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Regularization of Filippov systems near regular-tangential singularities and polycycles

Regularização de sistemas de Filippov próximos de singularidades tangenciais-regulares e policiclos

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

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Supervisor: Douglas Duarte Novaes

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To my family.

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"The LORD is my shepherd. I lack nothing".
(Psalm 23:1)

Resumo

Compreender como as singularidades tangenciais evoluem em processos de regularização foi um dos primeiros problemas relacionados à regularização dos sistemas de Filippov. Neste trabalho, estamos interessados em C^n-regularizações de sistemas Filippov em torno de singularidades tangenciais regulares visíveis de multiplicidade par. Mais especificamente, usando a Teoria de Fenichel e os Métodos de Blow-up, nosso objetivo é entender como as trajetórias do sistema regularizado transitam pela região de regularização. Aplicamos nossos resultados para investigar as C^n-regularizações dos ciclos limites de fronteira com contato de multiplicidade par com a variedade de deslize.

Além disso, estamos interessados na regularização de sistemas de Filippov em torno de conexões homoclínicas com singularidades tangenciais regulares. Fornecemos condições para garantir a existência de ciclos limites bifurcando-se de tais conexões. Condições adicionais também são fornecidas para garantir a estabilidade e unicidade de tais ciclos limites. Todas as provas são baseadas na construção do mapa de primeiro retorno do sistema de Filippov regularizado em torno de conexões homoclínicas. Tal mapa é obtido usando a nossa caracterização do comportamento local do sistema de Filippov regularizado em torno de singularidades tangenciais regulares. Teoremas de ponto fixo e argumentos de Poincaré-Bendixson também são usados.

Palavras-chave: Regularização. Sistemas de Filippov. Teoria de Fenichel. Método de Blow-up. Ciclos Limites. Policiclos.

Abstract

Understanding how tangential singularities evolves under smoothing processes was one of the first problem concerning regularization of Filippov systems. In this work, we are interested in C^n-regularizations of Filippov systems around visible regular-tangential singularities of even multiplicity. More specifically, using Fenichel Theory and Blow-up Methods, we aim to understand how the trajectories of the regularized system transits through the region of regularization. We apply our results to investigate C^n-regularizations of boundary limit cycles with even multiplicity contact with the switching manifold.

Moreover, we are concerned about smoothing of Filippov systems around homoclinic-like connections to regular-tangential singularities. We provide conditions to guarantee the existence of limit cycles bifurcating from such connections. Additional conditions are also provided to ensured the stability and uniqueness of such limit cycles. All the proofs are based on the construction of the first return map of the regularized Filippov system around homoclinic-like connections. Such a map is obtained by using our characterization of the local behaviour of the regularized Filippov system around regular-tangential singularities. Fixed point theorems and Poincaré-Bendixson arguments are also employed.

Keywords: Regularization. Filippov systems. Fenichel Theory. Blow-up Method. Limit cycles. $\Sigma-$Polycycles.

Contents

1 Introduction

The analysis of differential equations with discontinuous right-hand side dates back to the work of Andronov et. al [1] in 1937. Recently, the interest in such systems has increased significantly, mainly motivated by its wide range of applications in several areas of applied sciences. Piecewise smooth differential systems are used for modeling phenomena presenting abrupt behavior changes such as impact and friction in mechanical systems [4], refugee and switching feeding preference in biological systems [20, 28], gap junctions in neural networks [9], and many others.

In this work, we are interested in planar piecewise smooth systems. Formally, let M be an open subset of $\mathbb{R}^2$ and let $N \subset M$ be a codimension 1 submanifold of M. Denote by C_i, $i = 1, 2, \ldots, k$, the connected components of $M \backslash N$ and let $X_i : M \to \mathbb{R}^2$, for $i = 1, 2, \ldots, k$, be vector fields defined on M. A piecewise smooth vector field Z on M is defined by

$$Z(p) = X_i(p) \text{ if } p \in \overline{C_i}, \text{ for } i = 1, 2, \ldots, k. \tag{1.1}$$

Since N is a codimension 1 submanifold of M, for each $p \in N$ there exists a neighborhood $\mathcal{D} \subset M$ of p and a function $h : \mathcal{D} \to \mathbb{R}$, having 0 as a regular value, such that $\Sigma = N \cap \mathcal{D} = h^{-1}(0)$. Moreover, the neighborhood $\mathcal{D}$ can be taken sufficiently small in order that $\mathcal{D} \backslash \Sigma$ is composed by two disjoint regions $\Sigma^+ = \{q \in \mathcal{D} : h(q) \geqslant 0\}$ and $\Sigma^- = \{q \in \mathcal{D} : h(q) \leqslant 0\}$ such that $X^+ = Z|_{\Sigma^+}$ and $X^- = Z|_{\Sigma^-}$ are smooth vector fields. Accordingly, the piecewise smooth vector field (1.1) can be locally described as follows:

$$Z(p) = (X^+, X^-)_\Sigma = \begin{cases} X^+(p), & \text{if } \quad p \in \Sigma^+, \\ X^-(p), & \text{if } \quad p \in \Sigma^-, \end{cases} \quad \text{for} \quad p \in \mathcal{D}.$$

Throughout this work we will denote the components of $X^\pm$ by $X_i^\pm$ for $i \in \{1, 2\}$, i.e. $X^\pm = (X_1^\pm, X_2^\pm)$.

1.1 Filippov Systems

The notion of local trajectories of piecewise smooth vector fields (1.1) was stated by Filippov [13] as solutions of the following differential inclusion

$$\dot{p} \in \mathcal{F}_Z(p) = \frac{X^+(p) + X^-(p)}{2} + \text{sign}(h(p)) \frac{X^+(p) - X^-(p)}{2}, \tag{1.2}$$

where

$$\text{sign}(s) = \begin{cases} -1 & \text{if } s < 0, \\ [-1, 1] & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases}$$

This approach is called Filippov's convention. The piecewise smooth vector field (1.1) is called Filippov system when it is ruled by the Filippov's convention. For more informations on differential inclusions see, for instance, [29].

The solutions of the differential inclusion (1.2) are well described in the literature (see, for instance, [13]) and have a simple geometrical interpretation. In order to illustrate this convention we define the following open regions on Σ :

$$\begin{aligned}\Sigma^c &= \{p \in \Sigma : \; X^+h(p) \cdot X^-h(p) > 0\},\\ \Sigma^s &= \{p \in \Sigma : \; X^+h(p) < 0, \; X^-h(p) > 0\},\\ \Sigma^e &= \{p \in \Sigma : \; X^+h(p) > 0, \; X^-h(p) < 0\}.\end{aligned}$$

Here, $X^{\pm}h(p) = \langle \nabla h(p), X^{\pm}(p)\rangle$ denotes the Lie derivative of h in the direction of the vector fields $X^{\pm}$. Usually, they are called *crossing*, *sliding*, and *escaping* region, respectively. Notice that the points on Σ where both vectors fields X^+ and X^- simultaneously point outward or inward from Σ constitute, respectively, the *escaping* Σ^e and *sliding* Σ^s regions, and the complement of its closure in Σ constitutes the *crossing region*, Σ^c. The complement of the union of those regions in Σ constitutes the *tangency* points between X^+ or X^- with Σ, Σ^t.

For $p \in \Sigma^c$ the trajectories either side of the discontinuity Σ, reaching p, can be joined continuously, forming a trajectory that crosses Σ^c. Alternatively, for $p \in \Sigma^{s,e} = \Sigma^s \cup \Sigma^e$ the trajectories either side of the discontinuity Σ, reaching p, can be joined continuously to trajectories that slide on $\Sigma^{s,e}$ following the sliding vector field,

$$Z^s(p) = \frac{X^-h(p)X^+(p) - X^+h(p)X^-(p)}{X^-h(p) - X^+h(p)}, \text{ for } p \in \Sigma^{s,e}. \tag{1.3}$$

In addition, a singularity of the sliding vector field Z^s is called pseudo-equilibrium.

In the Filippov context, the notion of Σ-*singular points* also comprehends the tangential points Σ^t constituted by the contact points between X^+ and X^- with Σ, i.e. $\Sigma^t = \{p \in \Sigma : \; X^+h(p) \cdot X^-h(p) = 0\}$. In this work, we are interested in contact points of finite degeneracy. Recall that p is a *contact of order* $k-1$ (or multiplicity k) between $X^{\pm}$ and Σ if 0 is a root of multiplicity k of $f(t) := h \circ \varphi_{X^{\pm}}(t,p)$, where $t \mapsto \varphi_{X^{\pm}}(t,p)$ is the trajectory of $X^{\pm}$ starting at p. Equivalently,

$$X^{\pm}h(p) = (X^{\pm})^2h(p) = \ldots = (X^{\pm})^{k-1}h(p) = 0, \text{ and } (X^{\pm})^k h(p) \neq 0.$$

In addition, an even multiplicity contact, say $2k$, is called *visible* for X^+ (resp. X^-) when $(X^+)^{2k}h(p) > 0$ (resp. $(X^-)^{2k}h(p) < 0$). Otherwise, it is called *invisible.*

In this work, we shall focuses our attention in *visible regular-tangential singularities of multiplicity* $2k$, which are formed by a visible even multiplicity of X^+ and a regular point of X^-, or vice versa (see Figure 1). In the above definitions, the higher order Lie derivatives X^ih are defined, inductively, by $Xh(p) = \langle \nabla h(p), X(p)\rangle$ and $X^ih(p) = X(X^{i-1}h)(p)$ for $i > 1$.

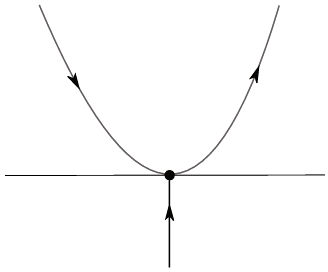

Figure 1 – Visible regular-tangential singularity.

1.2 Sotomayor-Teixeira Regularization

Frequently, non-smooth mathematical models are discontinuous idealizations of regular phenomena. Thus, in order to investigate a regular phenomenon with non-smooth mathematical model, it seems natural to inquire how the solutions of smooth systems, which converge to the non-smooth one, behave in the limit. Therefore, it is usual to consider a non-smooth system as a singular limit of a 1-parameter family of smooth vector fields.

Roughly speaking, a smoothing process of a piecewise smooth vector field Z consists in obtaining a one–parameter family of continuous vector fields Z_ε converging to Z when $\varepsilon \to 0$. A well known smoothing process is the Sotomayor-Teixeira regularization, which was introduced in [31]. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function satisfying $\phi(\pm 1) = \pm 1$, $\phi^{(i)}(\pm 1) = 0$ for $i = 1, 2, \ldots, n$, and $\phi'(s) > 0$ for $s \in (-1, 1)$. Then, a C^n-Sotomayor-Teixeira regularization (or just C^n-regularization for short) takes

$$Z_\varepsilon^\Phi(p) = \frac{1 + \Phi_\varepsilon(h(p))}{2} X^+(p) + \frac{1 - \Phi_\varepsilon(h(p))}{2} X^-(p), \ \ \Phi_\varepsilon(h) = \Phi(h/\varepsilon), \tag{1.4}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is defined as the following C^n function

$$\Phi(s) = \begin{cases} \phi(s) & \text{if} \quad |s| \leqslant 1, \\ \operatorname{sign}(s) & \text{if} \quad |s| \geqslant 1. \end{cases} \tag{1.5}$$

We call Φ a C^n*-monotonic transition function.* Proposition 9 provides examples of transition functions.

In addition, we can define the set of C^{n-1}-monotonic transition functions which are not C^n at ± 1 as follows.

Definition 1. *Denote by C^{n-1}_{ST} the set of C^{n-1}-monotonic transition functions Φ which are not C^n at ± 1. That is, for a $\Phi \in C^{n-1}_{ST}$ given as (1.5), then $\phi^{(i)}(\pm 1) = 0$, for $i = 1, 2, \ldots, n-1$, and $\phi^{(n)}(\pm 1) \neq 0$. Moreover, one can easily see that* $\operatorname{sign}\big(\phi^{(n)}(\pm 1)\big) = (\mp 1)^{n+1}$.

Notice that the vector field $Z_\varepsilon^\Phi(p)$ coincides with $X^+(p)$ or $X^-(p)$ whether $h(p) \geqslant \varepsilon$ or $h(p) \leqslant -\varepsilon$, respectively. In the region $|h(p)| \leqslant \varepsilon$, the vector $Z_\varepsilon^\Phi(p)$ is a linear combination of $X^+(p)$ and $X^-(p)$.

The Sotomayor-Teixeira regularization is the most widespread smoothing process. That is mainly because its intrinsic relation with Filippov's convention. Indeed, in [5], it was shown that the Sotomayor-Teixeira regularization of Filippov systems gives rise to *Singular Perturbation Problems*, for which the corresponding reduced dynamics is conjugated to the sliding dynamics (1.3). This kind of relation has been further investigated in [22] for more general transition functions. For more informations on Singular Perturbation Problems see, for instance, [12, 16].

1.3 Σ-Polycycles in Filippov Systems

A polycycle is a simple closed curve composed by a collection of singularities and regular orbits, inducing a first return map. In these last decades, the study of polycycles in Filippov systems have been considered in several papers. For instant, in [19] the authors introduced the critical crossing cycle bifurcation, which is defined as a one-parameter family Z_α of Filippov systems, where Z_0 has a homoclinic-like connection to a fold-regular singularity. In [14], Freire et al. proved that the unfolding of a critical crossing cycle bifurcation provided in [19] holds in a generic scenario.

More degenerate homoclinic-like connections to Σ-singularities has also been considered. In [25], the authors studied a codimension-two homoclinic-like connection to a visible-visible fold-fold singularity. In [24] its unfolding under non-autonomous periodic perturbation has been studied. Recently, in [10], Andrade et. al. developed a rather general method to investigate the unfolding of Σ-polycycles in Filippov systems. This method was applied to describe bifurcation diagrams of Filippov systems around several Σ-polycycles. The readers are referred to [6, 7, 15, 30, 32] for more studies on $\Sigma-$polycycles.

The interest in studying polycycles is due to the fact that they are non-local invariant sets that provide information on the dynamics of the system.

In what follows, we shall introduce some basic concepts for this work. First, we define the local separatrix at a point $p \in \Sigma$.

Definition 2. *If $p \in \Sigma$, the **asymptotically stable (resp. unstable) separatrix** $W^s_{t,\text{ʇ}}(p)$, (resp. $W^u_{t,\text{ʇ}}(p)$) of $Z = (X^+, X^-)$ at a visible regular-tangential singularity p in $\Sigma^\pm$ is defined as*

$$\begin{aligned} W^{s,u}_t(p) &= \{q = \varphi_{X^+}(t(q), p) : \varphi_{X^+}(I(q), p) \subset \Sigma^+ \text{ and } \delta_{s,u} t(q) > 0\}, \\ W^{s,u}_{\text{ʇ}}(p) &= \{q = \varphi_{X^-}(t(q), p) : \varphi_{X^-}(I(q), p) \subset \Sigma^- \text{ and } \delta_{s,u} t(q) > 0\}, \end{aligned}$$

where, $\delta_u = 1$, $\delta_s = -1$, and $I(q)$ is the open interval with extrema 0 and $t(q)$.

Now, we introduce the concepts of regular orbit and $\Sigma-$polycycle for planar Filippov systems.

Definition 3. *Consider the Filippov system $Z = (X^+, X^-)$.*

*(i) We say that γ is a **regular orbit** of Z if it is a piecewise smooth curve such that $\gamma \cap \Sigma^+$ and $\gamma \cap \Sigma^-$ are unions of regular orbits of X^+ and X^-, respectively, and $\gamma \cap \Sigma \subset \Sigma^c$.*

*(ii) A closed curve Γ is said to be a $\Sigma-$**polycycle** of Z if it is composed by a finite number of $\Sigma-$singularities, $p_1, p_2, \cdots, p_n \in \Sigma$, and a finite number of regular orbits of Z, $\gamma_1, \gamma_2, \cdots, \gamma_n$, such that for each $1 \leqslant i \leqslant n$, γ_i has ending points p_i and p_{i+1} ($p_{n+1} = p_1$), and satisfies that*

a) Γ is a S^1-immersion and it is oriented by increasing time along the regular orbits;

b) there exists a non-constant first return map π_Γ defined, at least, in one side of Γ.

Remark 1. *Condition (a) in Definition 3 gives the minimality of polycycles of Z, i.e. a polycycle Γ can not be written as union of two or more polycycles. This condition also establishes the notion of sides of Γ, which is used in Condition (b).*

In what follows, we will establish the classes of $\Sigma-$polycycles that will be studied in this work. To define these classes, consider the Filippov system $Z = (X^+, X^-)$ and suppose that Z has a $\Sigma-$polycycle Γ having a unique visible regular-tangential singularity at p of multiplicity $2k$, for $k \geqslant 1$. Through a local change of coordinates, we can assume that $p = (0, 0)$ and $h(x, y) = y$. Now, we shall locally characterize Γ around p. For this, it is enough to divide the $\Sigma-$polycycles in the following 2 classes:

(a) $W^s_t(p) \cup W^u_t(p) \subset \Gamma$ and $X^- h(p) > 0$ or $X^- h(p) < 0$;

(b) $W^s_{\pitchfork}(p) \cup W^u_t(p) \subset \Gamma$ or $W^u_{\pitchfork}(p) \cup W^s_t(p) \subset \Gamma$.

Here, Σ-polycycles satisfying condition (a) or (b) will be called Σ-polycycles of type (a) or type (b), respectively (see Figure 2).

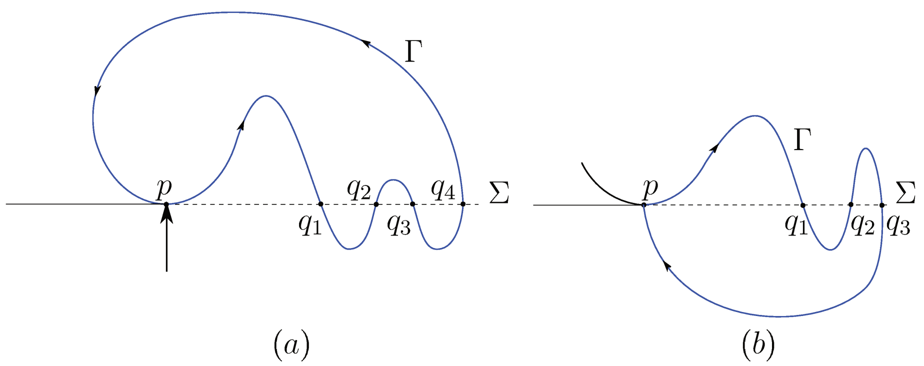

Figure 2 – Examples of $\Sigma-$polycycles Γ of type (a) and (b), respectively.

1.4 Main Goal

Understanding how tangential singularities evolves under smoothing processes was one of the first problem concerning smoothing of Filippov systems. Indeed, in the earlier work of Sotomayor and Teixeira [31], it is proved that around a regular-fold singularity of a Filippov system Z, the regularized system Z_ε^Φ possesses no singularities. Recently, based on the findings of [5], some works got deeper results by studying the corresponding slow-fast problems.

In [3] and [2], asymptotic methods [21] were used to study C^n-regularizations of generic regular-fold singularities and fold-fold singularities, respectively. In [18] and [17], the blow-up method introduced in [11] was adapted to study C^n-regularizations of fold-fold singularities and an analytic regularization of a regular-fold singularity, respectively.

In this work, we are interested in C^n-regularizations of Filippov systems around visible regular-tangential singularities of even multiplicity. More specifically, we aim to understand how the trajectories of the regularized system transits through the regions $h(p) \geqslant \varepsilon$, $|h(p)| \leqslant \varepsilon$, and $h(p) \leqslant -\varepsilon$. Accordingly, we characterize two transition maps, namely the *Upper Transition Map* $U_\varepsilon(y)$ and the *Lower Transition Map* $L_\varepsilon(y)$ (see Figure 3). The results are applied to study C^n-regularizations of $\Sigma-$polycycles with a unique even multiplicity contact with the switching manifold.

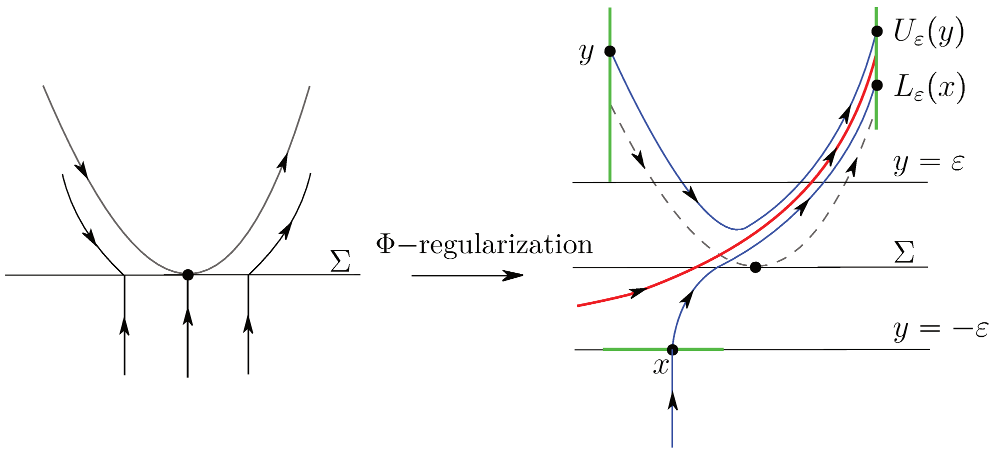

Figure 3 – Upper Transition Map $U_\varepsilon(y)$ and Lower Transition Map $L_\varepsilon(y)$ defined for C^n-regularizations of Filippov systems around visible regular-tangential singularities of even multiplicity.

Our first two main results, Theorems A and B, characterize the *Upper Transition Map* $U_\varepsilon(y)$ and the *Lower Transition Map* $L_\varepsilon(x)$, respectively. More specifically, the flow of the regularized system determines the maps $U_e(y)$ and $L_\varepsilon(x)$ and these are defined in the largest possible interval and the arrival of these maps are independent of the starting point. Theorem A generalizes to degenerate regular-tangential singularities the results obtained in [3] for regular-fold singularities. The main difference between our problem and the problem addressed in [3] is that a regular-fold singularity admits a normal form which simplify significantly the study, whilst here we have to deal with higher order terms.

Then, we will use Theorem A to prove Theorem C, which provides sufficient conditions for the existence of an asymptotically stable limit cycle of the regularized system bifurcating from a boundary limit cycle of a Filippov system with degenerated contact with the switching manifold. Theorems A, B and C were proved by the author in [26].

In addition, we study a case of uniqueness and nonexistence of limit cycles for the regularization of boundary limit cycles. More specifically, in Proposition 10 we provide a class of piecewise smooth vector fields having a boundary limit cycle for which its regularization either does not admit limit cycle or admit an unique limit cycle converging to the boundary limit cycle.

Since the boundary limit cycle is a $\Sigma-$polycycle, we would like to generalize Theorem C. For this, we will apply Theorems A and B to the study C^n-regularizations of $\Sigma-$polycycles with a unique even multiplicity contact with the switching manifold.

The next result of this work (Theorems D) is concerned with regularization of Filippov systems having a $\Sigma-$polycycle of type (a). It establishes sufficient conditions for the existence and nonexistence of limit cycles of the regularized system Z_ε^Φ passing through of a certain compact set with nonempty interior. When the limit cycle exists, its stability is characterized and its convergence to the $\Sigma-$polycycle is ensured.

Moreover, we would like to get a version of Proposition 10 for $\Sigma-$polycycles of type (a). More specifically, in Proposition 11 we provide a class of piecewise smooth vector fields having a $\Sigma-$polycycle of type (a) for which its regularization either does not admit limit cycle or admit an unique limit cycle converging to the $\Sigma-$polycycle of type (a).

The last result of this thesis (Theorem E) is concerned with regularization of Filippov systems having a $\Sigma-$polycycle of type (b). It establishes sufficient conditions for the existence of limit cycles of the regularized system Z_ε^Φ converging to the $\Sigma-$polycycle.

The proofs of Theorems C, D, and E are mainly based on the characterization of the upper and lower transition maps around regular-tangential singularities, fixed point theorems, and *Poincaré-Bendixson Theorem*. Theorems D and E were proved by the author in [23].

1.5 Structure of the thesis

In Chapter 2, we provide a simpler local expression for Filippov systems around visible regular-tangential singularities as well as some preliminary results. In addition, in Section 2.2 we apply blow-up methods to study the Fenichel Manifold associated to the singular perturbation problem arising from C^n-regularizations of visible regular-tangential singularities.

In Chapter 3, we give our first main results Theorems A and B, which characterize the transition maps near C^n-regularizations of visible regular-tangential singularities. Then, Theorems A and B are proven in Sections 3.2.5 and 3.3.4, respectively.

In Chapter 4, we state Theorem C regarding C^n-regularizations of boundary limit cycles. Then, Theorems C is proven in Section 4.3. Moreover, in Section 4.4 in light of our results, we perform an analysis of C^n-regularizations of piecewise polynomial examples admitting a boundary limit cycle.

In Chapter 5, we give Proposition 10, which is a case of uniqueness and nonexistence of limit cycles of the regularized system bifurcating from a boundary limit cycle of a Filippov system with degenerated contact with the switching manifold. For this, in Section 5.1 we introduce a special map called the Mirror map of the regularized system.

Finally, in Chapter 6 we state Theorems D and E regarding C^n-regularizations of $\Sigma-$polycycles of type (a) and (b), respectively. Then, Theorems D and E are proven in Sections 6.1.1 and 6.2.1, respectively. Furthermore, in section 6.1.2 we provide Proposition 11, which is a version of Proposition 10 for $\Sigma-$polycycles of type (a).

2 The Fenichel Manifold

One of our main goals in this work is to understand the C^n-regularization of Filippov systems around visible regular-tangential singularities of even multiplicity. The regularized system (1.4) can be studied through a slow-fast problem, which have associated a critical manifold. This manifold looses its normal hyperbolicity around a certain point. For this reason, around this point we cannot apply Fenichel Theorem 1 to find the Fenichel manifold associated with the slow-fast problem. This problem is overcome by means of the blow-up method. This method will be used to extend the Fenichel manifold to a certain transversal section. Firstly, we establish a canonical form of the Filippov system $Z = (X^+, X^-)$ that will be used throughout this work.

2.1 Canonical Form

Let $X^{\pm}$ be $\mathcal{C}^{2k}$, $k \geqslant 1$, vector fields defined on an open subset V of $\mathbb{R}^2$ and let Σ be a $\mathcal{C}^{2k}$ embedded codimension one submanifold of V. Suppose that X^+ has a visible $2k$-multiplicity contact with Σ at $(0,0)$ and that X^- is pointing towards Σ at $(0,0)$. Consider the Filippov system $Z = (X^+, X^-)_\Sigma$. Denote by $\varphi_{X^{\pm}}$ the flows of $X^{\pm}$.

First, we know that there exists a local $\mathcal{C}^{2k}$ diffeomorphism φ_1 defined on a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $\widetilde{\Sigma} = \varphi_1(\Sigma) = h^{-1}(0)$, with $h(x,y) = y$. Second, applying the *Tubular Flow Theorem* (see [27]) for $(\varphi_1)_* X^-$ at $(0,0)$ and considering the transversal section $\widetilde{\Sigma}$, there exists a local $\mathcal{C}^{2k}$ diffeomorphism φ_2 defined on U (taken smaller if necessary) such that $\widetilde{X}^- = (\varphi_2 \circ \varphi_1)_* X^- = (0,1)$ and $\varphi_2(\widetilde{\Sigma}) = \widetilde{\Sigma}$. Clearly, the transformed vector field $\widetilde{X}^+ = (\varphi_2 \circ \varphi_1)_* X^+$ still has a visible $2k$-multiplicity contact with $\widetilde{\Sigma}$ at $(0,0)$. Thus, without loss of generality, we can assume that the Filippov system $Z = (X^+, X^-)_\Sigma$ satifies

- **(A)** X^+ has a visible $2k$-multiplicity contact with Σ at $(0,0)$, $X_1^+(0,0) > 0$, and there exists a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $X^-\big|_U = (0,1)$ and $\Sigma \cap U = \{(x,0) : x \in (-x_U, x_U)\}$.

Now, we provide a simpler local expression for Filippov systems satisfying hypothesis **(A)** in a neighborhood of the visible regular-tangential singularity. Since $X_1^+(0,0) > 0$, we can take the neighborhood U smaller in order that $X_1^+(x,y) > 0$ for all $(x,y) \in U$. Performing a time rescaling in X^+, we get $\widehat{X}^+(x,y) = (1, f(x,y))$, with the function f given by $f(x,y) = X_2^+(x,y)/X_1^+(x,y)$. Clearly, the vector fields X^+ and $\widehat{X}^+$ have the same orbits in U with the same orientation. Notice that, for $(x,y) \in U$, we have

$$\begin{aligned} X^+h(x,y) = &\ X_2^+(x,y) \\ = &\ X_1^+(x,y) f(x,y) \\ = &\ X_1^+(x,y) \widehat{X}^+ h(x,y). \end{aligned}$$

In general, $(\widehat{X}^+)^i h(0,0) = 0$ if, and only if, $(X^+)^i h(0,0) = 0$, for all $i = 1, \ldots, 2k$. Moreover,

$$\widehat{X}^+ h(x,0) = f(x,0) \quad \text{and}$$
$$(\widehat{X}^+)^i h(0,0) = \frac{\partial^{i-1} f}{\partial x^{i-1}}(0,0),\ \forall i = 1, \ldots, 2k.$$

Therefore, expanding $f(x,0)$ around $x = 0$, we get

$$f(x,0) = \sum_{i=0}^{2k-1} \frac{1}{i!}\frac{\partial^i f}{\partial x^i}(0,0)x^i + g(x) = \alpha x^{2k-1} + g(x),$$

where $\alpha = \dfrac{(\widehat{X}^+)^{2k} h(0,0)}{(2k-1)!} > 0$ and $g(x) = \mathcal{O}(x^{2k})$ is a $\mathcal{C}^{2k}$ function. Consequently, the function $f(x,y)$ writes

$$f(x,y) = \alpha x^{2k-1} + g(x) + y\vartheta(x,y),$$

where ϑ is a $\mathcal{C}^{2k}$ function. Finally, dropping the hat, the Filippov system $Z = (X^+, X^-)_\Sigma$ on U becomes

$$\begin{aligned} X^+(x,y) &= (1, \alpha x^{2k-1} + g(x) + y\vartheta(x,y)), \\ X^-(x,y) &= (0,1), \end{aligned} \tag{2.1}$$

with $\alpha > 0$. Moreover, $\partial_y X_2^+(0,0) = \vartheta(0,0)$.

The next result establishes the intersection between the trajectory of X^+ (satisfying **(A)**) starting at $(0,0)$ with some sections (see Figure 4).

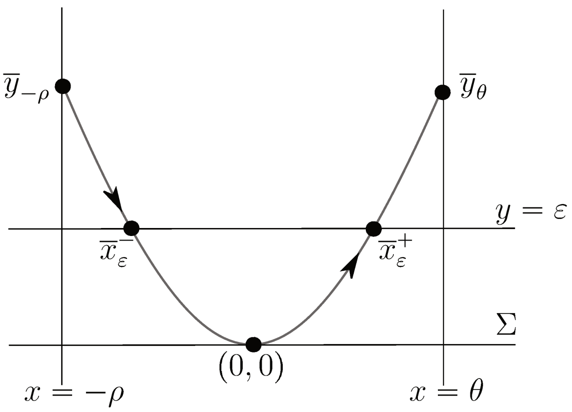

Figure 4 – Transversal intersections of the trajectory of X^+ passing through the $2k$-multiplicity contact $(0,0)$ with the transversal sections $\{x = -\rho\}$, $\{x = \theta\}$, and $\{y = \varepsilon\}$.

Lemma 1. *Assume that X^+ satisfies hypothesis* **(A)**. *For $\rho > 0$, $\theta > 0$, and $\varepsilon > 0$ sufficiently small, the trajectory of X^+ starting at $(0,0)$ intersects transversally the sections $\{x = -\rho\}$, $\{x = \theta\}$, and $\{y = \varepsilon\}$, respectively, at $(-\rho, \overline{y}_{-\rho})$, $(\theta, \overline{y}_\theta)$, and $(\overline{x}_\varepsilon^\pm, \varepsilon)$, where*

$$\overline{y}_x = \frac{\alpha\, x^{2k}}{2k} + \mathcal{O}(x^{2k+1}) \quad \text{and} \quad \overline{x}_\varepsilon^\pm = \pm\varepsilon^{\frac{1}{2k}} \left(\frac{2k}{\alpha}\right)^{\frac{1}{2k}} + \mathcal{O}(\varepsilon^{1+\frac{1}{2k}}). \tag{2.2}$$

Proof. Let us consider the differential equation induced by the vector field X^+

$$\begin{cases} x' = & 1, \\ y' = & \alpha x^{2k-1} + g(x) + y\vartheta(x,y). \end{cases} \tag{2.3}$$

Denote by $(x(t), y(t))$ the solution of system (2.3) satisfying $x(0) = 0$ and $y(0) = 0$. Thus, $x(t) = t$ and $y(t)$ satisfies the following differential equation

$$y' = \alpha t^{2k-1} + g(t) + y\vartheta(t,y).$$

Therefore, $y^{(i)}(0) = 0$ for $i = 0, 1, \ldots, 2k-1$ and $y^{(2k)}(0) = (2k-1)!\alpha$. Thus, the Taylor series of $y(t)$ around $t = 0$ writes

$$y(t) = \frac{\alpha t^{2k}}{2k} + \mathcal{O}(t^{2k+1}).$$

Hence, taking $\rho > 0$ and $\theta > 0$ sufficiently small, we conclude that the trajectory of X^+ starting at $(0,0)$ intersects the sections $\{x = -\rho\}$ and $\{x = \theta\}$ at the points defined in (2.2) $(-\rho, \overline{y}_{-\rho})$ and $(\theta, \overline{y}_\theta)$, respectively. These intersections are transversal, because $X_1^+(x,y) = 1$ for every $(x,y) \in U$.

Now, we shall study the intersection $y(t) = \varepsilon$, so define $\kappa(t,\varepsilon) = y(t) - \varepsilon$. Consider the change of variables $s = t^{2k}$ and define the function

$$\zeta(s,\varepsilon) = \kappa(s^{\frac{1}{2k}}, \varepsilon) = \frac{\alpha s}{2k} - \varepsilon + \mathcal{O}(s^{\frac{2k+1}{2k}}).$$

Since $\zeta(0,0) = 0$ and $\dfrac{\partial \zeta}{\partial s}(0,0) = \dfrac{\alpha}{2k} > 0$, by the *Implicit Function Theorem*, there exists a unique smooth function $s(\varepsilon)$ such that $\zeta(s(\varepsilon),\varepsilon) = 0$ and $s(0) = 0$. Moreover,

$$s'(0) = -\frac{\dfrac{\partial \zeta}{\partial \varepsilon}(0,0)}{\dfrac{\partial \zeta}{\partial s}(0,0)} = \frac{2k}{\alpha}.$$

Thus, the Taylor expansion of $s(\varepsilon)$ around $\varepsilon = 0$ writes

$$s(\varepsilon) = \varepsilon \frac{2k}{\alpha} + \mathcal{O}(\varepsilon^2).$$

Since, $s(\varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small, we can defined $t^{\pm}(\varepsilon) = \pm(s(\varepsilon))^{\frac{1}{2k}}$. Therefore,

$$t^{\pm}(\varepsilon) = \pm \varepsilon^{\frac{1}{2k}} \left(\frac{2k}{\alpha}\right)^{\frac{1}{2k}} + \mathcal{O}(\varepsilon^{1+\frac{1}{2k}}).$$

Hence, the trajectory of X^+ starting at $(0,0)$ intersects the section $\{y = \varepsilon\}$ at the points $\left(\overline{x}_\varepsilon^{\pm}, \varepsilon\right)$ defined in (2.2). We conclude this proof by showing that these intersections are transversal for $\varepsilon > 0$ small enough. Indeed, suppose that $X_2^+\left(\overline{x}_\varepsilon^{\pm}, \varepsilon\right) = 0$. Thus,

$$\alpha(\overline{x}_\varepsilon^{\pm})^{2k-1} + (\overline{x}_\varepsilon^{\pm})^{2k-1}\widetilde{g}(\overline{x}_\varepsilon^{\pm}) + \varepsilon\vartheta(\overline{x}_\varepsilon^{\pm}, \varepsilon) = 0,$$

and, consequently, $(\overline{x}_\varepsilon^{\pm})^{2k-1} = -\dfrac{\varepsilon\vartheta(\overline{x}_\varepsilon^{\pm},\varepsilon)}{\alpha+\widetilde{g}(\overline{x}_\varepsilon^{\pm})}$, where $\widetilde{g} = \mathcal{O}(x)$ is a continuous function such that $g(x) = x^{2k-1}\widetilde{g}(x)$. Thus,

$$\left|(\overline{x}_\varepsilon^{\pm})^{2k-1}\right| = \left|\frac{\vartheta(\overline{x}_\varepsilon^{\pm},\varepsilon)}{\alpha+\widetilde{g}(\overline{x}_\varepsilon^{\pm})}\right|\varepsilon \leqslant \max_{\varepsilon\in[0,\varepsilon_0],x\in\overline{B}}\left|\frac{\vartheta(x,\varepsilon)}{\alpha+\widetilde{g}(x)}\right|\varepsilon = C\varepsilon,$$

which implies that $\overline{x}_\varepsilon^{\pm} = \mathcal{O}(\varepsilon^{\frac{1}{2k-1}})$ and, therefore, $2k/\alpha = 0$. This is an absurd. Here, $B\subset\mathbb{R}$ is a neighbourhood of 0. Hence, $X_2^+\left(\overline{x}_\varepsilon^{\pm},\varepsilon\right)\neq 0$ for $\varepsilon>0$ sufficiently small. □

The next lemma is a technical result which will be useful for proving our main Theorems.

Lemma 2. *Let σ be a real number. The trajectory $(u(t), v(t))$ of the planar vector field $F(u,v) = (1, -u^{2k-1} - v^n + \sigma)$ satisfying $u(0) = u_0$ and $v(0) = v_0 > 0$ intersects $v = 0$ at the point $(u^*, 0)$ with $u^* > \sigma^{\frac{1}{2k-1}}$.*

Proof. For each positive real number μ, with $\mu^n > \sigma$, let $\mathcal{B}_\mu\subset\mathbb{R}^2$ be defined as the following compact region,

$$\mathcal{B}_\mu = \left\{(u,v)\middle|(-\mu^n+\sigma)^{\frac{1}{2k-1}} \leqslant u \leqslant -v+\mu+(1+\sigma+\delta)^{\frac{1}{2k-1}}, 0\leqslant v\leqslant \mu\right\},$$

where $\delta > 0$ is such that $1+\sigma+\delta > 0$ (see Figure 5).

First, we shall see that the trajectories of F enter the region $\mathcal{B}_\mu$ through $\partial\mathcal{B}_\mu\backslash\mathcal{L}_\mu$, where $\mathcal{L}_\mu = \{(u,v)|\sigma^{\frac{1}{2k-1}} \leqslant u \leqslant \mu+(1+\sigma+\delta)^{\frac{1}{2k-1}}, v = 0\}$. Denote

$$\begin{aligned}
\mathcal{B}_\mu^+ &= \left\{(u,v)\middle|u = -v+\mu+(1+\sigma+\delta)^{\frac{1}{2k-1}}, 0\leqslant v\leqslant\mu\right\},\\
\mathcal{B}_\mu^- &= \left\{(u,v)\middle|u = (-\mu^n+\sigma)^{\frac{1}{2k-1}}, 0\leqslant v<\mu\right\},\\
\mathcal{B}_\mu^* &= \left\{(u,v)\middle|(-\mu^n+\sigma)^{\frac{1}{2k-1}} < u \leqslant (1+\sigma+\delta)^{\frac{1}{2k-1}}, v = \mu\right\},\\
\mathcal{B}_\mu^\# &= \left\{(u,v)\middle|(-\mu^n+\sigma)^{\frac{1}{2k-1}} \leqslant u < \sigma^{\frac{1}{2k-1}}, v = 0\right\}.
\end{aligned}$$

Notice that $\partial\mathcal{B}_\mu\backslash\mathcal{L}_\mu = \mathcal{B}_\mu^+\cup\overline{\mathcal{B}_\mu^-}\cup\overline{\mathcal{B}_\mu^*}\cup\mathcal{B}_\mu^\#$.

Let $n^+ = (1,1)$ be a normal vector to $\mathcal{B}_\mu^+$. Since $F\big|_{\mathcal{B}_\mu^+} = (1, -u^{2k-1} - v^n + \sigma)$, we get

$$\begin{aligned}
\langle n^+, F\rangle &= \langle(1,1),(1,-u^{2k-1}-v^n+\sigma)\rangle\\
&\leqslant 1-u^{2k-1}+\sigma\\
&= 1-(-v+\mu+(1+\sigma+\delta)^{\frac{1}{2k-1}})^{2k-1}+\sigma\\
&\leqslant 1+(\mu-\mu-(1+\sigma+\delta)^{\frac{1}{2k-1}})^{2k-1}+\sigma\\
&= -\delta\\
&< 0.
\end{aligned}$$

Hence, F points inward $\mathcal{B}_\mu$ along $\mathcal{B}_\mu^+$. Now, let $n^- = (1,0)$ be a normal vector to $\mathcal{B}_\mu^-$. Since, $F\big|_{\mathcal{B}_\mu^-} = (1, \mu^n - v^n)$ we get $\langle F, n^-\rangle = 1 > 0$ and, then, F also points inward $\mathcal{B}_\mu$ along $\mathcal{B}_\mu^-$. Let $n^* = (0,1)$ be a normal vector to $\mathcal{B}_\mu^*$. Since $F\big|_{\mathcal{B}_\mu^*} = (1, -u^{2k-1} - \mu^n + \sigma)$, we get $\langle F, n^*\rangle = -u^{2k-1} - \mu^n + \sigma < 0$ and, then, F points inward $\mathcal{B}_\mu$ along $\mathcal{B}_\mu^*$. Finally, let $n^{\#} = (0,1)$ be a normal vector to $\mathcal{B}_\mu^{\#}$. Since $F|_{\mathcal{B}_\mu^{\#}} = (1, -u^{2k-1} + \sigma)$, we get $\langle F, n^{\#}\rangle = -u^{2k-1} + \sigma > 0$ and, then, F points inward $\mathcal{B}_\mu$ along $\mathcal{B}_\mu^{\#}$.

It remains to study the behavior of the trajectory of F passing through the point $p_1 = \left((-\mu^n + \sigma)^{\frac{1}{2k-1}}, \mu\right)$. Consider the function $h_1(u,v) = v - \mu$, then

$$\begin{aligned} Fh_1(p_1) = & \ \langle \nabla h_1(p_1), F(p_1)\rangle = 0, \\ F^2h_1(p_1) = & \ \langle \nabla Fh_1(p_1), F(p_1)\rangle = -(2k-1)\,(-\mu^n + \sigma)^{\frac{2(k-1)}{2k-1}} < 0. \end{aligned}$$

Consequently, F has a quadratic contact with the straight line $v = \mu$ at p_1 and the trajectory passing through p_1 stays, locally, below this line. Given that $\dot{u} = 1$, we conclude that the flow enters the region $\mathcal{B}_\mu$ through p_1 (see Figure 5).

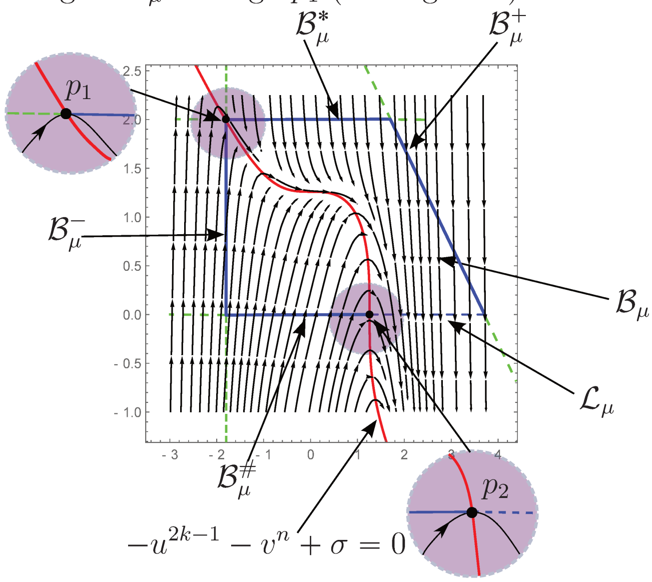

Figure 5 – The vector field F and the region $\mathcal{B}_\mu$. The red curve represents the isocline $-u^{2k-1} - v^n + \sigma = 0$.

Now, given $p = (u_0, v_0) \in \mathbb{R}^2$ with $v_0 > 0$ there exists μ_0 such that $p \in \mathcal{B}_{\mu_0}$. From the comments above, we known that the trajectory of F passing through p cannot leave the region $\mathcal{B}_\mu$ through $\partial\mathcal{B}_\mu \backslash \mathcal{L}_\mu$. Thus, assume by contradiction that the semi-orbit $\gamma_p^+ = \{(u(t), v(t)) | \, t \geqslant 0\}$ is contained in the compact region $\mathcal{B}_{\mu_0}$. From the *Poincaré–Bendixson Theorem* $\omega(p) \subset \mathcal{B}_{\mu_0}$ either contains a singularity of F or is a periodic orbit of F. In the last case, $\text{int}(\omega(p))$ contains a singularity of F. Both cases contradicts the fact that F does not admit singularities. Therefore, γ_p^+ must leave the region $\mathcal{B}_{\mu_0}$ through $\mathcal{L}_{\mu_0}$. In other words, there exists $t_0 > 0$ such that $(u(t_0), v(t_0)) = (u^*, 0)$ with $u^* \geqslant \sigma^{\frac{1}{2k-1}}$.

We conclude this proof by showing that $u^* > \sigma^{\frac{1}{2k-1}}$. Indeed, let $p_2 = \left(\sigma^{\frac{1}{2k-1}}, 0\right)$ and define the function $h_2(u,v) = v$. Then

$$\begin{aligned} Fh_2(p_2) = & \ \langle \nabla h_2(p_2), F(p_2)\rangle = 0, \\ F^2h_2(p_2) = & \ \langle \nabla Fh_2(p_2), F(p_2)\rangle \\ = & \ \begin{cases} -(2k-1) & if \quad k = 1, \\ -(2k-1)\sigma^{\frac{2(k-1)}{2k-1}} & if \quad k > 1. \end{cases} \end{aligned}$$

If $k = 1$ or $\sigma \neq 0$, then $F^2h_2(p_2) < 0$. Consequently, F has a quadratic contact with the straight line $v = 0$ at p_2 and the trajectory passing through p_2 stays, locally, below this line (see Figure 5). If $k > 1$ and $\sigma = 0$, then $F^2h_2(p_2) = 0$. In addition, one can see that $F^jh_2(p_2) = 0$ for $j \in \{1, \ldots, 2k-1\}$ and $F^{2k}h_2(p_2) = -(2k-1)! < 0$. Thus, F has an even multiplicity contact with the straight line $v = 0$ at p_2 and the trajectory passing through p_2 also stays, locally, below this line (see Figure 5). Hence, $p_2 \notin \gamma_p^+$ and, consequently, $u^* > \sigma^{\frac{1}{2k-1}}$. □

2.2 Extension of the Fenichel Manifold

Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ and assume that X^+ satisfies hypothesis **(A)** for some $k \geqslant 1$. For $n \geqslant \max\{2, 2k-1\}$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5). From the comments of the previous section, we can assume that Z, restricted to a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$, is given as (2.1). Thus, the regularized system Z_ε^Φ, defined in (1.4), leads to the following differential system

$$Z_\varepsilon^\Phi : \begin{cases} \dot{x} = \dfrac{1}{2}\left(1 + \Phi_\varepsilon(y)\right), \\ \dot{y} = \dfrac{1}{2}\left(\alpha x^{2k-1} + g(x) + y\vartheta(x,y)\right)\left(1 + \Phi_\varepsilon(y)\right) + \dfrac{1}{2}\left(1 - \Phi_\varepsilon(y)\right), \end{cases} \tag{2.4}$$

for $(x, y) \in U$ and $\varepsilon > 0$ sufficiently small. Recall that $\Phi_\varepsilon(y) = \Phi(y/\varepsilon)$.

Now, we shall study the regularized system (2.4) restricted to the band of regularization $|y| \leqslant \varepsilon$. Notice that $\Phi_\varepsilon(y) = \phi(y/\varepsilon)$ for $|y| \leqslant \varepsilon$. In this case, system (2.4) can be written as a *slow-fast problem*. Indeed, taking $y = \varepsilon\hat{y}$, we get the so-called *slow system*,

$$\begin{cases} \dot{x} = \dfrac{1}{2}\left(1 + \phi(\hat{y})\right), \\ \varepsilon\dot{\hat{y}} = \dfrac{1}{2}\left(\left(\alpha x^{2k-1} + g(x) + \varepsilon\hat{y}\vartheta(x,\varepsilon\hat{y})\right)\left(1 + \phi(\hat{y})\right) + \left(1 - \phi(\hat{y})\right)\right), \end{cases} \tag{2.5}$$

defined for $|\hat{y}| \leqslant 1$. Performing the time rescaling $t = \varepsilon\tau$, we obtain the so-called *fast system*,

$$\overline{Z}_\varepsilon^\Phi : \begin{cases} x' = \dfrac{\varepsilon}{2}\left(1 + \phi(\hat{y})\right), \\ \hat{y}' = \dfrac{1}{2}\left(\left(\alpha x^{2k-1} + g(x) + \varepsilon\hat{y}\vartheta(x,\varepsilon\hat{y})\right)\left(1 + \phi(\hat{y})\right) + \left(1 - \phi(\hat{y})\right)\right). \end{cases} \tag{2.6}$$

Clearly, systems (2.5) and (2.6) are equivalent for $\varepsilon \neq 0$. Taking $\varepsilon = 0$ in the fast system, we get the *layer problem*

$$\overline{Z}_0^\Phi : \begin{cases} x' = 0, \\ \hat{y}' = \dfrac{1}{2}\left(\left(\alpha x^{2k-1} + g(x)\right)\left(1 + \phi(\hat{y})\right) + \left(1 - \phi(\hat{y})\right)\right), \end{cases} \tag{2.7}$$

which has the following critical manifold

$$S_a = \left\{(x, \hat{y}) \middle| \hat{y} = m_0(x) := \phi^{-1}\left(\frac{1 + \alpha x^{2k-1} + g(x)}{1 - \alpha x^{2k-1} - g(x)}\right), -L \leqslant x \leqslant 0\right\}, \tag{2.8}$$

where L is a positive parameter satisfying $\alpha x^{2k-1} + g(x) < 0$ for $-L \leqslant x < 0$. Notice that, in this case,

$$-1 < \frac{1 + \alpha x^{2k-1} + g(x)}{1 - \alpha x^{2k-1} - g(x)} < 1, \text{ for } -L \leqslant x < 0.$$

Moreover,

$$\frac{\partial \pi_2 \overline{Z}_0^\Phi}{\partial \widehat{y}}(x, \widehat{y}) = \frac{\phi'(\widehat{y})}{2}(\alpha x^{2k-1} + g(x) - 1),$$

where $\pi_2 \overline{Z}_0^\Phi$ denote the second component of $\overline{Z}_0^\Phi$. Consequently, the critical manifold S_a is normally hyperbolic attracting on $S_a \backslash \{(0,1)\}$ and looses hyperbolicity at $(0,1)$. Indeed, $\phi'(\widehat{y})(\alpha x^{2k-1} + g(x) - 1) < 0$ for all $(x, \widehat{y}) \in S_a \backslash \{(0,1)\}$ and $\phi'(1) = 0$. Thus, the *Fenichel Theorem* [12, 16] can be applied for any compact subset of $S_a \backslash \{(0,1)\}$. In what follows we state the Fenichel Theorem for system (2.6) as it is stated in [3].

Theorema 1 (Fenichel Theorem)**.** *Consider L and N positive real numbers, $L > N$. There exist positive constants ε_0, K, and C, and a smooth function $m(x, \varepsilon)$, defined for $(x, \varepsilon) \in [-L, -N] \times [0, \varepsilon_0]$ and satisfying $m(x, 0) = m_0(x)$ (see (2.8)), such that the following statements hold.*

(i) $S_{a,\varepsilon} = \{(x, \widehat{y}) | \widehat{y} = m(x, \varepsilon), -L \leqslant x \leqslant -N\}$ *is a normally hyperbolic attracting locally invariant manifold of system (2.6), for* $0 < \varepsilon < \varepsilon_0$.

(ii) *There exists a neighborhood W of $S_{a,\varepsilon}$, which does not depend on ε, such that for any $z_0 \in W$ there exists $z^* \in S_{a,\varepsilon}$ satisfying*

$$|\varphi_{\overline{Z}_\varepsilon^\Phi}(t, z_0) - \varphi_{\overline{Z}_\varepsilon^\Phi}(t, z^*)| \leqslant K e^{-\frac{Ct}{\varepsilon}}, \, t \geqslant 0,$$

where $\varphi_{\overline{Z}_\varepsilon^\Phi}$ is the flow of system (2.5).

The invariant manifold $S_{a,\varepsilon}$ is called *Fenichel Manifold.*

In the sequel, in order to extend $S_{a,\varepsilon}$ until $\widehat{y} = 1$, we shall study system (2.6) around the degenerate point $(0,1)$. Notice that $1 + \phi(\widehat{y}) > 0$ for $\widehat{y}$ sufficiently close to 1. Thus, performing a time changing we can divide the right-hand side of the differential system (2.6) by $1 + \phi(\widehat{y})$, obtaining the following equivalent system

$$\begin{cases} x' = \varepsilon, \\ \widehat{y}' = \alpha x^{2k-1} + g(x) + \varepsilon \widehat{y} \vartheta(x, \varepsilon \widehat{y}) + \dfrac{1 - \phi(\widehat{y})}{1 + \phi(\widehat{y})}. \end{cases} \tag{2.9}$$

As an abuse of notation, we are still using the prime symbol $'$ to denote differentiation with respect to the new time variable. Denote $p(\widehat{y}) = (1 - \phi(\widehat{y}))/(1 + \phi(\widehat{y}))$. Computing the expansion of the function p around $\widehat{y} = 1$ we get

$$p(\widehat{y}) = \frac{1}{2}\phi^{[n]}(-(\widehat{y} - 1))^n(1 + (\widehat{y} - 1)\Upsilon(\widehat{y} - 1))$$

where

$$\phi^{[n]} = \frac{(-1)^{n+1}}{n!}\phi^{(n)}(1) > 0$$

and Υ is a smooth function defined in a neighborhood of 0. Taking $\widehat{y} = \widetilde{y} + 1$, system (2.9) becomes

$$\begin{cases} x' = \varepsilon, \\ \widetilde{y}' = \alpha x^{2k-1} + g(x) + \varepsilon(1+\widetilde{y})\vartheta(x, \varepsilon(1+\widetilde{y})) + \dfrac{1}{2}\phi^{[n]}(-\widetilde{y})^n(1+\widetilde{y}\Upsilon(\widetilde{y})). \end{cases}$$

Now, we consider the extended system

$$E: \begin{cases} x' = \widetilde{\varepsilon}, \\ \widetilde{y}' = \alpha x^{2k-1} + x^{2k-1}\widetilde{g}(x) + \widetilde{\varepsilon}(1+\widetilde{y})\vartheta(x, \widetilde{\varepsilon}(1+\widetilde{y})) + \dfrac{1}{2}\phi^{[n]}(-\widetilde{y})^n(1+\widetilde{y}\Upsilon(\widetilde{y})), \\ \widetilde{\varepsilon}' = 0, \end{cases} \tag{2.10}$$

where $g(x) = x^{2k-1}\widetilde{g}(x)$ with $\widetilde{g} = \mathcal{O}(x)$. Notice that, the above differential system keeps the planes $\widetilde{\varepsilon} =$ "constant" invariant. In addition, its restriction to $\widetilde{\varepsilon} = 0$ corresponds to the layer problem (2.7). Thus, once we have understood the orbits of (2.10) in an neighborhood of the origin $(x, \widetilde{y}, \widetilde{\varepsilon}) = (0, 0, 0)$, we can understand how the Fenichel manifold $S_{a,\varepsilon}$ of (2.6) behaves in a neighborhood of $(x, \widehat{y}) = (0, 1)$.

Notice that $\{(x, \widetilde{y}, 0) | \alpha x^{2k-1} + x^{2k-1}\widetilde{g}(x) + \dfrac{1}{2}\phi^{[n]}(-\widetilde{y})^n(1+\widetilde{y}\Upsilon(\widetilde{y})) = 0\}$ is a set of degenerate singularities of (2.10). Thus, in order to study the differential system (2.10) in a neighborhood of the origin, we shall apply the following blow-up

$$\begin{array}{rcl} \Psi : \mathbb{S}^2 \times \mathbb{R}_+ & \rightarrow & \mathbb{R}^3_* \\ (\overline{x}, \overline{y}, \overline{\varepsilon}, r) & \mapsto & (r^n\overline{x}, r^{2k-1}\overline{y}, r^{1+2k(n-1)}\overline{\varepsilon}). \end{array}$$

Here,

$$\mathbb{S}^2 = \{(\overline{x}, \overline{y}, \overline{\varepsilon}) \in \mathbb{R}^3 | \overline{x}^2 + \overline{y}^2 + \overline{\varepsilon}^2 = 1\} \text{ and } \mathbb{R}^3_* = \mathbb{R}^3 \backslash \{(0,0,0)\}.$$

Roughly speaking, the geometric idea of the blow-up method is to "change" the nonhyperbolic singularity $(0,0,0)$ by a sphere $\mathbb{S}^2$, leaving the dynamics away from the origin unchanged. This allow us to blow-up the dynamics around the origin. Formally, the map Ψ pulls back the vector field $E|_{\mathbb{R}^3_*}$, defined in (2.10), to a vector field Ψ^*E defined on $\mathbb{S}^2 \times \mathbb{R}_+$. Here, Ψ^* denotes the usual *pullback*,

$$\Psi^*E(p) = (D\Psi(p))^{-1} E(\Psi(p)), \; p = (\overline{x}, \overline{y}, \overline{\varepsilon}, r).$$

In order to study the behavior of Ψ^*E in a neighbourhood of $\mathbb{S}^2_0 = \mathbb{S}^2 \times \{0\}$, we have to extend its dynamics to $\mathbb{S}^2_0$ and desingularize it through a time rescaling. This provides a new vector field E^* which has its dynamics outside $\mathbb{S}^2_0$ equivalent to $E|_{\mathbb{R}^3_*}$. Then, we consider two charts of $\mathbb{S}^2 \times \mathbb{R}_{\geqslant 0}$, namely, $\kappa_1 = (\mathcal{U}_1, \psi_1)$ and $\kappa_2 = (\mathcal{U}_2, \psi_2)$, where

$$\mathcal{U}_1 = \{(\overline{x}, \overline{y}, \overline{\varepsilon}, r) \in \mathbb{S}^2 \times \mathbb{R}_{\geqslant 0} | \overline{y} < 0\}, \; \mathcal{U}_2 = \{(\overline{x}, \overline{y}, \overline{\varepsilon}, r) \in \mathbb{S}^2 \times \mathbb{R}_{\geqslant 0} | \overline{\varepsilon} > 0\},$$

and $\psi^{1,2} : \mathcal{U}_{1,2} \rightarrow \mathbb{R}^3$ are the following stereographic-like projections

$$\psi^1(\overline{x}, \overline{y}, \overline{\varepsilon}, r) = \left((-\overline{y})^{\alpha_1}\, \overline{x}, \; (-\overline{y})^{\beta_1}\, r, (-\overline{y})^{\gamma_1}\, \overline{\varepsilon}\right), \; \psi^2(\overline{x}, \overline{y}, \overline{\varepsilon}, r) = \left(\overline{\varepsilon}^{\alpha_2}\, \overline{x}, \; \overline{\varepsilon}^{\beta_2}\, \overline{y}, \overline{\varepsilon}^{\gamma_2}\, r\right),$$

with

$$\alpha_1 = \frac{-n}{2k-1}, \quad \beta_1 = \frac{1}{2k-1}, \quad \gamma_1 = \frac{-(1+2k(n-1))}{2k-1},$$

and

$$\alpha_2 = \frac{-n}{1+2k(n-1)}, \quad \beta_2 = \frac{-(2k-1)}{1+2k(n-1)}, \quad \gamma_2 = \frac{1}{1+2k(n-1)}.$$

The maps ψ^1 and ψ^2 are constructed by projecting the sets $\mathcal{U}_1$ and $\mathcal{U}_2$ into the planes $\overline{y} = -1$ and $\overline{\varepsilon} = 1$, respectively.

The above charts are used to push forward the vector fields $E_i^* = E^*|_{\mathcal{U}_i}$, $i = 1, 2$, to vector fields defined on $\mathbb{R}^3$, $F_i = \psi_*^i E_i^*$, $i = 1, 2$. Here, ψ_*^i denotes the usual *pushforward*,

$$\psi_*^i E_i^*(q) = D\psi^i\left((\psi^i)^{-1}(q)\right) E_i^*\left((\psi^i)^{-1}(q)\right), \; q \in \psi_i(\mathcal{U}_i).$$

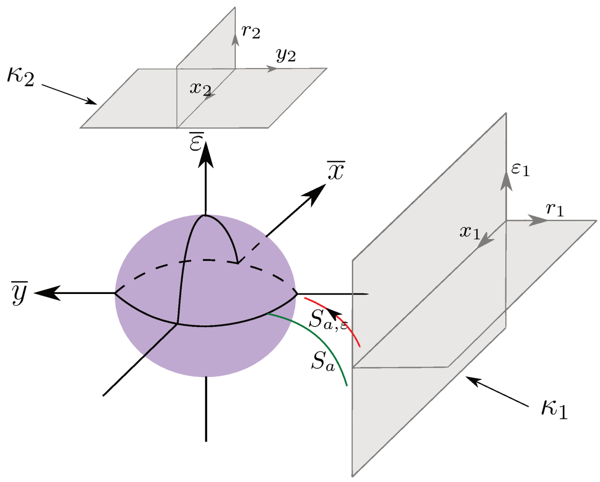

Figure 6 – The 2 charts of blow-up, critical manifold S_a, and Fenichel manifold $S_{a,\varepsilon}$.

Finally, consider the composition $\Psi_i = \Psi \circ (\psi^i)^{-1}$, $i = 1, 2$. Then,

$$\Psi_1(x_1, r_1, \varepsilon_1) = \left(r_1^n x_1, \; -r_1^{2k-1}, \; r_1^{1+2k(n-1)}\varepsilon_1\right),$$
$$\Psi_2(x_2, y_2, r_2) = \left(r_2^n x_2, \; r_2^{2k-1} y_2, \; r_2^{1+2k(n-1)}\right).$$

The vector field F_i, $i = 1, 2$, can be directly obtained as $F_i = \Psi_i^* E / r^{(n-1)(2k-1)}$. Notice that we are pulling back the vector field E through Ψ_i, extending $\Psi_i^* E$ to $r_i = 0$ and, then, desingularizing it by doing a time rescaling (i.e. dividing by $r^{(n-1)(2k-1)}$).

Moreover,

$$\mathcal{U}_{12} := \mathcal{U}_1 \cap \mathcal{U}_2 = \{(\overline{x}, \overline{y}, \overline{\varepsilon}, r) \in \mathbb{S}^2 \times \mathbb{R}_{\geqslant 0} | \overline{y} < 0 \text{ and } \overline{\varepsilon} < 0\}$$

and the change of coordinates $\psi_{12} : \psi_1(\mathcal{U}_{12}) \to \psi_2(\mathcal{U}_{12})$ which pushes forward $F_1|_{\psi_1(\mathcal{U}_{12})}$ to $F_2|_{\psi_2(\mathcal{U}_{12})}$ writes

$$\psi_{12}(x_1, r_1, \varepsilon_1) = \left(\varepsilon_1^{-\frac{n}{1+2k(n-1)}} x_1 \,, \; -\varepsilon_1^{-\frac{2k-1}{1+2k(n-1)}} \,, \; r_1 \varepsilon_1^{\frac{1}{1+2k(n-1)}}\right).$$

2.3 Chart κ_1

The differential system associated with the vector field F_1 writes

$$\left\{\begin{array}{rl} x_1' = & \dfrac{1}{2(2k-1)}[2(2k-1)\varepsilon_1 + \phi^{[n]} n x_1 + 2\alpha n x_1^{2k} - n x_1 \phi^{[n]} r_1^{2k-1}\Upsilon(-r_1^{2k-1}) \\ & +2n x_1^{2k}\widetilde{g}(r_1^n x_1) - n x_1 2(r_1^n - r_1^{1-2k+n})\varepsilon_1 \vartheta(r_1^n x_1, -r_1^{2k(n-1)}(-r_1 + r_1^{2k})\varepsilon_1)], \\ r_1' = & \dfrac{1}{2(2k-1)}[-2r_1 x_1^{2k-1}(\alpha + \widetilde{g}(r_1^n x_1)) - \phi^{[n]}(r_1 - r_1^{2k}\Upsilon(-r_1^{2k-1})) \\ & +2(r_1^{1+n} - r_1^{2-2k+n})\varepsilon_1 \vartheta(r_1^n x_1, -r_1^{2k(n-1)}(-r_1 + r_1^{2k})\varepsilon_1)], \\ \varepsilon_1' = & \dfrac{1}{2(2k-1)}(1 + 2k(n-1))\varepsilon_1[(2x_1^{2k-1}(\alpha + \widetilde{g}(r_1^n x_1)) + \phi^{[n]}(1 - r_1^{2k-1}\Upsilon(-r_1^{2k-1})) \\ & +2(-r_1^n + r_1^{1-2k+n})\varepsilon_1 \vartheta(r_1^n x_1, -r_1^{2k(n-1)}(-r_1 + r_1^{2k})\varepsilon_1)]. \end{array}\right. \tag{2.11}$$

First, taking $\varepsilon_1 = 0$ in (2.11), we get that the critical manifold S_a, in this coordinate system, is given by

$$S_{a,1} = \left\{(x_1, r_1, 0) | x_1^{2k-1}(\alpha + \widetilde{g}(r_1^n x_1)) = -\frac{\phi^{[n]}}{2}(1 - r_1^{2k-1}\Upsilon(-r_1^{2k-1})),\ -L \leqslant r_1^n x_1 \leqslant 0\right\}.$$

In what follows, we shall write the critical manifold $S_{a,1}$ locally as a graphic. For this, define $U_1 = \{(x_1, r_1) | -L \leqslant r_1^n x_1 \leqslant 0\}$ and consider the function $H : U_1 \to \mathbb{R}$ defined by

$$H(x_1, r_1) = -x_1^{2k-1}(\alpha + \widetilde{g}(r_1^n x_1)) - \frac{\phi^{[n]}}{2}(1 - r_1^{2k-1}\Upsilon(-r_1^{2k-1})).$$

Notice that $H(x_1, 0) = -\alpha x_1^{2k-1} - \dfrac{\phi^{[n]}}{2}$. Thus, for $x_1^* = \left(-\dfrac{\phi^{[n]}}{2\alpha}\right)^{\frac{1}{2k-1}}$ we get that $H(x_1^*, 0) = 0$. Furthermore,

$$\frac{\partial H}{\partial x_1}(x_1^*, 0) = -(2k-1)\alpha(x_1^*)^{2k-2} = -(2k-1)\alpha\left(-\frac{\phi^{[n]}}{2\alpha}\right)^{\frac{2k-2}{2k-1}} \neq 0.$$

From the *Implicit Function Theorem*, there exist open sets $W_1, V_1 \subseteq \mathbb{R}$ such that $(x_1^*, 0) \in W_1 \times V_1 \subseteq U_1$, and a unique smooth function $x_1 : V_1 \to W_1$ such that $x_1(0) = x_1^*$ and $H(x_1(r_1), r_1) = 0$, for all $r_1 \in V_1$. Moreover,

$$x_1'(0) = \left\{\begin{array}{ll} 0 & if \quad k > 1, \\ \dfrac{\phi^{[n]}\Upsilon(0)}{2\alpha} & if \quad k = 1. \end{array}\right.$$

Thus, expanding $x_1(r_1)$ around $r_1 = 0$, we have

$$\begin{array}{rcl} x_1(r_1) & = & x_1(0) + r_1 x_1'(0) + \mathcal{O}(r_1^2) \\ & = & \left\{\begin{array}{ll} x_1^* + \mathcal{O}(r_1^2) & \text{if} \quad k > 1, \\ x_1^* + r_1\dfrac{\phi^{[n]}\Upsilon(0)}{2\alpha} + \mathcal{O}(r_1^2) & \text{if} \quad k = 1. \end{array}\right. \end{array}$$

Consequently,

$$S_{a,1} \cap (W_1 \times V_1 \times \{0\}) = \{(x_1, r_1, 0) | \quad x_1 = x_1(r_1)\}.$$

Notice that $S_{a,1}$ intersects the plane $r_1 = 0$ (which is equivalent to the sphere $\mathbb{S}^2_0$) at the singularity $(x_1^*, 0, 0)$. Moreover,

$$DF_1(x_1^*, 0, 0) = \begin{pmatrix} -\dfrac{\phi^{[n]}n}{2} & \omega_k^{12} & 1 + \omega_{n,k}^{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} \omega_k^{12} &= \begin{cases} 0 & if \quad k > 1, \\ \dfrac{(\phi^{[n]})^2 n \Upsilon(0)}{4\alpha} & if \quad k = 1, \end{cases} \\ \omega_{n,k}^{13} &= \begin{cases} 0 & if \quad n > 2k-1, \\ x_1^* \vartheta(0,0) & if \quad n = 2k-1 \text{ and } k \neq 1. \end{cases} \end{aligned}$$

Hence, in the sequel, we shall use the *Center Manifold Theorem* [8] to study F_1 around the degenerated singularity $(x_1^*, 0, 0)$.

One can easily see that $\lambda_1 = -\phi^{[n]}n/2$, $\lambda_2 = 0$, and $\lambda_3 = 0$ are the eigenvalues of $DF(x_1^*, 0, 0)$ associated with the eigenvectors

$$v_1 = (1, 0, 0), \;\; v_2 = \left(\frac{2\omega_k^{12}}{\phi^{[n]}n}, 1, 0 \right), \text{ and } v_3 = \left(\frac{2}{\phi^{[n]}n}(1 + \omega_{n,k}^{13}), 0, 1 \right),$$

respectively. Thus, consider a box $\Omega = [\chi, 0] \times [0, \rho] \times [0, \nu]$ around $(x_1^*, 0, 0)$, where $\chi < x_1^*$ and $\rho, \nu > 0$ are small parameters. By the *Center Manifold Theorem* we know that within Ω there exists a center manifold $W^c = \{(x_1, r_1, \varepsilon_1) | \quad x_1 = k(r_1, \varepsilon_1)\}$ tangent to the eigenspace generated by v_2 and v_3 at the singularity $(x_1^*, 0, 0)$. Moreover, since $(x_1^*, 0, 0) \in W^c \cap S_{a,1} \neq \varnothing$, we conclude that W^c contains the critical manifold $S_{a,1}$. Assume that $W^c = \widehat{h}^{-1}(0)$, with $\widehat{h}(x_1, r_1, \varepsilon_1) = x_1 - k(r_1, \varepsilon_1)$ and $k(0,0) = x_1^*$. Since $\nabla \widehat{h}(0).v_2 = 0$ and $\nabla \widehat{h}(0).v_3 = 0$ we get

$$\frac{\partial k}{\partial r_1}(0,0) = \frac{2\omega_k^{12}}{\phi^{[n]}n} \quad \text{and} \quad \frac{\partial k}{\partial \varepsilon_1}(0,0) = \frac{2}{\phi^{[n]}n}(1 + \omega_{n,k}^{13}),$$

respectively. Therefore,

$$k(r_1, \varepsilon_1) = x_1^* + r_1 \frac{2\omega_k^{12}}{\phi^{[n]}n} + \varepsilon_1 \frac{2}{\phi^{[n]}n}(1 + \omega_{n,k}^{13}) + \mathcal{O}_2(r_1, \varepsilon_1).$$

Now, we shall see that the center manifold W^c is foliated by hyperbolas. Indeed, from (2.11) we have that

$$\frac{dr_1}{d\varepsilon_1} = -\frac{r_1}{\varepsilon_1(1 + 2k(n-1))}.$$

Thus, solving the above differential equation, we get that $\varepsilon_1 \mapsto \varepsilon_1 r_1(\varepsilon_1)^{1+2k(n-1)}$ is constant on ε_1. This means that, for each $\varepsilon > 0$, the surface

$$E_\varepsilon = \{(x_1, r_1, \varepsilon_1) | \quad \varepsilon_1 r_1^{1+2k(n-1)} = \varepsilon\},$$

is invariant through the flow of (2.11). Consequently, the manifold W^c is foliated by invariant hyperbolas $\gamma_\varepsilon = W^c \cap E_\varepsilon$, $\varepsilon > 0$, which correspond to orbits of (2.11). Thus, we can write $\gamma_\varepsilon = \{\varphi_{F_1}(t,\varepsilon) : t \in I_\varepsilon\}$ where $\varphi_{F_1}(t,\varepsilon)$ is a trajectory of (2.11) satisfying

$$\varphi_{F_1}(0,\varepsilon) = (k(\rho, \varepsilon\, \rho^{-(1+2k(n-1))}), \rho, \varepsilon\, \rho^{-(1+2k(n-1))}) \in W^c \cap \gamma_\varepsilon,$$

and I_ε is a neighborhood of the origin. Hence, $\Psi_1\gamma_\varepsilon$ is an orbit of E (2.10) lying in the plane $\widetilde{\varepsilon} = \varepsilon$. Therefore, (after the translation $\widehat{y} = 1 + \widetilde{y}$) we get it as an orbit (2.6).

Denote by $S^1_{a,\varepsilon}$ the Fenichel manifold $S_{a,\varepsilon}$ of (2.6) for $\widetilde{\varepsilon} = \varepsilon$ written in the coordinates $(x_1, r_1, \varepsilon_1)$. We claim that, for $\varepsilon > 0$ sufficiently small, the Fenichel manifold $S^1_{a,\varepsilon}$ can be continued as an orbit of F_1 in W^c, namely γ_ε. First, noticed that the orbit γ_ε is $\varepsilon-$close to $S_{a,1}$ at $r_1 = \rho$. Indeed, from the relation $\varepsilon_1 r_1^{1+2k(n-1)} = \varepsilon$ satisfied by γ_ε, we see that $\varphi_{F_1}(0,\varepsilon)$ approaches to $S_{a,1} = W^c \cap \{\varepsilon_1 = 0\}$ when ε goes to zero. Now, since $S^1_{a,\varepsilon}$ is also $\varepsilon-$close to $S_{a,1}$, we get that $S^1_{a,\varepsilon}$ and γ_ε are $\varepsilon-$ close to each other at $r_1 = \rho$. Noticing that γ_ε and $S^1_{a,\varepsilon}$ are related to orbits of (2.6), which are ε-close to each other, we get from item (ii) of Fenichel Theorem (1) that $d(\varphi_{F_1}(t,\varepsilon), S^1_{a,\varepsilon}) \leqslant Ke^{-\frac{Ct}{\varepsilon}}$. Hence, taking any positive time $t_0 \in I_\varepsilon$ we conclude that $S^1_{a,\varepsilon}$ and γ_ε are $\mathcal{O}(e^{-\frac{c}{\varepsilon}})$ close to each other at $r_1 = \rho' < \rho$, with $c = Ct_0 > 0$. Therefore, for each $\varepsilon > 0$, γ_ε can be seen as a continuation of $S^1_{a,\varepsilon}$ on W^c (see Figure 7).

Now, at $\varepsilon_1 = \nu$ we have

$$\begin{aligned}\gamma_\varepsilon \cap \{\varepsilon_1 = \nu\} &= \left(k\left((\varepsilon\,\nu^{-1})^{\frac{1}{1+2k(n-1)}}, \nu\right), (\varepsilon\,\nu^{-1})^{\frac{1}{1+2k(n-1)}}, \nu\right) \\ &= \left(k(0,\nu), 0, \nu\right) + \mathcal{O}(\varepsilon^{\frac{1}{1+2k(n-1)}}).\end{aligned}$$

Hence, we conclude that $S^1_{a,\varepsilon} \cap \{\varepsilon_1 = \nu\}$ is $\mathcal{O}(\varepsilon^{\frac{1}{1+2k(n-1)}})$ close to $(k(0,\nu), 0, \nu)$.

Remark 2. *Notice that $W^c \cap \{r_1 = 0\}$ is an orbit of (2.11) containing the point $(x_1, r_1, \varepsilon_1) = (k(0,\nu), 0, \nu)$ which the backward trajectory approaches asymptotically to $(x_1^*, 0, 0)$ (see Figure 7). Indeed,*

$$W^c \cap \{r_1 = 0\} = \left\{(x_1, 0, \varepsilon_1) \middle| \quad x_1 = x_1^* + \varepsilon_1 \frac{2}{\phi^{[n]} n}(1 + \omega^{13}_{n,k}) + \mathcal{O}(\varepsilon_1^2)\right\}$$

and, therefore, $W^c \cap \{r_1 = 0\} \cap \{\varepsilon_1 = 0\} = \{(x_1^*, 0, 0)\}$.

In what follows, we shall continue $S^1_{a,\varepsilon}$ in chart κ_2 by following the trajectory of $(x_2^*, y_2^*, 0) := \psi_{12}(k(0,\nu), 0, \nu)$ (see Figure 8).

2.4 Chart κ_2

The differential system associated with the vector field F_2 writes

$$\left\{\begin{aligned} x_2' &= 1, \\ y_2' &= x_2^{2k-1}(\alpha + \widetilde{g}(r_2^n x_2)) + \frac{\phi^{[n]}}{2}(-y_2)^n(1 + r_2^{2k-1} y_2 \Upsilon(r_2^{2k-1} y_2)) \\ &\quad + (r_2^{1-2k+n} + r_2^n y_2)\vartheta(r_2^n x_2, r_2^{2kn}(r_2^{1-2k} + y_2)), \\ r_2' &= 0. \end{aligned}\right. \tag{2.12}$$

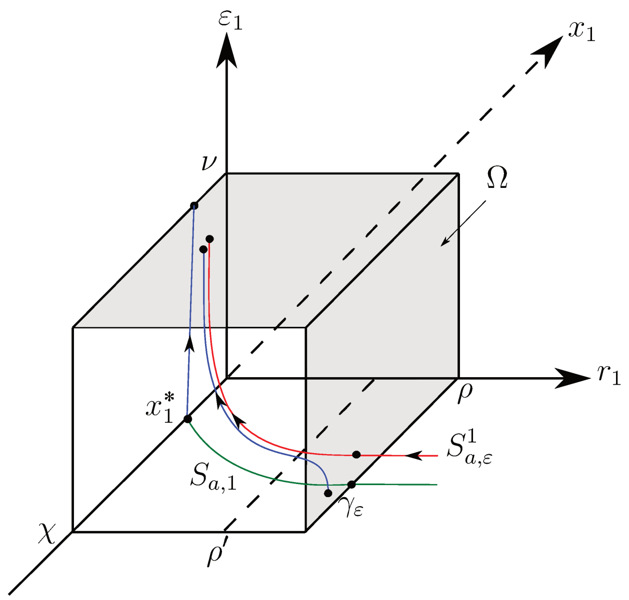

Figure 7 – Behavior of the vector field (2.11) around $(x_1^*, 0, 0)$.

Lemma 3. *The forward orbit of* (2.12) *starting at* $(x_2^*, y_2^*, 0)$ *intersects* $\{y_2 = 0\}$ *at*

$$(x_2, y_2, r_2) = (\eta, 0, 0)$$

where η *is a constant satisfying*

$$\eta > \sigma_{n,k} := \begin{cases} 0 & if \quad n > 2k-1, \\ -\left(\dfrac{\vartheta(0,0)}{\alpha}\right)^{\frac{1}{2k-1}} & if \quad n = 2k-1 \text{ and } k \neq 1. \end{cases} \tag{2.13}$$

Proof. Set $c_x = \left(2/(\phi^{[n]}\alpha^{n-1})\right)^{\frac{1}{1+2k(n-1)}} > 0$ and $c_y = -\alpha c_x^{2k} < 0$. Consider system (2.12) restricted to $r_2 = 0$. Applying the change of variables $(x_2, y_2) = (c_x u, c_y v)$, $v \leqslant 0$, and a time rescaling by the positive constant c_x, system $(2.12)|_{r_2=0}$ writes

$$\begin{cases} u' &= 1, \\ v' &= -u^{2k-1} - v^n + s_{n,k}, \end{cases} \tag{2.14}$$

where

$$s_{n,k} = \begin{cases} 0 & if \quad n > 2k-1, \\ -\dfrac{\vartheta(0,0)}{\alpha c_x^n} & if \quad n = 2k-1 \text{ and } k \neq 1. \end{cases}$$

Take $(u_0, v_0) = (x_2^*/c_x, y_2^*/c_y)$. Since $y_2^* = -\nu^{-\frac{2k-1}{1+2k(n-1)}}$, we have $v_0 > 0$. Thus, by Lemma 2, the forward trajectory of (2.14) starting at (u_0, v_0) intersects $\{v = 0\}$ at $(u^*, 0)$ with $u^* > s_{n,k}^{\frac{1}{2k-1}}$. Consequently, the forward flow of $(x_2^*, y_2^*, 0)$ intersects $\{y_2 = 0\}$ at the point $(x_2, y_2, r_2) = (\eta, 0, 0)$, where $\eta := c_x u^*$ is a constant satisfying $\eta > \sigma_{n,k}$. □

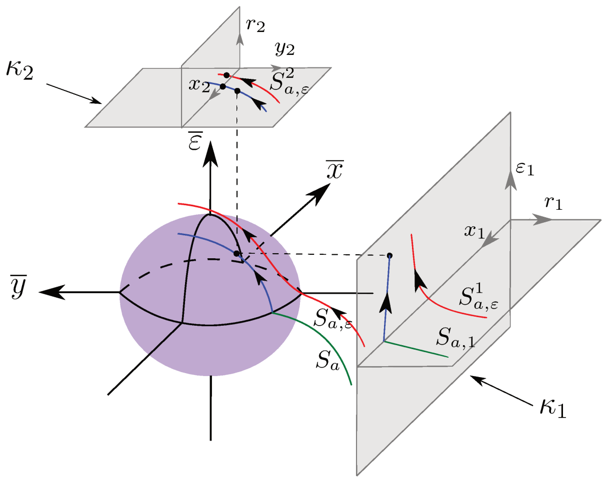

Figure 8 – The 2 charts of blow-up, critical manifold S_a, and Fenichel manifold $S_{a,\varepsilon}$.

Proposition 1. *There exist $a > 0$ and $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$, the forward trajectory of system (2.6), starting at any point in the set*

$$\left(\varepsilon^{\lambda^*}(x_2^* - a)\,,\, \varepsilon^{\lambda^*}(x_2^* + a)\right) \times \left(1 + \varepsilon^{\frac{2k-1}{1+2k(n-1)}}(y_2^* - a)\,,\, 1 + \varepsilon^{\frac{2k-1}{1+2k(n-1)}}(y_2^* + a)\right),$$

intersects the line $\{\widehat{y} = 1\}$. In particular, the Fenichel manifold $S_{a,\varepsilon}$ intersects $\{\widehat{y} = 1\}$ at $(x, \widehat{y}, \varepsilon) = (x_\varepsilon, 1, \varepsilon)$, where

$$x_\varepsilon = \varepsilon^{\lambda^*}\eta + \mathcal{O}\left(\varepsilon^{\lambda^* + \frac{1}{1+2k(n-1)}}\right), \tag{2.15}$$

with η satisfying (2.13) and $\lambda^* := \dfrac{n}{1 + 2k(n-1)}$.

Proof. Denote $S^2_{a,\varepsilon} := \psi_{12}(S^1_{a,\varepsilon})$. Notice that $S^2_{a,\varepsilon} \cap \{y_2 = y_2^*\}$ is $\mathcal{O}(\varepsilon^{\frac{1}{1+2k(n-1)}})$ close to $(x_2^*, y_2^*, 0)$. From Lemma 3, the forward orbit of $(x_2^*, y_2^*, 0)$ intersects $\{y_2 = 0\}$ transversally at $(x_2, y_2, r_2) = (\eta, 0, 0)$. Thus, from the *Implicit Function Theorem*, $S^2_{a,\varepsilon}$ also intersect $\{y_2 = y_2^*\}$ transversally at

$$\left(\eta + \mathcal{O}(\varepsilon^{\frac{1}{1+2k(n-1)}}), 0, \varepsilon^{\frac{1}{1+2k(n-1)}}\right),$$

for $\varepsilon > 0$ sufficiently small. Furthermore, there exist $a > 0$ and $b > 0$ sufficiently small such that any forward trajectory of (2.12), starting at the set

$$\Xi = (x_2^* - a, x_2^* + a) \times (y_2^* - a, y_2^* + a) \times [0, b),$$

also intersects the set $\{y_2 = 0\}$.

Going back to the original coordinates, we conclude that the forward flow of $S_{a,\varepsilon}$ intersect $\{\widehat{y} = 1\}$ at $(x, \widehat{y}, \varepsilon) = (x_\varepsilon, 1, \varepsilon)$ with

$$x_\varepsilon = \varepsilon^{\lambda^*}\left(\eta + \mathcal{O}\left(\varepsilon^{\frac{1}{1+2k(n-1)}}\right)\right) = \varepsilon^{\lambda^*}\eta + \mathcal{O}\left(\varepsilon^{\lambda^* + \frac{1}{1+2k(n-1)}}\right).$$

Moreover, writing the Ξ in the original coordinates we conclude that, for every $\varepsilon \in [0, \varepsilon^*)$, $\varepsilon^* = b^{1+2k(n-1)}$, any trajectory of system (2.6) starting at the set

$$\left(\varepsilon^{\lambda^*}(x_2^* - a)\,,\, \varepsilon^{\lambda^*}(x_2^* + a)\right) \times \left(1 + \varepsilon^{\frac{2k-1}{1+2k(n-1)}}(y_2^* - a)\,,\, 1 + \varepsilon^{\frac{2k-1}{1+2k(n-1)}}(y_2^* + a)\right)$$

intersects the line $\{\widehat{y} = 1\}$. $\square$

3 Transition maps of the regularized system

In this Chapter, we will continue with the study of the C^n-regularizations of Filippov systems around visible regular-tangential singularities of even multiplicity. More specifically, we shall understand how the trajectories of the regularized system transits around these singularities. Thus, we will characterize two transition maps, namely the *Upper Transition Map* $U_\varepsilon(y)$ and the *Lower Transition Map* $L_\varepsilon(y)$. For this, we will use the results obtained in the previous chapter with respect to the extension of the Fenichel manifold.

3.1 Main Results

Our first two main results guarantee that under some conditions the flow of the regularized system Z_ε^Φ near a visible regular-tangential singularity defines two distinct maps between transversal sections (see Figure 3). Before their statements, we need to establish some notations. Given $\Phi \in C_{ST}^{n-1}$ as (1.5), with $k \geqslant 1$, and $n \geqslant 2k-1$, define

$$x_\varepsilon = \varepsilon^{\lambda^*}\eta + \mathcal{O}\left(\varepsilon^{\lambda^* + \frac{1}{1+2k(n-1)}}\right),$$

where $\lambda^* := \dfrac{n}{1+2k(n-1)}$ and η is a constant satisfying

$$\eta > \begin{cases} 0 & if \quad n > 2k-1, \\ -\left(\dfrac{\vartheta(0,0)}{\alpha}\right)^{\frac{1}{2k-1}} & if \quad n = 2k-1 \text{ and } k \neq 1, \end{cases}$$

and

$$\begin{aligned} y_{\rho,\lambda}^\varepsilon &= \overline{y}_{-\rho} + \varepsilon + \mathcal{O}(\varepsilon\rho) + \beta\varepsilon^{2k\lambda} + \mathcal{O}(\varepsilon^{(2k+1)\lambda}) + \mathcal{O}(\varepsilon^{1+\lambda}), \\ y_\theta^\varepsilon &= \overline{y}_\theta + \varepsilon + \mathcal{O}(\varepsilon\theta) + \sum_{i=1}^{2k-1} \mathcal{O}(\theta^{2k+1-i}x_\varepsilon^i) + \mathcal{O}(x_\varepsilon^{2k}), \end{aligned} \tag{3.1}$$

where $\overline{y}_{-\rho}$ and $\overline{y}_\theta$ are given by Lemma 1 and β is a negative parameter which will be defined latter on.

In what follows, we establish the main results of this chapter, which will be proved in the Sections 3.2 and 3.3, respectively.

Theorem A. *Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ and assume that X^+ satisfies hypothesis* **(A)** *(see Section 2.1) for some $k \geqslant 1$. For $n \geqslant 2k-1$, let $\Phi \in C_{ST}^{n-1}$ be given as (1.5) and consider the regularized system Z_ε^Φ (1.4). Then, there exist $\rho_0, \theta_0 > 0$, and constants $\beta < 0$ and $c, r, q > 0$, for which the following statements hold for every $\rho \in (\varepsilon^\lambda, \rho_0]$, $\theta \in [x_\varepsilon, \theta_0]$, $\lambda \in (0, \lambda^*)$, with $\lambda^* := \dfrac{n}{2k(n-1)+1}$, and $\varepsilon > 0$ sufficiently small.*

(a) The vertical segments

$$\widehat{V}_{\rho,\lambda}^\varepsilon = \{-\rho\} \times [\varepsilon, y_{\rho,\lambda}^\varepsilon] \quad \textit{and} \quad \widetilde{V}_\theta^\varepsilon = \{\theta\} \times [y_\theta^\varepsilon, y_\theta^\varepsilon + re^{-\frac{c}{\varepsilon^q}}]$$

and the horizontal segments

$$\widehat{H}^{\varepsilon}_{\rho,\lambda} = [-\rho, -\varepsilon^{\lambda}] \times \{\varepsilon\} \quad and \quad \overleftarrow{H}_{\varepsilon} = [x_{\varepsilon} - re^{-\frac{c}{\varepsilon^q}}, x_{\varepsilon}] \times \{\varepsilon\}$$

are transversal sections for Z^{Φ}_{ε}.

(b) *The flow of* Z^{Φ}_{ε} *defines a map* U_{ε} *between the transversal sections* $\widehat{V}^{\varepsilon}_{\rho,\lambda}$ *and* $\widetilde{V}^{\varepsilon}_{\theta}$ *satisfying*

$$\begin{array}{rccl} U_{\varepsilon}: & \widehat{V}^{\varepsilon}_{\rho,\lambda} & \longrightarrow & \widetilde{V}^{\varepsilon}_{\theta} \\ & y & \longmapsto & y^{\varepsilon}_{\theta} + \mathcal{O}(e^{-\frac{c}{\varepsilon^q}}). \end{array}$$

(c) *the trajectories of* Z^{Φ}_{ε} *starting at the section* $\widehat{V}^{\varepsilon}_{\rho,\lambda}$ *intersect the line* $y = \varepsilon$ *only in two points before reaching the section* $\widetilde{V}^{\varepsilon}_{\theta}$. *Moreover, these intersections take place at* $\widehat{H}^{\varepsilon}_{\rho,\lambda} \cup \overleftarrow{H}_{\varepsilon}$.

The map U_{ε} *is called Upper Transition Map of the regularized system (see Figure 9).*

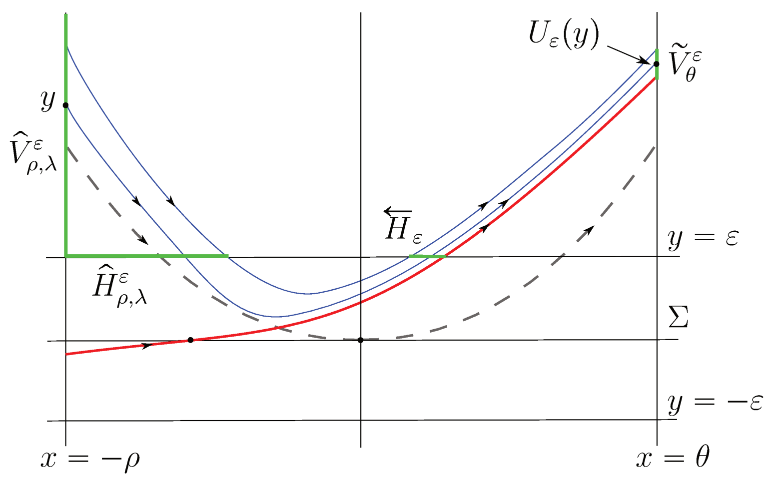

Figure 9 – Upper Transition Map U_{ε} of the regularized system Z^{Φ}_{ε}. The large domain $\widehat{V}^{\varepsilon}_{\rho,\lambda}$ is contracted into the small $\widetilde{V}^{\varepsilon}_{\theta}$. The dotted curve is the trajectory of X^+ passing through the visible $2k$-multiplicity contact with Σ with $(0,0)$.

Theorem B. *Consider a Filippov system* $Z = (X^+, X^-)_{\Sigma}$ *and assume that* X^+ *satisfies hypothesis* **(A)** *(see Section 2.1) for some* $k \geqslant 1$. *For* $n \geqslant 2k-1$, *let* $\Phi \in C^{n-1}_{ST}$ *be given as (1.5) and consider the regularized system* Z^{Φ}_{ε} *(1.4). Then, there exist* $\rho_0, \theta_0 > 0$, *and constants* $c, r, q > 0$, *for which the following statements hold for every* $\rho \in (\varepsilon^{\lambda}, \rho_0]$, $\theta \in [x_{\varepsilon} + re^{-\frac{c}{\varepsilon^q}}, \theta_0]$, $\lambda \in (0, \lambda^*)$, *with* $\lambda^* = \dfrac{n}{2k(n-1)+1}$, *and* $\varepsilon > 0$ *sufficiently small.*

(a) *The vertical segments*

$$\check{V}^{\varepsilon}_{\theta} = \{\theta\} \times [y^{\varepsilon}_{\theta} - re^{-\frac{c}{\varepsilon^q}}, y^{\varepsilon}_{\theta}]$$

and the horizontal segments

$$\check{H}^{\varepsilon}_{\rho,\lambda} = [-\rho, -\varepsilon^{\lambda}] \times \{-\varepsilon\} \quad and \quad \overrightarrow{H}_{\varepsilon} = [x_{\varepsilon}, x_{\varepsilon} + re^{-\frac{c}{\varepsilon^q}}] \times \{\varepsilon\}$$

are transversal sections for Z^{Φ}_{ε}.

(b) The flow of Z_ε^Φ *defines a map* L_ε *between the transversal sections* $\check{H}_{\rho,\lambda}^\varepsilon$ *and* $\check{V}_\theta^\varepsilon$*, namely*

$$\begin{array}{rccc} L_\varepsilon: & \check{H}_{\rho,\lambda}^\varepsilon & \longrightarrow & \check{V}_\theta^\varepsilon \\ & x & \longmapsto & y_\theta^\varepsilon + \mathcal{O}(e^{-\frac{c}{\varepsilon^q}}). \end{array}$$

The map L_ε *is called Lower Transition Map of the regularized system (see Figure 10).*

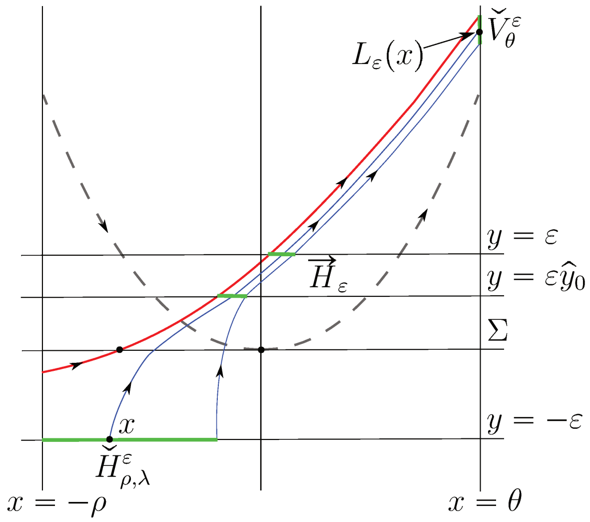

Figure 10 – Lower Transition Map L_ε of the regularized system Z_ε^Φ. The large domain $\check{H}_{\rho,\lambda}^\varepsilon$ is contracted into the small $\check{V}_\theta^\varepsilon$. The dotted curve is the trajectory of X^+ passing through the visible $2k$-multiplicity contact with Σ with $(0,0)$.

Remark 3. *The proofs of Theorems A and B is based on the analysis of the corresponding slow-fast problem associated with the regularized system* Z_ε^Φ (1.4)*. This analysis relies on the normal hyperbolicity of a related critical manifold. In previous chapter, when* $n \geqslant \max\{2, 2k-1\}$*, we saw that this critical manifold looses its normal hyperbolicity. This problem was overcome by means of blow-up methods. When* $k = n = 1$*, we do not face such a problem and the results are directly obtained from Fenichel Theory. In this case, Theorem A is already proved in [3] and Theorem B can be obtained analogously. Thus, through out the thesis, we shall assume that* $n \geqslant \max\{2, 2k-1\}$.

3.2 Upper Transition Map

This section is devoted to the proof of Theorem A. For this, we need to guarantee that under some conditions the flow of the regularized system Z_ε^Φ near a visible regular-tangential singularity defines a map between two vertical sections. Thus, it will be convenient to write this map as the composition of three maps, namely P^u, Q_ε^u and R^u (see Figure 11). The map Q_ε^u will be defined through the flow of Z_ε^Φ restricted to the band of regularization, and the maps P^u and R^u will be given by the flow of Z_ε^Φ defined outside the band of regularization. In what follows, we shall properly define these maps.

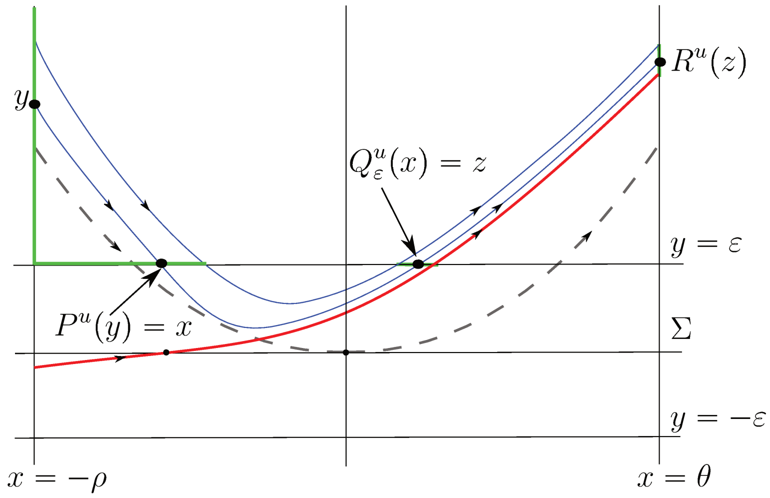

Figure 11 – Dynamics of the maps P^u, Q^u_ε and R^u. The dotted curve is the trajectory of X^+ passing through the visible $2k$-multiplicity contact with Σ with $(0, 0)$.

3.2.1 Tangential points and transversal sections

In Proposition 1, we have proved that the Fenichel manifold of system (2.4) intersects $\{y = \varepsilon\}$ at $(x_\varepsilon, \varepsilon)$ (see (2.15)). Now, we shall prove that if $(\psi(\varepsilon), \varepsilon)$ is a tangential contact of Z^Φ_ε with the line $\{y = \varepsilon\}$, then $x_\varepsilon > \psi(\varepsilon)$.

Lemma 4. *Let $\psi(\varepsilon)$ be a tangential contact of the vector field Z^Φ_ε (2.4) with $y = \varepsilon$. Then,*

(a) $\psi(\varepsilon) = \mathcal{O}(\varepsilon^{\frac{1}{2k-1}})$, and

(b) $x_\varepsilon > \psi(\varepsilon)$ for ε sufficiently small, where x_ε is given in Proposition 1.

Proof. First, we shall prove statement (a). Let $\psi : [0, \varepsilon_0] \to \mathbb{R}$ be a function, defined for $\varepsilon_0 > 0$ small, satisfying $\pi_2 Z^\Phi_\varepsilon(\psi(\varepsilon), \varepsilon) = 0$ for every $\varepsilon \in [0, \varepsilon_0]$. Here, $\pi_2 Z^\Phi_\varepsilon$ denote the second component of Z^Φ_ε. Then,

$$\begin{aligned} 0 &= \pi_2 Z^\Phi_\varepsilon(\psi(\varepsilon), \varepsilon) \\ &= \frac{1}{2}\Big(f(\psi(\varepsilon), \varepsilon)(1 + \Phi(1)) + (1 - \Phi(1))\Big) \\ &= f(\psi(\varepsilon), \varepsilon) \\ &= \alpha\psi(\varepsilon)^{2k-1} + \psi(\varepsilon)^{2k-1}\widetilde{g}(\psi(\varepsilon)) + \varepsilon\vartheta(\psi(\varepsilon), \varepsilon), \end{aligned}$$

where $\widetilde{g} = \mathcal{O}(x)$ is a continuous function such that $g(x) = x^{2k-1}\widetilde{g}(x)$. Then,

$$\psi(\varepsilon)^{2k-1} = -\frac{\varepsilon\vartheta(\psi(\varepsilon), \varepsilon)}{\alpha + \widetilde{g}(\psi(\varepsilon))} =: A(\varepsilon).$$

Notice that

$$|A(\varepsilon)| = \left|\frac{\vartheta(\psi(\varepsilon), \varepsilon)}{\alpha + \widetilde{g}(\psi(\varepsilon))}\right|\varepsilon \leqslant \max_{\varepsilon\in[0,\varepsilon_0], x\in\overline{B}}\left|\frac{\vartheta(x, \varepsilon)}{\alpha + \widetilde{g}(x)}\right|\varepsilon = C\varepsilon,$$

where $B \subset \mathbb{R}$ is a neighbourhood of 0. This implies that $A(\varepsilon) = \mathcal{O}(\varepsilon)$, i.e. $\psi(\varepsilon) = \mathcal{O}(\varepsilon^{\frac{1}{2k-1}})$.

Now, we shall prove statement (b). From Proposition 1,

$$x_\varepsilon = \varepsilon^{\lambda^*}\eta + \mathcal{O}\left(\varepsilon^{\lambda^* + \frac{1}{1+2k(n-1)}}\right),$$

where $\lambda^* = \dfrac{n}{1+2k(n-1)}$ and $\eta > \sigma_{n,k}$, where $\sigma_{n,k}$ satisfies (2.13).

First, suppose that $n > 2k-1$. Then, $\dfrac{1}{2k-1} > \lambda^*$ and $\eta > 0$. Hence, by statement (a), we conclude that $x_\varepsilon > \psi(\varepsilon)$.

Finally, suppose that $n = 2k-1$, with $k \neq 1$. In this case, $\lambda^* = 1/n$. Define

$$a(\varepsilon) = -\left(\frac{\vartheta(\psi(\varepsilon),\varepsilon)}{\alpha + \widetilde{g}(\psi(\varepsilon))}\right)^{\frac{1}{n}}.$$

Notice that $\psi(\varepsilon) = \varepsilon^{\frac{1}{n}} a(\varepsilon)$. Statement (a) implies that ψ is continuous at $\varepsilon = 0$ and $\psi(0) = 0$. Therefore, $a(\varepsilon)$ is also continuous at $\varepsilon = 0$ and $a(0) = -\big(\vartheta(0,0)/\alpha\big)^{\frac{1}{n}}$. Defining $r(\varepsilon) = a(\varepsilon) - a(0)$ we conclude that

$$\psi(\varepsilon) = \varepsilon^{\frac{1}{n}} a(\varepsilon) = \varepsilon^{\frac{1}{n}} a(0) + \varepsilon^{\frac{1}{n}} r(\varepsilon).$$

Since $\eta > \sigma_{n,\frac{n+1}{2}} = a(0)$ and $r(0) = 0$, we conclude $x_\varepsilon > \psi(\varepsilon)$. □

Statement (i) of Theorem A will follows from the next result.

Proposition 2. *Consider the Filippov system $Z = (X^+, X^-)_\Sigma$ given by (2.1), for some $k \geqslant 1$, and $y^\varepsilon_{\rho,\lambda}$ and y^ε_θ given in (3.1). For $n \geqslant \max\{2, 2k-1\}$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5) and consider the regularized system Z^Φ_ε (2.4). Then, there exist $\rho_0, \theta_0 > 0$ such that the vertical segments*

$$\widehat{V}^\varepsilon_{\rho,\lambda} = \{-\rho\} \times [\varepsilon, y^\varepsilon_{\rho,\lambda}] \ \ \text{and} \ \ \widetilde{V}^\varepsilon_\theta = \{\theta\} \times [y^\varepsilon_\theta, y^\varepsilon_\theta + re^{-\frac{c}{\varepsilon^q}}],$$

and the horizontal segments

$$\widehat{H}^\varepsilon_{\rho,\lambda} = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\} \ \ \text{and} \ \ \overleftarrow{H}_\varepsilon = [x_\varepsilon - re^{-\frac{c}{\varepsilon^q}}, x_\varepsilon] \times \{\varepsilon\},$$

are transversal sections for Z^Φ_ε for every $\rho \in (\varepsilon^\lambda, \rho_0]$, $\theta \in [x_\varepsilon, \theta_0]$, $\lambda \in (0, \lambda^*)$, with $\lambda^* = \dfrac{n}{2k(n-1)+1}$, constants $c, r, q > 0$, and $\varepsilon > 0$ sufficiently small.

Proof. First of all, we take $\rho_0, \theta_0 > 0$ sufficiently small in order that the points $(\rho_0, 0)$ and $(\theta_0, 0)$ are contained in U, domain of Z. Given $(-\rho, y_1) \in \widehat{V}^\varepsilon_{\rho,\lambda}$ and $(\theta, y_2) \in \widetilde{V}^\varepsilon_\theta$, we have

$$\left\langle Z^\Phi_\varepsilon(-\rho, y_1), (1,0)\right\rangle = \pi_1 Z^\Phi_\varepsilon(-\rho, y_1) = X^+_1(-\rho, y_1) = 1 \neq 0,$$
$$\left\langle Z^\Phi_\varepsilon(\theta, y_2), (1,0)\right\rangle = \pi_1 Z^\Phi_\varepsilon(\theta, y_2) = X^+_1(\theta, y_2) = 1 \neq 0,$$

respectively, where $\pi_1 Z^\Phi_\varepsilon$ denote the first component of Z^Φ_ε. Hence, $V^\varepsilon_{\rho,\lambda}$ and V^ε_θ are transversal sections for Z^Φ_ε.

From Lemma 4, we know that any branch of zeros $\psi(\varepsilon)$ of the equation $\pi_2 Z_\varepsilon^\Phi(x,\varepsilon) = 0$ satisfies $\psi(\varepsilon) = \mathcal{O}(\varepsilon^{\frac{1}{2k-1}})$. In other words, the zeros of $\pi_2 Z_\varepsilon^\Phi(x,\varepsilon)$ lie in an $\mathcal{O}(\varepsilon^{\frac{1}{2k-1}})$ neighbourhood of 0. Since $\rho \in (\varepsilon^\lambda, \rho_0]$, $\theta \in [x_\varepsilon, \theta_0]$, $\lambda \in (0, \lambda^*)$, the intervals $\widehat{H}^\varepsilon_{\rho,\lambda}$ and $\overleftarrow{H}_\varepsilon$ are always away from any $\mathcal{O}(\varepsilon^{\frac{1}{2k-1}})$ neighbourhood of 0 and, then, $\pi_2 Z_\varepsilon^\Phi(x,\varepsilon)$ does not admit zeros inside these sections. Consequently, given $(x_1,\varepsilon) \in \widehat{H}^\varepsilon_{\rho,\lambda}$ and $(x_2,\varepsilon) \in \overleftarrow{H}_\varepsilon$ we have

$$\left\langle Z_\varepsilon^\Phi(x_1,\varepsilon), (0,1)\right\rangle = \pi_2 Z_\varepsilon^\Phi(x_1,\varepsilon) \neq 0,$$
$$\left\langle Z_\varepsilon^\Phi(x_2,\varepsilon), (0,1)\right\rangle = \pi_2 Z_\varepsilon^\Phi(x_2,\varepsilon) \neq 0.$$

Hence, $\widehat{H}^\varepsilon_{\rho,\lambda}$ and $\overleftarrow{H}_\varepsilon$ are transversal sections for Z_ε^Φ. □

3.2.2 Construction of the map P^u

First, we shall see that the backward trajectory of X^+ (2.1) starting at $(-\varepsilon^\lambda, \varepsilon)$ reaches the straight line $\{x = -\rho\}$ at $(-\rho, y^\varepsilon_{\rho,\lambda})$ (see (3.1)). After that, the map will be obtained through Poincaré-Bendixson argument.

Accordingly, define $\mu : I_{(x,y)} \times U \times [0,\rho_0] \to \mathbb{R}$ by

$$\mu(t,x,y,\rho) = \varphi^1_{X^+}(t,x,y) + \rho,$$

where $\varphi_{X^+} = (\varphi^1_{X^+}, \varphi^2_{X^+})$ is the flow of X^+, $I_{(x,y)}$ is the interval of definition of $t \mapsto \varphi_{X^+}(t,x,y)$, and $U \subset \mathbb{R}^2$ is a neighbourhood of $(0,0)$. Since $\mu(0,0,0,0) = 0$ and $\dfrac{\partial}{\partial t}\mu(0,0,0,0) = 1$, by the *Implicit Function Theorem* there exists a unique smooth function $(x,y,\rho) \mapsto t_\rho(x,y)$, defined in a neighbourhood of $(x,y,\rho) = (0,0,0)$, such that $t_0(0,0) = 0$ and $\mu(t_\rho(x,y),x,y,\rho) = 0$, i.e. $\varphi^1_{X^+}(t_\rho(x,y),x,y) = -\rho$. Therefore, for $\rho > 0$ and $\varepsilon > 0$ sufficiently small, the backward trajectory of X^+ starting at $(-\varepsilon^\lambda, \varepsilon)$ reaches the straight line $\{x = -\rho\}$ at

$$\Big(-\rho, \varphi^2_{X^+}(t_\rho(-\varepsilon^\lambda,\varepsilon), -\varepsilon^\lambda, \varepsilon)\Big).$$

In order to prove that $\varphi^2_{X^+}(t_\rho(-\varepsilon^\lambda,\varepsilon), -\varepsilon^\lambda, \varepsilon) = y^\varepsilon_{\rho,\lambda}$, we shall compute the Taylor series expansion of the function $\varphi^2_{X^+}(t_\rho(x,y),x,y)$ around $(x,y,\rho) = (0,0,0)$. Notice that

$$\begin{aligned}
\varphi^2_{X^+}(t_\rho(x,y),x,y) = \;& \varphi^2_{X^+}(t_\rho(x,0),x,0) + y\frac{\partial}{\partial y}(\varphi^2_{X^+}(t_\rho(x,y),x,y))\Big|_{y=0} \\
& + \mathcal{O}(y^2), \\
= \;& \varphi^2_{X^+}(t_\rho(x,0),x,0) + y\left[\frac{\partial \varphi^2_{X^+}}{\partial t}(t_\rho(x,0),x,0))\frac{\partial t_\rho}{\partial y}(x,0)\right. \\
& \left. + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_\rho(x,0),x,0)\right] + \mathcal{O}(y^2) \\
= \;& \varphi^2_{X^+}(t_\rho(x,0),x,0) + y\left[\frac{\partial \varphi^2_{X^+}}{\partial t}(t_\rho(0,0),0,0))\frac{\partial t_\rho}{\partial y}(0,0)\right. \\
& \left. + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_\rho(0,0),0,0)\right] + \mathcal{O}(xy) + \mathcal{O}(y^2).
\end{aligned} \tag{3.2}$$

It is easy to see that

$$\frac{\partial \varphi^2_{X^+}}{\partial t}(t_\rho(0,0),0,0)) = f(-\rho,\overline{y}_{-\rho}) \text{ and } \frac{\partial t_\rho}{\partial y}(0,0) = -\frac{\partial \varphi^1_{X^+}}{\partial y}(t_\rho(0,0),0,0).$$

This last equality is obtained implicitly from $\varphi^1_{X^+}(t(0,y,\rho),0,y) = -\rho$ and using that $\frac{\partial \varphi^1_{X^+}}{\partial t}(t_\rho(0,0),0,0) = 1$. Thus, substituting the above relations into (3.2), we get

$$\begin{aligned}\varphi^2_{X^+}(t_\rho(x,y),x,y) = \quad & \varphi^2_{X^+}(t_\rho(x,0),x,0) + y\left[-f(-\rho,\overline{y}_{-\rho})\frac{\partial \varphi^1_{X^+}}{\partial y}(t_\rho(0,0),0,0)\right. \\ & \left. + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_\rho(0,0),0,0)\right] + \mathcal{O}(xy) + \mathcal{O}(y^2).\end{aligned} \tag{3.3}$$

Expanding the coefficient of y in (3.3) around $\rho = 0$, we have

$$-f(-\rho,\overline{y}_{-\rho})\frac{\partial \varphi^1_{X^+}}{\partial y}(t_\rho(0,0),0,0) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_\rho(0,0),0,0) = 1 + \mathcal{O}(\rho).$$

Thus, substituting the above equality into (3.3), we obtain that

$$\varphi^2_{X^+}(t_\rho(x,y),x,y) = \varphi^2_{X^+}(t_\rho(x,0),x,0) + y(1+\mathcal{O}(\rho)) + \mathcal{O}(xy) + \mathcal{O}(y^2).$$

Furthermore, from [10, Theorem *A*], we know that

$$\varphi^2_{X^+}(t_\rho(x,0),x,0) = \overline{y}_{-\rho} + \beta x^{2k} + \mathcal{O}(x^{2k+1}),$$

where $\text{sign}(\beta) = -\text{sign}((X^+)^{2k}h(0,0))$, i.e. $\beta < 0$. Thus, we conclude that

$$\varphi^2_{X^+}(t_\rho(x,y),x,y) = \overline{y}_{-\rho} + \beta x^{2k} + \mathcal{O}(x^{2k+1}) + y(1+\mathcal{O}(\rho)) + \mathcal{O}(xy) + \mathcal{O}(y^2).$$

Taking $x = -\varepsilon^\lambda$ and $y = \varepsilon$, we obtain

$$\varphi^2_{X^+}\Big(t(-\varepsilon^\lambda,\varepsilon,\rho),-\varepsilon^\lambda,\varepsilon\Big) = \overline{y}_{-\rho} + \varepsilon\Big(1+\mathcal{O}(\rho)\Big) + \beta\varepsilon^{2k\lambda} + \mathcal{O}(\varepsilon^{(2k+1)\lambda}) + \mathcal{O}(\varepsilon^{1+\lambda}), \tag{3.4}$$

which we have called by $y^\varepsilon_{\rho,\lambda}$.

Finally, consider the region

$$\mathcal{R} = \left\{(x,y) : -\rho \leqslant x \leqslant -\varepsilon^\lambda, \varepsilon \leqslant y \leqslant \varphi^2_{X^+}\Big(t,-\varepsilon^\lambda,\varepsilon\Big), \forall t \in [0, t(-\varepsilon^\lambda,\varepsilon,\rho)]\right\},$$

which is delimited by $\widehat{V}^\varepsilon_{\rho,\lambda}$, $\widehat{H}^\varepsilon_{\rho,\lambda}$, and the arc-orbit connecting $(-\rho, y^\varepsilon_{\rho,\lambda})$ with $(-\varepsilon^\lambda,\varepsilon)$. Since X^+ has no singularities inside $\mathcal{R}$, we conclude that the forward trajectory of X^+ starting at any point of the transversal section $\widehat{V}^\varepsilon_{\rho,\lambda}$ must leave $\mathcal{R}$ through the transversal section $\widehat{H}^\varepsilon_{\rho,\lambda}$. This naturally defines the map $P^u : \widehat{V}^\varepsilon_{\rho,\lambda} \longrightarrow \widehat{H}^\varepsilon_{\rho,\lambda}$.

3.2.3 Exponentially attraction and construction of the map Q^u_ε

As we saw in Chapter 2, the Fenichel manifold $S_{a,\varepsilon}$ of (2.5) is described as a graph

$$\widehat{y} = m(x,\varepsilon), \quad -L \leqslant x \leqslant -N, \quad 0 \leqslant \varepsilon \leqslant \varepsilon_0,$$

where $m(x,\varepsilon)$ is a smooth function, and $L > N > 0$ and $\varepsilon_0 > 0$ are small parameters. Notice that

$$m(x,0) = m_0(x) = \phi^{-1}\left(\frac{1+\alpha x^{2k-1}+g(x)}{1-\alpha x^{2k-1}-g(x)}\right), \tag{3.5}$$

which is the critical manifold of the system $(2.6)_{\varepsilon=0}$. Thus, we write

$$m(x,\varepsilon) = m_0(x) + \varepsilon m_1(x) + \mathcal{O}(\varepsilon^2),$$

for every $-L \leqslant x \leqslant -N$ and $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Since $S_{a,\varepsilon}$ is an invariant manifold for (2.6), the function $m(x,\varepsilon)$ satisfies

$$\varepsilon\frac{\partial m}{\partial x}(x,\varepsilon) = \frac{1+f(x,\varepsilon m(x,\varepsilon))+\phi(m(x,\varepsilon))(f(x,\varepsilon m(x,\varepsilon))-1)}{1+\phi(m(x,\varepsilon))}.$$

Hence, using that

$$\phi'(m_0(x)) = \frac{2\alpha(2k-1)x^{2k-2}+2g'(x)}{m_0'(x)(-1+\alpha x^{2k-1}+g(x))^2}, \tag{3.6}$$

we can compute

$$m_1(x) = \frac{-m_0'(x)(m_0'(x)-m_0(x)\vartheta(x,0))}{\alpha(2k-1)x^{2k-2}+g'(x)}. \tag{3.7}$$

The next result provides some estimations for $m_0(x)$.

Proposition 3. *For $-L \leqslant x < 0$ there exist positive constants C_1, C_2 such that*

$$\begin{aligned} C_1\sqrt[n]{|x|^{2k-1}} &\leqslant 1-m_0(x) \leqslant C_2\sqrt[n]{|x|^{2k-1}}, \\ \frac{C_1}{\sqrt[n]{|x|^{n-2k+1}}} &\leqslant m_0'(x) \leqslant \frac{C_2}{\sqrt[n]{|x|^{n-2k+1}}}, \end{aligned} \tag{3.8}$$

Proof. In order to obtain the above estimations, we consider the equation $\phi(\hat{y}) = \phi(m_0(x))$ for $-1 < \hat{y} < 1$ and $-L \leqslant x < 0$. Of course, $\hat{y} = m_0(x)$.

On the other hand, from (3.5),

$$\phi(m_0(x)) = 1 + 2\alpha x^{2k-1} + \mathcal{O}(x^{2k}). \tag{3.9}$$

In addition, expanding $\phi(\hat{y})$ around $\hat{y} = 1$ we get

$$\phi(\hat{y}) = 1 + \frac{\phi^{(n)}(1)}{n!}(\hat{y}-1)^n + \mathcal{O}((\hat{y}-1)^{n+1}). \tag{3.10}$$

Subtracting (3.10) from (3.9) we get that the equation $\phi(\hat{y}) = \phi(m_0(x))$ is equivalent to the system

$$\begin{cases} s = (\hat{y}-1)^n, \\ u = x^{2k-1}, \\ H(s,u) := \dfrac{\phi^{(n)}(1)}{n!}s - 2\alpha u + \mathcal{O}(s^{\frac{n+1}{n}}) + \mathcal{O}(u^{\frac{2k}{2k-1}}) = 0. \end{cases}$$

Since $H(0,0) = 0$ and $\dfrac{\partial H}{\partial s}(0,0) = \dfrac{\phi^{(n)}(1)}{n!} \neq 0$, the *Implicit Function Theorem* implies the existence of a unique function $s(u)$, defined in a small neighborhood of $u = 0$, such that $s(0) = 0$ and $H(s(u),u) = 0$. Moreover,

$$s(u) = \frac{2\alpha n!}{\phi^{(n)}(1)}u + \mathcal{O}(u^2).$$

Therefore, the equation $\phi(\hat{y}) = \phi(m_0(x))$ is solved as $\hat{y} = 1 - ((-1)^n s(x^{2k-1}))^{\frac{1}{n}}$. Recall that $\phi = \Phi\big|_{[-1,1]}$, where $\Phi \in C^{n-1}_{ST}$. Thus, from Definition 1, $\text{sign}\big(\phi^{(n)}(1)\big) = (-1)^{n+1}$. Consequently,

$$m_0(x) = \hat{y} = 1 - \sqrt[n]{\frac{2\alpha n!}{|\phi^{(n)}(1)|}}\sqrt[n]{|x|^{2k-1}} + \mathcal{O}(|x|^{1+\frac{2k-1}{n}}),\ -L \leqslant x \leqslant 0. \tag{3.11}$$

Finally, the inequalities (3.8) are obtained directly from (3.11). □

The next proposition is a technical result, which is used in the following result.

Proposition 4. *Consider $-L < -N < 0$ and $0 < \lambda \leqslant \lambda^* = \dfrac{n}{2k(n-1)+1}$. Then, there exist $K > 0$ and $\varepsilon_0 > 0$, such that, if $0 \leqslant \varepsilon \leqslant \varepsilon_0$ the invariant manifold $\hat{y} = m(x,\varepsilon)$ satisfies*

$$m_0(x) - \frac{\varepsilon K}{\sqrt[n]{x^{2k(n-2)+2}}} \leqslant m(x,\varepsilon) \leqslant m_0(x), \tag{3.12}$$

for $-L \leqslant x \leqslant -\varepsilon^\lambda$.

Proof. Consider the compact region

$$\mathcal{B} = \left\{(x,\hat{y}) : -L \leqslant x \leqslant -\varepsilon^\lambda, m_0(x) - \frac{\varepsilon K}{\sqrt[n]{x^{2k(n-2)+2}}} \leqslant \hat{y} \leqslant m_0(x)\right\}.$$

We shall prove that the vector field (2.6) points inwards $\mathcal{B}$ in the following three boundaries of $\mathcal{B}$,

$$\begin{aligned}\mathcal{B}^- &= \left\{(x,\hat{y}) : -L \leqslant x \leqslant -\varepsilon^\lambda, \hat{y} = \hat{y}_\varepsilon(x) = m_0(x) - \frac{\varepsilon K}{\sqrt[n]{x^{2k(n-2)+2}}}\right\},\\ \mathcal{B}^+ &= \left\{(x,\hat{y}) : -L \leqslant x \leqslant -\varepsilon^\lambda, \hat{y} = m_0(x)\right\}, \quad \text{and}\\ \mathcal{B}^l &= \left\{(-L,\hat{y}) : m_0(-L) - \frac{\varepsilon K}{\sqrt[n]{L^{2k(n-2)+2}}} \leqslant \hat{y} \leqslant m_0(-L)\right\}.\end{aligned}$$

On the border $\mathcal{B}^-$, the vector field (2.6) writes

$$\overline{Z}^\Phi_\varepsilon(x,\hat{y}_\varepsilon(x)) = \Big(\varepsilon\left(1 + \Phi\left(\hat{y}_\varepsilon(x)\right)\right), 1 + f(x,\varepsilon\hat{y}_\varepsilon(x)) + \Phi(\hat{y}_\varepsilon(x))(f(x,\varepsilon\hat{y}_\varepsilon(x)) - 1)\Big).$$

A normal vector of $\mathcal{B}^-$ is given by

$$n^-_\varepsilon(x) = \left(m_0'(x) - \frac{K\varepsilon(2k(n-2)+2)}{n\sqrt[n]{|x|^{(2k+1)(n-2)+4}}}, -1\right).$$

Thus, it is enough to see that

$$\begin{aligned}\langle Z^\Phi_\varepsilon(x,\hat{y}_\varepsilon(x)), n^-_\varepsilon(x)\rangle = &\left[\varepsilon(1+\Phi(\hat{y}_\varepsilon(x)))\left(m_0'(x) - \frac{K\varepsilon(2k(n-2)+2)}{n\sqrt[n]{|x|^{(2k+1)(n-2)+4}}}\right)\right]\\ &- \Big[1 + f(x,\varepsilon\hat{y}_\varepsilon(x)) + \Phi(\hat{y}_\varepsilon(x))(f(x,\varepsilon\hat{y}_\varepsilon(x)) - 1)\Big]\\ &< 0.\end{aligned} \tag{3.13}$$

Now, expanding in Taylor series $\Phi(\widehat{y}_\varepsilon(x))$ and $\vartheta(x, \varepsilon\widehat{y}_\varepsilon(x))$ around $\varepsilon = 0$, we have

$$\Phi(\widehat{y}_\varepsilon(x)) = \Phi(m_0(x)) - \frac{\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}}\varepsilon + \sum_{l=2}^{n-1} \frac{(-1)^l\Phi^{(l)}(m_0(x))K^l}{\sqrt[n]{x^{2lk(n-2)+2l}}}\frac{\varepsilon^l}{l!} + s(x,\varepsilon),$$

$$\vartheta(x, \varepsilon\widehat{y}_\varepsilon(x)) = \vartheta(x,0) + r(x,\varepsilon),$$

where $s(x,\varepsilon)$ and $r(x,\varepsilon)$ are the Lagrange remainders of $\Phi(\widehat{y}_\varepsilon(x))$ and $\vartheta(x, \varepsilon\widehat{y}_\varepsilon(x))$ respectively, i.e. for some $c, d \in (0,\varepsilon)$, we get

$$\begin{aligned} s(x,\varepsilon) &= \left[\frac{(-1)^n\Phi^{(n)}\left(\widehat{y}_c(x)\right)K^n}{x^{2k(n-2)+2}}\right]\frac{\varepsilon^n}{n!}, \quad \text{and} \\ r(x,\varepsilon) &= \left[\vartheta_y\left(x, d\widehat{y}_d(x)\right)\left(m_0(x) - \frac{2dK}{\sqrt[n]{x^{2k(n-2)+2}}}\right)\right]\varepsilon. \end{aligned} \tag{3.14}$$

Notice that, the inequality (3.13) can be written as

$$L(x,\varepsilon) + T(x,\varepsilon) + O(x,\varepsilon) < 0,$$

where

$$\begin{aligned} L(x,\varepsilon) = \; & \varepsilon\Big[m_0'(x)(1+\Phi(m_0(x))) + \frac{\Phi'(m_0(x))K(f(x,0)-1)}{\sqrt[n]{x^{2k(n-2)+2}}} \\ & -m_0(x)(1+\Phi(m_0(x)))\vartheta(x,0)\Big], \\ T(x,\varepsilon) = \; & -\varepsilon^2\frac{m_0'(x)\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}} + \varepsilon^3\frac{K^2(2k(n-2)+2)\Phi'(m_0(x))}{n\sqrt[n]{|x|^{(4k+1)(n-2)+6}}} \\ & +\varepsilon^2\frac{m_0(x)\vartheta(x,0)\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}} + \varepsilon^2\frac{K\vartheta(x,\varepsilon\widehat{y}_\varepsilon(x))(1+\Phi(m_0(x)))}{\sqrt[n]{x^{2k(n-2)+2}}} \\ & -\frac{\varepsilon^3K^2\vartheta(x,\varepsilon\widehat{y}_\varepsilon(x))\Phi'(m_0(x))}{\sqrt[n]{x^{4k(n-2)+4}}} - \varepsilon m_0(x)r(x,\varepsilon)(1+\Phi(m_0(x))) \\ & +\varepsilon^2\frac{m_0(x)r(x,\varepsilon)\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}} + \varepsilon\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^lm_0'(x)}{\sqrt[n]{x^{2lk(n-2)+2l}}}\frac{\varepsilon^l}{l!} \\ & -\varepsilon^2\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^{l+1}(2k(n-2)+2)}{n\sqrt[n]{x^{(2k(l+1)+1)(n-2)+2l+4}}}\frac{\varepsilon^l}{l!} \\ & -\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^l}{\sqrt[n]{x^{2lk(n-2)+2l}}}\frac{\varepsilon^l}{l!}\varepsilon\widehat{y}_\varepsilon(x)\vartheta(x,\varepsilon\widehat{y}_\varepsilon(x)) \\ & +\left(\varepsilon m_0'(x) - \varepsilon^2\frac{K(2k(n-2)+2)}{n\sqrt[n]{|x|^{(2k+1)(n-2)+4}}} - \varepsilon\widehat{y}_\varepsilon(x)\vartheta(x,\varepsilon\widehat{y}_\varepsilon(x))\right)s(x,\varepsilon), \\ O(x,\varepsilon) = \; & (-f(x,0)+1)\,s(x,\varepsilon) - \varepsilon^2\frac{K(2k(n-2)+2)(1+\Phi(m_0(x)))}{n\sqrt[n]{|x|^{(2k+1)(n-2)+4}}} \\ & +\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^l}{\sqrt[n]{x^{2lk(n-2)+2l}}}\frac{\varepsilon^l}{l!}(-f(x,0)+1). \end{aligned}$$

Now, we shall prove that the functions L, T and O can be bounded. Indeed, by (3.8) and (3.6), we have that, $L(x,\varepsilon)$ can be bounded, choosing K big enough depending on C_2, L, n, k, α, M, $M_{\min}$, and $\vartheta_{\min}$, where

- M is such that $|g(x)| \leqslant M|x|^{2k}$ for all $-L \leqslant x \leqslant 0$.
- $\widetilde{M}$ is such that $|\widetilde{g}'(x)| \leqslant \widetilde{M}|x|$ for all $-L \leqslant x \leqslant 0$ with $g'(x) = x^{2k-2}\widetilde{g}'(x)$.
- $M_{\min}$ is a positive constant such that $\alpha(2k-1) + \widetilde{g}'(x) \geqslant M_{\min}$, for all $x \in [-L, 0]$.
- $\vartheta_{\min} = \min\{\widehat{y}\vartheta(x,0) : -L \leqslant x \leqslant 0, -1 \leqslant \widehat{y} \leqslant 1\}$.

$$\begin{aligned}
& \varepsilon\left[m_0'(x)(1+\Phi(m_0(x))) + \frac{\Phi'(m_0(x))K(f(x,0)-1)}{\sqrt[n]{x^{2k(n-2)+2}}} - m_0(x)(1+\Phi(m_0(x)))\vartheta(x,0)\right] \\
= \; & \varepsilon\left[(m_0'(x) - m_0(x)\vartheta(x,0))(1+\Phi(m_0(x))) + \frac{\Phi'(m_0(x))K(f(x,0)-1)}{\sqrt[n]{x^{2k(n-2)+2}}}\right] \\
\leqslant \; & \varepsilon\left[(C_2|x|^{-\frac{n-2k+1}{n}} - \vartheta_{\min})\left(\frac{2}{1-f(x,0)}\right) - \frac{(2\alpha(2k-1)x^{2k-2} + 2g'(x))K}{C_2|x|^{-\frac{n-2k+1}{n}}(1-f(x,0))\sqrt[n]{x^{2k(n-2)+2}}}\right] \\
\leqslant \; & \varepsilon\left[\frac{2C_2|x|^{2k-2+\frac{n-2k+1}{n}}(C_2|x|^{-\frac{n-2k+1}{n}} - \vartheta_{\min}) - (2\alpha(2k-1)x^{2k-2} + 2x^{2k-2}\widetilde{g}'(x))K}{C_2(1-f(x,0))|x|^{2k-2+\frac{n-2k+1}{n}}}\right] \\
\leqslant \; & \varepsilon\left[\frac{2C_2(C_2 + |x|^{\frac{n-2k-1}{n}}|\vartheta_{\min}|) - (2\alpha(2k-1) + 2\widetilde{g}'(x))K}{C_2(1-f(x,0))|x|^{\frac{n-2k+1}{n}}}\right] \\
\leqslant \; & \frac{2C_2(C_2 + L^{\frac{n-2k+1}{n}}|\vartheta_{\min}|) - 2M_{\min}K}{C_2(1-f(x,0))} \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}} \\
\leqslant \; & \frac{2C_2(C_2 + L^{\frac{n-2k+1}{n}}|\vartheta_{\min}|) - 2M_{\min}K}{C_2(1+L^{2k-1}(\alpha+ML))} \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}} \\
\leqslant \; & -2\frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}}.
\end{aligned}$$

Now, we need to bound the function $T(x,\varepsilon)$. Using (3.6) and (3.8), we obtain

$$\begin{aligned}
\left|\varepsilon^2 \frac{m_0'(x)\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}}\right| &\leqslant d_\varepsilon^1 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}}, \\
\left|\varepsilon^3 \frac{K^2(2k(n-2)+2)\Phi'(m_0(x))}{n\sqrt[n]{|x|^{(4k+1)(n-2)+6}}}\right| &\leqslant d_\varepsilon^2 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}}, \\
\left|\frac{\varepsilon^2 m_0(x)\vartheta(x,0)\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}}\right| &\leqslant d_\varepsilon^3 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}}, \\
\left|\frac{\varepsilon^2 K\vartheta(x,\varepsilon\widehat{y}_\varepsilon)(1+\Phi(m_0(x)))}{\sqrt[n]{x^{2k(n-2)+2}}}\right| &\leqslant d_\varepsilon^4 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},,
\end{aligned}$$

$$\begin{aligned}
\left|\tfrac{\varepsilon^3 K^2 \vartheta(x,\varepsilon\widehat{y}_\varepsilon)\Phi'(m_0(x))}{\sqrt[n]{x^{4k(n-2)+4}}}\right| &\leqslant d_\varepsilon^5 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},\\
|\varepsilon m_0(x) r(x,\varepsilon)(1+\Phi(m_0(x))| &\leqslant d_\varepsilon^6 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},\\
\left|\varepsilon^2 \frac{m_0(x) r(x,\varepsilon)\Phi'(m_0(x))K}{\sqrt[n]{x^{2k(n-2)+2}}}\right| &\leqslant d_\varepsilon^7 \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},
\end{aligned}$$

where

$$\begin{aligned}
d_\varepsilon^1 &= (2\alpha(2k-1)+2\widetilde{M}L)K\varepsilon^{1-\lambda\left(\frac{n-2k+1}{n}\right)},\\
d_\varepsilon^2 &= \tfrac{K^2(2k(n-2)+2)(2\alpha(2k-1)+2\widetilde{M}L)}{nC_1}\varepsilon^{2-\lambda\left(\frac{(2k+1)(n-2)+4}{n}\right)},\\
d_\varepsilon^3 &= \tfrac{|\vartheta_{\max}|(2\alpha(2k-1)+2\widetilde{M}L)K}{C_1}\varepsilon,\\
d_\varepsilon^4 &= 2K|\vartheta_{\max}|\varepsilon^{1-\lambda\left(\frac{(2k-1)(n-1)}{n}\right)},\\
d_\varepsilon^5 &= \frac{K^2|\vartheta_{\max}|(2\alpha(2k-1)+2\widetilde{M}L)}{C_1}\varepsilon^{2-\lambda\left(\frac{2k(n-2)+2}{n}\right)},\\
d_\varepsilon^6 &= 2|\vartheta_{y_m}|(L^{\frac{2k(n-2)+2}{n}}+2dK)\varepsilon^{1-\lambda\left(\frac{(2k-1)(n-1)}{n}\right)},\\
d_\varepsilon^7 &= \frac{|\vartheta_{y_m}|(L^{\frac{2k(n-2)+2}{n}}+2dK)(2\alpha(2k-1)+2\widetilde{M}L)K}{C_1}\varepsilon^{2-\lambda\left(\frac{2k(n-2)+2}{n}\right)},\\
\vartheta_{y_m} &= \max\{\vartheta_y(x,\widehat{y}) : -L \leqslant x \leqslant 0, -1 \leqslant \widehat{y} \leqslant 1\},\\
\vartheta_{\max} &= \max\{\widehat{y}_1\vartheta(x,\widehat{y}_2) : -L \leqslant x \leqslant 0, -1 \leqslant \widehat{y}_1,\widehat{y}_2 \leqslant 1\}.
\end{aligned}$$

To bound the last terms of T, notice that by (3.10), we get

$$\Phi^{(l)}(\widehat{y}) = \frac{\Phi^{(n)}(1)}{(n-l)!}(\widehat{y}-1)^{n-l} + \mathcal{O}((\widehat{y}-1)^{n-l+1}), \quad 2 \leqslant l \leqslant n-1, \tag{3.15}$$

for $\widehat{y}$ sufficiently near to 1. In the particular case $\widehat{y} = m_0(x)$ for $x \in [-L, 0]$, we have

$$\Phi^{(l)}(m_0(x)) = (m_0(x)-1)^{n-l}\left(\frac{\Phi^{(n)}(1)}{(n-l)!} + \zeta(x)\right), \tag{3.16}$$

with $\zeta(x) = \mathcal{O}(m_0(x)-1)$, thus there exists a positive constant $\widehat{M}$ such that $|\zeta(x)| \leqslant \widehat{M}|m_0(x)-1|$. Therefore, by the above information about ζ and the first inequation in (3.8) for $-L \leqslant x \leqslant 0$, we obtain

$$\left|\Phi^{(l)}(m_0(x))\right| \leqslant C_2^{n-l}|x|^{\frac{(2k-1)(n-l)}{n}}\left(\frac{|\Phi^{(n)}(1)|}{(n-l)!} + \widehat{M}C_2L^{\frac{2k-1}{n}}\right),$$

i.e. for each $l \in [2, n-1]$ we have $|\Phi^{(l)}(m_0(x))| \leqslant C_l$ for all $-L \leqslant x \leqslant 0$, with $C_l = C_2^{n-l}L^{\frac{(2k-1)(n-l)}{n}}\widetilde{C}_l$ and $\widetilde{C}_l = \dfrac{|\Phi^{(n)}(1)|}{(n-l)!} + \widehat{M}C_2L^{\frac{2k-1}{n}}$. Consequently,

$$\begin{aligned}
\left|\varepsilon\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^l m_0'(x)}{\sqrt[n]{x^{2lk(n-2)+2l}}}\frac{\varepsilon^l}{l!}\right| &\leqslant d_\varepsilon^8\frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},\\
\left|\varepsilon^2\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^{l+1}(2k(n-2)+2)}{n\sqrt[n]{x^{(2k(l+1)+1)(n-2)+2l+4}}}\frac{\varepsilon^l}{l!}\right| &\leqslant d_\varepsilon^9\frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},\\
\left|\sum_{l=2}^{n-1}\frac{(-1)^l\Phi^{(l)}(m_0(x))K^l}{\sqrt[n]{x^{2lk(n-2)+2l}}}\frac{\varepsilon^l}{l!}\varepsilon\widehat{y}_\varepsilon(x)\vartheta(x,\varepsilon\widehat{y}_\varepsilon(x))\right| &\leqslant d_\varepsilon^{10}\frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},
\end{aligned}$$

where

- $d_\varepsilon^8 = \displaystyle\sum_{l=2}^{n-1} \frac{C_l K^l C_2}{l!} \varepsilon^{l-\lambda\left(\frac{2kl(n-2)+2l}{n}\right)}$,
- $d_\varepsilon^9 = \displaystyle\sum_{l=2}^{n-1} \frac{C_l K^{l+1}(2k(n-2)+2)}{l!n} \varepsilon^{l+1-\lambda\left(\frac{2k(l+1)(n-2)+2l+2k+1}{n}\right)}$,
- $d_\varepsilon^{10} = \displaystyle\sum_{l=2}^{n-1} \frac{C_2^{n-l} K^l |\vartheta_{\max}| \widetilde{C}_l}{l!} \varepsilon^{l-\lambda\left(\frac{(2k(n-1)+1)(l-1)}{n}\right)}$.

Finally, using that, for any $\eta_1 > 0$, there exists $C_n > 0$ such that

$$|\Phi^{(n)}(\widehat{y})| \leqslant C_n, \text{ for } 1-\eta_1 \leqslant \widehat{y} \leqslant 1,$$

and using that, for $\varepsilon > 0$ small enough

$$1 - \left(C_2 L^{\frac{2k-1}{n}} + \varepsilon^{1-\lambda\left(\frac{2k(n-2)+2}{n}\right)} K \right) \leqslant \widehat{y}_c(x) \leqslant 1, \tag{3.17}$$

if $-L \leqslant x \leqslant -\varepsilon^\lambda$ and also by (3.14), one has

$$\left| \left(\varepsilon m_0'(x) - \varepsilon^2 \frac{K(2k(n-2)+2)}{n\sqrt[n]{|x|^{(2k+1)(n-2)+4}}} - \varepsilon \widehat{y}_\varepsilon \vartheta(x, \varepsilon \widehat{y}_\varepsilon) \right) s(x,\varepsilon) \right| \leqslant d_\varepsilon^{11} \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}},$$

with

$$\begin{aligned} d_\varepsilon^{11} = & \left(C_2 \varepsilon^{1-\lambda\left(\frac{n-2k+1}{n}\right)} + \frac{K(2k(n-2)+2)}{n} \varepsilon^{2-\lambda\left(\frac{(2k+1)(n-2)+4}{n}\right)} + \varepsilon|\vartheta_{\max}| \right) \\ & \cdot \frac{C_n K^n}{n!} \varepsilon^{n-1-\lambda\left(\frac{(2kn+1)(n-2)+2k+1}{n}\right)}. \end{aligned}$$

Since $0 < \lambda \leqslant \lambda^*$ one has that $\lim\limits_{\varepsilon\to 0} d_\varepsilon^i = 0$, for all $i \in \{1, \ldots, 11\}$, hence for $\varepsilon > 0$ small enough, we get

$$|T(x,\varepsilon)| \leqslant \sum_{i=1}^{11} d_\varepsilon^i \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}} \leqslant \frac{1}{2} \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}}.$$

Now, we shall prove that the function $O(x,\varepsilon) < 0$ for all $x \in [-L, -\varepsilon^\lambda]$ and $\varepsilon > 0$ small enough. Indeed, since for each $n \geqslant 2$, we know that $(-1)^n \phi^{(n)}(1) < 0$, then $(-1)^n \phi^{(n)}(\widehat{y}) < 0$, for all $\widehat{y}$ sufficiently close to 1 and by (3.17) we obtain that $(-1)^n \phi^{(n)}(\widehat{y}_c(x)) < 0$, for all $x \in [-L, -\varepsilon^\lambda]$ and ε sufficiently enough. Hence, by (3.14) we have that $s(x,\varepsilon) < 0$ for all $x \in [-L, -\varepsilon^\lambda]$ and $\varepsilon > 0$ small enough. Therefore,

$$(-f(x,0)+1)s(x,\varepsilon) < 0,$$

for all $x \in [-L, -\varepsilon^\lambda]$ and $\varepsilon > 0$ small enough. After that, using (3.16) we can conclude that $(-1)^l \phi^{(l)}(m_0(x)) < 0$, for all $x \in [-L, 0]$ and $l \in \{2, \ldots, n-1\}$. Consequently,

$$\sum_{l=2}^{n-1} \frac{(-1)^l \Phi^{(l)}(m_0(x)) K^l}{\sqrt[n]{x^{2lk(n-2)+2l}}} \frac{\varepsilon^l}{l!} (-f(x,0)+1) < 0,$$

for all $x \in [-L, -\varepsilon^{\lambda}]$ and $\varepsilon > 0$ small enough. Last of all, as $1 + \Phi(m_0(x)) > 0$, for all $x \in [-L, 0]$, we get

$$-\varepsilon^2 \frac{K(2k(n-2)+2)(1+\Phi(m_0(x)))}{n\sqrt[n]{|x|^{(2k+1)(n-2)+4}}} < 0,$$

for all $x \in [-L, -\varepsilon^{\lambda}]$ and ε sufficiently small. Of this way, we obtained the result.

Finally, we conclude that

$$\begin{aligned}
\langle Z_\varepsilon^\Phi(x, \widehat{y}_\varepsilon(x)), n_\varepsilon^-(x)\rangle &\leqslant L(x,\varepsilon) + |T(x,\varepsilon)| + O(x,\varepsilon) \\
&\leqslant \left(-2 + \frac{1}{2}\right) \frac{\varepsilon}{\sqrt[n]{|x|^{n-2k+1}}} < 0.
\end{aligned}$$

Therefore, the vector field $\overline{Z}_\varepsilon^\Phi$ points inward $\mathcal{B}$ along $\mathcal{B}^-$.

In the border $\mathcal{B}^+$ the vector field $\overline{Z}_\varepsilon^\Phi$ in (2.6) is of the form

$$\overline{Z}_\varepsilon^\Phi = \left(\frac{\varepsilon(1+\Phi(m_0(x)))}{2}, \frac{\varepsilon m_0(x)\vartheta(x, \varepsilon m_0(x))(1+\Phi(m_0(x)))}{2}\right),$$

and the normal vector is $n^+(x) = (m_0'(x), -1)$, thus using the second inequation in (3.8) for $-L \leqslant x \leqslant -\varepsilon^{\lambda}$, we get

$$\begin{aligned}
\langle \overline{Z}_\varepsilon^\Phi, n^+(x)\rangle &= \frac{\varepsilon}{2}\Big(1+\Phi(m_0(x))\Big)\Big(m_0'(x) - m_0(x)\vartheta(x,\varepsilon m_0(x))\Big) \\
&\geqslant \frac{\varepsilon}{2}\left(\frac{2}{1-f(x,0)}\right)\Big(m_0'(x) - \vartheta_{\max}\Big) \\
&\geqslant \frac{\varepsilon}{2}\left(\frac{2}{1-f(x,0)}\right)\left(\frac{C_1}{L^{\frac{n-2k+1}{n}}} - \vartheta_{\max}\right) \\
&> 0,
\end{aligned}$$

for L enough small, therefore the flow points inward $\mathcal{B}$ along this border.

Finally, at the boundary $\mathcal{B}^l$ one has that $x' > 0$ thus the flow points inward $\mathcal{B}$.

Now, from the *Poincaré–Bendixson Theorem* we know that any orbit entering $\mathcal{B}$ stays in it until it reaches $x = -\varepsilon^{\lambda}$. Moreover, we know that the invariant manifold $S_{a,\varepsilon}$ at $x = -L$ is given by

$$m(-L,\varepsilon) = m_0(-L) + \varepsilon m_1(-L) + \mathcal{O}(\varepsilon^2).$$

Using (3.8) and since L is small enough one has that

$$m_0'(-L) - m_0(-L)\vartheta(-L,0) \geqslant \frac{C_1}{L^{\frac{n-2k+1}{n}}} - \vartheta_{\max} > 0,$$

thus from (3.7) $m_1(-L) < 0$. Therefore, adjusting the constants to have

$$K \geqslant -L^{\frac{2k(n-2)+2}{n}} m_1(-L),$$

the manifold enters $\mathcal{B}$ and satisfies (3.12) for $-L \leqslant x \leqslant -\varepsilon^{\lambda}$. □

From Theorem 1 (Fenichel Theorem), we know that, for $\varepsilon > 0$ sufficiently small, the Fenichel manifold $S_{a,\varepsilon}$ exponentially attracts all the solutions with initial conditions $(x_0, 1)$, with $-L \leqslant x_0 \leqslant -N$, for any small positive real numbers $L > N$. In the next result, we show that this exponential attraction holds for any $(x_0, 1)$ with $-L \leqslant x_0 \leqslant -\varepsilon^{\lambda}$. Consider the equation for the orbits of system (2.6)

$$\varepsilon\frac{d\widehat{y}}{dx} = \frac{1 + f(x, \varepsilon\widehat{y}) + \phi(\widehat{y})(f(x, \varepsilon\widehat{y}) - 1)}{1 + \phi(\widehat{y})}. \tag{3.18}$$

Proposition 5. *Fix* $0 < \lambda < \lambda^* = \dfrac{n}{2k(n-1)+1}$. *Let* $x_0 \in [-L, -\varepsilon^{\lambda}]$ *and consider the solution* $\widehat{y}(x, \varepsilon)$ *of the differential* (3.18) *satisfying* $\widehat{y}(x_0, \varepsilon) = 1$. *Then, there exist positive numbers* c *and* r *such that*

$$|\widehat{y}(x, \varepsilon) - m(x, \varepsilon)| \leqslant r e^{-\frac{c}{\varepsilon}\left(|x_0|^{\frac{1}{\lambda^*}} - |x|^{\frac{1}{\lambda^*}}\right)},$$

for $x_0 \leqslant x \leqslant -\varepsilon^{\lambda^*}$.

Proof. Performing the change of variables $\omega = \widehat{y} - m(x, \varepsilon)$ in equation (3.18), we get

$$\varepsilon\frac{d\omega}{dx} = -\xi(x, \varepsilon)\phi'(m(x, \varepsilon))\omega - \xi(x, \varepsilon)F(x, \omega, \varepsilon), \tag{3.19}$$

where,

$$F(x, \omega, \varepsilon) = \phi(m(x, \varepsilon) + \omega) - \phi(m(x, \varepsilon)) - \phi'(m(x, \varepsilon))\omega$$

and

$$\xi(x, \varepsilon) = \frac{2}{\Big(1 + \phi(m(x, \varepsilon))\Big)\Big(1 + \phi(m(x, \varepsilon) + \omega(x, \varepsilon))\Big)} + \frac{\varepsilon\Big(m(x, \varepsilon)\vartheta(x, \varepsilon m(x, \varepsilon)) - (\omega(x, \varepsilon) + m(x, \varepsilon))\vartheta(x, \varepsilon(\omega(x, \varepsilon) + m(x, \varepsilon)))\Big)}{\phi(m(x, \varepsilon) + \omega(x, \varepsilon)) - \phi(m(x, \varepsilon))}.$$

Here, we are denoting $\omega(x, \varepsilon) = \widehat{y}(x, \varepsilon) - m(x, \varepsilon)$, which is the solution of (3.19) with initial condition $\omega(x_0, \varepsilon) = 1 - m(x_0, \varepsilon)$. Therefore, we also have that

$$\omega(x, \varepsilon) = e^{-\frac{1}{\varepsilon}\int_{x_0}^{x} \xi(s,\varepsilon)\phi'(m(s,\varepsilon))ds}\widetilde{\omega}(x, \varepsilon),$$

where

$$\widetilde{\omega}(x, \varepsilon) = \omega(x_0, \varepsilon) - \frac{1}{\varepsilon}\int_{x_0}^{x} e^{\frac{1}{\varepsilon}\int_{x_0}^{\nu} \xi(s,\varepsilon)\phi'(m(s,\varepsilon))ds}\xi(\nu, \varepsilon)F(\nu, \omega(\nu, \varepsilon), \varepsilon)d\nu.$$

In what follows we shall estimate $|\omega(x, \varepsilon)|$. First, notice that F writes

$$F(x, \omega, \varepsilon) = A(x, \varepsilon)\omega, \tag{3.20}$$

where

$$A(x, \varepsilon) = \int_0^1 \phi'(m(x, \varepsilon) + s\omega(x, \varepsilon)) - \phi'(m(x, \varepsilon))ds.$$

We claim that $A(x,\varepsilon)$ is negative for $-L \leqslant x \leqslant 0$ and $L, \varepsilon > 0$ small enough. Indeed, from (3.10), we obtain

$$\phi''(\widehat{y}) = \frac{\phi^{(n)}(1)}{(n-2)!}(\widehat{y}-1)^{n-2} + \mathcal{O}((\widehat{y}-1)^{n-1}),\ \widehat{y} \leqslant 1. \tag{3.21}$$

Again, recall that $\phi = \Phi\big|_{[-1,1]}$, where $\Phi \in C^{n-1}_{ST}$. Hence, from Definition 1, $\text{sign}\big(\phi^{(n)}(1)\big) = (-1)^{n+1}$. Thus, from (3.21), we get the existence of $\eta > 0$ such that $\phi''(\widehat{y}) < 0$ for all $1-\eta < \widehat{y} < 1$. This means that ϕ' is decreasing for $1-\eta < \widehat{y} \leqslant 1$. Notice that

$$m(x,\varepsilon) \leqslant m(x,\varepsilon) + s\omega(x,\varepsilon) \leqslant (1-s)m(x,\varepsilon) + s \leqslant 1, \text{ for all } 0 \leqslant s \leqslant 1, \tag{3.22}$$

Thus, it remains to show that $m(x,\varepsilon) + s\omega(x,\varepsilon)$, $m(x,\varepsilon) > 1-\eta$ for $-L \leqslant x < 0$ and $L, \varepsilon > 0$ small enough. From Proposition 4 and (3.8), we have that

$$m(x,\varepsilon) \geqslant m_0(x) - \frac{\varepsilon K}{\sqrt[n]{x^{2k(n-2)+2}}} \geqslant 1 - C_2\sqrt[n]{L^{2k-1}} - \varepsilon^{1-\lambda^*\left(\frac{2k(n-2)+2}{n}\right)}K, \tag{3.23}$$

for $\varepsilon, L > 0$ small enough. Therefore, L and ε can be taking smaller, if necessary, in order that $C_2\sqrt[n]{L^{2k-1}} + \varepsilon^{1-\lambda^*\left(\frac{2k(n-2)+2}{n}\right)}KM < \eta$. This implies that

$$m(x,\varepsilon) + s\omega(x,\varepsilon) \geqslant m(x,\varepsilon) > 1-\eta.$$

Consequently, $A(x,\varepsilon)$ is negative.

Hence, by (3.20), we have that

$$\begin{aligned} |\widetilde{\omega}(x,\varepsilon)| &= |\omega(x_0)| + \frac{1}{\varepsilon}\int_{x_0}^{x} |\xi(\nu,\varepsilon)A(\nu,\varepsilon)\widetilde{\omega}(\nu,\varepsilon)|d\nu \\ &\leqslant |\omega(x_0)| - \frac{1}{\varepsilon}\int_{x_0}^{x} \xi(\nu,\varepsilon)A(\nu,\varepsilon)|\widetilde{\omega}(\nu,\varepsilon)|d\nu. \end{aligned}$$

Using Gronwall's Lemma, we get that

$$|\widetilde{\omega}(x,\varepsilon)| \leqslant |\omega(x_0)|e^{-\frac{1}{\varepsilon}\int_{x_0}^{x}\xi(\nu,\varepsilon)A(\nu,\varepsilon)d\nu}$$

and, therefore,

$$\begin{aligned} |\omega(x,\varepsilon)| &\leqslant |\omega(x_0)|e^{-\frac{1}{\varepsilon}\int_{x_0}^{x}\xi(\nu,\varepsilon)(A(\nu,\varepsilon)+\phi'(m(\nu,\varepsilon)))d\nu} \\ &\leqslant |\omega(x_0)|e^{-\frac{1}{\varepsilon}\int_{x_0}^{x}\xi(\nu,\varepsilon)(\int_0^1\phi'(m(\nu,\varepsilon)+s\omega(\nu,\varepsilon))ds)d\nu}. \end{aligned}$$

To conclude this proof, notice that

$$\xi(x,\varepsilon) = \frac{2}{\Big(1+\phi(m_0(x))\Big)\Big(1+\phi(\omega(x,0)+m_0(x))\Big)} + \mathcal{O}(\varepsilon).$$

Thus, $L, \varepsilon > 0$ can be taken small enough in order that $\xi(x,\varepsilon) \geqslant l > 0$, for every $x \in [-L,0]$. Moreover, from (3.10), given $0 < \eta < 1$, there exist positive constants $c_1, c_2 > 0$ such that

$$c_1(1-\widehat{y})^{n-1} \leqslant \phi'(\widehat{y}) \leqslant c_2(1-\widehat{y})^{n-1}, \quad \text{for} \quad |\widehat{y}-1| < \eta.$$

Finally, using (3.22) and (3.23), we obtain that $|m(\nu,\varepsilon)+s\omega(\nu,\varepsilon)-1|<\eta$. Hence, for $x\leqslant -\varepsilon^{\lambda^*}$, we have that

$$\begin{aligned}
|\omega(x,\varepsilon)| &\leqslant |\omega(x_0)|e^{-\frac{c_1 l}{\varepsilon}\int_{x_0}^{x}(\int_0^1(1-m(\nu,\varepsilon)-s\omega(\nu,\varepsilon))^{n-1}ds)d\nu}\\
&\leqslant |\omega(x_0)|e^{-\frac{c_1 l}{\varepsilon}\int_{x_0}^{x}(\int_0^1((1-m(\nu,\varepsilon))(1-s))^{n-1}ds)d\nu}\\
&\leqslant |\omega(x_0)|e^{-\frac{c_1 l}{n\varepsilon}\int_{x_0}^{x}(1-m(\nu,\varepsilon))^{n-1}d\nu}\\
&\leqslant |\omega(x_0)|e^{-\frac{c_1 l}{n\varepsilon}\int_{x_0}^{x}(1-m_0(\nu))^{n-1}d\nu}\\
&\leqslant |\omega(x_0)|e^{-\frac{c_1 l}{n\varepsilon}\int_{x_0}^{x}(C_1|\nu|^{\frac{2k-1}{n}})^{n-1}d\nu}\\
&\leqslant |\omega(x_0)|e^{-\frac{c}{\varepsilon}(|x_0|^{\frac{1}{\lambda^*}}-|x|^{\frac{1}{\lambda^*}})},
\end{aligned}$$

where $c=\dfrac{c_1 l C_1^{n-1}}{2k(n-1)+1}$ is a positive constant. The inequality (3.8) has also been used. $\square$

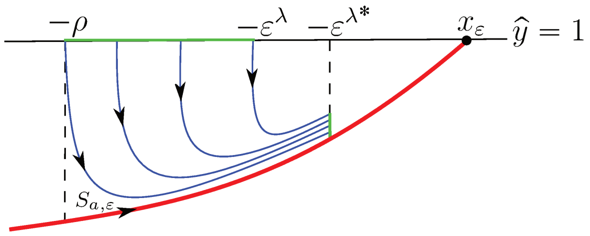

Figure 12 – The exponential attraction of $S_{a,\varepsilon}$.

Fix $0<\lambda<\lambda^*$. From Proposition 5, applied to $x_0=-\varepsilon^{\lambda}$ and $x=-\varepsilon^{\lambda^*}$, we know that there exist positive numbers $\tilde{r}$ and c such that

$$\begin{aligned}
|\hat{y}(-\varepsilon^{\lambda^*},\varepsilon)-m(-\varepsilon^{\lambda^*},\varepsilon)| &\leqslant \tilde{r}e^{-\frac{c}{\varepsilon}\left(|-\varepsilon^{\lambda}|^{\frac{1}{\lambda^*}}-|-\varepsilon^{\lambda^*}|^{\frac{1}{\lambda^*}}\right)}\\
&= re^{-\frac{c}{\varepsilon^q}},
\end{aligned}$$

where $r=\tilde{r}e^{c}$ and $q=1-\dfrac{\lambda}{\lambda^*}$ are positive constants. Thus,

$$\hat{y}(-\varepsilon^{\lambda^*},\varepsilon)=m(-\varepsilon^{\lambda^*},\varepsilon)+\mathcal{O}(e^{-c/\varepsilon^q}).$$

Hence, arguing analogously to the construction of map P^u (see Section 3.2.2), any solution of the system (2.6) with initial condition in the interval $[-\rho,-\varepsilon^{\lambda}]$, ε sufficiently small, reaches the section $x=-\varepsilon^{\lambda^*}$ exponentially close to the Fenichel manifold (see Figure 12). From Proposition 1, these solutions can be continued until the section $\hat{y}=1$. Going back through the rescaling $y=\varepsilon\hat{y}$, we get defined the following map through the flow of (2.4),

$$\begin{aligned}
Q^u_\varepsilon:\widehat{H}^{\varepsilon}_{\rho,\lambda} &\longrightarrow \overleftarrow{H}_\varepsilon\\
(x,\varepsilon) &\longmapsto \Big(x_\varepsilon+\mathcal{O}(e^{-c/\varepsilon^q}),\varepsilon\Big),
\end{aligned}$$

where $\widehat{H}^{\varepsilon}_{\rho,\lambda}=[-\rho,-\varepsilon^{\lambda}]\times\{\varepsilon\}$ and $\overleftarrow{H}_\varepsilon=[x_\varepsilon-re^{-\frac{c}{\varepsilon^q}},x_\varepsilon]\times\{\varepsilon\}$, for $\varepsilon>0$ small enough.

3.2.4 Construction of the map R^u

In order to define the map R^u, we first prove the following result.

Proposition 6. *Consider the Filippov system $Z = (X^+, X^-)_\Sigma$ given by (2.1), for some $k \geqslant 1$, and $y^\varepsilon_{\rho,\lambda}$ and y^ε_θ given in (3.1). For $n \geqslant \max\{2, 2k-1\}$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5) and consider the regularized system Z^Φ_ε (2.4). Then, there exists $\theta_0 > 0$ such that, for each $\theta \in [x_\varepsilon, \theta_0]$, the Fenichel manifold $S_{a,\varepsilon}$ intersects $\{x = \theta\}$ at $(\theta, y^\varepsilon_\theta)$.*

Proof. By Proposition 1 we know that the Fenichel manifold $S_{a,\varepsilon}$ intersects $\{y = \varepsilon\}$ at $(x_\varepsilon, \varepsilon)$. In order to continue $S_{a,\varepsilon}$ into $x = \theta$, consider the solutions $(x(t), y(t))$ of the differential system (2.3) with initial condition $x(0) = x_\varepsilon$ and $y(0) = \varepsilon$. Thus, $x(t) = t + x_\varepsilon$ and

$$y(t) = \varepsilon + \int_0^t \alpha(s + x_\varepsilon)^{2k-1} + g(s + x_\varepsilon) + y(s, \varepsilon)\vartheta(s + x_\varepsilon, y(s, \varepsilon))ds.$$

Therefore, the trajectory $(x(t), y(t))$ intersects $\{x = \theta\}$ at $(\theta, y^\varepsilon_\theta)$, with

$$y^\varepsilon_\theta := y(\theta - x_\varepsilon) = \frac{\alpha\theta^{2k}}{2k} - \frac{\alpha x_\varepsilon^{2k}}{2k} + \varepsilon + G_\varepsilon(x_\varepsilon, \theta),$$

where

$$G_\varepsilon(x, \theta) = \int_x^\theta [g(s) + y(s - x, \varepsilon)\vartheta(s, y(s - x, \varepsilon))]ds.$$

In what follows, we shall develop $G_\varepsilon(x_\varepsilon, \theta)$ in Taylor series around $(x, \theta, \varepsilon) = (0, 0, 0)$. First, notice that

$$G_\varepsilon(x, \theta) = G_\varepsilon(0, \theta) + \sum_{i=1}^{2k-1} \frac{\partial^i G_\varepsilon}{\partial x^i}(0, \theta)x^i + \mathcal{O}(x^{2k}), \tag{3.24}$$

and

$$G_\varepsilon(0, \theta) = G_0(0, \theta) + \varepsilon\frac{\partial}{\partial\varepsilon}G_\varepsilon(0, \theta)\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2). \tag{3.25}$$

Thus, substituting (3.25) into (3.24) and taking $x = x_\varepsilon$, we have

$$G_\varepsilon(x_\varepsilon, \theta) = G_0(0, \theta) + \varepsilon\frac{\partial}{\partial\varepsilon}G_\varepsilon(0, \theta)\Big|_{\varepsilon=0} + \sum_{i=1}^{2k-1} \frac{\partial^i G_0}{\partial x^i}(0, \theta)x_\varepsilon^i + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon x_\varepsilon) + \mathcal{O}(x_\varepsilon^{2k}). \tag{3.26}$$

Now, in order to estimate $G_0(0, \theta)$ and $\frac{\partial}{\partial\varepsilon}G_\varepsilon(0, \theta)\Big|_{\varepsilon=0}$ in (3.26), we compute

$$G_\varepsilon(0, \theta) = G_\varepsilon(0, 0) + \theta\frac{\partial G_\varepsilon}{\partial\theta}(0, 0) + \mathcal{O}(\theta^2) = \theta\frac{\partial G_\varepsilon}{\partial\theta}(0, 0) + \mathcal{O}(\theta^2). \tag{3.27}$$

We know that

$$G_0(0, \theta) = \int_0^\theta [g(s) + y_0(s)\vartheta(s, y_0(s))]ds,$$

where y_0 satisfies the following Cauchy problem

$$\begin{cases} y_0' &= \alpha t^{2k-1} + g(t) + y_0\vartheta(t, y_0), \\ y_0(0) &= 0. \end{cases}$$

Notice that $y_0^{(i)}(0) = 0$ for $i = 0, 1, \dots, 2k-1$ and $y_0^{(2k)}(0) = (2k-1)!\alpha$. Thus,

$$y_0(t) = \frac{\alpha}{2k} t^{2k} + \mathcal{O}(t^{2k+1}) \tag{3.28}$$

and

$$\begin{aligned} \frac{\partial G_0}{\partial \theta}(0,\theta) &= g(\theta) + y_0(\theta)\vartheta(s, y_0(\theta)) \\ &= g(\theta) + \frac{\alpha\theta^{2k}}{2k}\vartheta(s, y_0(\theta)) + \mathcal{O}(\theta^{2k+1}) \\ &= \mathcal{O}(\theta^{2k}). \end{aligned}$$

Hence, we conclude that

$$G_0(0,\theta) = \mathcal{O}(\theta^{2k+1}). \tag{3.29}$$

Analogously,

$$G_\varepsilon(0,\theta) = \int_0^\theta [g(s) + y(s,\varepsilon)\vartheta(s, y(s,\varepsilon))]ds$$

and, then, $\dfrac{\partial G_\varepsilon}{\partial \theta}(0,0) = \varepsilon\vartheta(0,\varepsilon)$. Therefore, by (3.27), $G_\varepsilon(0,\theta) = \theta\varepsilon\vartheta(0,\varepsilon) + \mathcal{O}(\theta^2)$. Hence,

$$\left.\frac{\partial G_\varepsilon}{\partial \varepsilon}(0,\theta)\right|_{\varepsilon=0} = \mathcal{O}(\theta). \tag{3.30}$$

Finally, in order to estimate the remainder terms in (3.26), we compute

$$G_0(x,\theta) = G_0(x,0) + \theta\frac{\partial G_0}{\partial \theta}(x,0) + \dots + \theta^{2k-1}\frac{\partial^{2k-1} G_0}{\partial \theta^{2k-1}}(x,0) + \mathcal{O}(\theta^{2k}). \tag{3.31}$$

Using the definition of $G_0(x,\theta)$ and (3.28), we get that

$$\frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial \theta^j}G_0(0,0) = 0, \tag{3.32}$$

for all $j \in \{0, \dots, 2k-1\}$ and $i \in \{1, \dots, 2k-j\}$. So, by (3.31) and (3.32), we obtain that

$$\frac{\partial^i G_0}{\partial x^i}(0,\theta) = \mathcal{O}(\theta^{2k+1-i}), \tag{3.33}$$

for all $i \in \{1, \dots, 2k-1\}$.

Substituting (3.29), (3.30), and (3.33) into (3.26), we get

$$\begin{aligned} G_\varepsilon(x_\varepsilon,\theta) = {} & G_0(0,\theta) + \mathcal{O}(\varepsilon\theta) + \mathcal{O}(\varepsilon^2) + \sum_{i=1}^{2k-1}\mathcal{O}(\theta^{2k+1-i}x_\varepsilon^i) + \mathcal{O}(\varepsilon x_\varepsilon) + \mathcal{O}(x_\varepsilon^{2k}) \\ = {} & \mathcal{O}(\theta^{2k+1}) + \mathcal{O}(\varepsilon\theta) + \sum_{i=1}^{2k-1}\mathcal{O}(\theta^{2k+1-i}x_\varepsilon^i) + \mathcal{O}(x_\varepsilon^{2k}). \end{aligned}$$

Consequently,

$$y_\theta^\varepsilon = \frac{\alpha\theta^{2k}}{2k} - \frac{\alpha x_\varepsilon^{2k}}{2k} + \varepsilon + \mathcal{O}(\theta^{2k+1}) + \mathcal{O}(\varepsilon\theta) + \sum_{i=1}^{2k-1}\mathcal{O}(\theta^{2k+1-i}x_\varepsilon^i) + \mathcal{O}(x_\varepsilon^{2k}),$$

Therefore, by Lemma 1 we can conclude that $y_\theta^0 = y(\theta) = \overline{y}_\theta$, i.e.

$$y_\theta^\varepsilon = \overline{y}_\theta + \varepsilon + \mathcal{O}(\varepsilon\theta) + \sum_{i=1}^{2k-1}\mathcal{O}(\theta^{2k+1-i}x_\varepsilon^i) + \mathcal{O}(x_\varepsilon^{2k}).$$

In particular, for $\theta = x_\varepsilon$, we obtain that

$$\begin{aligned} y^\varepsilon_{x_\varepsilon} &= \overline{y}_{x_\varepsilon} + \varepsilon + \mathcal{O}(\varepsilon x_\varepsilon) + \mathcal{O}(x_\varepsilon^{2k+1}) + \mathcal{O}(x_\varepsilon^{2k}) \\ &= \overline{y}_{x_\varepsilon} + \varepsilon + \mathcal{O}(\varepsilon x_\varepsilon) + \mathcal{O}(x_\varepsilon^{2k}). \end{aligned}$$

□

Finally, from Proposition 6 and arguing analogously to the construction of map P^u (see Section 3.2.2), we get defined the map

$$\begin{aligned} R^u : \overleftarrow{H}_\varepsilon &\longrightarrow \widetilde{V}^\varepsilon_\theta \\ (x,\varepsilon) &\longmapsto \Big(\theta, y^\varepsilon_\theta + \mathcal{O}(e^{-c/\varepsilon^q})\Big), \end{aligned}$$

where $\overleftarrow{H}_\varepsilon = [x_\varepsilon - re^{-\frac{c}{\varepsilon^q}}, x_\varepsilon] \times \{\varepsilon\}$ and $\widetilde{V}^\varepsilon_\theta = \{\theta\} \times [y^\varepsilon_\theta, y^\varepsilon_\theta + re^{-\frac{c}{\varepsilon^q}}]$, for all $\theta \in [x_\varepsilon, \theta_0]$ and $\varepsilon > 0$ small enough.

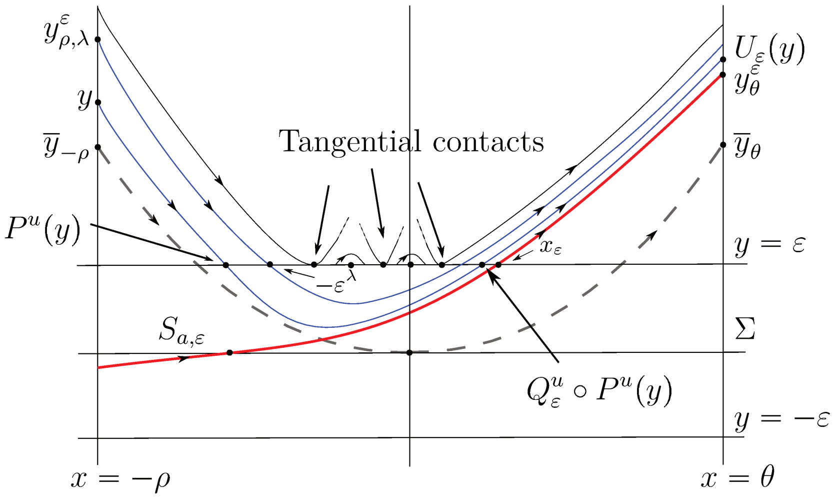

Figure 13 – The map $U_\varepsilon = R^u \circ Q^u_\varepsilon \circ P^u$ for the regularized system Z^Φ_ε. The dotted curve is the trajectory of X^+ passing through the visible $2k$-multiplicity contact with Σ with $(0,0)$. One can see the exponential attraction of the Fenichel manifold $S_{a,\varepsilon}$.

3.2.5 Proof of Theorem A

Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ satisfying hypothesis **(A)** for some $k \geqslant 1$. For $n \geqslant 2k-1$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5) and consider the regularized system Z^Φ_ε (1.4). As noticed in Remark 3, we shall assume that $n \geqslant \max\{2, 2k-1\}$.

From the comments of Section 2.1, we can assume that $Z\big|_U$ can be written as (2.1), which has its regularization given by (2.4). Thus, statement (a) of Theorem A follows from Proposition 2. Finally, statement (b) follows by taking the composition

$$\begin{aligned} U_\varepsilon : \quad \widehat{V}^\varepsilon_{\rho,\lambda} &\longrightarrow \widetilde{V}^\varepsilon_\theta \\ (-\rho, y) &\longmapsto R^u \circ Q^u_\varepsilon \circ P^u(-\rho, y), \end{aligned}$$

where P^u, Q^u_ε, and R^u are defined in Sections 3.2.2, 3.2.3, and 3.2.4, respectively. Indeed, the existence of ρ_0 and $\theta_0 > 0$ are guaranteed by the construction of the map P^u (see Section 3.2.2) and Proposition 6, respectively. The existence of constants $c, r, q > 0$, for which $U_\varepsilon(-\rho, y) = y^\varepsilon_\theta + \mathcal{O}(e^{-\frac{c}{\varepsilon^q}})$ is guaranteed by the construction of the map Q^u_ε (see Section 3.2.3). Furthermore, by construction of the maps P^u, Q^u_ε and R^u, we have that the trajectories of Z^Φ_ε starting at the section $\widehat{V}^\varepsilon_{\rho,\lambda}$ intersect the line $y = \varepsilon$ only in two points before reaching the section $\widetilde{V}^\varepsilon_{\theta,\lambda}$. Moreover, these intersections take place at $\widehat{H}^\varepsilon_{\rho,\lambda} \cup \overleftarrow{H}_\varepsilon$.

3.3 Lower Transition Map

This section is devoted to prove Theorem B. Analogously to the previous section, we need to guarantee that under some conditions the flow of the regularized system Z^Φ_ε near a visible regular-tangential singularity defines a map between two sections, in this case, a horizontal section and a vertical section. Again, it will be convenient to write this map as the composition of three maps, namely P^l, Q^l_ε and R^l. The maps P^l and Q^l_ε will be defined through the flow of Z^Φ_ε restricted to the band of regularization, and the map R^l will be given by the flow of Z^Φ_ε defined outside the band of regularization. In what follows, we shall define the maps P^l, Q^l_ε and R^l (see Figure 14).

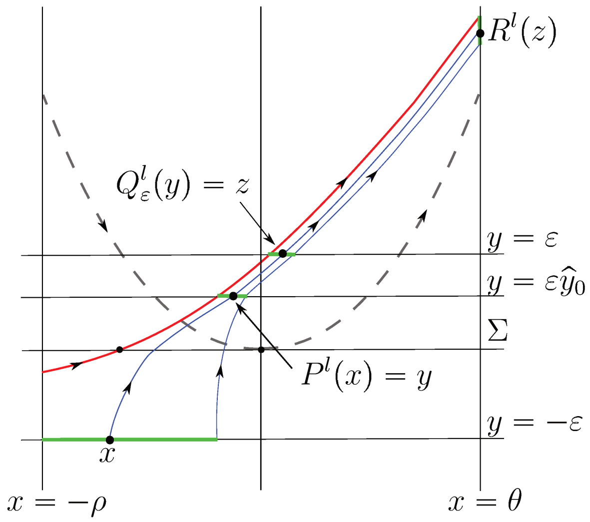

Figure 14 – Dynamics of the maps P^l, Q^l_ε and R^l. The dotted curve is the trajectory of X^+ passing through the visible $2k$-multiplicity contact with Σ with $(0, 0)$.

First of all, the next result is obtained following the same argument than Proposition 2.

Proposition 7. *Consider the Filippov system $Z = (X^+, X^-)_\Sigma$ given by (2.1), for some $k \geqslant 1$, and y^ε_θ given in (3.1). For $n \geqslant \max\{2, 2k-1\}$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5) and consider the regularized system Z^Φ_ε (2.4). Then, there exist $\rho_0, \theta_0 > 0$, such that the vertical segment*

$$\check{V}^\varepsilon_\theta = \{\theta\} \times [y^\varepsilon_\theta - re^{-\frac{c}{\varepsilon^q}}, y^\varepsilon_\theta],$$

and the horizontal segments

$$\breve{H}^{\varepsilon}_{\rho,\lambda} = [-\rho, -\varepsilon^{\lambda}] \times \{-\varepsilon\} \quad and \quad \overrightarrow{H}_{\varepsilon} = [x_{\varepsilon}, x_{\varepsilon} + re^{-\frac{c}{\varepsilon^{q}}}] \times \{\varepsilon\}$$

are transversal sections for every $\rho \in (\varepsilon^{\lambda}, \rho_0]$, $\theta \in [x_{\varepsilon}, \theta_0]$, $\lambda \in (0, \lambda^*)$, *with* $\lambda^* = \dfrac{n}{2k(n-1)+1}$, *constants* $c, r, q > 0$, *and* $\varepsilon > 0$ *sufficiently small.*

As before, statement (i) of Theorem B will follows from Proposition 7.

3.3.1 Construction of the map P^l

First, we shall see that the forward trajectory of $\overline{Z}^{\Phi}_{\varepsilon}$ (2.6) starting at $(-\varepsilon^{\lambda}, -1)$ reaches the straight line $\{\hat{y} = \hat{y}_0\}$, with $\hat{y}_0 \in (1-\eta, 1)$, for some $\eta > 0$ small enough. After that, the map will be obtained through Poincaré-Bendixson argument.

Accordingly, consider a function $\widetilde{\mu} : I_{(x,\widehat{y})} \times \widehat{U} \times [0, \varepsilon_0] \to \mathbb{R}$ given by

$$\widetilde{\mu}(\tau, x, -1, \varepsilon) = \varphi^2_{\overline{Z}^{\Phi}_{\varepsilon}}(\tau, x, -1) - \widehat{y}_0,$$

where $\varphi_{\overline{Z}^{\Phi}_{\varepsilon}} = (\varphi^1_{\overline{Z}^{\Phi}_{\varepsilon}}, \varphi^2_{\overline{Z}^{\Phi}_{\varepsilon}})$ denotes the flow of $\overline{Z}^{\Phi}_{\varepsilon}$, $I_{(x,\widehat{y})}$ is the maximal interval of definition of $\tau \mapsto \varphi_{\overline{Z}^{\Phi}_{\varepsilon}}(\tau, x, \hat{y})$, $\varepsilon_0 > 0$ is sufficiently small, and $\widehat{U}$ is the domain of the vector field Z in the $(x, \widehat{y})$-coordinates.

Now, for each $\widehat{y} \in [-1, \widehat{y}_0]$ and $\varepsilon = 0$, we have

$$\varphi_{\overline{Z}^{\Phi}_{0}}(0, 0, \widehat{y}) = (0, \widehat{y}) \text{ and } \frac{\partial \varphi^2_{\overline{Z}^{\Phi}_{0}}}{\partial \tau}(0, 0, \widehat{y}) = \frac{1 - \Phi(\widehat{y})}{2} > 0.$$

Then, there exists $\tau_0 > 0$ such that $\varphi_{\overline{Z}^{\Phi}_{0}}(\tau_0, 0, -1) = (0, \widehat{y}_0)$. In this way,

$$\widetilde{\mu}(\tau_0, 0, -1, 0) = 0 \text{ and } \frac{\partial \widetilde{\mu}}{\partial \tau}(\tau_0, 0, -1, 0) = \frac{1 - \Phi(\widehat{y}_0)}{2} \neq 0.$$

Thus, from *Implicit Function Theorem* there exists a unique smooth function $\tau(x, \varepsilon)$, such that, $\varphi^2_{\overline{Z}^{\Phi}_{\varepsilon}}(\tau(x, \varepsilon), x, -1) = \widehat{y}_0$ and $\tau(0, 0) = \tau_0$. Therefore, for $\varepsilon > 0$ sufficiently small, the forward trajectory of $\overline{Z}^{\Phi}_{\varepsilon}$ starting at $(-\varepsilon^{\lambda}, -1)$ reaches the straight line $\{\hat{y} = \hat{y}_0\}$ at

$$\Big(\varphi^1_{\overline{Z}^{\Phi}_{\varepsilon}}(\tau(-\varepsilon^{\lambda}, -1), -\varepsilon^{\lambda}, -1), \hat{y}_0\Big).$$

In what follows we shall compute the Taylor expansion of $\varphi^1_{\overline{Z}^{\Phi}_{\varepsilon}}(\tau(x, \varepsilon), x, -1)$ around $(x, \varepsilon) = (0, 0)$. Notice that

$$\begin{aligned}
\varphi^1_{\overline{Z}^{\Phi}_{\varepsilon}}(\tau(x, \varepsilon), x, -1) = {} & \varphi^1_{\overline{Z}^{\Phi}_{0}}(\tau(x, 0), x, -1) + \mathcal{O}(\varepsilon) \\
= {} & \varphi^1_{\overline{Z}^{\Phi}_{0}}(\tau(0, 0), 0, -1) + x \frac{\partial}{\partial x}\Big(\varphi^1_{\overline{Z}^{\Phi}_{0}}(\tau(x, 0), x, -1)\Big)\Big|_{x=0} \\
& + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon) \\
= {} & \varphi^1_{\overline{Z}^{\Phi}_{0}}(\tau_0, 0, -1) + x \left[\frac{\partial \varphi^1_{\overline{Z}^{\Phi}_{0}}}{\partial \tau}(\tau(x, 0), x, -1) \frac{\partial \tau}{\partial x}(x, 0) \right. \\
& \left. + \frac{\partial \varphi^1_{\overline{Z}^{\Phi}_{0}}}{\partial x}(\tau(x, 0), x, -1) \right]\Big|_{x=0} + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon).
\end{aligned} \tag{3.34}$$

Substituting

$$\varphi^1_{\overline{Z}^\Phi_\varepsilon}(\tau_0, 0, -1) = 0 \quad \text{and} \quad \frac{\partial \varphi^1_{\overline{Z}^\Phi_0}}{\partial \tau}(\tau_0, 0, -1) = 0$$

into (3.34), we have

$$\begin{aligned} \varphi^1_{\overline{Z}^\Phi_\varepsilon}(\tau(x,\varepsilon), x, -1) = {} & x\left[\frac{\partial \varphi^1_{\overline{Z}^\Phi_0}}{\partial \tau}(\tau_0,0,-1)\frac{\partial \tau}{\partial x}(0,0) + \frac{\partial \varphi^1_{\overline{Z}^\Phi_0}}{\partial x}(\tau_0,0,-1)\right] \\ & + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon) \\ = {} & x\frac{\partial \varphi^1_{\overline{Z}^\Phi_0}}{\partial x}(\tau_0,0,-1) + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon). \end{aligned} \tag{3.35}$$

Now, notice that $\dfrac{\partial \varphi_{\overline{Z}^\Phi_0}}{\partial x}(\tau, 0, -1)$ is solution of the differential equation

$$u' = D\overline{Z}^\Phi_0(0, \varphi^2_{\overline{Z}^\Phi_0}(\tau, 0, -1))u,$$

with

$$D\overline{Z}^\Phi_0(0, \varphi^2_{\overline{Z}^\Phi_0}(\tau, 0, -1)) = \begin{bmatrix} 0 & 0 \\ * & -\dfrac{\Phi'\left(\varphi^2_{\overline{Z}^\Phi_0}(\tau,0,-1)\right)}{2} \end{bmatrix}.$$

Consequently,

$$\begin{aligned} \begin{bmatrix} u_1'(\tau) \\ u_2'(\tau) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ * & -\dfrac{\Phi'\left(\varphi^2_{\overline{Z}^\Phi_0}(\tau,0,-1)\right)}{2} \end{bmatrix} \begin{bmatrix} u_1(\tau) \\ u_2(\tau) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ ** \end{bmatrix}, \end{aligned}$$

which implies that $u_1(\tau)$ is constant. Since

$$\frac{\partial \varphi^1_{\overline{Z}^\Phi_0}}{\partial x}(\tau_0, 0, -1) = \frac{\partial \varphi^1_{\overline{Z}^\Phi_0}}{\partial x}(0, 0, -1) = 1,$$

we conclude, by (3.35), that

$$\varphi^1_{\overline{Z}^\Phi_\varepsilon}(\tau(x,\varepsilon), x, -1) = x + \mathcal{O}(x^2) + \mathcal{O}(\varepsilon).$$

Taking $x = -\varepsilon^\lambda$, we get

$$\varphi^1_{\overline{Z}^\Phi_\varepsilon}(\tau(-\varepsilon^\lambda, \varepsilon), -\varepsilon^\lambda, -1) = -\varepsilon^\lambda + \mathcal{O}(\varepsilon^{2\lambda}) + \mathcal{O}(\varepsilon) =: x^\varepsilon_\lambda.$$

Finally, consider the region $\mathcal{K}$ delimited by the curves $y = -\varepsilon$, $y = \varepsilon\widehat{y}_0$, $y = m(x,\varepsilon)$, $y = -\dfrac{x}{\varepsilon} - (\dfrac{\rho}{\varepsilon} + \varepsilon)$ and the arc-orbit connecting $(-\varepsilon^\lambda, -\varepsilon)$ and $(x^\varepsilon_\lambda, \varepsilon\,\hat{y}_0)$. Since $\overline{Z}^\Phi_\varepsilon$ has no singularities inside $\mathcal{K}$, one can easily see that the forward trajectory of $\overline{Z}^\Phi_\varepsilon$ starting at any point of the transversal section $\breve{H}^\varepsilon_{\rho,\lambda}$ must leave $\mathcal{K}$ through the transversal section $\{(x,y) \in U : y = \varepsilon\widehat{y}_0\}$. This naturally defines a map

$$P^l : \breve{H}^\varepsilon_{\rho,\lambda} \longrightarrow \{(x,y) \in U : y = \varepsilon\widehat{y}_0\}.$$

3.3.2 Exponentially attraction and construction of the map Q^l_ε

As we saw in Section 3.2.3, for $L, N > 0$ and $\varepsilon_0 > 0$ small enough, the Fenichel manifold $S_{a,\varepsilon}$ is described as

$$m(x,\varepsilon) = m_0(x) + \varepsilon m_1(x) + \mathcal{O}(\varepsilon^2),$$

for $-L \leqslant x \leqslant -N$ and $0 \leqslant \epsilon \leqslant \varepsilon_0$, where m_0 and m_1 were defined in (3.5) and (3.7).

Now, we shall compute the intersection of $m(x,\varepsilon)$ with the straight line $\{\hat{y} = \hat{y}_0\}$ with $1-\eta < \widehat{y}_0 < 1$, for some $\eta > 0$ small enough. Indeed, since $m_0(0) = 1$ and

$$\lim_{x\to-\infty} m_0(x) = -1,$$

then there exists a negative number $\widehat{x}_0$ such that $m_0(\widehat{x}_0) = \widehat{y}_0$. Moreover, $\widehat{x}_0$ is close to zero because $\widehat{y}_0$ is near to 1 and $m_0(0) = 1$. After that, consider the function

$$\widehat{\mu}(x,\varepsilon) = m(x,\varepsilon) - \widehat{y}_0,$$

and notice that $\widehat{\mu}(\widehat{x}_0, 0) = m_0(\widehat{x}_0) - \widehat{y}_0 = 0$ and $\dfrac{\partial\widehat{\mu}}{\partial x}(\widehat{x}_0,0) = \dfrac{\partial m}{\partial x}(\widehat{x}_0,0) = m_0'(\widehat{x}_0) \neq 0$, where we have used equation (3.11). Thus, there exists a smooth function $\widehat{x}(\varepsilon)$, such that $\widehat{x}(0) = \widehat{x}_0$ and $m(\widehat{x}(\varepsilon),\varepsilon) = \widehat{y}_0$. Accordingly, from (3.7), we have

$$\widehat{x}'(0) = -\frac{\dfrac{\partial m}{\partial \varepsilon}(\widehat{x}_0,0)}{\dfrac{\partial m}{\partial x}(\widehat{x}_0,0)} = -\frac{m_1(\widehat{x}_0)}{m_0'(\widehat{x}_0)} = \frac{m_0'(\widehat{x}_0) - m_0(\widehat{x}_0)\vartheta(\widehat{x}_0,0)}{\alpha(2k-1)\widehat{x}_0^{2k-2} + g'(\widehat{x}_0)}.$$

The last expression is positive, because $m_0'(x) \to \infty$ when $x \to 0$ and $m_0(x)\vartheta(x,0)$ is bounded in the interval $[-L, 0]$, with L sufficiently small. Therefore, the Taylor expansion of $\widehat{x}(\varepsilon)$ around $\varepsilon = 0$ writes

$$\widehat{x}(\varepsilon) = \widehat{x}_0 + \varepsilon\widehat{x}'(0) + \mathcal{O}(\varepsilon^2)$$

and, consequently, $\widehat{x}_0 < \widehat{x}(\varepsilon) < 0$ for ε sufficiently small.

Proposition 8. *Fix* $0 < \lambda < \lambda^* = \dfrac{n}{2k(n-1)+1}$. *Let* $x_0 \in [\widehat{x}(\varepsilon), -\kappa\varepsilon^{\lambda}]$, *with* $0 < \kappa < 1$, *and consider the solution* $\widehat{y}(x,\varepsilon)$ *of system* (3.18) *satisfying* $\widehat{y}(x_0,\varepsilon) = \widehat{y}_0$. *Then, there exist positive numbers* C *and* $\widetilde{r}$ *such that*

$$|m(x,\varepsilon) - \widehat{y}(x,\varepsilon)| \leqslant \widetilde{r}e^{-\frac{C}{\varepsilon}\left(|x_0|^{\frac{1}{\lambda^*}} - |x|^{\frac{1}{\lambda^*}}\right)},$$

for $x_0 \leqslant x \leqslant -\varepsilon^{\lambda^*}$.

Proof. Performing the change of variables $\omega = m(x,\varepsilon) - \widehat{y}$ in equation (3.18), we have

$$\varepsilon\frac{d\omega}{dx} = \xi(x,\varepsilon)\phi'(m(x,\varepsilon))\omega + \xi(x,\varepsilon)F(x,\omega,\varepsilon), \tag{3.36}$$

where

$$F(x,\omega,\varepsilon) = \phi(m(x,\varepsilon)-\omega) - \phi(m(x,\varepsilon)) - \phi'(m(x,\varepsilon))\omega$$

and

$$\xi(x,\varepsilon) = \frac{2}{\Big(1+\phi(m(x,\varepsilon))\Big)\Big(1+\phi(m(x,\varepsilon)-\omega(x,\varepsilon))\Big)} + \frac{\varepsilon\Big(m(x,\varepsilon)\vartheta(x,\varepsilon m(x,\varepsilon)) - (m(x,\varepsilon)-\omega(x,\varepsilon))\vartheta(x,\varepsilon(\omega(x,\varepsilon)-m(x,\varepsilon)))\Big)}{\phi(\omega(x,\varepsilon)-m(x,\varepsilon)) - \phi(m(x,\varepsilon))}.$$

Here, we are denoting $\omega(x,\varepsilon) = m(x,\varepsilon) - \widehat{y}(x,\varepsilon)$ which is the solution of (3.36) with initial condition $\omega(x_0,\varepsilon) = m(x_0,\varepsilon) - \hat{y}_0$.

Notice that F writes

$$F(x,\omega,\varepsilon) = A(x,\varepsilon)\omega, \tag{3.37}$$

where

$$A(x,\varepsilon) = -\int_0^1 \phi'(m(x,\varepsilon)+(s-1)\omega(x,\varepsilon)) + \phi'(m(x,\varepsilon))ds.$$

Here, as in the proof of Proposition 5, we also claim that $A(x,\varepsilon)$ is negative for $-L \leqslant x \leqslant 0$ and ε sufficiently small. Indeed, we know that $\phi' > 0$ on the interval $(-1,1)$. In addition, since for $\varepsilon > 0$ small enough we have $m(x,\varepsilon) > \widehat{y}(x,\varepsilon)$ and $\dfrac{d\widehat{y}}{dx}(x) > 0$ for $x \geqslant x_0$, then the solution $\omega(x,\varepsilon)$ satisfies

$$0 \leqslant \omega(x,\varepsilon) \leqslant m(x,\varepsilon) - \widehat{y}_0.$$

Hence, from Proposition 4 and (3.8) we get

$$m(x,\varepsilon)+(s-1)\omega(x,\varepsilon) \leqslant m(x,\varepsilon) \leqslant m_0(x,\varepsilon) \leqslant 1 - C_1\sqrt[n]{|x|^{2k-1}} \leqslant 1 - C_1\sqrt[n]{\epsilon^{\lambda^*(2k-1)}} < 1, \tag{3.38}$$

and

$$m(x,\varepsilon)+(s-1)\omega(x,\varepsilon) \geqslant m(x,\epsilon)s - (s-1)\widehat{y}_0 \geqslant \widehat{y}_0 > 1-\eta, \tag{3.39}$$

for $0 \leqslant s \leqslant 1$ and $\eta,\varepsilon > 0$ small enough. Therefore, we conclude that $A(x,\varepsilon)$ is negative.

In this way, by (3.36) and (3.37), we obtain

$$\begin{aligned}
\epsilon\frac{d\omega}{dx} &= \xi(x,\epsilon)(\phi'(m(x,\epsilon))\omega + F(x,\omega,\epsilon)) \\
&= \xi(x,\epsilon)(\phi'(m(x,\epsilon)) + A(x,\epsilon))\omega \\
&= -\xi(x,\epsilon)\Big(\int_0^1 \phi'(m(x,\varepsilon)+(s-1)\omega(x,\varepsilon))ds\Big)\omega,
\end{aligned}$$

which has its solution with initial condition $\omega(x_0)$ given by

$$\omega(x,\varepsilon) = \omega(x_0)e^{-\frac{1}{\varepsilon}\int_{x_0}^x \xi(\nu,\varepsilon)(\int_0^1 \phi'(m(\nu,\varepsilon)+(s-1)\omega(\nu,\varepsilon))ds)d\nu}.$$

Thus,

$$|\omega(x,\varepsilon)| = |\omega(x_0)|e^{-\frac{1}{\varepsilon}\int_{x_0}^x \xi(\nu,\varepsilon)(\int_0^1 \phi'(m(\nu,\varepsilon)+(s-1)\omega(\nu,\varepsilon))ds)d\nu}.$$

To conclude this proof, we shall estimate $|\omega(x,\varepsilon)|$. For this, notice that

$$\xi(x,\varepsilon) = \frac{2}{\Big(1+\phi(m_0(x))\Big)\Big(1+\phi(m_0(x)-\omega(x,0))\Big)} + \mathcal{O}(\varepsilon).$$

Hence, $L, \varepsilon > 0$ can be taken sufficiently small in order that $\xi(x,\varepsilon) \geqslant l > 0$, for all $-L \leqslant x \leqslant 0$. Moreover, given $0 < \eta < 1$, there exist positive constants c_1, c_2 such that for $|\widehat{y} - 1| < \eta$ one has

$$c_1(1-\widehat{y})^{n-1} \leqslant \phi'(\widehat{y}) \leqslant c_2(1-\widehat{y})^{n-1}.$$

Finally, using (3.38) and (3.39), we obtain that $|m(\nu,\varepsilon) + (s-1)\omega(\nu,\varepsilon) - 1| < \eta$. Therefore, for $x \leqslant -e^{\lambda^*}$, we get that

$$\begin{aligned}
|\omega(x,\varepsilon)| &\leqslant |\omega(x_0)|e^{-\frac{c_1}{\varepsilon}\int_{x_0}^{x}\xi(\nu,\epsilon)(\int_0^1(1-m(\nu,\varepsilon)-(s-1)\omega(\nu,\varepsilon))^{n-1}ds)d\nu} \\
&\leqslant |\omega(x_0)|e^{-\frac{lc_1}{\varepsilon}\int_{x_0}^{x}(\int_0^1(1-m(\nu,\varepsilon))^{n-1}ds)d\nu} \\
&\leqslant |\omega(x_0)|e^{-\frac{lc_1}{\varepsilon}\int_{x_0}^{x}(1-m(\nu,\varepsilon))^{n-1}d\nu} \\
&\leqslant |\omega(x_0)|e^{-\frac{lc_1}{\varepsilon}\int_{x_0}^{x}(1-m_0(\nu))^{n-1}d\nu} \\
&\leqslant |\omega(x_0)|e^{-\frac{lc_1}{\varepsilon}\int_{x_0}^{x}(C_1|\nu|^{\frac{2k-1}{n}})^{n-1}d\nu} \\
&\leqslant |\omega(x_0)|e^{-\frac{C}{\varepsilon}(|x_0|^{\frac{1}{\lambda^*}}-|x|^{\frac{1}{\lambda^*}})},
\end{aligned}$$

where $C = \dfrac{nlc_1C_1^{n-1}}{2k(n-1)+1}$ is a positive constant. The inequality (3.8) has also been used. □

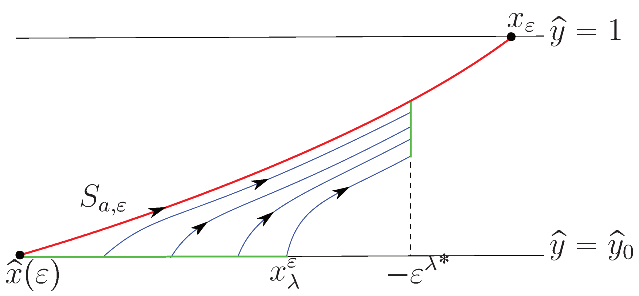

Figure 15 – The exponential attraction of $S_{a,\varepsilon}$.

Fix $0 < \lambda < \lambda^*$. From Proposition 8, applied to $x_0 = x_\lambda^\varepsilon$ and $x = -\varepsilon^{\lambda^*}$, where $x_0 \leqslant -\kappa\varepsilon^\lambda$ for some $\kappa \in (0,1)$, we know that there exist positive numbers $\widetilde{r}$ and C such that

$$\begin{aligned}
|m(-\varepsilon^{\lambda^*},\varepsilon) - \widehat{y}(-\varepsilon^{\lambda^*},\varepsilon)| &\leqslant \widetilde{r}e^{-\frac{C}{\varepsilon}\left(|x_\varepsilon^\lambda|^{\frac{1}{\lambda^*}}-|-\varepsilon^{\lambda^*}|^{\frac{1}{\lambda^*}}\right)} \\
&\leqslant re^{-\frac{c}{\varepsilon^q}},
\end{aligned}$$

where $c = C\kappa^{\frac{1}{\lambda^*}}$, $r = \widetilde{r}e^C$ and $q = 1 - \dfrac{\lambda}{\lambda^*}$ are positive constants. Hence,

$$\widehat{y}(-\varepsilon^{\lambda^*},\varepsilon) = m(-\varepsilon^{\lambda^*},\varepsilon) + \mathcal{O}(e^{-c/\varepsilon^q}).$$

Thus, arguing analogously to the construction of map P^l (see Section 3.3.1), any solution of the system (2.6) with initial condition in the interval $[\widehat{x}(\varepsilon), x_\lambda^\varepsilon]$, ε sufficiently small, reaches the section $x = -\varepsilon^{\lambda^*}$ exponentially close to the Fenichel manifold (see Figure 15).

From Proposition 1, these solutions can be continued until the section $\hat{y} = 1$. Going back through the rescaling $y = \varepsilon\hat{y}$, we get defined the following map through the flow of (2.4),

$$\begin{array}{rcl} Q^l_\varepsilon : [\widehat{x}(\varepsilon), x^\varepsilon_\lambda] \times \{y = \varepsilon\widehat{y}_0\} & \longrightarrow & \overrightarrow{H}_\varepsilon \\ (x, \varepsilon) & \longmapsto & \Big(x_\varepsilon + \mathcal{O}(e^{-c/\varepsilon^q}), \varepsilon\Big), \end{array}$$

where $\overrightarrow{H}_\varepsilon = [x_\varepsilon, x_\varepsilon + re^{-\frac{c}{\varepsilon^q}}] \times \{\varepsilon\}$, for $\varepsilon > 0$ small enough.

3.3.3 Construction of the map R^l

Finally, from Proposition 6 and arguing analogously to the construction of map P^l (see Section 3.3.1), we get defined the map

$$\begin{array}{rcl} R^l : \overrightarrow{H}_\varepsilon & \longrightarrow & \check{V}^\varepsilon_\theta \\ (x, \varepsilon) & \longmapsto & \Big(\theta, y^\varepsilon_\theta + \mathcal{O}(e^{-c/\varepsilon^q})\Big), \end{array}$$

where $\overrightarrow{H}_\varepsilon = [x_\varepsilon, x_\varepsilon + re^{-\frac{c}{\varepsilon^q}}] \times \{\varepsilon\}$ and $\check{V}^\varepsilon_\theta = \{\theta\} \times [y^\varepsilon_\theta - re^{-\frac{c}{\varepsilon^q}}, y^\varepsilon_\theta]$, for every $\theta \in [x_\varepsilon + re^{-\frac{c}{\varepsilon^q}}, \theta_0]$, and $\varepsilon > 0$ small enough.

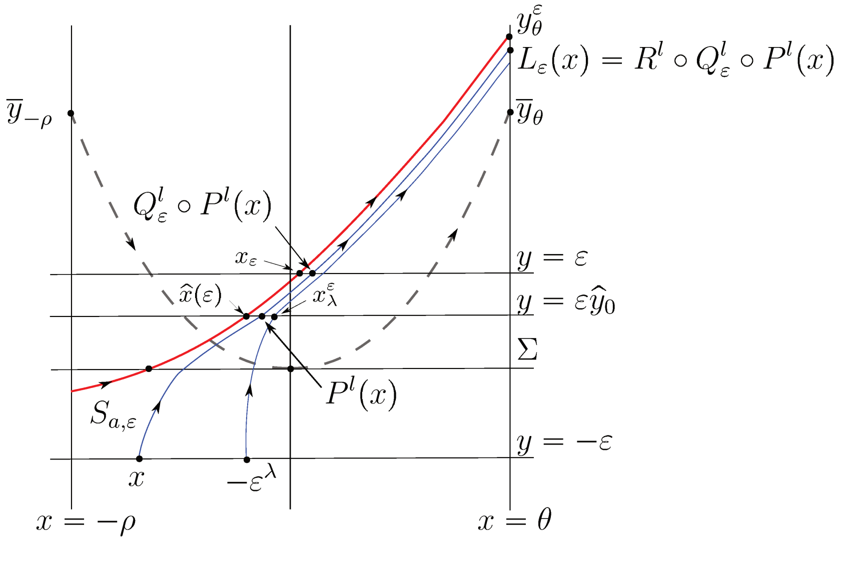

Figure 16 – The map $L_\varepsilon = R^l \circ Q^l_\varepsilon \circ P^l$ for the regularized system Z^Φ_ε. The dotted curve is the trajectory of X^+ passing through the visible $2k$-multiplicity contact with Σ with $(0, 0)$. One can see the exponential attraction of the Fenichel manifold $S_{a,\varepsilon}$.

3.3.4 Proof of Theorem B

Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ satisfying hypothesis **(A)** for some $k \geqslant 1$. For $n \geqslant 2k - 1$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5) and consider the regularized system Z^Φ_ε (1.4). As noticed in Remark 3, we shall assume that $n \geqslant \max\{2, 2k - 1\}$.

From the comments of Section 2.1, we can assume that $Z\big|_U$ can be written as (2.1), which has its regularization given by (2.4). Thus, statement (a) of Theorem B follows from Proposition 7. Finally, statement (b) follows by taking the composition

$$\begin{array}{rccl} L_\varepsilon : & \breve{H}^\varepsilon_{\rho,\lambda} & \longrightarrow & \breve{V}^\varepsilon_\theta \\ & (x,-\varepsilon) & \longmapsto & R^l \circ Q^l_\varepsilon \circ P^l(x,-\varepsilon). \end{array}$$

where P^l, Q^l_ε, and R^l are defined in Sections 3.3.1, 3.3.2, and 3.3.3, respectively. Indeed, the existence of ρ_0 and $\theta_0 > 0$ are guaranteed by the construction of the map P^l (see Section 3.3.1) and Proposition 6, respectively. The existence of constants $c, r, q > 0$, for which $L_\varepsilon(x,-\varepsilon) = y^\varepsilon_\theta + \mathcal{O}(e^{-\frac{c}{\varepsilon^q}})$ is guaranteed by the construction o the map Q^l_ε (see Section 3.3.2).

4 Regularization of boundary limit cycles

In this chapter, we shall state and prove the third main result of this work (Theorem C), which in particular establishes sufficient conditions under which the regularized vector field Z_ε^Φ has a limit cycle Γ_ε converging to a boundary limit cycle. Since a boundary limit cycle is a $\Sigma-$polycycle, we will generalize Theorem C in Chapter 6.

4.1 Main Result

Consider a Filippov system $Z = (X^+, X^-)$ and assume that

(B) X^+ has a hyperbolic limit cycle Γ, which has a $2k$-multiplicity contact with Σ at $(0,0)$ and X^- is pointing towards Σ at $(0,0)$. In other words, $(0,0)$ is a visible regular-tangential singularity of Z (see Figure 17).

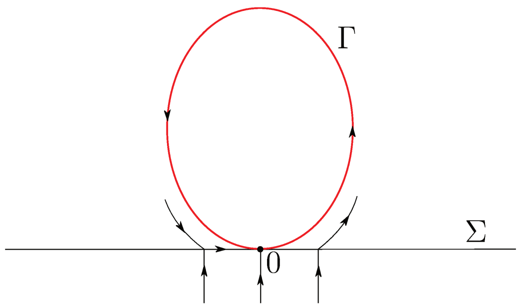

Figure 17 – Boundary limit cycle of Z.

Our third main result establishes conditions under which the regularized vector field Z_ε^Φ has an asymptotically stable limit cycle Γ_ε converging to Γ. This result will be proved in Section 4.3.

Theorem C. *Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ and assume that X^+ satisfies hypothesis* **(B)** *for some $k \geqslant 1$. For $n \geqslant 2k-1$, let $\Phi \in C_{ST}^{n-1}$ be given as (1.5). Then, the following statements hold.*

(a) Given $0 < \lambda < \lambda^* = \dfrac{n}{1+2k(n-1)}$, if the limit cycle Γ is unstable, then there exists $\rho > 0$ such that the regularized system Z_ε^Φ (1.4) does not admit limit cycles passing through the section $\widehat{H}_{\rho,\lambda}^\varepsilon = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, for $\varepsilon > 0$ sufficiently small.

(b) Given $\dfrac{1}{2k} < \lambda < \lambda^* = \dfrac{n}{1+2k(n-1)}$, if the limit cycle Γ is asymptotically stable, then there exists $\rho > 0$ such that the regularized system Z_ε^Φ (1.4) admits a unique limit cycle Γ_ε passing through the section $\widehat{H}_{\rho,\lambda}^\varepsilon = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, for $\varepsilon > 0$ sufficiently small. Moreover, Γ_ε is asymptotically stable and ε-close to Γ.

Remark 4. *Statement (a) and (b) of Theorem C guarantee, respectively, the nonexistence and uniqueness of limit cycles in a specific compact set with nonempty interior. However, since this set degenerates into* Γ *when* ε *goes to* 0*, it is not ensured, in general, the nonexistence and uniqueness of limit cycles converging to* Γ*. Nevertheless, if we assume, in addition, that* X^+ *has locally a unique isocline* $x = \psi(y)$ *of* $2k-$*multiplicity contacts with the straight lines* $y = cte$*, then we get the nonexistence and uniqueness of limit cycles converging to* Γ *(see Chapter 5).*

4.2 The first return map π

From the comments of Section 2.1, we can assume that, for some neighborhood $U \subset \mathbb{R}^2$ of the origin, $Z\big|_U$ is written as (2.1), which has its regularization given by (2.4).

Consider the transversal section $S = \{(x, y) \in U : x = 0\}$. From hypothesis **(B)**, the flow of Z defines a Poincaré map $\pi : S' \longrightarrow S$ around the limit cycle Γ. Here, $S' \subset S$ is an open set (in the topology induced by S) containing $(0, 0)$. Accordingly, $\pi(0) = 0$ and, since Γ is hyperbolic, $\pi'(0) = K \neq 0$. Moreover, one can easily see that $K > 0$.

Denote by F the saturation of S' through the flow of X^+ until S. For each $\theta > 0$ and $\rho > 0$ small enough, we know from (2.1) that $\Sigma_\theta := \{x = \theta\} \cap F$ and $\Sigma_{-\rho} := \{x = -\rho\} \cap F$ are transversal to X^+. Thus, the flow of X^+ induces a $\mathcal{C}^{2k}$ diffeomorphism $D : \Sigma_\theta \longrightarrow \Sigma_{-\rho}$, it is called exterior map. Accordingly, from Lemma 1 and hypothesis **(B)**, $D(\overline{y}_\theta) = \overline{y}_{-\rho}$ and $r_{\theta,\rho} := \dfrac{dD}{dy}(\overline{y}_\theta) \neq 0$. Moreover, one can easily see that $r_{\theta,\rho} > 0$. Thus, expanding D around $y = \overline{y}_\theta$, we get

$$D(y) = \overline{y}_{-\rho} + r_{\theta,\rho}(y - \overline{y}_\theta) + \mathcal{O}((y - \overline{y}_\theta)^2). \tag{4.1}$$

In order to prove Theorem C, we shall first establish the relationship between the derivative of the first return map K and the derivative of exterior map $r_{\theta,\rho}$ as follows.

Lemma 5. $\lim\limits_{\theta,\rho\to 0} r_{\theta,\rho} = K.$

Proof. Notice that, for $\rho > 0$ and $\theta > 0$ small enough, the flow of X^+ induces the following $\mathcal{C}^{2k}$ maps,

$$\lambda_\theta : S' \to \{x = \theta\} \cap F \quad \text{and} \quad \lambda_\rho : \{x = -\rho\} \cap F \to S \cap F,$$

which satisfies $\lambda_\rho(\overline{y}_{-\rho}) = 0$ and $\lambda_\theta(0) = \overline{y}_\theta$. Indeed, consider the functions

$$\mu_1(t, y, \theta) = \varphi^1_{X^+}(t, 0, y) - \theta, \quad \text{for} \quad (0, y) \in S',$$

and

$$\mu_2(t, y, \rho) = \varphi^1_{X^+}(t, -\rho, y), \quad \text{for} \quad (-\rho, y) \in \{x = -\rho\} \cap F.$$

Since, $\mu_1(0, 0, 0) = 0 = \mu_2(0, 0, 0)$,

$$\frac{\partial \mu_1}{\partial t}(0, 0, 0) = \frac{\partial \varphi^1_{X^+}}{\partial t}(0, 0, 0) = 1 \neq 0, \quad \text{and} \quad \frac{\partial \mu_2}{\partial t}(0, 0, 0) = \frac{\partial \varphi^1_{X^+}}{\partial t}(0, 0, 0) = 1 \neq 0,$$

we get, by the *Implicit Function Theorem*, the existence of unique smooth functions $t_1(y,\theta)$ and $t_2(y,\rho)$ such that $t_1(0,0) = 0 = t_2(0,0)$,

$$\mu_1(t_1(y,\theta), y, \theta) = 0, \quad \text{and} \quad \mu_2(t_2(y,\rho), y, \rho) = 0,$$

i.e. $\varphi^1_{X^+}(t_1(y,\theta), 0, y) = \theta$ and $\varphi^1_{X^+}(t_2(y,\rho), -\rho, y) = 0$. Thus,

$$\lambda_\theta(y) = \varphi^2_{X^+}(t_1(y,\theta), 0, y) \quad \text{and} \quad \lambda_\rho(y) = \varphi^2_{X^+}(t_2(y,\rho), -\rho, y).$$

Notice that

$$\frac{d\lambda_\theta}{dy}(0) = \frac{\partial \varphi^2_{X^+}}{\partial t}(t_1(0,\theta),0,0)\frac{\partial t_1}{\partial y}(0,\theta) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_1(0,\theta),0,0)$$

and

$$\frac{d\lambda_\rho}{dy}(\overline{y}_{-\rho}) = \frac{\partial \varphi^2_{X^+}}{\partial t}(t_2(\overline{y}_{-\rho},\rho),-\rho,\overline{y}_{-\rho})\frac{\partial t_2}{\partial y}(\overline{y}_{-\rho},\rho) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_2(\overline{y}_{-\rho},\rho),-\rho,\overline{y}_{-\rho}).$$

Since

$$\frac{\partial t_1}{\partial y}(0,0) = -\frac{\frac{\partial \varphi^1_{X^+}}{\partial y}(0,0,0)}{\frac{\partial \varphi^1_{X^+}}{\partial t}(0,0,0)} = -\frac{\partial \varphi^1_{X^+}}{\partial y}(0,0,0) = 0,$$

$$\frac{\partial t_2}{\partial y}(0,0) = -\frac{\frac{\partial \varphi^1_{X^+}}{\partial y}(0,0,0)}{\frac{\partial \varphi^1_{X^+}}{\partial t}(0,0,0)} = -\frac{\partial \varphi^1_{X^+}}{\partial y}(0,0,0) = 0,$$

and

$$\frac{\partial \varphi^2_{X^+}}{\partial y}(0,0,0) = 1,$$

we get that

$$\begin{aligned} \lim_{\theta\to 0}\frac{d\lambda_\theta}{dy}(0) &= \frac{\partial \varphi^2_{X^+}}{\partial t}(t_1(0,0),0,0)\frac{\partial t_1}{\partial y}(0,0) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_1(0,0),0,0) \\ &= \frac{\partial \varphi^2_{X^+}}{\partial t}(0,0,0)\left[-\frac{\partial \varphi^1_{X^+}}{\partial y}(0,0,0)\right] + \frac{\partial \varphi^2_{X^+}}{\partial y}(0,0,0) \\ &= 1 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \lim_{\rho\to 0}\frac{d\lambda_\rho}{dy}(\overline{y}_{-\rho}) &= \frac{\partial \varphi^2_{X^+}}{\partial t}(t_2(0,0),0,0)\frac{\partial t_2}{\partial y}(0,0) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_2(0,0),0,0) \\ &= \frac{\partial \varphi^2_{X^+}}{\partial t}(0,0,0)\left[-\frac{\partial \varphi^1_{X^+}}{\partial y}(0,0,0)\right] + \frac{\partial \varphi^2_{X^+}}{\partial y}(0,0,0) \\ &= 1. \end{aligned} \tag{4.3}$$

Finally, since $\pi = \lambda_\rho \circ D \circ \lambda_\theta$, we conclude that

$$\begin{aligned} \frac{d\pi}{dy}(0) &= \frac{d\lambda_\rho}{dy}(D\circ\lambda_\theta(0))\frac{dD}{dy}(\lambda_\theta(0))\frac{d\lambda_\theta}{dy}(0) \\ &= \frac{d\lambda_\rho}{dy}(\overline{y}_{-\rho})r_{\theta,\rho}\frac{d\lambda_\theta}{dy}(0). \end{aligned}$$

Therefore,

$$\frac{K}{r_{\theta,\rho}} = \frac{d\lambda_\rho}{dy}(\overline{y}_{-\rho})\frac{d\lambda_\theta}{dy}(0). \tag{4.4}$$

The result follows by taking the limit of (4.4) and using (4.2) and (4.3). □

4.3 Proof of Theorem C

By Theorem A for $\theta = x_\varepsilon$, there exist $\rho_0 > 0$, and constants $\beta < 0$ and $c, r, q > 0$ such that for every $\rho \in (\varepsilon^\lambda, \rho_0]$, $\lambda \in (0, \lambda^*)$, and $\varepsilon > 0$ small enough, the flow of Z_ε^Φ defines a map U_ε between the transversal sections $\widehat{V}^\varepsilon_{\rho,\lambda} = \{-\rho\} \times [\varepsilon, y^\varepsilon_{\rho,\lambda}]$ and $\widetilde{V}^\varepsilon_{x_\varepsilon} = \{x_\varepsilon\} \times [y^\varepsilon_{x_\varepsilon}, y^\varepsilon_{x_\varepsilon} + re^{-\frac{c}{\varepsilon^q}}]$ satisfying

$$\begin{array}{rccl} U_\varepsilon : & \widehat{V}^\varepsilon_{\rho,\lambda} & \longrightarrow & \widetilde{V}^\varepsilon_{x_\varepsilon} \\ & y & \longmapsto & y^\varepsilon_{x_\varepsilon} + \mathcal{O}(e^{-\frac{c}{\varepsilon^q}}), \end{array} \tag{4.5}$$

where $y^\varepsilon_{x_\varepsilon} = \overline{y}_{x_\varepsilon} + \varepsilon + \mathcal{O}(\varepsilon^{2k\lambda^*})$ (see Figure 18). Now, notice that there exists $\varepsilon_0 > 0$ such

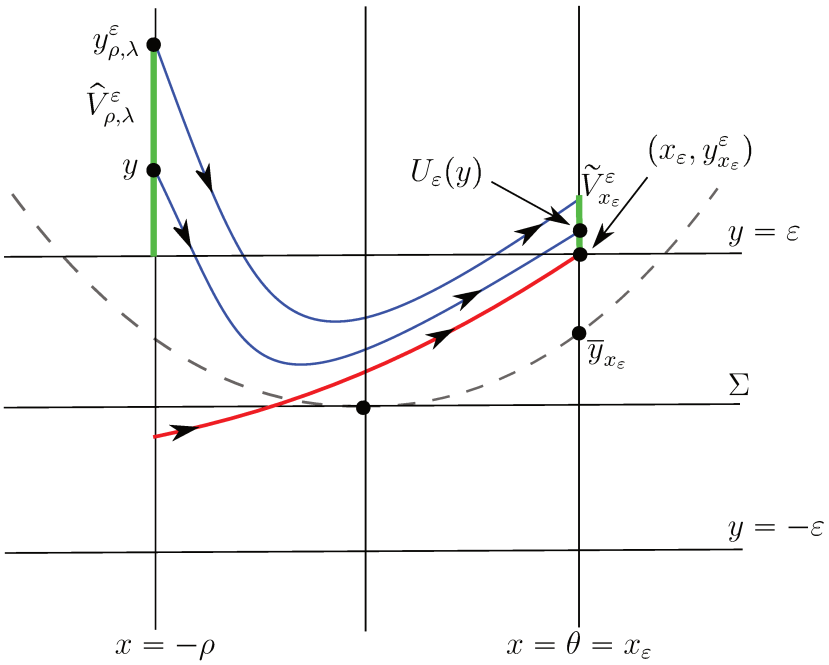

Figure 18 – Upper Transition Map U_ε of the regularized system Z_ε^Φ.The dotted curve is the trajectory of X^+ passing through the visible regular-tangential singularity of multiplicity $2k$. The red curve is the Fenichel manifold.

that

$$\widetilde{V}^\varepsilon_{x_\varepsilon} = \{x_\varepsilon\} \times [y^\varepsilon_{x_\varepsilon}, y^\varepsilon_{x_\varepsilon} + re^{-\frac{c}{\varepsilon^q}}] \subset \{x = x_\varepsilon\} \cap F,$$

for all $\varepsilon \in [0, \varepsilon_0]$. In this way, we define the function $\pi_\varepsilon(y) = D \circ U_\varepsilon(y)$. Thus, from (4.1) and (4.5), we have

$$\begin{aligned} \pi_\varepsilon(y) &= D\Big(\overline{y}_{x_\varepsilon} + \varepsilon + \mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big) + \mathcal{O}(e^{-c/\varepsilon^q})\Big) \\ &= D\Big(\overline{y}_{x_\varepsilon} + \varepsilon + \mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big)\Big) \\ &= \overline{y}_{-\rho} + r_{x_\varepsilon,\rho}\Big(\varepsilon + \mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big)\Big) + \mathcal{O}\Big(\varepsilon + \mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big)\Big)^2 \\ &= \overline{y}_{-\rho} + r_{x_\varepsilon,\rho}\varepsilon + \mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big). \end{aligned} \tag{4.6}$$

Using (3.4) and (4.6), we get

$$\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda} = (r_{x_\varepsilon,\rho} - 1)\varepsilon + \mathcal{O}(\varepsilon\rho) - \beta\varepsilon^{2k\lambda} + \mathcal{O}(\varepsilon^{(2k+1)\lambda}) + \mathcal{O}(\varepsilon^{1+\lambda}) + \mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big),$$

where $\beta < 0$. Recall that $0 < \lambda < \lambda^*$. Thus, we shall study the limit $\lim\limits_{\rho,\varepsilon\to 0} \dfrac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon}$ in three distinct cases.

First, suppose that $\lambda > \dfrac{1}{2k}$. Then,

$$\frac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon} = r_{x_\varepsilon,\rho} - 1 + \mathcal{O}(\rho) + \mathcal{O}(\varepsilon^{2k\lambda-1}).$$

Hence, by Lemma (5),

$$\lim_{\rho,\varepsilon\to 0} \frac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon} = K - 1. \tag{4.7}$$

Now, suppose that $\lambda < \dfrac{1}{2k}$. Then,

$$\frac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon^{2k\lambda}} = (r_{x_\varepsilon,\rho} - 1)\varepsilon^{1-2k\lambda} + \mathcal{O}(\varepsilon^{1-2k\lambda}\rho) - \beta + \mathcal{O}(\varepsilon^{\lambda}) + \mathcal{O}\Big(\varepsilon^{2k\lambda^* - 2k\lambda}\Big).$$

Hence, by Lemma (5),

$$\lim_{\rho,\varepsilon\to 0} \frac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon^{2k\lambda}} = -\beta > 0. \tag{4.8}$$

Finally, suppose that $\lambda = \dfrac{1}{2k}$. Then,

$$\frac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon} = r_{x_\varepsilon,\rho} - 1 - \beta + \mathcal{O}(\rho) + \mathcal{O}(\varepsilon^{\lambda}) + \mathcal{O}\Big(\varepsilon^{2k\lambda^* - 1}\Big).$$

Hence, by Lemma (5),

$$\lim_{\rho,\varepsilon\to 0} \frac{\pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda}}{\varepsilon} = K - 1 - \beta, \tag{4.9}$$

Now, we prove statement (a) of Theorem C. Since Γ is an unstable hyperbolic limit cycle, we know that $K > 1$. Consequently, all the above limits,(4.7), (4.8) and (4.9), are strictly positive and, since $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$0 < \rho, \varepsilon < \delta_0 \;\Rightarrow\; \pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda} > 0.$$

Hence, $\pi_\varepsilon([\varepsilon, y^\varepsilon_{\rho,\lambda}]) \cap [\varepsilon, y^\varepsilon_{\rho,\lambda}] = \varnothing$, for all $\varepsilon > 0$ small enough. This means that π_ε has no fixed points in $[\varepsilon, y^\varepsilon_{\rho,\lambda}]$ and, equivalently, the regularized system Z^Φ_ε does not admit limit cycles passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda}$.

Now, we prove statement (b) of Theorem C. In this case, $\lambda > \dfrac{1}{2k}$. Since Γ is an asymptotically stable hyperbolic limit cycle, we know that $K < 1$. Thus, the limit (4.7) is strictly negative and, since $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$0 < \rho, \varepsilon < \delta_0 \;\Rightarrow\; \pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda} < 0.$$

Hence, $\pi_\varepsilon(y) < y^\varepsilon_{\rho,\lambda}$. Moreover, from (4.6), we get

$$\lim_{\rho,\varepsilon\to 0} \frac{\pi_\varepsilon(y) - \overline{y}_{-\rho}}{\varepsilon} = K > 0.$$

Since $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$0 < \rho, \varepsilon < \delta_1 \Rightarrow \pi_\varepsilon(y) - \overline{y}_{-\rho} > 0.$$

Hence, $\pi_\varepsilon(y) > \overline{y}_{-\rho}$, for all $\varepsilon > 0$ sufficiently small. This means that $\pi_\varepsilon([\varepsilon, y^\varepsilon_{\rho,\lambda}]) \subset [\varepsilon, y^\varepsilon_{\rho,\lambda}]$. From the *Brouwer Fixed Point Theorem*, we conclude that π_ε admits fixed points in $[\varepsilon, y^\varepsilon_{\rho,\lambda}]$ and, equivalently, the regularized system Z^Φ_ε admits limit cycles passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda}$.

In what follows, we prove the uniqueness of the fixed point in $[\varepsilon, y^\varepsilon_{\rho,\lambda}]$. Indeed, expanding D in Taylor series around $y = y^\varepsilon_{x_\varepsilon}$, we have that

$$D(y) = D(y^\varepsilon_{x_\varepsilon}) + \frac{dD}{dy}(y^\varepsilon_{x_\varepsilon})(y - y^\varepsilon_{x_\varepsilon}) + \mathcal{O}((y - y^\varepsilon_{x_\varepsilon})^2).$$

Thus,

$$\begin{aligned} \pi_\varepsilon(y) &= D(y^\varepsilon_{x_\varepsilon} + \mathcal{O}(e^{-c/\varepsilon^q})) \\ &= D(y^\varepsilon_{x_\varepsilon}) + \frac{dD}{dy}(y^\varepsilon_{x_\varepsilon})\mathcal{O}(e^{-c/\varepsilon^q}) + \mathcal{O}(e^{-2c/\varepsilon^q}) \\ &= D(y^\varepsilon_{x_\varepsilon}) + \mathcal{O}(e^{-c/\varepsilon^q}), \end{aligned}$$

and, consequently, $|\pi_\varepsilon(y_1) - \pi_\varepsilon(y_2)| = \mathcal{O}(e^{-c/\varepsilon^q})$, for all $y_1, y_2 \in [\varepsilon, y^\varepsilon_{\rho,\lambda}]$. Now, consider the following function

$$\begin{aligned} \nu_\varepsilon : [\varepsilon, y^\varepsilon_{\rho,\lambda}] &\longrightarrow [0,1] \\ y &\longmapsto \frac{y}{y^\varepsilon_{\rho,\lambda} - \varepsilon} + \frac{\varepsilon}{\varepsilon - y^\varepsilon_{\rho,\lambda}}. \end{aligned}$$

Notice that $\nu_\varepsilon^{-1}(u) = (y^\varepsilon_{\rho,\lambda} - \varepsilon)u + \varepsilon$. Hence, if $\widetilde{\pi}_\varepsilon(u) = \pi_\varepsilon \circ \nu_\varepsilon^{-1}(u)$, then

$$|\widetilde{\pi}_\varepsilon(u_1) - \widetilde{\pi}_\varepsilon(u_2)| = \mathcal{O}(e^{-c/\varepsilon^q}),$$

for all $u_1, u_2 \in [0,1]$. Fix $l \in (0,1)$, take $u_1, u_2 \in [0,1]$, and define the function $\ell(\varepsilon) = (y^\varepsilon_{\rho,\lambda} - \varepsilon)l$. There exists $\varepsilon(u_1, u_2) > 0$ and a neighborhood $U(u_1, u_2) \subset [0,1]^2$ of (u_1, u_2) such that

$$|\widetilde{\pi}_\varepsilon(x) - \widetilde{\pi}_\varepsilon(y)| < \ell(\varepsilon)|x - y|,$$

for all $(x, y) \in U(u_1, u_2)$ and $\varepsilon \in (0, \varepsilon(u_1, u_2))$. Since $\{U(u_1, u_2) : (u_1, u_2) \in [0,1]^2\}$ is an open cover of the compact set $[0,1]^2$, there exists a finite sequence $(u_1^i, u_2^i) \in [0,1]^2$, $i = 1, \ldots, s$, for which $\{U^i := U(u_1^i, u_2^i) : i = 1, \ldots, s\}$ still covers $[0,1]^2$. Taking $\breve{\varepsilon} = \min\{\varepsilon(u_1^i, u_2^i) : i = 1, \ldots, s\}$, we obtain that

$$|\widetilde{\pi}_\varepsilon(x) - \widetilde{\pi}_\varepsilon(y)| < \ell(\varepsilon)|x - y|,$$

for all $\varepsilon \in (0, \breve{\varepsilon})$ and $(x, y) \in [0,1]^2$. Finally, since $\pi_\varepsilon(z) = \widetilde{\pi}_\varepsilon \circ \nu(z)$, we get

$$\begin{aligned} |\pi_\varepsilon(x) - \pi_\varepsilon(y)| &< \ell(\varepsilon)|\nu_\varepsilon(x) - \nu_\varepsilon(y)| \\ &= l|x - y|, \end{aligned}$$

for all $\varepsilon \in (0, \breve{\varepsilon})$ and $x, y \in [\varepsilon, y^\varepsilon_{\rho,\lambda}]$. Thus, we have concluded that π_ε is a contraction for $\varepsilon > 0$ small enough. By the *Banach Fixed Point Theorem*, π_ε admits a unique asymptotically stable fixed point for ε small enough. Therefore, the regularized system Z^Φ_ε admits a unique asymptotically stable limit cycle Γ_ε passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda}$, for ε sufficiently small. Moreover, since $y^\varepsilon_{\rho,\lambda} - \overline{y}_{-\rho} = \mathcal{O}(\varepsilon)$ and $x_\varepsilon - \overline{x}^+_\varepsilon = \mathcal{O}(\varepsilon^{\frac{1}{2k}})$, we get from differentiable dependency results on parameters and initial condition that Γ_ε is ε-close to Γ.

4.4 Piecewise Polynomial Example

This section is devoted to provide examples of piecewise polynomial transition functions and piecewise polynomial vector fields satisfying the hypotheses of Theorem C.

Proposition 9. *For $n \geqslant 1$, consider*

$$\phi_n(x) = (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2} \int_0^x (s-1)^n (s+1)^n ds.$$

Define $\Phi_n : \mathbb{R} \to \mathbb{R}$ as $\Phi_n(x) = \phi_n(x)$ for $x \in (-1,1)$, and $\Phi_n(x) = \text{sign}(x)$ for $|x| \geqslant 1$. Then, $\Phi_n \in C^n_{ST}$ for every positive integer n.

Proof. Notice that $\phi_n(\pm 1) = \pm 1$ and

$$\phi_n'(x) = (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2}(x-1)^n(x+1)^n.$$

Thus, $\phi_n'(x) > 0$ for all $x \in (-1,1)$, $\phi_n^{(i)}(\pm 1) = 0$ for $i = 1, \ldots, n$, and

$$\phi_n^{(n+1)}(\pm 1) = \prod_{i=1}^{n} (\mp 1)^n (2i+1) \neq 0.$$

Consequently, $\Phi_n \in C^n_{ST}$. □

Now, consider the planar vector field $Z = (X^+, X^-)$, with $X^+ = (X_1^+, X_2^+)$ and $X^-(x,y) = (0,1)$, where

$$X_1^+(x,y) = -x(-1+x^{2k}) + (-1+y)^{2k-1}(-1+x-xy),$$

and

$$X_2^+(x,y) = x^{2k-1} - (-1+x^{2k} + (-1+y)^{2k})(-1+y), \text{ for } k > 1.$$

Define $\Sigma = h^{-1}(0)$, with $h(x,y) = y$. Notice that the vector field Z has a $2k$-multiplicity contact with Σ at $(0,0)$. Indeed, $(X^+)^i h(0,0) = 0$, for $i = 1, \ldots, 2k-1$, and $(X^+)^{2k} h(0,0) = (2k-1)!$. Let $H(x,y) = 1 - x^{2k} - (y-1)^{2k}$ and consider the level curve $\Gamma = H^{-1}(0)$. Notice that

$$\langle DH(x,y), X^+(x,y) \rangle \Big|_{H^{-1}(0)} = 0,$$

thus, Γ is invariant through the flow of X^+. Moreover, X^+ has no singularities in $H^{-1}(0)$. Then, by the *Poincaré Bendixson Theorem*, Γ is a periodic orbit of X^+. Furthermore, for $(x,y) \in \Gamma$, we get

$$divX^+(x,y) = \frac{\partial X_1^+}{\partial x}(x,y) + \frac{\partial X_2^+}{\partial y}(x,y) = -2k < 0.$$

Thus, given γ any parametrization of Γ, T its period, and S a transversal section of X^+ at $0 \in \gamma$, we have that the derivative of Poincaré map $\pi : S_0 \subset S \to S$ is given by

$$\frac{d\pi}{dt}(0) = exp\Big[\int_0^T divX^+(\gamma(t))\Big] = e^{-2kT}.$$

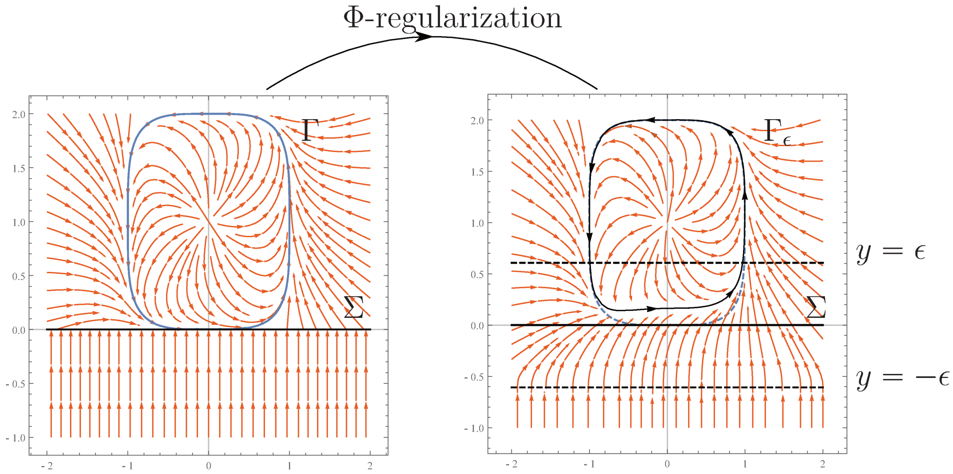

Figure 19 – Vector field Z and its regularized system Z_ε^Φ. The figure on the left shows the hyperbolic limit cycle Γ passing through the visible $2k$-multiplicity contact with Σ at $(0,0)$ and the figure on the right shows the limit cycle Γ_ϵ, for $n = 6$, $k = 2$, and $\Phi \in C^5_{ST}$ with $\phi(u) = -\frac{63}{319}u^{11} + \frac{35}{29}u^9 - \frac{90}{29}u^7 + \frac{126}{29}u^5 - \frac{105}{29}u^3 + \frac{63}{29}u$.

Consequently, we conclude that Γ is an asymptotically stable hyperbolic limit cycle of X^+.

Hence, by Theorem C, we conclude that the regularized system Z_ε^Φ with $\Phi \in C^{n-1}_{ST}$ admits a unique asymptotically stable limit cycle Γ_ε passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda} = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, for ε small enough (see Figure 19). Moreover, Γ_ε is ε-close to Γ.

Remark 5. *If we multiply the polynomial vector field X^+ by -1, we have that X^+ has an unstable hyperbolic limit cycle. Then, by Theorem C we conclude that Z_ε^Φ does not admit limit cycles passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda}$, for ε sufficiently small (see Figure 20).*

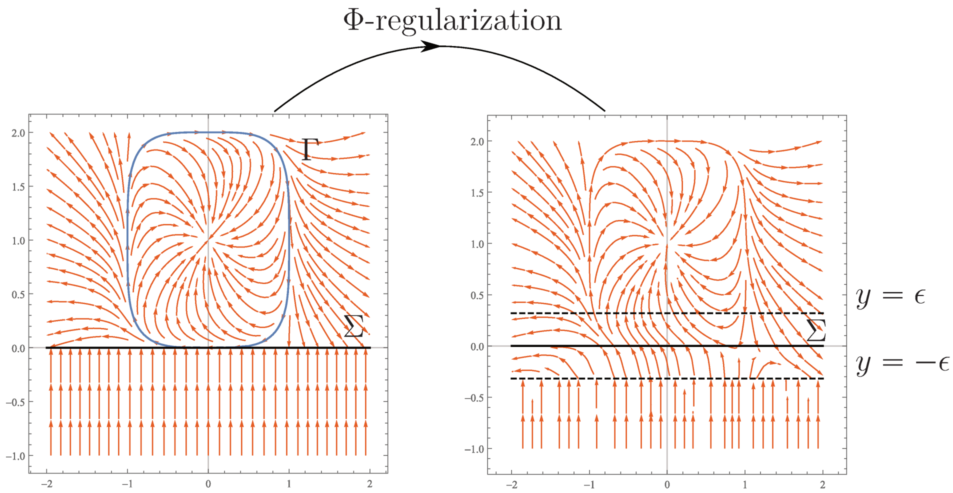

Figure 20 – The figure on the left shows the unstable hyperbolic limit cycle Γ of Z and the figure on the right shows that Z_ε^Φ has no limit cycles for ε small enough, $n = 6$, $k = 2$, and $\Phi \in C^5_{ST}$ with $\phi(u) = -\frac{63}{319}u^{11} + \frac{35}{29}u^9 - \frac{90}{29}u^7 + \frac{126}{29}u^5 - \frac{105}{29}u^3 + \frac{63}{29}u$.

5 Uniqueness and nonexistence of limit cycles

In the previous chapter, we proved statement (a) and (b) of Theorem C, which guaranteed, respectively, the nonexistence and uniqueness of limit cycles in a specific compact set with nonempty interior. Nevertheless, it is not ensured, in general, the nonexistence and uniqueness of limit cycles converging to Γ, because this compact set degenerates into Γ when $\varepsilon \to 0$. However, if we suppose, in addition, that X^+ has locally a unique isocline $x = \psi(y)$ of $2k-$multiplicity contacts with the straight lines $y = cte$, then we have the nonexistence and uniqueness of limit cycles converging to Γ, respectively. For this, we shall first introduce a special map called the Mirror map of the regularized system Z_ε^Φ (1.4).

5.1 Mirror maps in the regularized system

Consider the nonsmooth vector field $Z = (X^+, X^-)$ and assume that X^+ satisfies the following hypotheses:

- **(A)** X^+ has a visible $2k$-multiplicity contact with Σ at $(0,0)$, $X_1^+(0,0) > 0$, and there exists a neighborhood $U \subset \mathbb{R}^2$ of $(0,0)$ such that $X^-\big|_U = (0,1)$ and $\Sigma \cap U = \{(x,0) : x \in (-x_U, x_U)\}$;
- **(H)** The limit cycle Γ of X^+ has locally a unique isocline $x = \psi(y)$ of $2k-$multiplicity contacts with the straight lines $y = \varepsilon$, $\varepsilon > 0$ small enough;

for some $k \geqslant 1$. For $n \geqslant \max\{2, 2k-1\}$, let $\Phi \in C_{ST}^{n-1}$ be given as (1.5) and consider the regularized system Z_ε^Φ (1.4). In what follows, we shall see that, for each $(x,\varepsilon) \in \{y = \varepsilon\}$ near to $(\psi(\varepsilon), \varepsilon)$ there exists a unique small time $t(x,\varepsilon)$ satisfying $t(x,\varepsilon) = 0$ if, and only if, $x = \psi(\varepsilon)$ and $\varphi_{Z_\varepsilon^\Phi}(t(x,\varepsilon), x, \varepsilon) \in \{y = \varepsilon\}$. In this case, we can define the following map

$$\begin{array}{rcl} \rho_\varepsilon : V_{\psi(\varepsilon)}^- \subset \{y = \varepsilon\} & \longrightarrow & V_{\psi(\varepsilon)}^+ \subset \{y = \varepsilon\} \\ (x,\varepsilon) & \longmapsto & \varphi_{Z_\varepsilon^\Phi}(t(x,\varepsilon), x, \varepsilon). \end{array}$$

where $V_{\psi(\varepsilon)}^- = (\psi(\varepsilon) - \delta_\varepsilon^-, \psi(\varepsilon)] \times \{\varepsilon\}$ and $V_{\psi(\varepsilon)}^+ = [\psi(\varepsilon), \psi(\varepsilon) + \delta_\varepsilon^+) \times \{\varepsilon\}$, for some positive real numbers $\delta_\varepsilon^-, \delta_\varepsilon^+$. Notice that $\rho_\varepsilon(\psi(\varepsilon), \varepsilon) = (\psi(\varepsilon), \varepsilon)$. The map ρ_ε is called *Mirror Map* associated with Z_ε^Φ at $\psi(\varepsilon)$ (see Figure 21).

First, consider the horizontal and vertical translations $u = x - \psi(\varepsilon)$ and $v = y - \varepsilon$, respectively. Notice that $(u,v) = (0,0)$ is a point on the isocline $u = \psi(v+\varepsilon) - \psi(\varepsilon)$ in the $(u,v)-$coordinates. Define the vector fields $X_\varepsilon^+(u,v) := X^+(u + \psi_\varepsilon(\varepsilon), v + \varepsilon)$ and $\widetilde{Z}_\varepsilon^\Phi(u,v) := Z_\varepsilon^\Phi(u + \psi(\varepsilon), v + \varepsilon)$. Expanding $\pi_2 \circ \varphi_{\widetilde{Z}_\varepsilon^\Phi}(t,u,0)$ in Taylor series around $t = 0$, we get

$$\pi_2 \circ \varphi_{\widetilde{Z}_\varepsilon^\Phi}(t,u,0) = \sum_{i=1}^{2k} \frac{(X_\varepsilon^+)^i h(u,0)}{i!} t^i + \mathcal{O}(t^{2k+1}). \tag{5.1}$$

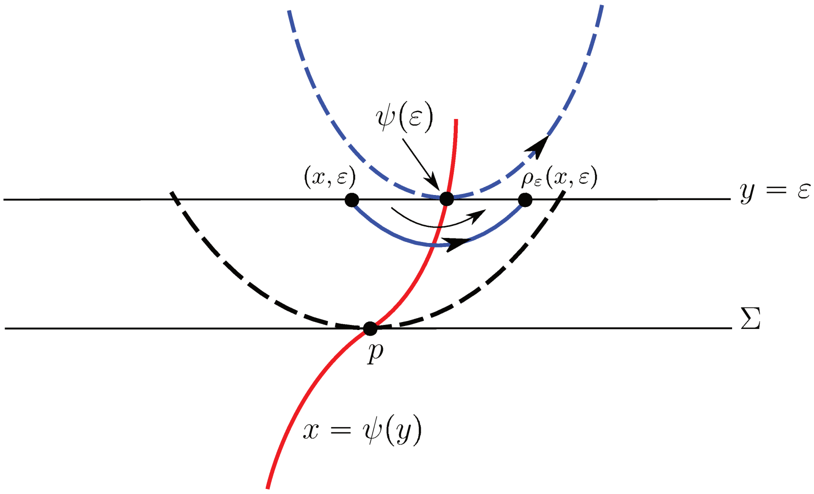

Figure 21 – Mirror Map ρ_ε of Z_ε^Φ at $\psi(\varepsilon)$.

From the construction of Section 2.1, it is easy to see that

$$(X_\varepsilon^+)^i h(u,0) = \frac{\alpha_\varepsilon(2k-1)!}{(2k-i)!}u^{2k-i} + \mathcal{O}(u^{2k-i+1}), \tag{5.2}$$

for each $i \in \{1, \cdots, 2k\}$, where

$$\alpha_\varepsilon = \frac{1}{(2k-1)!}\frac{\partial^{2k-1}f_\varepsilon}{\partial u^{2k-1}}(0,0) > 0 \text{ and } f_\varepsilon(u,v) = \frac{\pi_2 \circ X^+(u+\psi(\varepsilon), v+\varepsilon)}{\pi_1 \circ X^+(u+\psi(\varepsilon), v+\varepsilon)}.$$

Notice that $\alpha_0 = \alpha > 0$, which is given in (2.1). Now, we define the map

$$S(s,u,\varepsilon) = \frac{2k}{\alpha_\varepsilon u^{2k}}\pi_2 \circ \varphi_{\widetilde{Z}_\varepsilon^\Phi}(su,u,0).$$

Using (5.1) and (5.2) we can rewrite S as

$$S(s,u,\varepsilon) = -1 + (1+s)^{2k} + \mathcal{O}(u,\varepsilon).$$

Since $S(-2,0,0)=0$ and $\dfrac{\partial S}{\partial s}(-2,0,0) = -2k < 0$, by *Implicit Function Theorem* we get the existence of a smooth function $s(u,\varepsilon)$ such that $s(0,0)=-2$ and $S(s(u,\varepsilon),u,\varepsilon)=0$. From the definition of S, for $t(u,\varepsilon) = us(u,\varepsilon)$ we have that $\pi_2 \circ \varphi_{\widetilde{Z}_\varepsilon^\Phi}(t(u,\varepsilon),u,0) = 0$. Finally, expanding s around $(u,\varepsilon)=(0,0)$ we get that $s(u,\varepsilon) = -2+\mathcal{O}(u,\varepsilon)$. Consequently, we can define the map $\widetilde{\rho}_\varepsilon$ in a neighborhood $V_0 \subset \Sigma$ of $(0,0)$ by

$$\widetilde{\rho}_\varepsilon(u,0) = u + t(u,\varepsilon) = -u + \mathcal{O}(u^2, \varepsilon u).$$

Therefore, going back to the original coordinates, we conclude that

$$\rho_\varepsilon(x,\varepsilon) = -x + 2\psi(\varepsilon) + \mathcal{O}\left((x-\psi(\varepsilon))^2, \varepsilon(x-\psi(\varepsilon))\right).$$

5.2 The first return map π_ε

In what follows, we state the following proposition, which will be proved in Section 5.3.

Proposition 10. *Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ and assume that X^+ satisfies hypotheses* **(B)** *(see Section 4.1) and* **(H)** *for some $k \geqslant 1$. For $n \geqslant 2k-1$, let $\Phi \in C^{n-1}_{ST}$ be given as (1.5). Then, the following statements hold.*

(a) If the limit cycle Γ is unstable, then for $\varepsilon > 0$ sufficiently small the regularized system Z^Φ_ε (1.4) does not admit limit cycles converging to Γ.

(b) If the limit cycle Γ is asymptotically stable, then for $\varepsilon > 0$ sufficiently small the regularized system Z^Φ_ε (1.4) admits a unique limit cycle Γ_ε converging to Γ. Moreover, Γ_ε is hyperbolic and asymptotically stable.

Remark 6. *Notice that a boundary limit cycle is a $\Sigma-$polycycle, thus we will generalize Proposition 10 in Section 6.1.2.*

To prove Proposition 10 we need to define the first return map π_ε of Z^Φ_ε, for $\varepsilon > 0$ sufficiently small.

First of all, take $\rho, \varepsilon > 0$ small enough in order that the intersections of the trajectory of Z^Φ_ε starting at $(\psi(\varepsilon), \varepsilon)$ with the sections $\{x = -\rho\}$ and $\{x = x_\varepsilon\}$ are contained in U, namely $(-\rho, \overline{y}^\varepsilon_{-\rho})$ and $(x_\varepsilon, \overline{y}^\varepsilon_{x_\varepsilon})$, respectively. Since $\pi_1 \circ X^+(-\rho, \overline{y}^\varepsilon_{-\rho}) \neq 0$ and $\pi_1 \circ X^+(x_\varepsilon, \overline{y}^\varepsilon_{x_\varepsilon}) \neq 0$, then $\{x = -\rho\}$ and $\{x = x_\varepsilon\}$ are transversal sections of X^+ at the points $(-\rho, \overline{y}^\varepsilon_{-\rho})$ and $(x_\varepsilon, \overline{y}^\varepsilon_{x_\varepsilon})$, respectively. Hence, by [10, Theorem A] we know that there exist the transition maps $T^u_\varepsilon : [\psi(\varepsilon), x_\varepsilon] \times \{\varepsilon\} \longrightarrow \{x = x_\varepsilon\}$ and $T^s_\varepsilon : [-\rho, \psi(\varepsilon)] \times \{\varepsilon\} \longrightarrow \{x = -\rho\}$ satisfying

$$\begin{aligned} T^u_\varepsilon(x) &= \overline{y}^\varepsilon_{x_\varepsilon} + \kappa^u_{x_\varepsilon,\varepsilon}(x - \psi(\varepsilon))^{2k} + \mathcal{O}\left((x - \psi(\varepsilon))^{2k+1}\right), \\ T^s_\varepsilon(x) &= \overline{y}^\varepsilon_{-\rho} + \kappa^s_{\rho,\varepsilon}(x - \psi(\varepsilon))^{2k} + \mathcal{O}\left((x - \psi(\varepsilon))^{2k+1}\right), \end{aligned} \tag{5.3}$$

where $\text{sign}(\kappa^u_{x_\varepsilon,\varepsilon}) = -\text{sign}((X^+)^{2k}h(\psi(\varepsilon))) = \text{sign}(\kappa^s_{\rho,\varepsilon})$, i.e. $\kappa^u_{x_\varepsilon,\varepsilon}, \kappa^s_{\rho,\varepsilon} < 0$. Using the *Implicit Function Theorem*, it is easy to see that

$$(T^s_\varepsilon)^{-1}(y) = \psi(\varepsilon) - \sqrt[2k]{\frac{1}{-\kappa^s_{\rho,\varepsilon}}}(\overline{y}^\varepsilon_{-\rho} - y)^{\frac{1}{2k}} + \mathcal{O}\left((\overline{y}^\varepsilon_{-\rho} - y)^{1+\frac{1}{2k}}\right).$$

Now, we know that there exists a diffeomorphism $P^e_\varepsilon : \{x = x_\varepsilon\} \longrightarrow \{x = -\rho\}$ given by

$$P^e_\varepsilon(y) = \overline{y}^\varepsilon_{-\rho} + K^\varepsilon_{x_\varepsilon,\rho}(y - \overline{y}^\varepsilon_{x_\varepsilon}) + \mathcal{O}((y - \overline{y}^\varepsilon_{x_\varepsilon})^2).$$

Finally, we get the first return map $\pi_\varepsilon : \{x = -\rho\} \longrightarrow \{x = -\rho\}$ defined as

$$\begin{aligned} \pi_\varepsilon(y) &:= P^e_\varepsilon \circ T^u_\varepsilon \circ \rho_\varepsilon \circ (T^s_\varepsilon)^{-1}(y) \\ &= \overline{y}^\varepsilon_{-\rho} - \frac{K^\varepsilon_{x_\varepsilon,\rho}\kappa^u_{x_\varepsilon,\varepsilon}}{\kappa^s_{\rho,\varepsilon}}(\overline{y}^\varepsilon_{-\rho} - y) + \mathcal{O}((\overline{y}^\varepsilon_{-\rho} - y)^p) + \mathcal{O}(\varepsilon), \end{aligned} \tag{5.4}$$

for some $p > 1$.

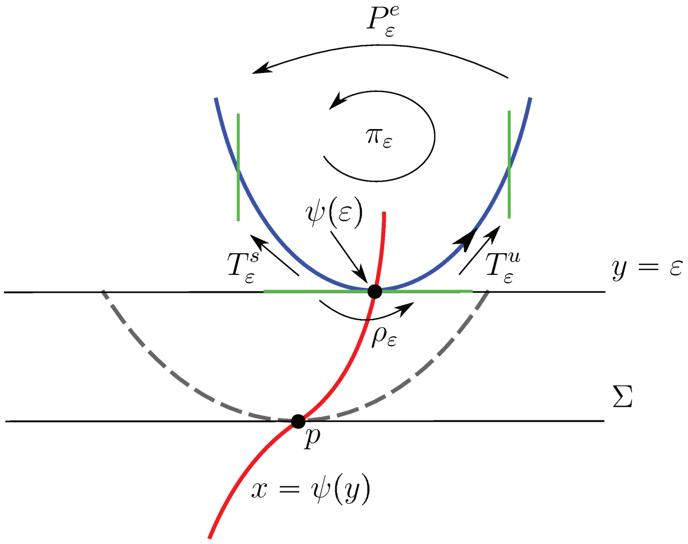

Figure 22 – The first return map π_ε of Z^Φ_ε.

5.3 Proof of Proposition 10

First of all, if Γ_ε is a limit cycle of the regularized system Z^Φ_ε (1.4) converging to Γ (i.e. there exists a fixed point $(-\rho, y^\rho_\varepsilon) \in \{x = -\rho\}$ of π_ε such that $\lim_{\varepsilon \to 0} y^\rho_\varepsilon = \overline{y}_{-\rho}$), then by (5.4) we get

$$\frac{d\pi_\varepsilon}{dy}(y) = \frac{K^\varepsilon_{x_\varepsilon,\rho} \kappa^u_{x_\varepsilon,\varepsilon}}{\kappa^s_{\rho,\varepsilon}} + \mathcal{O}\left((\overline{y}^\varepsilon_{-\rho} - y)^{p-1}\right).$$

It is easy to see that $\lim_{\varepsilon,\rho \to 0} \frac{\kappa^u_{x_\varepsilon,\varepsilon}}{\kappa^s_{\rho,\varepsilon}} = 1$. Thus, using Lemma 5 we have that

$$\lim_{\varepsilon,\rho \to 0} \frac{d\pi_\varepsilon}{dy}(y^\rho_\varepsilon) = K.$$

Hence, since Γ is hyperbolic, then $K > 1$ (resp. $K < 1$) provided that Γ is unstable (resp. asymptotically stable). Consequently, Γ_ε is hyperbolic and unstable (resp. asymptotically stable), for $\varepsilon > 0$ sufficiently small.

The proof of the first statement is by contradiction. Suppose that there exists a limit cycle Γ_ε of Z^Φ_ε such that Γ_ε converging to Γ, for $\varepsilon > 0$ small enough. Consider the region $\mathcal{B}_\varepsilon$ delimited by the curves $x = -\rho$, the limit cycle Γ_ε and the Fenichel manifold $S_{a,\varepsilon}$ associated with Z^Φ_ε, (see Figure 23). Since Γ_ε converges to the regular orbit Γ then $\mathcal{B}_\varepsilon$ has no singular points. In addition, it is easy to see that $\mathcal{B}_\varepsilon$ is positively invariant compact set, for $\varepsilon > 0$ small enough. For $\varepsilon > 0$ choose $q_\varepsilon \in \mathcal{B}_\varepsilon$, from the *Poincaré–Bendixson Theorem* $\omega(q_\varepsilon) \subset \mathcal{B}_\varepsilon$ is a limit cycle of Z^Φ_ε that is not unstable, absurd.

Now, we shall prove the second statement. Indeed, from Theorem C, for $\varepsilon > 0$ small enough, we know that Z^Φ_ε admits a asymptotically stable limit cycle Γ_ε converging to Γ. Moreover, from above we have that Γ_ε is hyperbolic. Finally, we claim that Γ_ε is the unique limit cycle with these properties. Indeed, suppose that there exists another limit cycle $\widetilde{\Gamma}_\varepsilon$ converging to Γ, hyperbolic and asymptotically stable. Now, consider the region $\mathcal{R}_\varepsilon$ delimited by the limit cycles Γ_ε and $\widetilde{\Gamma}_\varepsilon$. Notice that $\mathcal{R}_\varepsilon$ is negatively invariant compact

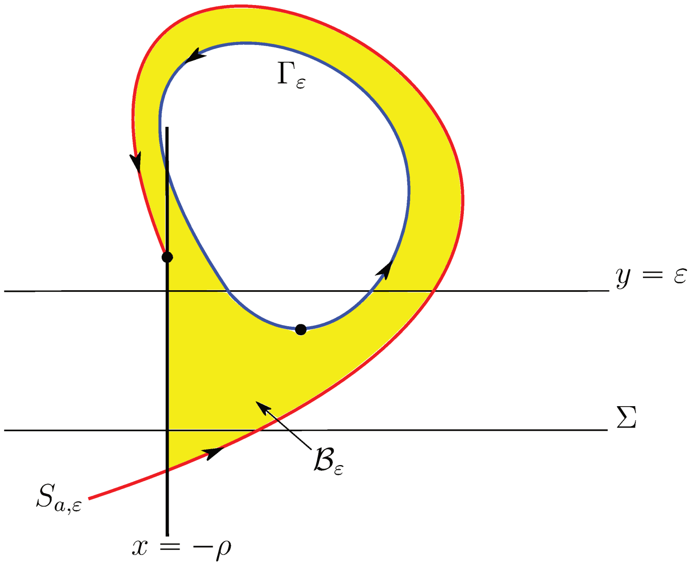

Figure 23 – The region $\mathcal{B}_\varepsilon$.

set. Furthermore, since Γ_ε and $\widetilde{\Gamma}_\varepsilon$ converges to the regular orbit Γ, we can conclude that $\mathcal{R}_\varepsilon$ has no singular points for $\varepsilon > 0$ small enough. For $\varepsilon > 0$ choose $q_\varepsilon \in \mathcal{R}_\varepsilon$, from the *Poincaré–Bendixson Theorem* we can conclude that $\alpha(q_\varepsilon) \subset \mathcal{R}_\varepsilon$ is a limit cycle of Z_ε^Φ that is not asymptotically stable, absurd.

6 Regularization of Σ-Polycycles

In Chapter 4, we stated Theorem C which studied C^n-regularizations of boundary limit cycles with a even multiplicity contact with the switching manifold. Notice that the simplest $\Sigma-$polycycle is the boundary limit cycle. Thus, our main interest here is to generalize Theorem C for homoclinic-like $\Sigma-$polycycles through $\Sigma-$singularities of planar Filippov systems.

6.1 Regularization of Σ-Polycycles of type (a)

In this section, we shall state and prove the fourth main result of this work, which in particular establishes sufficient conditions under which the regularized vector field Z_ε^Φ has a limit cycle Γ_ε converging to a $\Sigma-$polycycle of type (a). For this, suppose that a Filippov system $Z = (X^+, X^-)$ has a $\Sigma-$polycycle Γ of type (a). Through a local change of coordinates, we can assume that $p = (0, 0)$ and $h(x, y) = y$. Without loss of generality, assume that:

(a.1) $X_1^+(p) > 0$;

(a.2) the trajectory of Z through p crosses Σ transversally $m-$times at $q_1, \cdots, q_m$, i.e. if $m \neq 0$, then for each $i = 1, \cdots, m$, there exists $t_i > 0$ such that $\varphi_Z(t_i, q_i) = q_{i+1}$, where $q_{m+1} = p$. Moreover, $\Gamma \cap \Sigma = \{q_1, \cdots, q_m, p\}$.

We shall also assume that

(a.3) $X^-h(p) > 0$.

The case $X^-h(p) < 0$ is obtained from this case multiplying the vector field Z by -1 (see Remark 7 below).

Notice that assumption $(a.2)$ above guarantees the existence of an exterior map D associated to Z. In what follows, we characterize such a map D. Since $X_1^+(p) > 0$, implies that there exists an open set U such that $X_1^+(x, y) > 0$, for all $(x, y) \in U$. Take $\rho, \theta > 0$ small enough in order that the points $q^u = (\theta, \overline{y}_\theta) \in W_t^u(p)$ and $q^s = (-\rho, \overline{y}_{-\rho}) \in W_t^s(p)$ are contained in U. Hence, there exist $\delta^{u,s}$ and η positive numbers such that

$$\begin{aligned} \tau_t^u &= \{(\theta, y) : y \in (\overline{y}_\theta - \delta^u, \overline{y}_\theta + \delta^u)\}, \\ \tau_t^s &= \{(-\rho, y) : y \in (\overline{y}_{-\rho} - \delta^s, \overline{y}_{-\rho} + \delta^s)\}, \quad \text{and} \\ \sigma_p &= \{0\} \times [0, \eta) \end{aligned} \tag{6.1}$$

are transversal sections of X^+. In addition, we know by the *Tubular Flow Theorem* that there exist the $C^{2k}-$diffeomorphisms $T^{u,s} : \sigma_p \longrightarrow \tau_t^{u,s}$ and $D : \tau_t^u \longrightarrow \tau_t^s$ such that $T^{u,s}(p) = q^{u,s}$ and $D(q^u) = q^s$ (see Figure 24). Thus, expanding D around $y = \overline{y}_\theta$, we get

$$D(y) = \overline{y}_{-\rho} + r_{\theta,\rho}(y - \overline{y}_\theta) + \mathcal{O}((y - \overline{y}_\theta)^2), \tag{6.2}$$

where $r_{\theta,\rho} = \dfrac{dD}{dy}(\overline{y}_\theta)$.

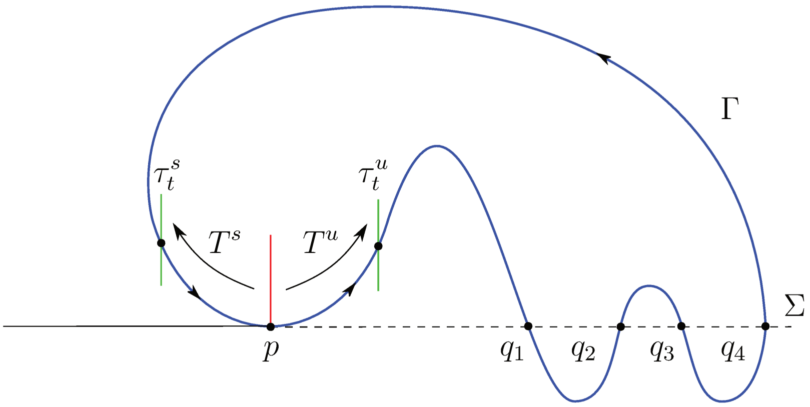

Figure 24 – Σ−polycycle Γ of type (a) satisfying (i) and (ii).

Now, from Definition 3, we know that there exists a first return map π_Γ defined, at least, in one side of Γ. In what follows, we shall see that there exist positive constants η and K such that the first return map $\pi_\Gamma : \sigma_p \to \sigma_p$ is given by

$$\pi_\Gamma(y) = Ky + \mathcal{O}(y^2), \text{ where } K > 0. \tag{6.3}$$

Indeed, expanding $T^{u,s}$ around $y = 0$, we have

$$\begin{aligned} T^s(y) &= \overline{y}_{-\rho} + \kappa^s_\rho y + \mathcal{O}(y^2) \quad \text{and} \\ T^u(y) &= \overline{y}_\theta + \kappa^u_\theta y + \mathcal{O}(y^2), \end{aligned} \tag{6.4}$$

with $\kappa^{s,u}_{\rho,\theta} = \dfrac{dT^{s,u}}{dy}(0)$. In addition, using (6.2) and the *Implicit Function Theorem*, we get

$$D^{-1}(y) = \overline{y}_\theta + \frac{1}{r_{\theta,\rho}}(y - \overline{y}_{-\rho}) + \mathcal{O}((y - \overline{y}_{-\rho})^2). \tag{6.5}$$

Moreover, one can easily see that $r_{\theta,\rho}, \kappa^{s,u}_{\rho,\theta} > 0$ for all θ, ρ sufficiently small. Hence, we can define the first return map $\pi_\Gamma : \sigma_p \longrightarrow \sigma_p$ by

$$\pi_\Gamma(y) = (D^{-1} \circ T^s)^{-1} \circ T^u(y).$$

From (6.4) and (6.5), we obtain $D^{-1} \circ T^s(y) = \overline{y}_\theta + \dfrac{\kappa^s_\rho}{r_{\theta,\rho}} y + \mathcal{O}(y^2)$. Thus, using the *Implicit Function Theorem*, we have that

$$(D^{-1} \circ T^s)^{-1}(y) = \frac{r_{\theta,\rho}}{\kappa^s_\rho}(y - \overline{y}_\theta) + \mathcal{O}((y - \overline{y}_\theta)^2). \tag{6.6}$$

Therefore, using (6.4) and (6.6) we conclude that

$$\pi_\Gamma(y) = \frac{r_{\theta,\rho}\kappa^u_\theta}{\kappa^s_\rho} y + \mathcal{O}(y^2). \tag{6.7}$$

Consequently, taking $K := \dfrac{r_{\theta,\rho}\kappa_\theta^u}{\kappa_\rho^s} > 0$, we get (6.3).

Now, since D is a diffeomorphism induced by a regular orbit, we can easily see that the regularized system Z_ε^Φ also admits an exterior map $D_\varepsilon : \tau_t^u \longrightarrow \tau_t^s$. If we denote $S := \left.\dfrac{\partial D_\varepsilon(0)}{\partial \varepsilon}\right|_{\varepsilon=0}$, then

$$D_\varepsilon(y) = D(y) + S\varepsilon + \mathcal{O}(\varepsilon^2, \varepsilon y), \tag{6.8}$$

See Section 6.1.3 for more details about the exterior map and how to estimate S.

In what follows, we state our first main result of this chapter, which will be proven in Section 6.1.1.

Theorem D. *Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ and assume that Z has a $\Sigma-$polycycle Γ of type (a) satisfying (a.1), (a.2), and (a.3). For $n \geqslant 2k-1$, let $\Phi \in C^{n-1}_{ST}, K, S$ be given as (1.5), (6.3) and (6.8) respectively and consider the regularized system Z_ε^Φ (1.4). If $K + S - 1 \neq 0$, then the following statements hold:*

(a) Given $0 < \lambda < \lambda^* = \dfrac{n}{1+2k(n-1)}$, if $K + S - 1 > 0$, then there exists $\rho > 0$ such that the regularized system Z_ε^Φ does not admit limit cycles passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda} = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, for $\varepsilon > 0$ sufficiently small.

(b) Given $\dfrac{1}{2k} < \lambda < \lambda^* = \dfrac{n}{1+2k(n-1)}$, if $K + S - 1 < 0$, then there exists $\rho > 0$ such that the regularized system Z_ε^Φ admits a unique limit cycle Γ_ε passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda} = [-\rho, -\varepsilon^\lambda] \times \{\varepsilon\}$, for $\varepsilon > 0$ sufficiently small. Moreover, Γ_ε is asymptotically stable and ε-close to Γ.

Remark 7. *In Theorem D we are assuming (a.3), i.e. $X^-h(p) > 0$. If $X^-h(p) < 0$, Theorem D can be applied to $-Z$. Consequently, the limit cycle obtained for Z, under the suitable assumptions, would be unstable.*

In order to prove this theorem, we shall first establish the relationship between the derivative of the first return map K and the derivative of exterior map $r_{\theta,\rho}$ as follows.

Lemma 6. *Consider $r_{\theta,\varepsilon}$ given as in (6.2). Then, $\lim\limits_{\theta,\rho\to 0} r_{\theta,\rho} = K$.*

Proof. Using equations (6.3) and (6.7) we get for $\theta, \rho > 0$ small enough that

$$r_{\theta,\rho} = \frac{K\kappa_\rho^s}{\kappa_\theta^u}.$$

To prove this lemma, we just need to prove that $\lim\limits_{\theta,\rho\to 0} \dfrac{\kappa_\rho^s}{\kappa_\theta^u} = 1$, because in this case we have

$$\begin{aligned} \lim_{\theta,\rho\to 0} r_{\theta,\rho} &= \lim_{\theta,\rho\to 0} \frac{K\kappa_\rho^s}{\kappa_\theta^u} \\ &= K \lim_{\theta,\rho\to 0} \frac{\kappa_\rho^s}{\kappa_\theta^u} \\ &= K. \end{aligned}$$

For this, we shall prove that the flow of X^+ induces $\mathcal{C}^{2k}$ maps,

$$\lambda^s_\rho : \sigma'_p \subset \sigma_p \longrightarrow \tau^s_t \quad \text{and} \quad \lambda^u_\theta : \sigma'_p \subset \sigma_p \longrightarrow \tau^u_t,$$

between the transversal sections defined in (6.1) and satisfying $\lambda^s_\rho(0) = q^s$ and $\lambda^u_\theta(0) = q^u$, respectively. Indeed, consider the functions

$$\mu_1(t, y, \theta) = \varphi^1_{X^+}(t, 0, y) - \theta, \text{ for } (0, y) \in \sigma_p, \; t \in I_{(0,y)},$$

and

$$\mu_2(t, y, \rho) = \varphi^1_{X^+}(t, 0, y) + \rho, \text{ for } (0, y) \in \sigma_p, \; t \in I_{(0,y)},$$

where $\varphi_{X^+} = (\varphi^1_{X^+}, \varphi^2_{X^+})$ is the flow of X^+ and $I_{(0,y)}$ is the maximal interval of existence of $t \mapsto \varphi_{X^+}(t, 0, y)$. Since,

$$\mu_1(0, 0, 0) = 0 = \mu_2(0, 0, 0) \text{ and}$$

$$\frac{\partial \mu_{1,2}}{\partial t}(0, 0, 0) = \frac{\partial \varphi^1_{X^+}}{\partial t}(0, 0, 0) = X^+_1(p) \neq 0,$$

by the *Implicit Function Theorem* there exist $\eta_0 > 0$ and smooth functions $t_1(y, \theta)$ and $t_2(y, \rho)$, with $(0, y) \in \sigma'_p := \{0\} \times [0, \eta_0) \subset \sigma_p$ and $\theta, \rho > 0$ sufficiently small, such that

$$t_1(0, 0) = 0 = t_2(0, 0),$$

$$\mu_1(t_1(y, \theta), y, \theta) = 0, \text{ and } \mu_2(t_2(y, \rho), y, \rho) = 0,$$

i.e. $\varphi^1_{X^+}(t_1(y, \theta), 0, y) = \theta$ and $\varphi^1_{X^+}(t_2(y, \rho), 0, y) = -\rho$. Thus, we can define the functions

$$\lambda^u_\theta(y) = \varphi^2_{X^+}(t_1(y, \theta), 0, y) \text{ and } \lambda^s_\rho(y) = \varphi^2_{X^+}(t_2(y, \rho), 0, y).$$

Notice that

$$\frac{d\lambda^u_\theta}{dy}(0) = \frac{\partial \varphi^2_{X^+}}{\partial t}(t_1(0, \theta), 0, 0)\frac{\partial t_1}{\partial y}(0, \theta) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_1(0, \theta), 0, 0),$$

and

$$\frac{d\lambda^s_\rho}{dy}(0) = \frac{\partial \varphi^2_{X^+}}{\partial t}(t_2(0, \rho), 0, 0)\frac{\partial t_2}{\partial y}(0, \rho) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_2(0, \rho), 0, 0).$$

Since,

$$\frac{\partial t_1}{\partial y}(0, 0) = -\frac{\frac{\partial \varphi^1_{X^+}}{\partial y}(0, 0, 0)}{\frac{\partial \varphi^1_{X^+}}{\partial t}(0, 0, 0)}, \quad \frac{\partial t_2}{\partial y}(0, 0) = -\frac{\frac{\partial \varphi^1_{X^+}}{\partial y}(0, 0, 0)}{\frac{\partial \varphi^1_{X^+}}{\partial t}(0, 0, 0)}$$

and

$$\frac{\partial \varphi^1_{X^+}}{\partial y}(0, 0, 0) = 0, \; \frac{\partial \varphi^2_{X^+}}{\partial y}(0, 0, 0) = 1, \; \kappa^{s,u}_{\theta,\rho} = \frac{dT^{s,u}\big|_{\sigma'_p}}{dy}(0) = \frac{d\lambda^{s,u}_{\rho,\theta}}{dy}(0),$$

we get that

$$\begin{aligned} \lim_{\theta \to 0} \kappa^u_\theta &= \lim_{\theta \to 0} \frac{d\lambda^u_\theta}{dy}(0) \\ &= \frac{\partial \varphi^2_{X^+}}{\partial t}(t_1(0, 0), 0, 0)\frac{\partial t_1}{\partial y}(0, 0) + \frac{\partial \varphi^2_{X^+}}{\partial y}(t_1(0, 0), 0, 0) \\ &= 1, \end{aligned} \tag{6.9}$$

and

$$\begin{aligned}\lim_{\rho\to 0}\kappa^s_\rho &= \lim_{\rho\to 0}\frac{d\lambda^s_\rho}{dy}(0)\\ &= \frac{\partial\varphi^2_{X^+}}{\partial t}(t_2(0,0),0,0)\frac{\partial t_2}{\partial y}(0,0)+\frac{\partial\varphi^2_{X^+}}{\partial y}(t_2(0,0),0,0)\\ &= 1.\end{aligned} \tag{6.10}$$

The result follows from (6.9) and (6.10). □

6.1.1 Proof of Theorem D

First, from Theorem A for $\theta = x_\varepsilon$, there exist $\rho_0 > 0$, and constants $\beta < 0$ and $c, r, q > 0$ such that for every $\rho \in (\varepsilon^\lambda, \rho_0]$, $\lambda \in (0, \lambda^*)$, and $\varepsilon > 0$ sufficiently small, the flow of Z^Φ_ε defines a map U_ε between the transversal sections $\widehat{V}^\varepsilon_{\rho,\lambda} = \{-\rho\} \times [\varepsilon, y^\varepsilon_{\rho,\lambda}]$ and $\widetilde{V}^\varepsilon_{x_\varepsilon} = \{x_\varepsilon\} \times [y^\varepsilon_{x_\varepsilon}, y^\varepsilon_{x_\varepsilon} + re^{-\frac{c}{\varepsilon^q}}]$ satisfying

$$\begin{aligned} U_\varepsilon :\ \widehat{V}^\varepsilon_{\rho,\lambda} &\longrightarrow \widetilde{V}^\varepsilon_{x_\varepsilon}\\ y &\longmapsto y^\varepsilon_{x_\varepsilon} + \mathcal{O}(e^{-\frac{c}{\varepsilon^q}}),\end{aligned} \tag{6.11}$$

where $y^\varepsilon_{x_\varepsilon} = \overline{y}_{x_\varepsilon} + \varepsilon + \mathcal{O}(\varepsilon^{2k\lambda^*})$ (see Figure 25).

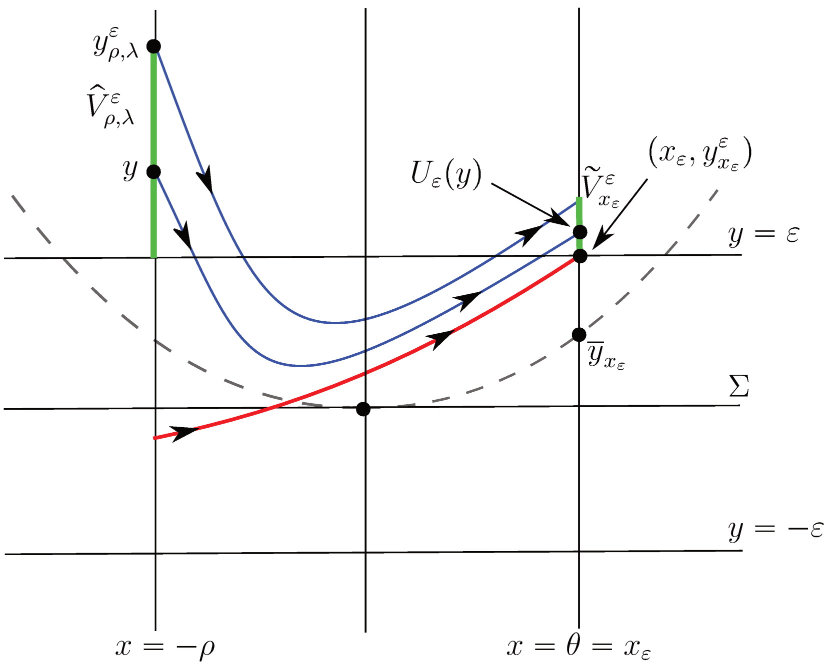

Figure 25 – Upper Transition Map U_ε of the regularized system Z^Φ_ε.The dotted curve is the trajectory of X^+ passing through the visible regular-tangential singularity of multiplicity $2k$ $p = (0, 0)$. The red curve is the well-known Fenichel manifold.

Now, notice that there exists $\varepsilon_0 > 0$ such that $\widetilde{V}^\varepsilon_{x_\varepsilon} \subset \tau^u_t$, for all $\varepsilon \in [0, \varepsilon_0]$. In this way, for $\varepsilon \in [0, \varepsilon_0]$, we define the function $\pi_\varepsilon(y) = D_\varepsilon \circ U_\varepsilon(y)$. Hence, from (6.8) and

(6.11), we get

$$\begin{array}{rcl}\pi_\varepsilon(y) & = & D_\varepsilon\Big(y^\varepsilon_{x_\varepsilon}+\mathcal{O}(e^{-c/\varepsilon^q})\Big)\\ & = & D\Big(\overline{y}_{x_\varepsilon}+\varepsilon+\mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big)\Big)+\varepsilon S+\mathcal{O}(\varepsilon^2)\\ & = & \overline{y}_{-\rho}+r_{x_\varepsilon,\rho}\Big(\varepsilon+\mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big)\Big)+\mathcal{O}\Big(\varepsilon+\mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big)\Big)^2+\varepsilon S+\mathcal{O}(\varepsilon^2)\\ & = & \overline{y}_{-\rho}+\Big(r_{x_\varepsilon,\rho}+S\Big)\varepsilon+\mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big).\end{array} \tag{6.12}$$

Using (6.12) and (3.1), we have

$$\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}=(r_{x_\varepsilon,\rho}+S-1)\,\varepsilon+\mathcal{O}(\varepsilon\rho)-\beta\varepsilon^{2k\lambda}+\mathcal{O}(\varepsilon^{(2k+1)\lambda})+\mathcal{O}(\varepsilon^{1+\lambda})+\mathcal{O}\Big(\varepsilon^{2k\lambda^*}\Big).$$

Hence, we must study the limit $\lim\limits_{\rho,\varepsilon\to 0}\dfrac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon}$ in three distinct cases. First, assume that $\lambda>\dfrac{1}{2k}$. Then,

$$\frac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon}=r_{x_\varepsilon,\rho}+S-1+\mathcal{O}(\rho)-\beta\varepsilon^{2k\lambda-1}+\mathcal{O}(\varepsilon^{(2k+1)\lambda-1})+\mathcal{O}(\varepsilon^{\lambda})+\mathcal{O}\Big(\varepsilon^{2k\lambda^*-1}\Big).$$

Thus, by Lemma 6,

$$\lim_{\rho,\varepsilon\to 0}\frac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon}=K+S-1. \tag{6.13}$$

Now, suppose that $\lambda<\dfrac{1}{2k}$. Then,

$$\frac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon^{2k\lambda}}=(r_{x_\varepsilon,\rho}+S-1)\varepsilon^{1-2k\lambda}+\mathcal{O}(\varepsilon^{1-2k\lambda}\rho)-\beta+\mathcal{O}(\varepsilon^{\lambda})+\mathcal{O}\Big(\varepsilon^{2k(\lambda^*-\lambda)}\Big).$$

Hence, by Lemma 6,

$$\lim_{\rho,\varepsilon\to 0}\frac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon^{2k\lambda}}=-\beta>0. \tag{6.14}$$

Finally, assume that $\lambda=\dfrac{1}{2k}$. Then,

$$\frac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon}=r_{x_\varepsilon,\rho}+S-1-\beta+\mathcal{O}(\rho)+\mathcal{O}(\varepsilon^{\lambda})+\mathcal{O}\Big(\varepsilon^{2k\lambda^*-1}\Big).$$

Thus, by Lemma 6,

$$\lim_{\rho,\varepsilon\to 0}\frac{\pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}}{\varepsilon}=K+S-1-\beta. \tag{6.15}$$

Now, we prove statement (a) of Theorem D. As $K+S-1>0$, then all the above limits (6.13), (6.14) and (6.15), are strictly positive and, since $\varepsilon>0$, there exists $\delta_0>0$ such that

$$0<\rho,\varepsilon<\delta_0 \;\Rightarrow\; \pi_\varepsilon(y)-y^\varepsilon_{\rho,\lambda}>0.$$

Therefore, $\pi_\varepsilon([\varepsilon,y^\varepsilon_{\rho,\lambda}])\cap[\varepsilon,y^\varepsilon_{\rho,\lambda}]=\varnothing$, for all $\varepsilon\in(0,\delta_0)$. This means that π_ε has no fixed points in $[\varepsilon,y^\varepsilon_{\rho,\lambda}]$, that is, the regularized system Z^Φ_ε does not admit limit cycles passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda}$.

Now, we prove statement (b) of Theorem D. In this case, $\lambda > \dfrac{1}{2k}$. As $K+S-1 < 0$, then the limit (6.13) is strictly negative and, since $\varepsilon > 0$, there exists $\delta_0 > 0$ such that

$$0 < \rho, \varepsilon < \delta_0 \ \Rightarrow \ \pi_\varepsilon(y) - y^\varepsilon_{\rho,\lambda} < 0.$$

Consequently, $\pi_\varepsilon(y) < y^\varepsilon_{\rho,\lambda}$. Moreover, from (6.12), we get

$$\lim_{\varepsilon\to 0} \pi_\varepsilon(y) - \varepsilon = \overline{y}_{-\rho} > 0,$$

for all $\rho > 0$. Since $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$0 < \varepsilon < \delta_1 \ \Rightarrow \ \pi_\varepsilon(y) - \varepsilon > 0.$$

Accordingly, $\pi_\varepsilon(y) > \varepsilon$, for $\varepsilon > 0$ sufficiently small. This means that $\pi_\varepsilon([\varepsilon, y^\varepsilon_{\rho,\lambda}]) \subset [\varepsilon, y^\varepsilon_{\rho,\lambda}]$. By the *Brouwer Fixed Point Theorem*, we can conclude that π_ε admits fixed points in $[\varepsilon, y^\varepsilon_{\rho,\lambda}]$, that is, the regularized system Z^Φ_ε has limit cycles passing through the section $\widehat{H}^\varepsilon_{\rho,\lambda}$.

In what follows, we will prove the uniqueness of the fixed point in $[\varepsilon, y^\varepsilon_{\rho,\lambda}]$. Indeed, expanding D_ε in Taylor series around $y = y^\varepsilon_{x_\varepsilon}$, we have that

$$D_\varepsilon(y) = D_\varepsilon(y^\varepsilon_{x_\varepsilon}) + \frac{dD_\varepsilon}{dy}(y^\varepsilon_{x_\varepsilon})(y - y^\varepsilon_{x_\varepsilon}) + \mathcal{O}((y - y^\varepsilon_{x_\varepsilon})^2).$$

Hence,

$$\begin{aligned} \pi_\varepsilon(y) &= D_\varepsilon(y^\varepsilon_{x_\varepsilon} + \mathcal{O}(e^{-c/\varepsilon^q})) \\ &= D_\varepsilon(y^\varepsilon_{x_\varepsilon}) + \frac{dD_\varepsilon}{dy}(y^\varepsilon_{x_\varepsilon})\mathcal{O}(e^{-c/\varepsilon^q}) + \mathcal{O}(e^{-2c/\varepsilon^q}) \\ &= D_\varepsilon(y^\varepsilon_{x_\varepsilon}) + \mathcal{O}(e^{-c/\varepsilon^q}), \end{aligned}$$

and, therefore, $|\pi_\varepsilon(y_1) - \pi_\varepsilon(y_2)| = \mathcal{O}(e^{-c/\varepsilon^q})$, for all $y_1, y_2 \in [\varepsilon, y^\varepsilon_{\rho,\lambda}]$. Now, let ν_ε be the function given by

$$\begin{aligned} \nu_\varepsilon : [\varepsilon, y^\varepsilon_{\rho,\lambda}] &\longrightarrow [0,1] \\ y &\longmapsto \frac{y - \varepsilon}{y^\varepsilon_{\rho,\lambda} - \varepsilon}. \end{aligned}$$

Notice that $\nu_\varepsilon^{-1}(u) = (y^\varepsilon_{\rho,\lambda} - \varepsilon)u + \varepsilon$. Thus, if $\widetilde{\pi}_\varepsilon(u) = \pi_\varepsilon \circ \nu_\varepsilon^{-1}(u)$, then

$$|\widetilde{\pi}_\varepsilon(u_1) - \widetilde{\pi}_\varepsilon(u_2)| = \mathcal{O}(e^{-c/\varepsilon^q}),$$

for all $u_1, u_2 \in [0,1]$. Fix $l \in (0,1)$, take $u_1, u_2 \in [0,1]$, and define the function $\ell(\varepsilon) := (y^\varepsilon_{\rho,\lambda} - \varepsilon)l$. There exists $\varepsilon(u_1,u_2) > 0$ and a neighborhood $U(u_1,u_2) \subset [0,1]^2$ of (u_1,u_2) such that

$$|\widetilde{\pi}_\varepsilon(x) - \widetilde{\pi}_\varepsilon(y)| < \ell(\varepsilon)|x - y|,$$

for all $(x,y) \in U(u_1,u_2)$ and $\varepsilon \in (0, \varepsilon(u_1,u_2))$. Since $\{U(u_1,u_2) : (u_1,u_2) \in [0,1]^2\}$ is an open cover of the compact set $[0,1]^2$, there exists a finite sequence $(u_1^i, u_2^i) \in [0,1]^2$, $i = 1, \dots, s$, for which $\{U^i := U(u_1^i, u_2^i) : i = 1, \dots, s\}$ still covers $[0,1]^2$. Taking $\breve{\varepsilon} = \min\{\varepsilon(u_1^i, u_2^i) : i = 1, \dots, s\}$, we get

$$|\widetilde{\pi}_\varepsilon(x) - \widetilde{\pi}_\varepsilon(y)| < \ell(\varepsilon)|x - y|,$$

for all $\varepsilon \in (0, \breve{\varepsilon})$ and $(x, y) \in [0,1]^2$. Finally, since $\pi_\varepsilon(z) = \widetilde{\pi}_\varepsilon \circ \nu(z)$, we have that

$$\begin{aligned} |\pi_\varepsilon(x) - \pi_\varepsilon(y)| &= |\widetilde{\pi}_\varepsilon \circ \nu_\varepsilon(x) - \widetilde{\pi}_\varepsilon \circ \nu_\varepsilon(y)| \\ &< \ell(\varepsilon)|\nu_\varepsilon(x) - \nu_\varepsilon(y)| \\ &= \frac{\ell(\varepsilon)}{y^{\varepsilon}_{\rho,\lambda} - \varepsilon}|x - y| \\ &= l|x - y|, \end{aligned}$$

for all $\varepsilon \in (0, \breve{\varepsilon})$ and $x, y \in [\varepsilon, y^{\varepsilon}_{\rho,\lambda}]$. Therefore, π_ε is a contraction for $\varepsilon > 0$ small enough. From the *Banach Fixed Point Theorem*, π_ε admits a unique asymptotically stable fixed point for $\varepsilon > 0$ small enough. Therefore, the regularized system Z^{Φ}_{ε} has a unique asymptotically stable limit cycle Γ_ε passing through the section $\widehat{H}^{\varepsilon}_{\rho,\lambda}$, for $\varepsilon > 0$ sufficiently small. Moreover, since $\pi_\varepsilon(y) - \overline{y}_{-\rho} = \mathcal{O}(\varepsilon)$ for all $y \in [\varepsilon, y^{\varepsilon}_{\rho,\lambda}]$ and $x_\varepsilon - \overline{x}^{+}_{\varepsilon} = \mathcal{O}(\varepsilon^{\frac{1}{2k}})$, we get from differentiable dependency results on parameters and initial condition that Γ_ε is ε-close to Γ.

6.1.2 A case of uniqueness and nonexistence of limit cycles

The goal of this session is to obtain a version of Proposition 10 for $\Sigma-$polycycles of type (a). More specifically, we know that statement (a) and (b) of Theorem D guarantee, respectively, the nonexistence and uniqueness of limit cycles in a specific compact set with nonempty interior. However, it is not ensured, in general, the nonexistence and uniqueness of limit cycles converging to Γ, because this compact set degenerates into Γ when $\varepsilon \to 0$. Nevertheless, if we suppose, in addition, that X^+ has locally a unique isocline $x = \psi(y)$ of $2k-$multiplicity contacts with the straight lines $y = cte$ and $K > 1$ or $K < 1$, then we can establish the nonexistence and uniqueness of limit cycles converging to Γ, respectively. More precisely, consider the following proposition.

Proposition 11. *Let $Z = (X^+, X^-)_\Sigma$ be a Filippov system and assume that Z has a $\Sigma-$polycycle Γ of type $a)$ satisfying $(a.1)$, $(a.2)$, and $(a.3)$. For $n \geqslant 2k-1$, let $\Phi \in C^{n-1}_{ST}$, K, S be given as (1.5), (6.3) and (6.8) respectively and consider the regularized system Z^{Φ}_{ε} (1.4). If $K + S - 1 \neq 0$ and X^+ has locally a unique isocline $x = \psi(y)$ of $2k-$multiplicity contacts with the straight lines $y = \varepsilon$, then the following statements hold.*

- *(a) If $K + S - 1 > 0$ and $K > 1$, then for $\varepsilon > 0$ sufficiently small the regularized system Z^{Φ}_{ε} does not admit limit cycles converging to Γ.*
- *(b) If $K + S - 1 < 0$ and $K < 1$, then for $\varepsilon > 0$ sufficiently small the regularized system Z^{Φ}_{ε} admits a unique limit cycle Γ_ε converging to Γ. Moreover, Γ_ε is hyperbolic and asymptotically stable.*

Remark 8. *In Proposition 11 we are assuming $(a.3)$, i.e. $X^-h(p) > 0$. If $X^-h(p) < 0$, Proposition 11 can be applied to $-Z$. Consequently, the unique limit cycle obtained for Z, under the suitable assumptions, would be unstable.*

Proof. We will do this demonstration in 3 steps:

Step 1. First of all, we shall construct the first return map π_ε of Z_ε^Φ, for $\varepsilon > 0$. Since $X_1^+(p) > 0$, implies that there exist an open set U, such that $X_1^+(x,y) \neq 0$, for all $(x,y) \in U$. Take $\rho, \varepsilon > 0$ small enough in order that the points $q_\varepsilon^u = (x_\varepsilon, \overline{y}^\varepsilon_{x_\varepsilon}) \in W_t^u(\psi(\varepsilon))$ and $q_\varepsilon^s = (-\rho, \overline{y}^\varepsilon_{-\rho}) \in W_t^s(\psi(\varepsilon))$ are contained in U. Thus, there exist $\delta_\varepsilon^{u,s}$ positive numbers, such that

$$\tau^u_{t,\varepsilon} = \{(x_\varepsilon, y) : y \in (\overline{y}^\varepsilon_{x_\varepsilon} - \delta^u_\varepsilon, \overline{y}^\varepsilon_{x_\varepsilon} + \delta^u_\varepsilon)\},$$
$$\tau^s_{t,\varepsilon} = \{(-\rho, y) : y \in (\overline{y}^\varepsilon_{-\rho} - \delta^s_\varepsilon, \overline{y}^\varepsilon_{-\rho} + \delta^s_\varepsilon)\},$$

are transversal sections of X^+. Thus, by [10, Theorem A] we know that there exist the transition maps $T^u_\varepsilon : \sigma^+_{p,\varepsilon} := [\psi(\varepsilon), x_\varepsilon] \times \{\varepsilon\} \longrightarrow \tau^u_{t,\varepsilon}$ and $T^s_\varepsilon : \sigma^-_{p,\varepsilon} := [-\rho, \psi(\varepsilon)] \times \{\varepsilon\} \longrightarrow \tau^s_{t,\varepsilon}$ satisfying

$$\begin{aligned} T^u_\varepsilon(x) &= \overline{y}^\varepsilon_{x_\varepsilon} + \kappa^u_{x_\varepsilon,\varepsilon}(x-\psi(\varepsilon))^{2k} + \mathcal{O}\left((x-\psi(\varepsilon))^{2k+1}\right), \\ T^s_\varepsilon(x) &= \overline{y}^\varepsilon_{-\rho} + \kappa^s_{\rho,\varepsilon}(x-\psi(\varepsilon))^{2k} + \mathcal{O}\left((x-\psi(\varepsilon))^{2k+1}\right), \end{aligned} \tag{6.16}$$

where $\text{sign}(\kappa^u_{x_\varepsilon,\varepsilon}) = -\text{sign}((X^+)^{2k}h(\psi(\varepsilon))) = \text{sign}(\kappa^s_{\rho,\varepsilon})$, i.e. $\kappa^u_{x_\varepsilon,\varepsilon}, \kappa^s_{\rho,\varepsilon} < 0$. Furthermore, from *Implicit Function Theorem*, it is easy to see that

$$(T^s_\varepsilon)^{-1}(y) = \psi(\varepsilon) - \sqrt[2k]{\frac{1}{-\kappa^s_{\rho,\varepsilon}}}(\overline{y}^\varepsilon_{-\rho} - y)^{\frac{1}{2k}} + \mathcal{O}\left((\overline{y}^\varepsilon_{-\rho} - y)^{1+\frac{1}{2k}}\right).$$

Now, we know that there exists a diffeomorphism $P^e_\varepsilon : \tau^u_{t,\varepsilon} \longrightarrow \tau^s_{t,\varepsilon}$ defined as

$$P^e_\varepsilon(y) = \overline{y}^\varepsilon_{-\rho} + r^\varepsilon_{x_\varepsilon,\rho}(y - \overline{y}^\varepsilon_{x_\varepsilon}) + \mathcal{O}((y - \overline{y}^\varepsilon_{x_\varepsilon})^2) + \mathcal{O}(\varepsilon).$$

Consequently, we get the map $\pi_\varepsilon : \tau^s_{t,\varepsilon} \longrightarrow \tau^s_{t,\varepsilon}$ given by

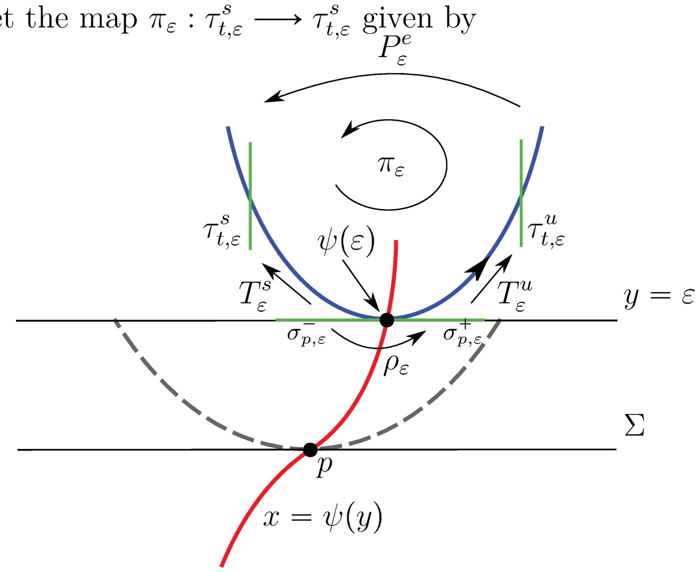

Figure 26 – The first return map π_ε of Z^Φ_ε.

$$\begin{aligned} \pi_\varepsilon(y) &= P^e_\varepsilon \circ T^u_\varepsilon \circ \rho_\varepsilon \circ (T^s_\varepsilon)^{-1}(y) \\ &= \overline{y}^\varepsilon_{-\rho} - \frac{r^\varepsilon_{x_\varepsilon,\rho}\kappa^u_{x_\varepsilon,\varepsilon}}{\kappa^s_{\rho,\varepsilon}}(\overline{y}^\varepsilon_{-\rho} - y) + \mathcal{O}((\overline{y}^\varepsilon_{-\rho} - y)^p) + \mathcal{O}(\varepsilon) \end{aligned} \tag{6.17}$$

for some $p > 1$, where ρ_ε is the *mirror map* associated with Z^Φ_ε at $\psi(\varepsilon)$ (see Section 5.1).

Step 2. Now, if Γ_ε is a limit cycle of the regularized system Z_ε^Φ converging to Γ (i.e. there exists a fixed point $(-\rho, y_\varepsilon^\rho) \in \tau_{t,\varepsilon}^s$ of π_ε such that $\lim_{\varepsilon\to 0} y_\varepsilon^\rho = \overline{y}_{-\rho}$), then by (6.17) we have

$$\frac{d\pi_\varepsilon}{dy}(y) = \frac{r^\varepsilon_{x_\varepsilon,\rho}\kappa^u_{x_\varepsilon,\varepsilon}}{\kappa^s_{\rho,\varepsilon}} + \mathcal{O}\left((\overline{y}^\varepsilon_{-\rho} - y)^{p-1}\right).$$

It is easy to see that $\lim_{\varepsilon,\rho\to 0} \frac{\kappa^u_{x_\varepsilon,\varepsilon}}{\kappa^s_{\rho,\varepsilon}} = 1$. Hence, using Lemma 6, we get

$$\lim_{\varepsilon,\rho\to 0} \frac{d\pi_\varepsilon}{dy}(y_\varepsilon^\rho) = K.$$

Therefore, if $K > 1$ (resp. $K < 1$), then Γ_ε is hyperbolic and unstable (resp. asymptotically stable), for $\varepsilon > 0$ small enough.

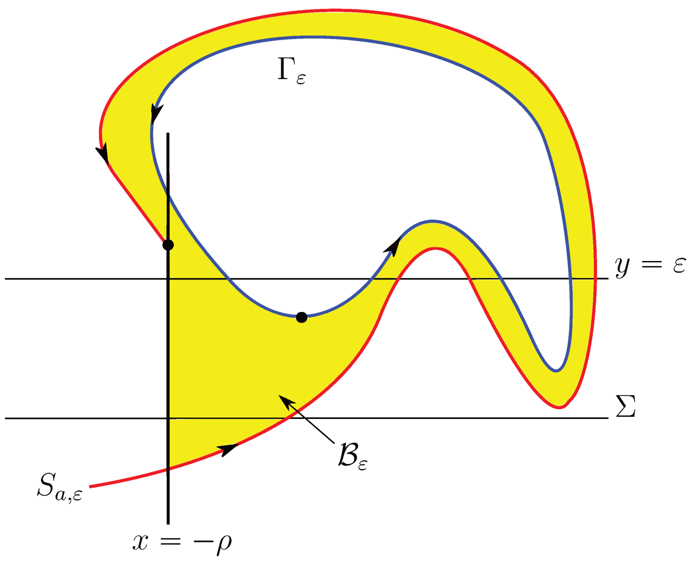

Figure 27 – The region $\mathcal{B}_\varepsilon$.

Step 3. Finally, we are ready to prove this proposition. The proof of the first statement is by contradiction. Assume that there exists a limit cycle Γ_ε of Z_ε^Φ such that Γ_ε converging to Γ, for $\varepsilon > 0$ sufficiently small. Let $\mathcal{B}_\varepsilon$ be the region delimited by the curves $x = -\rho$, the limit cycle Γ_ε and the Fenichel manifold $S_{a,\varepsilon}$ associated with Z_ε^Φ, (see Figure 27). Since Γ_ε converges to the regular orbit Γ then $\mathcal{B}_\varepsilon$ has no singular points. Moreover, it is easy to see that $\mathcal{B}_\varepsilon$ is positively invariant compact set, for $\varepsilon > 0$ small enough. For $\varepsilon > 0$ choose $q_\varepsilon \in \mathcal{B}_\varepsilon$, from the *Poincaré–Bendixson Theorem* $\omega(q_\varepsilon) \subset \mathcal{B}_\varepsilon$ is a limit cycle of Z_ε^Φ which is not unstable, but this contradicts step 2.

Now, we prove the second statement. By Theorem D, for $\varepsilon > 0$ sufficiently small, we know that Z_ε^Φ has a asymptotically stable limit cycle Γ_ε converging to Γ. In addition, from step 2 we have that Γ_ε is hyperbolic. Finally, we claim that Γ_ε is the unique limit cycle with these properties. In fact, assume that there exists another limit cycle $\widetilde{\Gamma}_\varepsilon$ converging to Γ, hyperbolic and asymptotically stable. Now, let $\mathcal{R}_\varepsilon$ be the region delimited by the limit cycles Γ_ε and $\widetilde{\Gamma}_\varepsilon$. Notice that $\mathcal{R}_\varepsilon$ is negatively invariant compact set. Furthermore, since Γ_ε and $\widetilde{\Gamma}_\varepsilon$ converges to the regular orbit Γ, we can conclude that

$\mathcal{R}_\varepsilon$ has no singular points for $\varepsilon > 0$ small enough. For $\varepsilon > 0$ choose $q_\varepsilon \in \mathcal{R}_\varepsilon$, from the *Poincaré–Bendixson Theorem* we can conclude that $\alpha(q_\varepsilon) \subset \mathcal{R}_\varepsilon$ is a limit cycle of Z_ε^Φ that is not asymptotically stable, but this contradicts step 2. □

6.1.3 The exterior map of the regularized system

In this section, we shall study the exterior map of the regularized system Z_ε^Φ and its derivative.

Let $Z = (X, Y)$ be a Filippov system with $X, Y : V \subset \mathbb{R}^2 \to \mathbb{R}^2$ vector fields of class $\mathcal{C}^{2k}$ defined on an open set V of $q \in \mathbb{R}^2$ and Z_ε^Φ the regularized system associated with Z. Assume that the switching manifold Σ of Z is a $\mathcal{C}^{2k}$ embedded codimension one submanifold of V and let $q \in \Sigma^c$.

Notice that there exists a local $\mathcal{C}^{2k}$ diffeomorphism $\psi_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined on an open set U of $q \in \mathbb{R}^2$ such that $\widetilde{\Sigma} = \psi_1(\Sigma) = h^{-1}(0)$, with $h(x, y) = x$. Now, applying the *Tubular Flow Theorem* for $(\psi_1)_* Y$ at q and considering the transversal section $\widetilde{\Sigma}$, there exists a local $\mathcal{C}^{2k}$ diffeomorphism ψ_2 defined on U (taken smaller if necessary) such that $\widetilde{Y} = (\psi_2 \circ \psi_1)_* Y = (1, 0)$ and $\psi_2(\widetilde{\Sigma}) = \widetilde{\Sigma}$.

Since $X_1(q) > 0$ then $X_1(x, y) > 0$ for all $(x, y) \in U$ (taken smaller if necessary). Consequently, we can perform a time rescaling in X, thus we get $\widehat{X}(x, y) = (1, X_2(x, y)/X_1(x, y))$, for all $(x, y) \in U$. Hence, without loss of generality, we can assume that there exists an open set $U \subset \mathbb{R}^2$ of $q = (0, q_2)$ such that the Filippov system $Z = (X, Y)_\Sigma$ satisfies that $X\big|_U = (1, X_2)$, $Y\big|_U = (1, 0)$, and $\Sigma \cap U = \{(0, y) : y \in (q_2 - \delta_U, q_2 + \delta_U)\}$.

Take $\rho > 0$ and $\theta > 0$ sufficiently small in order that the sections $\{x = -\rho\}$ and $\{x = \theta\}$ is contained in U and without loss of generality assume that the points $(-\rho, 0)$ and q are connected by a trajectory of X. Thus, we can consider the following maps induced by the flow of Z_ε^Φ:

$$\begin{aligned} P_\varepsilon : \{x = -\rho\} &\to \{x = -\varepsilon\}, \\ T_\varepsilon : \{x = -\varepsilon\} &\to \{x = \varepsilon\}, \\ Q_\varepsilon : \{x = \varepsilon\} &\to \{x = \theta\}, \end{aligned}$$

where $Q_\varepsilon = Id$ (see Figure (28)). The exterior map $D_\varepsilon : \{x = -\rho\} \to \{x = \theta\}$ of Z_ε^Φ is defined as

$$D_\varepsilon(y) := Q_\varepsilon \circ T_\varepsilon \circ P_\varepsilon(y).$$

Now, we shall compute the first derivative of D_ε at $\varepsilon = 0$ and $y = 0$. Expanding P_ε and T_ε around $\varepsilon = 0$, we get that

$$P_\varepsilon = P + P_1 \varepsilon + \mathcal{O}(\varepsilon^2),$$

$$T_\varepsilon = Id + T_1 \varepsilon + \mathcal{O}(\varepsilon^2)$$

where $P : \{x = -\rho\} \to \Sigma$ is induced from the flow of Z, $P_1 = \dfrac{\partial P_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}$, and $T_1 = \dfrac{\partial T_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}$. Hence, expanding D_ε around $\varepsilon = 0$, we have that

$$D_\varepsilon(y) = P(y) + [(P_1(y) + T_1(P(y))]\varepsilon + \mathcal{O}(\varepsilon^2).$$

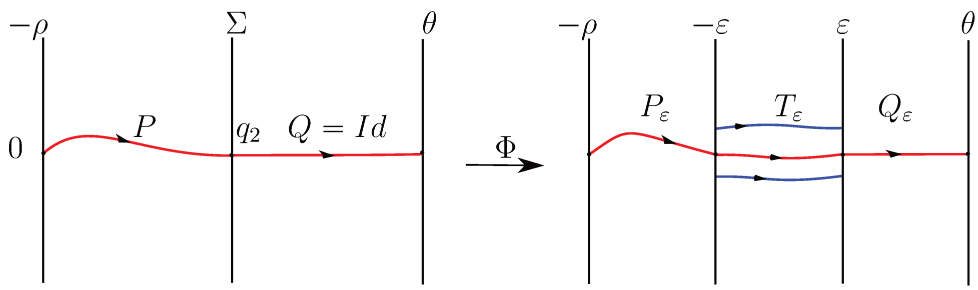

Figure 28 – Exterior map $D_\varepsilon = Q_\varepsilon \circ T_\varepsilon \circ P_\varepsilon$ of the regularized system Z_ε^Φ.

In addition, developing in Taylor series $P_1 + T_1 \circ P$ around $y = 0$, we get

$$\begin{aligned} D_\varepsilon(y) &= P(y) + [(P_1(0) + T_1(P(0))]\varepsilon + \mathcal{O}(\varepsilon^2, \varepsilon y) \\ &= P(y) + [(P_1(0) + T_1(q_2)]\varepsilon + \mathcal{O}(\varepsilon^2, \varepsilon y) \\ &= P(y) + S\varepsilon + \mathcal{O}(\varepsilon^2, \varepsilon y) \end{aligned}$$

where $S := P_1(0) + T_1(q_2)$. In order to determine S, we shall study the maps P_1 and T_1 at $y = 0$ and $y = q_2$, respectively.

First, we study the map P_1. Let φ_X be the flow of X, and assume that $I_{(-\rho,y)}$ the maximal interval of existence of $t \mapsto \varphi_X(t, -\rho, y)$. Is easy to see that there exists a smooth function $t_\rho(y)$ such that $t_0(0) = 0$ and $\varphi_X^1(t_\rho(y), -\rho, y) = 0$. Thus, we can write P as follows

$$P(y) = \varphi_X^2(t_\rho(y), -\rho, y), \;\text{ for }\; y \in \{x = -\rho\}.$$

From *Implicit Function Theorem*, there exists a smooth function $\tau_\rho(y, \varepsilon)$ such that $\tau_\rho(y, 0) = t_\rho(y)$, $\tau_0(0, 0) = 0$, and $\varphi_X^1(\tau_\rho(y, \varepsilon), -\rho, y) = -\varepsilon$, for $|\varepsilon| \neq 0$ sufficiently small. Hence, we can write P_ε as follows

$$P_\varepsilon(y) = \varphi_X^2(\tau_\rho(y, \varepsilon), -\rho, y), \;\text{ for }\; y \in \{x = -\rho\}.$$

Notice that

$$P_1(y) = \frac{\partial P_\varepsilon}{\partial \varepsilon}(y)\Big|_{\varepsilon=0} = \frac{\varphi_X^2}{\partial t}(t_\rho(y), -\rho, y)\frac{\partial \tau_\rho}{\partial \varepsilon}(y, 0)$$

and

$$\frac{\partial \tau_\rho}{\partial \varepsilon}(y, 0) = -\frac{1}{\frac{\varphi_X^1}{\partial t}(t_\rho(y), -\rho, y)}.$$

Consequently,

$$P_1(0) = -\frac{\frac{\varphi_X^2}{\partial t}(t_\rho(0), -\rho, 0)}{\frac{\varphi_X^1}{\partial t}(t_\rho(0), -\rho, 0)} = -\frac{X_2(q)}{1} = -X_2(q).$$

Second, we study the map T_1. For this, consider the differential system associated to the regularized system Z_ε^Φ (1.4) given by

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = \dfrac{1}{2}X_2(x, y)\left(1 + \Phi(x/\varepsilon)\right), \end{cases} \tag{6.18}$$

for $(x, y) \in U$ and $\varepsilon > 0$ small enough. Notice that system (6.18), restricted to the band of regularization $-\varepsilon \leqslant x \leqslant \varepsilon$, is as a *slow-fast problem*. Indeed, taking $v = x/\varepsilon$, we get the so-called *slow system*,

$$\begin{cases} \varepsilon \dot{v} = 1, \\ \dot{y} = \dfrac{1}{2} X_2(\varepsilon v, y)\left(1 + \phi(v)\right), \end{cases} \tag{6.19}$$

defined for $-1 \leqslant v \leqslant 1$. Performing the time rescaling $t = \varepsilon\tau$, we obtain the so-called *fast system*,

$$\begin{cases} v' = 1, \\ y' = \dfrac{\varepsilon}{2} X_2(\varepsilon v, y)\left(1 + \phi(v)\right). \end{cases} \tag{6.20}$$

Now, we know that the equation for the orbits of system (6.20) is given by

$$\frac{dy}{dv} = \varepsilon F(v, y, \varepsilon) \tag{6.21}$$

where

$$F(v, y, \varepsilon) = \frac{X_2(\varepsilon v, y)\left(1 + \phi(v)\right)}{2}.$$

Expanding F around $(v, y, \varepsilon) = (v, y, 0)$, we get

$$F(v, y, \varepsilon) = \frac{X_2(0, y)\left(1 + \phi(v)\right)}{2} + \mathcal{O}(\varepsilon).$$

In addition, the solution of differential equation (6.21) with initial condition $y_\varepsilon(-1, q_2) = q_2$ is given by

$$\begin{aligned} y_\varepsilon(v, q_2) &= q_2 + \varepsilon \int_{-1}^{v} F(s, y_\varepsilon(s, q_2), \varepsilon) ds \\ &= q_2 + \varepsilon \int_{-1}^{v} F(s, y_0(s, q_2), 0) ds + \mathcal{O}(\varepsilon^2) \\ &= q_2 + \frac{\varepsilon}{2} \int_{-1}^{v} X_2(q)\left(1 + \phi(s)\right) ds + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Notice that

$$\begin{aligned} T_\varepsilon(q_2) &= y_\varepsilon(1, q_2) \\ &= q_2 + \varepsilon \int_{-1}^{1} F(s, y_0(s, q_2), 0) ds + \mathcal{O}(\varepsilon^2) \\ &= q_2 + \frac{\varepsilon}{2} \int_{-1}^{1} X_2(q)\left(1 + \phi(s)\right) ds + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Accordingly,

$$\begin{aligned} T_1(q_2) &= \frac{1}{2} \int_{-1}^{1} X_2(q)\left(1 + \phi(s)\right) ds \\ &= \frac{X_2(q)}{2} \left(2 + \int_{-1}^{1} \phi(s) ds\right). \end{aligned}$$

Therefore,

$$\begin{aligned} S &= P_1(0) + T_1(q_2) \\ &= -X_2(q) + \frac{X_2(q)}{2} \left(2 + \int_{-1}^{1} \phi(s) ds\right) \\ &= \frac{X_2(q)}{2} \int_{-1}^{1} \phi(s) ds. \end{aligned}$$

6.2 Regularization of $\Sigma-$Polycycles of type (b)

In this section, we shall state and prove the fifth main result of this work, which in particular establishes sufficient conditions under which the regularized vector field Z_ε^Φ has a limit cycle Γ_ε converging to a $\Sigma-$polycycle of type (b). For this, suppose that a Filippov system $Z = (X^+, X^-)$ has a $\Sigma-$polycycle Γ of type (b). Without loss of generality, assume that:

(b.1) $X_1^+(p) > 0$;

(b.2) the trajectory of Z through p crosses Σ transversally $m-$times at $q_1, \cdots, q_m$, satisfying that for each $i = 1, \cdots, m$, there exists $t_i > 0$ such that $\varphi_Z(t_i, q_i) = q_{i+1}$, where $q_{m+1} = p$. Moreover, $\Gamma \cap \Sigma = \{q_1, \cdots, q_m, p\}$.

We shall also assume that

(b.3) $W_{\pitchfork}^s(p) \cup W_t^u(p) \subset \Gamma$, i.e. $X^-h(p) > 0$.

The case $W_{\pitchfork}^u(p) \cup W_t^s(p) \subset \Gamma$ is obtained from the previous case multiplying the vector field Z by -1 (see Remark 10 bellow).

Since $X^-h(p) > 0$, by the *Tubular Flow Theorem* for X^- at $p = (0,0)$, there exist an open set U and a local $\mathcal{C}^{2k}$ diffeomorphism ψ defined on U such that $\widetilde{X}^- = \psi_* X^- = (0,1)$. Clearly, the transformed vector field $\widetilde{X}^+ = \psi_* X^+$ still has a visible $2k$-multiplicity contact with Σ at $p = (0,0)$ and $\psi(\Sigma) = \Sigma$. Thus, without loss of generality, we can assume that there exists a neighborhood $U \subset \mathbb{R}^2$ of $p = (0,0)$ such that $X^-\big|_U = (0,1)$.

Notice that assumption $(b.2)$ above guarantees the existence of an exterior map D associated to Z. In what follows, we characterize such a map. Since $X_1^+(p), X_2^-(p) > 0$, then $X_1^+(x,y), X_2^-(x,y) > 0$, for all $(x,y) \in U$ (taken smaller if necessary). Take $\varepsilon, \theta > 0$ small enough in order that the points $q^u = (\theta, \overline{y}_\theta) \in W_t^u(p)$ and $q^s = (0, -\varepsilon) \in W_{\pitchfork}^s(p)$ are contained in U. Then there exist positive numbers $\delta^{u,s}$ such that

$$\begin{array}{l} \tau_t^u = \{(\theta, y) : y \in (\overline{y}_\theta - \delta^u, \overline{y}_\theta + \delta^u)\} \quad \textit{and} \\ \tau_{\pitchfork}^s = \{(x, -\varepsilon) : x \in (-\delta^s, \delta^s)\} \end{array} \tag{6.22}$$

are transversal sections of X^+ and X^-, respectively. In addition, $\sigma_p = [0, \theta] \times \{0\}$ is a transversal section of X^-. Moreover, by the *Tubular Flow Theorem* there exist the $C^{2k}-$diffeomorphism $T^s : \sigma_p \longrightarrow \tau_{\pitchfork}^s$ and $D : \tau_t^u \longrightarrow \tau_{\pitchfork}^s$ such that $T^s(p) = q^s$ and $D(q^u) = q^s$ (see Figure 29). Thus, expanding D around $y = \overline{y}_\theta$, we get

$$D(y) = r_{\theta,\varepsilon}(y - \overline{y}_\theta) + \mathcal{O}((y - \overline{y}_\theta)^2), \tag{6.23}$$

where $r_{\theta,\varepsilon} = \dfrac{dD}{dy}(\overline{y}_\theta)$.

Now, by the Definition 3, we know that there exists a first return map π_Γ defined, at least, in one side of Γ. In what follows, we shall see that there exist positive

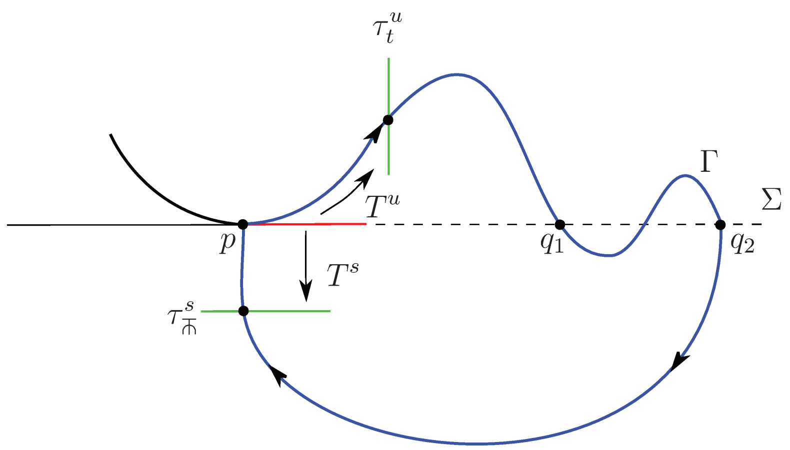

Figure 29 – Σ−polycycle Γ satisfying $(i) - (ii)$.

constants θ and K such that the first return map $\pi_\Gamma : \sigma_p \to \sigma_p$ is given by

$$\pi_\Gamma(x) = Kx^{2k} + \mathcal{O}(x^{2k+1}), \text{ where } K > 0. \tag{6.24}$$

Indeed, expanding T^s around $x = 0$, we have

$$T^s(x) = \kappa_\varepsilon^s x + \mathcal{O}(x^2), \tag{6.25}$$

with $\kappa_\varepsilon^s = \dfrac{dT^s}{dx}(0)$. In addition, using (6.23) and the *Implicit Function Theorem*, we get

$$D^{-1}(x) = \overline{y}_\theta + \frac{1}{r_{\theta,\varepsilon}} x + \mathcal{O}(x^2). \tag{6.26}$$

Even more, one can easily see that $r_{\theta,\varepsilon} < 0$ and $\kappa_\varepsilon^s > 0$ for all $\theta, \varepsilon > 0$ sufficiently small.

The same way, by Theorem A in [10] we know that there exists a transition map $T^u : \sigma_p \longrightarrow \tau_t^u$ defined as

$$T^u(x) = \overline{y}_\theta + \kappa_\theta^u x^{2k} + \mathcal{O}(x^{2k+1}), \tag{6.27}$$

where $\text{sign}(\kappa_\theta^u) = -\text{sign}((X^+)^{2k} h(p))$, i.e. $\kappa_\theta^u < 0$.

Thus, we can define the first return map $\pi_\Gamma : \sigma_p \longrightarrow \sigma_p$ by

$$\pi_\Gamma(x) = (D^{-1} \circ T^s)^{-1} \circ T^u(x).$$

From (6.25) and (6.26), we have that $D^{-1} \circ T^s(x) = \overline{y}_\theta + \dfrac{\kappa_\varepsilon^s}{r_{\theta,\varepsilon}} x + \mathcal{O}(x^2)$. Hence, using the *Implicit Function Theorem* we conclude that

$$(D^{-1} \circ T^s)^{-1}(x) = \frac{r_{\theta,\varepsilon}}{\kappa_\varepsilon^s}(x - \overline{y}_\theta) + \mathcal{O}((x - \overline{y}_\theta)^2).$$

Thus,

$$\pi_\Gamma(x) = \frac{r_{\theta,\varepsilon}\kappa_\theta^u}{\kappa_\varepsilon^s}x^{2k} + \mathcal{O}(x^{2k+1}). \tag{6.28}$$

Accordingly, taking $K := \dfrac{r_{\theta,\varepsilon}\kappa_\theta^u}{\kappa_\varepsilon^s} > 0$, we get (6.24).

Remark 9. *Notice that for $x \in [0,\theta]$ we have that $\pi_\Gamma(x) < x$, for some small $\theta > 0$. This means that Γ is always asymptotically stable provided that (b.3) is satisfied, i.e. $X^- h(p) > 0$. If $X^- h(p) < 0$, Γ would be unstable.*

Now, since D is a diffeomorphism induced by a regular orbit, we can easily see that the regularized system Z_ε^Φ also admits an exterior map given by

$$D_\varepsilon(y) = D(y) + \mathcal{O}(\varepsilon). \tag{6.29}$$

In what follows, we state our second main result of this chapter, which will be proven in Section 6.2.1.

Theorem E. *Consider a Filippov system $Z = (X^+, X^-)_\Sigma$ and assume that Z has a $\Sigma-$polycycle Γ of type (b) satisfying (b.1), (b.2), and (b.3). For $n \geqslant 2k-1$, let $\Phi \in C_{ST}^{n-1}$ be given as (1.5) and consider the regularized system Z_ε^Φ (1.4). Then the regularized system Z_ε^Φ (1.4) admits at least a limit cycle Γ_ε, for $\varepsilon > 0$ sufficiently small. Moreover, Γ_ε converges to Γ.*

Remark 10. *In Theorem E we are assuming (b.3), i.e. $W_t^u(p) \cup W_{\pitchfork}^s(p) \subset \Gamma$. If $W_{\pitchfork}^u(p) \cup W_t^s(p) \subset \Gamma$, Theorem E can be applied to $-Z$. Consequently, we get a limit cycle for Z.*

In order to prove this theorem, we shall first establish the relationship between the derivative of the first return map K and the derivative of exterior map $r_{\theta,\varepsilon}$ as follows.

Lemma 7. *Consider α, $r_{\theta,\varepsilon}$, and κ_θ^u given as in (2.1), (6.23), and (6.27), respectively. Then,*

i) $\displaystyle\lim_{\theta\to 0} \kappa_\theta^u = -\frac{\alpha}{2k}.$

ii) $\displaystyle\lim_{\theta,\varepsilon\to 0} r_{\theta,\varepsilon} = -\frac{2kK}{\alpha}.$

Proof. First, we prove the statement *i*). Indeed, using (6.27) and that $T^u(\theta) = 0$ for all $\theta > 0$ small enough, we have $\overline{y}_\theta + \kappa_\theta^u\theta^{2k} + \mathcal{O}(\theta^{2k+1}) = 0$. By Lemma 1 we know that $\overline{y}_\theta = \dfrac{\alpha\theta^{2k}}{2k} + \mathcal{O}(\theta^{2k+1})$, thus

$$\frac{\alpha\theta^{2k}}{2k} + \mathcal{O}(\theta^{2k+1}) + \kappa_\theta^u\theta^{2k} + \mathcal{O}(\theta^{2k+1}) = 0,$$

that is,

$$\frac{\alpha}{2k} + \kappa_\theta^u + \mathcal{O}(\theta) = 0,$$

consequently, when θ tends to 0 we conclude the result.

Now, we shall prove statement *ii*). For this, notice that using equations (6.24) and (6.28) we get for $\theta, \varepsilon > 0$ small enough that

$$r_{\theta,\varepsilon} = \frac{K\kappa_\varepsilon^s}{\kappa_\theta^u}.$$

To prove statement *ii*), we just need to prove that $\lim\limits_{\varepsilon\to 0} \kappa_\varepsilon^s = 1$, because in this case we have

$$\begin{aligned}\lim_{\theta,\varepsilon\to 0} r_{\theta,\varepsilon} &= \lim_{\theta,\varepsilon\to 0} \frac{K\kappa_\varepsilon^s}{\kappa_\theta^u} \\ &= K\frac{\lim\limits_{\varepsilon\to 0}\kappa_\varepsilon^s}{\lim\limits_{\theta\to 0}\kappa_\theta^u} \\ &= -\frac{2kK}{\alpha}.\end{aligned}$$

where we have used item *i*). In what follows, we shall prove that the flow of X^- induces $\mathcal{C}^{2k}$ map,

$$\lambda_\varepsilon^s : \sigma_p' \subset \sigma_p \longrightarrow \tau_{\overline{\text{ф}}}^s,$$

between the transversal sections defined in (6.22) and satisfying $\lambda_\varepsilon^s(0) = q^s$. Indeed, consider the function

$$\mu(t,x,\varepsilon) = \varphi_{X^-}^2(t,x,0) + \varepsilon, \text{ for } (x,0) \in \sigma_p, \ t \in I_{(0,y)},$$

where φ_{X^-} is the flow of X^- and $I_{(x,0)}$ is the maximal interval of existence of $t \mapsto \varphi_{X^-}(t,x,0)$. Since,

$$\mu(0,0,0) = 0 \text{ and}$$

$$\frac{\partial\mu}{\partial t}(0,0,0) = \frac{\partial\varphi_{X^-}^2}{\partial t}(0,0,0) = X_2^-(p) \neq 0,$$

by the *Implicit Function Theorem* there exist $\theta_0 > 0$ and smooth function $t(x,\varepsilon)$ with $(x,0) \in \sigma_p' := [0,\theta_0) \times \{0\} \subset \sigma_p$ and $\theta, \varepsilon > 0$ sufficiently small such that

$$t(0,0) = 0 \text{ and}$$

$$\mu(t(x,\varepsilon),x,\varepsilon) = 0,$$

i.e. $\varphi_{X^-}^2(t(x,\varepsilon),x,0) = -\varepsilon$. Thus, we can define the function

$$\lambda_\varepsilon^s(x) = \varphi_{X^-}^1(t(x,\varepsilon),x,0).$$

Notice that

$$\frac{d\lambda_\varepsilon^s}{dx}(0) = \frac{\partial\varphi_{X^-}^1}{\partial t}(t(0,\varepsilon),0,0)\frac{\partial t}{\partial x}(0,\varepsilon) + \frac{\partial\varphi_{X^-}^1}{\partial x}(t(0,\varepsilon),0,0).$$

Since,

$$\frac{\partial t}{\partial x}(0,0) = -\frac{\frac{\partial\varphi_{X^-}^2}{\partial x}(0,0,0)}{\frac{\partial\varphi_{X^-}^2}{\partial t}(0,0,0)},$$

and

$$\frac{\partial\varphi_{X^-}^1}{\partial x}(0,0,0) = 1, \ \frac{\partial\varphi_{X^-}^2}{\partial x}(0,0,0) = 0, \ \kappa_\varepsilon^s = \frac{dT^u\Big|_{\sigma_p'}}{dx}(0) = \frac{d\lambda_\varepsilon^s}{dx}(0),$$

we get that

$$\begin{aligned}\lim_{\varepsilon\to 0}\kappa^s_\varepsilon &= \lim_{\varepsilon\to 0}\frac{d\lambda^s_\varepsilon}{dx}(0)\\ &= \frac{\partial\varphi^1_{X^-}}{\partial t}(t(0,0),0,0)\frac{\partial t}{\partial x}(0,0)+\frac{\partial\varphi^1_{X^-}}{\partial x}(t(0,0),0,0)\\ &= 1.\end{aligned} \tag{6.30}$$

The result follows from (6.30). □

6.2.1 Proof of Theorem E

Let $\mathcal{U}$ be an open set such that $\Gamma\subset\mathcal{U}$. Since Γ is a $\Sigma-$polycycle of Z, then it is easy to see that there exists an open set $V\subset\mathcal{U}$ such that V has no singular points of Z^Φ_ε. Taking $(\delta,0)\in V$, we get that $(\delta,\varepsilon)\in V$, for $\varepsilon>0$ small enough.

Now, define the map $\widetilde{D}_\varepsilon(x)=\pi_\Gamma(x)+\mathcal{O}(\varepsilon)$, for all $x\in I^\varepsilon_\delta$, where I^ε_δ is a neighborhood of δ. By Remark 9 we know that $\pi_\Gamma(\delta)<\delta$, then we can conclude that $\widetilde{D}_\varepsilon(\delta)<\delta$, for $\varepsilon>0$ sufficiently small.

By Theorem B for $\theta=\overline{x}^+_\varepsilon$, there exist $\rho_0>0$, and constants $c,r,q>0$ such that for every $\rho\in(\varepsilon^\lambda,\rho_0]$, $\lambda\in(0,\lambda^*)$, and $\varepsilon>0$ sufficiently small, the flow of Z^Φ_ε defines a map L_ε between the transversal sections $\check{H}^\varepsilon_{\rho,\lambda}=[-\rho,-\varepsilon^\lambda]\times\{-\varepsilon\}$ and $\check{V}^\varepsilon_{\overline{x}^+_\varepsilon}=\{\overline{x}^+_\varepsilon\}\times[y^\varepsilon_{\overline{x}^+_\varepsilon}-re^{-\frac{c}{\varepsilon^q}},y^\varepsilon_{\overline{x}^+_\varepsilon}]$ satisfying

$$\begin{array}{rccl} L_\varepsilon: & \check{H}^\varepsilon_{\rho,\lambda} & \longrightarrow & \check{V}^\varepsilon_{\overline{x}^+_\varepsilon}\\ & x & \longmapsto & y^\varepsilon_{\overline{x}^+_\varepsilon}+\mathcal{O}(e^{-\frac{c}{\varepsilon^q}}),\end{array} \tag{6.31}$$

where $y^\varepsilon_{\overline{x}^+_\varepsilon}=\overline{y}_{\overline{x}^+_\varepsilon}+\varepsilon+\mathcal{O}(\varepsilon^{1+\frac{1}{2k}})+\mathcal{O}(\varepsilon^{1+\lambda^*})+\mathcal{O}(\varepsilon^{2k\lambda^*})$ (see Figure 30).

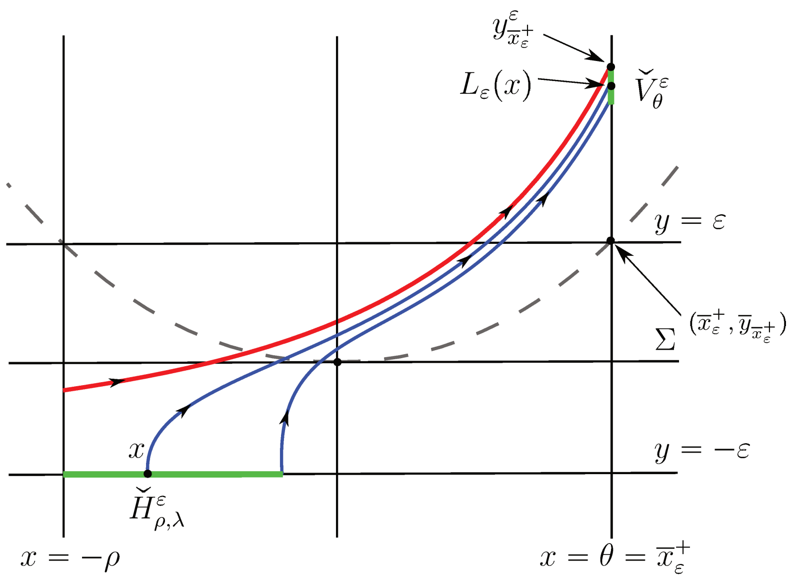

Figure 30 – Lower Transition Map L_ε of the regularized system Z^Φ_ε. The dotted curve is the trajectory of X^+ passing through the visible regular-tangential singularity of multiplicity $2k$ $p=(0,0)$. The red curve is the well-known Fenichel manifold.

Since $\check{V}^{\varepsilon}_{\overline{x}^+_{\varepsilon}} \subset \tau^u_t$, for $\varepsilon > 0$ sufficiently small, then we can consider the map $D_\varepsilon \circ L_\varepsilon : \check{H}^{\varepsilon}_{\rho,\lambda} \longrightarrow \tau^s_{\overline{\pitchfork}}$. We claim that $D_\varepsilon \circ L_\varepsilon(-\varepsilon^\lambda) > -\varepsilon^\lambda$. Indeed, from (6.29) and (6.31), we get

$$\begin{aligned} D_\varepsilon(L_\varepsilon(-\varepsilon^\lambda)) &= D_\varepsilon\Big(y^{\varepsilon}_{\overline{x}^+_{\varepsilon}} + \mathcal{O}(e^{-c/\varepsilon^q})\Big) \\ &= D\Big(\overline{y}_{\overline{x}^+_{\varepsilon}} + \varepsilon + \mathcal{O}(\varepsilon^{1+\frac{1}{2k}}) + \mathcal{O}(\varepsilon^{1+\lambda^*}) + \mathcal{O}(\varepsilon^{2k\lambda^*})\Big) + \mathcal{O}(\varepsilon) \\ &= r_{\overline{x}^+_{\varepsilon},\varepsilon}\varepsilon + \mathcal{O}(\varepsilon^{1+\frac{1}{2k}}) + \mathcal{O}(\varepsilon^{1+\lambda^*}) + \mathcal{O}(\varepsilon^{2k\lambda^*}) + \mathcal{O}(\varepsilon). \end{aligned} \tag{6.32}$$

Hence,

$$D_\varepsilon(L_\varepsilon(-\varepsilon^\lambda)) + \varepsilon^\lambda = \varepsilon^\lambda + r_{\overline{x}^+_{\varepsilon},\varepsilon}\varepsilon + \mathcal{O}(\varepsilon^{1+\frac{1}{2k}}) + \mathcal{O}(\varepsilon^{1+\lambda^*}) + \mathcal{O}(\varepsilon^{2k\lambda^*}) + \mathcal{O}(\varepsilon),$$

that is,

$$\begin{aligned} \frac{D_\varepsilon(L_\varepsilon(-\varepsilon^\lambda)) + \varepsilon^\lambda}{\varepsilon^\lambda} &= 1 + r_{\overline{x}^+_{\varepsilon},\varepsilon}\varepsilon^{1-\lambda} + \mathcal{O}(\varepsilon^{1+\frac{1}{2k}-\lambda}) + \mathcal{O}(\varepsilon^{1+\lambda^*-\lambda}) + \mathcal{O}(\varepsilon^{2k\lambda^*-\lambda}) \\ &\quad + \mathcal{O}(\varepsilon^{1-\lambda}). \end{aligned}$$

Using Lemma 7, we get

$$\lim_{\varepsilon \longrightarrow 0} \frac{D_\varepsilon(L_\varepsilon(-\varepsilon^\lambda)) + \varepsilon^\lambda}{\varepsilon^\lambda} = 1 > 0.$$

Since $\varepsilon > 0$, there exists $\varepsilon_0 > 0$ such that

$$0 < \varepsilon < \varepsilon_0 \implies D_\varepsilon(L_\varepsilon(-\varepsilon^\lambda)) + \varepsilon^\lambda > 0. \tag{6.33}$$

Hence, $D_\varepsilon \circ L_\varepsilon(-\varepsilon^\lambda) > -\varepsilon^\lambda$, for $\varepsilon > 0$ sufficiently small. Now, let $\mathcal{R}_\varepsilon$ be the region

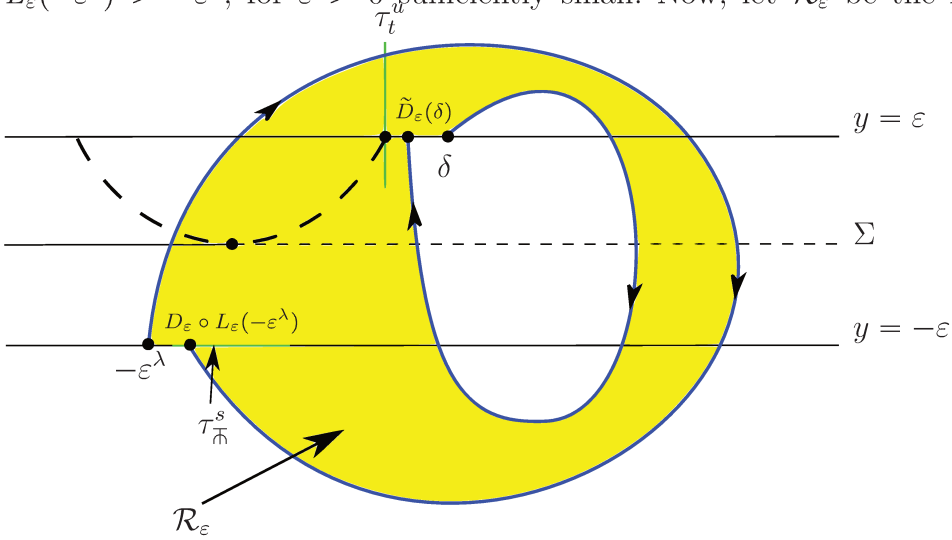

Figure 31 – The region $\mathcal{R}_\varepsilon$.

delimited by the curves $y = \varepsilon$, $y = -\varepsilon$ and the arc-orbits connecting (δ, ε) with $(\widetilde{D}_\varepsilon(\delta), \varepsilon)$ and $(-\varepsilon^\lambda, -\varepsilon)$ with $(D_\varepsilon \circ L_\varepsilon(-\varepsilon^\lambda), -\varepsilon)$, respectively (see Figure 31). It is easy to see that $\mathcal{R}_\varepsilon$ is positively invariant compact set of V, and has no singular points for $\varepsilon > 0$ small enough. For $\varepsilon > 0$ choose $q_\varepsilon \in \mathcal{R}_\varepsilon$ from the *Poincaré–Bendixson Theorem* $\Gamma_\varepsilon := \omega(q_\varepsilon) \subset \mathcal{R}_\varepsilon$ is a limit cycle of Z^{Φ}_{ε}. Furthermore, $\Gamma_\varepsilon \subset V \subset \mathcal{U}$ for $\varepsilon > 0$ small enough, hence Γ_ε converges to Γ.

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