorphic vector bundles: if E and E' are holomorphic, the vector bundles E^* and $E \otimes E'$ have natural holomorphic structures. For a complex manifold M, the tangent and cotangent bundles also have natural holomorphic structures, and it follows that the tensor bundles are all holomorphic. However, the bundle of forms $\Omega^{(p,q)}(M)$ is *not* holomorphic in general (although it is complex, i.e., the fibers are complex manifolds); in fact, only the bundle of $(p,0)$-forms $\Omega^{(p,0)}(M)$ is holomorphic. Holomorphic sections of $\Omega^{(p,0)}(M)$ are called *holomorphic p-forms*. The Dolbeault differential $\bar{\partial} : \Omega^{(p,q)}(M;E) \to \Omega^{(p,q+1)}(M;E)$ can be considered a natural exterior derivative operator in this context, since one can prove that a section $s \in \Omega^{(p,0)}(M;E)$ is holomorphic if and only if $\bar{\partial}s = 0$. Also, since $\bar{\partial}^2 = 0$, clearly the space of holomorphic $(p,0)$-forms is the same as the Dolbeault cohomology group $H^{(p,0)}_{\bar{\partial}}(M)$.

Every complex vector bundle is oriented as a real vector bundle. In particular, every complex manifold is oriented. This is a consequence of the fact that the group $GL(n,\mathbb{C})$ is connected; indeed, a complex basis $\{v_1 = u_1 + iw_1, ..., v_n = u_n + iw_n\}$ for a complex vector space determines a ordered real basis $\{u_1, iw_1, ..., u_n, iw_n\}$ for the underlying real vector space. Once a complex basis is chosen, one can pass to any other basis by a continuous deformation, which can't change the orientation. So the orientation does not depend on the choice of complex basis, and this generalizes to any complex bundle since every fiber has a canonical orientation.

2.2 Kähler Geometry

Definition 2.2.1. Let (X, J) be a complex manifold. A Riemannian metric g on X is called *Hermitian* if it satisfies $g(Jv, Jw) = g(v, w)$ for all vector fields v, w in M. A complex Riemannian manifold whose metric is Hermitian is called a *Hermitian* manifold.

The *Hermitian 2-form* is defined by $\omega(v, w) = g(Jv, w)$ for vector fields v, w. We will say that a Hermitian manifold (X, J, g) is a *Kähler manifold* if $d\omega = 0$, and we call ω the Kähler form.

Any complex manifold admits a Hermitian structure, by the usual partition of unity argument for constructing Riemannian metrics on real manifolds. However, the same result is not true for Kähler structures.

Notice that $\omega \in \Omega^{(1,1)}(X)$; indeed, for $v, w \in \Gamma(TX)$

$$\begin{aligned} \omega(Jv, Jw) &= g(-v, Jw) = -g(Jw, v) \\ &= -\omega(w, v) = \omega(v, w) \end{aligned}$$

But $\omega_{ab} J^a_c J^b_d = -\omega_{cd}$ for forms of type $(0,2)$ and $(2,0)$; thus, ω is a $(1,1)$-form.

From a Hermitian metric on a complex manifold one can define the adjoint operators ∂^*, $\bar{\partial}^*$ and the Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$. These operators satisfy $(\partial^*)^2 = 0$ and $(\bar{\partial}^*)^2 = 0$ since the same identities hold for their adjoints. There are also analogues of Hodge's theorems for complex forms in the context of Kähler geometry. The reader is referred to appendix $A.3.2$ for more on these issues.

The following fundamental result establishes when a given almost complex manifold equipped with a Hermitian metric is actually Kähler:

Proposition 2.2.2. Let (X, J, g) be an almost complex manifold of dimension $2m$ with Hermitian metric. Let ω be the associated Hermitian form, and let ∇ be the Levi-Civita connection. The following conditions are equivalent:

1. $\nabla J = 0$
2. $\nabla \omega = 0$
3. $Hol(g) \subset U(m)$
4. (X, J) is complex and $d\omega = 0$

Proof. The implication (1) $\Leftrightarrow$ (2) follows from the formula $\omega_{ab} = g_{cb} J^c_a$. Also, (1) $\Leftrightarrow$ (3) by theorem 1.2.4. We have that (1) $\Rightarrow$ (4) because if $\nabla J = 0$ then $N_J = 0$, and since (1) $\Rightarrow$ (2), we have that $\nabla \omega = 0$ and hence $d\omega = 0$. Finally, to prove (4) $\Rightarrow$ (1) first notice that $0 = \nabla_X(-Id) = \nabla_X J^2 = J\nabla_X J + (\nabla_X J)J$ for $X \in \Gamma(TX)$. Now define a tensor T by:

$$T(X, Y, Z) = g((\nabla_X J)Y, Z)$$

Thus, $T(X, Y, JZ) = T(X, JY, Z)$. A computation shows that the condition $N_J = 0$ is equivalent to:

$$(\nabla_{JX} J)Y = J(\nabla_X J)Y$$

for all $X, Y \in \Gamma(TX)$. From this we get that $T(X, Y, JZ) + T(JX, Y, Z) = 0$, and therefore $T(X, JY, Z) + T(JX, Y, Z) = 0$.

Now, from the condition $d\omega = 0$ applies to (X, JY, Z) and then to (X, Y, JZ) we get the identities:

$$T(X, JY, Z) + T(JY, Z, X) + T(Z, X, JY) = 0$$

$$T(X, Y, JZ) + T(Y, JZ, X) + T(JZ, X, Y) = 0$$

By summing these two formulas and using the previously proved properties one concludes that $T(X, Y, JZ) = 0$, which implies that $\nabla J = 0$. $\square$

Example 2.2.3. Complex projective space $\mathbb{CP}^n$ can be given a Kähler structure as follows: we view $\mathbb{CP}^n \cong S^{2n-1}/S^1$, and endow S^{2n-1} with the round metric. This is invariant by the action of S^1 through rotations, so that we get a metric in the quotient, called the *Fubini-Study* metric. It is in fact a Kähler metric, as can be proven by a computation in local coordinates. In homogeneous coordinates $[z_0, ..., z_n]$ the Kähler form can be written as $\omega = i\partial\overline{\partial}(\log |z|^2)$.

In this final part of this section we will summarize a few facts about Kähler manifolds. The reader interested in more informaton on this material should consult sections 4.4, 4.5 and 4.6 of [16].

A complex submanifold $Y \subset X$ of a Kähler manifold (X, g, J, ω) inherits the Kähler structure from X. Indeed, the restriction $g|_N$ is again a Riemannian metric, and the tangent spaces T_pY are invariant by the complex structure J, so the restriction $J|_Y$ is a complex structure for Y such that $g|_Y$ and $J|_Y$ are compatible. By definition, the Hermitian 2-form ω_N is given by $\omega_Y(v, w) = g|_Y(J|_Y v, w) = g(Jv, w)|_Y = \omega|_Y(v, w)$ for $v, w \in \Gamma(TY)$. Thus, $d_Y\omega_Y = d_Y\omega|_Y = (d_X\omega)|_Y = 0$.

A direct consequence is that the restriction of the Fubini-Study metric to complex submanifolds of $\mathbb{CP}^n$ endows these submanifolds with a Kähler structure. Thus, all projective manifolds are Kähler.

A useful result in complex geometry is a holomorphic version of the Poincaré lemma; it says that every closed real $(1, 1)$-form α can be written locally as $\alpha = i\partial\overline{\partial}f$ for a real function f. When α is the Kähler form for a Kähler metric on the manifold, we call f a *Kähler potential*. However, the result is not true globally in general, since there are topological obstructions. For a compact Kähler manifold, the Kähler form ω gives rise to a volume form ω^n, so that the cohomology class $[\omega] \neq 0$; on the other hand, $\partial\overline{\partial}f$ is always exact, so the result is never true globally. Nevertheless, it *is* true that *exact* real $(1,1)$-forms on compact Kähler manifolds are globally of the form $\partial\overline{\partial}f$. In particular, it follows that the Kähler potentials parametrize Kähler metrics inside a given Kähler cohomology class, which is very useful for studying variations of Kähler structures on a complex manifold.

The Kähler condition imposes many remarkable restrictions in the curvature. For example, since $Hol(g) \subset U(n)$, it follows by theorem 1.2.14 and symmetries of the Riemann tensor that the curvature is completely determined by a single component $R^{\alpha}_{\beta\gamma\overline{\delta}}$. This also imposes restrictions on the Ricci tensor - it is a real $(1,1)$-tensor, and therefore a Hermitian metric. The Hermitian form associated to it is called the *Ricci form* and is denoted by ρ. The interesting fact is that one can show (by doing a computation in local holomorphic coordinates) that ρ is actually a closed form, so it determines an element $[\rho]$ in the de Rham cohomology of X. We will say more about $[\rho]$ in the next section.

2.3 Calabi-Yau Geometry

In theorem 1.2.19 we called a manifold (X,g) *Calabi-Yau* if $Hol(g) \subset SU(n)$. In particular, Calabi-Yau manifolds are Kähler. It is not obvious at first that Calabi-Yau manifolds should exist - this result is the *Calabi-Yau theorem*, which we will address shortly.

First, let us discuss a few properties these manifolds should have. Suppose X is a Calabi-Yau n-fold with Kähler form ω and Kähler metric g. We can introduce complex coordinates $z_1, ..., z_n$ in X such that we have, in a point $p \in X$:

$$g_p = dz_1^2 + ... + dz_n^2 \tag{2.3.1}$$

and

$$\omega_p = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + ... + dz_n \wedge d\overline{z}_n) \tag{2.3.2}$$

Now, these forms are preserved by the holonomy group $SU(n)$. Therefore, there are constant tensors in M that restrict to g_p and ω_p at p; these are, of course, just the Kähler metric and Kähler form. However, there is another tensor that is preserved by $SU(n)$; define:

$$\Theta_p = dz_1 \wedge ... \wedge dz_n \tag{2.3.3}$$

From this we obtain a tensor Θ in X satisfying $\nabla\Theta = 0$. This is the *holomorphic volume form* (it is clear that $\overline{\partial}\Theta = 0$). Notice that Θ and ω are related by:

$$\omega^n = k\Theta \wedge \overline{\Theta} \tag{2.3.4}$$

where k is a non-zero constant. Thus, since ω^n is a volume form for the real manifold X (remember that ω is non-degenerate) we obtain that Θ is a nowhere-vanishing global section of the *canonical line bundle* $K_X := \Omega^{3,0}(X)$. Therefore, Calabi-Yau manifolds have trivial canonical line bundles, and in particular, $c_1(X) = 0$ [1]. There is a sort of converse to this result: if (X,g) is a compact Kähler manifold with a Ricci-flat metric and trivial canonical bundle, then $Hol(g) \subset SU(n)$. This is a consequence of theorem 2.3.1 below, applied to $(n,0)$-forms where n is the complex dimension of X.

We mentioned in section 1.2 that Calabi-Yau manifolds are always Ricci-flat. It is also true that a Ricci-flat Kähler manifold X has holonomy inside $SU(n)$. This is due to the fact that the Ricci form ρ is

[1] See appendix A.2 for the definition of the first Chern class $c_1(X)$.

actually the curvature form of the connection ∇' induced on K_X, and it vanishes if and only if $Hol^0(\nabla')$ is trivial. However, $Hol^0(\nabla') = \det(Hol^0(\nabla))$ (since $U(n)$ acts on volume forms through determinants). Therefore, $\rho = 0 \Leftrightarrow Hol^0(\nabla') = \{e\} \Leftrightarrow Hol^0(\nabla) \subset SU(n)$. Since X is Ricci flat if and only if $\rho = 0$, this proves the assertion.

The reason why these manifolds are called Calabi-Yau is the following: suppose (X, g, ω) is a compact Kähler manifold. One might consider the question of under which conditions M admits a Kähler form for which the associated metric is Ricci-flat?

Calabi conjectured that if $c_1(X) = 0$, then in each Kähler cohomology class there is a metric of zero Ricci curvature. More generally, suppose that one is given a compact Kähler manifold (X, ω) with Kahler metric g and Ricci form ρ, and that $\overline{\rho} \in \Omega^{1,1}(X)$ is a closed form such that $[\overline{\rho}] = 2\pi c_1(X)$. Then the *Calabi Conjecture* says there is a (unique) Kähler form $\overline{\omega}$ in the same homology class of ω such that the Ricci form of the Kähler metric of $\overline{\omega}$ is $\overline{\rho}$. In particular, if $c_1(X) = 0$, we can "deform" our original Kähler metric to obtain a Ricci-flat Kähler manifold. In particular, the space of Ricci-flat Kähler metrics on X has dimension equal to the Hodge number $h^{1,1}(X)$.[2]

The Calabi conjecture was proved by Yau and is nowadays called the *Calabi-Yau* theorem. The idea of the proof consists in rewriting the conjecture as a statement about existence of solutions to a non-linear PDE, and then use the continuity method to prove a solution exists. For a detailed account of the proof, see chapter 5 of [17].

Theorem 1.2.25 shows that the fundamental group of Calabi-Yau manifolds is always finite, and the homology group $H^1(X)$ is always trivial. We also have the following result:

Proposition 2.3.1. If α is a closed form in a compact Ricci-flat Kähler manifold (X, g, J, ω), then α is also parallel. Hence $H^{p,0}(X)$ parametrizes parallel $(p, 0)$-forms on X.

Proof. We only sketch the proof. Since $d\alpha = 0$ we have that $\bar{\partial}\alpha = 0$ and due to α being of type $(p, 0)$ we also have $\bar{\partial}^*\alpha = 0$. It follows that $\Delta_{\bar{\partial}}\alpha = 0$, i.e., α is $\bar{\partial}$-harmonic. We have the Weitzenböck formula (1.2.7) for forms:

$$\Delta_d \alpha = \nabla^*\nabla\alpha - 2S(\alpha)$$

Joyce [16] shows (proposition 6.2.4) that due to Ricci-flatness and the way the curvature tensor of Kähler metrics decomposes that in fact $S(\alpha) = 0$ for $(p, 0)$-forms. Now, as stated on the appendix, on Kähler manifolds we have the Kähler identity (A.3.9). Therefore, α is also Δ_d-harmonic. It follows that $\nabla^*\nabla\alpha = 0$. But then:

$$0 = \int_X g(\nabla^*\nabla\alpha, \alpha) = \int_X g(\nabla\alpha, \nabla\alpha) = ||\nabla\alpha||^2_{L^2}$$

since X is compact. Thus, $\nabla\alpha = 0$. □

Calabi-Yau manifolds have many other interesting properties. For example, their deformation theory (in the sense of Kodaira-Spencer) is unobstructed: the celebrated *Bogomolov-Tian-Todorov theorem* states that the local moduli space of deformations of the complex structure of a compact Kähler m-fold X is a complex manifold of (complex) dimension $h^{m-1,1}(X)$ and that each point defines a complex structure in which X is also Kähler. As a consequence the local moduli space of deformations of the Calabi-Yau structure of a Calabi-Yau manifold is a real manifold of dimension $h^{1,1}(X) + 2h^{m-1,1}(X)$. These result depend on an equivalence between the sheaf cohomology and the Dolbeault cohomology of M which ultimately comes from the existence of a holomorphic volume form (see section 6.8 of [16]).

[2]For the definition of Hodge numbers, see appendix A.3.2.

However, a complete description of this theory would take us too far from our purposes, so in what follows we will give a brief exposition of the theory of *special Lagrangian submanifolds* and will sketch a few facts about the simplest family of non-trivial Calabi-Yau manifolds, the $K3$ surfaces.

2.3.1 Special Lagrangean Geometry

As we explained, a Calabi-Yau n-fold X comes equipped with a holomorphic volume form Θ. An important feature is that both $Re(\Theta)$ and $Im(\Theta)$ are calibrations, in the sense of section 1.3.

Definition 2.3.2. A real n-dimensional submanifold L of X that is calibrated by $Re(\Theta)$ is called a *special Lagrangean submanifold* (or *sLag submanifold* for short).

In general it is possible to have submanifolds calibrated by $aRe(\Theta) + bIm(\Theta)$ for any real a, b such that $a^2 + b^2 = 1$. Nevertheless, we will only work with submanifolds calibrated by $Re(\Theta)$.

An alternative characterization of sLag submanifolds was given by Harvey and Lawson in their foundational paper [13]:

Proposition 2.3.3. Let $(X, g, J, \omega, \Theta)$ be a Calabi-Yau n-fold and $M \subset X$ a real n-dimensional submanifold. Then there exists an orientation of M making it into a sLag submanifold if and only if $\omega|_M = 0$ and $Im(\Theta)|_M = 0$

In particular, special Lagrangean manifolds are Lagrangean with respect to the symplectic structure given by ω.

It follows from this characterization that, in order for a submanifold M to be special Lagrangean, one must have $[\omega|_M] = [Im(\Theta)|_M] = 0$ in the de Rham cohomology. Now these are topological invariants, and are therefore invariant under a continuous deformation of the embedding map $f : M \to X$. Thus these cohomology classes are obstructions for submanifolds isotopic to M to be special Lagrangean too.

We would like to sketch a proof of one of the most famous results in the theory of sLag manifolds. It is a theorem of McLean [25] that shows that the deformation theory of sLag manifolds is unobstructed:

Theorem 2.3.4. Let $(X, g, J, \omega, \Theta)$ be a Calabi-Yau n-fold and M a compact special Lagrangean submanifold of X. Then the moduli space $\mathcal{M}_M$ of deformations of M by special Lagrangean manifolds is a smooth manifold of dimension $b^1(M)$.[3].

Proof. We will only give a rough idea of how the proof works, following [16]. For a complete account, the reader should consult [25].

The idea of the proof is to identify submanifolds close to M to certain 1-forms over M, and then study which 1-forms are associated to sLag submanifolds.

First of all, since M is an embedded submanifold, we can write $TX|_M = TM \oplus NM$, where NM is the normal bundle to M in X. We have isomorphisms $NM \cong TM \cong T^*M$. The first one is given as follows: since M is Lagrangean, $\omega|_M = 0$ and since $\omega_{ac} = J_a^b g_{bc}$, it follows that $J|_M = 0$ (since the metric is nondegenerate). Therefore J maps NM injectively into TM. The second isomorphism is of course given by the metric itself.

We take a tubular neighborhood U of M and try to understand submanifolds lying inside U. But U can be naturally identified with a neighborhood of the zero section in NM, and therefore a neighborhood of T^*M. In this way we identify submanifolds close to M with small 1-forms on M.

[3] Here $b_1(M)$ is the first Betti number of M.

Now, if one of these submanifolds N is special Lagrangean, then $\omega|_N = Im(\Theta)|_N = 0$. Suppose that N is associated to a 1-form α. The projection $\pi : U \to M$ restricts to a diffeomorphism $\pi|_N : N \to M$, which we can use to pushforward the forms $\omega|_N$ and $Im\Theta|_N$. McLean shows that the result is:

$$\begin{aligned} \pi_*(\omega|_N) &= d\alpha \\ \pi_*(Im(\Theta)|_N) &= f(\alpha, \nabla\alpha) \end{aligned} \tag{2.3.5}$$

for f a certain non-linear function, that nevertheless satisfies $f(\alpha, \nabla\alpha) \approx d(*\alpha)$ for forms of small norm.

Now, forms satisfying $d\alpha = d(*\alpha) = 0$ are the *harmonic* 1-forms, $\mathcal{H}^1(M)$. So, what we have shown is that $\mathcal{M}_M$ is locally isomorphic to $\mathcal{H}^1(M)$. By Hodge theory this space is isomorphic to the de Rham cohomology group $H^1(M)$. Thus, $\dim \mathcal{M}_M = b^1(M)$. □

2.3.2 $K3$ Surfaces

An important class of examples of Calabi-Yau surfaces is provided by the $K3$ surfaces. These are defined to be compact manifolds X of complex dimension 2 with trivial canonical bundle and $h^{1,0}(X) = 0$. $K3$ surfaces have holonomy $SU(2)$, as we will show in a moment. Since $SU(2) \cong Sp(1)$, $K3$ surfaces are also hyperkähler[4]. There are two main methods for constructing $K3$ surfaces: the first is through the methods of algebraic geometry, by constructing projective algebraic varieties which are shown to have trivial canonical bundles. For example, the *Fermat quartic* $Q = \{[z_1, ..., z_4] \in \mathbb{CP}^3; z_1^4 + ... + z_4^4 = 0\}$ can be shown to be a $K3$ surface.

The second method consists in resolving singularities of quotients of flat spaces to enhance the holonomy. For instance, in the *Kummer construction* one starts with a complex torus $\mathbb{T}^4 = \mathbb{C}^2/\mathbb{Z}^4$. Coordinates z_1, z_2 in $\mathbb{C}^2$ induce coordinates $\bar{z}_1$, $\bar{z}_2$ in $\mathbb{T}^4$. We define an involution

$$\begin{aligned} f : \mathbb{T}^4 &\to \mathbb{T}^4 \\ (\bar{z}_1, \bar{z}_2) &\mapsto (-\bar{z}_1, -\bar{z}_2) \end{aligned} \tag{2.3.6}$$

which can be seen as an action of $\mathbb{Z}^2$ on $\mathbb{T}^4$. This map fixes the 16 points $(a \bar{+} bi, c \bar{+} di)$ where a, b, c, d are either 1 or 0. Thus the quotient $\mathbb{T}^4/\mathbb{Z}^2$ has 16 singular points. The blowup of this space along the singular points replaces every point by a copy of $\mathbb{CP}^1$, and it can be shown that the resulting space is a $K3$, called the Kummer surface, which is not algebraic. For more details on both constructions, see section 7.3 of [16].

An important fact is that all $K3$ surfaces are diffeomorphic - they only differ in their complex structures. Thus a major concern in the study of $K3$ surfaces is to understand the moduli space of complex structures. A corollary is that all $K3$ surfaces are simply connected, since the Fermat quartic is. Also, all $K3$ surfaces are Kähler manifolds. Both these results are difficult to prove - they rely on the Kodaira-Spencer theory of deformations of complex structures, which is developed in Kodaira's book [21].

Let us prove that $K3$ surfaces have indeed holonomy $SU(2)$:

Proposition 2.3.5. Let (X, J, ω) be a $K3$ surface. Then each Kähler class has a unique metric of holonomy $SU(2)$. Conversely, a compact Riemannian 4-manifold (X, g) with holonomy $SU(2)$ admits a unique complex structure J satisfying $\nabla J = 0$ and making (X, J) a $K3$ surface.

[4] Joyce [16] points out that the geometry of $K3$ surfaces has much more in common with hyperkähler rather than with Calabi-Yau geometry.

Proof. The first assertion follows of course from the Calabi-Yau theorem: since $h^{1,0}(X) = 0$, it follows from the definition of the Chern classes (c.f. appendix A.2) that $c_1(X) = 0$ and therefore there is a unique Ricci-flat metric with $Hol^0(g) \subset SU(2)$ in each Kähler class. Now, X is simply connected, so $Hol(g) = Hol^0(g)$. The only possibilities are that $Hol(g)$ is either $SU(2)$ or trivial, due to the Berger classification theorem 1.2.19. Since X is compact, we must have $Hol(g) = SU(2)$.

Let us prove now the converse. The result is clearly true when $X = \mathbb{C}^2$, since $SU(2)$ preserves the standard complex structure J_0 and the holomorphic volume form $dz_1 \wedge dz_2$ of $\mathbb{C}^2$. Since $\mathbb{C}^2$ can be identified with the tangent spaces of X, by theorem 1.2.4 there is a complex structure J and a $(2,0)$-form Θ on X satisfying $\nabla J = 0$ and $\nabla\Theta = 0$. So X is complex and has trivial canonical bundle. It also follows by theorem 1.2.24 that $\pi_1(X)$ is finite, so that $h^{1,0}(X) = 0$, i.e., (X, J) is a $K3$ surface. □

2.4 G_2 Geometry

The Lie algebra $\mathfrak{g}_2$ is one of the exceptional simple Lie algebras that appear in the classification by Dynkin diagrams. There is up to isomorphism only one simply connected Lie group with this Lie algebra, the Lie group G_2. We have that $dim(G_2) = 14$. The group G_2 is one of the Lie groups in Berger's list, and as follows from previous discussions (see Wang's theorem 1.2.20), manifolds with holonomy exactly G_2 are always Ricci-flat, spin, and have finite fundamental group.

Let $GL(\mathbb{R}^7)$ act on $\mathbb{R}^7$ in the usual way. This induces an action in the space $\Lambda^3(\mathbb{R}^7)$ by pullback. We fix a basis $\{dx^{123}, ..., dx^{567}\}$[5] and define the following 3-form Ω:

$$\Omega_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \tag{2.4.1}$$

Proposition 2.4.1. The group $G_2 \subset GL(\mathbb{R}^7)$ is the stabilizer of Ω_0: $G_2 = \{T \in GL(\mathbb{R}^7); T^*\Omega_0 = \Omega_0\}$.

Proof. See lemma 11.1 of [30]. □

In fact, we have that $G_2 \subset SO(7)$ - indeed, G_2 is connected and acts transitively on S^6, so it preserves the metric and the orientation on $\mathbb{R}^7$. Therefore, G_2 also preserves the 4-form $*\Omega$. A different proof of this fact is given by Salamon [30] - he shows that $*\Omega_0$ is, up to a multiplicative constant, a natural 4-form coming from the holonomy reduction to G_2.

It is well-known that $\mathfrak{so}(n) \cong \Lambda^2(\mathbb{R}^n)$ - the isomorphism is given by associating to a basic bivector $v \wedge w$ the antisymmetric matrix $v^* \otimes w - w^* \otimes v$, where v^* denotes the covector associated to v. In the case $n = 7$, since $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ this gives a splitting $\Lambda^2(\mathbb{R}^7) \cong \Lambda^2_{14} \oplus \Lambda^2_7$, where[6] $\mathfrak{g}_2 \cong \Lambda^2_{14}$.

This splitting can be seen more explicitly as follows: consider the map

$$*_{\Omega_0} : \Lambda^i(\mathbb{R}^7) \to \Lambda^{4-i}(\mathbb{R}^7) \tag{2.4.2}$$

defined by $*_{\Omega_0}(\alpha) = *(\Omega_0 \wedge \alpha)$. This is an endomorphism when restricted to 2-forms, and the splitting above corresponds exactly to the decomposition of $\Lambda^2(\mathbb{R}^7)$ into eigenspaces of $*_{\Omega_0}$.

Now, due to theorem 1.2.4, in a Riemannian 7-manifold (M, g) with $Hol(g) \subset G_2$, the tensor Ω_0 gives rise to a 3-form Ω such that $\nabla\Omega = 0$. For the same reason, $*\Omega_0$ gives rise to the 4-form $*\Omega$, and $\nabla * \Omega = 0$. Therefore, these forms are closed, and in fact they are calibrations,as we have discussed in the last paragraph of section 1.3. Submanifolds calibrated by Ω are called *associative submanifolds* and submanifolds calibrated by $*\Omega$ are not suprisingly called *coassociative submanifolds*.

In complete analogy with the theory of sLag submanifolds of Calabi-Yau manifolds (see theorem

[5]In the next two sections we will write a differential form $dx^{i_1} \wedge ... \wedge dx^{i_k}$ as $dx^{i_1 \cdots i_k}$ to simplify the notation.

[6]In this notation, the subscript index labels the dimension of the space.

Proposition 2.4.2. A 4-dimensional submanifold N of a G_2-manifold (M, g, Ω) admits an orientation such that N is a coassociative submanifold if and only if $\Omega|_N = 0$

There is also a version of McLean's theorem 2.3.4 for the deformation theory of coassociative submanifolds:

Theorem 2.4.3. Let (M, g, Ω) be a G_2-manifold with a coassociative submanifold N. Then the moduli space of deformations of N by coassociative manifolds is a smooth manifold of dimension $b^2_+(N)$.

Here $b^2_+(N)$ is the dimension of the space of harmonic selfdual 2-forms.

Proof. The proof is similar to theorem 2.3.4. See [16] for details. □

The deformation theory of associative submanifolds is more complicated since these manifolds cannot be characterized in a manner similar to theorem 2.4.2. Roughly speaking, what one does is to study the normal bundle of an associative submanifold N inside a G_2-manifold. One can identify this bundle with a spinor bundle over N and define a certain *twisted Dirac operator* D acting on sections of the normal bundle[7], such that $Ker(D)$ measures infinitesimal deformations of N, and $Coker(D)$ measures obstructions to deformations of N. The dimension of the moduli space of associative deformations of N is then equal to the index $ind(D) = Ker(D) - Coker(D)$, which is always zero for compact odd-dimensional manifolds.

2.5 $Spin(7)$ Geometry

$Spin(7)$ is, of course, the universal cover of $SO(7)$. The covering map $p : Spin(7) \to SO(7)$ is 2 to 1, as $\pi_1(SO(7)) \cong \mathbb{Z}_2$. This group can be shown to be the stabilizer of a 4-form Γ in $\mathbb{R}^8$ which can be written as:

$$\Gamma_0 = \Omega_0 \wedge dx^8 + *\Omega_0 \tag{2.5.1}$$

where we are now seeing the 3-form Ω_0 (defined in the last section) as a 3-form on $\mathbb{R}^8$. For the proof that $Spin(7)$ is indeed the stabilizer, see lemma 12.2 of [30].

As $\mathfrak{spin}(7) \cong \mathfrak{so}(7)$, it follows that $\dim(Spin(7)) = 28$. Also, $Spin(7)$ acts transitively on S^7 with stabilizer G_2, so that $Spin(7) \subset SO(8)$. We can use once more the identification $\mathfrak{so}(8) \cong \Lambda^2(\mathbb{R}^8)$ to split the space of 2-forms as $\Lambda^2(\mathbb{R}^8) = \Lambda^2_{21} \oplus \Lambda^2_7$, where once again the subscripts label the dimension of the space. We can see this splitting more explicitly by considering the operator:

$$*_{\Gamma_0} : \Lambda^i(\mathbb{R}^8) \to \Lambda^{4-i}(\mathbb{R}^8) \tag{2.5.2}$$

and noticing that the splitting above corresponds to the decomposition into eigenspaces of the operator $*_{\Gamma_0}$ acting on 2-forms. In fact, by writing the Hodge star operator in local coordinates and going through a lenghty computation, one finds the characteristic polynomial of $*\Gamma$ to be $(x-3)^7(x+1)^{21}$.

The group $Spin(7)$ is on Berger's list of holonomy groups of Riemannian manifolds, so there are 8-manifolds with holonomy $Spin(7)$. From previous discussions, we know that any such manifold is Ricci-flat, spin and has finite fundamental group. Also, from theorem 1.2.4, we know that on any $Spin(7)$-manifold the Γ_0 can be extended to a 4-form Γ such that $\nabla\Gamma = 0$. Hence $d\Gamma = 0$, and the discussion in section 1.3 shows that Γ is in fact a calibration. The 4-dimensional submanifolds $N \subset M$ calibrated by Γ are called *Cayley submanifolds*.

[7] See appendix A.1 for the relevant definitions.

The deformation theory of Cayley submanifolds behaves much in the same way as that for associative submanifolds of G_2-manifolds, since there is also no criterion similar to theorem 2.4.2 to characterize Cayley submanifolds. In general the dimension of the moduli space of deformations will be equal to the index of a certain elliptic operator, which can be identified with a twisted Dirac operator whenever the Cayley submanifold is spin.

Chapter 3

Gauge Theory

3.1 Yang-Mills equations

Gauge theory is the study of connections on principal bundles and its associated vector bundles. We have already seen in chapter 1 that we can study connections in the context of both principal and vector bundles, and these give equivalent theories. Thus, in what follows we will change from one language to the other according to convenience.

Let us also fix some notation: $P \to M$ is a principal G-bundle over M, and $\mathcal{H} \subset TP$ is a connection on P. We denote the connection one-form by ω. Also, $E \to M$ will be a vector bundle associated to P - so its structure group is G and it is endowed with a connection ∇. We fix a local trivialization $\{U_\alpha\}$ of E so that locally, we write $\nabla = d + A_\alpha$ for $A_\alpha \in \Omega^1(U_\alpha) \otimes Ad(P)$. The curvature of ∇ is denoted by F; this is an element of $\Omega^2(M) \otimes Ad(E)$. We also have an exterior covariant derivative that extends the action of the connection to bundle-valued forms, which we denote by d_∇ as in section 1.1.1. In this notation, the Bianchi identity reads $d_\nabla F = 0$. Finally, we denote the Hodge star operator again by $*$.

In this notation, the *Yang-Mills equation* is:

$$d_\nabla * F = 0 \tag{3.1.1}$$

When written in local form, this equation translates as a set of second-order equations for the connection coefficients A_α.

3.2 Gauge transformations

Now, denote by $\mathcal{A}$ the space of all connections on the vector bundle E.

The action of G on E induces an action on the sections of a trivializing neighborhood U of E in the following way: if $s \in \Gamma(U)$ and $\sigma : U \to G$, then $\sigma.s(x) = \sigma(x)s(x)$. This action in turn induces an action of the functions $\sigma : U \to G$ in the space of connections. In fact, we would like our covariant derivative ∇ to behave nicely with respect to the this action; this means that we should have $\nabla(\sigma s) = (\sigma.\nabla)s$ for all $s \in \Gamma(E)$. This formula implies that the action induced on the connection coefficients is:

$$\sigma.A = \sigma^{-1}A\sigma + \sigma^{-1}d\sigma \tag{3.2.1}$$

The curvature coefficients then satisfy:

$$\sigma.F = \sigma^{-1}F\sigma \tag{3.2.2}$$

However, this is only a local form for the action on the space of connections. To understand how it looks like globally, let us switch to the framework of connections on principal bundles. From a principal G-bundle $P \to M$ we can form an associated fiber bundle $C(P) = P \times_G G$ through the conjugate representation of G, $g \mapsto C_g \in \mathrm{Diff}(G)$. Now, local sections of $C(P)$ are just maps $\sigma : U \to G$. The point of introducing this bundle is the following theorem:

Theorem 3.2.1. $\Gamma(C(P))$ is isomorphic (as a group) to $\mathcal{G} := Aut(P)$, the group of automorphisms of the principal bundle P.

Note: The group structure on $\Gamma(C(P))$ is just the one given by pointwise multiplication.

Proof. If $\alpha : P \to P$ is an automorphism of the bundle P, then $\pi \circ \alpha = \pi$ and therefore there exists $\sigma : P \to G$ such that:

$$\phi(p) = p\sigma(p) \tag{3.2.3}$$

(since the action is transitive on the fibers). The map σ is G-invariant in the sense that $\sigma(pg) = g^{-1}\sigma(p)g$. There is an isomorphism between the space of G-invariant maps $P \to G$ and sections of $C(P)$; this associates to a section $s(m) = [p(m), g(m)]$ the map $f(p) = g(\pi(p))$. Conversely, if we start with such a σ , we can get an automorphism α of P by formula 3.2.3. It is also clear that this formula is compatible with the group structures. □

So, what we are actually seeking is an action of $\mathcal{G}$ on the space of connections, which globalizes 3.2.1.

Now, if we take a connection 1-form ω and $\phi \in \mathcal{G}$, then $\phi^*\omega$ is also a connection 1-form. Indeed, since ϕ commutes with the G-action, $\phi^*\omega$ also transforms by $Ad(G)$, and $p^*(\phi^*\omega) = \phi(p)^*(\omega) = \omega_{MC}$. Therefore, $\mathcal{G}$ acts on $\mathcal{A}$ on the right by pulling back connections, and formula 3.2.1 is the local version of this action.

The quotient space $\mathcal{A}/\mathcal{G}$ is called the moduli space of connections. Actually, things are not this simple, since one should make many restrictions to get a reasonable quotient space. For instance, in the case of $SU(2)$ connections on a four-manifold, one should restrict attention to L^2_2 connections and L^3_2 gauge transformations. This is enough, for example, to assure that the quotient space is Hausdorff. We will not describe this (important) part of the theory, since it relies on hard analysis of the relevant Sobolev spaces (see Morgan's lectures in [9]). Instead, from now on we simply assume that all connections live in L^2_2 and gauge transformations live in L^3_2. The main point of this assumption is to guarantee that the action has local slices (see theorem (3.3.1) below).

3.3 Four Dimensions

In four dimensions, the Hodge star operator maps 2-forms to 2-forms. Hence we have a special class of solutions to the Yang-Mills equations, the *anti-selfdual* connections (which are also called *instantons*). These satisfy:

$$*F = -F \tag{3.3.1}$$

i.e., the selfdual part of the curvature vanishes identically. Notice that these connections automatically satisfy the Yang-Mills equation, due to the Bianchi identity 1.1.12. Nevertheless, this is a first-order (although non-linear) equation for ∇, so supposedly it should be easier to solve than the Yang-Mills equation.

The set of all instantons $\mathcal{I}$ is invariant by the group of gauge transformations, so we get a well-defined *moduli space of instantons*:

$$\mathcal{M} = \mathcal{I}/\mathcal{G}$$

3.3.1 The Moduli Space of Instantons

An important fact is that the subgroup of $\mathcal{G}$ that stabilizes a given connection A consists exactly of the sections $\sigma : M \to Ad(P)$ which are horizontal with respect to A. Furthermore, this is also the centralizer of the holonomy representation of A in the fiber.

In the case of $SU(2)$ connections, the subgroup $\{-1, +1\}$ stabilizes every connection, but some connections might have larger stabilizers (it can be proved that it must be either a S^1-subgroup or all of $SU(2)$, see Morgan's lectures in [9]). Connections whose stabilizer is larger than $\{-1, +1\}$ are called *reducible*.

The moduli space of instantons is, by definition, the set of orbits of connections for the action of the group of gauge transformations. To get a feeling of how it looks like, we need to study the self-duality map:

$$F_+ : \mathcal{A} \to \Omega^2_+(M; Ad(P)) \tag{3.3.2}$$

The moduli space is just $F_+^{-1}(0)$. To get a local model for this space, we would need some version of the implicit function theorem for operators on infinite-dimensional manifolds. It is known that such a result holds for the so-called *Fredholm operators*, i.e., those linear operators T satisfying:

$$ind(T) := \dim(Ker(T)) - \dim(Coker(T)) < \infty \tag{3.3.3}$$

The number in this equation is called the *index* of the operator T.

A computation shows that $d_A F_+ = \nabla_+$, the self-dual part of the covariant derivative map acting on 1-forms. Unfortunately, the operator ∇_+ does not define a Fredholm map. However, by using hard analysis one can get a better understanding of the action of $\mathcal{G}$ in $\mathcal{A}$, and as a consequence one can prove that this action admits local slices modelled on $Ker(\nabla^*)$. Then it follows that:

Theorem 3.3.1. The map $F_+ : Ker\nabla^* \to \Omega^2(M; Ad(P))$ is a Fredholm operator.

Proof. Consider the complex:

$$\mathcal{C}(A) : 0 \to \Omega^0(M; Ad(P)) \xrightarrow{\nabla} \Omega^1(M; Ad(P)) \xrightarrow{\nabla_+} \Omega^2_+(M; Ad(P)) \to 0 \tag{3.3.4}$$

This is a complex when the connection is ASD (since $\nabla_+\nabla = F_+$). We can form the symbol complex:

$$0 \to \pi^*(\Omega^0(M; Ad(P))) \xrightarrow{\alpha} \pi^*(\Omega^1(M; Ad(P))) \xrightarrow{\beta} \pi^*(\Omega^2_+(M; Ad(P))) \to 0 \tag{3.3.5}$$

where the maps α and β are given fiberwisely by $\alpha(\omega) = (\wedge\omega) \otimes Id_{Ad(P)}$ and $\beta(\omega) = F_+((\wedge\omega) \otimes Id_{Ad(P)})$, for every $\omega \in T^*M$.

The symbol complex is exact away from the zero section [9][1], so that the original complex is elliptic. This implies (see appendix A of [5]) that the images of the operators are closed and that the associated cohomology groups are finite-dimensional.

[1]Essentially, this follows from the exactness of the complex obtained by replacing $Ad(P)$ by the trivial bundle.

Now, the first map is just the linearization of the $\mathcal{G}$-action on $\mathcal{A}$, i.e., ∇ is just the differential of $A : \mathcal{G} \to \mathcal{A}$ defined by $\sigma \mapsto \sigma^{-1}A\sigma + \sigma d\sigma^{-1}$. Therefore, its image is just the tangent space of the gauge orbit through A inside $\mathcal{A}$. The second map is the differential of the curvature map $A \mapsto F_A$. We have a splitting:

$$\Omega^1(M; Ad(P)) = \nabla(\Omega^0(M; Ad(P))) \oplus Ker(\nabla^*) \tag{3.3.6}$$

Therefore, $Ker(F_+|_{Ker(\nabla^*)})$ can be identified with the first cohomology group, and is thus finite-dimensional.

Moreover, it is easy to see that $Coker(F_+)$ is just the second cohomology of the complex, and thus it is also finite-dimensional. It follows that $F_+|_{Ker(\nabla^*)}$ is a Fredholm map.

□

The Euler characteristic of the complex 3.3.4 is just the index of F_+ and can be computed via the Atiyah-Singer index theorem in terms of topological invariants of $Ad(P)$:

$$\chi(\mathcal{C}(A)) = 3(b_0(M) - b_1(M) + b_2^+(M)) - 8c_2(P) \tag{3.3.7}$$

and therefore it is independent of the choice of (irreducible) connection defining $\mathcal{C}(A)$. Here, $b_0(M)$ and $b_1(M)$ are the first and second Betti numbers of M, $c_2(P)$ is the second Chern class of the bundle and $b_2^+(M)$ is the dimension of the maximal positive subspace of $H^2(M)$ under the intersection pairing $I : H_2(M) \times H_2(M) \to \mathbb{R}$ defined by:

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$

This number is equal to minus the dimension of $\mathcal{M}$ provided that $H^2(\mathcal{C}(A)) = 0$; in fact, by the implicit function theorem we have a map:

$$K : H^1(\mathcal{C}(A)) \to H^2(\mathcal{C}(A)) \tag{3.3.8}$$

called the *Kuranishi map*, such that $K^{-1}(0)$ defines a local model for $\mathcal{M}$ around an irreducible connection A. If the codomain is zero, then the dimension of $K^{-1}(0)$ is the same of $H^1(\mathcal{C}(A))$, which is the same as the Euler characteristic of the complex. Thus, near an irreducible connection the moduli space can be described as a smooth manifold of dimension $-\chi(\mathcal{C}(A))$. Moreover, the map F_+ can be seen as a Fredholm section of the vector bundle $\mathcal{A} \times_{\mathcal{G}} \Omega^2_+(M; Ad(P))$ defined over the space of connections, and that the moduli space is just the zero set of this section.

We won't go further into the theory beyond this point, since it starts to get intricate. There are issues related to existence of reducible connections, and about the behavior of the moduli space near these points. For many of the applications, one also has to study the behavior of $\mathcal{M}$ under variations of the metric, and to prove that it is orientable. Finally, it is important to understand the behavior of limiting connections and their neighborhoods in $\mathcal{M}$; there are two main results in this direction: Uhlenbeck's compactness theorem and Taubes' gluing theorem. From these one can define a natural compactification of the moduli space, which allows one, for instance, to integrate cohomology classes on the resulting compactification - this actually is how the Donaldson polynomial invariants are defined. The reader interested in pursuing these matters in depth should consult [5] and Morgan's lectures in [9].

3.3.2 Perspectives

Gauge theory in low dimensions can also be described as a *Topological Quantum Field Theory* (TQFT). In general, a n-dimensional $TQFT$ is a monoidal functor from the category of n-dimensional cobordisms[2] to a linear category (usually the category of vector spaces with the tensor product structure), satisfying some naturality axioms [1]. For 3-manifolds, we can study instantons using *Floer homology*; this works roughly as follows: for any 3-manifold M we can construct the *Chern-Simons functional* defined over the space of connections modulo gauge transformations:

$$CS(A) = \frac{k}{8\pi} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{3.3.9}$$

The critical points of this functional are the flat connections, so applying methods of Morse theory we can construct a homology complex, at least in the case when M has the same homology as the sphere[3]: the homology groups are free groups generated by flat connections and the boundary map is constructed from the Chern-Simons flow. Physically, these so-called *Floer homology groups* correspond are state spaces of vacuum configurations of a physical system. Floer homology behaves nicely through topological operations: if one gives M by a handlebody decomposition $M = L_1 \#_\Sigma L_2$ through a Riemann surface Σ, then L_1 and L_2 define Lagrangian submanifolds in the moduli space of unitary flat bundles over Σ.

An important technical point is that the Chern-Simons functional is not gauge invariant, but changes by an integer multiple of 2π by a gauge transformation. However, this is not a problem in the quantum Chern-Simons theory, where the basic object is $S = \int D\mathcal{A} \exp(iCS(A))$. Although ill-defined, this Feynman integral can be computed due to a relation to a conformal field theory called the Wess-Zumino-Witten model: roughly speaking, locally M looks like $\Sigma \times I$ where Σ is a Riemann surface and I is the unit interval, and the statement is that the Feynman integral should give the dimension of conformal blocks in the WZW model of Σ. This is described in [35] and is the most famous TQFT to date: it calculates the Jones polynomial of knots in M, and more recently it was discovered to be a particular case of a more general theory known as Khovanov homology [19] through a certain string duality.

Leung [23] sketches a definition of a Topological Quantum Field Theory for any Calabi-Yau or G_2-manifold, based on the work of Donaldson and Thomas [7].

3.3.3 Dimensional Reduction from Four Dimensions

We will now show how 4-dimensional gauge theory interacts with lower dimensional theories by exhibiting the dimensional reductions of the anti-selfduality equations.

Suppose that M is a 4-manifold and introduce local coordinates $\{x^0, x^1, x^2, x^3\}$. Let A be the local form for a connection on M, and suppose A is independent of x^0. We can then write $A = A_i dx^i + \phi dx^0 := A' + \Phi$, where ϕ is called a *Higgs field*. The A_i's and ϕ are $\mathfrak{g}$-valued functions independent of the x^0

[2]Here, a *cobordism* between two oriented n-dimensional closed manifolds M_1, M_2 is an oriented $n+1$-dimensional compact manifold N such that $\partial N = M_1 \cup -M_2$, where the minus sign means "opposite orientation". The category of n-dimensional cobordisms has as its objects all oriented n-dimensional closed manifolds and the morphisms are cobordisms. It has a natural monoidal structure given by disjoint union of manifolds.

[3]Actually, all that is required is that $H_1(M, \mathbb{Z}) = \{0\}$; the reason for this is a little subtle, but in a nutshell this group labels reducible representations of $\pi_1(M)$ which should be disregarded. See page 216 of Floer's paper [8] for more on this.

coordinate. In components, the ASD equations read:

$$\begin{aligned} F_{01} &= -F_{23} \\ F_{02} &= F_{34} \\ F_{03} &= -F_{12} \end{aligned} \tag{3.3.10}$$

This translates to:

$$\begin{aligned} &-\partial_1\phi + [\phi, A_1] + F_{23} = 0 \\ &-\partial_2\phi + [\phi, A_2] - F_{13} = 0 \\ &-\partial_3\phi + [\phi, A_3] + F_{12} = 0 \end{aligned} \tag{3.3.11}$$

If our manifold is locally of the form $M = N \times \mathbb{R}$, with x^0 a coordinate in $\mathbb{R}$, A' can be though of as connection on N. Then, $B = F_{ij}dx^i \wedge dx^j$ is just the curvature of A'. The above 3 equations can then be written as:

$$\nabla_{A'}\phi = *B \tag{3.3.12}$$

where the subscript in ∇ means that this is the covariant derivative in N associated to A'. These are the so-called *Bogomol'nyi monopole equations*.

By following the same procedure, we can get the analogous of instantons in fewer dimensions. Suppose now that A only depends on x^0. Defining the new connection as $A' = A_0 dt$ we have that:

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = \partial_0 A_1 + [A_0, A_1] = \nabla_{A'} A_1 \tag{3.3.13}$$

Analogously:

$$F_{23} = [A_2, A_3] \tag{3.3.14}$$

So the first ASD equation is:

$$\nabla_{A'} A_1 + [A_2, A_3] = 0 \tag{3.3.15}$$

and the other two equations are:

$$\nabla_{A'} A_2 + [A_3, A_1] = 0 \tag{3.3.16}$$

$$\nabla_{A'} A_3 + [A_1, A_2] = 0 \tag{3.3.17}$$

These are called the *Nahm equations*. One of the nice things about them is that there is a *Nahm transform* connecting solutions of these (rather simple) equations to solutions to the monopole equations.

Finally, let us see what happens in two dimensions. Following the same procedure, we suppose that A is independent of x^2 and x^3, and write $A' = A_0 dx^0 + A_1 dx^1$. If we suppose $M = \Sigma \times C$ with x^0, x^1 coordinates in Σ, the ASD equations read:

$$F_{01} = -[A_2, A_3] \tag{3.3.18}$$

$$\nabla^0_{A'} A_2 = \nabla^1_{A'} A_3 \tag{3.3.19}$$

$$\nabla^0_{A'} A_3 = -\nabla^1_{A'} A_2 \tag{3.3.20}$$

where the $\nabla^0_{A'}$ is the component of $\nabla_{A'}$ in the x^0-direction, etc.

These equations can be cast in a more illuminating form if we use complex coordinates $dz = dx^0 + idx^1$ (all orientable 2-manifolds are complex). Then, we write $\phi = \frac{1}{2}(A_2 + iA_3)dz$. This is called the *Higgs field*; we have $\phi \in \Omega^1(M, End(E))$. Notice that:

$$[\phi, \phi^*] = \phi\phi^* + \phi^*\phi = -\frac{i}{2}[A_2, A_3]dz \wedge d\overline{z} = -[A_2, A_3]dx^0 \wedge dx^1 \tag{3.3.21}$$

So the first equation is simply:

$$F_{A'} = [\phi, \phi^*] \tag{3.3.22}$$

Now, consider the Dolbeault operator $\overline{\partial}_{A'} = \frac{1}{2}(\nabla^0_{A'} + i\nabla^1_{A'})d\overline{z}$. Then:

$$\overline{\partial}_{A'}\phi = 0 \tag{3.3.23}$$

Equations 3.3.22 and 3.3.23 are called the *Hitchin equations*. They were introduced by Hitchin in [15], where he showed that the moduli space of solutions has the structure of a completely integrable system.

3.4 Gauge Theory on Special Holonomy Manifolds

3.4.1 Calabi-Yau Manifolds

Remark: In what follows, a complex manifold of complex dimension n is called a n-fold.

Gauge Theory on Calabi-Yau manifolds was first described by Donaldson and Thomas in [7] and developed in more detail in Thomas' PhD thesis [31]. We recall that in the context of real manifolds, gauge theory is the study of solutions to the Yang-Mills equations (3.1.1). This requires the manifold to be endowed with an orientation and a (semi)Riemannian metric, because one needs a volume form to define the Hodge star operator. Thomas argues that in the context of complex manifolds, the analogous to a real oriented Riemannian manifold is a Calabi-Yau manifold X. Indeed, the section $\Theta \in \Lambda^{(n,0)}(T^*X)$ that trivializes the canonical bundle K_X is a holomorphic volume form, and allows us to define a "holomorphic" Hodge operator:

$$\begin{gathered} *_\Theta : \Lambda^{(0,p)}(T^*X) \to \Lambda^{(0,n-p)}(T^*X) \\ \alpha \mapsto *(\Theta \wedge \alpha) \end{gathered} \tag{3.4.1}$$

where $*$ is the usual anti-linear Hodge operator:

$$* : \Lambda^{(p,q)}(T^*X) \to \Lambda^{(n-p,n-q)}(T^*X)$$

such that for (p,q)-forms α and β, $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \overline{dV}$.

The philosophy is then that, by replacing real coordinates by complex coordinates, d by $\bar{\partial}$ and by wedging all relevant equations/functionals with Θ, one should get a sensible "complex gauge theory". A natural generalization of the Yang-Mills equations is:

$$\bar{\partial}^*_A F^{(0,2)} = 0 \tag{3.4.2}$$

for the $(0,2)$ part of the curvature of a connection A on X.

In the particular case where X is a Calabi-Yau 4-fold we have a complex version of the ASD equations (3.3.1):

$$*_\Theta F_A^{(0,2)} = -F_A^{(0,2)} \tag{3.4.3}$$

In principle this equation only depends on the conformal class of the metric, so they are often supplemented with an Hermitian-Einstein type of condition:

$$\Lambda F := ig^{b\bar{a}} F_{b\bar{a}}^{(1,1)} = \lambda Id \tag{3.4.4}$$

As Thomas explains, this also has the additional benefit of making the equations elliptic, so that one could hope to get a sensible moduli space of solutions. In fact, the main point of Thomas' work is to define a "holomorphic Casson invariant" counting holomorphic bundles over a Calabi-Yau 3-fold, i.e., solutions to the equation

$$F_A^{(0,2)} = 0 \tag{3.4.5}$$

in analogy to the real case in which one counts flat $SU(2)$-connections. Flat connections are, of course, stationary points of the gradient flow of the Chern-Simons functional; Thomas defines a "holomorphic Chern-Simons functional":

$$CS(A) = \frac{1}{8\pi^2} \int_X tr(\bar{\partial}_{\tilde{A}} a \wedge a + \frac{2}{3} a \wedge a \wedge a) \wedge \Theta \tag{3.4.6}$$

whose gradient flow coincides with equations (3.4.3). Counting stationary points should then give the number of holomorphic bundles. In the real case, one computes the number of flat connections by taking CS to be a Morse function in the space of connections and calculating the Euler characteristic of the associated Morse complex[4]. For the holomorphic case, Thomas proposes that, with a suitable compactness theorem for the moduli space, one should be able to use Picard-Lefschetz theory - a complex analogue of Morse theory - to count the number of holomorphic bundles.

3.4.2 Exceptional Holonomy Manifolds

The existence of a generalized ASD equation is not a special feature of Calabi-Yau 4-folds. In general, if we have a n-dimensional manifold M $(n > 4)$ with a closed $(n-4)$-form Ω, then for any vector bundle with connection over M we can consider the equation:

$$*(\Omega \wedge F_A) = -F_A \tag{3.4.7}$$

Tian [32] studies a more general version of the special holonomy ASD equations: In fact, Tian proves this for a more general version of these special holonomy ASD equations:

$$\Omega \wedge (F_A - \frac{1}{r} tr(F_A) Id) = - * (F_A - \frac{1}{r} tr(F_A) Id) \tag{3.4.8}$$

where r is the rank of the bundle. He then proceeds to prove that, if one considers the decomposition induced on the space of 2-forms by the special holonomy structure, then each piece of the curvature F_A on this decomposition satisfies a Hermitian-Yang-Mills type of condition. This is proved by giving a formula for the Yang-Mills action in terms of the Chern classes of the bundle and the cohomology class of the special holonomy structure.

[4] This is the idea behind the celebrated Floer theory.

3.5 Dimensional Reduction from Higher Dimensions

We will work out a few dimensional reductions of instantons in higher dimensional manifolds. We will reduce a $Spin(7)$-instanton to a G_2-manifold and to a $K3$ surface, and a G_2-instanton to a Calabi-Yau manifold. We will also reduce a $SU(4)$-instanton to a Calabi-Yau manifold of complex dimension 3 following the work of [2]; in this case, by seeing $SU(4)$ as a subgroup of $Spin(7)$, this last example fits in the same picture as the first one; furthermore, we will show how some of these reductions furnish links with advanced topics such as a certain non-abelian version of the Seiberg-Witten equations, and also to the 3 twists of $N = 4$ Super Yang-Mills theory.

As far as we know, this is the first work to present the dimensional reduction from a G_2 manifold to a Calabi-Yau manifold, and in particular to derive the perturbed Kähler instanton equations (3.5.11). Nevertheless, it should be pointed out that the calculation is completely analogous to the dimensional reduction from a $Spin(7)$ manifold to a G_2 manifold presented in the Diploma Thesis of B. Jurke [18], which results in the perturbed G_2-instanton equations (3.5.16).

3.5.1 G_2 to Calabi-Yau

Let X be a Calabi-Yau manifold with holomorphic volume form Θ and Kähler form ω. Then $Y = X \times \mathbb{R}$ can be given a G_2-structure by defining:

$$\Omega = dt \wedge \pi^*\omega + \pi^* Re\Theta \tag{3.5.1}$$

where t denotes a coordinate in $\mathbb{R}$ and $\pi : Y \to X$ is the projection.

The G_2-instanton equation is:

$$\Omega \wedge F = - *_Y F \tag{3.5.2}$$

where $*_Y$ is the Hodge star operator on Y. By writing down Ω in local coordinates one can write the operator $\alpha \mapsto *(\Omega \wedge \alpha)$ (for α a 2-form) in matrix form; its only eigenvalues are -1 and 2. So the G_2-instanton equation can also be written as:

$$\pi_2(F) = 0 \tag{3.5.3}$$

where π_2 is of course projection onto the eigenspace associated to the eigenvalue 2.

We wish to see what happens to G_2-instantons in temporal gauge, i.e., with vanishing component in the t-direction. For this, we need the following lemma relating the Hodge star of X to the one on Y:

Lemma 3.5.1. Let $\alpha \in \Omega^k(Y)$ and $\pi : Y = X \times \mathbb{R} \to X$ be the projection. Then we can write $\alpha = dt \wedge \pi^*\alpha_1 + \pi^*\alpha_2$.

Denote by $*$ the Hodge operator on X. Therefore:

$$*_Y\alpha = \pi^*(*\alpha_1) + dt \wedge \pi^*(*\alpha_2) \tag{3.5.4}$$

Proof. If $\alpha, \beta \in \Lambda^2(Y)$, then $\beta \wedge *_Y\alpha = \langle\beta, \alpha\rangle_Y dV_Y$. A short computations shows that $\langle\beta, \alpha\rangle_Y =$

$\langle\beta_1,\alpha_1\rangle_X + \langle\beta_2,\alpha_2\rangle_X$. Also, clearly $dV_Y = dt\wedge\pi^*(dV_X)$. Therefore:

$$\begin{aligned}\beta\wedge *_Y\alpha &= dt\wedge\pi^*(\langle\beta_1,\alpha_1\rangle_X + \langle\beta_2,\alpha_2\rangle_X)dV_X\\ &= dt\wedge\pi^*(\beta_1\wedge *\alpha_1 + \beta_2\wedge *\alpha_2)\\ &= (dt\wedge\pi^*\beta_1)\wedge\pi^*(*\alpha_1) + \pi^*\beta_2\wedge(dt\wedge\pi^*(*\alpha_2))\\ &= \underbrace{(dt\wedge\pi^*\beta_1 + \pi^*\beta_2)}_{=\beta}\wedge(dt\wedge\pi^*(*\alpha_2) + \pi^*(*\alpha_1))\end{aligned} \tag{3.5.5}$$

Thus, $*_Y\alpha = (dt\wedge\pi^*(*\alpha_2) + \pi^*(*\alpha_1))$ as stated. □

Now, if A_Y is a G_2-instanton in temporal gauge, we can write $A_Y = \pi^*A + \phi$ for A a $\mathfrak{g}$-valued 1-form in X. The curvature then satisfies $F_Y = \pi^*F + \pi^*\nabla\phi\wedge dt$. From this, the G_2-instanton equation 3.5.2 can be rewritten:

$$(dt\wedge\pi^*\omega + \pi^*Re\Theta)\wedge(\pi^*F + \pi^*\nabla\phi\wedge dt) = - *_Y(\pi^*F + \pi^*\nabla\phi\wedge dt) \tag{3.5.6}$$

The RHS is, by the lemma, the same as:

$$-\pi^*(*F) - dt\wedge\pi^*(*\nabla\phi) \tag{3.5.7}$$

On the other hand, the LHS can be rearranged so as to be written as:

$$\pi^*[*(\omega\wedge F - Re\Theta\wedge\nabla\phi)] + dt\wedge\pi^*[*(Re\Theta\wedge F)] \tag{3.5.8}$$

Therefore, we get the following two equations for the dimensional reduction:

$$*(\omega\wedge F) + F = *(Re\Theta\wedge\nabla\phi) \tag{3.5.9}$$

$$*(Re\Theta\wedge F) = -\nabla\phi \tag{3.5.10}$$

Also, plugging the second equation in the first, we get:

$$*(\omega\wedge F) + F = - * (Re\Theta\wedge *(Re\Theta\wedge F)) \tag{3.5.11}$$

The RHS of this equation can be seen intuitively as some kind of Kähler version of the anti-self duality condition. In fact, on a 6-manifold M with a 2-form ω, the equation:

$$*(\omega\wedge F_A) = -F_A \tag{3.5.12}$$

which can be called a *"Kähler ASD equation"*, makes sense for a connection A on a principal bundle over M. Equation 3.5.11 is a generalization of this including a perturbation term depending on the Calabi-Yau structure through Θ. Notice also that when the Higgs field vanishes, we recover 3.5.12, and also get:

$$Re\Theta\wedge F = 0 \tag{3.5.13}$$

3.5.2 $Spin(7)$ to G_2

Let X be a G_2-manifold with G_2-structure given by $\phi \in \Omega^3(X)$. Then the manifold $Y = X \times S^1$ can be given a $Spin(7)$-structure by defining:

$$\Gamma = d\theta \wedge \pi^*(\Omega) + \pi^*(*\Omega) \tag{3.5.14}$$

where $\pi : Y \to X$ is the usual projection, and θ is a coordinate on S^1. This defines a $Spin(7)$-structure for the product metric $g_Y = g_X \times d\theta^2$ on Y. This follows from our definition of a $Spin(7)$-structure in $\mathbb{R}^8$ in terms of a G_2-structure in $\mathbb{R}^7$.

The dimensional reduction of a $Spin(7)$-instanton to a G_2 manifold is completely analogous to the reduction of a G_2-instanton to a Calabi-Yau manifold done in the last section. We recall that the G_2-instanton equation can be written as

$$\pi_7(F) = 0 \tag{3.5.15}$$

where π_7 is the projection into the 7-dimensional eigenspace of the operator $*_\Omega$ associated to the eigenvalue 2. Upon dimensional reduction to a G_2 factor, the $Spin(7)$ instanton equation associated to the $Spin(7)$ structure 3.5.14 becomes:

$$*(\Omega \wedge F) + F = 9 * (*\Phi \wedge *(*\Phi \wedge F)) \tag{3.5.16}$$

For the detailed calculation, see section 4.1 of [18].

3.5.3 Calabi-Yau 4-fold to Calabi-Yau 3-fold

Let us now work out the dimensional reduction from a Calabi-Yau manifold $(X_1, g_1, \omega_1, \Theta_1)$ of complex dimension 4 to another Calabi-Yau $(X_2, g_2, \omega_2, \Theta_2)$ of complex dimension 3. This is a particular case of a $Spin(7)$ reduction, since $SU(4) \cong Spin(6)$ ao that we have a natural inclusion $SU(4) \subset Spin(7)$. We suppose X_1 is of the form $X_1 = X_2 \times S$, for S a Riemann surface. The good thing about complex dimension 4 is that the holomorphic volume form has the correct rank to induce a Calabi-Yau instanton equation:

$$*(\Theta_1 \wedge \tilde{F}) = -\tilde{F} \tag{3.5.17}$$

for the curvature $\tilde{F}$ of a connection $\tilde{A}$ over a vector bundle E on X_1. We remind the reader that the complex Hodge star is defined by the condition that $\alpha \wedge *\alpha = |\alpha|^2 \overline{dV}$ for every complex form α.

We can choose complex coordinates $z_1, \ldots, z_4$ in which Θ_1 is given by

$$\Theta_1 = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4 \tag{3.5.18}$$

Then the only component of $\tilde{F}$ that matters for equation 3.5.17 is $F^{(0,2)}$, i.e., the component in $\Lambda^2(T^*(0,1)X_1)$. We will write the components of this tensor as $F_{\bar{a}\bar{c}}$ to emphasize it is associated to the anti-holomorphic eigenspace.

We introduce a complex coordinate w on S and suppose that $\tilde{A}$ is independent of w. We write $\tilde{A} = A + \Phi$, where $A = \sum_{i=1}^{3} A_i dz_i$. Then equation 3.5.17 becomes:

$$\begin{aligned} \tilde{F}_{\bar{1}\bar{2}} &= -\tilde{F}_{\bar{3}\bar{4}} = -\nabla_{\bar{3}}\Phi \\ \tilde{F}_{\bar{1}\bar{3}} &= \tilde{F}_{\bar{2}\bar{4}} = \nabla_{\bar{2}}\Phi \\ \tilde{F}_{\bar{2}\bar{3}} &= -\tilde{F}_{\bar{1}\bar{4}} = -\nabla_{\bar{1}}\Phi \end{aligned} \tag{3.5.19}$$

One should compare these equations with 3.3.10. The equations are exactly the same, except that in the Calabi-Yau context we only care about anti-holomorphic coordinates. This reinforces the point addressed in [7] that a Calabi-Yau 4-fold should be seen as the complex analogue of an oriented real four-manifold in the context of gauge theory.

Now, $\nabla = \partial_{\tilde{A}} + \bar{\partial}_{\tilde{A}}$, and since we only care about anti-holomorphic components we can replace ∇ by $\bar{\partial}$. Our equations then become:

$$\bar{\partial}_A \Phi = *F^{(0,2)} \tag{3.5.20}$$

where now $*$ is the Hodge star on X_2 and F is the curvature of A seen as a connection over the bundle $E|_{X_2}$. Notice that this is exactly a holomorphic version of the Bogomol'nyi equations 3.3.12.

We can refine this result as follows: define $\phi \in \Lambda^{(0,2)}(X_2)$ by $\Phi = *\phi$. Then, due to the (anti-holomorphic part of) Bianchi identity $\bar{\partial} F^{(0,2)} = 0$ it follows from 3.5.20 that $\bar{\partial}_A *^{-1} \bar{\partial}_A * \phi = 0$, i.e.:

$$\bar{\partial}_A \bar{\partial}_A^* \phi = 0 \tag{3.5.21}$$

which implies, when X_2 is closed and compact, that $\bar{\partial}_A^* \phi = 0$. So we finally get:

$$\bar{\partial}_A \Phi = 0 \tag{3.5.22}$$

i.e., the Higgs field Φ is *holomorphic*. As a consequence of 3.5.19, we have that $F_{\bar{a}\bar{c}} = 0$. Also, we could have done the same trick with an anti-holomorphic volume form $\bar{\Theta} = d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4$, so that we also have $F_{ac} = 0$. Thus:

$$F_{ac} = F_{\bar{a}\bar{c}} = 0 \tag{3.5.23}$$

which means that F is a $(1,1)$-form, i.e., the bundle is holomorphic and A is its metric connection.

There is an extra set of equations that should be imposed in this situation. They are related to questions of stability which we have not addressed in this work. Singer et al. [2] write these as:

$$F.\omega = 0 \tag{3.5.24}$$

where the dot means the inner product induced by the metric on the space of $(1,1)$-forms. This is a moment map condition that is imposed to account for gauge invariance. In general, the orbit space of the space of unitary connections under the action of $\mathcal{G}^c$ (the group of complex gauge transformations) is identified with the symplectic quotient $\left\{F^{(1,1)}.\omega = 0\right\}/\mathcal{G}$, at least for stable bundles. In this case the right notion of stability is that E should be Hermitian-Einstein. The reader is refered to [7] and section 2.3 of [2] for more on this subject.

Equations 3.5.23 and 3.5.24 are called the *Donaldson-Uhlenbeck-Yau equations*. They describe the moduli space of stable vector bundles on a Kähler manifold (this is the celebrated Hitchin-Kobayashi correspondence) and are also relevant for Calabi-Yau compactifications in heterotic string theory.

For more on the DUY equations, the reader is encouraged to consult section 4.1.1 of [2], where the authors discuss them from the point of view of the BRST cohomology of supersymmetric Yang-Mills theory on a Calabi-Yau manifold.

3.5.4 $Spin(7)$ to a $K3$ surface

In what follows, we will make use of the quaternionic structure on $\mathbb{R}^4$, as well as results about spin geometry. For background, see appendix A.1.

Now we would like to perform a dimensional reduction from a $Spin(7)$-manifold to four dimensions. It is well-known folklore that, by taking one of the factors in the product structure to be $\mathbb{R}^4$, and using an appropriate identification between spinors and differential forms[5], the dimensional reduction results in equations very closely related to the celebrated Seiberg-Witten equations. This statement can be found, for instance, in [6] and [2].

As far as we know, this dimensional reduction was originally performed in the Diploma Thesis of Benjamin Jurke, reference [18]. Following his work, we will consider the case in which the 8-manifold is the total space of the positive spinor bundle over a $K3$ surface X with Kähler metric g.[6] The good thing about choosing a $K3$ is that, due to Wang's theorem 1.2.20, the bundle of positive spinors $\mathcal{S}^+$ is trivial:

$$\mathcal{S}^+ = X \times \mathbb{R}^4 \tag{3.5.25}$$

so that, denoting by h the Euclidean metric on $\mathbb{R}^4$, it follows that the holonomy of the product metric $g \times h$ is $SU(2) \subset Spin(7)$. In fact, the $Spin(7)$-structure can be written explicitly; using the notation of section 2.5:

$$\begin{aligned}\Gamma_0 = dx^{1234} &+ (dx^{13} - dx^{24}) \wedge Re\Theta - (dx^{14} + dx^{23}) \wedge Im\Theta \\ &+ (dx^{12} + dx^{34} + \frac{1}{2}\omega) \wedge \omega\end{aligned} \tag{3.5.26}$$

in appropriate coordinates at a point p. Here ω is the Kähler form and Θ the holomorphic volume form on X.

Then, by introducing complex coordinates z_1, z_2 such that $\Theta = dz_1 \wedge dz_2$ and $\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ and rewriting Γ_0 in terms of the real and imaginary parts of z_1 and z_2, we obtain exactly formula 2.5.1. This form is of course preserved by $Spin(7)$, so by theorem 1.2.4 it defines a covariantly constant tensor Γ, the $Spin(7)$-structure.

The $Spin(7)$ self-duality equation is:

$$F = *(\Gamma \wedge F) \tag{3.5.27}$$

where F is the curvature of a connection A on a principal bundle over $\mathcal{S}^+$. We can also write this equation as:

$$\pi_7(F) = 0 \tag{3.5.28}$$

[5]This can be done in spin manifolds by choosing a spin structure and a covariantly constant unitary spinor field, or for complex manifolds in general by choosing a $Spin^c$-structure - see Nicolaescu's notes on Seiberg-Witten theory [27] for more on this.

[6]In what follows, we will omit some lenghty calculations. The reader is encouraged to consult [18] for a complete exposition.

where $\pi_7 : \Lambda^2(\mathbb{R}^8) \to \Lambda^2_7(\mathbb{R}^8)$ is the projection into the eigenspace of $*_\Gamma$ corresponding to the eigenvalue $x = 3$, as in section 2.5.

Now, suppose we have a connection A on a principal bundle $P \to X$. Then we get a connection $\tilde{A}$ on the pullback bundle $\tilde{P}$ over $\mathcal{S}^+ \cong X \times \mathbb{R}^4$. Assume that $\tilde{A}$ is independent of the $\mathbb{R}^4$ coordinates $\{x_1, ..., x_4\}$. We can then write it as[7] $\tilde{A} = \pi^* A + \Phi$, where $\pi : X \times \mathbb{R}^4 \to X$ is the projection and the *Higgs field* Φ is of the form $\Phi = \sum_{i=1}^{4} \phi_i(z_1, z_2) dx^i$.

In other words, Φ can be seen as an adjoint section of the conormal bundle $N^*(X)$. Actually, we need to be more precise about this statement, since Φ is a field over $\mathcal{S}$, and not X. What happens is that, since the component fields ϕ_i do not depend on the fiber coordinates, they can be represented by pullbacks of sections ψ^i of $ad(P) \xrightarrow{p} X$ by the projection map $\pi : \mathcal{S}^+ \to X$. Also, the fact that $\mathcal{S}^+$ is trivial means that $N^*(X)$ is also trivial, $N^*(X) = X \times (\mathbb{R}^4)^*$. Thus, $\Phi \in \Gamma(\pi^* N^*(X) \otimes ad\tilde{P})$ and Φ is the pullback of a section $\Psi \in \Gamma(N^*(X) \otimes adP)$ by the projection π. Now notice that $N^*(X) \cong \mathcal{S}^+$, so that $\Psi \in \Gamma(\mathcal{S}^+ \otimes adP)$ is an *adjoint spinor*.

In quaternionic notation we can write $\Psi = \psi_0 + i\psi_1 + j\psi_2 + k\psi_3$. The good thing about passing from Φ to Ψ is that, as we will see, in the dimensional reduction of the $Spin(7)$-instanton equation the field Ψ will turn out to be a zero mode for the Dirac equation (i.e., a harmonic spinor).

The main thing we will need for the dimensional reduction is a decomposition of the projection π_7. For this purpose, we identify $\mathbb{R}^8$ with two copies of $\mathbb{H}$, $\mathbb{R}^8 \cong \mathbb{H}_1 \oplus \mathbb{H}_2$ and introduce the quaternionic forms:

$$\begin{aligned} dq^1 &= dx^1 + idx^2 + jdx^3 + kdx^4 \\ dq^2 &= dx^5 + idx^6 + jdx^7 + kdx^8 \end{aligned} \tag{3.5.29}$$

and their quaternionic conjugates:

$$\begin{aligned} d\bar{q}^1 &= dx^1 - idx^2 - jdx^3 - kdx^4 \\ d\bar{q}^2 &= dx^5 - idx^6 - jdx^7 - kdx^8 \end{aligned} \tag{3.5.30}$$

We also recall that there is an identification of $Im\mathbb{H}$ with $\Lambda^2_+(\mathbb{R}^4)$, the self-dual 2-forms on $\mathbb{R}^4$. The usual way to do this is by picking the standard self-dual basis $\omega_1 = dx^{12} + dx^{34}$, $\omega_1 = dx^{13} + dx^{24}$, $\omega_1 = dx^{14} + dx^{23}$ and defining the identification map by $a\omega_1 + b\omega_2 + c\omega_3 \mapsto ai + bj + ck$.

Now, a quick calculation shows that $\frac{1}{2}(dq_i \wedge d\bar{q}_i) = i\omega^l_1 + j\omega^l_2 + k\omega^l_3$, for $l = 1, 2$. The upperscripts here are to remind us that the ω^1_m's are a selfdual basis related to $\mathbb{H}_1$, and the ω^2_m's are related to $\mathbb{H}_2$.

Define the 2-forms $\alpha_1, ..., \alpha_3$ and $\beta_1, ..., \beta_4$ by:

$$\frac{1}{2}(dq_2 \wedge d\bar{q}_2 - dq_1 \wedge d\bar{q}_1) = i\alpha_1 + j\alpha_2 + k\alpha_3 \tag{3.5.31}$$

$$d\bar{q}_1 \wedge dq_2 = \beta_1 + i\beta_2 + j\beta_3 + k\beta_4 \tag{3.5.32}$$

Then a computation with the ω^k_m's (lemma 4.13 of [18]) shows that $\Lambda^2_7 = span\,\{\alpha_1, ..., \beta_4\}$, i.e., these forms span the eigenspace of $*_\Gamma$ associated to the eigenvalue 3.

We define $\Lambda^2_3 = span\,\{\alpha_1, \alpha_2, \alpha_3\}$ and $\Lambda^2_4 = span\,\{\beta_1, \beta_2, \beta_3, \beta_4\}$. Then $\Lambda^2_7 = \Lambda^2_3 \oplus \Lambda^2_4$. Notice also that $\Lambda^2_3 \cong Im\mathbb{H}$ and that $\Lambda^2_4 \cong \mathbb{H}$.

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[7] For simplicity, from now on we will write A' for $\pi^*(A)$.

Now, from the identity[8] $\Lambda^k(V \oplus W) \cong \bigoplus_{p+q=k} \Lambda^p(V) \otimes \Lambda^q(W)$ it follows that:

$$\Lambda^2(\mathbb{H}_1 \oplus \mathbb{H}_2)^* \cong \Lambda^2(\mathbb{H}_1)^* \oplus (\mathbb{H}_1^* \otimes \mathbb{H}_2^*) \oplus \Lambda^2(\mathbb{H}_2)^*$$

The desired decomposition of $\pi_7 : \Lambda^2(\mathbb{H}_1 \oplus \mathbb{H}_2)^* \to \Lambda^2_7$ is then $\pi_7 = \pi_3 \oplus \pi_4$ with:

$$\pi_3 : \Lambda^2(\mathbb{H}_1)^* \oplus \Lambda^2(\mathbb{H}_2)^* \to \Lambda^2_3 \tag{3.5.33}$$

and

$$\pi_4 : \mathbb{H}_1^* \otimes \mathbb{H}_2^* \to \Lambda^2_4 \tag{3.5.34}$$

These projections can of course be described in a more explicit way: if we let $\pi_+^{\mathbb{H}_i} : \Lambda^2(\mathbb{H}_i) \to \Lambda^2_+(\mathbb{H}_i)$ be the projections into the space of self-dual 2-forms, then by using once again the identifications $\Lambda^2_+(\mathbb{R}^4) \cong Im\mathbb{H} \cong \Lambda^2_3$, the projection π_3 can be written as $\pi_+^{\mathbb{H}_1} - \pi_+^{\mathbb{H}_2}$.

The projection π_4 can be described as follows: there is a natural projection $\pi_{\mathbb{H}} : Hom_{\mathbb{R}}(\mathbb{H}_1, \mathbb{H}_2) \to Hom_{\mathbb{H}}(\mathbb{H}_1, \mathbb{H}_2)$. Now, with the identifications $\mathbb{H}_1^* \otimes \mathbb{H}_2^* \cong Hom_{\mathbb{R}}(\mathbb{H}_1, \mathbb{H}_2)$ and $\Lambda^2_4 \cong \mathbb{H} \cong Hom_{\mathbb{H}}(\mathbb{H}_1, \mathbb{H}_2)$ taken into account, we can write $\pi_4 = \pi_{\mathbb{H}}$. We refer the reader to lemmas 4.17 and 4.19 of [18] for the proofs that π_3 and π_4 are indeed given by the maps we have described.

Now, let us return to the $Spin(7)$-instanton equation $\pi_7(F_{\tilde{A}}) = 0$. We are assuming that $\tilde{A} = A' + \Phi$ is independent of the fiber coordinates, so that $F_{\tilde{A}} = F_{A'} + \nabla\Phi = F_{A'} + d_{A'}\Phi + \frac{1}{2}[\Phi, \Phi]$, where in the last equality we used the structure equation. The instanton equation then becomes:

$$\begin{aligned} &\pi_3(F_{A'} + \frac{1}{2}[\Phi, \Phi]) = 0 \\ &\pi_4(d_{A'}\Phi) = 0 \end{aligned} \tag{3.5.35}$$

Let us try to understand the first equation. First of all, since $\pi_3 = \pi_+^{\mathbb{H}_1} - \pi_+^{\mathbb{H}_2}$ projects into the subspace of self-dual forms, the result should be $F^+_{A'} - \frac{1}{2}[\Phi, \Phi]^+$ (since, in our notation, $F_{A'}$ lives in $\Lambda^2(\mathbb{H}_1)$ and $[\Phi, \Phi]$ lives in $\Lambda^2(\mathbb{H}_2)$). We write $F^+_{A'} = \pi^* F^+_A$ and notice that:

$$\begin{aligned} [\Phi, \Phi]^+ = &([\phi_1, \phi_2] + [\phi_3, \phi_4])\omega^2_1 \\ &+ ([\phi_1, \phi_3] + [\phi_4, \phi_2])\omega^2_2 \\ &+ ([\phi_1, \phi_4] + [\phi_2, \phi_3])\omega^2_3 \end{aligned} \tag{3.5.36}$$

On the other hand, the adjoint spinor $\Psi = \pi^*\Phi$ satisfies:

$$\begin{aligned} \pi^*[\Psi, \bar{\Psi}] &= \pi^*[\psi_1 + i\psi_2 + j\psi_3 + k\psi_4, \psi_1 - i\psi_2 - j\psi_3 - k\psi_4] \\ &= -2[\Phi, \Phi]^+ \end{aligned} \tag{3.5.37}$$

where we used once again the identification between imaginary quaternions and self-dual 2-forms. The first equation in 3.5.35 becomes, then:

$$\pi^* F^+_A + \frac{1}{4}\pi^*[\Psi, \bar{\Psi}] = 0 \tag{3.5.38}$$

[8]This follows from the exactness of the functor $\wedge$.

Suppressing the pullback we finally get:

$$F_A^+ = -\frac{1}{4}[\Psi, \bar{\Psi}] \tag{3.5.39}$$

Let us work out now the second equation. First we notice that $d_{A'}\Phi$ is, of course, an $ad\tilde{P}$-valued 2-form, to which we can associate an $ad\tilde{P}$-valued matrix[9]. With this understood, we can now see $d_{A'}\Phi$ as an element of $Hom_{\mathbb{R}}(TX, T\mathbb{H})$, to which we can apply $\pi_{\mathbb{H}}$. But first we introduce the *quaternionic covariant derivative*:

$$\nabla^{\mathbb{H}} = \nabla_1 + i\nabla_2 + j\nabla_3 + k\nabla_3 \tag{3.5.40}$$

and its conjugate:

$$\overline{\nabla}^{\mathbb{H}} = \nabla_1 - i\nabla_2 - j\nabla_3 - k\nabla_4 \tag{3.5.41}$$

Here ∇_l means the component of ∇ in the l-th direction. A short calculation then gives:

$$\begin{aligned} \pi_{\mathbb{H}}(d_{A'}\Phi) &= \frac{1}{4}\overline{\nabla}^{\mathbb{H}}\Phi \\ &= \frac{1}{4}\pi^*\overline{\nabla}^{\mathbb{H}}\Psi \end{aligned} \tag{3.5.42}$$

where we are using again the identification $Hom_{\mathbb{H}}(\mathbb{H}_1, \mathbb{H}_2) \cong \mathbb{H}$.

Therefore, the equation $\pi_{\mathbb{H}}(d_{A'}\Phi) = 0$ becomes $\frac{1}{4}\pi^*\overline{\nabla}^{\mathbb{H}}\Psi = 0$ which can only happen if:

$$\overline{\nabla}^{\mathbb{H}}\Psi = 0 \tag{3.5.43}$$

Now, as we explain in appendix A.1 the Clifford algebra $Cl(4)$ is isomorphic to $M_2(\mathbb{H})$, the space of 2×2 quaternionic matrices. The isomorphism is given in terms of the generators $c_1, ..., c_4$ of $Cl(4)$ by:

$$c_1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad c_2 \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad c_3 \mapsto \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad c_4 \mapsto \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

One can easily check that these matrices indeed satisfy the Clifford algebra relations (A.1.1).

In this notation, the Dirac operator $\not{D} = \sum_{l=1}^{4} e_l.\nabla_l$ as applied to $s = (s^+, s^-) \in \Gamma(\mathcal{S})$ is:

$$\not{D}s(x) = (-\overline{\nabla}^{\mathbb{H}}\sigma^-(x), \nabla^{\mathbb{H}}\sigma^+(x)) \tag{3.5.44}$$

so we see that $\not{D}^+ = \overline{\nabla}^{\mathbb{H}}$. Thus, we finally arrive at the promised formula

$$\not{D}^+\Psi = 0 \tag{3.5.45}$$

which means that Ψ is a *harmonic spinor*. Thus, the final result of the dimensional reduction are the two equations:

$$\begin{aligned} F_A^+ &= -\frac{1}{4}[\Psi, \bar{\Psi}] \\ \not{D}^+\Psi &= 0 \end{aligned} \tag{3.5.46}$$

[9]That is, the matrix of the coefficients of $d_{A'}\Phi$, which is obtained by dualization $T^*X \otimes T^*\mathbb{H} \to T^*X \otimes T\mathbb{H}$.

One should compare these to the *Seiberg-Witten* equations [27]:

$$\begin{aligned} F_A^+ &= \frac{1}{2} q(\Psi) \\ \not{D}\Psi &= 0 \end{aligned} \tag{3.5.47}$$

where $q : \Gamma(\mathcal{S}^+) \to End_0(\mathcal{S}^+)$ is given by $q(\psi) = \bar{\psi} \otimes \psi - \frac{1}{2}|\psi|^2 Id$, and the subspace $End_0(\mathcal{S}^+)$ is identified with Λ_+^2 through $End(\mathcal{S}^+) \cong \mathbb{H}$.

Appendix A

Appendix

A.1 Spin Geometry

In this section we give a brief introduction to spin geometry. We will only sketch the basic facts needed in the main text. The interested reader is encouraged to consult [22] for a complete exposition.

We start by summarizing a few facts from the theory of Lie groups. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then every representation $\rho : G \to GL(V)$ induces a so-called "infinitesimal representation" $d\rho_1 : \mathfrak{g} \to \mathfrak{gl}(V)$. The universal covering $\tilde{G}$ of G inherits a Lie group structure by lifting the smooth and product structures from G, and since these groups are locally isomorphic, the Lie algebra of $\tilde{G}$ is also $\mathfrak{g}$. An important fact is that every representation $\sigma : \mathfrak{g} \to \mathfrak{gl}(V)$ lifts to a representation of $\tilde{G}$. This is not true for non simply-connected groups, since in the universal covering elements in the same fiber can have different images under the lifted representation, so that the induced map on the base would be multi-valued. Nevertheless, any connected Lie group is a quotient of its universal cover by a discrete central subgroup, and thus any representation of the universal covering whose kernel contains the given subgroup descend to a representation of the group.

Let us apply this theory to the special orthogonal groups $SO(n)$ for $n > 2$; this group is a compact Lie group of dimension $n(n+1)/2$. It is the connected component of the identity in $O(n)$. It has a standard representation in $\mathbb{R}^n$ is given by rotation matrices. Thus, a rotation angle and direction suffice to specify an element in $SO(n)$, and this data can be encoded in a vector in $\mathbb{R}^n$ pointing in the direction of the rotation axis, and with norm equal to the rotation angle measured in radians. So the set of rotations is identified with a ball of radius π in $\mathbb{R}^n$, but since rotations by π in opposite directions are the same, we have to identify antipodal points. This proves that $SO(n)$ is topologically $\mathbb{RP}^n$. Thus, its fundamental group is $\mathbb{Z}_2$ (loops given by an odd number of full rotations give rise to a non-trivial homotopy class).

We define the *spin group* $Spin(n)$ to be a double covering of $SO(n)$ (and thus, it is also the universal covering). Representations of $Spin(n)$ are then classified by the representation theory of $\mathfrak{so}(n)$. Some of these spin representations descend to representations of $SO(n)$, and some descend to maps $\rho : SO(n) \to GL(V)$ which are "anti-representations", in the sense that $\rho(1) = -Id_V$.

There is another useful way to describe the group $Spin(n)$, in terms of Clifford algebras. We will briefly explain how this works.

Let V be a n-dimensional Euclidean vector space with an orthonormal basis $\{e_1, \ldots, e_n\}$, and $\mathcal{T}(V)$ its tensor algebra[1]. We define the *Clifford algebra* of V to be:

[1]I.e., $\mathcal{T}(V) = \bigotimes_{i=0}^{\infty} V^{\otimes i}$, where $V^{\otimes 0} = \mathbb{R}$ by definition.

$$Cl(n) = \mathcal{T}(V)/\mathcal{R}$$

where the relations $\mathcal{R}$ are given by

$$e_i \otimes e_j + e_j \otimes e_i = -2\delta_{ij} \tag{A.1.1}$$

for $i, j \in \{1, \ldots, n\}$.

This definition makes $Cl(n)$ a unital associative 2^n dimensional algebra. There is also a linear map $i : V \hookrightarrow Cl(n)$ such that $i(v)^2 = 1$. We define $Cl^{\times}(n)$ to be the multiplicative subgroup of invertible elements in Cl(n). It can be proven that $Cl^{\times}(n)$ has a natural structure of a Lie group of dimension 2^n, with the Lie algebra being the Clifford algebra itself.

The Clifford algebra satisfies the following universal property: whenever we have a linear map $f : V \to A$ to a real associative algebra A satisfying $f(v)f(w) + f(w)f(v) = -2Id_A$, there is a unique algebra homomorphism $F : Cl(n) \to A$ such that $F \circ i = f$. This means that linear maps from V satisfying the relations can be extended to the entire Clifford algebra.

An important example is the map $\alpha : V \to V$ defined by $\alpha(v) = -v$. It extends uniquely to a map $\tilde{\alpha} : Cl(n) \to Cl(n)$ such that $\tilde{\alpha}^2 = Id$. Therefore, it induces a decomposition $Cl(n) = Cl^0(n) \oplus Cl^1(n)$, where $Cl^0(n) = Eig_{\tilde{\alpha}}(+1)$ is the *even part* of $Cl(n)$ and $Cl^1(n) = Eig_{\tilde{\alpha}}(-1)$ is the *odd* part of $Cl(n)$. One can prove that in general, $Cl^0(n) \cong Cl(n-1)$.

An important fact about Clifford algebras is that they can always be described as a graded matrix algebra. In fact, all Clifford algebras are isomorphic to either $M_{k \times k}(F)$ or $M_{k \times k}(F) \oplus M_{k \times k}(F)$ for some $k \in \mathbb{N}$, where the field F is either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Thus, every Clifford algebra has a standard representation on F^k for some k. Important examples for this work are $Cl(4) \cong M_{2\times 2}(\mathbb{H})$ and $Cl^0(4) \cong Cl(3) \cong \mathbb{H} \oplus \mathbb{H}$.

The group $Spin(n)$ can be identified with the subgroup of $Cl(n)$ defined by:

$$Spin(n) = \left\{ c_1 c_2 \ldots c_k \in Cl^{\times}(n); ||c_i|| = 1, \forall c_i \in V \right\}$$

The representations of $Spin(n)$ can be described in terms of restrictions of representations of $Cl(n)$; in fact, *all* representations of $Spin(n)$ are restrictions of representations of the Clifford algebra, and for this reason whenever $Spin(n)$ acts on a vector space, this space is automatically endowed with a Clifford multiplication.

The important point is that $Spin(n)$ admits a natural representation Δ^n, called the *spin reresentation*, which is simply the restriction of the standard representation of $Cl(n)$ to $Spin(n)$. This representation has the following properties:

- If n is even, Δ^n is a complex representation of complex dimension $2^{\frac{n}{2}}$. It is a *reducible* representation, and it splits as a direct sum of two irreducible representations of complex dimension $2^{\frac{n}{2}-1}$, $\Delta^n = \Delta^n_+ \oplus \Delta^n_-$.
- If n is odd, Δ^n is a complex irreducible representation of complex dimension $2^{\frac{n}{2}}$.

Now we proceed to discuss spin strcutures on oriented Riemannian manifolds. Suppose (M, g) is such a manifold, with $\dim(M) = n$. Then the Riemannian metric and orientation automatically reduce the frame bundle $F(M)$ to a principal $SO(n)$-bundle. The question we are interested is this: under what conditions can we "lift" this $SO(n)$-structure to a $Spin(n)$-structure such that the tanget spaces of M can be regarded as representations of $Spin(n)$? This motivates the following defintion:

Definition A.1.1. Let (M, g) be a n-dimensional oriented Riemannian manifold with frame bundle $F(M) \xrightarrow{\pi} M$, and let $\rho : Spin(n) \to SO(n)$ be the covering map. We say that M is a *spin manifold* if there is a principal bundle $P \xrightarrow{\bar{pi}} M$ with structure group $Spin(n)$ and a map of bundles $f : P \to F(M)$ which is a 2-fold covering map satisfying:

1. $\bar{\pi} = \pi \circ f$
2. $f(pg) = f(p)\rho(g)$

The first condition guarantees the two bundles are linked together by their projections, while the second condition ensures that the map f is locally modelled on the covering ρ.

Not every oriented Riemannian manifold admits a spin structure. This can be proven by studying the *Čech cohomology* of these groups. Let us explain roughly how this works. For a principal G-bundle $P \to M$ and an open cover $\mathcal{U}$ of M that trivializes P, we can define a simplicial complex such that the p-cochains are maps $\sigma \mathcal{U}^p \to G$. The coboundary map is defined in such a way that a 2-cocycle σ satisfies:

$$\sigma(U_j, U_k) = \sigma(U_j, U_i)\sigma(U_i, U_k)$$

Notice that this is exactly the cocycle condition for transition functions on a principal bundle. Also, two 2-cochains σ, $\tilde{\sigma}$ differ by a 1-coboundary α if and only if

$$\tilde{\sigma}(U_i, U_j) = \alpha(U_i)\sigma(U_i, U_j)\alpha(U_j)^{-1}$$

Again, if σ and $\tilde{\sigma}$ are thought as transition functions on principal bundles over M, this condition means that the bundles defined by them are isomorphic. We define $H^1(M, \mathcal{U})$ to be the quotient of the space of 2-cocycles by the space of 1-coboundaries.

Define the *Čech cohomology group* $H^1(M, G)$ as the direct limit of the groups $H^1(M, G)$ with respect to refinements of $\mathcal{U}$. By construction, Čech cohomology classes are in $1-1$ correspondence with equivalence classes of principal G-bundles over M.

The important thing is that an exact sequence of groups

$$0 \to H \to G \to K \to 0$$

induces a long exact sequence in the Čech cohomology:

$$\ldots \to H^1(M, H) \to H^1(M, G) \to H^1(M, K) \to H^2(M, H) \to \ldots$$

If we start with the short exact sequence:

$$0 \to \mathbb{Z}_2 \xrightarrow{i} Spin(n) \xrightarrow{p} SO(n) \to 0$$

what we get is:

$$\ldots \to H^1(M, \mathbb{Z}_2) \xrightarrow{i^*} H^1(M, Spin(n)) \xrightarrow{p^*} H^1(M, SO(n)) \xrightarrow{\partial^*} H^2(M, \mathbb{Z}_2) \to \ldots$$

Now, if we fix a $SO(n)$-structure $[P] \in H^1(M, SO(n))$, we can define $w_2(P) = \partial^*[P]$. This is called the *second Stiefel-Whitney class of* M. From the exactness of the sequence, we see that $w_2(P) = 0$ if and only if there is $[\mathcal{S}] \in H^1(M, Spin(n))$ such that $p^*[\mathcal{S}] = [P]$, which means that $[\mathcal{S}]$ defines an equivalence class of $Spin(n)$-structures over M. In fact, by working out how p^* acts on the transition functions, one can show directly that $\mathcal{S}$ is a lift of the $SO(n)$-structure P in the sense of definition A.1.1.

From a $Spin(n)$-structure on $\mathcal{S} \to M$ we can construct the vector bundles S associated with the spin representations Δ^n called *spinor bundles*. These are complex vector bundles, and sections of S are called *spinors*. When n is even, we know that Δ^n is reducible, so the spinor bundle splits as a direct sum $S = S^+ \oplus S^-$ of bundles of *positive* and *negative* spinors.

Moreover, the Levi-Civita connection on TM may be lifted by the map of bundles $f : \mathcal{S} \to F(M)$, since f, being a covering map, is locally an isomorphism. We can then induce a connection ∇^S on the spinor bundle. A spinor $\psi \in \Gamma(S)$ satisfying $\nabla^S \gamma$ is called a *parallel spinor*.

From this setup one can define the *Dirac operator* $\not{D} : \Gamma(S) \to \Gamma(S)$ as the composition:

$$\not{D} : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) \xrightarrow{c} \Gamma(S)$$

where the map c is Clifford multiplication. In terms of a local orthonormal frame $\{e_1, \dots, e_n\}$ of sections of S we can write

$$\not{D}\psi = \sum_{i=1}^{n} e_i.\nabla^S_{e_i}\psi$$

A spinor ψ satisfying $\not{D}\psi = 0$ is called a *harmonic spinor*.

In even dimensions, the Dirac operators splits as $\not{D} = \not{D}^+ \oplus \not{D}^-$, with $\not{D}^+ : \Gamma(S^+) \to \Gamma(S^-)$ and $\not{D}^- : \Gamma(S^-) \to \Gamma(S^+)$.

A.2 Chern Classes

Characteristic classes are topological invariants of vector bundles. They are the main objects used to distinguish isomorphism classes of bundles, since the usual topological invariants (homology, cohomology) are not sufficiently strong for this - indeed, since vector spaces are homotopic to a point, every vector bundle is homotopically equivalent to the basis manifold. In this brief section we will define the *Chern classes*, which are cohomology classes associated to a complex vector bundle.

Let (M, J) be a $2n$-dimensional complex manifold and $E \to M$ a rank k complex vector bundle over M. The Chern classes are defined in terms of a connection ∇ on E, but it can be shown that they are in fact independent of the choice of connection (i.e., they only depend on the topology of the bundle).

Definition A.2.1. The *total Chern class* of E is:

$$c(E) = \det(Id + \frac{iF}{2\pi}) \tag{A.2.1}$$

where F is the curvature of a connection ∇ on E. We can write:

$$c(E) = 1 + c_1(E) + c_2(E) + ... + c_n(E)$$

with $c_i(E) \in \Omega^{2i}(M)$ being the i-th *Chern class* of E.

In the above definition, all the Chern classes are forms of even rank since F is a 2-form.

We can expand $c(E)$ in a Taylor series in F to write the Chern classes of E as functions of $tr(F \wedge \ldots \wedge F)$. As a particular case, the first and second Chern classes are:

$$c_1(E) = tr\left(\frac{iF}{2\pi}\right)$$

$$c_2(E) = -\frac{1}{8\pi^2}(tr(F) \wedge tr(F) - tr(F \wedge F))$$

One has to show that these forms are indeed closed, so that they define cohomology classes. This is due to the fact that $dtr(\alpha \wedge \beta) = tr(d_\nabla(\alpha \wedge \beta))$, which can be verified by a direct computation. Since in the Taylor expansion of formula A.2.1 we only get terms of the form $tr(f(F))$, where $f(F)$ is some function of the curvature, it follows from the Bianchi identity that all of these terms define closed forms.

For more about characteristic classes and Chern classes in particular, see [26].

A.3 Hodge Theory

In this section we introduce Hodge theory on (semi)Riemannian manifolds. The highlight of this theory is the Hodge theorem regarding the splitting of the space of differential forms induced by the exterior differential and the metric, and also the identification between the de Rham cohomology and a certain space of "harmonic forms". We will also briefly develop the same theory in the context of Kähler manifolds, where the results can be refined with the aid of the complex structure.

If we start with a compact Riemannian manifold (M, g), we get an induced inner product on $\Omega^1(M)$ defined for $\alpha, \beta \in \Omega^1(M)$ by $g(\alpha, \beta) = g^{ab}\alpha_a\beta_b$. This, in turn, can be extended to all forms in the following way: for $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_k$, $\beta = \beta_1 \wedge \ldots \wedge \beta_k \in \Omega^k(M)$, $g(\alpha, \beta) = \det(g(\alpha_i, \beta_j))$. This is then extended by linearity to all k-forms.

Definition A.3.1. Let (M, g) be a semi-Riemannian manifold of signature $(r, n-r)$, and dV_g the volume form of g. The *Hodge star* or *duality* operator $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is an isomorphism of vector bundles induced by the Riemannian metric g. It is defined to be the unique linear operator satisfying $\alpha \wedge *\beta = g(\alpha, \beta)dV_g$. for α and β k-forms.

One of the first things to notice is that the Hodge star acting on k-forms satisfies $*^2 = (-1)^r(-1)^{k(n-k)}$. This can be proved by finding an explicit formula for the action of $*$ on an orthonormal frame $\{e_1, ..., e_n\}$ of T^*M. In this setting, it is easy to show directly from the definition that $*$ satisfies:

$$*(e_{a_1} \wedge \ldots \wedge e_{a_k}) = g^{a_1a_1} \ldots g^{a_ka_k}\epsilon_{a_1\ldots a_n}e_{a_{k+1}} \wedge \ldots e_{a_n} \tag{A.3.1}$$

where $(a_1, \ldots, a_n)$ is a permutation of $(1, \ldots, n)$ and $\epsilon_{a_1\ldots a_n}$ is the completely anti-symmetric Levi-Civita symbol, i.e., $\epsilon_{a_1\ldots a_n}$ is either 1 or -1 according to whether $(a_1 \ldots a_n)$ is an even or odd permutation of $(1, \ldots, n)$, and it is 0 whenever there are repeated indices.

From this, one easily computes:

$$\begin{aligned} **(e_{a_1} \wedge \ldots \wedge e_{a_k}) &= *(g^{a_1a_1} \ldots g^{a_ka_k}\epsilon_{a_1\ldots a_n}e_{a_{k+1}} \wedge \ldots e_{a_n}) \\ &= g^{a_1a_1} \ldots g^{a_ka_k}\epsilon_{a_1\ldots a_n}g^{a_{k+1}a_{k+1}} \ldots g^{a_na_n}\epsilon_{a_{k+1}\ldots a_na_1\ldots a_k}e_{a_1} \wedge \ldots \wedge e_{a_k} \\ &= (-1)^r(-1)^{k(n-k)}e_{a_1} \wedge \ldots \wedge e_{a_k} \end{aligned} \tag{A.3.2}$$

where we have used the identity $\epsilon_{a_{k+1}\ldots a_na_1\ldots a_k} = (-1)^{k(n-k)}\epsilon_{a_1\ldots a_n}$. This proves that $*^2 = (-1)^r(-1)^{k(n-k)}$ when $*$ acts on k-forms.

We can introduce another inner product in $\Omega^\cdot(M)$ using the volume form dV_g; for k-forms α and β define:

$$(\alpha, \beta) = \int_M g(\alpha, \beta)dV_g \tag{A.3.3}$$

This is called the L^2 inner product. We also define the operator $d^* : \Omega^k(M) \to \Omega^{k-1}(M)$ to be the formal adjoint of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with respect to the L^2 inner product; that is, for $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$ we have:

$$(d\alpha, \beta) = (\alpha, d^*\beta) \tag{A.3.4}$$

Now, since M is compact it follows that $\int_M d(\alpha \wedge *\beta) = 0$. But $d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{k-1}\alpha \wedge d * \beta$ so that:

$$(d\alpha, \beta) = (-1)^k \int_M \alpha \wedge d * \beta$$

As a consequence:

$$d^* = (-1)^k *^{-1} d*$$

and thus:

$$d^*\alpha = (-1)^{kn+n+s+1} * d(*\alpha) \tag{A.3.5}$$

We define the *Laplaci-Beltrami* operator $\Delta : \Omega^k(M) \to \Omega^k(M)$:

$$\Delta = dd^* + d^*d$$

If a k-form α is such that $\Delta\alpha = 0$, we call α a *harmonic form*. Clearly, if $d\alpha = d^*\alpha = 0$ (i.e., α is closed and coclosed) then α is harmonic. The converse is also true: suppose $\nabla\alpha = 0$. Then:

$$0 = (\alpha, \nabla_k\alpha) = (d^*\alpha, d^*\alpha) + (d\alpha, d\alpha)$$

so that $d\alpha = d^*\alpha = 0$.

We will sometimes write Δ_k for the operator Δ acting on k-forms, and similarly d_k and d_k^* for the operators d and d^*.

Now, let $\mathfrak{H}^k = Ker(\Delta_k)$ be the space of harmonic k-forms.

Theorem A.3.2 (Hodge Decomposition Theorem). Let M be a smooth compact oriented Riemannian manifold. Write d_q for the operator d acting on q-forms and similarly for d^*. Then:

$$\Omega^k(M) = Im(d_{k-1}) \oplus Im(d_{k+1}^*) \oplus \mathfrak{H}^k$$

Proof. We can define an operator $P : \Omega^k(M) \to \Omega^k(M)$ given by projection into the space of harmonic forms. Thus, for α a k-form, $\Delta(P\alpha) = 0$. Due to he fact that Δ admits a Green function[2], there exists a k-form β such that $\Delta\beta = \alpha - P\alpha$ (the Green function allows this equation to be "solved for β"). Therefore, the Hodge decomposition of α is:

$$\alpha = dd^*\beta + d^*d\beta + P\alpha$$

□

Since $Im(d_{k-1}) \oplus \mathfrak{H}^k = Ker(d_k)$, we get an isomorphism $\mathfrak{H}^k \cong H^k(M)$. This is the content of *Hodge's theorem*:

[2]For a proof, see [11].

Corollary A.3.3 (Hodge's Theorem)**.** Every de Rham cohomology class $\omega \in H^q(M)$ admits a (unique) harmonic representative; this means we can find $\beta \in \Omega^q(M)$ such that $[\beta] = \omega$ and $\Delta\beta = 0$. Thus we have a vector space isomorphism $H^q(M) \cong \mathfrak{H}^q$.

A.3.1 Hodge Star in Four Dimensions

When $n = 4$ the Hodge star restricted to 2-forms can be regarded as an endomorphism of $\Omega^2(M)$. Since in this case $*^2 = Id$, it follows that $*$ has the eigenvalues 1 and -1. If we fix a coordinate basis $\{dx_1, \dots dx_n\}$ of T^*M, it is easy to check the identities:

$$\begin{aligned} *\,(dx_1 \wedge dx_2 + dx_3 \wedge dx_4) &= dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \\ *\,(dx_1 \wedge dx_3 - dx_2 \wedge dx_4) &= dx_1 \wedge dx_3 - dx_2 \wedge dx_4 \\ *\,(dx_1 \wedge dx_4 + dx_2 \wedge dx_3) &= dx_1 \wedge dx_4 + dx_2 \wedge dx_3 \end{aligned} \tag{A.3.6}$$

and

$$\begin{aligned} *\,(dx_1 \wedge dx_2 - dx_3 \wedge dx_4) &= -(dx_1 \wedge dx_2 - dx_3 \wedge dx_4) \\ *\,(dx_1 \wedge dx_3 + dx_2 \wedge dx_4) &= -(dx_1 \wedge dx_3 + dx_2 \wedge dx_4) \\ *\,(dx_1 \wedge dx_4 - dx_2 \wedge dx_3) &= -(dx_1 \wedge dx_4 - dx_2 \wedge dx_3) \end{aligned} \tag{A.3.7}$$

So we get a decomposition $\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)$ where $\Omega^2_+(M) = Eig(*; +1)$ is the space of *selfdual* 2-forms and $\Omega^2_-(M) = Eig(*; -1)$ is the space of *anti-selfdual* 2-forms.

A.3.2 Hodge Theory on Kähler Manifolds

Now, let (X, g, J, ω) be a compact Hermitian manifold (defined in section 2.2). We define a complex version of the Hodge star operator as follows: $* : \Omega^{(p,q)} \to \Omega^{(mn-p,n-q)}$ is the unique anti-linear operator such that, for complex forms α, $\beta \in \Omega^{(p,q)}$, we have:

$$\alpha \wedge *\bar{\beta} = g(\alpha, \beta) dV$$

We can define a Hermitian inner product on the exterior algebra of X by:

$$(\alpha, \beta) = \int_X g(\alpha, \beta) dV$$

Of course, this inner product respects the decomposition of the exterior algebra into forms of type (p, q), since the metric itself does (if α is of type (p, q) and β of type (p', q'), then $g(\alpha, \beta) = 0$ unless $p = p'$ and $q = q'$).

We can define the adjoint operators ∂^* and $\bar{\partial}^*$ in the usual way; for $\alpha \in \Omega^{(p-1,q)}$, $\beta \in \Omega^{(p,q)}$

$$(\partial\alpha, \beta) = (\alpha, \partial^*\beta)$$

and for $\omega \in \Omega^{(p,q-1)}$ and $\eta \in \Omega^{(p,1)}$:

$$(\bar{\partial}\omega, \eta) = (\omega, \bar{\partial}^*\eta)$$

It follows, of course, that $d^* = \partial^* + \bar{\partial}^*$, and from the analogous formulas for real manifolds we have

$$\partial^* = - * \bar{\partial} *$$

and

$$\bar{\partial}^* = - * \partial *$$

We can also define the associated Laplacians:

$$\Delta_\partial = \partial\partial^* + \partial^*\partial$$

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

A (p,q)-form α is said to be $\Delta_{\bar{\partial}}$-harmonic if $\Delta_{\bar{\partial}} = 0$. The set of $\Delta_{\bar{\partial}}$-harmonic (p,q) -forms is denoted by $\mathfrak{H}^{p,q}_{\bar{\partial}}$. We have then a Hodge decomposition theorem for complex forms:

Theorem A.3.4 (Hodge decoposition theorem)**.** Let (M, g, J, ω) be a compact Hermitian manifold. Then:

$$\Omega^{(p,q)}(M) = Im(\bar{\partial}_{p,q-1}) \oplus Im(\bar{\partial}^*_{p,q+1}) \oplus \mathfrak{H}^{p,q} \tag{A.3.8}$$

This decomposition can be described in the same way as in the real case: we have a projection $P : \Omega^{(p,q)}(M) \to \mathfrak{H}^{p,q}$ and due to the existence of a Green function for $\Delta_{\bar{\partial}}$, for every (p,q)-form α there is a (p,q)-form β such that $\Delta_{\bar{\partial}}\beta = \alpha - P\alpha$. Then the Hodge decomposition of α is simply $\alpha = \bar{\partial}\bar{\partial}^*\beta + \bar{\partial}^*\bar{\partial}\beta + P\alpha$.

Furthermore, since $Ker\bar{\partial}_{p,q} = Im(\bar{\partial}_{p,q-1}) \oplus \mathfrak{H}^{p,q}$, it follows that $H^{p,q}_{\bar{\partial}}(M) \cong \mathfrak{H}^{p,q}$.

Up to now we haven't required the metric to be Kähler, but only the compatibility between the metric and the complex structure (i.e., a Hermitian structure). However, the Kähler condition is necessary for the splitting of the complexified de Rham cohomology, as we now explain.

The crucial fact is that on a Kähler manifold one has the identity:

$$\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta_d \tag{A.3.9}$$

This is one of the well-known *Kähler identities*. For a proof, the reader should consult either **??** or **??**.

If we forget about the decomposition of the exterior algebra into (p,q)-forms, we can define:

$$\mathfrak{H}^k = Ker(\bar{\partial}_k)$$

i.e., the space of harmonic k-forms with respect to $\Delta_{\bar{\partial}}$. Of course, this includes all (p,q)-forms such that $p + q = k$. Then, the real version of Hodge's theorem A.3.3 implies that $\mathfrak{H}^k \cong H^k(X, \mathbb{C})$, the complexified de Rham cohomology of X (recall that we are still dealing with forms on the *complexified* tangent bundle, so that $\Delta_{\bar{\partial}}$ is really seen as a map $\Lambda^k T^*M \otimes \mathbb{C} \to \Lambda^k T^*M \otimes \mathbb{C}$). From this, one can consider the space $H^{p,q}(X, \mathbb{C}) \subset H^{p+q}(X, \mathbb{C})$, defined to be the subspace with representatives in $\mathfrak{H}^{p,q} \subset \mathfrak{H}^{p+q}$. We have then the following version of Hodge's theorem:

Theorem A.3.5 (Hodge's theorem)**.** Let (X, g, J, ω) be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. Then there exists a decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C})$$

which does not depend on the choice of Kähler structure.

Furthermore, every element of $H^{p,q}(X,\mathbb{C})$ is reresented by a unique harmonic (p,q)-form, so that $H^{p,q}(X,\mathbb{C}) \cong \mathfrak{H}^{p,q} \cong H^{p,q}_{\bar{\partial}}(X)$. We also have that:

$$\overline{H^{p,q}(X,\mathbb{C})} \cong H^{q,p}(X,\mathbb{C})$$

and

$$H^{p,q}(X,\mathbb{C}) \cong H^{n-p,n-q}(X,\mathbb{C})^*$$

The same statements hold of course for $\mathfrak{H}^{p,q}$ and $H^{p,q}_{\bar{\partial}}(X)$, since these are all isomorphic. The statement $H^{p,q}_{\bar{\partial}}(X) \cong H^{n-p,n-q}_{\bar{\partial}}(X)^*$ is known in the literature as *Serre duality* (in analogy with Poincaré duality for the de Rham cohomology of real manifolds).

Definition A.3.6. The *Hodge number* of X are defined by:

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X,\mathbb{C})$$

From the previous results we have the identities:

$$\begin{gathered} b^k = \sum_{i=0}^{k} h^{i,k-i} \\ h^{p,q} = h^{q,p} = h^{n-q,n-p} = h^{n-p,n-q} \end{gathered} \tag{A.3.10}$$

where $b^k = \dim_{\mathbb{R}} H^k(X)$ is the kth Betti number of X.

The Hodge numbers provide more refined topological information about X than the Betti numbers. They are invariants of the hermitian structure and they also provide information about the existence of Kähler metrics on complex manifolds. For example, the manifold $M = S^3 \times S^1$ admits a complex structure, but no Kähler metric. This follows from the fact that $b^1(M) = 1$, and that for a Kähler manifold X, $b^1(X) = h^{0,1} + h^{1,0} = 2h^{0,1}$ is even.

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