Kaio Karam Galvão

"Developments of Fulkerson's Conjecture"

"*Desenvolvimentos da Conjetura de Fulkerson*"

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Kaio Karam Galvão

"Developments of Fulkerson's Conjecture"

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Orientador(a): **Profa. Dra. Christiane Neme Campos**

"Desenvolvimentos da Conjetura de Fulkerson"

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Developments of Fulkerson's Conjecture

Kaio Karam Galvão[1]

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Abstract

In 1971, Fulkerson proposed a conjecture that states that every bridgeless cubic graph has six perfect matchings such that each edge of the graph belongs to precisely two of these matchings. Fulkerson's Conjecture has been challenging researchers since its publication. It is easily verified for 3-edge-colourable cubic graphs. Therefore, the difficult task is to settle the conjecture for non-3-edge-colourable bridgeless cubic graphs, called snarks.

In this dissertation, Fulkerson's Conjecture and snarks are presented with emphasis in their history and remarkable results. We selected some results related to Fulkerson's Conjecture, emphasizing their reach and connections with other conjectures. It is also presented a brief history of the Four-Colour Problem and its connections with snarks.

In the second part of this work, we verify Fulkerson's Conjecture for some infinite families of snarks constructed with Loupekine's method using subgraphs of the Petersen Graph. More specifically, we first show that the family of LP_0-snarks satisfies Fulkerson's Conjecture. Then, we generalise this result by proving that Fulkerson's Conjecture holds for the broader family of LP_1-snarks. We also extend these results to even more general Loupekine Snarks constructed with subgraphs of snarks other than the Petersen Graph.

Resumo

Em 1971, Fulkerson propôs a seguinte conjetura: todo grafo cúbico sem arestas de corte admite seis emparelhamentos perfeitos tais que cada aresta do grafo pertence a exatamente dois destes emparelhamentos. A Conjetura de Fulkerson tem desafiado pesquisadores desde sua publicação. Esta conjetura é facilmente verificada para grafos cúbicos 3-aresta-coloráveis. Portanto, a dificuldade do problema reside em estabelecer a conjetura para grafos cúbicos sem arestas de corte que não possuem 3-coloração de arestas. Estes grafos são chamados snarks.

Nesta dissertação, a Conjetura de Fulkerson e os snarks são introduzidos com ênfase em sua história e resultados mais relevantes. Alguns resultados relacionados à Conjetura de Fulkerson são apresentados, enfatizando suas conexões com outras conjeturas. Um breve histórico do Problema das Quatro Cores e suas relações com snarks também são apresentados.

Na segunda parte deste trabalho, a Conjetura de Fulkerson é verificada para algumas famílias infinitas de snarks construídas com o método de Loupekine, utilizando subgrafos do Grafo de Petersen. Primeiramente, mostramos que a família dos LP_0-snarks satisfaz a Conjetura de Fulkerson. Em seguida, generalizamos este resultado para a família mais abrangente dos LP_1-snarks. Além disto, estendemos estes resultados para Snarks de Loupekine construídos com subgrafos de snarks diferentes do Grafo de Petersen.

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"To myself I am only a child playing
on the beach, while vast oceans of truth
lie undiscovered before me."

Isaac Newton

Contents

Chapter 1

Introduction

> The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.
>
> G. H. Hardy

Graph Theory is a very fertile field of mathematics. Graphs are flexible structures for representing binary relations on an arbitrary set of objects. This characteristic allows graphs to be used in modeling many real-life situations and systems. Graph Theory presents an enormous diversity of problems to be explored. The great intellectual appeal of the area, and why not say its aesthetics, have attracted many prominent researchers. Applicability of Graph Theory to practical problems also creates broad interest in professionals of other areas. Fulkerson's Conjecture, which is the main subject of this work, is a problem of primarily theoretical interest. Before presenting Fulkerson's Conjecture, it is necessary to introduce some basic concepts.

A graph G is an ordered triple $(V(G), E(G), \psi_G)$ composed of a nonempty set of *vertices* $V(G)$, a set of *edges* $E(G)$ disjoint from $V(G)$, and an *incidence function* ψ_G that associates each edge with an unordered pair of not necessarily distinct vertices. A graph is *finite* if $V(G)$ and $E(G)$ are finite sets. In this text, all graphs are finite. An *element* of G is either a vertex or an edge of G. Let e be an edge of G, and let u and v be vertices of G. If $\psi_G(e) = \{u, v\}$, then u and v are the *ends* of e. An edge is *incident* with its ends, and vice versa. Two vertices incident with a common edge are *adjacent*. Similarly, two edges incident with a common vertex are also *adjacent*. If $\psi_G(e) = \{u, u\} = \{u\}$, then

e is a *loop*. Distinct edges with the same ends are *multiple* or *parallel*. A graph with no loops and no multiple edges is a *simple graph*.

In this work, a *drawing* of a graph G is a graphical representation of G in the plane. In a drawing of a graph, each vertex is a different point of the plane, and each edge is a simple curve joining the points that represent its ends. Figure 1.1 shows drawings of some graphs. Except for the first graph, all graphs of the figure are simple graphs.

Let v be a vertex of G. The *degree* of v, denoted by $d_G(v)$, or simply $d(v)$, is the number of edges incident with v, loops counted twice. A graph is *k-regular* if all its vertices have degree equal to k. Sometimes, the term *valency* is used for degree and *k-valent* is used instead of k-regular. A 3-regular graph is also called a *cubic* graph. Observe that the last three graphs of Figure 1.1 are cubic. Cubic graphs became very important in the context of the Four-Colour Problem, as discussed in Chapter 2.

An edge e of G is a *cut edge* or *bridge* if there is a partition of $V(G)$ into two sets U and W such that e is the only edge with one end in U and one end in W. The leftmost graph of Figure 1.1 has a bridge.

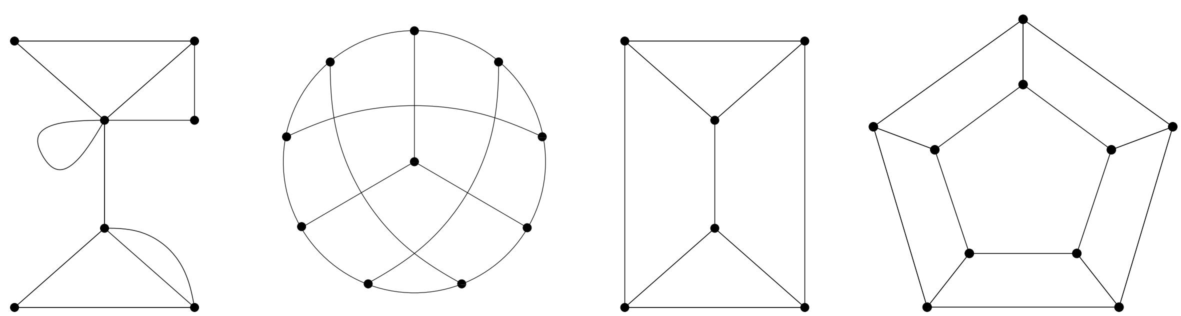


Figure 1.1: Examples of graphs. The first graph has a loop and parallel edges. The others are simple graphs.

A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. Graph G is said to *contain* graph H, and H is said to be *contained* in G. If $H \subseteq G$ and $V(H) = V(G)$, then H is a *spanning subgraph* of G. Moreover, if spanning subgraph H of G is k-regular, then H is a k-factor of G. A graph with vertex set $X \subseteq V(G)$ and edge set comprising every edge of G with both ends in X is an *induced subgraph* of G, denoted by $G[X]$.

A simple graph is a *path* if its vertices can be arranged in a (linear) sequence such that two vertices are adjacent if and only if they are consecutive in the sequence. The first and last vertices of the sequence are the *ends* of the path. If x and y are the ends of a path P, then P is an *xy-path* or a *path from x to y*. Moreover, the vertices of $V(P) \setminus \{x, y\}$ are the *internal vertices* of P. A *cycle* on three or more vertices is a simple graph such that its vertices can be arranged in a cyclic sequence with the same conditions as those of a path. A cycle on one vertex is comprised by a single vertex and a loop, and a cycle on two vertices consists of a pair of vertices joined by two parallel edges. The *length* of a path or a cycle is its number of edges. The *distance* between two vertices u and v of a graph G is the minimum length of a uv-path contained in G. The *girth* of G is the minimum length of a cycle contained in G.

A *matching* M of a graph G is a subset of $E(G)$ such that, for every pair $e, f \in M$, e and f are not adjacent. Set M is a *perfect matching* if every vertex of G is incident with an edge of M. Figure 1.2 exhibits some examples of matchings. Matchings have attracted continuous interest in Graph Theory. The first important result in the area, due to Julius Petersen [74], states that every bridgeless cubic graph has a perfect matching.

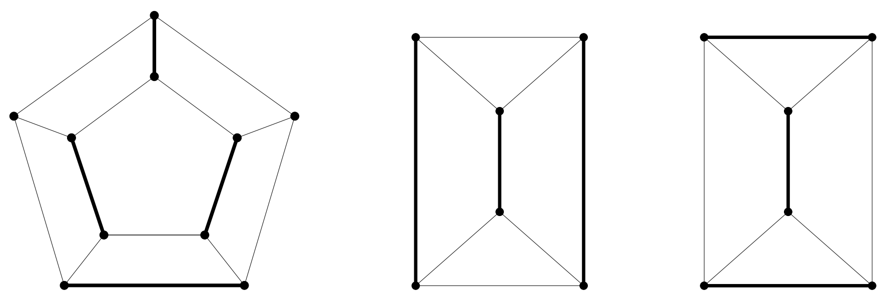


Figure 1.2: Examples of matchings. The two rightmost drawings show perfect matchings.

There is a class of Graph Theory problems that use the notion of colouring elements of a graph. Colouring problems are extremely common and there is a large variety of them. A *k-edge-colouring* of a graph G is an assignment of k *colours* to the edges of G such that every two adjacent edges receive distinct colours. The *chromatic index* of G, denoted by $\chi'(G)$, is the least number k for which G has a k-edge-colouring. Suppose an edge-colouring of G, and let c be one of its colours. The set of edges of G with colour c is a *colour class*. Note that every colour class is a matching. Figure 1.3 shows an example of an edge-colouring.

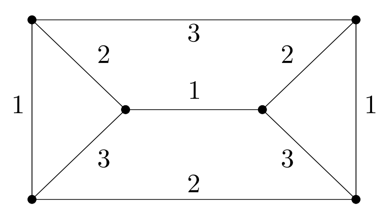


Figure 1.3: A 3-edge-colouring of a cubic graph on colours $1, 2, 3$.

Notice that the chromatic index of a cubic graph is at least three, since all its vertices have degree three. Moreover, in a cubic graph with a 3-edge-colouring, each colour class is a perfect matching.

A *cover* of a graph G is a family $\mathcal{F}$ of subgraphs of G such that

$$\bigcup_{H\in\mathcal{F}} E(F) = E(G).$$

If every subgraph of $\mathcal{F}$ is a path, then $\mathcal{F}$ is a *path cover*. Similarly, if every subgraph of $\mathcal{F}$ is a cycle, then $\mathcal{F}$ is a *cycle cover*. A cover of a graph G is *uniform* if each edge of G belongs to the same number of subgraphs of the cover. If this number is k, then the cover is a *k-cover*. A 1-cover, which is a cover such that any two of its subgraphs are edge-disjoint, is also called a *decomposition*. A 2-cover is also called a *double cover*. Given a cover $\mathcal{F}$, if $E(F)$ is a perfect matching for every $F \in \mathcal{F}$, then $\mathcal{F}$ is a *cover by perfect matchings*. The notion of cover is frequent in Graph Theory. For instance, the *Cycle Double Cover Conjecture* [44], independently formulated by George Szekeres [85] and Paul Seymour [82], is a famous unsolved problem. This conjecture states that every bridgeless graph has a double cover by cycles.

A 3-edge-colouring of a cubic graph is a decomposition by perfect matchings. A cubic graph G with $\chi'(G) > 3$ does not admit such a decomposition. However, every edge of a bridgeless cubic graph belongs to a perfect matching, as shown in Section 1.2. Hence, G admits a cover by perfect matchings. Generalising the fact that every bridgeless cubic graph with chromatic index three has a 1-cover by perfect matchings, it is natural to question whether every bridgeless cubic graph has a uniform cover by perfect matchings. A result due to Jack Edmonds [23], published in 1965, and a result due to Seymour [81], published in 1979, show that this question has positive answer. As an example, the Petersen Graph admits a double cover by perfect matchings, as shown in Figure 1.4.

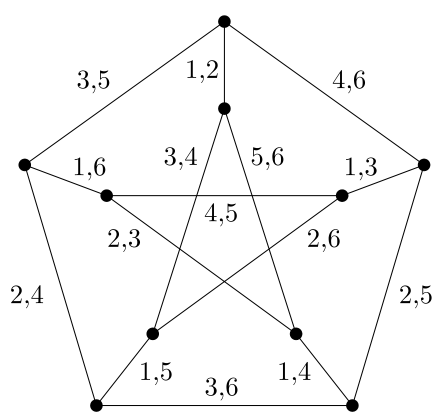


Figure 1.4: A double cover by perfect matchings. Each number represents a matching.

The above discussion leads to a conjecture published by Delbert Fulkerson [30] in the Mathematical Programming journal in 1971. *Fulkerson's Conjecture* states that every bridgeless cubic graph admits a double cover by six perfect matchings.

At the end of the 1970s, a few partial results on Fulkerson's Conjecture were achieved [16, 81]. In the 1980s and 1990s, the problem remained mostly unexplored. During the decade of 2000 to 2010, new partial results and several results on related conjectures were published [28, 38, 65, 71]. Some of these results are presented in Chapter 3. Due to its great difficulty, Fulkerson's Conjecture is one of the most celebrated open problems in Graph Theory.

In the remaining of this chapter, we give further definitions and results in Graph Theory, required for a deeper discussion of our problem. Definitions used and not given in this text can be found in the book *Graph Theory*, by Bondy and Murty [8]. At the end of this chapter, we present Fulkerson's Conjecture in greater detail. Fulkerson's Conjecture is restricted to a special class of graphs called snarks, which became important in the context of the Four-Colour Problem. Chapter 2 provides a brief presentation of the Four-Colour Problem, establishing the initial motivation for the study of snarks. It includes a short introduction to topological maps and planar graphs. Then, the first snarks are presented, followed by descriptions of other families and methods for the construction of snarks. Chapter 3 presents results, as well as other conjectures, related to Fulkerson's Conjecture. The results achieved in this work are described in Chapter 4, and final conclusions are the subject of Chapter 5.

1.1 Basic concepts

In this section, we introduce the definitions and results explicitly used in this text. The material presented here is basic and can be found in any textbook of Graph Theory.

Let $G = (V(G), E(G), \psi_G)$ be a graph. Whenever there is no ambiguity, $V(G)$ may also be denoted simply by V, $E(G)$ by E, and ψ_G by ψ. The *order* of G is the number of vertices of G, and the *size* of G is the number of edges of G. Let e be an edge of G with distinct ends u and v. Then, u and v are *neighbours* and e is said to *link* or *join* u and v.

If G is a simple graph, each edge of G is uniquely identified by its ends. Thus, an edge e with ends u and v is also denoted by uv or vu, leaving ψ_G implicit. Many problems in Graph Theory are solely concerned with or can be reduced to simple graphs. Fulkerson's Conjecture is an example of them.

The *maximum degree* of a graph G, denoted by $\Delta(G)$, is the number $\max\{d_G(v) \colon v \in V(G)\}$. The *minimum degree* of G, denoted by $\delta(G)$, is the number $\min\{d_G(v) \colon v \in V(G)\}$. If every vertex of G has even (odd) degree, G is an *even (odd) graph*.

The following theorems are simple yet fundamental results of Graph Theory, and are used in this text. They are presented here without proofs.

Theorem 1.1. *Let G be a graph. Then,*

$$\sum_{v \in V(G)} d(v) = 2|E(G)|. \qquad \square$$

Theorem 1.2. *Let G be a graph. If $\delta(G) \geq 2$, then G has a cycle.* $\square$

Theorem 1.3. *If a connected graph G has no cycles, then $|E(G)| = |V(G)| - 1$.* $\square$

Theorem 1.4. *An edge e of a graph G is a bridge if and only if e does not belong to a cycle of G.* $\square$

Let G be a graph and let $S \subseteq V(G)$. The graph obtained by deleting the vertices of S and all their incident edges from G is denoted by $G \setminus S$. If $S = \{v\}$, then $G \setminus S$ is also denoted by $G - v$. Similarly, if $F \subseteq E(G)$, then $G \setminus F$ is the graph obtained by removing all edges of F from G. If $F = \{e\}$, then $G \setminus F$ is also written as $G - e$. In order to *identify* nonadjacent vertices $u, v \in V(G)$, first remove them and add a vertex w. Then, for each loop of G incident with u or v, add a loop incident with w; and for each nonloop $e = xy$ of G, $x \in \{u, v\}$, add e' incident with w and y. Edges e and e' are *equivalent*. To *contract* a nonloop $e \in E(G)$, remove e and identify its ends. To *shrink* a set of vertices $X \subseteq V(G)$, remove every edge with both ends in X and then identify the vertices of X into a single vertex. Finally, to *suppress* a degree-two vertex $v \in V(G)$, remove v and add an edge incident with the two neighbours of v.

The set of all edges of G with exactly one end in $X \subseteq V(G)$ is an *edge cut* of G and is denoted by $\partial(X)$. Figure 1.5 shows examples of edge cuts. If $X = \{v\}$, then $\partial(X)$ is also denoted by $\partial(v)$, and it is a *trivial* edge cut. An edge cut with k edges is a *k-edge cut*. Observe that if $e \in E(G)$ is a bridge, then $\{e\}$ is an edge cut of G. A graph G is *connected* if $\partial(X) \neq \emptyset$ for every $X \subset V(G)$. A *connected component*, or simply a *component*, of G is a maximal connected subgraph of G. Note that if $\partial(X)$ is a nonempty edge cut of G, then $G \setminus \partial(X)$ has more components than G.

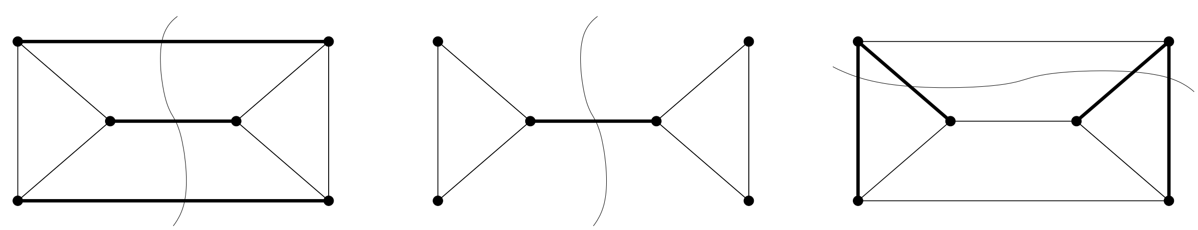


Figure 1.5: Examples of edge cuts.

Theorem 1.5. *Let G be an odd graph. If $X \subseteq V(G)$, then $|\partial(X)| \equiv |X| \pmod 2$.*

Proof. Let G and X be as stated in the hypothesis, and let $H = G[X]$. By Theorem 1.1, $\sum_{v \in V(H)} d_H(v) = 2|E(H)|$. This sum is equal to

$$\sum_{v \in X} d_G(v) - |\partial(X)|. \tag{1.1}$$

Because every vertex of X has odd degree, term $\sum_{v \in X} d_G(v)$ has the same parity as $|X|$. Since expression (1.1) is even, its two terms have the same parity. Henceforth, $|\partial(X)| \equiv |X| \pmod 2$. □

1.2 Matchings

Recall that a matching M of a graph G is a subset of pairwise nonadjacent edges of $E(G)$. The two ends of an edge of M are *matched* by M or *M-matched*. If a vertex v of G is incident with an edge of M, then v is *saturated* by M or *M-saturated*. Otherwise, v is *M-unsaturated*. Note that M is a perfect matching if every vertex of G is M-saturated. A matching M^* of G is *maximum* if for every matching M of G, $|M| \leq |M^*|$. Since each edge of M is incident with exactly two vertices, the number of vertices saturated by M is even and equal to $2|M|$. The following theorem relates matchings and edge cuts of cubic graphs.

Theorem 1.6. *Let G be a cubic graph and $X \subseteq V$. Let M be a matching of G such that every vertex of X is saturated by M. Then $|M \cap \partial(X)| \equiv |\partial(X)| \pmod 2$.*

Proof. Let G, X and M be defined as in the hypothesis. Let M_X be the set of edges of M with both ends in X. Thus, $|X| = 2|M_X| + |M \cap \partial(X)|$. Hence, $|X| \equiv |M \cap \partial(X)| \pmod 2$. By Theorem 1.5, $|X| \equiv |\partial(X)| \pmod 2$. Thus, $|M \cap \partial(X)| \equiv |\partial(X)| \pmod 2$. □

An *odd component* of a graph G is a connected component of G with an odd number of vertices. An *even component* is defined analogously. The number of odd components of G is denoted by $o(G)$. Graph G is *matchable* if it has a perfect matching, and G is *hypomatchable* if $G - v$ is matchable for every $v \in V(G)$. A vertex of a graph is *essential* if it is saturated by every maximum matching of the graph.

In this section, we present the characterisation of matchable graphs due to William Tutte [88]. The proof of Tutte's Theorem given here was taken from the book of Bondy and Murty [8]. Other interesting proofs of Tutte's Theorem are provided by László Lovász [64] and Ian Anderson [1].

Let M be a matching of G, and $S \subseteq V(G)$. If an odd component of graph $G \setminus S$ has all of its vertices saturated by M, then at least one vertex of the component is matched with a vertex of S. Moreover, at most $|S|$ vertices of $G \setminus S$ are matched with vertices of S. Therefore, if U denotes the set of M-unsaturated vertices of G, then for all $S \subseteq V(G)$,

$$|U| \geq o(G \setminus S) - |S|. \tag{1.2}$$

If equality holds in (1.2), i.e.,

$$|U| = o(G \setminus S) - |S|, \tag{1.3}$$

then M is a maximum matching and S is a *barrier* of G.

Lemma 1.7. *Let G be a graph with an essential vertex v. Let B be a barrier of $G - v$. Then, $B \cup \{v\}$ is a barrier of G.* □

Lemma 1.8. *Every connected graph without essential vertices is hypomatchable.* □

Lemma 1.9. *Every graph has a barrier.* □

Now we are ready to present Tutte's Theorem.

Theorem 1.10 (Tutte's Theorem)**.** *A graph G has a perfect matching if and only if*

$$o(G \setminus S) \leq |S| \text{ for all } S \subseteq V(G).$$

Proof. Suppose that G has a perfect matching. By (1.2), $o(G \setminus S) \leq |S|$ for all $S \subseteq V(G)$. Conversely, suppose that G has no perfect matching. By Lemma 1.9, G has a barrier B. Let M be a maximum matching of G. If U denotes the set of vertices of G not saturated by M, then $|U| = o(G \setminus B) - |B|$. Thus, $o(G \setminus B) = |B| + |U|$. Because M is not perfect, $|U|$ is greater than zero. Therefore, $o(G \setminus B) > |B|$ and the result follows. □

In 1891, Petersen [74] provided the first significant result on perfect matchings, which is presented here as a corollary of Tutte's Theorem.

Theorem 1.11 (Petersen's Theorem)**.** *Every bridgeless cubic graph has a perfect matching.*

Proof. Let G be a bridgeless cubic graph. By Tutte's Theorem, it is sufficient to show that $o(G \setminus S) \leq |S|$ for every $S \subseteq V(G)$. Let $S \subseteq V(G)$, and let $V_1, V_2, \ldots, V_k$ be the vertex sets of the odd components of $G \setminus S$. By Theorem 1.5, $|\partial(V_i)|$ is odd, $1 \leq i \leq k$. Since G is bridgeless, $|\partial(V_i)| \geq 3$. Also, $\partial(V_i) \subseteq \partial(S)$. Therefore, $|\partial(S)| \geq 3k$. Since each vertex of S has degree three, $|\partial(S)| \leq 3|S|$. Thus, $3o(G \setminus S) = 3k \leq |\partial(S)| \leq 3|S|$. □

Theorem 1.12. *Every bridge of a cubic graph belongs to every perfect matching of the graph.*

Proof. Let $e = uv$ be a bridge of a cubic graph G. Let M be a perfect matching of G. Consider graph $G - e$, and let G_u be the component of $G - e$ containing u. The sum of the degrees of all vertices of G_u is even. Since u is the only vertex of G_u with degree two, this sum is equal to $3(|V(G_u)| - 1) + 2$. Thus, $|V(G_u)|$ is odd. At least one vertex of G_u is matched with a vertex not in G_u. Therefore, $e \in M$, and the result follows. $\square$

Theorem 1.13. *Every edge of a bridgeless cubic graph belongs to a perfect matching.*

Proof. Let G be a bridgeless cubic graph, and let uv be an edge of G. It is sufficient to show that $G' = G \setminus \{u, v\}$ has a perfect matching M' since, in this case, $M' \cup \{uv\}$ is a perfect matching of G.

Suppose that G' does not have a perfect matching. By Tutte's Theorem, there exists $S' \subseteq V(G')$ such that $o(G' \setminus S') > |S'|$. Let $S = S' \cup \{u, v\}$. By Theorem 1.11, G has a perfect matching. This implies that $o(G \setminus S) \leq |S|$. Moreover, $o(G' \setminus S') = o(G \setminus S) = k$ since $G' \setminus S' = G \setminus S$. We conclude that

$$|S| - 2 < k \leq |S|. \tag{1.4}$$

By the same reasoning in the proof of Theorem 1.11, $3k \leq |\partial(S)|$. Moreover, $|\partial(S)| \leq 3|S| - 2$, since G is cubic, and $uv \in E(G)$. Therefore,

$$3k \leq |\partial(S)| \leq 3|S| - 2. \tag{1.5}$$

By (1.4) and (1.5), we conclude that $k = |S| - 1$ and $3|S| - 3 \leq |\partial(S)| \leq 3|S| - 2$. By Theorem 1.5, $|\partial(S)| \equiv |S| \pmod 2$, which implies that $|\partial(S)| \neq 3|S| - 3$. Therefore,

$$|\partial(S)| = 3|S| - 2.$$

Let $V_1, \ldots, V_k$ be the vertex sets of odd components of $G \setminus S$. By Theorem 1.5, each $|\partial(V_i)|$ is odd. Since G is bridgeless, each $|\partial(V_i)| \geq 3$. Suppose there exists V_i with $|\partial(V_i)| \geq 5$. Thus,

$$|\partial(S)| \geq |\bigcup_{i=1}^{k} \partial(V_i)| = \sum_{i=1}^{k} |\partial(V_i)| \geq 3(k-1) + 5 = 3k + 2 = 3|S| - 1,$$

contradicting the fact that $|\partial(S)| = 3|S| - 2$. Therefore, $|\partial(V_i)| = 3$ for every i and $|\partial(S)| - |\bigcup_{i=1}^{k} \partial(V_i)| = 1$. This implies that there exists exactly one edge of $\partial(S)$ connecting S and an even component of $G \setminus S$, which contradicts the fact that G is bridgeless. Therefore, G' has a perfect matching, and the result follows. $\square$

1.3 Graph colourings

Colouring problems are extraordinarily frequent in Graph Theory. Graph colourings were in great part motivated and developed due to the Four-Colour Problem. In this section, we present some basic definitions and a few fundamental results related to the idea of colouring vertices and edges of a graph.

A *vertex-colouring* of a graph G is an assignment of *colours* to the vertices of G such that adjacent vertices receive different colours. In other words, a vertex-colouring of a graph G is a mapping

$$\pi \colon V(G) \to \mathcal{C},$$

with $\mathcal{C}$ a set of colours, such that for all adjacent $u, v \in V(G)$, $\pi(u) \neq \pi(v)$. If $|\mathcal{C}| = k$, then π is a *k-vertex-colouring*. A graph is *k-vertex-colourable* if it has a k-vertex-colouring. Only loopless graphs admit vertex-colourings. Set $\mathcal{C}$ of colours is commonly taken as $\{1, \ldots, k\}$. A vertex-colouring of a graph G can be seen as a partition $\{V_1, \ldots, V_k\}$ of $V(G)$, where *colour class* V_i is the set of vertices assigned colour i. Vertices in the same colour class are pairwise nonadjacent.

The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum k for which G admits a k-vertex-colouring. If G is a cycle with odd order or a complete graph, then $\chi(G) = \Delta(G) + 1$. Brooks' Theorem [10] states that all other connected graphs have chromatic number bounded by $\Delta(G)$.

Recall that an edge-colouring of G is a mapping

$$\pi \colon E(G) \to \mathcal{C},$$

with $\mathcal{C}$ a set of colours, such that for all adjacent $e, f \in E(G)$, $\pi(e) \neq \pi(f)$. Moreover, recall that the chromatic index $\chi'(G)$ of G is the minimum k for which G is k-edge-colourable. Consider $u \in V(G)$, with $d(u) = \Delta(G)$. In any edge-colouring of G, the $\Delta(G)$ edges incident with u receive distinct colours. Therefore, $\chi'(G) \geq \Delta(G)$. Vizing's Theorem [90] gives the upper bound for the chromatic index of simple graphs.

Theorem 1.14 (Vizing's Theorem)**.** *For any simple graph G, $\chi'(G) \leq \Delta(G) + 1$.* □

A simple graph G is *Class 1* if $\chi'(G) = \Delta(G)$. Otherwise, $\chi'(G) = \Delta(G) + 1$, and G is *Class 2*. The *Petersen Graph*, depicted in Figure 1.6 with a 4-edge-colouring, is a very well known Class 2 cubic graph. It is the smallest bridgeless cubic simple graph with chromatic index greater than three.

Let $\{M_1, \ldots, M_k\}$ be a k-edge-colouring of a graph G. If G is k-regular, then each vertex of G is incident with an edge of matching M_i, with $1 \leq i \leq k$. Therefore, a k-edge-colouring of a k-regular graph is a partition of its edge set into k perfect matchings. The following lemma is due to Danilo Blanuša [7], Blanche Descartes [21], and Petersen [75].

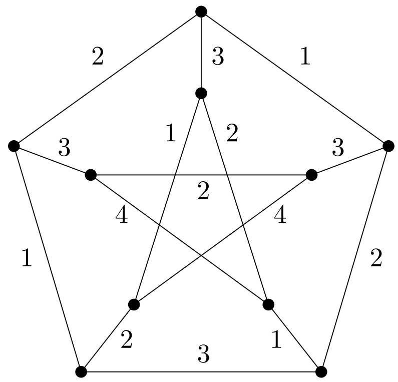


Figure 1.6: The Petersen Graph, with a 4-edge-colouring.

It is very useful in the construction of Class 2 cubic graphs, as shown in Chapter 2, and it is a consequence of Theorem 1.6.

Lemma 1.15 (The Parity Lemma)**.** *Let G be a cubic graph with a 3-edge-colouring π, and let $\partial(X)$, $X \subseteq V(G)$, be an edge cut of G. Denote by m_i the number of edges $e \in \partial(X)$ with $\pi(e) = i$. Then, $m_i \equiv |\partial(X)| \pmod 2$ for all i.*

Proof. Let G be a cubic graph with a 3-edge-colouring π. Consider an edge cut $\partial(X)$, $X \subseteq V(G)$. Set $M_i = \{e \in E(G) \colon \pi(e) = i\}$ is a perfect matching. Thus, M_i saturates every vertex of X, and $m_i = |M_i \cap \partial(X)|$. By Theorem 1.6, $m_i \equiv |\partial(X)| \pmod 2$. □

1.4 Fulkerson's Conjecture

As it was presented in the beginning of this chapter, a natural question which arises in the context of edge-colourings of regular graphs is: if a k-regular graph G has $\chi'(G) > k$, does G admit a uniform cover by perfect matchings? Or, in general: is it true that every k-regular graph admits a uniform cover by perfect matchings? The answer to this question is not positive in general, since there are regular graphs with odd number of vertices (e.g. cycles with odd order). However, the problem can be limited to regular odd graphs, since these graphs have even order.

The problem of determining whether every regular odd graph has a uniform cover by perfect matchings can be further restricted to cubic graphs. By Theorem 1.12, a bridge of a cubic graph belongs to every perfect matching of the graph. Thus, cubic graphs with a bridge do not have a cover by perfect matchings (and in particular are not 3-edge-colourable). On the other hand, Theorem 1.13 asserts that every edge of a bridgeless cubic graph belongs to a perfect matching. Henceforth, every bridgeless cubic graph has a cover by perfect matchings. As a corollary of a result due to Edmonds [23], every bridgeless cubic graph has a uniform cover by perfect matchings. A different form of Edmonds' result was stated and proved by Seymour [81], and it is discussed in Chapter 3.

This fact naturally leads to a quest for the least integer k such that every bridgeless cubic graph has a k-cover by perfect matchings. Observe that $k > 1$. Moreover, a k-cover by perfect matchings of a cubic graph is composed of exactly $3k$ perfect matchings.

Problem 1.16. *What is the least integer k such that every bridgeless cubic graph has a k-cover by perfect matchings?*

In 1971, Fulkerson [30] exhibited a double cover by six perfect matchings of the Petersen Graph (Figure 1.7) and posed the question whether every bridgeless cubic graph would have a double cover by six perfect matchings.[1]

Conjecture 1.17 (Fulkerson's Conjecture)**.** *Every bridgeless cubic graph admits a double cover by six perfect matchings.*

Seymour [81] claimed that this was first conjectured by Claude Berge, although it was first published by Fulkerson. For this reason, this conjecture is also called *Berge-Fulkerson Conjecture* [50]. It is a very challenging problem for which there are not many partial results [16, 28, 38, 65, 71, 81].

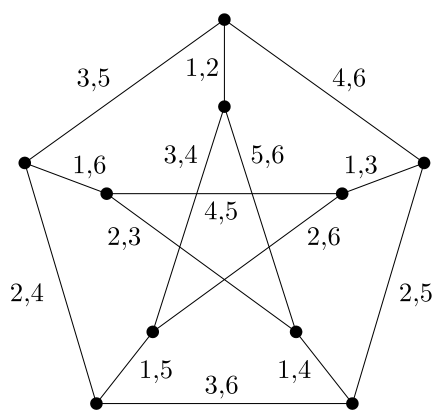


Figure 1.7: A double cover by six perfect matchings of the Petersen Graph.

Suppose G is a 3-edge-colourable cubic graph. Then, G has a 1-cover by three perfect matchings. A double cover by six perfect matchings of G can be obtained by duplicating each perfect matching of the 1-cover. Thus, it remains to verify Fulkerson's Conjecture for brigdeless cubic graphs with chromatic index greater than three. These graphs relate to various important problems in Graph Theory and have a special denomination: snarks. The interest for these graphs began in the context of the Four-Colour Problem. Snarks and the Four-Colour Problem deserve special attention and are introduced in Chapter 2.

[1]Bondy and Murty [8] argued that Fulkerson's enquiry was motivated by questions concerning the polyhedra defined by the incidence vectors of perfect matchings.

Chapter 2

The Four-Colour Problem and Snarks

Snarks are a special class of graphs related to important problems in Graph Theory. Snarks appeared and acquired great importance in the context of one of the most famous mathematical enigmas, the Four-Colour Problem. In its original version, the problem deals with colouring geographical maps. In a simple way, a geographical map can be seen as a drawing that partitions the plane into a number of non-overlapping regions (countries). A region is delimited by its frontier. Colouring a map involves attributing a colour to every region of the map. Figure 2.1 shows an example.

Figure 2.1: A colouring of the map of South America.

The *Four-Colour Problem* consists in determining whether four colours are sufficient to colour every map such that any two neighbouring regions, that is, two regions sharing a portion of their frontiers, have different colours. The assertion that four colours suffice became known as the *Four-Colour Conjecture*.

> **The Four-Colour Conjecture:** At most four colours are required to colour every map in a way that any two neighbouring regions have different colours.

Despite its very simple statement, the history of the Four-Colour Problem has made clear its great difficulty, with many frustrated attempts to solve it and several incorrect proofs. It puzzled mathematicians for over a century before it was settled. In 1879, Alfred Kempe [57] published the first wrong proof. In 1880, Peter Tait [87] provided another one of the most famous incorrect proofs of the conjecture. Although Tait's proof was incomplete, it contained a reduction of the problem to a new one: that of 3-edge-colouring bridgeless cubic graphs. His work gave origin to the term *Tait-colouring* (3-edge-colouring) of a cubic graph, as well as to the concept of edge-colouring. Tait's reduction motivated the search for bridgeless cubic graphs that do not admit a Tait-colouring, which were later called snarks. The Petersen Graph, depicted in Figure 2.2 with an optimal edge-colouring, is the smallest snark and was the first discovered. A more precise definition of snarks is given and discussed further in this chapter. Fulkerson's Conjecture, as well as the Cycle Double Cover Conjecture, Tutte's 5-Flow Conjecture, and the Petersen Colouring Conjecture, are some other important problems in Graph Theory related to snarks.

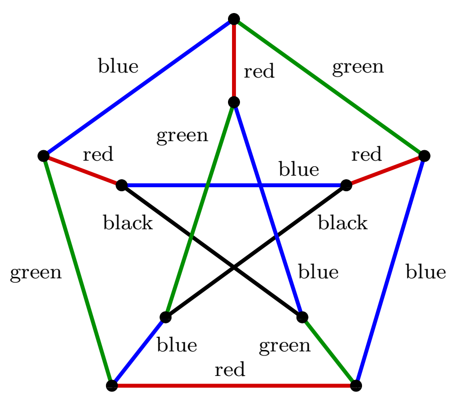


Figure 2.2: A 4-edge-colouring of the Petersen Graph, which is the smallest snark.

The origin of the Four-Colour Conjecture can be traced back to 1852, when it was posed by Francis Guthrie [35, 70], a former student of Augustus De Morgan. The problem became very popular in 1878, when Arthur Cayley [14] asked, at a meeting of the London Mathematical Society, whether a proof had been given to the statement that no more than four distinct colours are required when colouring any map [20, 70]. It was not solved until

1976, almost a hundred years after Cayley's enquiry, when Kenneth Appel and Wolfgang Haken [2] announced a computer-assisted proof of the Four-Colour Conjecture.

Appel and Haken's proof of the Four-Colour Conjecture was received with controversy. It was not fully accepted, due to its heavy usage of computational resources and the impossibility of manual verification in a reasonable time, among other reasons. Another proof, also supported by computers, yet simpler than the first, was published about twenty years later by Robertson, Sanders, Seymour, and Thomas [77]. Although the conjecture is now widely accepted as true, scientists continue to puzzle over the Four-Colour Problem, searching for a simpler and not computer-dependent proof. As an example, more recently a proof by Georges Gonthier [34] was published, although it is also supported by computers. Frank Harary [39] said that "the Four Color Conjecture can truly be renamed the 'Four Color Disease', for it exhibits so many properties of an infection. It is highly contagious."

This chapter starts with a short introduction to maps and plane graphs, including a few concepts, definitions, and fundamental results necessary for a precise discussion of the Four-Colour Problem. Section 2.2 briefly presents the early history of the problem, from its discovery by Guthrie until Tait's proof. Special attention is given to Kempe's wrong proof and Tait's reduction, since both gave rise to important concepts and techniques in Graph Theory. As Tait's work motivated the study of snarks, Section 2.3 describes the early discoveries of snarks, culminating in the discovery of the first infinite families. This section also contains a discussion of the meaning of a trivial snark. The last section presents some of the best-known methods of construction of snarks. There are good texts with more details on the motivation, history, and construction of snarks [13, 19, 93, 92]. Concerning the history and solution of the Four-Colour Problem, we refer to the books *Four-Colours Suffice: How the Map Problem was Solved*, by Robin Wilson [95], and *The Four-Color Problem: Assaults and Conquest*, by Thomas Saaty and Paul Kainen [80].

2.1 Maps and plane graphs

The primary aim of this section is to give a precise formulation of the Four-Colour Conjecture and to present it as a Graph Theory problem, as well as to show some of its Graph Theory equivalent formulations. Additional concepts and definitions are necessary for this task. The initial ideas emerge in a geometrical and topological context. Thus, we first give a mathematical definition based on the intuitive notion of geographical maps. We formalize the problem in this context, and then give a formulation in terms of graph-theoretical elements. In the topological part, we use mostly the ideas and notation from the book *The Four Color Theorem: History, Topological Foundations, and Idea of Proof* of Rudolf and Gerda Fritsch [29]. When changing to the graph-theoretical context, we use the notation introduced by Reinhard Diestel [22] in his book *Graph Theory*. We refer to these books and to the book *The Four-Color Problem* of Øystein Ore [73] for detailed coverings of the Four-Colour Problem and plane graphs.

Before giving a precise definition of a map, some concepts are required. A subset C of $\mathbb{R}^2$ is

(i) an *arc* if there exists an injective continuous mapping $\phi\colon [0,1] \to \mathbb{R}^2$ such that $C = \mathrm{Im}(\phi)$;

(ii) a *simple closed curve* if there exists a continuous mapping $\phi\colon [0,1] \to \mathbb{R}^2$ for which $C = \mathrm{Im}(\phi)$ and such that $\phi(0) = \phi(1)$ and the restriction of ϕ to $[0,1)$ is injective;

(iii) a *simple curve* if it is either an arc or a simple closed curve.

If C is an arc, then points $x = \phi(0)$ and $y = \phi(1)$ are the *endpoints* of C. Arc C *links* or *joins* x and y, and C is said to run *between* x and y. The *interior* of C is the set of points of C that are distinct from its endpoints. If C is a simple closed curve, then $\phi(0)$, which is the same as $\phi(1)$, is the only *endpoint* of C.

Let X be a subset of the plane. A point p is a *boundary point* of X if each disk centered at p has nonempty intersection with both X and $\mathbb{R}^2 \setminus X$. The *frontier* or *boundary* of X, denoted by $B(X)$, is the set of all boundary points of X. A point x is an *interior point* of X if it belongs to X and is not a boundary point of X. The set X is *open* if it contains none of its boundary points. In other words, for each point $x \in X$, there is a disk centered at x that is contained in X. The set X is *closed* if it contains all of its boundary points. The set X is *bounded* if, for some point $x \in X$, there exists a disk centered at x with finite radius which completely encompasses X. Otherwise, X is *unbounded*.

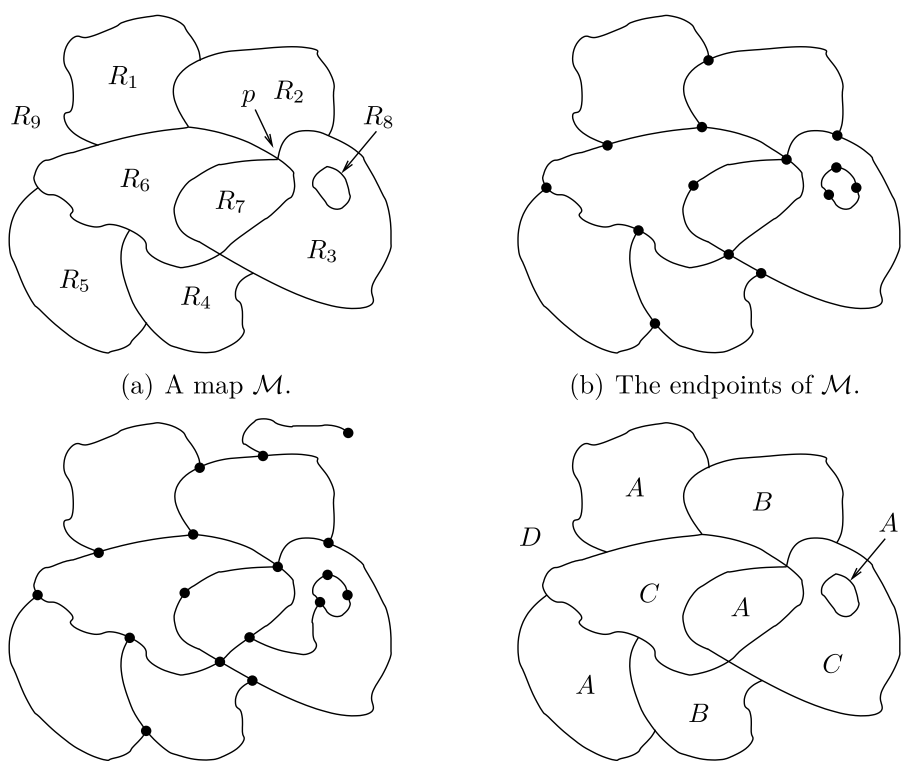
Let X be an open set. The boundary of X is contained in $\mathbb{R}^2 \setminus X$. The boundary of X is also the boundary of $\mathbb{R}^2 \setminus X$, and $\mathbb{R}^2 \setminus X$ is a closed set. Conversely, if X is a closed set, then $\mathbb{R}^2 \setminus X$ is an open set. A set $X \subseteq \mathbb{R}^2$ is *connected* if every pair of points of X is linked by an arc lying entirely in X. A set L is a *component* of X if L is a maximal nonempty connected subset of X.

In order to have a precise statement of the Four-Colour Conjecture, we formalize the notion of a simple geographical map.

> A *map* $\mathcal{M}$ is a finite set of arcs such that, for any distinct arcs C_1 and C_2, $C_1 \cap C_2$ is either empty or is a common endpoint of C_1 and C_2.

Every point of an arc C is a boundary point of C, and vice-versa. Thus, C is a closed set. The *border set* of $\mathcal{M}$, denoted by $C(\mathcal{M})$, is the union of all arcs of $\mathcal{M}$. The border set $C(\mathcal{M})$ is a closed set and thus the set $\mathbb{R}^2 \setminus C(\mathcal{M})$ is an open set. The components of $\mathbb{R}^2 \setminus C(\mathcal{M})$ are the *regions* or *countries* of the map $\mathcal{M}$. The frontier of a region of $\mathcal{M}$ is contained in $C(\mathcal{M})$. Since arcs are bounded and $\mathcal{M}$ is finite, $C(\mathcal{M})$ is a bounded set. A *corner* of $\mathcal{M}$ is a point of $C(\mathcal{M})$ which is an endpoint of at least three different arcs of $\mathcal{M}$. Figure 2.3(a) depicts a map, indicating a corner p which is an endpoint of four arcs.

The regions of this map are $R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8$, and the external unbounded region R_9. The endpoints of all arcs of the map are indicated in Figure 2.3(b). Notice that the boundary of region R_8 contains three arcs. This is necessary in order to comply with the definition of maps, which says that a map contains only arcs and two distinct arcs have at most one common endpoint. Given a border set, the map that originates this border set is not unique, since different sets of arcs may produce the same border set. In the example of Figure 2.3(b), the endpoints in the boundary of R_8 can been chosen differently, and yet define the same border set. However, all possibilities are essentially the same map.



(a) A map $\mathcal{M}$. (b) The endpoints of $\mathcal{M}$.

(c) A map with bridges. (d) A 4-colouring of $\mathcal{M}$.

Figure 2.3: An example of a map.

An arc contained in the frontier of precisely one region is a *bridge*. As an example, we add a bridge to the map of Figure 2.3(a) by adding an arc linking a point of $B(R_3) \cap B(R_7)$ to a point of $B(R_8)$. Another bridge is created by adding an arc linking a point of $B(R_2) \cap B(R_9)$ to a point of R_9. The result is shown in Figure 2.3(c). None of the added arcs divides a region of the map into smaller regions. Thus, the first arc is contained only in the new frontier of R_3, while the second arc is part only of the new frontier of R_9. Bridges do not make sense in geographical maps. Thus, throughout this chapter we assume that every map is bridgeless, unless otherwise stated. Note that in a bridgeless map every endpoint of an arc is also an endpoint of at least another arc.

Two distinct regions of a map $\mathcal{M}$ are *adjacent* if the intersection of their frontiers contains an arc. The set of all regions of $\mathcal{M}$ is denoted by $\mathcal{R}(\mathcal{M})$. A *colouring* of map $\mathcal{M}$ is a mapping $\varphi\colon \mathcal{R}(\mathcal{M}) \to \mathcal{C}$, with $\mathcal{C}$ a set of *colours*, such that any two adjacent regions have different colours. If $\mathcal{C}$ has k colours, then φ is a *k-colouring* of $\mathcal{M}$ and the map $\mathcal{M}$ is *k-colourable*. Figure 2.3(d) shows a 4-colouring of the map of Figure 2.3(a).

The Four-Colour Conjecture can be precisely enunciated as follows.

Conjecture 2.1 (The Four-Colour Conjecture)**.** *Every map is* 4*-colourable.*

A region R of a map is an *enclave* if its frontier is contained in the frontier of another region S of the map. Region S *encloses* region R. Thus, S is the only region adjacent to R. In Figure 2.3(a), region R_3 encloses region R_8. Let R be an enclave of a map $\mathcal{M}$ such that S encloses R. Let $\mathcal{L}$ be a map whose border set is $C(\mathcal{M}) \setminus B(R)$. In $\mathcal{L}$, the region R does not exist, and the region S is expanded to contain R. Suppose $\mathcal{L}$ has a 4-colouring and let c be the colour of S. We get a 4-colouring of $\mathcal{M}$ by extending the 4-colouring of $\mathcal{L}$ by simply assigning to R one of the three colours different from c. Thus, it is enough to consider maps without enclaves. Hence, throughout the remaining of this chapter, every map is also assumed to have no enclaves.

A *plane graph* is a pair (V, E), with V a finite set of *vertices* and E a finite set of *edges*, satisfying the following properties:

(i) $V \subseteq \mathbb{R}^2$;

(ii) every edge is either an arc between two vertices or a simple closed curve containing exactly one vertex (its only endpoint);

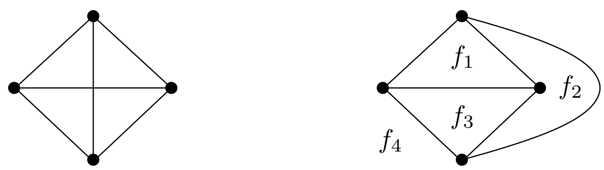
(iii) apart from its endpoints, an edge contains no vertex and no point of another edge.

A plane graph (V, E) defines an abstract graph G on V in a natural way. Terms and notation defined for graphs are also used for plane graphs. Following Diestel's notation, as long as no confusion arises, we use the name G of the abstract graph also for the plane graph (V, E).

A *simple plane graph* is a plane graph such that its abstract graph is simple. Note that in a simple plane graph (V, E), the edge set E has no simple closed curve and different edges have at most one common endpoint.

Let G be a plane graph. Since $G \subseteq \mathbb{R}^2$ is a union of a finite number of arcs and a finite number of points, the set G is closed and thus $\mathbb{R}^2 \setminus G$ is an open set. The components of $\mathbb{R}^2 \setminus G$ are the *faces* of the plane graph G. Each face of G is an open subset of $\mathbb{R}^2$. Since the vertex and edge sets of G are finite, and each edge is bounded, G is a bounded set. Hence, for a point $p \in G$, there is a disk D centered at p such that $G \subseteq D$. The only unbounded face of G is the face that contains $\mathbb{R}^2 \setminus D$. This is the *outer face* of G. The other faces of G are its *inner faces*. The set of faces of G is denoted by $F(G)$.

A graph is *embeddable in the plane*, or *planar*, if it has a drawing in the plane such that its edges intersect only at their ends. Such a drawing is a *planar embedding* of the graph. A planar embedding of a graph G can be regarded as a plane graph, with vertex set equal to the set of vertices of the drawing and edge set equal to the set of edges of the drawing. Figure 2.4 exhibits the complete graph on four vertices K_4 and one of its planar embeddings, with faces f_1, f_2, f_3, and f_4.



(a) (b)

Figure 2.4: (a) The planar graph K_4 and (b) one of its planar embeddings.

A face is *incident* with the edges and vertices in its boundary, and vice-versa. Let e be an edge of a plane graph G. When e is not a bridge, it is incident with precisely two faces. If e is a bridge, then it is in the boundary of only one face f. In this case, e is considered to be incident with f twice, and vice-versa. Figure 2.5 depicts a plane graph whose only bridge is incident twice with the outer face. The *degree* of a face f, denoted by $d(f)$, is the number of edges incident with f, bridges counted twice. For example, every face of Figure 2.4(b) has degree three. This is also true for every face of Figure 2.5, apart from the outer face, which has degree eight. The above definitions imply the following equality, similar to that stated by Theorem 1.1 for the vertex degrees of a graph:

$$\sum_{f \in F(G)} d(f) = 2|E(G)|. \tag{2.1}$$

Two faces are *adjacent* if they are incident with a common edge. Face f_2 of the plane graph of Figure 2.4(b) is adjacent to faces f_1, f_3, and f_4. A face incident with a bridge is adjacent to itself. The outer face of the plane graph of Figure 2.5 is adjacent to all other faces and to itself.

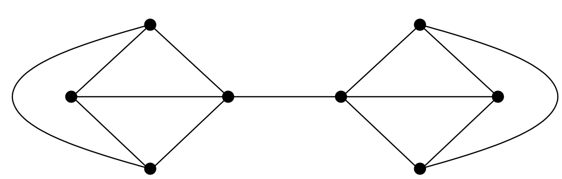


Figure 2.5: An example of a plane graph with a bridge.

A *face-colouring* of a plane graph G is an assigment of colours to the faces of G such that no two adjacent faces receive the same colour. That is, a face-colouring is a mapping $\pi\colon F(G) \to \mathcal{C}$, with $\mathcal{C}$ a set of *colours*, such that $\pi(f_1) \neq \pi(f_2)$ for all adjacent $f_1, f_2 \in F(G)$. If $|\mathcal{C}| = k$, then π is a *k-face-colouring*. A plane graph is *k-face-colourable* if it admits a k-face-colouring. If we regard f_1, f_2, f_3, and f_4 as colours, then the labelling of the faces of the plane graph of Figure 2.4(b) correspond to a 4-face-colouring. It is an optimal face-colouring, since each face is adjacent to all others.

A map $\mathcal{M}$ can be regarded as a simple plane graph in the following way. Let V be the set of points $p \in \mathbb{R}^2$ such that p is an endpoint of an arc of $\mathcal{M}$, and let E be the set of arcs of $\mathcal{M}$. By the definition of map, every arc of E has its endpoints in V, and two arcs do not intersect but in a possible common point of V. Therefore, $G = (V, E)$ is a simple plane graph, and it is the *associated plane graph* of map $\mathcal{M}$. Figure 2.3(b) exhibits the associated plane graph of the map of Figure 2.3(a). If we suppose that $\mathcal{M}$ contains bridges, then every bridge of $\mathcal{M}$ is a bridge in G. Recall that two arcs of a map have at most one common endpoint, and that the endpoints of an arc do not coincide. Thus, if we also suppose that $\mathcal{M}$ has enclaves, the frontier of each enclave is a 2-regular component of G. Because we assumed previously that maps have neither bridges nor enclaves, all associated plane graphs of a map are bridgeless and do not contain 2-regular components. Lastly, note that each face of G is a region of $\mathcal{M}$, and two faces of G are adjacent if and only if the two corresponding regions of $\mathcal{M}$ are adjacent.

In summary, given a map $\mathcal{M}$, a k-colouring of $\mathcal{M}$ naturally defines a k-face-colouring of its associated plane graph, and vice-versa. Hence, the Four-Colour Conjecture can be stated as a graph theoretical problem in terms of plane graphs and face-colourings. Furthermore, given a plane graph, the problem can be solved for each connected component separately.

Conjecture 2.2 (The Four-Colour Conjecture – Face version)**.** *Every connected bridgeless simple plane graph is* 4*-face-colourable.*

Conjecture 2.2 has an equivalent form for all connected bridgeless plane graphs, not necessarily simple. To see that, suppose Conjecture 2.2 is true and take a connected bridgeless plane graph G. Subdivision of all multiple edges and loops of G produces a connected bridgeless simple plane graph G'. This process preserves the faces of G, i.e., $F(G) = F(G')$, and the adjacency relation between them. Thus, a 4-face-colouring of G' naturally determines a 4-face-colouring of G. Conversely, if every connected bridgeless plane graph is 4-face-colourable, then Conjecture 2.2 is also true.

Conjecture 2.3. *Every connected bridgeless plane graph is* 4*-face-colourable.*

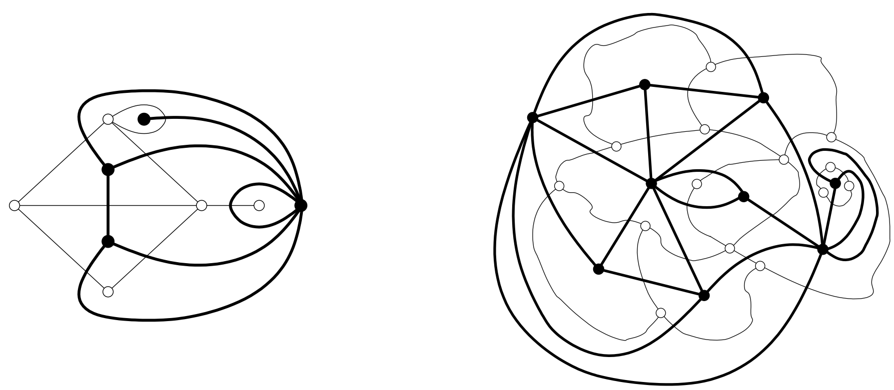
Additionally, we can consider only plane graphs with minimum degree at least three. To see that, let G be a connected bridgeless plane graph. Consider the suppression of all degree-two vertices of G. This operation produces a connected bridgeless plane graph

G' with minimum degree greater than two, while preserving the faces of G. Therefore, a 4-face-colouring of G' implies the existence of a 4-face-colouring of G.

Conjecture 2.4. *Every connected bridgeless plane graph with minimum degree at least three is* 4*-face-colourable.*

2.1.1 Dual graphs

Let G be a plane graph. The plane graph G^* is obtained as follows. For every face f of G, place a new vertex f^* inside f. For every edge e of G, if e is incident with two distinct faces f and g, then link vertices f^* and g^* by a new edge e^*. Otherwise, if e is incident with only one face f, then add a loop e^* containing f^*. Edge e^* must intersect just e and in a single point. The resulting plane graph G^* is a *plane dual* of G. Figure 2.6 shows examples of plane graphs together with their plane duals drawn in thick lines. The *dual* of a plane graph G is the abstract graph of a plane dual of G. To simplify notation, the dual of a plane graph G will also be denoted by G^*. Observe that, if G is connected and G^* is its plane dual, then G is also a plane dual of G^*.



(a) A plane graph and its plane dual. (b) Plane graph of Figure 2.3(b) and its dual.

Figure 2.6: Examples of plane duals.

Pairs of adjacent faces of a plane graph correspond to pairs of adjacent vertices in its dual. Therefore, a plane graph is 4-face-colourable if and only if its dual is 4-vertex-colourable. Let G be a plane graph with a dual G^*. Note that G has a bridge e if and only if e^* is a loop in G^*. This is illustrated in Figure 2.6(a). Therefore, the dual of a bridgeless plane graph is a loopless planar graph. For this reason, Conjecture 2.2 is also equivalent to the statement that every loopless planar graph is 4-vertex-colourable.

Conjecture 2.5 (The Four-Colour Conjecture – Vertex version)**.** *Every loopless planar graph is* 4*-vertex-colourable.*

2.1.2 Euler's Formula

For every connected plane graph, there is a simple relation between its number of vertices, edges, and faces. It was established by Euler and it is known as *Euler's Formula.* Its proof is given here as presented by Bondy and Murty [8].

Theorem 2.6 (Euler's Formula)**.** *If G is a connected plane graph, then*

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

Proof. By induction on the number of faces of G. If $|F(G)| = 1$, then G has no cycle. By Theorem 1.3, $|E(G)| = |V(G)| - 1$. Thus,

$$|V(G)| - |E(G)| + |F(G)| = |V(G)| - |V(G)| + 1 + |F(G)| = 2.$$

Let G be a connected plane graph with $|F(G)| \geq 2$. Therefore, graph G has at least one cycle. Let e be an edge of a cycle of G. By Theorem 1.4, e is not a bridge. Thus, $G - e$ is a connected graph. Moreover, by removing e from G, the two faces incident with e become one unique face. Therefore, $G - e$ has $|F(G)| - 1$ faces. By induction hypothesis, $|V(G - e)| - |E(G - e)| + |F(G - e)| = 2$. By construction,

$$|V(G - e)| = |V(G)|, \quad |E(G - e)| = |E(G)| - 1, \quad \text{and} \quad |F(G - e)| = |F(G)| - 1.$$

Therefore,

$$|V(G)| - |E(G)| + |F(G)| = |V(G - e)| - |E(G - e)| - 1 + |F(G - e)| + 1 = 2,$$

and the result follows. □

The next statement was proved by Kempe [57, 58], and used by him in an attempt to prove the Four-Colour Conjecture. Percy Heawood [40] also used it to prove the Five-Colour Theorem (Theorem 2.12). The proof presented here relies on Euler's Formula and is different from the proof given by Kempe.

Corollary 2.7. *Every connected bridgeless simple plane graph has a face adjacent to at most five other faces.*

Proof. First, we prove the statement for plane graphs with minimum degree at least three. Then, we show that this implies the result.

Let G be a plane graph with minimum degree at least three. This implies that $\sum_{v \in V(G)} d(v) \geq 3|V(G)|$. Then, by Theorem 1.1 we have that

$$2|E(G)| \geq 3|V(G)|. \tag{2.2}$$

Suppose G does not have a face adjacent to at most five other faces. Then, each face of G is adjacent to at least six other faces, and $\sum_{f \in F(G)} d(f) \geq 6|F(G)|$. Together with equality (2.1), it implies that

$$2|E(G)| \geq 6|F(G)|, \text{ and}$$
$$|F(G)| \leq |E(G)|/3.$$

Using this inequality and Euler's Formula, we have $2|E(G)| \leq 3|V(G)| - 6$, which implies that $2|E(G)| < 3|V(G)|$. This contradicts inequality (2.2). Therefore, G has at least one region adjacent to at most five regions.

Now, let G be a simple bridgeless connected plane graph. If G is 2-regular, then G has only two faces and the result follows. Suppose G is not 2-regular. Since G is bridgeless, it does not have vertices of degree one. Therefore, every vertex of G has degree at least two. By suppressing all degree-two vertices of G, we obtain a connected bridgeless plane graph G' with minimum degree at least three. Moreover, the face sets of G and G' are the same. The adjacency relation on the face sets is also the same in both graphs, and the result follows for G since it follows for G'. □

The following result is similar to the previous one, but it is applied to vertices of planar graphs instead of faces of plane graphs (in other words, it is a dual version of Corollary 2.7). It is useful when discussing the vertex version of the Four-Colour Conjecture.

Corollary 2.8. *Every simple planar graph has a vertex with degree at most five.* □

2.2 The Four-Colour Problem

The Four-Colour Problem is one of the mathematical problems that has received greatest attention from scientists. One of the reasons for its enormous appeal is its very simple statement, which can be well understood even by non-mathematicians, in spite of the great difficulty of solving it. Before its solution by Appel and Haken was announced, Harary [39] wrote the following humourous piece of text at the begining of Chapter 12 of his classic book *Graph Theory*.

> *The Four Color Conjecture (4CC) can truly be renamed the "Four Color Disease" for it exhibits so many properties of an infection. It is highly contagious. Some cases are benign and others malignant or chronic. There is no known vaccine, but men with a sufficiently strong constitution have achieved life-long immunity after a mild bout. It is recurrent and has been known to cause exquisite pain although there are no terminal cases on record. At least one case of the disease was transmitted from father to son, so it may be hereditary.*

A letter from De Morgan to Sir William Hamilton [70] on October 23, 1852, and a note by Frederick Guthrie [35] in the Proceedings of the Royal Society of Edinburgh in 1880 are the two most important sources of information about the origin of the Four-Colour Problem. They were analysed by Kenneth May [70] in his 1965 article. From these documents, it is known that, in 1852, Francis Guthrie showed his brother Frederick Guthrie that four is the greatest number of colours required to colour a map of England such that neighbouring counties have different colours. In his note, Frederick Guthrie says that his brother Francis provided a proof of this statement for arbitrary maps, which Francis himself did not find satisfactory. Francis was a former student of Professor De Morgan at University College, London, and Frederick was De Morgan's student at the time. Frederick reported to De Morgan the fact discovered by his brother. De Morgan then wrote a letter to Hamilton describing the problem and asking whether it had already been noticed by anyone. In the letter, De Morgan observes that, in order to draw four pairwise adjacent regions, thus forcing the use of four colours, one of the regions must be surrounded by the other three. This fact prevents a fifth region to have a common boundary with the enclosed one, thus making it impossible to draw five mutually adjacent regions. Hamilton replied to De Morgan's letter on October 16, 1852, stating no further interest in De Morgan's question.

Brendan McKay [72] discovered that the Four-Colour Problem was published on June 10, 1854, in a letter in the Miscellanea Section of *The Athenæum*. A photocopy of the letter appears in McKay's paper. It is a note of ten lines, where the author describes the problem, states to have found out empirically that four colours are necessary and sufficient, and also claims a proof of this fact. The letter is signed "F. G.", which is not identified in the magazine but most probably stands for Francis Guthrie. This is the earliest currently known publication of the Four-Colour Conjecture.

Another appearence of the conjecture in print was discovered by John Wilson [94]. It was in the April 14, 1860 issue of *The Athenæum*, in an anonymous review of William Whewell's book *The Philosophy of Discovery, Chapters Historical and Critical*. The author of the review outlines the Four-Colour Problem, claiming that the fact that four different colours are enough must have been always known to map-colourers. This claim is what most probably started the tradition that the Four-Colour Problem was already known among cartographers. However, this idea was pointed out by May [70] as having no foundation. John Wilson [94] and Robin Wilson [95] both identify De Morgan as the author of the anonymous review.

May [70] affirms that De Morgan was not successful in trying to attract other mathematicians to attempt a solution to the Four-Colour Conjecture. Also, De Morgan communicated the problem to his students and to several other mathematicians, and one of them brought the problem back almost thirty years later. Indeed, the problem remained forgotten until 1878, when Cayley [14], in a meeting of the London Mathematical Society, asked if anyone knew a proof for the conjecture. In 1879, Cayley [15] published a short article in the Proceedings of the Royal Geographical Society, explaining where the difficulty in proving the conjecture resides. More on Cayley's relation to the Four-Colour Problem can be found in the article *Arthur Cayley FRS and the four-colour map problem* by Tony Crilly [20]. After Cayley's enquiry, the problem gained popularity and, since then, many mathematicians have tried to solve it. A number of incorrect proofs appeared, some of them bringing important new concepts.

The first important wrong proof of the Four-Colour Conjecture was published by Kempe [57] in 1879, shortly after Cayley's article. A flaw in Kempe's proof was discovered only eleven years later, in 1890, by Heawood [40]. Although incorrect, Kempe's proof introduced an important technique, largely used in other proofs. Heawood himself used Kempe's idea to prove that five colours are sufficient to colour any map. Another important incorrect proof was published in 1890 by Tait [87]. As a consequence of Tait's work, the Four-Colour Problem was reduced to an edge-colouring problem. More details on Kempe's, Heawood's, and Tait's works are provided in the following sections.

The several attempts to solve the Four-Colour Problem motivated the development of new concepts and techniques in Graph Theory. For instance, the Four-Colour Conjecture was proved to be equivalent to other mathematical statements, many of them apparently unrelated to the subject of map colouring. A description of these statements can be found in the article *Thirteen colorful variations on Guthrie's Four-Color Conjecture*, by Saaty [79], as well as in the book *The Four-Color Problem: Assaults and Conquest*, coauthored by Saaty [80].

In 1976, almost a century after Cayley's enquiry at the London Mathematical Society meeting, a proof of the Four-Colour Conjecture was announced by Appel and Haken [2]. Appel and Haken's [3, 4] proof heavily relies on computational resources. This fact contributed to the lack of acceptance among mathematicians. In 1996, Robertson, Sanders, Seymour, and Thomas [77] announced another proof of the Four-Colour Conjecture, in great part motivated by the enormous difficulty of manual verification of Appel and Haken's proof. This new proof was published the year after its announcement [78] and was well received by the academic community. It uses the same approach as Appel and Haken's proof and is also computer assisted, but it is simpler. After these proofs, the Four-Colour Problem was finally considered settled. As an example of the continuous interest in the Four-Colour Problem, more recently, a new computer-based proof was given by Georges Gonthier [33, 34].

In summary, the Four-Colour Conjecture and its equivalent forms finally became theorems, as stated below.

Theorem 2.9 (The Four-Colour Theorem – Map version)**.** *Every map is* 4*-colourable.*

Theorem 2.10 (The Four-Colour Theorem – Face version)**.** *Every simple connected bridgeless plane graph is* 4*-face-colourable.*

Theorem 2.11 (The Four-Colour Theorem – Vertex version)**.** *Every loopless planar graph is* 4*-vertex-colourable.*

The next sections describe Kempe's proof, Heawood's exposition of its flaw, a reduction of the problem to cubic maps, and Tait's reduction to an edge-colouring problem.

2.2.1 Kempe's wrong proof and Heawood's counterexample

Kempe [57] published the first famous incorrect proof of the Four-Colour Conjecture in 1879, shortly after Cayley's note in the Proceedings of the London Mathematical Society. Although Kempe uses a process he called *patching*, his proof is essentially a mathematical induction. We present an adapted version of the proof, making the induction explicit, based on the work of Timothy Sipka [83]. Kempe's proof relies on Corollary 2.7 as an essential argument in the inductive step. In 1880, Kempe [58] published an article where he claims to present a simpler version of his proof. Both versions of Kempe's proof contain the same incorrect argument.

We start by describing an observation made by Kempe that plays an important role in his proof. Consider a map with a 4-colouring on colours red, green, blue and yellow. Select any two of these colours, say blue and yellow. Let R be a region coloured blue. There is a maximal closed connected subset X of the plane such that X is the union of regions coloured blue or yellow and their boundaries, and such that X contains R. The bicoloured set of all regions contained in X is a *blue-yellow chain*. Let S be a region coloured either blue or yellow, and not contained in X. Since X is maximally connected, S is not adjacent to any region of X. Thus, if we interchange the colours of the regions of X, by turning the blue regions yellow and the yellow regions blue, we end up with a different 4-colouring of the map, this time with R coloured yellow.

Kempe's proof is by induction on the number of regions of a map. All maps with at most four regions are 4-colourable. Let $\mathcal{M}$ be a map with n regions, where $n > 4$. Induction hypothesis states that every map with less than n regions has a 4-colouring. Let R be a region of $\mathcal{M}$ such that R is adjacent to at most five other regions. The existence of R is asserted by Corollary 2.7. Name these regions $R_1, \ldots, R_k$, with $k \leq 5$, in cyclic order as shown in Figure 2.7. Take a point v of R. For each corner y lying in the boundary of R, add an arc C linking v and y such that C do not intersect any other arc. Then, remove all arcs contained in the boundary of R. This process is illustrated in Figure 2.7. Denote the resulting map by $\mathcal{M}'$. In $\mathcal{M}'$, region R does not exist and each of regions $R_1, \ldots, R_k$ is expanded. All other regions are the same in both maps. Thus, $\mathcal{M}'$ has $n - 1$ regions.

By induction hypothesis, map $\mathcal{M}'$ has a 4-colouring. Suppose regions $R_1, \ldots, R_k$ together have at most three different colours. This is always true when R is adjacent to at most three regions. To construct a 4-colouring of map $\mathcal{M}$, transfer the colouring of $\mathcal{M}'$ to $\mathcal{M}$ by assigning the colour of each region of $\mathcal{M}'$ to the corresponding region of $\mathcal{M}$. There is at least one colour that is not assigned to any of $R_1, \ldots, R_k$. Assign this colour to R. The resulting assignment is a 4-colouring of map $\mathcal{M}$.

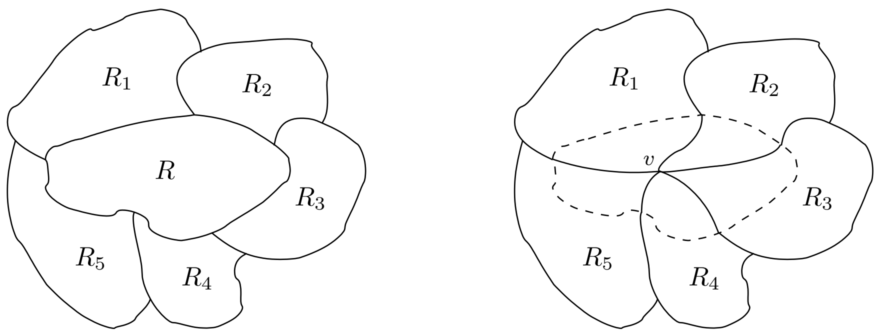


Figure 2.7: Reduction of the number of regions of a map by one.

Now suppose $R_1, \ldots, R_k$ have four different colours. Two cases must be considered.

<u>Case 1:</u> $k = 4$. Let the colours of $R_1, \ldots, R_4$ be red, blue, green, and yellow, respectively. Transfer the colouring of $\mathcal{M}'$ to $\mathcal{M}$, as illustrated in Figure 2.8(a). Suppose R_1 (red) and R_3 (green) are contained in different red-green chains. By interchanging the colours of the regions of the chain containing R_1, we end up with an assignment of colours where none of $R_1, \ldots, R_4$ has colour red. Then, it is enough to assign red to R. Now, suppose R_1 and R_3 are in the same red-green chain. If R_2 and R_4 are in the same blue-yellow chain, then this chain has a nonempty intersection with the red-green chain. But this is impossible, since the two chains use different colours. Thus, R_2 and R_4 are in distinct blue-yellow chains. Similarly to the previous case, we interchange the colours of the regions of one of these chains to obtain a 4-colouring for $\mathcal{M}$.

<u>Case 2:</u> $k = 5$. One colour is assigned to two of $R_1, \ldots, R_5$, while each of the other three colours appears in exactly one region. Transfer this colouring to $\mathcal{M}$, as shown in Figure 2.8(b). Suppose the colours are assigned to $R_1, \ldots, R_5$ as in the figure, with blue assigned to R_2 and R_5. If R_1 and R_4 are contained in different green-red chains, or R_1 and R_3 are contained in different green-yellow chains, we proceed as in Case 1. Thus, suppose that:

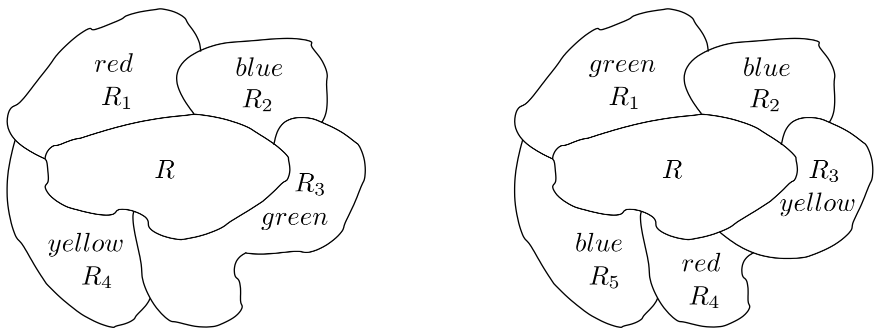
(i) R_1 and R_4 are in the same green-red chain, and

(ii) R_1 and R_3 are in the same green-yellow chain.

In this case, Kempe argumented that:

(iii) the green-red chain prevents R_3 and R_5 from being in the same blue-yellow chain,

(iv) the green-yellow chain prevents R_2 and R_4 from being in the same blue-red chain.



(a) Case 1. (b) Case 2.

Figure 2.8: The two cases of Kempe's proof.

Supposing this is always valid, interchange colours in the blue-red chain containing R_2 and interchange colours in the blue-yellow chain containing R_5. Thus, the regions adjacent to R are coloured green, red, and yellow, leaving colour blue available for R. This case completes the proof.

The process introduced by Kempe of interchanging the colours of regions in bicoloured chains is an important technique employed in many other proofs. This idea is used not only in colouring regions of a map, but also in colouring vertices and edges of a graph. The bicoloured chains are known as *Kempe chains*.

In Case 2, Kempe did not consider that the interchange of colours in one chain may modify another chain. In 1890, thus eleven years later, Heawood [40] exhibited a map in which Kempe's process does not work. Figure 2.9(a) shows the map used by Heawood. Notice that, in this map, the blue-red chain containing R_2 has a red region adjacent to a yellow region of the blue-yellow chain containing R_5. Suppose we interchange colours in the blue-red chain containing R_2. After that, as shown in Figure 2.9(b), the blue-yellow chain containing R_5 is modified, thus being the same blue-yellow chain containing R_3. By interchanging colours in this chain, region R will still be adjacent to a region coloured blue, thus not reducing the number of colours of $R_1, \ldots, R_5$. Another way of exposing the flaw in Kempe's proof is the following. Consider the blue-red chain containing R_2 and the blue-yellow chain containing R_5. Now, interchange colours in each of these chains. As a result, there will be two adjacent regions coloured blue (the regions above R_1).

Using Kempe's technique, Heawood also proved that five colours are sufficient to colour any map. Or, equivalently, that any loopless planar graph is 5-vertex-colourable. The proof presented here is due to Harary [39]. It shows an example of the use of Kempe chains in a vertex-colouring context.

Theorem 2.12 (The Five-Colour Theorem)**.** *Every loopless planar graph is* 5*-vertex-colourable.*

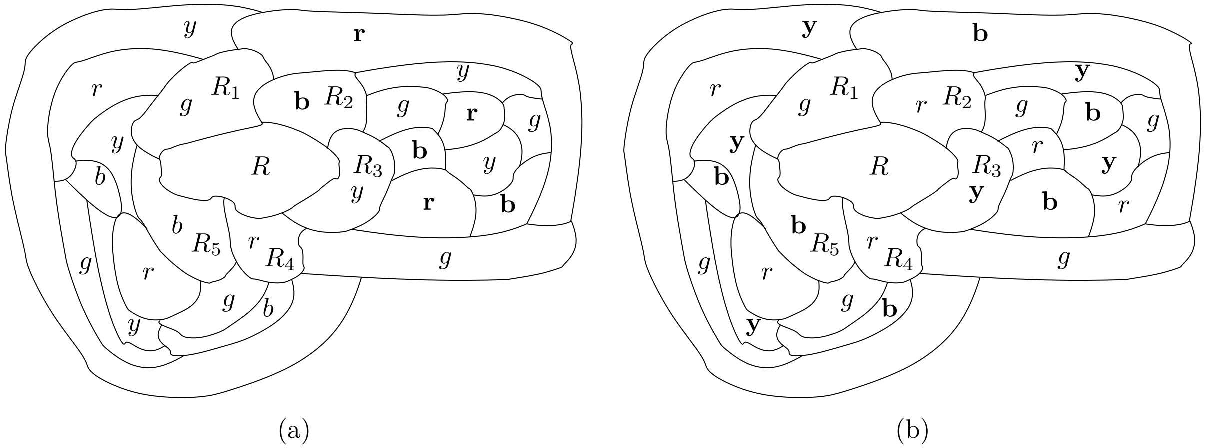


Figure 2.9: Heawood's map before (a) and after (b) interchanging colours in the blue-red chain containing R_2.

Proof. Note that it is enough to show that every simple planar graph is 5-vertex-colourable, since multiple edges do not play any special role in vertex-colourings. The proof is by induction on the number of vertices. The result trivially follows for every simple planar graph with at most five vertices.

The inductive hypothesis states that every simple planar graph with less than n vertices is 5-vertex-colourable. Let G be a simple planar graph with n vertices. By Corollary 2.8, G has a vertex v with degree at most five. Consider the planar graph $G-v$. By the induction hypothesis, $G-v$ has a 5-vertex-colouring. In this colouring, if some of the five colours is not assigned to any of the neighbours of v, then it is enough to assign this colour to v. Thus, we only have to consider the case when the degree of v is five and five colours are used for the neighbours of v. Let $\{1, 2, 3, 4, 5\}$ be the set of colours used. Label the vertices adjacent to v as v_1, v_2, v_3, v_4, and v_5 cyclically about v (see Figure 2.10(a)). Adjust the colours so that vertex v_i is assigned colour i, with $1 \leq i \leq 5$.

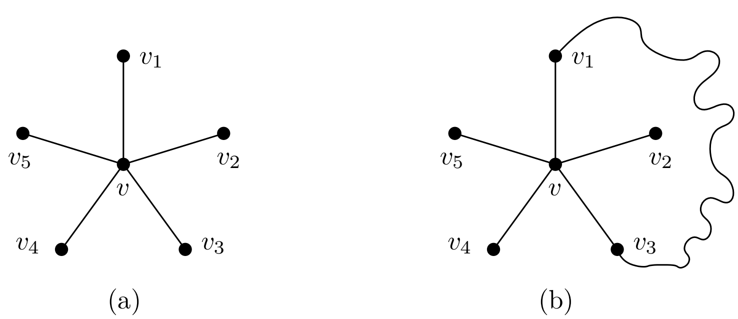


Figure 2.10: Steps in the proof of Five-Colour Theorem.

Let G_{13} be the subgraph of $G - v$ induced by the vertices assigned colours 1 or 3. Suppose v_1 and v_3 belong to different components of G_{13} (each component is a Kempe chain). Exchange the colours of the vertices of the component of G_{13} containing v_1. This results in a 5-vertex-colouring of $G - v$ such that no vertex adjacent to v is assigned colour 1. Extend this colouring by assigning 1 to v, resulting a 5-vertex-colouring of G.

On the other hand, suppose that v_1 and v_3 belong to the same component of G_{13}. Therefore, there exists a path in $G - v$ from v_1 to v_3 with each vertex assigned either colour 1 or 3 (see Figure 2.10(b)). This path, together with path $v_1 v v_3$, forms a cycle enclosing either both vertices v_4 and v_5, or vertex v_2. Let G_{24} be the subgraph of $G - v$ induced by the vertices assigned colour 2 or 4. Then, v_2 and v_4 do not belong to the same component of G_{24}. Interchange the colours of the vertices of the component of G_{24} that contains v_2. This produces a 5-vertex-colouring of $G - v$ with no neighbour of v assigned colour 2. Therefore, a 5-vertex-colouring of G is obtained by assigning colour 2 to v. □

2.2.2 Reduction to cubic plane graphs

Cayley [14] showed that the Four-Colour Problem can be reduced to cubic maps. Saaty [79] stated that William Story [84] used Kempe's work to show the same reduction. Here, we present the proof of this equivalence given in Harary's book *Graph Theory* [39].

Theorem 2.13. *Every bridgeless plane graph is* 4*-face-colourable if and only if every bridgeless cubic plane graph is* 4*-face-colourable.*

Proof. If every bridgeless plane graph is 4-face-colourable, then every bridgeless cubic plane graph is 4-face-colourable. It remains to prove the converse. To do that, consider the complete graph on four vertices K_4 and an edge $e \in E(K_4)$. Let $K_4' = K_4 - e$, as shown in Figure 2.11. Now, suppose that every bridgeless cubic plane graph is 4-face-colourable and let G be an arbitrary bridgeless plane graph. Since G is bridgeless, it has no vertex of degree one. For every vertex v of G such that $d(v) \neq 3$, we proceed as follows.

If $d(v) = 2$, let e and f be the edges incident with v. Subdivide e with a new vertex x and f with a new vertex y. Then, remove v, take a copy of K_4', and identify each x and y with a different degree-two vertex of K_4'. This operation is shown in Figure 2.12(a). We say that the vertices of K_4' added in the process are *associated* with v, and observe that all of them have degree three.

Now, suppose v has degree $d > 3$. Denote the edges incident with v cyclically by $e_1, \ldots, e_d$. Then, subdivide each e_i with a new vertex x_i, and remove v. We say that vertices $x_1, \ldots, x_d$ are associated with v. Finally, add d edges so as to have cycle $x_1 x_2 \ldots x_d x_1$. This process is illustrated in Figure 2.12(b) for $d = 5$. Observe again that every newly added vertex has degree three.

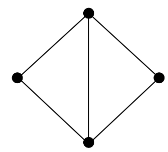
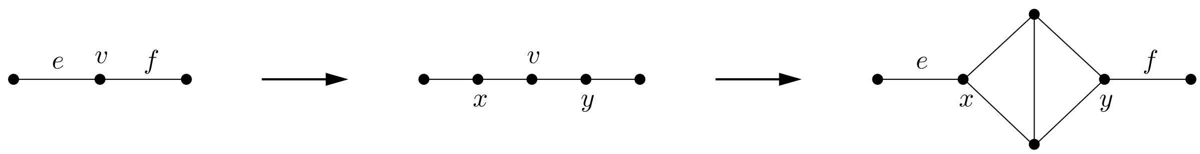
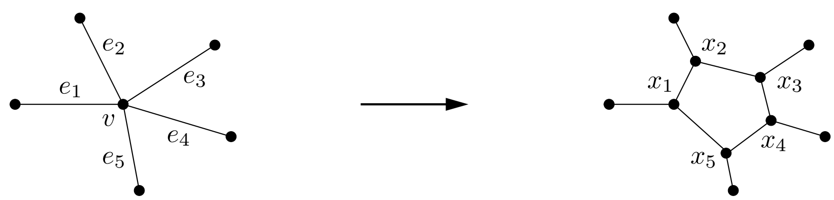


Figure 2.11: Graph $K_4 - e$, with $e \in E(K_4)$.



(a)



(b)

Figure 2.12: Conversion of a bridgeless plane graph into a cubic bridgeless plane graph.

After proceeding on G as described above, we end up with a bridgeless cubic plane graph G'. By hypothesis, G' has a 4-face-colouring. For every $v \in V(G)$ with $d_G(v) \neq 3$, shrink the vertices associated with v in G'. This process restores G from G'. Observe that the shrinking operation eliminates two faces of G' if $d_G(v) = 2$, or one face of G' if $d_G(v) > 3$. Moreover, if $d_G(v) = 2$, then two faces are expanded, which are the faces incident with v in G. These faces are adjacent in both G' and G. Otherwise, if $d_G(v) > 3$, then $d_G(v)$ faces get expanded, which are also the faces incident with v in G. Two of these faces are adjacent in G' if and only if they are adjacent in G. Thus, by restricting a 4-face-colouring of G' to the faces of G, we get a 4-face-colouring of G. □

2.2.3 Tait's reduction to edge-colouring

In the context of snarks, a work by Tait is of special interest. Tait was first told about the Four-Colour Conjecture by Cayley, and had his attention recalled to the problem after reading the article [58] published in 1880 containing Kempe's wrong proof of the Four-Colour Theorem. In the same year, Tait [86] published a note in the Proceedings of the Royal Society of Edinburgh containing what he claimed to be another proof of the Four-Colour Theorem, later shown incorrect. In this note, Tait observes that his process also shows that every bridgeless cubic plane graph is 3-edge-colourable. In another note published in 1880, Tait [87] shows that the statement that every bridgeless cubic plane graph is 3-edge-colourable follows from the Four-Colour Theorem. He also says that this statement is elementary and that its proof, without using the Four-Colour Theorem, is easily given by induction. In summary, by proving the theorem below, Tait thought he had given a new proof of the Four-Colour Theorem, simpler than Kempe's proof.

In the first part of the next proof, we use Tait's [87] technique so as to obtain an edge-colouring of a plane graph from a face-colouring of the same graph. In the second part of the proof, we do the reverse process: obtain a face-colouring from a given edge-colouring. However, we use a different set of colours, which makes the proof technically simpler. The second part of the proof is essentially the same presented by Harary [39].

Theorem 2.14 (Tait's Theorem)**.** *A bridgeless cubic plane graph is* 4*-face-colourable if and only if it is* 3*-edge-colourable.*

Proof. Let G be a bridgeless cubic plane graph. Suppose G has a 4-face-colouring π on colour set $\mathcal{C}$, with $\mathcal{C} = \{A, B, C, D\}$. Let $\mathcal{D} = \{\alpha, \beta, \gamma\}$ and let $t\colon \mathcal{C}\times\mathcal{C} \to \mathcal{D}$ be a mapping such that

$$t(A, B) = t(B, A) = t(C, D) = t(D, C) = \alpha;$$
$$t(A, C) = t(C, A) = t(B, D) = t(D, B) = \beta;$$
$$t(A, D) = t(D, A) = t(B, C) = t(C, B) = \gamma.$$

Let $\theta\colon E(G) \to \{\alpha, \beta, \gamma\}$ be defined as

$$\theta(e) = t(\pi(f), \pi(g)),$$

where f, g are the faces incident with e. We claim that θ is a 3-edge-colouring of G. To see that, let e_1 and e_2 be adjacent edges of G. Let f_i and g_i be the faces incident with e_i, for $i = 1, 2$. Since G is cubic, we can assume that $g_1 = g_2$. By construction,

$$\theta(e_1) = t(\pi(f_1), \pi(g_1)) \neq t(\pi(f_2), \pi(g_1)) = t(\pi(f_2), \pi(g_2)) = \theta(e_2).$$

Therefore, θ assigns distinct colours to e_1 and e_2.

Conversely, suppose G has a 3-edge-colouring $\theta\colon E(G) \to \mathcal{B}$, with $\mathcal{B} = \{(0,1), (1,0), (1,1)\}$. For $(p,q), (r,s) \in \mathcal{B} \cup \{(0,0)\}$, define

$$(p,q) + (r,s) = ((p+r) \bmod 2, (q+s) \bmod 2).$$

We construct a 4-face-colouring π of G as follows. Choose a face f_0 and make $\pi(f_0) = (0,0)$. For every $f \in F(G)$, take an arc $C_f \subseteq \mathbb{R}^2$ joining an interior point of f_0 and an interior point of f such that C_f does not contain any vertex of G. Let $e_0, \dots, e_k$ be the sequence of edges that intersect C_f. Set $\pi(f) = \sum_{i=0}^{k} \theta(e_i)$.

To prove that π is a 4-face-colouring, we need an intermediary result. Let C be a simple closed curve not containing a vertex of G. Suppose $e_0, \dots, e_k$ is the cyclic sequence of edges that intersect C. We show that $\sum_{i=0}^{k} \theta(e_i) = (0,0)$. To see this, consider the bounded region $R \subseteq \mathbb{R}^2$ with boundary C, and let $e_{k+1}, \dots, e_l$ be the edges contained in R. If $v \in V(G)$, with $\partial(v) = \{e_v^1, e_v^2, e_v^3\}$, then $h(v) = \theta(e_v^1)+\theta(e_v^2)+\theta(e_v^3) = (0,0)$. Denote by V_C the set of vertices contained in R. Therefore, $\sum_{v \in V_C} h(v) = (0,0)$. Notice that, in this sum, each edge contained in R is considered twice, and each edge that intersects C is considered once. Thus,

$$\begin{aligned}
\sum_{v \in V_C} h(v) &= [\theta(e_1) + \cdots + \theta(e_k)] \quad + \quad [(\theta(e_{k+1}) + \theta(e_{k+1}) + \cdots + \theta(e_l) + \theta(e_l)] \\
&= \theta(e_1) + \cdots + \theta(e_k),
\end{aligned}$$

since for every $b \in \mathcal{B}$, $b + b = (0,0)$. Therefore, $\theta(e_1) + \cdots + \theta(e_k) = (0,0)$, as claimed.

Take $f \in F(G)$. Let C be another arbitrary arc running between an interior point of f_0 and an interior point of f such that C does not contain any vertex of G. By what we just proved, the sum of the colours of the edges intersecting $C \cup C_f$ is $(0,0)$. Hence, the sum of the colours of the edges that intersect C is $\pi(f)$.

Now we are ready to show that π is a 4-face-colouring. Take adjacent $f_1, f_2 \in F(G)$, with $e \in E(G)$ incident with f_1 and f_2. We show that $\pi(f_1) \neq \pi(f_2)$. Let C be an arc between an interior point of f_0 and an interior point of f_2 such that C does not contain any vertex of G. Moreover, suppose e is the last edge that intersects C. Let $e_0, \dots, e_k$ be the sequence of edges that intersect C, with $e_k = e$. Observe that $\pi(f_1) = \sum_{i=1}^{k-1} \theta(e_i)$ and, thus, $\pi(f_2) = \pi(f_1) + \theta(e)$. Because $\theta(e) \neq (0,0)$, $\pi(f_1) \neq \pi(f_2)$. $\square$

Tait was incorrect when he affirmed that the statement that every bridgeless cubic plane graph is 3-edge-colourable can be easily proved. Actually, this is a problem as difficult as the Four-Colour Problem, since they are equivalent. Nevertheless, Tait revealed a very important equivalent formulation of the Four-Colour Conjecture. Furthermore, his ideas led to the concept of edge-colouring, which is largely used in Graph Theory.

Theorem 2.15. *Every bridgeless plane graph is* 4*-face-colourable if and only if every bridgeless cubic planar graph is* 3*-edge-colourable.*

Proof. By Tait's Theorem (Theorem 2.14) and by Theorem 2.13. $\square$

Thus, the Four-Colour Conjecture gained a new equivalent formulation.

Conjecture 2.16 (Four-Colour Conjecture – Edge version)**.** *Every bridgeless cubic planar graph is* 3*-edge-colourable.*

Finally, the Four-Colour Theorem can be enunciated in the same way.

Theorem 2.17 (Four-Colour Theorem – Edge version)**.** *Every bridgeless cubic planar graph is* 3*-edge-colourable.*

In face of this reduction, the existence of a bridgeless cubic planar graph that is not 3-edge-colourable would disprove the Four-Colour Conjecture. This motivated the pursuit of such a graph. To find a bridgeless cubic graph with chromatic index four was a very difficult task at the time. The next sections give more details on these special graphs, which are called snarks.

2.3 Early history of snarks

As described in the previous section, a work by Tait on the Four-Colour Conjecture showed that it is equivalent to the statement that every bridgeless cubic planar graph is 3-edge-colourable. This equivalence motivated a search for bridgeless cubic planar graphs with chromatic index four, since the existence of such a graph would disprove the Four-Colour Conjecture. On the other hand, a proof that every bridgeless cubic graph with chromatic index four is not planar would prove the conjecture. In 1948, Blanche Descartes [21] (a pseudonym used by mathematicians Rowland Brooks, Arthur Stone, Cedric Smith and William Tutte) wrote:

> *I wonder why problems about map-colourings are so fascinating? [sic] I know several people who have made more or less serious attempts to prove the Four-Colour Theorem, and I suppose many more have made collections of maps in the hope of hitting upon a counter-example. I like P. G. Tait's approach myself; he removed the problem from the plane so that it could be discussed in terms of more general figures. He showed that the Four-Colour Theorem is equivalent to the proposition that if N is a connected cubical network, without isthmus, in the plane, then the edges of N can be coloured in three colours so that the colours of the three meeting at any vertex are all different. (...) It was at first conjectured that every cubical network having no isthmus could be "three-coloured" in this way, but this was disproved (...).*
>
> *I have often tried to find other cubical networks which cannot be three-coloured. I do think that the right way to attack the Four-Colour Theorem is to classify the exceptions to Tait's Conjecture and see if any correspond to networks in the plane.*

The first discoveries of non-3-edge-colourable bridgeless cubic graphs were very few and sporadic. Rufus Isaacs [42] expressed the great difficulty he experimented when seeking these graphs. Motivated by this difficulty, Martin Gardner [31] proposed calling these graphs snarks. Gardner's inspiration came from Lewis Carrol's [12] nonsense poem *The Hunting of the Snark*, which describes the voyage of a crew chasing a fantastic, rare and unconceivable creature named snark. Lewis Carrol is the pen name of Charles Lutwidge Dodgson, an English writer and mathematician, best-known for his very famous book *Alice's Adventures in Wonderland*.

In order to avoid trivial cases, which are explained later in this chapter, Gardner defined a snark as follows.

> A *snark* is a bridgeless cubic graph with chromatic index four and without cycles of length two or three.

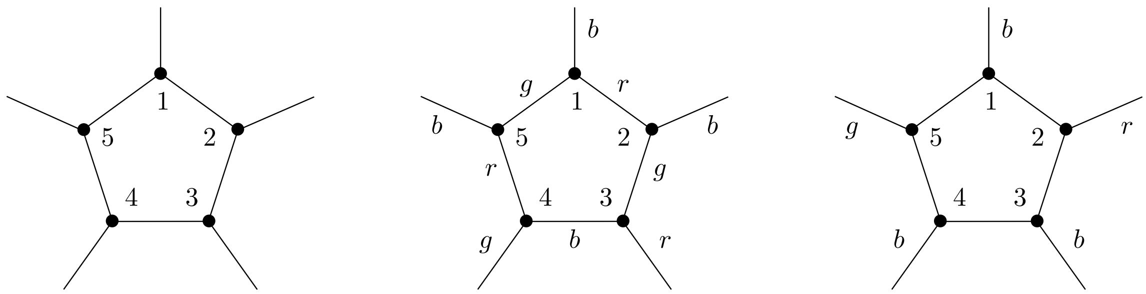
Besides having two-length and three-length cycles, other properties of cubic graphs were also considered trivial by Isaacs [42]. These properties are discussed in Section 2.3.2.

The next subsection presents the first four discoveries of snarks, namely: the Petersen Graph (1898), the Blanuša Snark (1946), the Descartes Snarks (1948), and the Szekeres Snark (1973). Whenever possible, we describe the constructions of these snarks provided by their discoverers. The first infinite families of snarks were found by Issacs [42] in 1975. One of them contains all snarks previously found. These families are presented in Section 2.3.3.

2.3.1 First discoveries

The first discovery of a snark was made by Petersen [75] in 1898. The snark found by him became known as the *Petersen Graph*, and it is usually denoted by P. It was the first counterexample to the conjecture that states that every bridgeless cubic graph is 3-edge-colourable, mentioned by Blanche Descartes [21]. The Petersen Graph had appeared before, although in a different context, in an 1886 article written by Kempe [59]. Here we describe its construction as given by Petersen.

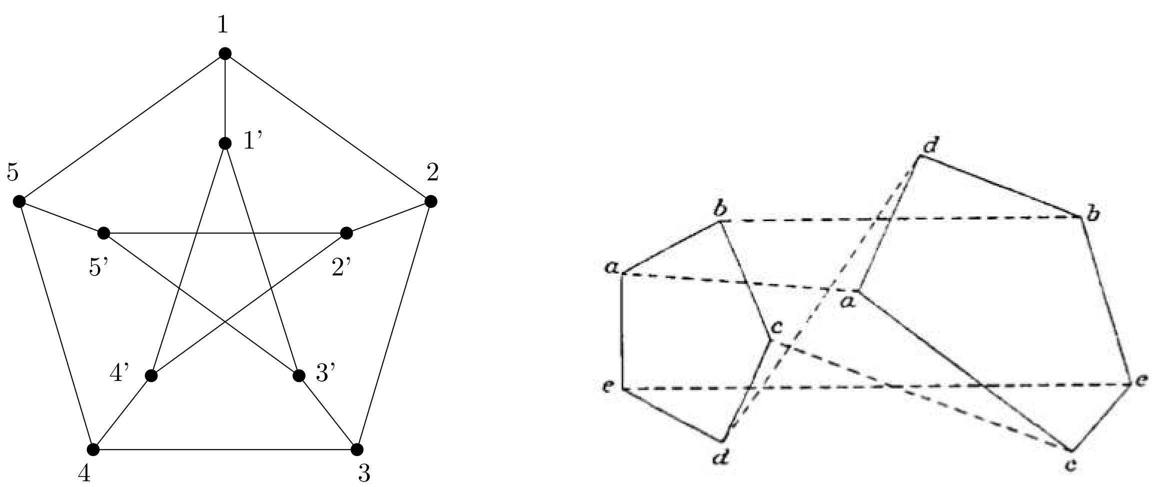
Petersen showed a result equivalent to the Parity Lemma (Lemma 1.15), which he used to construct the Petersen Graph. To see his construction, consider a cycle $C = 12345$, with each vertex also incident with an edge not in C, as shown in Figure 2.13(a). The edges not in C form an edge cut of a cubic graph G. Suppose G has a 3-edge-colouring on colours blue, red, and green. By the Parity Lemma, three edges of the edge cut are coloured blue, while one edge is coloured red and the last edge is coloured green. Thus, there are only two ways of colouring the edges of the edge cut: either the three blue edges are incident with consecutive vertices of C, or not. Figure 2.13(b) shows the first case, together with a 3-edge-colouring of C, while Figure 2.13(c) shows the second case, where a 3-edge-colouring of C is not possible. Therefore, in any 3-edge-colouring of G, the three edges with the same colour are always incident with three consecutive vertices of C.



(a) (b) (c)

Figure 2.13: Construction of the Petersen Graph.

Now suppose the remaining vertices of G induce another cycle $C' = 1'3'5'2'4'$. Moreover, the two cycles are joined by edges $11', 22', 33', 44'$, and $55'$, as in Figure 2.14. Assume that G has a 3-edge-colouring. The three edges of edge cut $\{11', 22', 33', 44', 55'\}$ which have the same colour are incident with consecutive vertices in cycle C. Without loss of generality, suppose that these vertices are 1, 2, and 3, that is, edges $11'$, $22'$, and $33'$ have the same colour. By construction of the graph, the edges with the same colour are not incident with consecutive vertices in cycle C', a contradiction.



(a) A common drawing. (b) Drawing from Petersen's note.

Figure 2.14: The Petersen Graph.

Figure 2.15 shows other drawings of the Petersen Graph. The depiction of Figure 2.15(a) is due to Kempe [59], while the drawing of Figure 2.15(c) is due to Danilo Blanuša [7]. Isaacs [42] gave a simple proof of the fact that the Petersen Graph is the smallest snark.

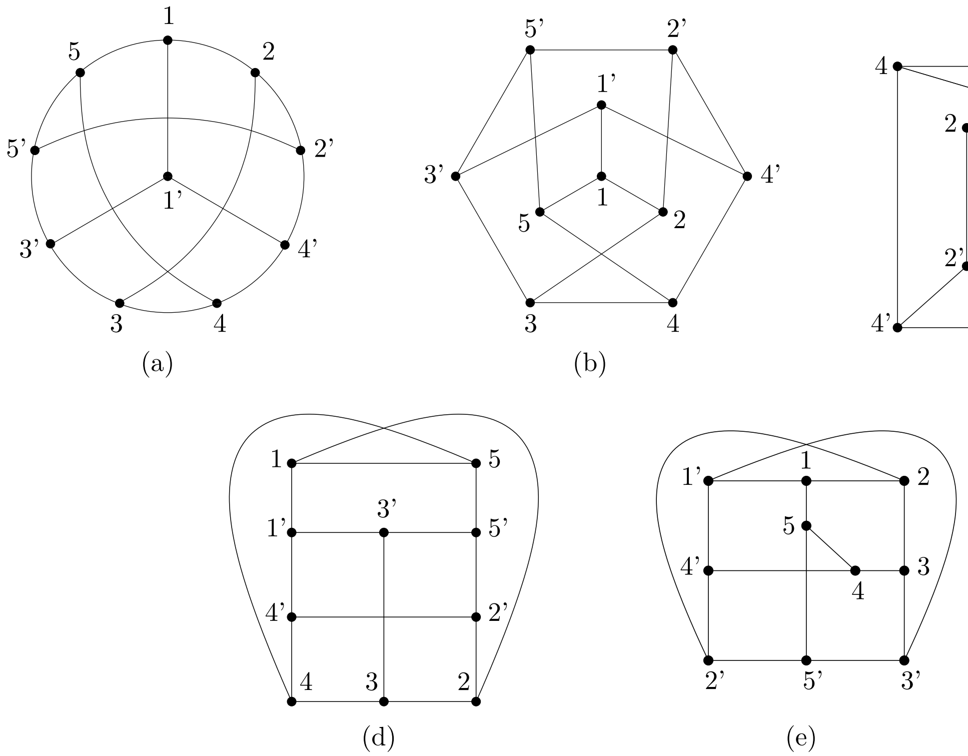


Figure 2.15: Other drawings of the Petersen Graph.

The second snark was found by Blanuša [7] in 1964 and is called *Blanuša Snark*. It has 18 vertices and two of its drawings are shown in Figure 2.16. It is clear from these figures that the Blanuša Snark can be obtained from two copies of the Petersen Graph. Blanuša observed that his graph has no cycles of length less than five and also that it does not have nontrivial edge cuts with less than four edges. These observations anticipate the concerns about the additional properties considered trivial by Isaacs [42] when searching for non-Tait-colourable graphs.

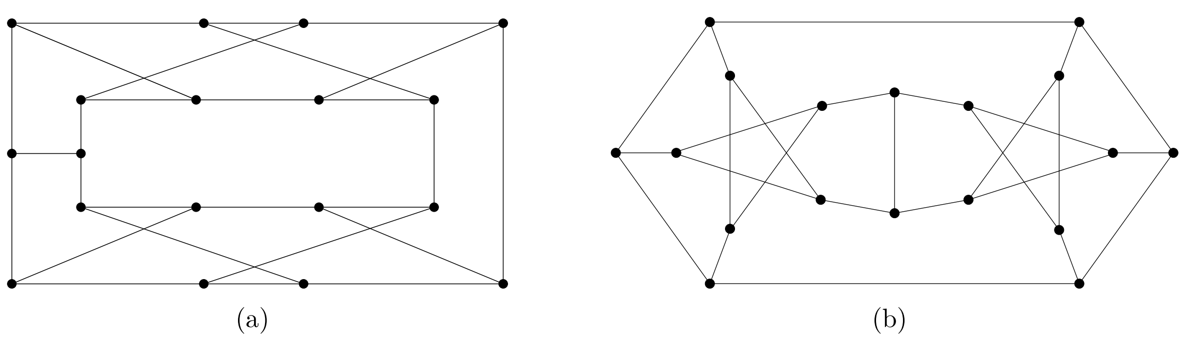


Figure 2.16: The Blanuša Snark (a) as depicted in Blanuša's article and (b) in one of its common drawings.

The proof that the Blanuša Snark is not 3-edge-colourable is given in Section 2.3.3. An operation known as dot product is shown to generate non-3-edge-colourable cubic graphs. The Blanuša Snark is a particular case of graphs obtained from this operation.

In the article quoted in the beginning of this section, Descartes [21] described a trivial modification performed on the Petersen Graph to produce non-Tait-colourable bridgeless cubic graphs. This modification consists of removing a vertex from P and adding new edges and vertices to obtain a different bridgeless cubic graph. The graph of Figure 2.17 appears in Descartes' article as an example of such a trivial modification.

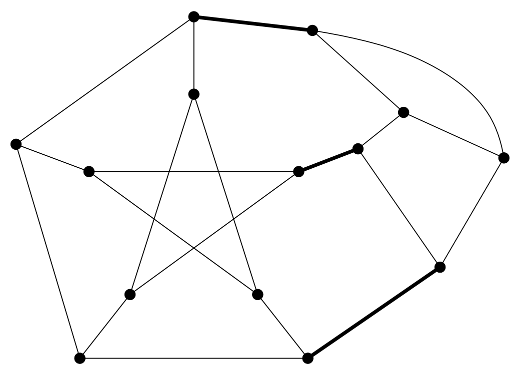


Figure 2.17: An example of a trivial modification of the Petersen Graph. The bold edges form a 3-edge cut.

Every trivial modification of the Petersen Graph has a nontrivial edge cut of size three. Descartes also proved the Parity Lemma (Lemma 1.15) and observed that the edges of every nontrivial 3-edge cut have different colours in any 3-edge-colouring of a cubic graph. It implies that these cuts play the same role as a trivial edge cut. Thus, Descartes constructed snarks without nontrivial 3-edge cuts.

We reproduce Descartes' construction. Denote the Petersen Graph by P, and denote the graph of Figure 2.18 by L. Associate with every vertex $v \in V(P)$ a 9-cycle C_v. Let $\partial(v) = \{e, f, g\}$ and define $\{P_v^e, P_v^f, P_v^g\}$, a partition of $V(C_v)$ such that each part has three vertices. For every edge $e \in E(P)$, with $e = uw$, add a copy L_e of L. Join cycles C_u and C_w to L_e as follows. Identify the vertices of P_u^e with vertices a, b, and c of L_e, and identify the vertices of P_w^e with vertices a', b', and c'. Each 9-cycle is joined with exactly three copies of L, and each vertex of the resulting graph has degree three. The new graph has 210 vertices and is a *Descartes Snark*. Notice that it is not specified how to make the partition of the vertices of each 9-cycle. Hence, this procedure does not lead to a unique graph. An example is exhibited in Figure 2.19. Every Descartes Snark has no edge cut with less than four edges and no cycle of length less than five.

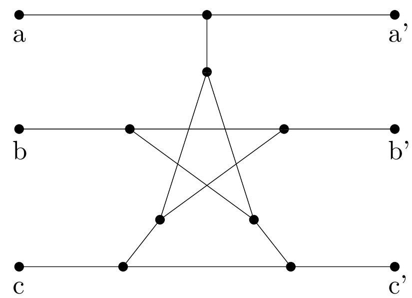


Figure 2.18: Graph L used in Descartes construction.

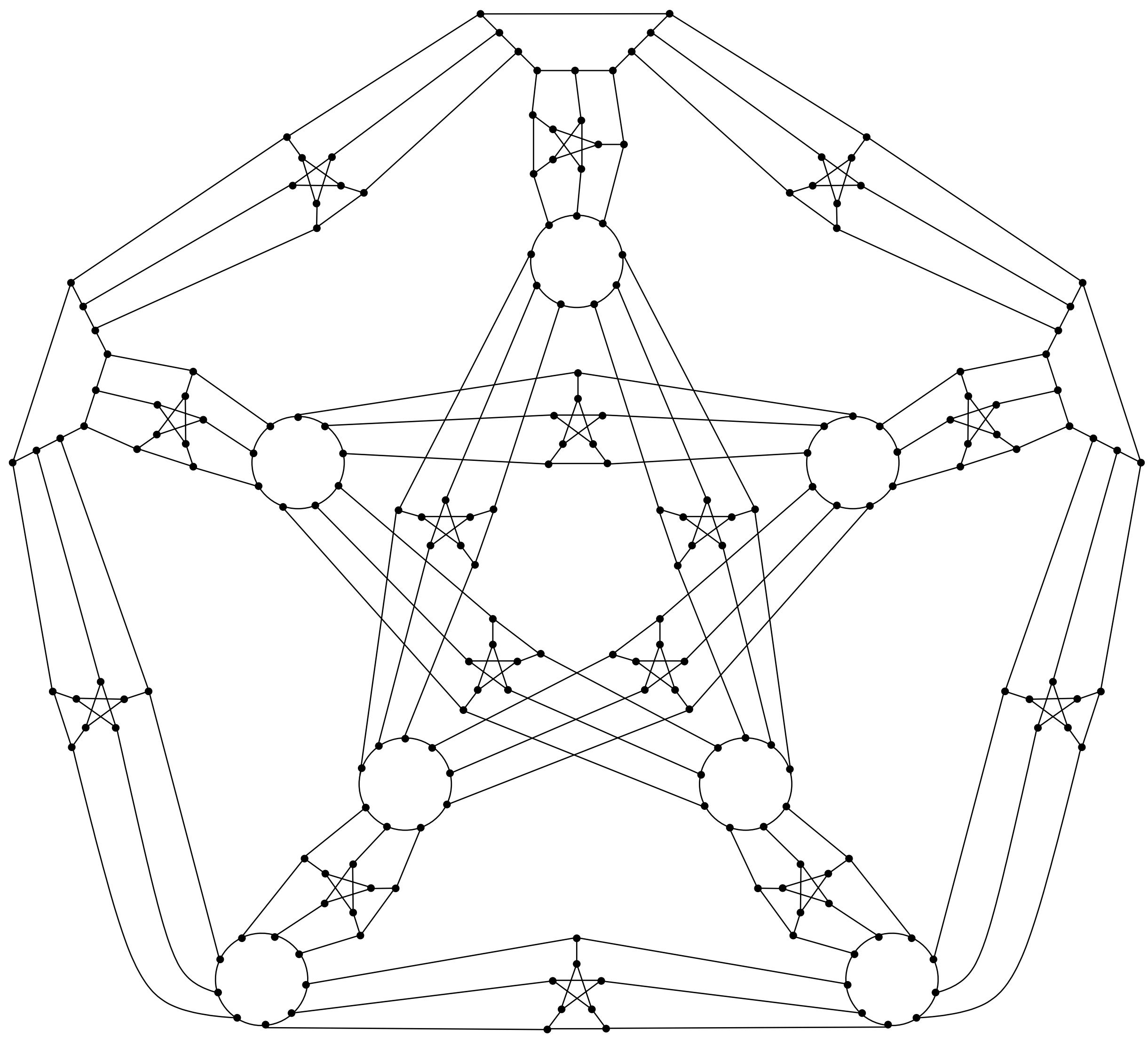


Figure 2.19: A Descartes Snark.

Theorem 2.18. *Every Descartes Snark is non-*3*-edge-colourable.*

Proof. Let G be a Descartes Snark. Suppose G has a 3-edge-colouring π with colour set $\{1,2,3\}$. Let $\mathcal{L} = \{L_e \colon e \in E(P)\}$, where L_e is the copy of graph L (see Figure 2.18) replacing edge e. Define mapping $\phi\colon \mathcal{L} \to \{0,1,2,3\}$ in the following manner. If any two of the edges of L_e incident with vertices a, b, and c have the same colour, let $\phi(L_e)$ be the colour of the third edge. Otherwise, set $\phi(L_e) = 0$. Note that, if the three considered edges have the same colour c, then $\phi(L_e) = c$. Moreover, if ϕ' is defined analogously to ϕ, but using the colours of the eges of L_e incident with a', b', and c' instead, then $\phi' = \phi$.

Note that if $\phi(L_e) = 0$ for some $e \in E(P)$, then the three edges of L_e incident with a, b, and c have different colours, and the same is true for the three edges of L_e incident with a', b', and c'. In this case, by identifying a, b, and c, and identifying a', b', and c', while preserving the colours of the edges of L_e, we obtain a copy of P with a 3-edge-colouring. Therefore, $\phi(L_e) \neq 0$ for all $e \in E(P)$.

Let $v \in V(P)$, with $\partial_P(v) = \{e, f, g\}$. By construction, G has a 9-cycle C_v, and L_e, L_f, L_g are the copies of L connected to C_v. Suppose that for two of L_e, L_f, L_g, say L_e and L_f, $\phi(L_e) = \phi(L_f) = x$. Since $x \neq 0$, by construction of ϕ and by the Parity Lemma, $\phi(L_g) = 0$. Hence, $\phi(L_e)$, $\phi(L_f)$, and $\phi(L_g)$ are different. In this case, a 3-edge-colouring of P is obtained by assigning $\phi(L_e)$ to each $e \in E(P)$. Therefore, G is not 3-edge-colourable. $\square$

The fourth discovery of a snark was made by Szekeres [85] in 1973. Initially, he defined the Petersen Graph P as the graph with vertex set $\{0,\dots,9\}$ and edge set $\{01, 12, 23, 34, 40, 56, 67, 78, 89, 95, 05, 17, 29, 36, 48\}$. Graph H is obtained from P by removing edges 23 and 78, as shown in Figure 2.20. Szekeres constructed the *Szekeres Snark* as follows. Take graphs H_0, H_1, H_2, H_3, and H_4 isomorphic to H. For each H_i, let θ_i be an isomorphism from H to H_i. For every $j \in V(H)$, denote vertex $\theta(j) \in V(H_i)$ by $x_{i,j}$. Add edges $\{x_{i,2}x_{(i+1) \bmod 5,3}\}$ and $\{x_{i,7}x_{(i+2) \bmod 5,8}\}$, with $0 \le i \le 4$, adding precisely one edge between each pair of graphs H_i. The Szekeres Snark is shown in Figure 2.21. This snark can also be obtained by the dot product operation, described in Section 2.3.3.

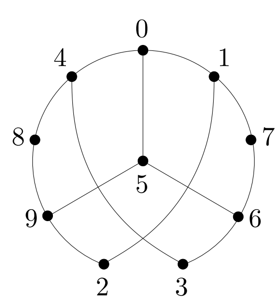


Figure 2.20: Graph H defined by Szekeres.

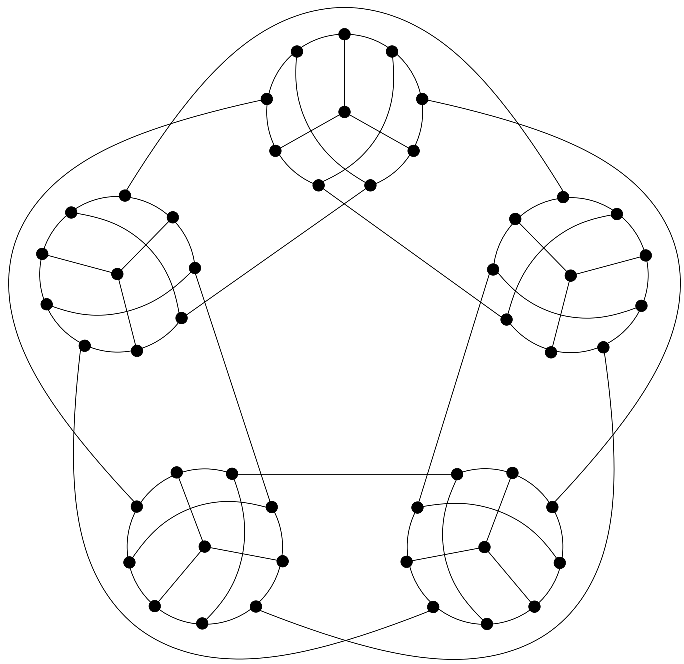


Figure 2.21: The Szekeres Snark.

2.3.2 Trivial snarks

In the context of 3-edge-colourings, some snarks are commonly considered trivial variations of other snarks. In this section, we present a formalization of this idea.

The Four-Colour Conjecture motivated the study of bridgeless cubic graphs. Respectively to edge-colourings, there is another essential reason for not considering cubic graphs with bridges. Let G be a cubic graph with a bridge e. Recall that a 3-edge-colouring is a partition of $E(G)$ into three disjoint perfect matchings. By Theorem 1.12, e belongs to every perfect matching of G. Thus, G does not have three pairwise disjoint perfect matchings. Therefore, every cubic graph with a bridge is non-3-edge-colourable. Unless otherwise stated, every cubic graph considered in this subsection is bridgeless.

In a 3-edge-colouring, cycles of length at most four and nontrivial edge cuts with size at most three do not play any special role. In order to see this, consider initially cycles of length at most four. Let G be a bridgeless cubic graph with a cycle C of length two, and let e and f be the edges with only one end in $V(C)$. In every 3-edge-colouring of G, the same colour is assigned to e and f, as illustrated in Figure 2.22(a). As indicated in the figure, if C, e, and f are replaced by a single edge, resulting a smaller cubic graph H, then G is 3-edge-colourable if and only if H is 3-edge-colourable. If cycle C has length three, then in any 3-edge-colouring of G, each edge of $\partial(C)$ has a different colour. As depicted in Figure 2.22(b), the vertices of C can be shrinked to produce a graph with the same chromatic index of G. Suppose now that the length of C is four. By the Parity Lemma, there are just two cases to be considered for a 3-edge-colouring of G, illustrated in Figure 2.23. In both of them, C can be replaced by two edges, as shown in the figure, without changing the chromatic index of G.

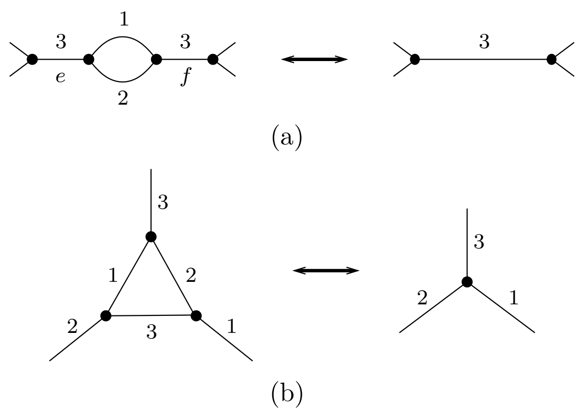


Figure 2.22: Configurations with cycles of length two and three.

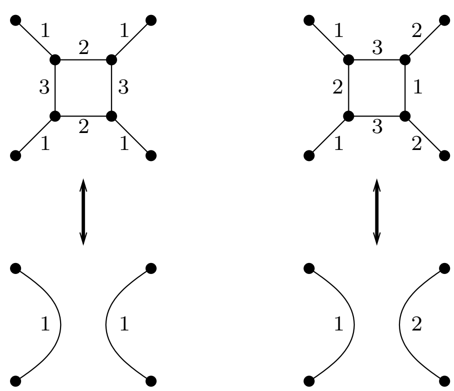


Figure 2.23: Configurations with cycles of length four.

Before proceeding with the analysis of 2-edge and 3-edge cuts, we introduce two simple operations described by Isaacs [42]. Let G_1 and G_2 be cubic graphs. Let $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$. The first Isaacs' operation builds a graph G composed of $G_1 - e_1$ and $G_2 - e_2$, connected by edges u_1u_2 and v_1v_2. Note that G has a 2-edge cut comprised of edges u_1u_2 and v_1v_2. For the second Isaacs' operation, let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, with x_1, y_1, z_1 the neighbours of v_1, and x_2, y_2, z_2 the neighbours of v_2. This operation produces a new graph G comprised of $G_1 - v_1$, $G_2 - v_2$, and edges x_1x_2, y_1y_2, and z_1z_2. In this case, G has the 3-edge cut $\{x_1x_2, y_1y_2, z_1z_2\}$. Both operations are illustrated in Figure 2.24.

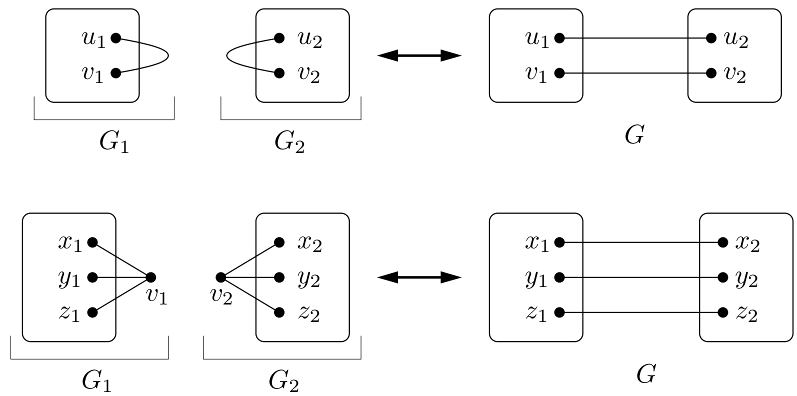


Figure 2.24: Isaacs' operations for constructing trivial cubic graphs.

Theorem 2.19. *Let G_1 and G_2 be cubic graphs. Let G be a graph obtained from G_1 and G_2 applying one of Isaacs operations. Thus, G is 3-edge-colourable if and only if G_1 and G_2 are 3-edge-colourable.*

Proof. Let G_1, G_2, and G be defined as in the hypothesis. Suppose G has a 3-edge-colouring. If G is obtained with the first Isaacs operation, then, by the Parity Lemma, the two edges of edge cut $\{u_1u_2, v_1v_2\}$ of G have the same colour c. By restricting the 3-edge-colouring of G to the edges of G_i, for $i = 1, 2$, and assigning colour c to edge u_iv_i, we have a 3-edge-colouring of G_i. If G is constructed with the second Isaacs operation, then the Parity Lemma asserts that each edge of edge cut $\{x_1x_2, y_1y_2, z_1z_2\}$ of G has a different colour. Also in this case, we restrict the 3-edge-colouring of G to the edges of G_i, for $i = 1, 2$. Then, we assign the colours of x_1x_2, y_1y_2, z_1z_2 respectively to x_iv_i, y_iv_i, z_iv_i. The result is a 3-edge-colouring of G_i.

Conversely, suppose G_1 and G_2 each has a 3-edge-colouring. Assume that G is constructed with the first Isaacs operation. We permute the colours of the edges of G_1 and G_2 so that u_1v_1 and u_2v_2 have the same colour c. Then, we transfer the colours of the edges of G_1 and G_2 to the equivalent edges of G, and assign colour c to both edges u_1u_2, v_1v_2. The result is a 3-edge-colouring of G. Now, assume that G is constructed with the second Isaacs operation. In this case, we permute the colours of the edges of G_1 and G_2 so that each of x_1v_1, y_1v_1, and z_1v_1 has respectively the same colour as x_2v_2, y_2v_2, and z_2v_2. By transfering the colours of the eges of G_1 and G_2 to the equivalent edges of G, we get a 3-edge-colouring of G. $\square$

The next proposition shows that nontrivial edge cuts of size at most three in a cubic graph are cyclic edge cuts. This relation is useful for defining nontrivial snarks.

Proposition 2.20. *If G is a cubic graph, then every nontrivial edge cut of G with at most three edges is a cyclic edge cut.*

Proof. Let G be a cubic graph with a nontrivial k-edge cut $\partial(X)$, $X \subseteq V(G)$ and $k \leq 3$. Let H be a connected component of $G \setminus \partial(X)$. At most three edges of $\partial(X)$ are incident with vertices of H. By Theorem 1.2, it suffices to show that there is a subgraph of H with minimum degree at least two.

Suppose just one edge of $\partial(X)$ is incident with a vertex of H. Thus, every vertex of H has degree at least two. If exactly two edges of $\partial(X)$ are incident with vertices $u, v \in V(H)$, $u \neq v$, then every vertex of H has degree at least two. If $u = v$, then v has degree one in H, but every vertex of $H - v$ has degree at least two.

As the last case, suppose three edges of $\partial(X)$ are incident with vertices of H. Since $\partial(X)$ is nontrivial, these edges are incident with more than one vertex of H. If they are incident with three different vertices, then every vertex of H has degree at least two. Thus, consider that the three edges of $\partial(X)$ are incident with two vertices u and v of H. Without loss of generality, suppose u is incident with two edges of $\partial(X)$. In H, u has degree one and v has degree two. First, suppose u and v are not adjacent. Thus, $H - u$ has two vertices of degree two while all other vertices have degree three. Suppose u and v are adjacent. In this case, v is the only degree-one vertex of $H - u$. Therefore, every vertex of $H \setminus \{u, v\}$ has degree at least two. □

As a consequence of Proposition 2.20, a cubic graph is cyclically 4-edge-connected if and only if all its edge cuts with less than four edges are trivial.

Considering what is discussed in this section, we provide a more detailed classification of cubic graphs with chromatic index four. A *snark* is defined as a bridgeless cubic graph with chromatic index four. If a snark has a nontrivial k-edge cut, $k \leq 3$, or a cycle of length at most four, then it is a *trivial snark*. Otherwise, it is a *nontrivial snark*. In the light of Proposition 2.20, we can restate this definition as follows.

> A *nontrivial snark* is a cyclically 4-edge-connected cubic graph with chromatic index four and girth at least five.

The Petersen Graph, the Blanuša Snark, the Descartes Snarks and the Szekeres Snark are nontrivial snarks.

Recall that the Four-Colour Theorem is equivalent to the statement that every planar bridgeless cubic graph is 3-edge-colourable. This statement, in turn, is equivalent to the assertion that every snark is nonplanar. To show that every snark is nonplanar, it is enough to verify that every nontrivial snark is nonplanar. For this reason, the search for snarks is usually restricted to nontrivial cases.

Conjecture 2.21 (Four-Colour Conjecture – Snark version)**.** *Every nontrivial snark is nonplanar.*

Theorem 2.22 (Four-Colour Theorem – Snark version)**.** *Every nontrivial snark is nonplanar.*

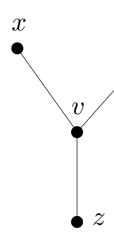
2.3.3 Isaacs snarks

After a very slow progress in finding new snarks, an article published by Isaacs [42] in 1975 changed dramatically the history of the discoveries. Isaacs found two infinite families of snarks: the Flower Snarks and the BDS Class. According to John Watkins [92] and Mark Goldberg [32], Flower Snarks were independently discovered by Grinberg, but never published. Goldberg also pointed that Grinberg presented Flower Snarks at the Symposium on Graph Theory held in Vaivary, Latvia, in 1972.

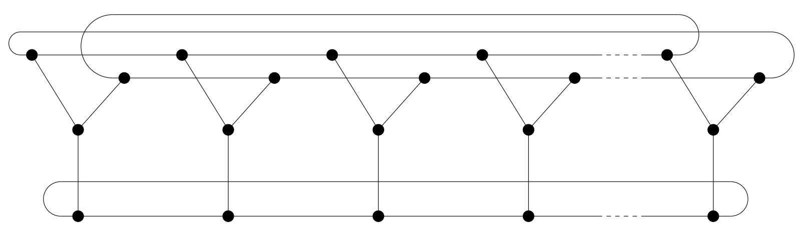
The BDS Class is constructed by an operation called dot product. According to Isaacs, the letters in BDS stand for Blanuša, Descartes and Szekeres, whose snarks belong to this family and inspired its construction. Isaacs defined an additional family, called Q Class, and described one new member of this class, known as Double Star Snark. In what follows, we describe Isaacs' infinite families of snarks and the Double Star Snark.

Flower Snarks

Isaacs described a sequence of snarks $\{J_k\}$, k odd and $k \geq 3$, known as *Flower Snarks.* Let T be the graph shown in Figure 2.25(a). In graph T, v is the central vertex and x, y, and z are the vertices adjacent to v. For k an odd integer, $k \geq 3$, Flower Snark J_k is constructed using $T_1, T_2, \ldots, T_k$, k copies of T. For each T_i, let its vertices be v_i, x_i, y_i, and z_i. Graphs $T_1, \ldots, T_k$ are linked by two cycles: $C_0 = z_1 z_2 \ldots z_k$ and $C_1 = x_1 x_2 \ldots x_k y_1 y_2 \ldots y_k$, as indicated in Figure 2.25(b). Common drawings of J_5 and J_7 are shown in Figure 2.26. Graph J_3 is a trivial snark, since its cycle C_0 has length three. It is worth noting that the contraction of C_0 in J_3 produces the Petersen Graph.



(a)



(b)

Figure 2.25: Construction of Flower Snarks.

Theorem 2.23. *The Flower Snarks are not* 3*-edge-colourable.*

Proof. We describe Isaacs' proof [42]. Let J_k be a Flower snark. Let $Y_i = V(T_i)$, $T_i \subseteq J_k$, with $1 \leq i \leq k$. Let edge cut $\partial(Y_i)$ be partitioned into two sets: $\partial^-(Y_i)$, comprising the three edges with one end in Y_i and another end in Y_{i-1}, and $\partial^+(Y_i)$, comprising the three edges joining Y_i and Y_{i+1}. In order to see that J_k is not 3-edge-colourable, suppose it has

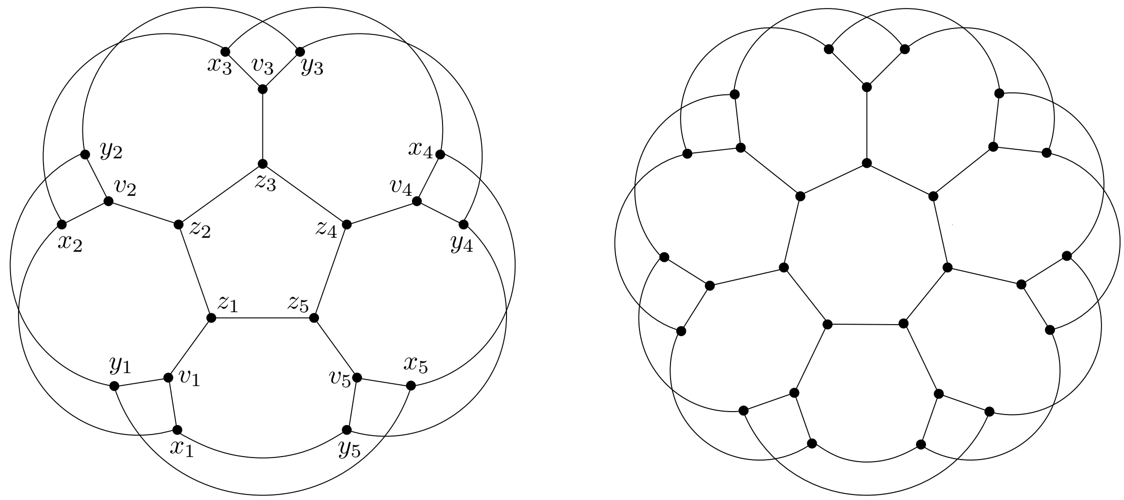


Figure 2.26: The Flower Snarks J_5 and J_7.

a 3-edge-colouring with colour set $\{1, 2, 3\}$. There are three cases to analyse, depending on the colours of the edges of $\partial^-(Y_1)$.

Case 1. The three edges of $\partial^-(Y_1)$ receive the same colour. In this case, it is not possible to assign distinct colours to the three edges of $E(T_1)$.

Case 2. The three edges of $\partial^-(Y_1)$ are coloured with only two colours, say 1 and 2, with colour 1 used twice. By inspection, we conclude that edges of $\partial^+(Y_1)$ receive colours 2 and 3, with colour 3 used twice. Had we considered $2, 3, 3$ as the colours of the edges of $\partial^-(Y_1)$, the inspection would yield colours $1, 1, 2$ for the edges of $\partial^+(Y_1)$. Thus, the edges of the sequence $\partial^-(Y_1), \partial^-(Y_2), \partial^-(Y_3), \ldots, \partial^-(Y_k)$, receive alternately colours $1, 1, 2$ and $2, 3, 3$. Since k is odd, the edges of $\partial^-(Y_k)$ and the edges of $\partial^-(Y_1)$ receive the same colours $1, 1, 2$, resulting in a pair of adjacent edges with the same colour.

Case 3. The three edges of $\partial^-(Y_1)$ are coloured with distinct colours. For the edges in each $\partial^-(Y_i)$ and $\partial^+(Y_i)$, consider the following cyclic order: the edge incident with x_i, the edge incident with y_i, and the edge incident with z_i.

Suppose the edges of $\partial^-(Y_i)$ have colours $1, 2, 3$ in cyclic order. This implies that the edges of $\partial^+(Y_i)$ also have colours $1, 2, 3$ in cyclic order, up to shifts, as illustrated in Figure 2.27. Therefore, if the edges of $\partial^-(Y_1)$ have colours $1, 2, 3$ in cyclic order, then the edges of $\partial^+(Y_k)$ also have colours $1, 2, 3$ in cyclic order. Observe that T_1 and T_k are joined by edges x_1y_k, y_1x_k, and z_1z_k, as illustrated in Figure 2.28. Thus, the edges of $\partial^-(Y_1)$ have colours $1, 2, 3$ in reverse cyclic order, contradicting the hypothesis that the edges of $\partial^-(Y_1)$ have colours $1, 2, 3$ in cyclic order. $\square$

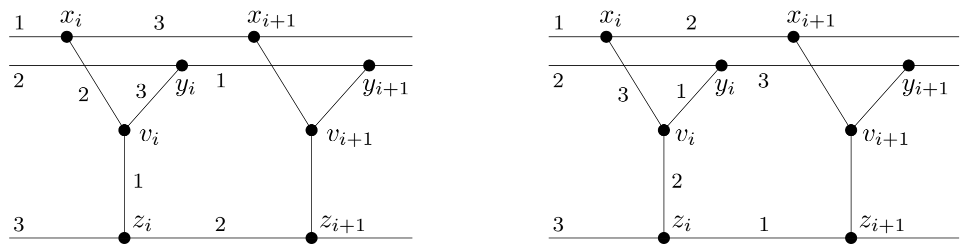


Figure 2.27: Case 3: Two ways of colouring the edges incident with T_i.

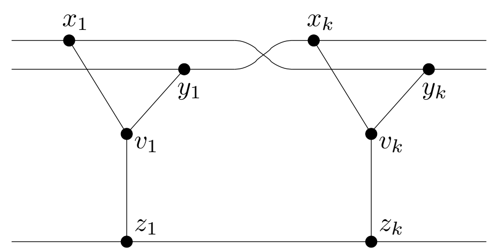


Figure 2.28: Edges connecting T_k and T_1.

Dot product (BDS Class)

The second infinite family of snarks described by Isaacs [42] is the *BDS Class*. As already mentioned, this family contains the Blanuša Snark, the Descartes Snarks and the Szekeres Snark. Each graph of this class is the result of successive applications of an operation called dot product. Let U and W be any cubic graphs. Remove from U two non-adjacent edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$. Remove from W two adjacent vertices x_1 and x_2, with x_1 also adjacent to w_1 and z_1, and x_2 also adjacent with w_2 and z_2. The *dot product* $U \cdot W$ is a cubic graph obtained by linking vertices u_1 and v_1 with w_1 and z_1, and linking vertices u_2 and v_2 with w_2 and z_2, after removing e_1 and e_2 from U and x_1 and x_2 from W. Figure 2.29 illustrates the operation.

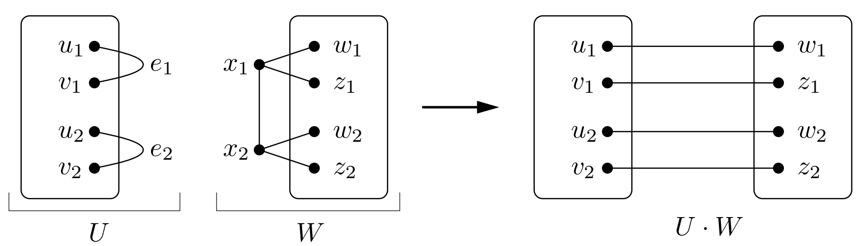


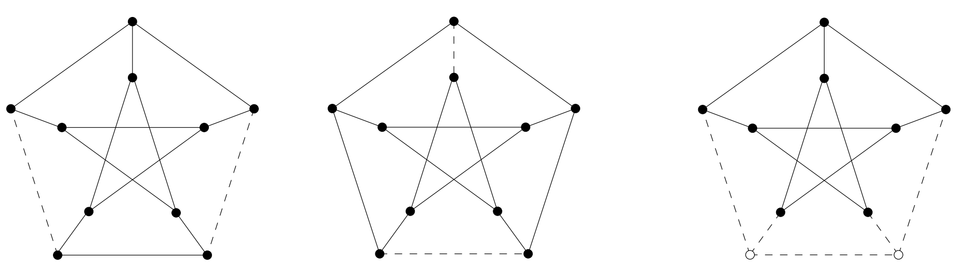
Figure 2.29: Isaacs' dot product.

Theorem 2.24. *If U and W are non-3-edge-colourable cubic graphs, dot product $U \cdot W$ produces a non-3-edge-colourable cubic graph.*

Proof. Let U and W be non-3-edge-colourable cubic graphs. Suppose that $U \cdot W$ has a 3-edge-colouring $\pi \colon E(U \cdot W) \to \mathcal{C}$. Let the vertices and edges of U, W, and $U \cdot W$ be denoted as in Figure 2.29. Let E_C be the edge cut $\{u_1w_1, v_1z_1, u_2w_2, v_2z_2\}$ of $U \cdot W$. By the Parity Lemma, each colour $c \in \mathcal{C}$ is assigned to an even number of edges of E_C. Therefore, either the four edges of E_C have the same colour, or two of them have one colour, while the other two have another colour.

Suppose that all edges of E_C have the same colour c. Thus, by restricting π to $E(U)$ and assigning c to edges u_1v_1 and u_2v_2 of U, we have a 3-edge-colouring of U, contradicting the fact that U is non-3-edge-colourable. Hence, the edges of E_C have colours c_1 and c_2. If u_1w_1 and v_1z_1 have colour c_1, while u_2w_2 and v_2z_2 both have colour c_2, then a 3-edge-colouring of U is obtained, similarly to the previous case. Thereby, assume that u_1w_1 and v_1z_1 have different colours c_1, c_2, as well as u_2w_2 and v_2z_2. In this case, we construct a 3-edge-colouring of W by restricting π to $E(W)$, then transferring the colours of u_1w_1 and v_1z_1 respectively to x_1w_1 and x_1z_1, transferring the colours of u_2w_2 and v_2z_2 respectively to x_2w_2 and x_2z_2, and assigning to x_1x_2 the colour of $\mathcal{C} \setminus \{c_1, c_2\}$. We conclude that, in any case, a 3-edge-colouring of $U \cdot W$ cannot exist. □

Due to the high symmetry of the Petersen Graph P, there are only two essentially different ways of choosing two non-adjacent edges: either the two edges are adjacent to a common third edge, or not. These ways are shown in Figure 2.30(a). Still by the symmetry of P, there is only one way of choosing two adjacent vertices, as illustrated in Figure 2.30(b). Hence, there are only two manners of performing the dot product on two copies of the Petersen Graph [91]. One of these, where the first operand of $P \cdot P$ is modified as indicated in the leftmost drawing of Figure 2.30(a), yields the Blanuša Snark, as noted by Isaacs. This snark, shown in Figure 2.16, is also known as the *First Blanuša Snark*. When modifying the first operand of $P \cdot P$ as indicated in the rightmost drawing of Figure 2.30(a), the result of $P \cdot P$ is the *Second Blanuša Snark*, depicted in Figure 2.31.



(a) (b)

Figure 2.30: The Petersen Graph as the first (a) and as the second (b) operand of the dot product $P \cdot P$. Dashed lines are removed edges and white vertices are removed vertices.

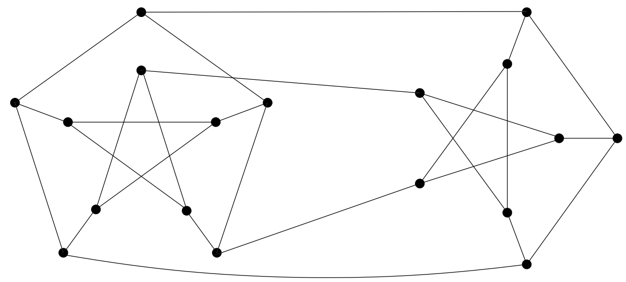
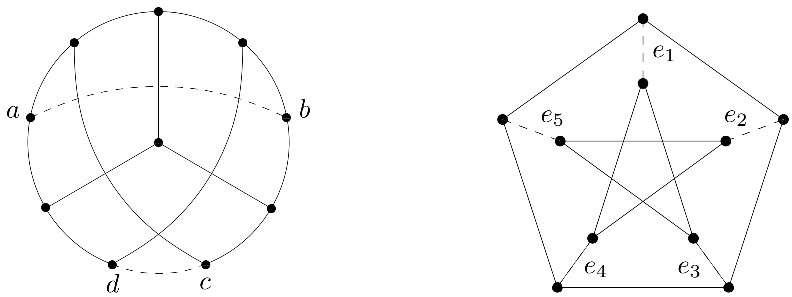


Figure 2.31: The Second Blanuša Snark.

Isaacs, as well as Uldis Celmins and Edward Swart [18], Myriam Preissmann [76], and Watkins [91, 92], observed that $P \cdot P$ has two possible results. Preissmann showed that the two Blanuša Snarks are the only nontrivial snarks on 18 vertices. In Stanley Fiorini and Robin Wilson's book [26], it is stated that Fiorini, using a computer-generated list by Bussemaker et al. [11], has shown that nontrivial snarks of order 12 or 14 do not exist. Later, Celmins and Swart [18] gave a proof of the same fact, while Jean-Luc Fouquet [27] showed that a nontrivial snark on 16 vertices does not exist. Furthermore, Fiorini and Wilson claimed that it can be shown that there are nontrivial snarks of any even order greater than 16.

The discovery of the Second Blanuša Snark is sometimes credited to Blanuša. However, in his 1946 paper [7], we only found evidence of the First Blanuša Snark, and the piece of information given by Isaacs [42] and Watkins [92] provides evidence that the Second Blanuša Snark was not known until the definition of the dot product. There is some difficulty in providing more precise information, since Blanuša's paper is written in Croatian and it has only a summary in French. On the other hand, the credit given to Blanuša by some authors may be justified by the fact that the Second Blanuša Snark is a simple variation of the first.

The Szekeres Snark, exhibited in Figure 2.21, is a result of the sequence of dot products $P_5 \cdot (P_4 \cdot (P_3 \cdot (P_2 \cdot (P_1 \cdot P_0))))$, where $P_0, \ldots, P_5$ are copies of P. Let $G_i = P_i \cdot G_{i-1}$, with $1 \leq i \leq 5$, where $G_0 = P_0$. There are multiple ways of carrying out this sequence of products. To produce the Szekeres Snark, in each product the first operand P_i is modified as indicated in Figure 2.32(a). For the second operand, consider edges $e_1, \ldots, e_5$ shown in Figure 2.32(b). When performing $P_i \cdot G_{i-1}$, $1 \leq i \leq 5$, remove the ends of e_i from G_{i-1} and preserve the labels of edges $e_{i+1}, \ldots, e_5$ in G_i. For every i, let v_i^{out} be the end of e_i in the outermost cycle of Figure 2.32(b), and let v_i^{in} be the end of e_i in the innermost cycle. In each product, vertices a and b are joined respectively to v_{i+3}^{in} and v_{i+2}^{in}, and vertices c and d are joined respectively to v_{i+4}^{out} and v_{i+1}^{out}. Indexes greater than five are taken modulo five. The Szekeres Snark is the result of the last product, $P_5 \cdot G_4$.

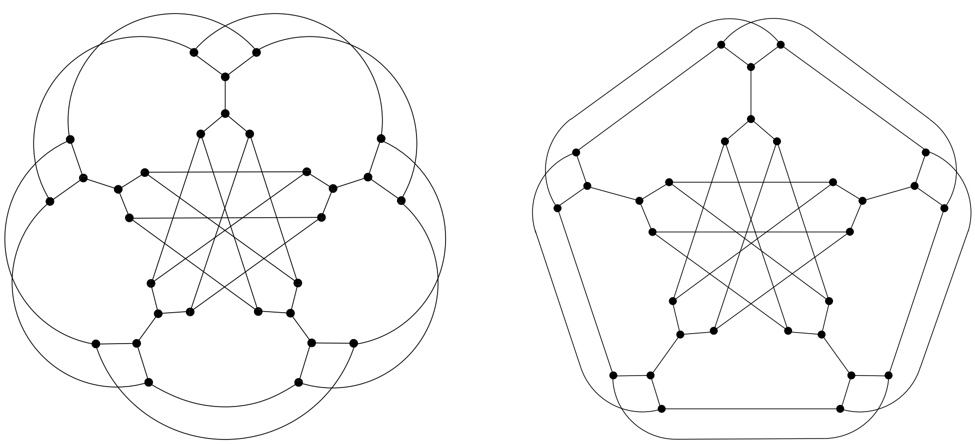


(a) $P_1, \ldots, P_5$ (b) P_0

Figure 2.32: Construction of Szekeres Snark.

The Q Class and the Double Star Snark

Isaacs [42] described another class of snarks, called *Q Class*, which we do not describe here. This family includes the Petersen Graph, the Flower Snark J_5, and the *Double Star Snark*, depicted in Figure 2.33. The Double Star Snark is constructed as follows. Take two copies of J_5, remove cycle $C_0 = z_1z_2z_3z_4z_5$ from each of them, and denote the resulting graphs by J_5' and J_5''. Then, add edges $v_1'v_1''$, $v_3'v_2''$, $v_5'v_3''$, $v_2'v_4''$, and $v_4'v_5''$.



(a) A drawing by Isaacs. (b) A drawing by Celmins [16].

Figure 2.33: The Double Star Snark.

2.4 Additional constructions of snarks

In the previous section, we described the first discoveries of snarks. These discoveries are summarised in Table 2.1.

After Isaacs' remarkable paper, many other families and methods for constructing snarks were described. Some of them are presented here. Examples not covered here

Snarks	**Order**	**Author**	**Year**
Petersen Graph	10	Petersen	1979
First Blanuša Snark	18	Blanuša	1946
Descartes Snarks	210	Descartes	1948
Szekeres Snark	50	Szekeres	1973
Flower Snarks	$4(2k+1)$, $k \geq 1$	Grinberg	1972
		Isaacs	1975
BDS Class (dot product)	–	Isaacs	1975
Double Star Snark	30	Isaacs	1975

Table 2.1: First discoveries of snarks

are: the square product operation, due to Cavicchioli et. al. [13]; two infinite families, due to Jonas Hägglund [37]; constructions described by Celmins and Swart [18]; and constructions described by Watkins [91, 92], including a generalization of the Szekeres Snark. There are also works that present algorithms for generating snarks of a given size, such as the algorithm by Brinkmann et al. [9], which was used to generate all non-isomorphic snarks with up to 36 vertices.

2.4.1 Generalised Blanuša Snarks

Watkins [91, 92] generalised the construction of First and Second Blanuša Snarks, defining two infinite families of *generalised Blanuša Snarks*. Let $\mathcal{B}^1 = \{B_1^1, B_2^1, B_3^1, \dots\}$ be the first family of generalised Blanuša Snarks and let $\mathcal{B}^2 = \{B_1^2, B_2^2, B_3^2, \dots\}$ be the second family of generalised Blanuša Snarks. The first member of $\mathcal{B}^1$ is the First Blanuša Snark, while the first member of $\mathcal{B}^2$ is the Second Blanuša Snark.

In order to construct the first family, we consider the drawing of the Petersen Graph shown in the left of Figure 2.34. The First Blanuša Snark B_1^1 is the result of product $P \cdot P$ performed as indicated in Figure 2.34. In this figure, the leftmost drawing shows the first operand, and the middle drawing shows the second operand. In order to construct B_1^1, remove edges u_1v_1 and u_2v_2 from the first operand, remove vertices x_1 and x_2 from the second operand, and add edges $u_1w_1, v_1z_1, u_2w_2, v_2z_2$. The rightmost drawing of the figure shows graph B_1^1. Finally, assign labels u_1, u_2, v_1, v_2 to vertices of B_1^1 as in the figure. Each member B_i^1 of $\mathcal{B}^1$, with $i > 1$, is obtained by the dot product $B_{i-1}^1 \cdot P$, with B_{i-1}^1 playing the role of P in the construction of B_1^1. Then, add edges $u_1w_1, v_1z_1, u_2w_2, v_2z_2$, and assign labels u_1, u_2, v_1, v_2 to vertices of B_i^1 as in Figure 2.35.

The construction of family $\mathcal{B}^2$ is similar to the construction of family $\mathcal{B}^1$. The leftmost drawing of Figure 2.36 shows the depiction of the Petersen Graph considered when building the first member B_1^2 of $\mathcal{B}^2$. Figure 2.37 shows the scheme of a member B_i^2 of $\mathcal{B}^2$.

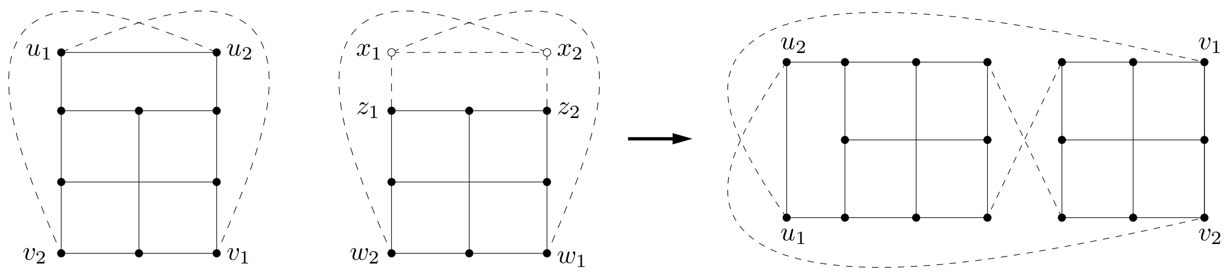


Figure 2.34: Construction of the First Blanuša Snark.

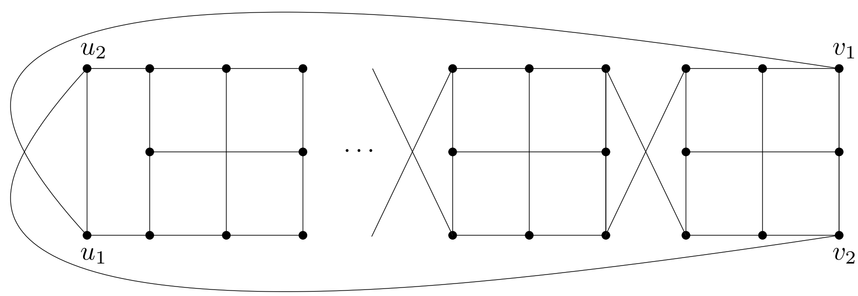


Figure 2.35: Generalised Blanuša Snark B_i^1 of the first family $\mathcal{B}^1$.

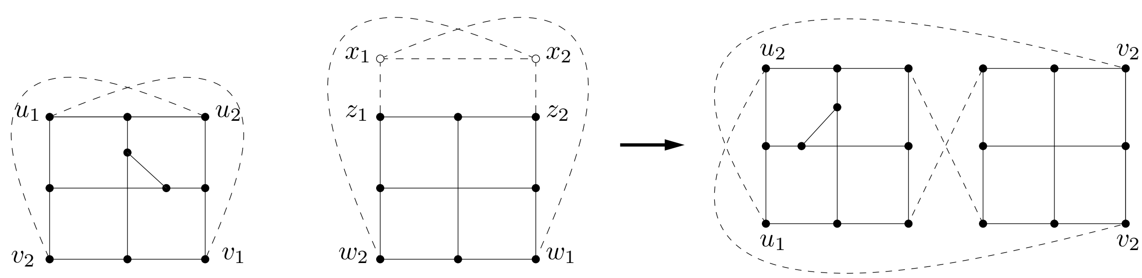


Figure 2.36: Construction of the Second Blanuša Snark.

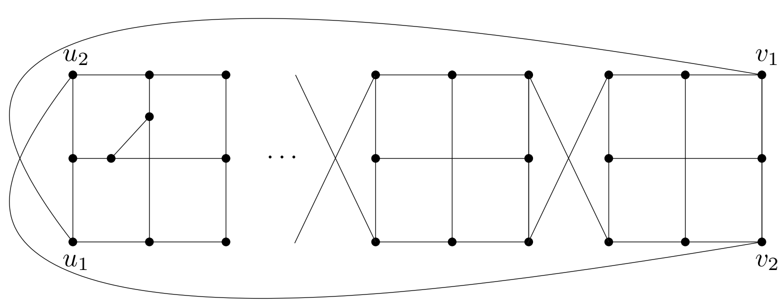
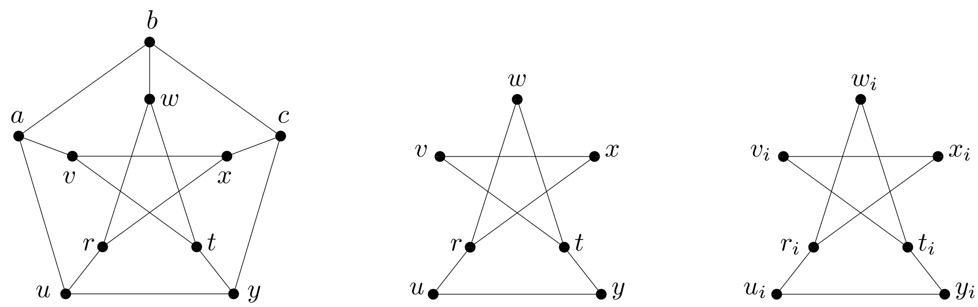


Figure 2.37: Generalised Blanuša Snark B_i^2 of the second family $\mathcal{B}^2$.

2.4.2 Loupekine Snarks

In 1976, Feodor Loupekine proposed a construction of two infinite families of snarks, using other known snarks. We present Loupekine's construction as described by Isaacs [43].

Let G be a nontrivial snark. A subgraph $B(G)$ of G is obtained by removing a path of three vertices from G. Graph $B(G)$ is a *block* and it is used in the construction of a new snark. Since the girth of G is at least five, $B(G)$ has five different vertices with degree two, named u, v, w, x, y relatively to the vertices of the removed path. These vertices are called *border vertices*. Figure 2.38 illustrates this operation for the Petersen Graph P. By the symmetries of the Petersen Graph, all blocks constructed in this way are isomorphic.



(a) Petersen Graph P. (b) Graph $B(P)$. (c) A block B_i.

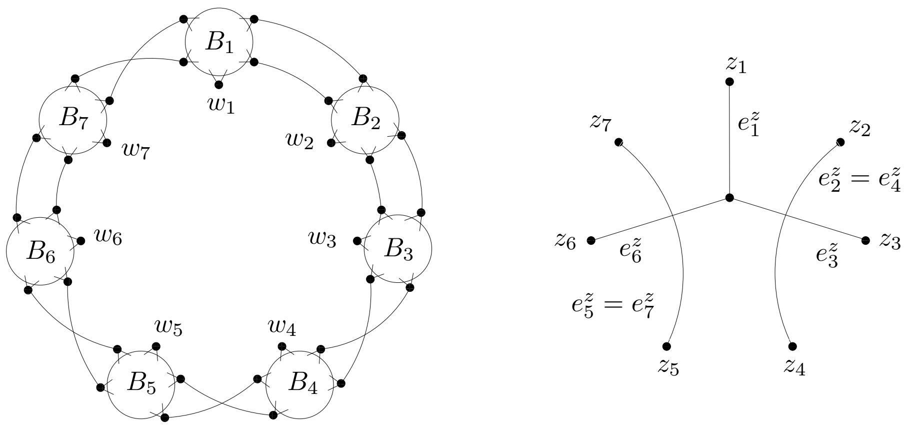
Figure 2.38: Construction of a block from the Petersen Graph.

Let $G_1, G_2, \ldots, G_k$ be snarks. We construct a cubic graph G_k^L using *blocks* $B(G_i)$, $1 \leq i \leq k$. Denote each block $B(G_i)$ by B_i, and attach index i to its vertex names, as in Figure 2.38(c). For each i, add a pair of edges linking each vertex of $\{u_i, v_i\}$ to a different vertex of $\{x_{i+1}, y_{i+1}\}$. In this section, indexes greater than k are taken modulo k. The resulting graph G_B is the *block subgraph* of G_k^L. Figure 2.39(a) shows an example.

Notice that G_B has exactly k vertices with degree two: $w_1, \ldots, w_k$. Let G_C be a graph with k vertices of degree one, namely $z_1, \ldots, z_k$, and all other vertices of degree three. The edge incident with z_i is denoted by e_i^z. Graph G_C is the *central subgraph* of G_k^L. An example of a central subgraph is depicted in Figure 2.39(b). For each i, identify vertices w_i and z_i. The resulting graph is G_k^L.

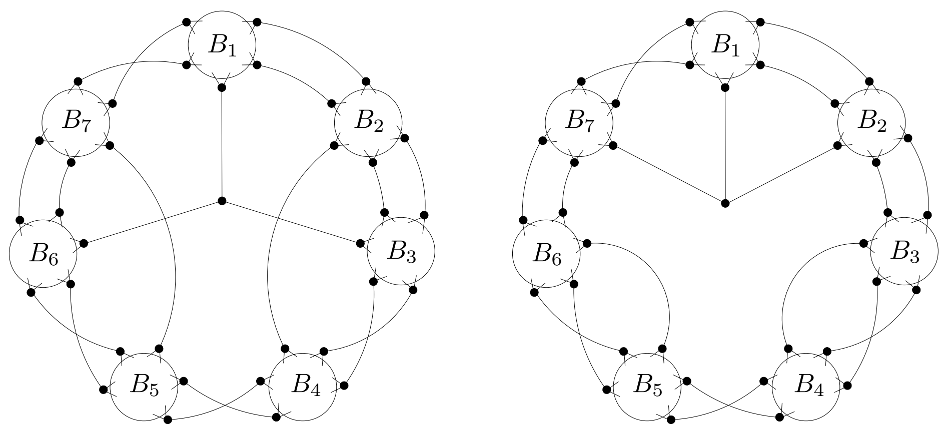
Isaacs [43] showed that, if k is odd, then G_k^L is a snark. If k is even, with $k \geq 6$, an additional constraint on subgraph G_C is necessary. This constraint is described in Isaacs' report. These two families of snarks, one with odd $k \geq 3$ and the other with even $k \geq 6$, are Loupekine's families, which are called *Loupekine Snarks*. A Loupekine Snark with an odd number of blocks is an *L-snark*. Figure 2.40 shows examples of Loupekine's construction with k odd. Notice that G_B and G_C form a decomposition of an L-snark.

Let G be an L-snark such that each connected component of G_C is isomorphic to one of graphs K_2 and S_3, where K_2 is the complete graph with two vertices, and S_3 is the



(a) A graph G_B. (b) A graph G_C.

Figure 2.39: Examples of block subgraph and central subgraph.

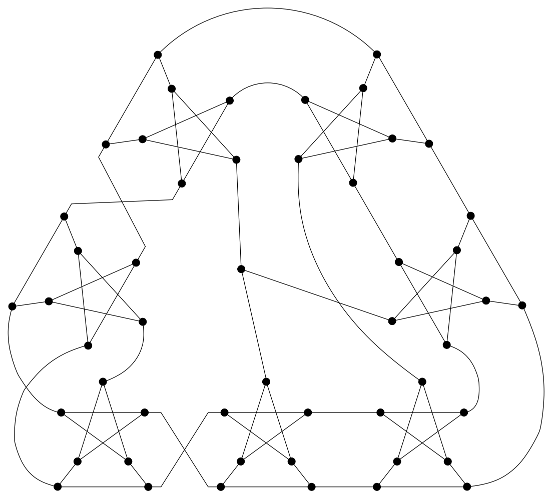


(a) An L_1-snark. (b) An L_0-snark.

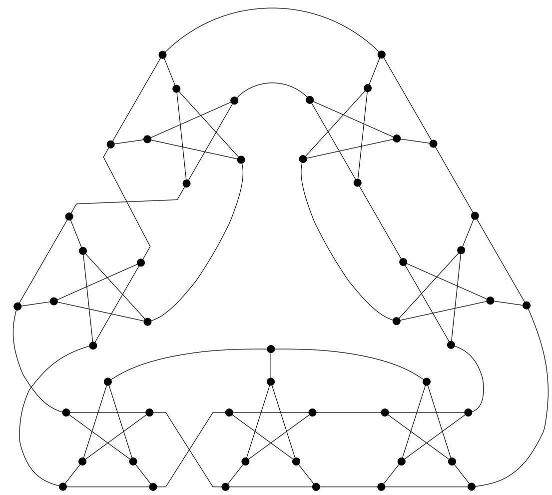
Figure 2.40: Examples of L-snarks.

star with three vertices of degree one. Graph G is an L_1-*snark*. Examples are depicted in Figure 2.40. Suppose that each connected component of G_C is connected to consecutively indexed blocks. Then, G is an L_0-*snark*, as in Figure 2.40(b). Now, suppose G is an L-snark such that each of its blocks is isomorphic to $B(P)$, depicted in Figure 2.38(b). Then, G is an LP-*snark*. An L_1-snark that is also an LP-snark is called an LP_1-*snark*. Moreover, an L_0-snark that is also an LP-snark is called an LP_0-*snark*[1]. Figure 2.41 shows examples of LP-snarks.

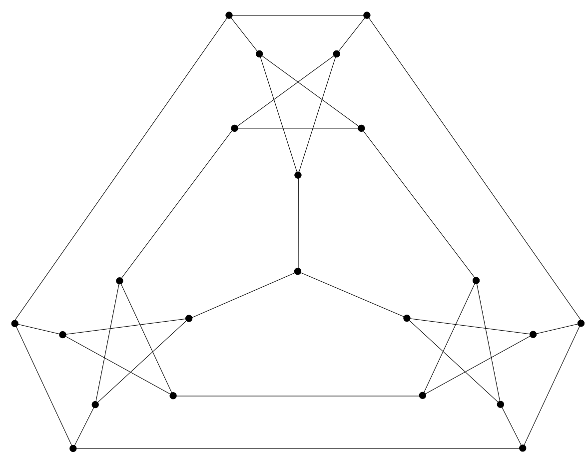
[1]The names of subfamilies of Loupekine Snarks used here were taken from L. Vaux's work [89].



(a) An LP_1-snark.



(b) An LP_0-snark.



(c) The smallest L-snark.

Figure 2.41: Examples of Loupekine Snarks constructed from the Petersen Graph.

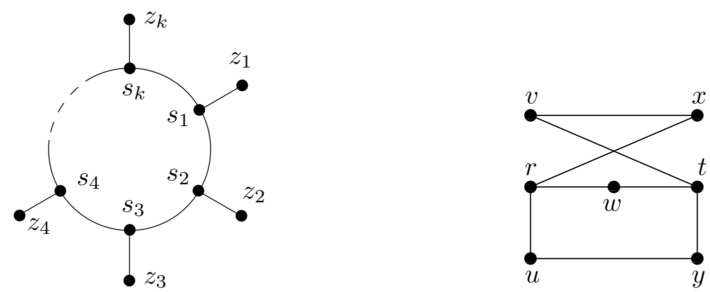
2.4.3 Goldberg Snarks

In 1981, Mark K. Goldberg [32] described a method for constructing Class 2 graphs with maximum degree three. Using Goldberg's method, it is also possible to construct snarks. As an example, Goldberg constructed the family of Flower Snarks. His method is more general than Loupekine's construction, and every L-snark can be obtained by this method.

A graph G is *edge-colour-critical* if every graph obtained by removing an edge of G has chromatic index less than $\chi'(G)$. Another important result provided by Goldberg is the disproval of the *Edge-Colour-Critical Graph Conjecture* [5, 48], which says that every edge-colour-critical graph has an odd number of vertices. Goldberg used his method to produce an infinite family of edge-colour-critical graphs, each with an even number of vertices. Each of these graphs has maximum degree three and chromatic index four.

We do not describe Goldberg's method. Instead, we present the family of *Goldberg Snarks*, built with his method. Each Goldberg Snark is also an LP-snark, which is a Loupekine Snark constructed with blocks derived from the Petersen Graph. Thereby, we use Loupekine's construction to describe Goldberg Snarks.

Let $\{G_k\}$, with k odd and $k \geq 3$, be the family of Goldberg Snarks. For each k, where $k \geq 5$, graph G_k is the LP-snark whose central subgraph has vertex set $\{s_1, s_2, \ldots, s_k\} \cup \{z_1, z_2, \ldots, z_k\}$ and edge set $\{s_1s_2, s_2s_3, \ldots, s_{k-1}s_k, s_ks_1\} \cup \{s_1z_1, s_2z_2, \ldots, s_kz_k\}$, as shown in Figure 2.42(a). A drawing of block $B(P)$ used in this construction is shown in Figure 2.42(b). For all i, $1 \leq i \leq k$, blocks B_i and B_{i+1} are linked by edges $\{v_ix_{i+1}, u_iy_{i+1}\}$, with indexes greater than k taken modulo k, and vertices z_i and w_i are identified. Figure 2.43 depicts G_k. Construction of G_3 differs only in the central subgraph, which is isomorphic to the star S_3, with vertex set $\{s, z_1, z_2, z_3\}$ and edge set $\{sz_1, sz_2, sz_3\}$. Goldberg Snark G_3 is the smallest Loupekine Snark, exhibited in Figure 2.41(c).



(a) Central subgraph. (b) Block $B(P)$.

Figure 2.42: Construction of Goldberg Snark G_k, $k \geq 5$.

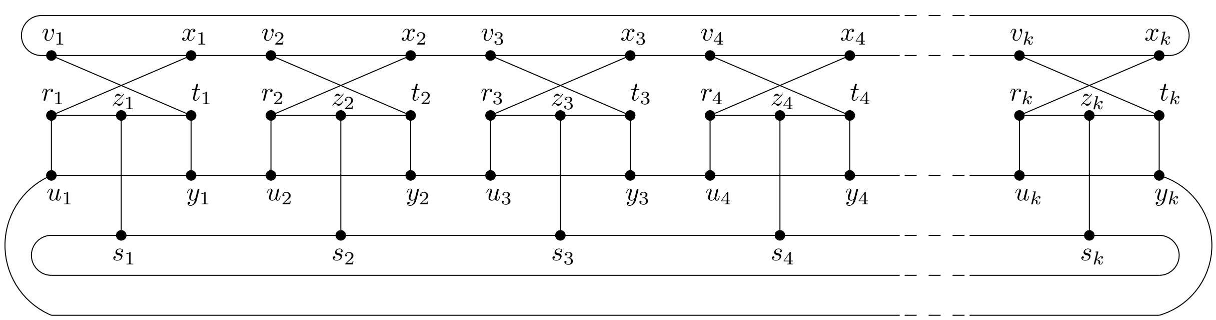


Figure 2.43: Goldberg Snark G_k, with $k \geq 5$.

2.4.4 Kochol's superposition

In 1993, Martin Kochol [61, 63] described a method called *superposition* for constructing new snarks from other ones. In order to describe Kochol's method, some preliminary definitions and results are necessary. We adopt the same notation used by Kochol [63].

A *multipole* is a triple $M = (V(M), E(M), S(M))$, where $V(M)$, or simply V, is a set of vertices, $E(M)$, or E, is a set of non-loop edges, and $S(M)$, or S, is a set of *semiedges*. The terms relative to vertices and edges used in the context of graphs are equally applied to multipoles. In this sense, a multipole is an extension of a graph. A semiedge is *incident* with either a vertex or another semiedge. A semiedge that is incident with a vertex v is denoted by (v), and v is also said to be *incident* with semiedge (v). Two mutually incident semiedges form an *isolated edge*. The degree of a vertex v of a multipole M, denoted by $d_M(v)$ or $d(v)$, is the cardinality of set $\{e \in E(M) \cup S(M) \colon e \text{ is incident with } v\}$. A multipole M is *cubic* if every vertex of M has degree three. Figure 2.44 shows examples of cubic multipoles.

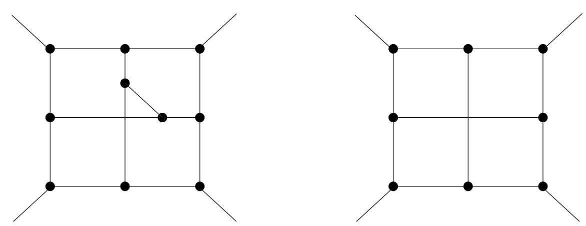


Figure 2.44: Examples of cubic multipoles, each with four semiedges.

Let $M = (V, E, S)$ be a multipole and let $e = uv$ be an edge of M. To *cut* edge e means to remove it from M and add two semiedges (u) and (v). Figure 2.45 shows an example where two edges of a multipole are cut, thus adding four semiedges. This operation can also be performed on a graph, thus creating a multipole. Let $s = (u)$ and $t = (v)$ be semiedges of M. To *join* s and t means to remove them from M and add a new edge with ends u and v. Figure 2.45 also illustrates this operation, which can be performed on semiedges of different multipoles, thus creating a new unique multipole.

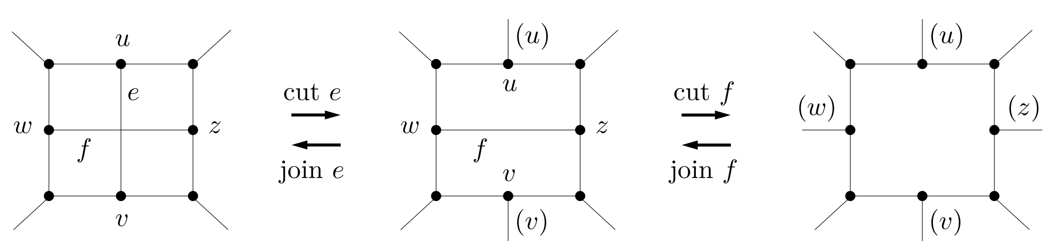


Figure 2.45: Cutting edges (from left to right) and joining semiedges (from right to left) of a multipole.

Let $\mathcal{C}$ be the set of colours $\{(0,1),(1,0),(1,1)\}$. A 3-edge-colouring of a multipole M is a mapping $\rho\colon E(M)\cup S(M)\to\mathcal{C}$ satisfying the following conditions:

- $\rho(e_1)\neq\rho(e_2)$ for any adjacent $e_1,e_2\in E(M)\cup S(M)$;
- $\rho(s_1)=\rho(s_2)$ for any mutually incident $s_1,s_2\in S(M)$.

The *addition* of two elements (a,b) and (c,d) of $\mathcal{C}\cup\{(0,0)\}$ is defined as

$$(a,b)+(c,d)=((a+c) \bmod 2,(b+d) \bmod 2).$$

For any $X\subseteq S(M)$, let

$$\overline{\rho}(X)=\sum_{e\in X}\rho(e).$$

The Parity Lemma (Lemma 1.15) can be restated in the following form.

Lemma 2.25. *Let $M=(V,E,S)$ be a cubic multipole with a 3-edge-colouring. If m_c denotes the number of semiedges of M with colour c, then $m_c\equiv|S| \pmod 2$.* □

For $c\in\mathcal{C}$ and an integer k greater than zero, the sum $\sum_{i=1}^{k}c$ is equal to $(0,0)$ when k is even, and is equal to c when k is odd. Moreover, the addition $(0,1)+(1,0)+(1,1)$ is equal to $(0,0)$. Lemma 2.25 is equivalent to the following statement.

Lemma 2.26. *Let $M=(V,E,S)$ be a cubic multipole and let ρ be a 3-edge-colouring of M. Then, $\overline{\rho}(S)=(0,0)$.*

Proof. Let M be a cubic multipole. In order to see that Lemma 2.25 implies Lemma 2.26, notice that m_c, as defined in the statement of Lemma 2.25, has the same parity for every colour c. To show the converse, it is sufficient to consider the case when m_c does not have the same parity for every colour c. In this case, the sum $\overline{\rho}(S(M))$ is not equal to $(0,0)$. □

Let $M=(V(M),E(M),S(M))$ be a multipole with a partition of $S(M)$ into n pairwise disjoint nonempty sets $S_1,S_2,\dots,S_n$. Sets S_i, with $1\le i\le n$, are the *connectors* of multipole M. Figure 2.46 shows a multipole with two connectors. A *superedge* is a multipole with two connectors, while a *supervertex* is a multipole with three connectors. The following two statements are corollaries of Lemma 2.26.

Corollary 2.27. *If $M=(V,E,S)$ is a multipole with two connectors S_1 and S_2, and ρ is a 3-edge-colouring of M, then $\overline{\rho}(S_1)=\overline{\rho}(S_2)$.*

Proof. Since M has a 3-edge-colouring, then, by Lemma 2.26, $\overline{\rho}(S)=(0,0)$. Thus, $\overline{\rho}(S_1)+\overline{\rho}(S_2)=(0,0)$. This can happen only if $\overline{\rho}(S_1)=\overline{\rho}(S_2)$. □

Corollary 2.28. *Let $M=(V,E,S)$ be a multipole with three connectors S_1, S_2 and S_3. Let ρ be a 3-edge-colouring of M such that $\overline{\rho}(S_i)\neq(0,0)$ for each $i\in\{1,2,3\}$. Then, $\overline{\rho}(S_1)$, $\overline{\rho}(S_2)$, and $\overline{\rho}(S_3)$ are pairwise distinct.*

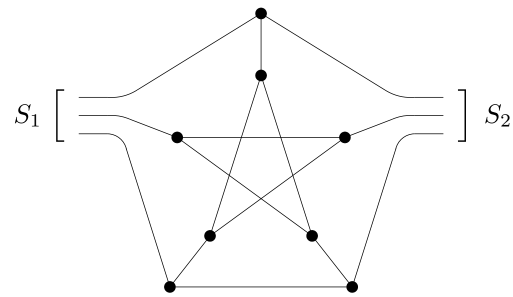


Figure 2.46: A multipole with two connectors.

Proof. Suppose that for two connectors of M, say S_1 and S_2, $\overline{\rho}(S_1) = \overline{\rho}(S_2)$. Then, $\overline{\rho}(S_1) + \overline{\rho}(S_2) = (0, 0)$. Hence, $\overline{\rho}(S) = \overline{\rho}(S_1) + \overline{\rho}(S_2) + \overline{\rho}(S_3) = (0, 0) + \overline{\rho}(S_3)$. By hypothesis, $\overline{\rho}(S_3) \neq (0, 0)$. Therefore, $\overline{\rho}(S) \neq (0, 0)$, contradicting Lemma 2.26. □

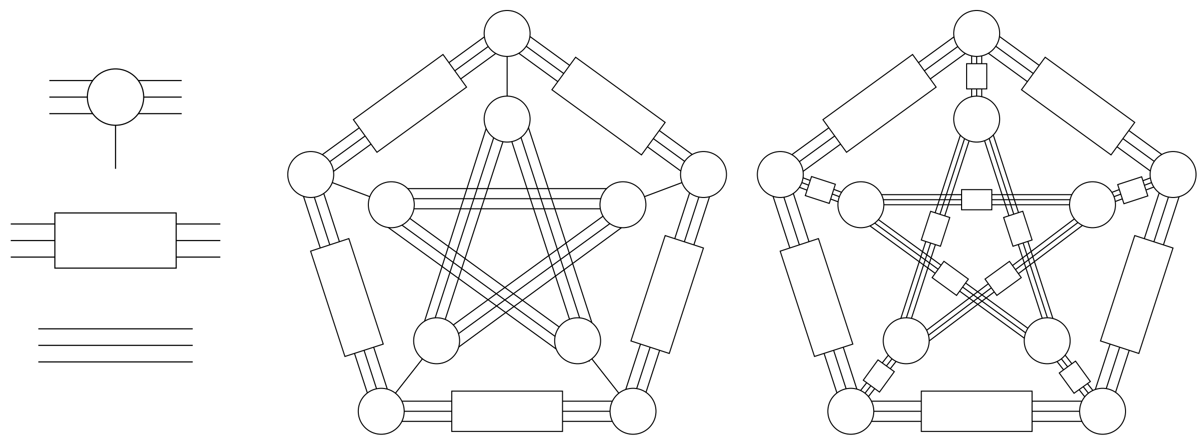
Let G be a cubic graph with two nonadjacent vertices u and v. Denote by $G_{u,v}$ the superedge constructed in the following manner. First, cut the edges of $\partial(u)$ and remove vertex u and its incident semiedges. Denote by S_1 the set of semiedges created. Repeat the operation for vertex v and make S_2 the new set of semiedges obtained. The resulting multipole, together with connectors S_1 and S_2, is superedge $G_{u,v}$. The multipole of Figure 2.46 is a superedge $P_{u,v}$ obtained from the Petersen Graph.

A multipole $M = (V, E, S)$ with connectors $S_1, \ldots, S_n$ is *proper* if $\overline{\rho}(S_i) \neq (0, 0)$ for every 3-edge-colouring ρ of M, and for each i, with $1 \leq i \leq n$. The following lemma introduces a way of constructing proper superedges.

Lemma 2.29. *Let G be a snark. Let $u, v \in V(G)$, u and v nonadjacent. Then, multipole $G_{u,v}$ is a proper superedge.*

Proof. Let S_1 and S_2 be the connectors of $G_{u,v}$. Suppose $G_{u,v}$ is not a proper superedge. Thus, there exists a 3-edge-colouring ρ of $G_{u,v}$ such that for one of the connectors of $G_{u,v}$, say S_1, $\overline{\rho}(S_1) = (0, 0)$. By Corollary 2.27, we have that $\overline{\rho}(S_2) = \overline{\rho}(S_1)$. Since S_1 has three semiedges, each semiedge has a different colour. Similarly for S_2. Thus, ρ induces a 3-edge-colouring of G, contradicting the hypothesis that G is a snark. We conclude that $G_{u,v}$ is a proper superedge. □

Let M and N be multipoles. Let S be a connector of M and let T be a connector of N. If $|S| = |T|$, we can associate S and T by saying that they are *paired*, and we *join* S and T by joining each semiedge of S with a semiedge of T. Let G be a loopless cubic graph. Associate with each edge e of G a superedge $\mathcal{E}(e)$, and with each vertex v of G, a supervertex $\mathcal{V}(v)$, such that for all $e \in E(G)$ and $v \in V(G)$, with e incident with v, a connector of $\mathcal{E}(e)$ is paired with a connector of $\mathcal{V}(v)$. Moreover, each connector must be paired with exactly one other connector. Join every two paired connectors. The resulting cubic graph is a *superposition* of G, denoted by $G(\mathcal{V}, \mathcal{E})$. Figure 2.47 illustrates examples of superposition. If $\mathcal{E}(e)$ is proper for every $e \in E(G)$, then $G(\mathcal{V}, \mathcal{E})$ is a *proper superposition*. Theorem 2.30 shows that this method can be used to construct snarks.



(a) A supervertex and two superedges.

(b) Superpositions of the Petersen Graph.

Figure 2.47: Examples of superposition.

Theorem 2.30. *Let G be a snark. Then, every proper superposition $G(\mathcal{V}, \mathcal{E})$ is a snark.*

Proof. Let G be a snark, and Let $G(\mathcal{V}, \mathcal{E})$ be a proper superposition of G. Suppose that $G(\mathcal{V}, \mathcal{E})$ has a 3-edge-colouring ρ. Let $e \in E(G)$, and let S_1 and S_2 be the connectors of $\mathcal{E}(e)$. By Corollary 2.27, $\overline{\rho}(S_1) = \overline{\rho}(S_2)$. Let $\theta(e) = \overline{\rho}(S_1)$. Since $\mathcal{E}(e)$ is a proper superedge, $\theta(e) \neq (0, 0)$. Let $v \in V(G)$, and let e_1, e_2, e_3 be the edges incident with v. By Corollary 2.28, $\theta(e_1), \theta(e_2)$, and $\theta(e_3)$ are pairwise distinct. Therefore, θ is a 3-edge-colouring of G, contradicting the hypothesis that G is a snark. $\square$

Each Descartes Snark [21], presented in Section 2.3, is a special case of superposition. Let P be the Petersen Graph. For each $v \in V(P)$, let $\mathcal{V}_D(v)$ be a supervertex composed of a cycle of length nine, with a semiedge incident with each vertex of the cycle. The set of semiedges is partitioned into three parts of size three, each part being a connector of $\mathcal{V}_D(v)$. For every $e \in E(P)$, let the proper superedge $\mathcal{E}_D(e)$ be a copy of the multipole of Figure 2.46. Then, each Descartes Snark is a superposition $P(\mathcal{V}_D, \mathcal{E}_D)$. This superposition has the same general structure as the illustration in the right of Figure 2.47(b).

Using superposition, Kochol constructed infinite families of cyclically 4-edge-connected and cyclically 5-edge-connected snarks with arbitrarily large girth. With these families, he disproved the *Girth Conjecture*, a conjecture by Jaeger and Swart [46] which says that every snark has girth at most six. More precisely, Kochol proved the following theorems.

Theorem 2.31 (Kochol [63])**.** *For any given $c \geq 5$, there exists an infinite family of cyclically 5-edge-connected snarks with girth c.*

Theorem 2.32 (Kochol [63])**.** *Let G be a cyclically 4-edge-connected cubic graph of order n and with girth $c \geq 4$. Then, there exists a cyclically 4-edge-connected snark $T(G)$ of order $12n + 10$ and with girth c.*

Kochol also constructed infinite families of cyclically 6-edge-connected snarks [60, 62], although these families do not contain graphs with arbitrarily large girth. The Flower Snarks J_k, $k \geq 5$, are cyclically 6-edge-connected as well. Jaeger and Swart [47] conjectured that a snark with cyclic edge-connectivity greater than six does not exist. This conjecture is mentioned by Kochol and it remains open.

Conjecture 2.33 (Jaeger and Swart)**.** *There is no cyclically 7-edge-connected snark.*

Chapter 3

Fulkerson's Conjecture and related results

Research on cubic graphs has been very much motivated by the Four-Colour Problem. Moreover, other important problems, such as the Cycle Double Cover Conjecture, have contributed to increase the interest in cubic graphs. Matchings in cubic graphs have received special attention. The earliest remarkable result in this subject is Petersen's Theorem (Theorem 1.11), which states that every bridgeless cubic graph has a perfect matching. Another fundamental result, stronger than Petersen's Theorem, asserts that every edge of any bridgeless cubic graph belongs to a perfect matching (Theorem 1.13).

Several results and conjectures about matchings in cubic graphs were published after Petersen's Theorem. One of the most celebrated problems in this field was proposed by Fulkerson [30] in 1971, in the following quote:

> "can one assign pairs of colors from six colors to the edges of such a [3-connected cubic] graph so that all six colors appear at each vertex, i.e., does such a graph have a "Tait bicoloring" in six colors? The Petersen graph does" (see Figure 3.1).

Fulkerson was the first to publish this question. For this reason, the conjecture that its answer is positive is usually called *Fulkerson's Conjecture*. Seymour [81] claims that Claude Berge had posed the conjecture before, although he never published it. Thus, it is also referred to as the *Berge-Fulkerson Conjecture* [50].

Conjecture 3.1 (Berge-Fulkerson Conjecture)**.** *Every bridgeless cubic graph has six perfect matchings such that every edge belongs to exactly two of them.*

Fulkerson's Conjecture is considered a very challenging open problem in Graph Theory. Not much progress has been done towards its solution. The conjecture is satisfied by cubic

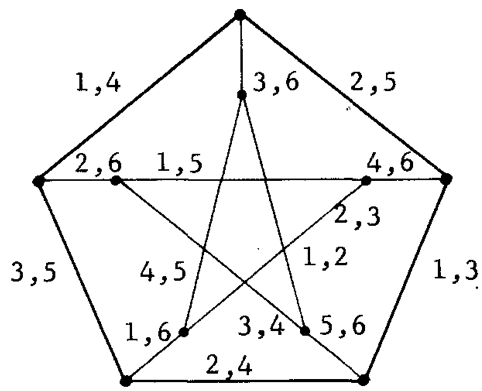


Figure 3.1: Figure taken from Fulkerson's paper [30], exhibiting what Fulkerson called a Tait bicolouring of the Petersen Graph with six colors.

graphs with chromatic index three, as discussed in Section 1.4. As detailed in Section 3.2, a necessary condition for a k-regular graph G to be k-edge-colourable is that $|\partial(X)| \geq k$, for every $X \subseteq V(G)$ with $|X|$ odd. This fact motivates the next definition. A graph G is an *r-graph* if it is r-regular and $|\partial(X)| \geq r$, for all $X \subseteq V(G)$ with $|X|$ odd. In 1965, Edmonds [23] provided a result which implies that every r-graph has a uniform cover by perfect matchings. In 1979, Seymour [81] gave a different proof of this fact, and proposed a generalization of Fulkerson's Conjecture, stating that every r-graph has a k-cover by perfect matchings with $k = 1$ or $k = 2$. Seymour provided another result closely related to Fulkerson's Conjecture, shown in Section 3.2.

Still in 1979, Celmins proved that Fulkerson's Conjecture holds for an infinite class of snarks. This class comprises the Petersen Graph, the Flower Snarks, the Double Star Snark, and other snarks obtained from them by successive applications of Isaacs' dot product. Celmins' results are detailed in Section 3.2.

In 1994, Genghua Fan and André Raspaud [25] showed that the statement of Fulkerson's Conjecture implies that every brigdeless graph G admits a cover by three even subgraphs G_1, G_2, G_3, with $|E(G_1)| + |E(G_2)| + |E(G_3)| \leq \frac{22}{15}|E(G)|$. They proved that this upper bound is tight over all bridgeless graphs, and conjectured that every bridgeless cubic graph has three perfect matchings such that no edge belongs to all of them.

More recently, some results and other conjectures related to Fulkerson's Conjecture have appeared. For instance, Giuseppe Mazzuoccolo [71] showed that a conjecture formulated by Claude Berge in the beginning of years 1990, which was believed to be weaker than Fulkerson's Conjecture, is actually equivalent to it. *Berge's Conjecture* states that every bridgeless cubic graph has a cover by five perfect matchings. Furthermore, Hao et. al. [38] provided necessary and sufficient conditions for a bridgeless cubic graph to satisfy Fulkerson's Conjecture. Additionally, Jean-Luc Fouquet and Jean-Marie Vanherpe [28] provided other sufficient conditions. The results by Hao and by Fouquet and Vanherpe can be used, for example, to verify Fulkerson's Conjecture for classes of snarks.

In the next section we show that no trivial snark is a minimum counterexample to Fulkerson's Conjecture. In Section 3.2, we introduce the mentioned results due to Seymour and Celmins. Section 3.3 presents some equivalent formulations of Fulkerson's Conjecture, such as Berge's Conjecture, as well as other related conjectures. Section 3.4 presents alternative ways of investigating Fulkerson's Conjecture by means of necessary and sufficient conditions discovered by Hao et al. and by Fouquet and Vanherpe.

3.1 Fulkerson's Conjecture and trivial snarks

Recall from Section 2.3.2 that a snark is a trivial snark if it has a nontrivial edge cut of size at most three, or a cycle of length at most four. Moreover, recall that some simple operations can be applied to a trivial snark to produce a smaller nontrivial snark. In this section, it is shown that no trivial snark is a minimum counterexample to Fulkerson's Conjecture.

Let G be a minimum counterexample to Fulkerson's Conjecture. Suppose that G is a trivial snark. We consider the following cases.

Case 1: G has a 2-edge cut $\partial(X)$, $X \subseteq V(G)$.
Suppose $\partial(X) = \{x_1x_2, y_1y_2\}$, with $x_1, y_1 \in X$ and $x_2, y_2 \in V(G) \setminus X$. Note that $x_1 \neq y_1$ and $x_2 \neq y_2$ since G is bridgeless. Define G_1 as $G[X]$ plus edge x_1y_1, and G_2 as $G[V(G) \setminus X]$ plus edge x_2y_2. Fulkerson's Conjecture holds for G_1 and G_2. We claim that G also satisfies Fulkerson's Conjecture. In order to see that, let $\{M_i^1, \ldots, M_i^6\}$ be a double cover by perfect matchings of G_i, for $i = 1, 2$. Suppose $x_iy_i \in M_i^1, M_i^2$. Let $M^j = M_1^j \cup M_2^j$, for $1 \leq j \leq 6$. Then, replace x_1y_1 and x_2y_2 in both M^1 and M^2 by x_1x_2 and y_1y_2. Thus, $\{M^1, \ldots, M^6\}$ is a double cover by perfect matchings of G, a contradiction.

Case 2: G has a nontrivial 3-edge cut $\partial(X)$, $X \subseteq V(G)$.
Suppose $\partial(X) = \{x_1x_2, y_1y_2, z_1z_2\}$, with $x_1, y_1, z_1 \in X$ and $x_2, y_2, z_2 \in V(G) \setminus X$. Define G_1 as $G[X]$ plus vertex v_1 and edges v_1x_1, v_1y_1, v_1z_1, and define G_2 as $G[V(G) \setminus X]$ plus vertex v_2 and edges v_2x_2, v_2y_2, v_2z_2. Fulkerson's Conjecture holds for G_1 and G_2. To see that it also holds for G, let $\{M_i^1, \ldots, M_i^6\}$ be a double cover by perfect matchings of G_i, for $i = 1, 2$. Without loss of generality, suppose $v_ix_i \in M_i^1, M_i^2$, $v_iy_i \in M_i^3, M_i^4$, and $v_iz_i \in M_i^5, M_i^6$. Let $M^j = (M_1^j \setminus \partial_{G_1}(v_1)) \cup (M_2^j \setminus \partial_{G_2}(v_2))$, for $1 \leq j \leq 6$. Then, add x_1x_2 to M^1, M^2, add y_1y_2 to M^3, M^4, and add z_1z_2 to M^5, M^6. This process yields a double cover by six perfect matchings of G.

<u>Case 3</u>: G has a k-cycle, with $k = 2, 3$.
In this case, G has a nontrivial k-edge cut comprised by the edges incident with vertices of the k-cycle. Therefore, G can be analysed as in the previous cases.

<u>Case 4</u>: G has a 4-cycle C.
Let G' be the graph obtained from G by replacing C with two edges e, f as illustrated in Figure 3.2. Since G is a minimum counterexample, G' satisfies Fulkerson's Conjecture. Therefore, e and f together belong to either two, three, or four matchings. Considering each case separately, as depicted in Figure 3.2, it is possible to show that G satisfies Fulkerson's Conjecture, a contradiction.

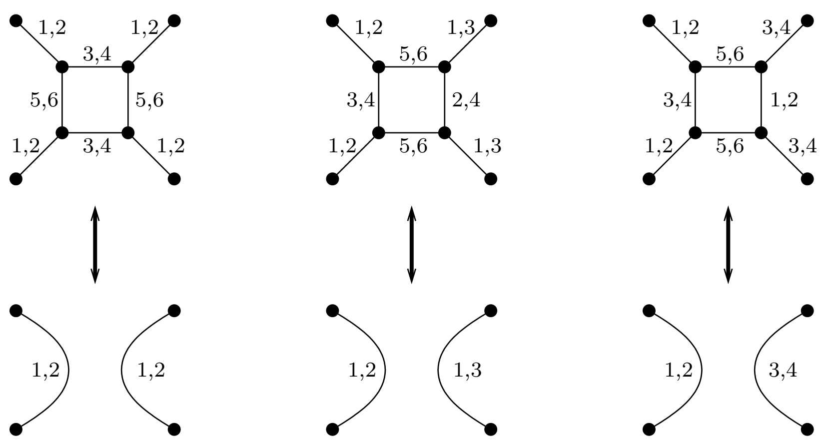


Figure 3.2: A 4-cycle of a trivial snark. Each $1, \ldots, 6$ indicates a distinct perfect matching.

From the above discussion we conclude that it is sufficient to consider only nontrivial snarks when investigating Fulkerson's Conjecture.

3.2 Early results

In 1979, Seymour [81] and Celmins [16] published some of the first partial results directly related to Fulkerson's Conjecture. Seymour also proposed a generalisation of Fulkerson's Conjecture. Celmins showed that an infinite class of snarks constructed with the dot product operation satisfies Fulkerson's Conjecture. The current section details part of these authors' work.

We start by discussing Seymour's contribution. His work is described in an extensive paper, with many intermediate and related results. We selected those which are most relevant and directly related to the subject of this text.

Let G be an r-regular graph and suppose G is r-edge-colourable. Let $X \subseteq V(G)$, with $|X|$ odd. Take a perfect matching M of G and denote by M_X the intersection between

M and the edge set of $G[X]$. The number of M_X-saturated vertices is equal to $2|M_X|$. Since $|X|$ is odd, there is at least one M_X-unsaturated vertex $x \in X$. Therefore, since M is a perfect matching, M saturates x through an edge of $\partial(X)$. Given that G has r pairwise disjoint perfect matchings, $\partial(X)$ has at least r edges. This discussion motivates the following definition.

> A graph G is an *r-graph* if it is r-regular and $|\partial(X)| \geq r$ for all $X \subseteq V(G)$, with $|X|$ odd.

Thus, a necessary condition for an r-regular graph to be r-edge-colourable is that it be an r-graph. However, this condition is not sufficient. For instance, the Petersen Graph is a 3-graph, although it is not 3-edge-colourable.

Proposition 3.2. *If G is an r-graph, with $r > 0$, then $|V(G)|$ is even.*

Proof. Let G be an r-graph, with $r > 0$. Suppose $|V(G)|$ is odd. Thus, $|\partial(V(G))| \geq r$. Since $r > 0$, $|\partial(V(G))| > 0$, a contradiction. ☐

Proposition 3.3. *A cubic graph is bridgeless if and only if it is a 3-graph.*

Proof. Let G be a bridgeless cubic graph. Let $X \subseteq V(G)$, with $|X|$ odd. By hypothesis, $|\partial(X)| > 1$. By Theorem 1.5, $|\partial(X)|$ is odd. Therefore, $|\partial(X)| \geq 3$. Conversely, let G be a 3-graph. Suppose that G has a bridge. Thus, $|\partial(X)| = 1$ for some $X \subseteq V(G)$. By Theorem 1.5, $|X|$ is odd, a contradiction. ☐

The next theorem generalises Petersen's Theorem (Theorem 1.11) for every r-graph.

Theorem 3.4. *If G is an r-graph, with $r > 0$, then G has a perfect matching.*

Proof. Let G be an r-graph, $r > 0$. Let $S \subseteq V(G)$ and $V_1, \ldots, V_k$ be the vertex sets of the odd components of $G \setminus S$. Edge cuts $\partial(V_1), \ldots, \partial(V_k)$ are pairwise disjoint and, since G is an r-graph, $|\partial(V_i)| \geq r$, for $1 \leq i \leq k$. Moreover, $\bigcup_{i=1}^{k} \partial(V_i) \subseteq \partial(S)$. Thus, $kr \leq |\partial(S)|$. Since G is r-regular, $|\partial(S)| \leq |S|r$. Therefore, $k \leq |S|$. By Tutte's Theorem (Theorem 1.10), G has a perfect matching. ☐

Consider a perfect matching of a graph G as a function $f \colon E(G) \to \{0, 1\}$ where $f(e) = 1$ if and only if e belongs to the perfect matching. In this case, addition and subtraction of perfect matchings are naturally defined as addition and subtraction of functions. If k is an integer, then $\mathbf{k}$ denotes the constant function that associates k to every edge of G. Using this terminology, Seymour provided the following formulation of Fulkerson's Conjecture.

Conjecture 3.5. *For any bridgeless cubic graph, constant function $\mathbf{2}$ can be obtained by addition of perfect matchings.*

A *p-multiple edge-colouring* of an r-graph G is a p-cover by perfect matchings of G. Note that the cover has exactly pr perfect matchings. A p-multiple edge-colouring of an r-graph G can be seen as a special edge-colouring where each edge receives p different colours from a set of pr colours and any two adjacent edges do not receive the same colour.

For every r-graph G, let $\mathscr{P}(G)$ be the set of integers $p > 0$ such that G has a p-multiple edge-colouring. Denote by $p_*(G)$ the smallest member of $\mathscr{P}(G)$. If G is r-edge-colourable, then $p_*(G) = 1$. If P is the Petersen Graph, then $p_*(P) = 2$. Fulkerson's Conjecture statement is equivalent to the assertion that $p_*(G) = 1$ or 2 for every 3-graph G. Seymour conjectured that $p_*(G) = 1$ or 2 for every r-graph G. This assertion became known as the *Generalised Berge-Fulkerson Conjecture*.

Conjecture 3.6 (Generalised Berge-Fulkerson Conjecture)**.** *If G is an r-graph, then there exist $2r$ perfect matchings of G with the property that every edge of G belongs to exactly two of them.*

Seymour proved that every r-graph admits a uniform cover by perfect matchings. He argued that this result is a corollary of the Matching Polytope Theorem due to Edmonds [23]. In what follows, we present Seymour's proof.

Lemma 3.7. *Let G be an r-graph. If G has a p-multiple edge-colouring, then G has a kp-multiple edge-colouring, for every positive integer k.*

Proof. Let G be an r-graph. Suppose that G has a p-multiple edge-colouring $\mathcal{M} = \{M_i \colon 1 \leq i \leq rp\}$. Let k be a positive integer, and let

$$\mathcal{M}' = \{M_{ij} \colon 1 \leq i \leq rp \text{ and } 1 \leq j \leq k\},$$

where M_{ij} is a copy of M_i. Every edge of G is in kp members of $\mathcal{M}'$, and $|\mathcal{M}'| = k(rp) = r(kp)$. Thus, $\mathcal{M}'$ is a kp-multiple edge-colouring of G. $\square$

Theorem 3.8 (Edmonds [23], Seymour [81])**.** *If G is an r-graph, then G has a p-multiple edge-colouring for some integer $p > 0$.*

Proof. Let G be an r-graph. Recall that $\mathscr{P}(G)$ is the set of integers $p > 0$ such that G has a p-multiple edge-colouring. The proof is by induction on $|V(G)|$ and r, and considers three cases.

<u>Case 1</u>: There exists a perfect matching M such that $G \setminus M$ is an $(r-1)$-graph.
By induction hypothesis, $G \setminus M$ has a p-multiple edge-colouring $\mathcal{M}$. Let $M_1, \ldots, M_p$ be p copies of M. Hence, $\mathcal{M} \cup \{M_1, \ldots, M_p\}$ is a p-multiple edge-colouring of G.

<u>Case 2</u>: There is a nontrivial edge cut $\partial_G(S)$, with $|\partial_G(S)| = r$ and $|X|$ odd.
Let $S_1 = S$ and $S_2 = V(G) \setminus S_1$. By Proposition 3.2, $|S_2|$ is also odd. Let G_1 be obtained

from G by shrinking S_2 into a vertex v_1. Since $|\partial_G(S_1)| = r$, v_1 has degree r. Moreover, shrinking S_2 preserves the degrees of vertices of S_1. Therefore, G_1 is r-regular. Take $X \subseteq V(G_1)$, with $|X|$ odd. If $v_1 \notin X$, then $\partial_{G_1}(X) = \partial_G(X)$. Thus, $|\partial_{G_1}(X)| \geq r$. If $v_1 \in X$, then $\partial_{G_1}(X) = \partial_G((X - v_1) \cup S_2)$. Since $|X|$ and $|S_2|$ are odd, $|(X - v_1) \cup S_2|$ is also odd. In this case, $|\partial_{G_1}(X)| \geq r$ as well. Therefore, G_1 is an r-graph. Define G_2 similarly to G_1.

By the induction hypothesis, $\mathscr{P}(G_1) \neq \emptyset$ and $\mathscr{P}(G_2) \neq \emptyset$. Choose $p_1 \in \mathscr{P}(G_1)$ and $p_2 \in \mathscr{P}(G_2)$. Let $p = p_1p_2$. By Lemma 3.7, $p \in \mathscr{P}(G_1) \cap \mathscr{P}(G_2)$. Define p-multiple edge-colourings $\mathcal{M}_1$ for G_1 and $\mathcal{M}_2$ for G_2 as

$$\mathcal{M}_1 = \{M_1^{ij} \colon 1 \leq i \leq r,\ 1 \leq j \leq p\} \quad \text{and} \quad \mathcal{M}_2 = \{M_2^{ij} \colon 1 \leq i \leq r,\ 1 \leq j \leq p\}.$$

Let $\partial_G(S) = \{e^1, \ldots, e^r\}$. Denote by e_k^i the edge of G_k equivalent to e^i, for $k = 1, 2$ and $1 \leq i \leq r$. Adjust notation so that $e_k^i \in M_k^{ij}$.

Let $M^{ij} = M_1^{ij} \cup M_2^{ij}$, replacing each edge e_k^i by its equivalent e^i. Let $\mathcal{M} = \{M^{ij} \colon 1 \leq i \leq r,\ 1 \leq j \leq p\}$. Every M^{ij} is a perfect matching of G, and each edge of $E(G)$ is contained in precisely p matchings of $\mathcal{M}$. Thus, $\mathcal{M}$ is a p-multiple edge-colouring of G.

<u>Case 3</u>: Case 1 and Case 2 do not apply.
By Theorem 3.4, G has a perfect matching M. We construct a p-multiple edge-colouring $\mathcal{M}$ of G, for some integer $p > 0$, such that $M \in \mathcal{M}$.

For every $X \subseteq V(G)$, $|X|$ odd, such that $|M \cap \partial_G(X)| > 1$, define

$$t_X = \frac{|\partial_G(X)| - r}{|M \cap \partial_G(X)| - 1}. \tag{3.1}$$

Given that $|M \cap \partial_G(X)| > 1$, $\partial_G(X)$ is not trivial. Because Case 2 does not apply, $|\partial_G(X)| > r$. Thus, $t_X > 0$. Since Case 1 does not apply, there exists $S \subseteq V(G)$, with $|X|$ odd, such that $|\partial_{G \setminus M}(S)| < r - 1$, which implies $|\partial_G(S)| - |M \cap \partial_G(S)| < r - 1$. Since $|\partial_G(S)| \geq r$, we conclude that $|M \cap \partial_G(S)| > 1$. Hence, t_S is well defined and, additionally, $t_S < 1$. Thus, let

$$t_* = \min\{t_X \colon X \subseteq V(G),\ |X| \text{ odd},\ |M \cap \partial_G(X)| > 1\}. \tag{3.2}$$

Since $t_S < 1$, we conclude that $0 < t_* < 1$.

We claim that $|\partial_G(X)| - r \geq t_*(|M \cap \partial_G(X)| - 1)$, that is,

$$|\partial_G(X)| - t_*|M \cap \partial_G(X)| \geq r - t_*, \text{ for every } X \subseteq V(G), |X| \text{ odd}. \tag{3.3}$$

If $|M \cap \partial_G(X)| > 1$, then (3.4) follows by the definition of t_X and t_*. Otherwise, $|M \cap \partial_G(X)| \leq 1$, which implies $t_*(|M \cap \partial_G(X)| - 1) \leq 0$. Since $|\partial_G(X)| - r \geq 0$, (3.4) is satisfied.

Let x and y be non-negative integers such that $t_* = x/y$. Note that $x \neq 0$ and $x < y$. Let $r' = ry - x$. We construct a graph G' from G by replacing each edge of $E(G) \setminus M$ by y parallel edges, and each edge of M by $y - x$ parallel edges. We show that G' is an r'-graph.

Let $v \in V(G')$. Since v has degree r in G and M is a perfect matching, the degree of v in G' is $(r-1)y + (y-x)$, that is, r'. Therefore, G' is r'-regular. Consider $S \subseteq V(G')$, with $|S|$ odd. Thus,

$$\begin{aligned}
|\partial_{G'}(S)| &= y|\partial_G(S) \setminus M| + (y-x)|M \cap \partial_G(S)| \\
&= y|\partial_G(S)| - y|M \cap \partial_G(S)| + (y-x)|M \cap \partial_G(S)| \\
&= y|\partial_G(S)| - x|M \cap \partial_G(S)| \\
&= y|\partial_G(S)| - yt_*|M \cap \partial_G(S)| \\
&= y \cdot (|\partial_G(S)| - t_*|M \cap \partial_G(S)|). \qquad (3.4)
\end{aligned}$$

Therefore, by (3.4),

$$|\partial_{G'}(S)| \geq y(r - t_*) = yr - yt_* = ry - x = r'.$$

Consider $S \subseteq V(G)$, with $|S|$ odd and $|M \cap \partial_G(S)| > 1$, such that $t_S = t_*$. By (3.1), we have $|\partial_G(S)| - t_*|M \cap \partial_G(S)| = r - t_*$. Therefore, by (3.4), $|\partial_{G'}(S)| = yr - yt_* = r'$. By Case 2, G' has a p'-multiple edge-colouring $\mathcal{M}'$, for some integer p' greater than zero. Construct a $p'y$-multiple edge-colouring of G as follows. For each $M' \in \mathcal{M}'$, replace every edge of M' by its parallel edge of $E(G)$. The result is a collection of perfect matchings of G such that each edge of $E(G) \setminus M$ is contained in exactly $p'y$ perfect matchings, and each edge of M is contained in exactly $p'(y-x)$ perfect matchings. By adding $p'x$ copies of M to this collection, we get a $p'y$-multiple edge-colouring of G. □

Seymour's main result related to Fulkerson's Conjecture states that for every 3-graph G, the constant function **2** can be obtained by addition and subtraction of perfect matchings of G. In order to prove this result, stated as Corollary 3.13, additional definitions and auxiliary results are needed. First, we show that $\mathscr{P}(G)$ is eventually periodic. Denote by $p_T(G)$ the period of $\mathscr{P}(G)$. Then, we prove that $p_T(G)$ is at most two. Finally, Corollary 3.13 is presented as a consequence of this last fact.

The following lemma is a well known property of a greatest common divisor, and it is used in the proof of Lemma 3.10.

Lemma 3.9. *Suppose that S is a non-empty set of positive integers, with the property that if $p, q \in S$, then $p + q \in S$. Let d be the greatest common divisor of the members of S. Then, S consists of all positive multiples of d, except for a finite number of them.* □

Lemma 3.10. *Let G be an r-graph. Then, $\mathscr{P}(G)$ is eventually periodic. Moreover, $\mathscr{P}(G)$ has period $p_T(G)$ at most two, if and only if $\mathscr{P}(G)$ contains all even integers greater than n_0, for n_0 sufficiently large.*

Proof. Let G be an r-graph. Notice that, if positive integers p and q belong to $\mathscr{P}(G)$, then so does $p + q$. Moreover, by Theorem 3.8, $\mathscr{P}(G) \neq \emptyset$. Thus, by Lemma 3.9, $\mathscr{P}(G)$ is eventually periodic and its period $p_T(G)$ is equal to the greatest common divisor of all its members.

Suppose $p_T(G) \leq 2$. If $p_T(G) = 1$, then $\mathscr{P}(G)$ contains all sufficiently large integers, and the result follows. Thus, we assume that $p_T(G) = 2$. Because $p_T(G)$ is the greatest common divisor of all members of $\mathscr{P}(G)$, every member of $\mathscr{P}(G)$ is even, and the result follows.

Conversely, suppose that $\mathscr{P}(G)$ contains all sufficiently large even integers. It is straightforward to see that $p_T(G) \leq 2$. $\square$

For a graph G, let $\Omega(G)$ be the set of non-zero mappings $w\colon E(G) \to \{-1, 0, +1\}$ such that for every $v \in V(G)$, $\sum_{\partial(v)} w(e) = 0$. Moreover, graph $G(w)$ is obtained from G by removing every edge e with $w(e) = -1$, and adding a new edge parallel to every edge e with $w(e) = +1$. Notice that, for every $v \in V(G)$, the degree of v is the same in G and $G(w)$. The next lemma relates 3-graphs and $\Omega(G)$, and it is used in the proof of Theorem 3.12. Lemma 3.11 has a very long proof, which can be found in Seymour's paper.

Lemma 3.11. *Let G be a 3-graph which is not the Petersen Graph. Then, one of the following holds:*

(i) *G has a nontrivial k-edge cut, with $k \leq 3$;*

(ii) *G has a cycle of length at most four;*

(iii) *there exist $w_1, w_2, w_3 \in \Omega(G)$, such that $w_1 + w_2 + w_3 = \mathbf{0}$ and $G(w_1)$, $G(w_2)$, $G(w_3)$, $G(-w_1)$ are 3-graphs.* $\square$

Theorem 3.12 (Seymour)**.** *If G is a 3-graph, then $\mathscr{P}(G)$ is eventually periodic, with period $p_T(G)$ at most two.*

Proof. The proof is by induction on the number of vertices. By Lemma 3.10, $\mathscr{P}(G)$ is eventually periodic, and it suffices to show that $\mathscr{P}(G)$ contains all sufficiently large even integers. If $|V(G)| \leq 4$, then the theorem is true. Thus, we assume that $|V(G)| \geq 6$ and that the theorem is true for all 3-graphs with less than $|V(G)|$ vertices.

Case 1: G is not connected.
Let G_1 be a connected component of G, and $G_2 = G \setminus V(G_1)$. Thus, G_1 and G_2 are 3-graphs and both $|V(G_1)|$ and $|V(G_2)|$ are less than $|V(G)|$. By the induction hypothesis,

each of $\mathscr{P}(G_1)$ and $\mathscr{P}(G_2)$ has period at most two. Hence, by Lemma 3.10, each of $\mathscr{P}(G_1)$ and $\mathscr{P}(G_2)$ contains all sufficiently large even integers. Thus, this is also true for $\mathscr{P}(G_1) \cap \mathscr{P}(G_2)$. We show that

$$\mathscr{P}(G_1) \cap \mathscr{P}(G_2) \subseteq \mathscr{P}(G). \tag{3.5}$$

Suppose $p \in \mathscr{P}(G_1) \cap \mathscr{P}(G_2)$. Let $\{M_j^i \colon 1 \leq j \leq 3p\}$ be a p-multiple edge-colouring of G_i, for $i = 1, 2$. Thus, $\{M_j^1 \cup M_j^2 \colon 1 \leq j \leq 3p\}$ is a p-multiple edge-colouring of G. Therefore, $p \in \mathscr{P}(G)$. Thus, $\mathscr{P}(G)$ contains all sufficiently large even integers and, by Lemma 3.10, $p_T(G) \leq 2$.

<u>Case 2</u>: G has a 2-edge cut.
Let $X_1 \subseteq V(G)$, $\partial(X_1) = \{e_1, e_2\}$, and $X_2 = V(G) \setminus X_1$. Let G_1 be the graph obtained by shrinking X_2 and contracting e_2. Define G_2 analogously. Because G is bridgeless, G_i is also bridgeless, for $i = 1, 2$. Moreover, $|V(G_i)| < |V(G)|$ and G_i is cubic. Thus, by the induction hypothesis, $p_T(G_i) \leq 2$. We prove that $p_T(G) \leq 2$ by showing that

$$\mathscr{P}(G_1) \cap \mathscr{P}(G_2) \subseteq \mathscr{P}(G).$$

Let $p \in \mathscr{P}(G_1) \cap \mathscr{P}(G_2)$ and $\{M_j^i \colon 1 \leq j \leq 3p\}$ be a p-multiple edge-colouring of G_i, $i = 1, 2$, such that the edge of G_i equivalent to e_i belongs to M_j^i for $1 \leq j \leq p$. Make $M_j = M_j^1 \cup M_j^2$, replacing each edge not in $E(G)$ by its equivalent edge of $E(G)$. Then, $\{M_j \colon 1 \leq j \leq 3p\}$ is a p-multiple edge-colouring of G.

<u>Case 3</u>: G has a nontrivial 3-edge cut.
Let $X_1 \subseteq V(G)$, $\partial(X_1) = \{e_1, e_2, e_3\}$ a nontrivial edge cut, and $X_2 = V(G) \setminus X_1$. Let G_1 be obtained by shrinking X_2, and G_2 obtained by shrinking X_1. Because G is bridgeless, G_1 and G_2 are also bridgeless. Moreover, $|V(G_i)| < |V(G)|$ and G_i is cubic, for $i = 1, 2$. Thus, by the induction hypothesis, $p_T(G_i) \leq 2$. Let $p \in \mathscr{P}(G_1) \cap \mathscr{P}(G_2)$ and $\{M_j^i \colon 1 \leq j \leq 3p\}$ be a p-multiple edge-colouring of G_i. Let e_1^i, e_2^i, e_3^i be the edges of G_i respectively, equivalent to edges e_1, e_2, e_3 of G. Note that $\{e_1^i, e_2^i, e_3^i\}$ is a trivial edge cut of G_i. Thus, since M_j^i is a perfect matching, it contains precisely one of e_1^i, e_2^i, e_3^i. Adjust notation so that

$$e_1^i \in M_1^i, \ldots, M_p^i; \quad e_2^i \in M_{p+1}^i, \ldots, M_{2p}^i; \quad e_3^i \in M_{2p+1}^i, \ldots, M_{3p}^i.$$

Make $M_j = M_j^1 \cup M_j^2$, replacing each edge not in $E(G)$ by the equivalent edge of $E(G)$. Then, $\{M_j^1 \cup M_j^2 \colon 1 \leq j \leq 3p\}$ is a p-multiple edge-colouring of G.

<u>Case 4</u>: Previous cases do not apply and G has a cycle of length at most four.
Let C be a 4-cycle of G, with $E(C) = \{f_1, f_2, f_3, f_4\}$ and $\partial(C) = \{e_1, e_2, e_3, e_4\}$ as shown in Figure 3.3. Construct G' modifying G as follows. Remove edges e_2 and e_4. Because G

has no 3-edge cut, the resulting graph is bridgeless. Now, contract edges f_4, e_1 and edges f_3, e_3. This process is illustrated in Figure 3.3. Note that, after contractions, f_1 and f_2 have no common end, otherwise G has a 3-cycle.

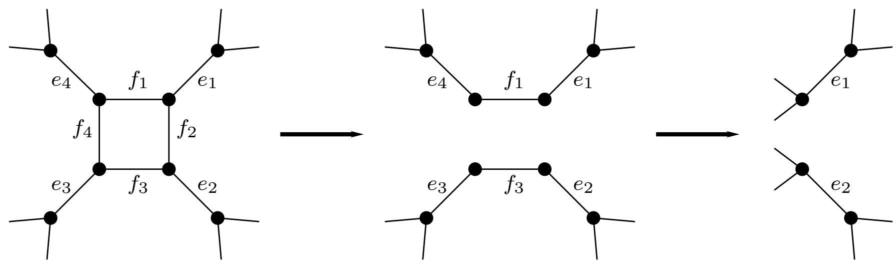


Figure 3.3: Case 4.

Graph G' is cubic and bridgeless. Therefore, by the induction hypothesis, $p_T(G') \leq 2$. We prove that $p_T(G) \leq 2$ by showing that

$$\mathscr{P}(G') \subseteq \mathscr{P}(G).$$

Let $p \in \mathscr{P}(G')$ and $\{M_i \colon 1 \leq i \leq 3p\}$ be a p-multiple edge-colouring of G' such that

$$\begin{aligned} &f_1 \in M_i, f_2 \in M_i \quad 1 \leq i \leq t_0,\\ &f_1 \in M_i, f_2 \notin M_i \quad t_0+1 \leq i \leq t_1,\\ &f_1 \notin M_i, f_2 \in M_i \quad t_1+1 \leq i \leq t_2,\\ &f_1 \notin M_i, f_2 \notin M_i \quad t_2+1 \leq i \leq 3p. \end{aligned}$$

Both f_1 and f_2 belong to p perfect matchings. Thus, $t_1 = p$ and $t_2 - t_1 + t_0 = p$. Thus, $t_2 = 2p - t_0$. Each M_i is a matching in G. To derive a perfect matching of G from M_i, we extend M_i by adding edges of $E(C) \cup \partial(C)$ to saturate the vertices of C that are not M_i-saturated as follows:

$$\begin{aligned} &\{M_i \cup \{f_3, f_4\} \colon 1 \leq i \leq t_0\}\\ \cup\, &\{M_i \cup \{f_4, e_3\} \colon t_0+1 \leq i \leq p\}\\ \cup\, &\{M_i \cup \{f_3, e_1\} \colon p+1 \leq i \leq 2p-t_0\}\\ \cup\, &\{M_i \cup \{e_1, e_3\} \colon 2p-t_0+1 \leq i \leq 2p\}\\ \cup\, &\{M_i \cup \{e_2, e_4\} \colon 2p+1 \leq i \leq 3p\}. \end{aligned}$$

Therefore, $p \in \mathscr{P}(G)$ and the result follows.

Case 5: Previous cases do not apply.
The Petersen Graph has a double edge-colouring, as shown in Figure 1.7, Section 1.4.

Thus, if G is the Petersen Graph, then $\{2,4,6,\dots\} \subseteq \mathscr{P}(G)$ and, by Lemma 3.10, the result follows. We assume that G is not the Petersen Graph.

By previous cases, we know that every nontrivial edge cut of G has at least four edges, and that every cycle has length at least five. Hence, by Lemma 3.11, there exist $w_1, w_2, w_3 \in \Omega(G)$ such that $w_1 + w_2 + w_3 = \mathbf{0}$ and $G(w_1)$, $G(w_2)$, $G(w_3)$, and $G(-w_1)$ are all 3-graphs. Each of $G(w_1)$, $G(w_2)$, $G(w_3)$, and $G(-w_1)$ has 2-cycles. Therefore, by Case 2, the theorem is true for all these graphs and, by Lemma 3.10, $\mathscr{P}(G(w))$ contains all sufficiently large even integers, for every $w \in \{w_1, w_2, w_3, -w_1\}$.

First, we prove the following statements:

(i) if $p \in \mathscr{P}(G(w_1)) \cap \mathscr{P}(G(-w_1))$, then $2p \in \mathscr{P}(G)$;

(ii) if $p \in \mathscr{P}(G(w_1)) \cap \mathscr{P}(G(w_2)) \cap \mathscr{P}(G(w_3))$, then $3p \in \mathscr{P}(G)$.

Suppose $p \in \mathscr{P}(G(w_1)) \cap \mathscr{P}(G(-w_1))$. Let $\mathcal{M}_+$ be a p-multiple edge-colouring of $G(w_1)$ and let $\mathcal{M}_-$ be a p-multiple edge colouring of $G(-w_1)$, defined as

$$\mathcal{M}_+ = \{M_{i+} \colon 1 \le i \le 3p\} \quad \text{and} \quad \mathcal{M}_- = \{M_{i-} \colon 1 \le i \le 3p\}.$$

For each M_{i+}, replace every edge $e \in M_{i+} \backslash E(G)$ by the edge of $E(G)$ parallel to e in $G(w_1)$. Denote by M'_{i+} the perfect matching of G obtained this way. Proceed similarly for each M_{i-} to obtain M'_{i-}. Thus, $\mathcal{M}'_+ = \{M'_{i+} \colon 1 \le i \le 3p\}$ is a collection of perfect matchings such that each $e \in E(G)$ is in exactly $(1+w_1(e))p$ of them, while $\mathcal{M}'_- = \{M'_{i-} \colon 1 \le i \le 3p\}$ is a collection of perfect matchings such that each $e \in E(G)$ is in exactly $(1 - w_1(e))p$ of them. Therefore, $\mathcal{M}'_+ \cup \mathcal{M}'_-$ is a $2p$-multiple edge-colouring of G. Thus, $2p \in \mathscr{P}(G)$, as claimed.

Statement (ii) is proved in a similar way. Suppose $p \in \mathscr{P}(G(w_1)) \cap \mathscr{P}(G(w_2)) \cap \mathscr{P}(G(w_3))$. Let $\mathcal{M}_i = \{M_{i,j} \colon 1 \le j \le 3p\}$ be a p-multiple edge-colouring of $G(w_i)$, for $i = 1, 2, 3$. In each $M_{i,j}$, replace every edge e not in $E(G)$ by the edge of $E(G)$ parallel to e in $G(w_i)$. Denote the resulting perfect matching of G by M'_{ij}. Thus, $\mathcal{M}'_i = \{M'_{ij} \colon 1 \le j \le 3p\}$ is a collection of perfect matchings such that each $e \in E(G)$ belongs to precisely $(1 + w_i(e))p$ of them. Because $w_1(e) + w_2(e) + w_3(e) = 0$ for every $e \in E(G)$, we have that $(1 + w_1(e))p + (1 + w_2(e))p + (1 + w_3(e))p = 3p$. Therefore, $\mathcal{M}'_1 \cup \mathcal{M}'_2 \cup \mathcal{M}'_3$ is a $3p$-multiple edge-colouring of G, and $3p \in \mathscr{P}(G)$, as claimed.

Now, (i) and (ii) are used to conclude the case. Let $Q = \mathscr{P}(G(w_1)) \cap \mathscr{P}(G(w_2)) \cap \mathscr{P}(G(w_3)) \cap \mathscr{P}(G(-w_1))$. By construction, Q contains all sufficiently large even integers. Let $p \in Q$. By statements (i) and (ii), $2p, 3p \in \mathscr{P}(G)$. By Lemma 3.9, $p_T(G)$ divides $2p$ and $3p$. Hence, $2p/p_T(G)$ is an integer, and so is $3p/p_T(G) = 2p/p_T(G) + p/p_T(G)$. Thus, $p_T(G)$ divides p. Therefore, $p_T(G)$ divides every $p \in Q$. Let $q, q+2 \in Q$. Then, $q/p_T(G)$ and $(q+2)/p_T(G) = q/p_T(G) + 2/p_T(G)$ are integers. Hence, $p_T(G) \le 2$. $\square$

Now, we are ready to establish Seymour's result related to Fulkerson's Conjecture.

Corollary 3.13 (Seymour)**.** *For every 3-graph G, constant function $\mathbf{2}$ is obtained by addition and, possibly, subtraction of perfect matchings of G.*

Proof. Let G be a 3-graph. By Theorem 3.12, there exist $p, p+2 \in \mathscr{P}(G)$. Let $q = p+2$. Consider $\mathcal{M}_p$ a p-multiple edge-colouring of G, and $\mathcal{M}_q$ a q-multiple edge-colouring of G. Consider each perfect matching as a function. By adding all perfect matchings of $\mathcal{M}_p$, we get constant function $\mathbf{p}$, and by adding all perfect matchings of $\mathcal{M}_q$, we get constant function $\mathbf{q}$. Then, $\mathbf{q} - \mathbf{p}$ results in constant function $\mathbf{2}$. ☐

We now describe Celmins' results supporting Fulkerson's Conjecture. These results were published in a technical report from 1979 [16].

Denote the Double Star Snark by Q. Let $\mathscr{B}$ be the family of snarks comprising the Petersen Graph, the Double Star Snark, and the nontrivial Flower Snarks, i.e., $\mathscr{B} = \{P, Q, J_5, J_7, J_9, \dots\}$. Celmins showed that every graph of $\mathscr{B}$ satisfies Fulkerson's Conjecture. It is already known that the conjecture is true for the Petersen Graph. Figure 3.4 exhibits the double cover by perfect matchings of Q defined by Celmins.

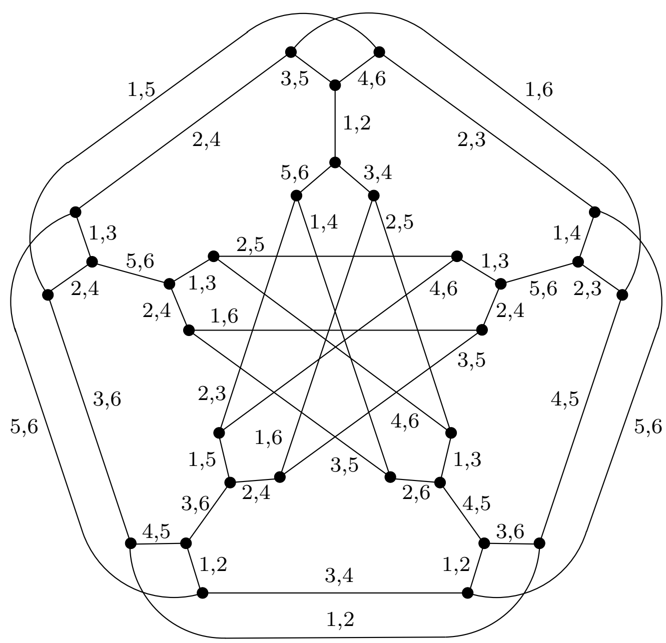
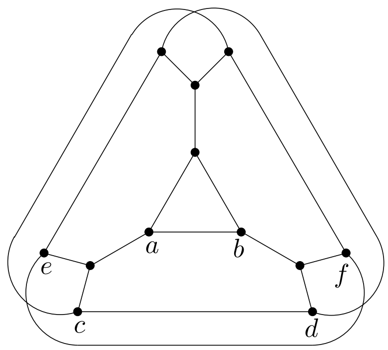


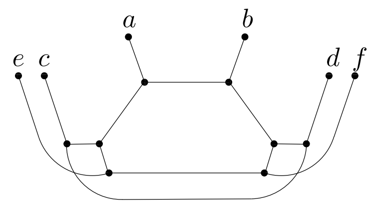
Figure 3.4: A double cover by perfect matchings of the Double Star Snark.

In Section 2.3.3, we construct each Flower Snark J_k, with k odd and $k \geq 3$, from k copies of graph T shown in Figure 2.25(a). Celmins' proof that Flower Snarks satisfy Fulkerson's Conjecture relies on a recursive construction of this family. Consider the trivial snark J_3 and the *link graph* J_L exhibited in Figure 3.5. Flower Snark J_k, with $k \geq 5$, is obtained from J_{k-2} as follows. Choose an integer i such that $1 \leq i \leq k$. Consider copies

T_i and T_{i+1} of T contained in J_{k-2} (indexes greater than k are taken modulo k). Label vertices of T_i and T_{i+1} with $a, \ldots, f$ as indicated in Figure 3.5(a). Then, remove edges ab, cd, and ef, and denote the resulting graph by J^-. Add a copy of J_L and identify each vertex $a, \ldots, f$ of J_L with a vertex of same label of J^-. The result is Flower Snark J_k.



(a) Trivial snark J_3.



(b) Link graph J_L.

Figure 3.5: Celmins' recursive construction of Flower Snarks.

Using the recursive construction just described and the covers of J_3 and J_L shown in Figure 3.6, it is possible to obtain double covers by perfect matchings for all Flower Snarks. Figure 3.6(c) exhibits the double cover obtained for J_5. Observe that each edge e removed from J_3 has the same label as the link graph edges whose ends are identified with the ends of e.

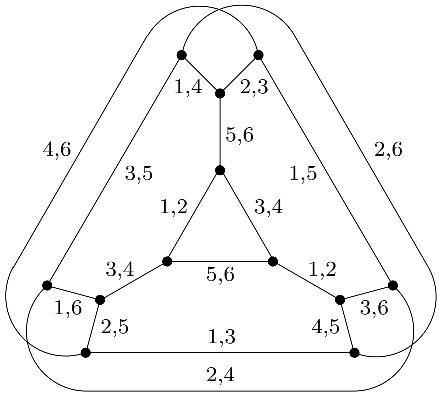
Theorem 3.14 (Celmins). *The Petersen Graph, the Double Star Snark, and the nontrivial Flower Snarks satisfy Fulkerson's Conjecture.* □

The drawings of Flower Snarks and the Double Star Snark presented in this section were taken from Celmin's technical report, and they are more symmetric than Isaacs' drawings presented in Section 2.3.3. However, Isaacs drawings are more common and also more attractive.

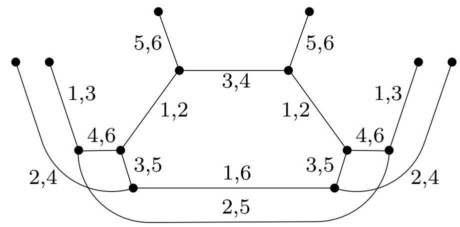
In the same work, Celmins showed that snarks obtained by successive applications of dot product on members of family $\mathscr{B}$ satisfy Fulkerson's Conjecture. It is interesting to note that the Szekeres Snarks and the generalised Blanuša Snarks belong to this extended family.

Theorem 3.15 (Celmins). *Let family $\mathscr{B}$ be comprised by the Petersen Graph, the Double Star Snark, and the nontrivial Flower Snarks. Every snark obtained by successive applications of dot product on members of $\mathscr{B}$ satisfies Fulkerson's Conjecture.* □

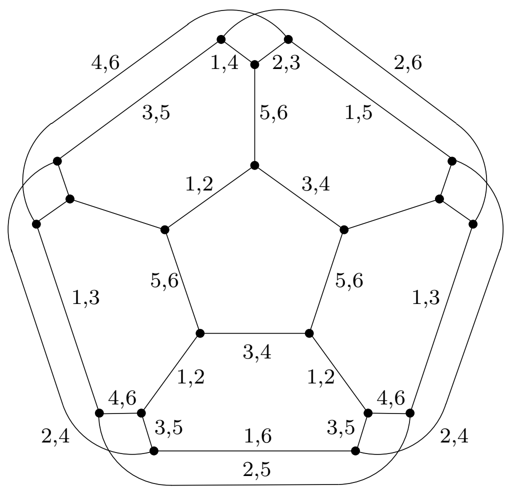
The complete proofs of Theorem 3.14 and Theorem 3.15 can be found in Celmins' technical report [16] or in his PhD thesis [17]. To conclude this section, we present some interesting properties, described by Celmins, of the members of family $\mathscr{B}$.



(a) Cover $\mathcal{C}_3$ of J_3.



(b) Cover $\mathcal{C}_L$ of link graph J_L.



(c) Cover of J_5 obtained from $\mathcal{C}_3$ and $\mathcal{C}_L$.

Figure 3.6: Celmins' construction of double covers by perfect matching of Flower Snarks. Each number indicates a perfect matching.

A *cycle pair* of a cubic graph G is a pair $\{C_1, C_2\}$ of vertex disjoint cycles of G such that $V(G) = V(C_1) \cup V(C_2)$. An *odd cycle pair* is a cycle pair such that each cycle has odd length. An *even cycle pair* is similarly defined. Let e and f be nonadjacent edges of G. An (e, f)-*odd cycle pair* of G is an odd cycle pair such that one cycle contains e and the other contains f. An (e, f)-*even cycle pair* is analogously defined.

Theorem 3.16 (Celmins)**.** *Let $\mathscr{B}$ be the family of snarks comprised by the Petersen Graph, the nontrivial Flower Snarks, and the Double Star Snark. Every $G \in \mathscr{B}$ has an (e, f)-odd cycle pair for any nonadjacent edges $e, f \in E(G)$.* □

Theorem 3.16 is important to several results in Celmins' work. In particular, it is extensively used in the proof of Theorem 3.15.

Celmins [17] described another interesting property of family $\mathscr{B}$. To present this property, some definitions are needed. Take a cubic graph G. Let $e_1, e_2 \in E(G)$ be nonadjacent edges. Subdivide e_1 by a new vertex v_1, subdivide e_2 by a new vertex v_2, and add edge v_1v_2. The resulting graph is an *edge-added version* of G. On the other hand, an *edge-suppressed version* of G is obtained by removing an edge uv from G and suppressing vertices u and v.

A snark G is *delicate* if all edge-added and edge-suppressed versions of G are 3-edge-colourable. A delicate snark can be seen as a snark as close to be 3-edge-colourable as possible.

Theorem 3.17. *Let $\mathscr{B}$ be the family of snarks comprised by the Petersen Graph, the nontrivial Flower Snarks, and the Double Star Snark. Every snark of $\mathscr{B}$ is delicate.*

Proof. Let $G \in \mathscr{B}$, with nonadjacent $e, f \in E(G)$. By Theorem 3.16, G has an (e, f)-odd cycle pair. Let G^+ be an edge-added version of G obtained by subdividing e and f. The subdivision of e replaces it by two adjacent edges, and similarly for f. Hence, G^+ has an even cycle pair $\{C_1, C_2\}$. Now, construct a 3-edge-colouring of G^+ by assigning two colours alternately to the edges of C_1 and C_2, and assigning a third colour to the edges not in $E(C_1) \cup E(C_2)$.

Let $e \in E(G)$, with $e = uv$, $\partial(u) = \{e, e_u, f_u\}$ and $\partial(v) = \{e, e_v, f_v\}$. Since G is a nontrivial snark, e_u and e_v are not adjacent. By Theorem 3.16, G has an (e_u, e_v)-odd cycle pair $\{C_u, C_v\}$, with $e_u \in E(C_u)$ and $e_v \in E(C_v)$. Let G^- be the edge-suppressed version of G obtained by removing e and suppressing u and v. The suppression of u replaces e_u and f_u by a single edge, and similarly for v. Therefore, G^- has an even cycle pair. Construct a 3-edge-colouring of G^- in the same way it was done for G^+. $\square$

Celmins proposed the search for other delicate snarks as a research problem. We do not have any news about progress on this problem. However, some authors have published works related to the number of odd cycles contained in 2-factors of cubic graphs.

Problem 3.18. *Are the Petersen Graph, the Double Star Snark and the nontrivial Flower Snarks the only delicate snarks?*

3.3 Related conjectures

Fulkerson's Conjecture is a very general and fundamental problem. In fact, this conjecture is related to several other important problems in Graph Theory. In this section, we present some conjectures related to Fulkerson's Conjecture. We start by presenting well-known Conjecture 3.19, about uniform covers by even subgraphs [24]. We also demonstrate its equivalence with Fulkerson's Conjecture.

Conjecture 3.19. *Every bridgeless cubic graph has a 4-cover by six even subgraphs.*

Theorem 3.20. *Fulkerson's Conjecture and Conjecture 3.19 are equivalent.*

Proof. Suppose Fulkerson's Conjecture is true. Let G be a bridgeless cubic graph with a double cover by perfect matchings $\mathcal{F} = \{M_1, \ldots, M_6\}$. Let $G_i = G \setminus M_i$ for every $i \in \{1, \ldots, 6\}$. Since M_i is a perfect matching, G_i is 2-regular. Thus G_i is an even subgraph of G. Let $\mathcal{G} = \{G_1, \ldots, G_6\}$. Consider $e \in E(G)$. The fact that e belongs to exactly two matchings of $\mathcal{F}$ implies that e belongs to exactly four even subgraphs of $\mathcal{G}$. Therefore, $\mathcal{G}$ is a 4-cover by six even subgraphs of G.

Now, suppose that Conjecture 3.19 is true. Let G be a bridgeless cubic graph with a 4-cover by even subgraphs $\mathcal{G} = \{G_1, \ldots, G_6\}$. Without loss of generality, assume that G_i, $1 \leq i \leq 6$, has no vertex of degree zero. Thus, G_i is 2-regular. We show that $V(G_i) = V(G)$. Note that $|E(G_i)| = |V(G_i)|$. Moreover, since $\mathcal{G}$ is a 4-cover, $4|E(G)| = \sum_{i=1}^{6} |E(G_i)| = \sum_{i=1}^{6} |V(G_i)|$. By Theorem 1.1, $2|E(G)| = 3|V(G)$ and, thus, $4|E(G)| = 6|V(G)$. Hence, $\sum_{i=1}^{6} |V(G_i)| = 6|V(G)|$. Since $G_i \subseteq G$, each term of the sum is at most $|V(G)|$. Therefore, $|V(G_i)| = |V(G)|$. In this case, $M_i = E(G) \setminus E(G_i)$ is a perfect matching of G. Let $\mathcal{F} = \{M_1, \ldots, M_6\}$. Since each $e \in E(G)$ belongs to exactly four even subgraphs of $\mathcal{G}$, e belongs to exactly two perfect matchings of $\mathcal{F}$. Therefore, $\mathcal{F}$ is a double cover by perfect matchings of G. $\square$

In 1988, Jaeger [45] generalised Conjecture 3.19 to all bridgeless graphs, as stated in Conjecture 3.21, and showed that the two conjectures are actually equivalent. We give a demonstration of this fact.

Conjecture 3.21 (Jaeger [45])**.** *Every bridgeless graph has a 4-cover by six even subgraphs.*

Theorem 3.22. *Conjecture 3.19 and Conjecture 3.21 are equivalent.*

Proof. If Conjecture 3.21 is true, then so is Conjecture 3.19. Suppose that Conjecture 3.19 is true. Let G be a bridgeless graph. Construct G' by suppressing every degree-two vertex of G. Each vertex of G' has degree at least three. Let $v \in V(G')$ with degree d. Label the edges of $\partial(v)$ with $e_1, \ldots, e_d$. Subdivide each e_i with a new vertex v_i, remove v, and add d edges to form cycle $v_1 v_2 \ldots v_d v_1$. Call it a *derived cycle*, denoted by C_v. This operation is illustrated in Figure 3.7. Observe that every new vertex v_i has degree three.

Denote by G'' the bridgeless cubic graph obtained from the previously described operation. By hypothesis, G'' has a 4-cover by six even subgraphs $\mathcal{G}'' = \{G''_1, \ldots, G''_6\}$. Return to graph G' by contracting each derived cycle $C_v = v_1 v_2 \ldots v_d$ to vertex v. Let $G''_i \in \mathcal{G}''$. Let G'_i be the subgraph of G' obtained from G''_i by contracting all derived cycles. Suppose that G''_i contains edges of a cycle C_v derived from $v \in V(G)$. Since every vertex of

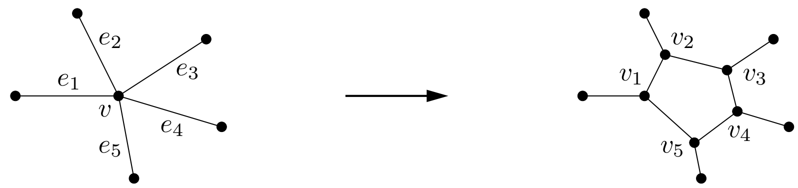


Figure 3.7: Conversion of a bridgeless graph into a cubic bridgeless graph.

$V(C_v) \cap V(G''_i)$ has degree two in G''_i, after contraction of $E(C_v) \cap E(G''_i)$ the degree of v is even in G'_i. Therefore, G'_i is an even subgraph of G'. Moreover, each edge of G' belongs to exactly four of $G'_1, \dots, G'_6$. Thus, we have a 4-cover by six even subgraphs of G'.

Now, return to G by subdividing edges of G' with each degree-two vertex of G. Let G_i be the subgraph of G obtained from G'_i in this process. The subdivision of an edge preserves the degrees of the ends of the subdivided edge, and introduces a new vertex of degree two. Therefore, G_i is an even subgraph of G. Furthermore, each edge of G_i obtained from a subdivision is contained in exactly four of $G_1, \dots, G_6$, since the original subdivided edge was contained in four of $G'_1, \dots, G'_6$. Therefore, $\{G_1, \dots, G_6\}$ is a 4-cover by six even subgraphs of G. $\square$

In 1983, Bermond, Jackson, and Jaeger [6] proved that every bridgeless graph has a 4-cover by seven even subgraphs, which is a weaker version of Conjecture 3.21.

Theorem 3.23 (Bermond, Jackson, Jaeger [6]). *Every bridgeless graph has a* 4*-cover by seven even subgraphs.*

Ury Jamshy and Michael Tarsi [49] showed that if every bridgeless graph has a cover by even subgraphs with length at most $\frac{21}{15}|E(G)|$, then the Cycle Double Cover Conjecture is true. The following result asserts that the statement of Fulkerson's Conjecture implies that every bridgeless graph has a cover by even subgraphs with length at most $\frac{22}{15}|E(G)|$.

Theorem 3.24 (Fan and Raspaud [25]). *If Fulkerson's Conjecture is true, then every bridgeless graph G has a cover $\mathcal{C}$ by three even subgraphs and the length of $\mathcal{C}$ is at most $\frac{22}{15}|E(G)|$.*

If Fulkerson's Conjecture is true, then any five perfect matchings taken from a double cover by six perfect matchings of a bridgeless cubic graph constitute a cover of the graph. In the beginning of years 1990, Berge conjectured that every bridgeless cubic graph has a cover by five perfect matchings [50]. Berge's Conjecture was long believed to be weaker than Fulkerson's Conjecture. Nevertheless, in 2011, Mazzuoccolo [71] showed that the conjectures actually are equivalent.

Conjecture 3.25 (Berge's Conjecture). *Every bridgeless cubic graph has a cover by five perfect matchings.*

Theorem 3.26 (Mazzuoccolo). *Fulkerson's Conjecture and Berge's Conjecture are equivalent.* □

In studying a hard problem, it is common to consider different and possibly easier versions of the problem. Related to Fulkerson's Conjecture, in 1994, Fan and Raspaud [25] proposed an alternative approach, involving the study of sets of simultaneously disjoint perfect matchings. They proposed the so-called *Fan-Raspaud Conjecture* in a paper where they investigated the relation between Fulkerson's Conjecture and problems of covers by even subgraphs.

Conjecture 3.27 (Fan-Raspaud Conjecture [25]). *Every bridgeless cubic graph has perfect matchings M_1, M_2, and M_3, such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset.$$

If Fulkerson's Conjecture is true, then so is Conjecture 3.27, since any three of the six perfect matchings provided by Fulkerson's Conjecture would have empty intersection. Some results supporting the Fan-Raspaud Conjecture were published [51, 52, 68, 69]. Section 3.4 gives more details on the relation between the Fan-Raspaud Conjecture and Fulkerson's Conjecture.

Thomáš Kaiser and André Raspaud [51] published a weaker version of the Fan-Raspaud Conjecture, stated here as Conjecture 3.28. Although Conjecture 3.28 was already known before this publication, its origin is not clear. Before stating this conjecture, an additional definition is required. A subgraph H of a graph G is a *parity subgraph* of G if $V(H) = V(G)$ and the degree of each vertex of $V(G)$ has the same parity in G and H.

Conjecture 3.28. *Every bridgeless cubic graph has two perfect matchings M_1 and M_2, and a parity subgraph with edge set J, such that*

$$M_1 \cap M_2 \cap J = \emptyset.$$

Conjecture 3.28 has another formulation due to Edita Máčajová and Martin Škoviera.

Conjecture 3.29 (Máčajová and Škoviera). *Every bridgeless cubic graph has perfect matchings M_1 and M_2, such that $M_1 \cap M_2$ does not contain an odd edge cut.*

Fulkerson's Conjecture, as well as the Fan-Raspaud Conjecture and Conjecture 3.28, are true for cubic graphs with chromatic index three. The *oddness* of a cubic graph G is the minimum number of odd cycles in a 2-factor of G. Note that cubic graphs with chromatic index three have oddness zero, while bridgeless cubic graphs with chromatic index four have even oddness and different from zero. Kaiser and Raspaud [51] showed that Conjecture 3.28 is true for bridgeless cubic graphs with oddness two. Later, Mácǎjová and Škoviera [68] improved this result by proving that Fan-Raspaud Conjecture holds for every bridgeless cubic graph with oddness two.

Máčajová and Škoviera [69] proposed a structured way of looking at these conjectures.

Conjecture 3.30 (k-Perfect-Matching Conjecture)**.** *Let k be an integer with $3 \leq k \leq 6$. Then every bridgeless cubic graph contains a family of k perfect matchings such that any three of them have empty intersection.*

This is a way of approaching the conjectures in apparent order of ascending difficulty. However, it is already known that the statements for $k = 5$ and $k = 6$ are equivalent. An interesting question is whether equivalence occurs among other values of k.

In order to finalize this section, consider an interesting and natural special case of Fulkerson's Conjecture proposed by Roland Häggkvist [36] in 2007. A *Hamiltonian cycle* of a graph is a cycle that contains all vertices of the graph. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. Note that every Hamiltonian cubic graph is 3-edge-colourable: two colours are assigned alternately to the edges of a Hamiltonian cycle, while a third colour is assigned to the remaining edges. Therefore, all Hamiltonian cubic graphs satisfy Fulkerson's Conjecture. A graph G is *hypohamiltonian* if $G - v$ is Hamiltonian for every $v \in V(G)$.

Conjecture 3.31 (Häggkvist)**.** *Every hypohamiltonian cubic graph admits a double cover by six perfect matchings.*

Using Kochol's superposition, Máčajová and Škoviera [66, 67] constructed cyclically 5-edge-connected and cyclically 6-edge-connected hypohamiltonian snarks. This family could be a starting point for investigating Conjecture 3.31.

3.4 Alternative approaches

More recently, some results on Fulkerson's Conjecture have appeared. In 2009, Hao et al. [38] provided necessary and sufficient conditions for a cubic graph satisfying Fulkerson's Conjecture. These conditions are stated in Lemma 3.32. Using this lemma, Hao et al. proved that Goldberg Snarks and Flower Snarks satisfy Fulkerson's Conjecture. In order to prove Lemma 3.32, some intermediate results are necessary.

If G is a graph, then *G-suppressed* is the graph obtained from G by removing all its 2-regular components and suppressing all its remaining degree-two vertices.

Lemma 3.32 (Hao et al. [38])**.** *A cubic graph G admits a double cover by six perfect matchings if and only if G has disjoint matchings M_A and M_B such that $M_A \cup M_B$ is the* *edge set of a union of disjoint cycles and both* $(G \backslash M_A)$*-suppressed and* $(G \backslash M_B)$*-suppressed are* 3*-edge-colourable.* □

As an example, Figure 3.8 shows matchings M_A and M_B of Flower Snark J_5 satisfying conditions stated in Lemma 3.32. Observe that in this case $J_5 \setminus M_A$ and $J_5 \setminus M_B$ are isomorphic.

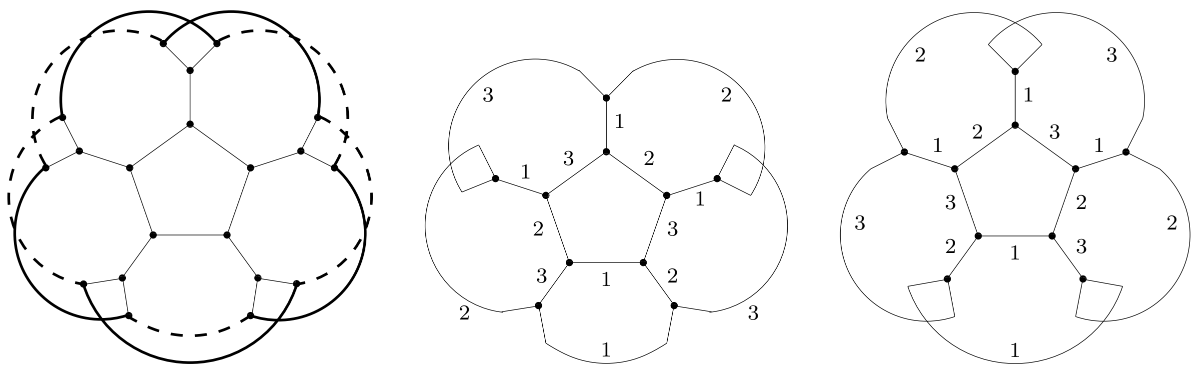


Figure 3.8: Lemma 3.32: Flower Snark J_5 with matchings M_A and M_B in solid and dashed heavy lines, and 3-edge-colourings of $(J_5 \setminus M_A)$-suppressed and $(J_5 \setminus M_B)$-suppressed.

In 2011, Fouquet and Vanherpe [28] published a different proof of Lemma 3.32. Their proof provides more details on the relation between Fulkerson's Conjecture and Fan-Raspaud Conjecture (Conjecture 3.27). Lemma 3.32 is an immediate corollary of Theorem 3.34 and Theorem 3.37.

Throughout this section, a *Fulkerson cover* of a cubic graph is a double cover by six perfect matchings. An *FR-triple* of a cubic graph G is a triple of perfect matchings of G,

$$\mathcal{T} = (M_1, M_2, M_3), \ \text{ such that } \ M_1 \cap M_2 \cap M_3 = \emptyset.$$

Set $T_i \subseteq E(G)$, with $0 \leq i \leq 2$, is the set of the edges of G that belong to exactly i perfect matchings of $\mathcal{T}$.

Proposition 3.33. *Let* G *be a bridgeless cubic graph with an FR-triple* $\mathcal{T}$*. Then,*

(i) T_0 *and* T_2 *are disjoint matchings;*

(ii) $T_0 \cup T_2$ *is the edge set of a 2-regular graph;*

(iii) $\forall v \in V(G)$*, either* $\partial(v) \subseteq T_1$*, or each edge of* $\partial(v)$ *belongs to one of* T_0*,* T_1*,* T_2*.*

Proof. Let G and $\mathcal{T}$ be as stated in the hypothesis. Let $v \in V(G)$. Suppose that an edge of $\partial(v)$ belongs to T_0. Since v is saturated by the three matchings of $\mathcal{T}$, one of the two remaining edges of $\partial(v)$ belongs to T_1, while the other belongs to T_2. Now, suppose that an edge of $\partial(v)$ belongs to T_2. Thus, one of the two remaining edges incident with v belongs to T_1, while the other belongs to T_0. We conclude that each $v \in V(G)$ is incident with an edge of T_0 if and only if v is incident with an edge of T_2, and that no two edges of $\partial(v)$ are both in T_0 neither in T_2. The result follows. $\square$

Two FR-triples $\mathcal{T}$ and $\mathcal{T}'$ are *compatible* if

$$T_2 = T_0' \quad \text{and} \quad T_0 = T_2'.$$

Theorem 3.34. *A bridgeless cubic graph has a Fulkerson cover if and only if it has two compatible FR-triples.*

Proof. Let G be a bridgeless cubic graph with a Fulkerson cover $\mathcal{F} = \{M_1, \ldots, M_6\}$. Let $\mathcal{T} = \{M_1, M_2, M_3\}$ and $\mathcal{T}' = \{M_4, M_5, M_6\}$. Because no edge of G belongs to three perfect matchings of $\mathcal{F}$, both $\mathcal{T}$ and $\mathcal{T}'$ are FR-triples. The edges of T_0 do not belong to any of M_1, M_2, M_3. Since $\mathcal{F}$ is a Fulkerson cover, each edge of T_0 belongs to exactly two of M_4, M_5, M_6. Conversely, each edge of G belonging to exactly two of M_4, M_5, M_6 does not belong to any of M_1, M_2, M_3. Thus, $T_0 = T_2'$. It is similarly shown that $T_0' = T_2$. Therefore, $\mathcal{T}$ and $\mathcal{T}'$ are compatible.

Conversely, suppose that G is a bridgeless cubic graph with compatible FR-triples $\mathcal{T}$ and $\mathcal{T}'$. Let $\mathcal{F} = \mathcal{T} \cup \mathcal{T}'$. Let $e \in E(G)$. If $e \in T_0$, then $e \in T_2'$. Hence, e belongs to exactly two perfect matchings of $\mathcal{F}$. In case $e \in T_2$, it implies that $e \in T_0'$ and, thus, e also belongs to exactly two perfect matchings of $\mathcal{F}$. As the last case, suppose $e \in T_1$. Note that $E(G) = T_0 \cup T_1 \cup T_2 = T_0' \cup T_1' \cup T_2'$. Since $T_0 \cup T_2 = T_0' \cup T_2'$, we conclude that $T_1 = T_1'$. Hence, e belongs to exactly two perfect matchings of $\mathcal{F}$. Therefore, $\mathcal{F}$ is a Fulkerson cover of G. $\square$

Proposition 3.35. *Let G be a bridgeless cubic graph with an FR-triple $\mathcal{T}$. Then, $(G\setminus T_0)$-suppressed is 3-edge-colourable.*

Proof. Consider G a bridgeless cubic graph with an FR-triple $\mathcal{T} = (M_1, M_2, M_3)$. Let G' be $(G \setminus T_0)$-suppressed. By Proposition 3.33, a vertex of G is incident with an edge of T_2 if and only if it is incident with an edge of T_0. Thus, every degree-two vertex of $G \setminus T_0$ is incident with an edge of T_2. Since G' does not have vertices of degree two, the edges of T_2 do not belong to $E(G')$. Then, every edge of G' is either an edge of T_1 or a new edge added by suppression of degree-two vertices.

Considering these two cases, we construct a mapping $\pi : E(G') \to \{1, 2, 3\}$ as follows. If $e \in E(G') \cap T_1$, set $\pi(e) = i$, where $e \in M_i$. Suppose $e \in E(G') \setminus T_1$, with $e = xy$. By construction of G', there exists an (x, y)-path P in $G \setminus T_0$ such that the internal vertices of P have degree two. In G', both x and y have degree three. Proposition 3.33 implies that the edges of P belong to T_1 and T_2 alternately. Consider an edge $f = uv$ of $E(P) \cap T_2$. Let $f_u \in T_1 \cap \partial_G(u)$ and let $f_v \in T_1 \cap \partial_G(v)$. Edge f belongs to two perfect matchings of $\mathcal{T}$. Consequently, both f_u and f_v belong to the remaining perfect matching of $\mathcal{T}$, say

M_i. Therefore, all edges of $E(P) \cap T_1$ belong to M_i, in particular the first and last edges of P. Thus, set $\pi(e) = i$.

By construction of π and by the fact that M_1, M_2, M_3 are perfect matchings, π is a 3-edge-colouring of G'. $\square$

Lemma 3.36. *Let G be a bridgeless cubic graph with disjoint matchings M_A and M_B, such that $M_A \cup M_B$ is the edge set of a 2-regular subgraph of G and $(G \setminus M_A)$-suppressed is 3-edge-colourable. Then, G has an FR-triple $\mathcal{T}$ with $T_0 = M_A$ and $T_2 = M_B$.*

Proof. Let G, M_A and M_B be defined as in the hypothesis. Denote $(G \setminus M_A)$-suppressed by G'. Let $\{M_1', M_2', M_3'\}$ be a 3-edge-coloring of G'. We modify M_1', M_2', M_3' to produce an FR-triple of G. Notice that the vertex set of G' is the set of degree-three vertices of $G \setminus M_A$. Initially, let $M_i = M_i' \setminus (E(G') \setminus E(G))$, for $i = 1, 2, 3$. Notice that M_1, M_2, M_3 are matchings of $G \setminus M_A$. We add edges to M_1, M_2, M_3 to produce perfect matchings of G.

Let $e \in E(G') \setminus E(G)$ and let P_e be the path of $G \setminus M_A$ replaced by e. The edges of P_e alternately belong to M_B. Moreover, the first and last edges of P_e do not belong to M_B, since they are incident with degree-three vertices in $G \setminus M_A$. Suppose $e \in M_i'$. Then, add to M_i every edge of $E(P_e) \setminus M_B$, and add every edge of $E(P_e) \cap M_B$ to both matchings of $\{M_1, M_2, M_3\} \setminus M_i$. Then, every vertex of P_e is saturated by the three matchings M_1, M_2, M_3.

For every 2-regular component C of $G \setminus M_A$, proceed as follows. Every vertex of C is M_B-saturated. Thus, C is an even cycle. For every $e \in E(C)$, add e to M_1 and to M_2 if $e \in M_B$. Otherwise, add e to M_3. Thus, every vertex of C is saturated by the three matchings M_1, M_2, M_3.

Now, since $V(G \setminus M_A) = V(G)$, every vertex of $V(G)$ is saturated by M_1, M_2, and M_3. Notice that no edge was assigned to more than two of M_1, M_2, M_3. Therefore, $\mathcal{T} = (M_1, M_2, M_3)$ is an FR-triple of G. Moreover, the edges of M_A were not assigned to any matchings, and each edge of M_B was assigned to exactly two matchings. Thus, $T_0 = M_A$ and $T_2 = M_B$. $\square$

Theorem 3.37. *Let G be a bridgeless cubic graph. Graph G has disjoint matchings M_A and M_B, such that $M_A \cup M_B$ is the edge set of a 2-regular subgraph of G, and both $(G \setminus M_A)$-suppressed and $(G \setminus B)$-suppressed are 3-edge-colourable, if and only if G has two compatible FR-triples.*

Proof. Conversely, suppose G, M_A and M_B defined as in the hypothesis. Since $(G \setminus M_A)$-suppressed and $(G \setminus M_B)$-suppressed are 3-edge-colourable, by Lemma 3.36, G has an FR-triple $\mathcal{T}$ with $T_0 = M_A$ and $T_2 = M_B$, and an FR-triple $\mathcal{T}'$ with $T_0' = M_B$ and $T_2' = M_A$. Moreover, given that $T_0 = M_A = T_2'$ and $T_2 = M_B = T_0'$, $\mathcal{T}$ and $\mathcal{T}'$ are compatible.

Let G be a bridgeless cubic graph with compatible FR-triples $\mathcal{T}$ and $\mathcal{T}'$. By Proposition 3.35, $(G \setminus T_0)$-suppressed and $(G \setminus T'_0)$-suppressed are 3-edge-colourable. Moreover, by Proposition 3.33, T_0 and T_2 are disjoint matchings of G and $T_0 \cup T_2$ is the edge set of a union of disjoint cycles. Since $\mathcal{T}$ and $\mathcal{T}'$ are compatible, $T'_0 = T_2$. Thus, the result follows by making $M_A = T_0$ and $M_B = T_2$. □

In the same work, Fouquet and Vanherpe provided another set of sufficient conditions for a bridgeless cubic graph satisfying Fulkerson's Conjecture.

Let M be a perfect matching of a bridgeless cubic graph G. A set $X \subseteq E(G)$ is an *M-balanced matching* if there exists a perfect matching M_X such that $X = M \cap M_X$. Let $\mathcal{M} = \{A, B, C, D\}$ be a set of pairwise disjoint M-balanced matchings. Thus, G has perfect matchings M_A, M_B, M_C, M_D such that

$$A = M \cap M_A, \quad B = M \cap M_B, \quad C = M \cap M_C, \quad D = M \cap M_D.$$

Figure 3.9 shows examples of M-balanced matchings for the Petersen Graph, the Flower Snark J_5, and the Blanuša Snarks.

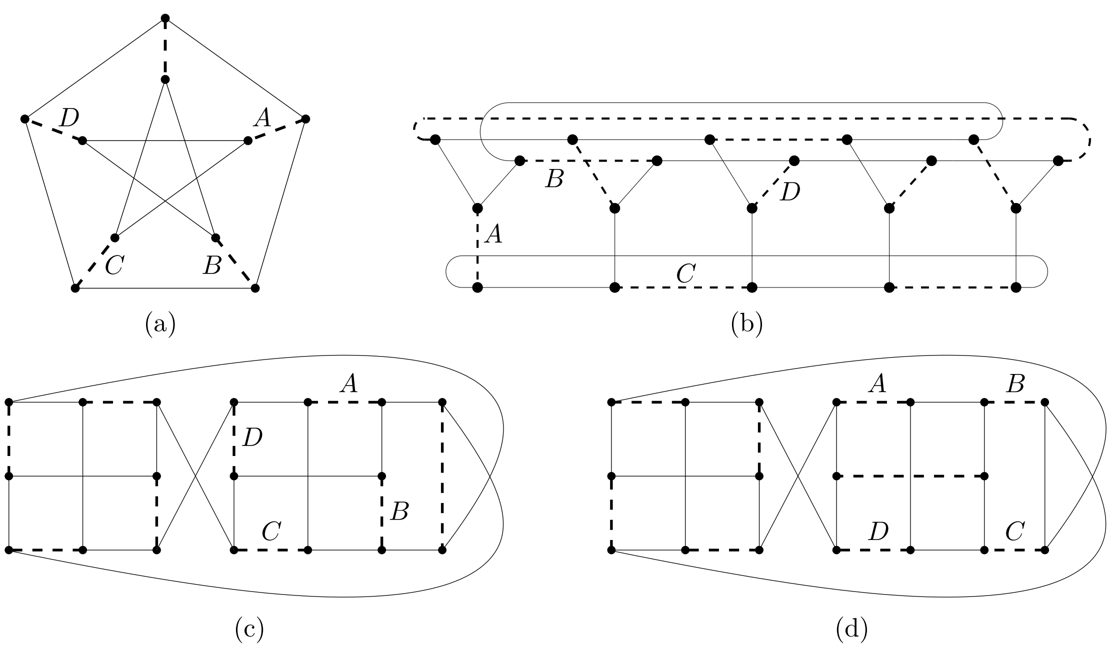


Figure 3.9: A perfect matching M, in bold dashed lines, and four pairwise disjoint M-balanced matchings A, B, C, D.

Since graph $G \setminus M$ is 2-regular, it can be seen as a union of disjoint cycles. Family $\mathcal{M}$ is an *F-family for M* if the following conditions are satisfied.

1. For every odd cycle C of $G \setminus M$ and every $X \in \mathcal{M}$, X saturates exactly one vertex of C.

2. For every even cycle C of $G \setminus M$, if a matching of $\mathcal{M}$ saturates a vertex of C, then C has four vertices such that one of the following is true:
   (a) the four vertices are saturated by the same matching of $\mathcal{M}$;
   (b) two of them are saturated by one matching of $\mathcal{M}$, while the remaining two are saturated by another matching of $\mathcal{M}$.
3. For every cycle C of $G \setminus M$ with four vertices saturated by matchings of $\mathcal{M}$ as determined in the previous items, the subgraph of G induced by these four vertices has a perfect matching.

Fouquet and Vanherpe proved, by construction of a Fulkerson cover, that every bridgeless cubic graph with a perfect matching M and an F-family for M satisfies Fulkerson's Conjecture.

Theorem 3.38 (Fouquet and Vanherpe [28]). *Let G be a bridgeless cubic graph. If G has a perfect matching M and an F-family $\mathcal{M}$ for M, then G has a Fulkerson cover.* □

The families exhibited in Figure 3.9 are examples of F-families. Fouquet and Vanherpe exhibited F-families for the Petersen Graph, the Szekeres Snark, the two families of generalised Blanuša Snarks, and the Flower Snarks. Moreover, they described conditions under which the result of a dot product on two snarks has an F-family. Using these results, they defined a family of snarks derived from members of $\{P, J_5, J_7, J_9, \dots\}$. The family constructed by them is contained in the family defined by Celmins [16], and described in Section 3.2. Proving that these graphs have an F-family is a different approach to prove Fulkerson's Conjecture. An interesting approach would be to investigate the relation between the results by Celmins and by Fouquet and Vanherpe.

Additionaly, Fouquet and Vanherpe observed that they do not know any F-family for the Goldberg Snarks. The search for an F-family for these snarks or the disproval of its existance is also an interesting problem. A more general question arises naturally: is the existence of an F-family a necessary condition for a snark to have a Fulkerson cover?

We close this section with the three just mentioned questions.

Problem 3.39. *What is the relation between Celmins' result (Theorem 3.15 and Theorem 3.16) and Fouquet and Vanherpe's result?*

Problem 3.40. *Do the Goldberg Snarks have an F-family?*

Problem 3.41. *If a snark G satisfies Fulkerson's Conjecture, does G have an F-family?*

Chapter 4

Results

This chapter presents the results we achieved while investigating Fulkerson's Conjecture considering some of the constructions of snarks presented in Chapter 2. The results are organized by the classes of snarks considered. Before presenting them, we introduce a few concepts and definitions.

Let G be a graph with $\Delta(G) \leq 3$, and let L be an *index set* with $|L| = 6$. A *Fulkerson cover* is a family $\mathcal{F} = \{M_l \colon l \in L\}$ of matchings of G such that each edge of $E(G)$ belongs to exactly two members of $\mathcal{F}$. By this definition, every vertex of G with degree three is saturated by the six matchings of $\mathcal{F}$. As a consequence, if G is cubic, then every matching of $\mathcal{F}$ is perfect.

Proposition 4.1. *A cubic graph satisfies Fulkerson's Conjecture if and only if it admits a Fulkerson cover.* $\square$

Let G be a graph such that $\Delta(G) \leq 3$, with a Fulkerson cover $\mathcal{F} = \{M_l \colon l \in L\}$. Denote by $L^{(2)}$ the set of two-element subsets of L. Function $\lambda \colon E(G) \to L^{(2)}$, defined as $\lambda(e) = \{l \in L \colon e \in M_l, M_l \in \mathcal{F}\}$, is *induced* by Fulkerson cover $\mathcal{F}$. By this definition, λ satisfies the following property.

F1 For all adjacent $e, f \in \mathrm{Dom}(\lambda)$, $\lambda(e) \cap \lambda(f) = \emptyset$.

Function λ can be seen as a labelling of the edges of G, where each $e \in E(G)$ has $\lambda(e)$ as its label. Furthermore, if L is considered a set of colours, then λ can be seen as a special colouring where each edge receives two different colours. By property F1, λ conforms with the general concept of colouring in the sense that adjacent edges do not have the same colour. This discussion motivates our next definitions.

A *Fulkerson function* of G, $\Delta(G) \leq 3$, is a function $\lambda \colon E' \to L^{(2)}$ satisfying F1, where $E' \subseteq E(G)$ and $|L| = 6$. The image of λ is a set of unordered pairs of elements of L. An unordered pair $\{p, q\}$ is also denoted by pq or p,q. A *Fulkerson colouring* is a Fulkerson function with domain $E(G)$.

Proposition 4.2. *Fulkerson covers and Fulkerson colourings are equivalent.*

Proof. Let G be a graph with $\Delta(G) \leq 3$. Suppose that G has a Fulkerson cover $\mathcal{F} = \{M_l \colon l \in L\}$, with $L = \{1, \ldots, 6\}$. Let $\lambda \colon E(G) \to L^{(2)}$ be the function induced by $\mathcal{F}$. By definition, λ satisfies property F1. Moreover, $|L| = 6$ and $\mathrm{Dom}(\lambda) = E(G)$. Thus, λ is a Fulkerson colouring.

Now, suppose that G has a Fulkerson colouring $\lambda \colon E(G) \to L^{(2)}$. Let $\mathcal{F} = \{M_l \colon l \in L\}$, with each $M_l = \{e \in E(G) \colon l \in \lambda(e)\}$. By F1, each M_l is a matching. By construction, every edge $e \in E(G)$ belongs to exactly two of these matchings. Therefore, $\mathcal{F}$ is a Fulkerson cover of G. $\square$

In this work, we use the terms Fulkerson cover and Fulkerson colouring indistinctly. Next lemma shows how to construct a Fulkerson colouring for a cubic graph G by making the union of Fulkerson functions of G. Figure 4.1 shows an example of application of Lemma 4.3.

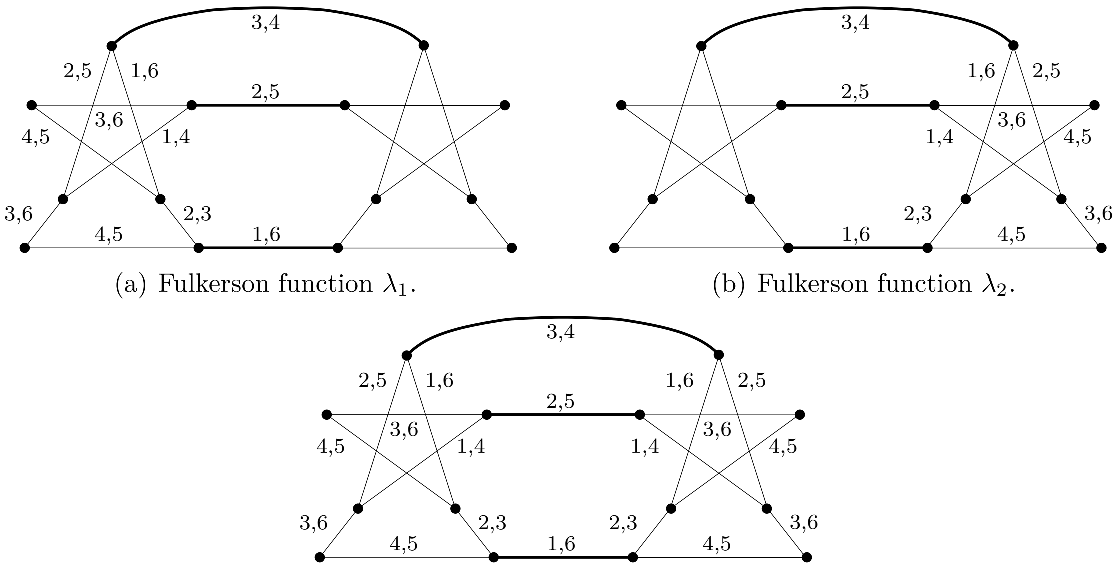
Lemma 4.3. *Let G be a cubic graph and let $X \subseteq V(G)$. If λ_1 and λ_2 are Fulkerson functions of G, with $Dom(\lambda_1) \cup Dom(\lambda_2) = E(G)$ and $Dom(\lambda_1) \cap Dom(\lambda_2) = \partial(X)$, such that $\lambda_1(e) = \lambda_2(e)$ for all $e \in \partial(X)$, then $\lambda_1 \cup \lambda_2$ is a Fulkerson colouring of G.*

Proof. Let G be a cubic graph, with $X \subseteq V(G)$. Let $G_1 = G[X]$, $G_2 = G[V(G) \setminus X]$, and $L = \{1, \ldots, 6\}$. Consider λ_1 and λ_2 as stated in the hypothesis. Let $\lambda = \lambda_1 \cup \lambda_2$. By definition, λ is a function from $E(G)$ to $L^{(2)}$. Consider adjacent $e, f \in E(G)$. Thus, both e and f belong to $\mathrm{Dom}(\lambda_i)$, for some $i \in \{1, 2\}$. Since λ_i is a Fulkerson function and is a restriction of λ, λ satisfies F1. Therefore, λ is a Fulkerson colouring of G. $\square$

4.1 Generalised Blanuša Snarks

The first and second families of generalised Blanuša Snarks are defined in Section 2.4.1. Recall that every member of these families is a result of repeated applications of the dot product using the Petersen Graph P. In 1979, Celmins [16] verified Fulkerson's Conjecture for a family of snarks generated by applications of the dot product. This family includes both generalised Blanuša Snarks families. In 2011, Fouquet and Vanherpe [28] showed that generalised Blanuša Snarks satisfy Fulkerson's Conjecture, using the F-family, as presented in Chapter 3.

In this section, we give a different proof that both families of generalised Blanuša Snarks satisfy Fulkerson's Conjecture. The proof consists of recursively labelling the edges of the members of each family so as to construct a Fulkerson colouring.



(a) Fulkerson function λ_1. (b) Fulkerson function λ_2.

(c) Fulkerson colouring $\lambda_1 \cup \lambda_2$.

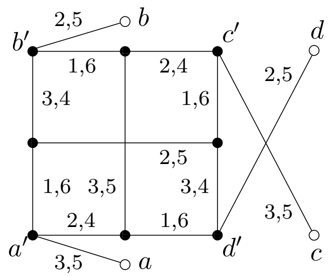
Figure 4.1: Example of Lemma 4.3. Edge cut $\partial(X)$ is indicated by thick edges.

Theorem 4.4. *Every graph of the first family of generalised Blanuša Snarks has a Fulkerson colouring.*

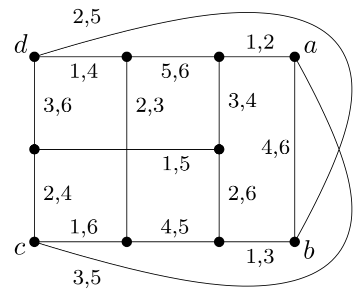
Proof. Let $\mathcal{B}^f = \{B_1^f, B_2^f, B_3^f, \dots\}$ be the first family of generalised Blanuša Snarks. Moreover, let B_0^f be a copy of the Petersen Graph as depicted in Figure 4.2(b), and let $\mathcal{B} = \{B_0^f\} \cup \mathcal{B}^f$. We prove, by induction on i, that every $B_i^f \in \mathcal{B}$, with $i \geq 0$, has a Fulkerson colouring λ_i.

Graph B_0^f has a Fulkerson function λ_0 exhibited in Figure 4.2(b), where the pair of labels of each edge e represents $\lambda_0(e)$. Also, B_0^f has four vertices a, b, c, and d, as indicated in the figure, such that edges ac and bd are adjacent to edge ab. Moreover, observe that $\lambda_0(ac) = \{3, 5\}$ and $\lambda_0(bd) = \{2, 5\}$.

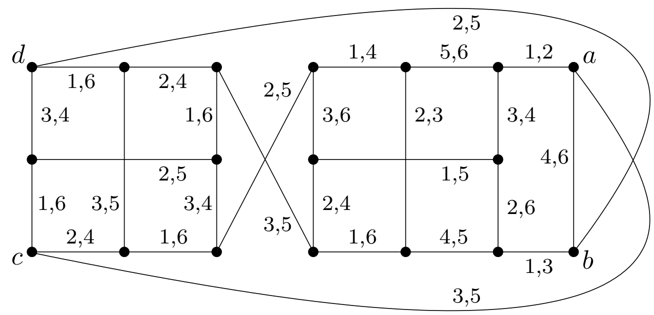
Let i be an integer, $i \geq 0$. By the induction hypothesis, graph B_i^f has vertices a, b, c, and d, with edges ac and bd adjacent to edge ab, and B_i^f has a Fulkerson function λ_i such that $\lambda_i(ac) = \{3, 5\}$ and $\lambda_i(bd) = \{2, 5\}$. Suppose that every edge e of B_i^f is labelled with $\lambda_i(e)$. Let B_L be the *link graph* depicted in Figure 4.2(a), with vertices a, a', b, b', c, c', d, and d' indicated. The figure also exhibits a Fulkerson function λ_L of B_L. In order to construct B_{i+1}^f with a Fulkerson colouring λ_{i+1}, first remove edges ac and bd from B_i^f. Add a copy of B_L and identify each of vertices a, b, c, and d of B_L with the vertex of same name of B_i^f. The resulting graph is B_{i+1}^f. Figure 4.2 shows an example of this operation when $i = 0$. The labels of B_i^f and of B_L are preserved in this operation. Notice that the edges incident with a, b, c, and d removed from B_i^f have exactly the same labels of the edges aa', bb', cc', and dd' of B_L, respectively. Therefore, the obtained labelling of B_{i+1}^f is a Fulkerson colouring λ_{i+1}. To complete the proof, rename the vertices of B^f_{i+1} as follows: call vertices a' and b' respectively as c and d, preserve the names of vertices a and b, and remove all other vertex names. Thus, we have a copy of B^f_{i+1} with vertices a, b, c, and d such that edges ac and bd are both adjacent to edge ab and that $\lambda_{i+1}(ac) = \{3, 5\}$ and $\lambda_{i+1}(bd) = \{2, 5\}$. □



(a) Function λ_L of B_L.



(b) Function λ_0 of B^f_0.

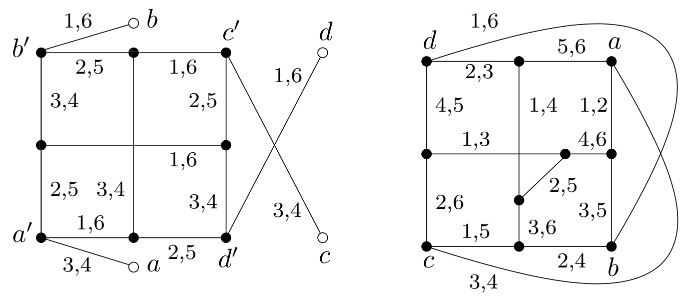


(c) Function λ_1 of B^f_1 constructed from λ_0 and λ_L as in Theorem 4.4.

Figure 4.2: Construction of a Fulkerson function for the First Blanuša Snark.

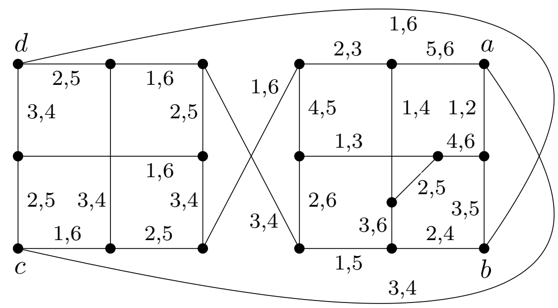
Theorem 4.5. *Every graph of the second family of generalised Blanuša Snarks has a Fulkerson colouring.*

Proof. Let $\mathcal{B}^s = \{B^s_1, B^s_2, B^s_3, \dots\}$ be the second family of generalised Blanuša Snarks, and let $\mathcal{B} = \{B^s_0\} \cup \mathcal{B}^s$, with B^s_0 the copy of the Petersen Graph shown in Figure 4.3(b). The proof is similar to the proof of Theorem 4.4, replacing function λ_0 of B^f_0 with function λ_0 of B^s_0 depicted in Figure 4.3(b), and using function λ_L of link graph B_L depicted in Figure 4.3(a). Also note that, in this case, $\lambda_0(ac) = \{3, 4\}$ and $\lambda_0(bd) = \{1, 6\}$. Figure 4.3(c) depicts Fulkerson function λ_1 constructed for the first member B^s_1 of $\mathcal{B}^s$. □



(a) Function λ_L of B_L.

(b) Function λ_0 of B_0^s.



(c) Function λ_1 of B_1^s constructed from λ_0 and λ_L as in Theorem 4.5.

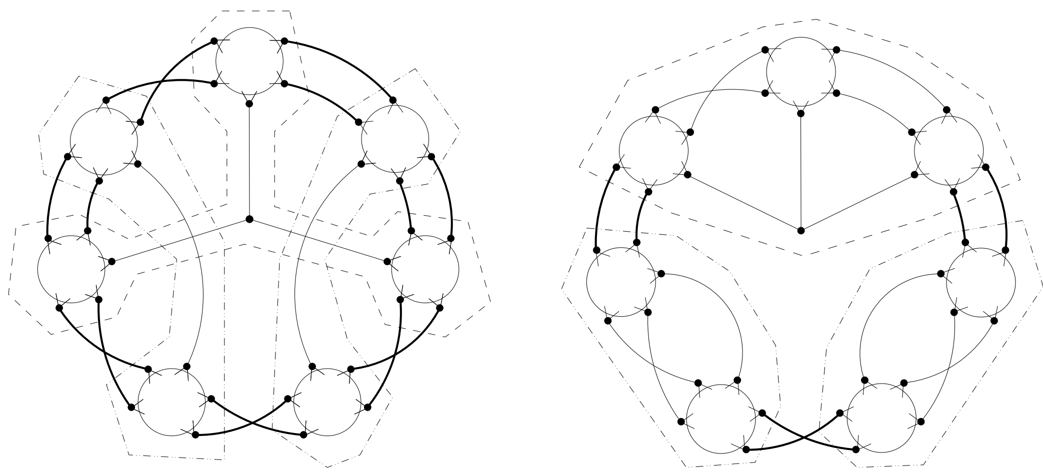
Figure 4.3: Construction of a Fulkerson function for the Second Blanuša Snark.

4.2 Loupekine Snarks

In the current section, we first prove that every LP_0-snark verifies Fulkerson's Conjecture. This proof uses a technique similar to that used in Section 4.1, although it does not rely on a recursive construction. We also prove that LP_1-snarks satisfy Fulkerson's Conjecture, by using a different technique. The later result is more general than the first one, since every LP_0-snark is an LP_1-snark. At the end of the section, it is shown how the technique applied to the LP_1 family can be extended to prove that Fulkerson's Conjecture is satisfied by other families of Loupekine Snarks. We start by showing some properties of LP_0-snarks which are useful in the proof of Theorem 4.8.

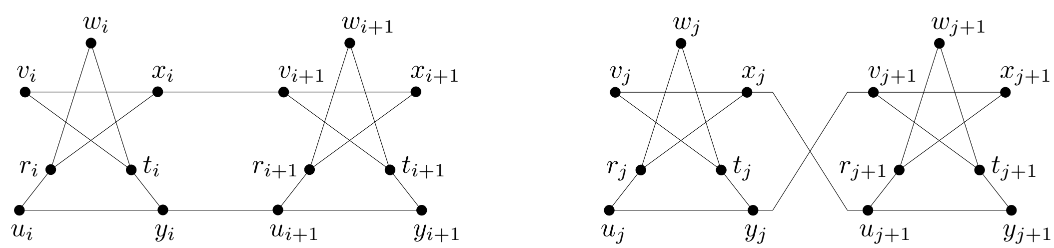
Let G be an L_1-snark, as defined in Section 2.4.2. A *gadget* of G is a subgraph of G composed by a connected component of G_C, plus the blocks attached to it and the edges between them. A gadget of G is an *m-gadget* if it comprises m blocks, $m = 2, 3$. Figure 4.4 marks gadgets of L_1-snarks. An edge is a *block link* if its ends lie in different blocks of the same gadget. Every edge of G not in a gadget is a *gadget link*. Two gadgets connected by gadget links are *adjacent*.

Proposition 4.6. *Let G be the graph comprised of blocks B_i and B_{i+1}, and edges y_iu_{i+1} and x_iv_{i+1}, shown in* Figure 4.5(a). *Let H be the graph comprised of blocks B_j and B_{j+1}, and edges u_jx_{j+1} and v_jy_{j+1}, shown in* Figure 4.5(b). *Graphs G and H are isomorphic.*



(a) An L_1-snark. (b) An L_1-snark which is an L_0-snark.

Figure 4.4: Two L_1-snarks. Only vertices u, v, w, x, y of each block and vertices of central subgraphs are represented. Each graph has two 2-gadgets and one 3-gadget. Thick lines represent gadget links.

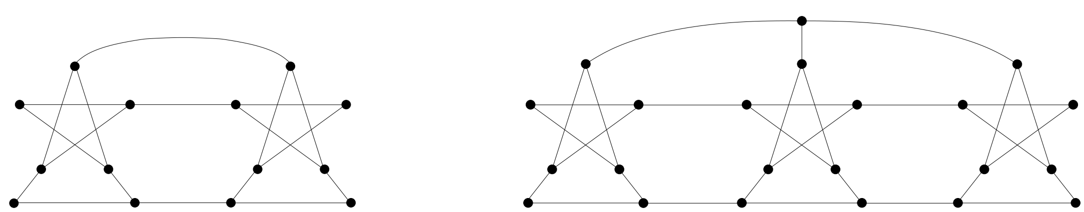


(a) Parallel block links. (b) Crossing block links.

Figure 4.5: Isomorphic graphs, each comprised by two adjacent blocks and their links.

Proof. Let ϕ be a bijection from $V(G)$ to $V(H)$ defined as: $\phi(x_i) = y_j; \phi(y_i) = x_j; \phi(u_i) = v_j; \phi(v_i) = u_j; \phi(r_i) = t_j; \phi(t_i) = r_j; \phi(w_i) = w_j$; and for all $z_{i+1} \in V(B_{i+1})$, $\phi(z_{i+1}) = z_{j+1}$. The result follows by inspection of the adjacency relations in both graphs. $\square$

Corollary 4.7. *Every gadget of an LP_0-snark is isomorphic to one of the graphs of* Figure 4.6*.*



(a) A 2-gadget. (b) A 3-gadget.

Figure 4.6: Gadgets of an LP_0-snark.

Proof. The result follows by applying Proposition 4.6 on each gadget with crossing block links. Note that this process may cross or uncross pairs of gadget links. □

Theorem 4.8. *Every LP_0-snark satisfies Fulkerson's Conjecture.*

Proof. Let G be an LP_0-snark comprised of l gadgets $G^1, \dots, G^l$. Recall that each gadget can be a 2-gadget or a 3-gadget, and that adjacent gadgets are connected by gadget links (see Figure 4.4(b)). Adjust notation so that gadgets G^i, G^{i+1} are adjacent, for all $i \in \{1, \dots, l\}$, with indexes greater than l taken modulo l.

Let G^i be a gadget of G. By Corollary 4.7, G^i is isomorphic to one of the graphs of Figure 4.6. Figure 4.7 exhibits Fulkerson colourings of these graphs, with $L = \{1, \dots, 6\}$. If G^i is a 2-gadget, then we assign to every edge of G^i the labels of the corresponding edge of the graph of Figure 4.7(a). If G^i is a 3-gadget, we proceed analogously using the graph of Figure 4.7(b). For all $j \in L$, let M^i_j be the set comprised of the edges of $E(G^i)$ with label j. We conclude, by inspection, that each M^i_j satisfies the following properties:

(a) $M^i_j, 3 \le j \le 6$, is a perfect matching;

(b) every border vertex of G^i (white vertices in Figure 4.7) is M^i_1-unsaturated and M^i_2-unsaturated.

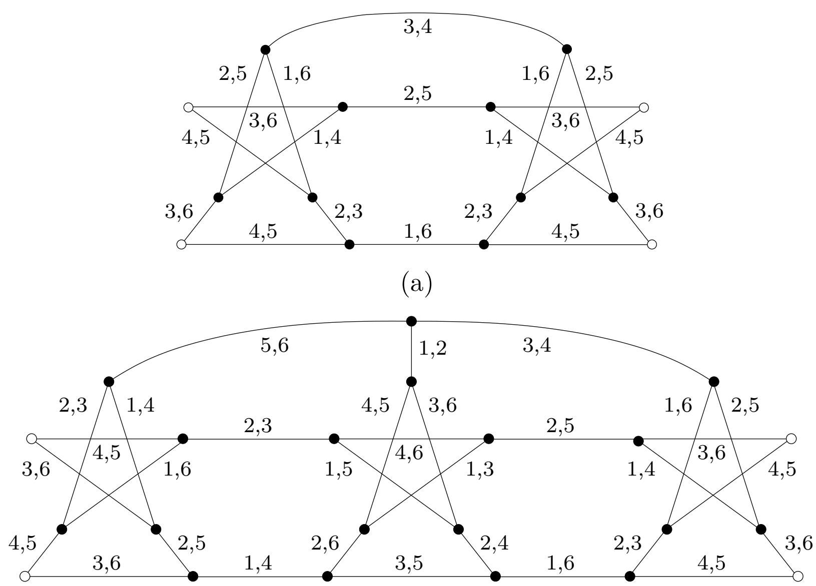


Figure 4.7: Fulkerson colourings of LP_0-snark gadgets.

For all $j \in L$, let $M_j = \bigcup_{i=1}^{l} M^i_j$. Since gadgets are pairwise vertex-disjoint, M_j is a matching. Moreover, by construction, every edge of each gadget is contained in exactly two of these matchings. In order to complete the proof, it is necessary to assign each gadget link to two of these six matchings. By (a), M_3, M_4, M_5, M_6 are perfect matchings of G. By (b), all border vertices of G are M_1-unsaturated and M_2-unsaturated. The border vertices are the ends of the gadget links. Since gadget links are pairwise nonadjacent, it is enough to assign each of them to M_1 and M_2. □

In order to prove that LP_1-snarks satisfy Fulkerson's Conjecture, a Fulkerson colouring is constructed for an arbitrary LP_1-snark. Before that, some notation is introduced and useful properties of Fulkerson functions of LP-snarks are shown.

Let B_i be a block of an LP-snark G, as defined in Section 2.4.2. Figure 4.8 shows B_i together with edge cut $\partial(V(B_i)) = \{e_i^u, e_i^v, e_i^w, e_i^x, e_i^y\}$, also denoted by ∂_{B_i}. The edges of ∂_{B_i} and their ends $u_i, u_i', v_i, v_i', w_i, w_i', x_i, x_i', y_i, y_i'$ are named as shown in the figure. The *extended block* B_i^+ is the graph with vertex set $V(B_i) \cup \{u_i', v_i', w_i', x_i', y_i'\}$ and edge set $E(B_i) \cup \partial_{B_i}$. For simplicity, indices are omitted whenever they are clear in the context.

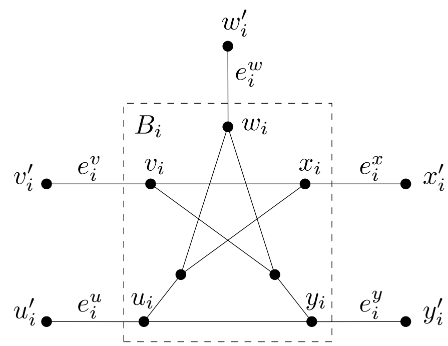


Figure 4.8: The edges of ∂_{B_i} and its end vertices in the graph B_i^+.

Let $\pi\colon E' \to L^{(2)}$, with $E' \subseteq E(G)$ and $|L| = 6$. Let $\mathcal{L} = \{L_1, L_2, L_3\}$ be a partition of L with each part of cardinality two. Consider edges e^u, e^v, e^w, e^x, e^y of an extended block B^+ of G and the following properties.

P1 $\pi(e^w) \in \mathcal{L}$.

P2 π satisfies exactly one of:

- **(a)** $\pi(e^x) = \pi(e^y) \in \mathcal{L}$
- **(b)** $\pi(e^x), \pi(e^y) \notin \mathcal{L}$;
  $\pi(e^x)$ and $\pi(e^y)$ are disjoint; and
  $\pi(e^x) \cup \pi(e^y) = L_i \cup L_j,\ \ L_i, L_j \in \mathcal{L}$.

P3 π satisfies exactly one of:

- **(a)** $\pi(e^u) = \pi(e^v) \in \mathcal{L}$;
- **(b)** $\pi(e^u), \pi(e^v) \notin \mathcal{L}$;
  $\pi(e^u)$ and $\pi(e^v)$ are disjoint; and
  $\pi(e^u) \cup \pi(e^v) = L_i \cup L_j,\ \ L_i, L_j \in \mathcal{L}$.

Let $\lambda\colon E' \to L^{(2)}$, with $E' \subseteq E(G)$, be a Fulkerson function of G. Function λ is $(B^+, \mathrm{P1})$-*strong* if it satisfies property P1 considering the edges of B^+. Furthermore, $(B^+, \mathrm{P2})$-*strong* and $(B^+, \mathrm{P3})$-*strong* are analogously defined.

Let B^+ be an extended block and let $L = \{a, b, c, d, e, f\}$. Figure 4.9 exhibits four different Fulkerson functions $\Lambda^j\colon E(B^+) \to L^{(2)}$, with $1 \leq j \leq 4$, called *models*. These models are later used in the construction of a Fulkerson colouring for an LP_1-snark. The next proposition relates properties P1, P2, and P3 to models $\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4$.

Proposition 4.9. *Let* $\mathcal{L} = \{ab, cd, ef\}$. *Each model* $\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4$ *is* $(B^+, \mathrm{P1})$*-strong,* $(B^+, \mathrm{P2})$*-strong, and* $(B^+, \mathrm{P3})$*-strong, where* B^+ *is an extended block of an* LP*-snark. Moreover, the following statements are true:*

(i) Λ^1 *satisfies* P2(a) *and* $\Lambda^1(e^w) = \Lambda^1(e^x)$*;*
(ii) Λ^2 *satisfies* P2(a) *and* $\Lambda^2(e^w) \cap \Lambda^2(e^x) = \emptyset$*;*
(iii) Λ^3 *satisfies* P2(b) *and* $\Lambda^3(e^w) \subset (\Lambda^3(e^x) \cup \Lambda^3(e^y))$*;*
(iv) Λ^4 *satisfies* P2(b) *and* $\Lambda^4(e^w) \not\subset (\Lambda^4(e^x) \cup \Lambda^4(e^y))$.

Proof. By inspection of Figure 4.9. ☐

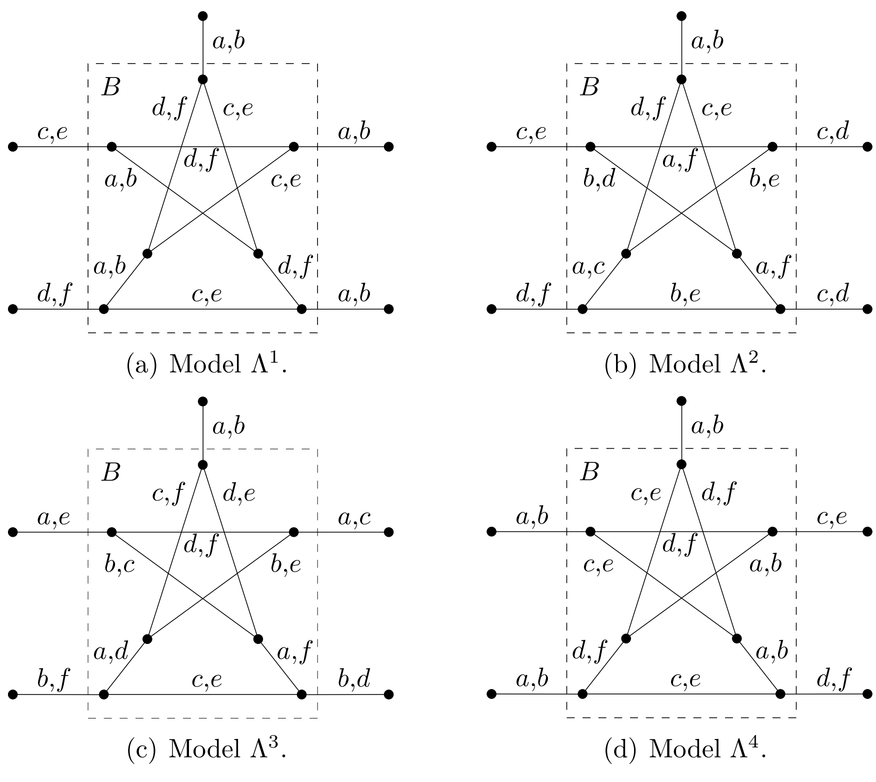


Figure 4.9: Fulkerson functions $\Lambda^j\colon E(B^+) \to L^{(2)}$, $1 \leq j \leq 4$, with $L = \{a, b, c, d, e, f\}$.

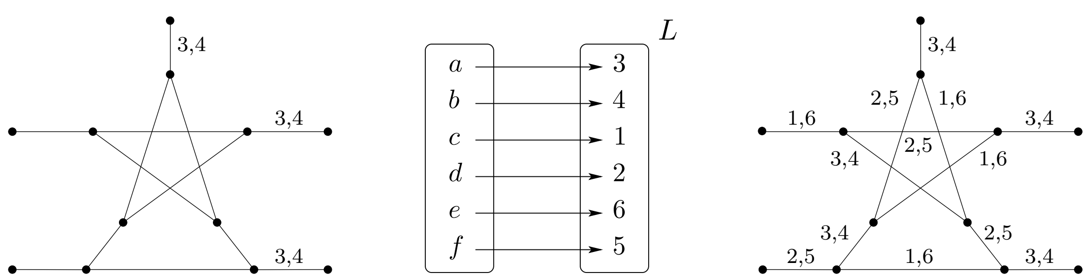
The following lemma is used in the constructive proof of Theorem 4.12.

Lemma 4.10. *Let B^+ be an extended block of an LP-snark, with $\partial_B = \{e^u, e^v, e^w, e^x, e^y\}$. Let $L = \{1,2,3,4,5,6\}$ and $\mathcal{L} = \{12, 34, 56\}$. If $\lambda\colon \{e^w, e^x, e^y\} \to L^{(2)}$ is a $(B^+, \mathrm{P1})$-strong and $(B^+, \mathrm{P2})$-strong Fulkerson function, then there exists a Fulkerson function $\lambda^+\colon E(B^+) \to L^{(2)}$ which is $(B^+, \mathrm{P3})$-strong and such that λ is a restriction of λ^+.*

Proof. Let B^+, L, and $\mathcal{L}$ be defined as in the hypothesis. Let $\lambda\colon \{e^w, e^x, e^y\} \to L^{(2)}$ be a Fulkerson function which is $(B^+, \mathrm{P1})$-strong and $(B^+, \mathrm{P2})$-strong. We construct a Fulkerson function $\lambda^+\colon E(B^+) \to L^{(2)}$ using one of the models in Figure 4.9. Then, we show that λ^+ is $(B^+, \mathrm{P3})$-strong and that λ is a restriction of λ^+.

By P1, $\lambda(e^w) \in \mathcal{L}$, and, by P2, λ satisfies either P2(a) or P2(b). Suppose λ satisfies P2(a). Thus, we have to consider two cases: either $\lambda(e^w) = \lambda(e^x)$ or $\lambda(e^w) \cap \lambda(e^x) = \emptyset$. If λ satisfies P2(b), it is also necessary to consider two cases: either $\lambda(e^w) \subset (\lambda(e^x) \cup \lambda(e^y))$ or $\lambda(e^w) \not\subset (\lambda(e^x) \cup \lambda(e^y))$. Notice that, by Property 4.9, each model $\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4$ falls into exactly one of these four cases. Using the appropriate model Λ^j, we define a function $\lambda^+\colon E(B^+) \to L^{(2)}$ such that λ is a restriction of λ^+. It is done by finding a suitable bijection ϕ from $\{a, b, c, d, e, f\}$ to L.

Let $\varphi(pq) = \phi(p)\phi(q)$. Bijection ϕ must satisfy the following: φ maps $\{ab, cd, ef\}$ to $\mathcal{L}$, and $\varphi(\Lambda^j(e)) = \lambda(e)$. Then, λ^+ is defined as $\lambda^+(e) = \varphi(\Lambda^j(e))$, for $e \in E(B^+)$. Since each of $\Lambda^1, \Lambda^2, \Lambda^3, \Lambda^4$ is $(B^+, \mathrm{P3})$-strong, λ^+ also is $(B^+, \mathrm{P3})$-strong. Figure 4.10 exhibits an example of a function λ, a bijection ϕ, and a function λ^+ for the extended block B^+. $\square$



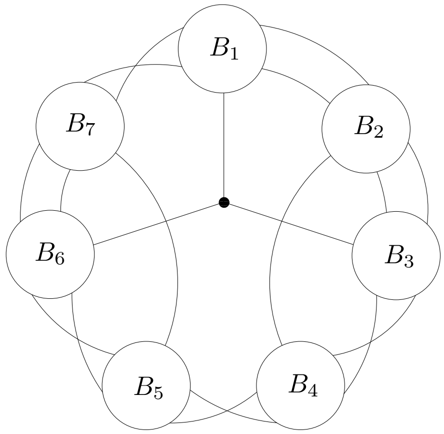
(a) Function $\lambda\colon \{e^w, e^x, e^y\} \to L^{(2)}$. (b) Bijection ϕ based on λ (a) and on Λ^1. (c) Function $\lambda^+\colon E(B^+) \to L^{(2)}$.

Figure 4.10: Construction of a Fulkerson function λ^+ for an extended block.

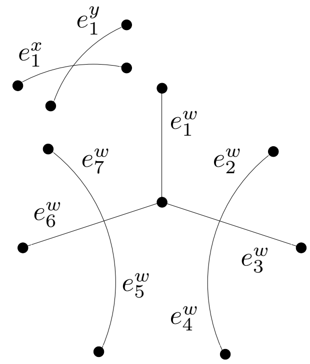
Definition 4.11. *Let G be an LP-snark with k blocks. The sequence $\{G_j\}, 0 \le j < k$, of subgraphs of G is defined as*

$$G_j = \begin{cases} G_C \cup G[\{x_1, x_1', y_1, y_1'\}], & j = 0 \\ G_{j-1} \cup B_j^+, & 1 \le j < k \end{cases}.$$

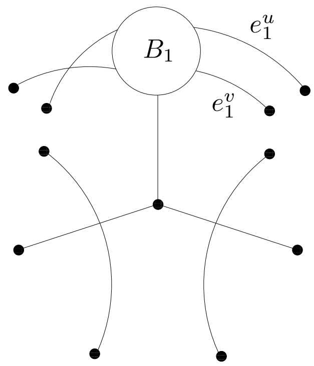
Figure 4.11 shows examples of subgraphs in the sequence $\{G_j\}$ of an LP_1-snark G.



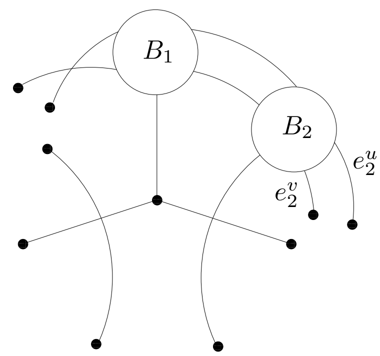
(a) LP_1-snark G.



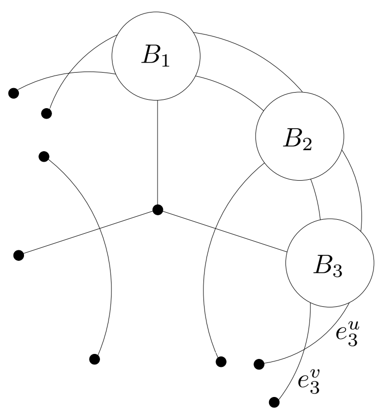
(b) Subgraph G_0.



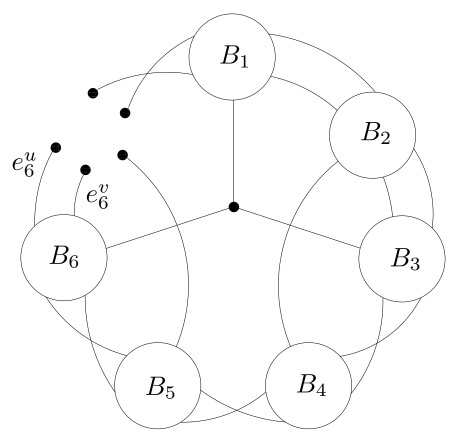
(c) Subgraph G_1.



(d) Subgraph G_2.



(e) Subgraph G_3.



(f) Subgraph G_6.

Figure 4.11: LP_1-snark G with 7 blocks and its subgraphs G_0, G_1, G_2, G_3, G_6 as established in Definition 4.11.

Theorem 4.12. *Every LP_1-snark has a Fulkerson colouring.*

Proof. Let G be an LP_1-snark with k blocks. Let $L = \{1, 2, 3, 4, 5, 6\}$. We construct a Fulkerson colouring $\lambda\colon E(G) \to L^{(2)}$. For this purpose, let $\mathcal{L} = \{12, 34, 56\}$ be a partition of L.

First, we construct a Fulkerson colouring λ_C for central subgraph G_C such that $\lambda_C(e_k^w) = 12$. Graph G_C is 3-edge-colourable, since its connected components are isomorphic to K_2 and S_3. Then, there exists a 3-edge-colouring $\lambda_C\colon E(G_C) \to \mathcal{L}$ such that $\lambda_C(e_k^w) = 12$. Notice that λ_C is a Fulkerson colouring of G_C.

As a second step, we prove, by induction on j, that there exists a Fulkerson colouring λ_j for subgraphs of sequence $\{G_j\}$, $0 \leq j < k$, such that:

(i) $\lambda_j(e_i^w) = \lambda_C(e_i^w)$, $1 \leq i \leq k$;
(ii) $\lambda_j(e_1^x) = \lambda_j(e_1^y) = 12$;
(iii) λ_j is $(B_{j+1}^+, \text{P2})$-strong.

For $j = 0$, define function λ_0 as: $\lambda_0(e) = \lambda_C(e)$ if $e \in E(G_C)$, $\lambda_0(e) = 12$ if $e \in \{e_1^x, e_1^y\}$. It is clear that λ_0 satisfies conditions (i) and (ii). Since $\lambda_0(e_1^x) = \lambda_0(e_1^y) = 12$ and $12 \in \mathcal{L}$, λ_0 is $(B_1^+, \text{P2})$-strong. Thus, λ_0 satisfies condition (iii). Moreover, e_1^x and e_1^y are not adjacent to any other edge of $E(G_0)$. We conclude that λ_0 is a Fulkerson colouring of G_0.

For $j > 0$, suppose that there exists a Fulkerson colouring $\lambda_{j-1}\colon E(G_{j-1}) \to L^{(2)}$ satisfying (i), (ii), and (iii). By (i) and by the construction of λ_C, function λ_{j-1} is $(B_j^+, \text{P1})$-strong. By (iii), λ_{j-1} is $(B_j^+, \text{P2})$-strong. Thus, we apply Lemma 4.10 by letting λ be the restriction of λ_{j-1} to $\{e_j^w, e_j^x, e_j^y\}$. Let $\lambda^+\colon E(B_j^+) \to L^{(2)}$ be the Fulkerson function obtained from Lemma 4.10. Let $\lambda_j = \lambda_{j-1} \cup \lambda^+$. By Lemma 4.3, with $X = V(B_j)$, we conclude that λ_j is a Fulkerson colouring of G_j.

Since λ_{j-1} satisfies (i) and (ii), and $\{e_i^w\colon 1 \leq i \leq k]\} \cup \{e_1^x, e_1^y\} \subseteq \text{Dom}(\lambda_{j-1})$, λ_j satisfies (i) and (ii). Observe that $\{e_j^u, e_j^v\} = \{e_{j+1}^x, e_{j+1}^y\}$. By Lemma 4.10, λ^+ is $(B_j^+, P3)$-strong. Moreover, P2 is essentially the same statement as P3, but applied to e_{j+1}^x and e_{j+1}^y. Therefore, λ_j satisfies (iii). This completes the induction.

Now, consider subgraph G_{k-1} and its Fulkerson colouring $\lambda_{k-1}\colon E(G_{k-1}) \to L^{(2)}$ previously constructed, satisfying conditions (i), (ii), and (iii). Figure 4.12 shows a representation of G with G_{k-1}. Note that $E(G_{k-1}) = E(G) \setminus E(B_k)$.

Let $\mathcal{F}_{k-1} = \{M_l\colon l \in L\}$, where each M_l is the set $\{e \in E(G_{k-1})\colon l \in \lambda_{k-1}(e)\}$. Remark that, by F1, M_l is a matching. Let X be the set of degree-three vertices of G_{k-1}. Note that every $v \in X$ is M_l-saturated, and $\partial(X) = \partial_{B_k}$. Therefore, by Theorem 1.6, $|M_l \cap \partial_{B_k}|$ is odd and, thus, $|M_l \cap \partial_{B_k}| \geq 1$. By construction of λ_{k-1}, $\lambda_{k-1}(e_1^x) = \lambda_{k-1}(e_1^y) = \lambda_{k-1}(e_k^w) = 12$. Therefore, $|M_1 \cap \partial_{B_k}| \geq 3$ and $|M_2 \cap \partial_{B_k}| \geq 3$. Moreover, considering the

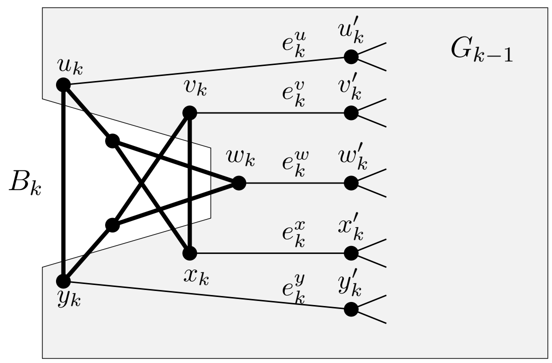


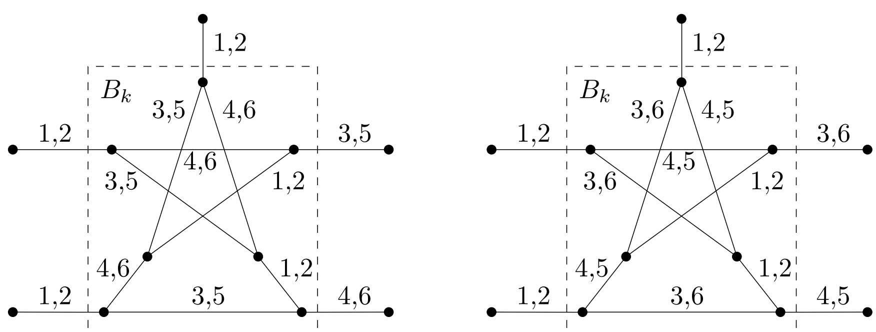
Figure 4.12: Decomposition of G into G_{k-1}, in the shaded part, and B_k, in bold lines.

fact that $|\partial_{B_k}| = 5$, we conclude that

$$|M_l \cap \partial_{B_k}| = 3, \qquad l = 1, 2 \tag{4.1}$$

$$|M_l \cap \partial_{B_k}| = 1, \qquad l = 3, 4, 5, 6. \tag{4.2}$$

By construction, λ_{k-1} is $(B_k^+, \mathrm{P2})$-strong. By P2 and by (4.2), $\{\lambda_{k-1}(e_k^x), \lambda_{k-1}(e_k^y)\}$ is either $\{35, 46\}$ or $\{36, 45\}$, satisfying P2(b). Since $\lambda_{k-1}(e_k^w) = 12$, we have that $\lambda_{k-1}(e_k^w) \not\subset (\lambda_{k-1}(e_k^x) \cup \lambda_{k-1}(e_k^y))$. Therefore, by Property 4.9(iv), it is possible to use model Λ^4 (Figure 4.9(d)) to define a Fulkerson function $\lambda_k^+ \colon E(B_k^+) \to L$. Figure 4.13 exhibits two examples. The other two possibilities, obtained by exchanging $\lambda_{k-1}(e_k^x)$ and $\lambda_{k-1}(e_k^y)$, are very similar. Let $\lambda = \lambda_{k-1} \cup \lambda_k^+$. By Lemma 4.3, with $X = V(B_k)$, we conclude that λ is a Fulkerson colouring of G. $\square$



(a) $\lambda_k^+(e_k^x) = 35$ and $\lambda_k^+(e_k^y) = 46$ (b) $\lambda_k^+(e_k^x) = 36$ and $\lambda_k^+(e_k^y) = 45$

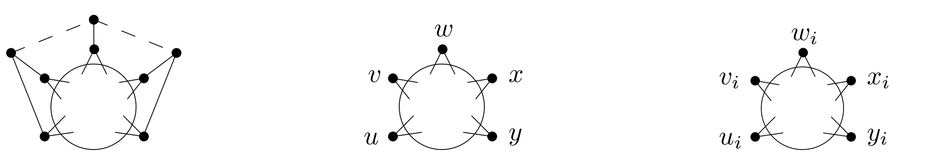
Figure 4.13: Fulkerson function $\lambda_k \colon E(B_k^+) \to L$ based on model Λ^4.

Corollary 4.13. *Every LP-snark with a 3-edge-colorable central subgraph has a Fulkerson colouring.* $\square$

Recall from Section 2.4.3 that the family of Goldberg Snarks [32] can be obtained by Loupekine's construction[1]. Every Goldberg Snark is also an LP-snark. Moreover, the central subgraph of a Goldberg Snark is 3-edge-colourable. Thus, Corollary 4.13 implies that Goldberg Snarks verify Fulkerson's Conjecture, as an alternative to the proofs given by Hao et. al. [38], and by Fouquet and Vanherpe [28].

4.2.1 Application to additional families of Loupekine Snarks

The technique used to show that every LP_1-snark satisfies Fulkerson's Conjecture can be adapted to show that the conjecture is verified by other families of Loupekine Snarks. For this purpose, we generalise the construction of a block $B(G)$, as sketched in Figure 4.14. In the graphs of Figure 4.4, only vertices u, v, w, x, y of each block are represented. Thus, each graph of the figure is the sketch of an L_1-snark whose blocks are derived from arbitrary snarks.



(a) A snark G. (b) A block $B(G)$. (c) Indexed block B_i.

Figure 4.14: Construction of a generic block.

Let B^+ be the extended block obtained from a block B, sketched in Figure 4.15. Let $L = \{a, b, c, d, e, f\}$. Consider the labels of edges e^u, e^v, e^w, e^x, e^y in the models of Figure 4.9. Using these labels in the extended block of Figure 4.15, we define the *generic models* $\Lambda_\partial^j \colon \partial_B \to L^{(2)}$, with $1 \le j \le 4$, shown in Figure 4.16.

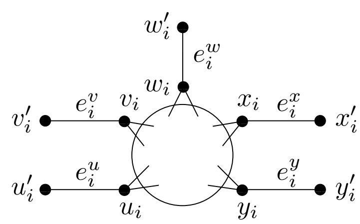
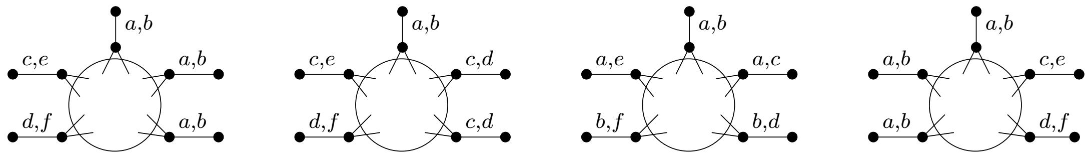


Figure 4.15: A generic extended block B_i^+.

It is possible to generalise Theorem 4.12 to families of L_1-snarks other than LP_1-snarks. Let $\mathcal{B}$ be a set of non-isomorphic blocks. An *$L\mathcal{B}$-snark* is a Loupekine Snark such

[1]It is also worth reminding that Goldberg's construction [32], which encompasses the so-called Goldberg Snarks family, is more general than Loupekine's construction.



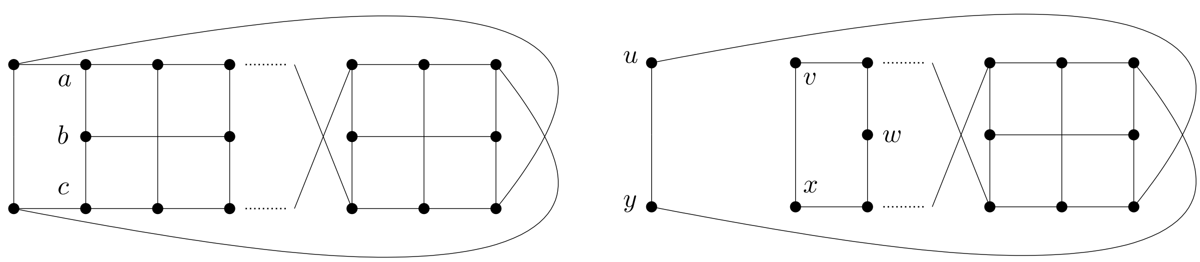
(a) Model Λ^1_∂. (b) Model Λ^2_∂. (c) Model Λ^3_∂. (d) Model Λ^4_∂.

Figure 4.16: Generic models $\Lambda^j_\partial \colon \partial_B \to L^{(2)}$, $1 \leq j \leq 4$, with $L = \{a, b, c, d, e, f\}$.

that each of its blocks is isomorphic to a block of $\mathcal{B}$. Furthermore, an $L\mathcal{B}_1$*-snark* is an $L\mathcal{B}$-snark which is also an L_1-snark.

For each $B \in \mathcal{B}$, suppose that there exist Fulkerson functions $\Lambda^j \colon E(B^+) \to L^{(2)}$, with $L = \{a, b, c, d, e, f\}$ and $1 \leq j \leq 4$, such that generic model Λ^j_∂ is a restriction of Λ^j. Thus, we have four models for each block of $\mathcal{B}$. Under these conditions, to prove that every $L\mathcal{B}_1$-snark admits a Fulkerson colouring, it is enough to use these models in the constructions of Lemma 4.10 and Theorem 4.12. Moreover, analogously to Corollary 4.13, this process shows that every $L\mathcal{B}$-snark with a 3-edge-colourable central subgraph has a Fulkerson colouring.

As an example, consider $\mathcal{B}^f = \{B^f_1, B^f_2, \dots\}$ the first family of generalised Blanuša Snarks, described in Section 2.4.1. Take $B^f_i \in \mathcal{B}^f$, and consider path abc, as shown in Figure 4.17(a). Define block B_i as the subgraph of B^f_i obtained by removing a, b, and c, as depicted in Figure 4.17(b). Observe that B_i is uniquely defined, since abc is a fixed path in B^f_i. Let $\mathcal{B}_{Blanuša}$ be the family of blocks $\{B_1, B_2, B_3, \dots\}$.



(a) A generalised Blanuša Snark B^f_i. (b) Block B_i obtained from Blanuša Snark B^f_i.

Figure 4.17: Construction of a block of family of blocks $\mathcal{B}_{Blanuša}$.

Lemma 4.14. *Every extended block B^+_i, with $B_i \in \mathcal{B}_{Blanuša}$, admits Fulkerson functions $\Lambda^j_i \colon E(B^+_i) \to L^{(2)}$, with $L = \{a, b, c, d, e, f\}$ and $1 \leq j \leq 4$, such that Λ^j_∂ is a restriction of Λ^j_i.*

Proof. The proof is by induction on i. Let Λ^1_1, Λ^2_1, Λ^3_1, Λ^4_1 be the Fulkerson functions from $E(B^+_1)$ to $L^{(2)}$ exhibited in Figure 4.18, with $L = \{a, b, c, d, e, f\}$. By inspection of

Figure 4.16 and Figure 4.18, we see that Λ^j_∂ is a restriction of Λ^j_1, $1 \leq j \leq 4$. Figure 4.18 also exhibits Fulkerson functions $\Lambda^1_L, \Lambda^2_L, \Lambda^3_L, \Lambda^4_L$ for link graph B_L.

Suppose that B^+_{i-1} admits Fulkerson functions $\Lambda^j_{i-1}, 1 \leq j \leq 4$, as stated in the hypothesis. Extended block B^+_i can be constructed by attaching a copy of link graph B_L to B^+_{i-1}: remove edges uu' and yy' from B^+_{i-1}, and identify each vertex u, u', y, y' of B^+_{i-1} with vertex of same name in B_L, preserving the labels of the edges of $E(B^+_{i-1})$ and B_L. Note that the edges incident with vertices u, u', y, y' have the same labels in B^+_{i-1} and in B_L. Thus, the labeling of B^+_i just constructed represents a Fulkerson function $\Lambda^j_i \colon E(B^+_i) \to L^{(2)}$. Moreover, notice that $\Lambda^j_i(e) = \Lambda^j_{i-1}(e)$, for every $e \in \partial_{B_i}$. Thus, by induction hypothesis, Λ^j_∂ is a restriction of Λ^j_i. At last, in block B^+_i just constructed, remove the labels of vertices y' and u', and rename vertices r and t of the copy of B_L respectively to y' and u', while keeping the labels of vertices u and v. □

Theorem 4.15. *Every Loupekine Snark such that each of its blocks is isomorphic to a block of* $\mathcal{B}_{Blanuša} \cup \{B(P)\}$ *and its central subgraph is* 3*-edge-colourable admits a Fulkerson colouring.* □

We extend the class of L-snarks which we know that satisfy Fulkerson's Conjecture with another example. Consider the Flower Snarks introduced in Section 2.3.3. Let B^J_k, k odd and $k \geq 5$, be the block derived from J_k by removing vertices a, b, c as illustrated in Figure 4.19. Let $\mathcal{B}_{Flower} = \{B^J_5, B^J_7, B^J_9, \ldots\}$. Figure 4.20 shows extended block B^{J+}_5 and link graph J_L, used in Lemma 4.16 to construct B^{J+}_{k+2} from B^{J+}_k.

Lemma 4.16. *Every extended block* B^{J+}_k*, with* $B^J_k \in \mathcal{B}_{Flower}$*, admits Fulkerson functions* $\Lambda^j_k \colon E(B^{J+}_k) \to L^{(2)}$*, with* $L = \{a, b, c, d, e, f\}$ *and* $1 \leq j \leq 4$*, such that* Λ^j_∂ *is a restriction of* Λ^j_k*.*

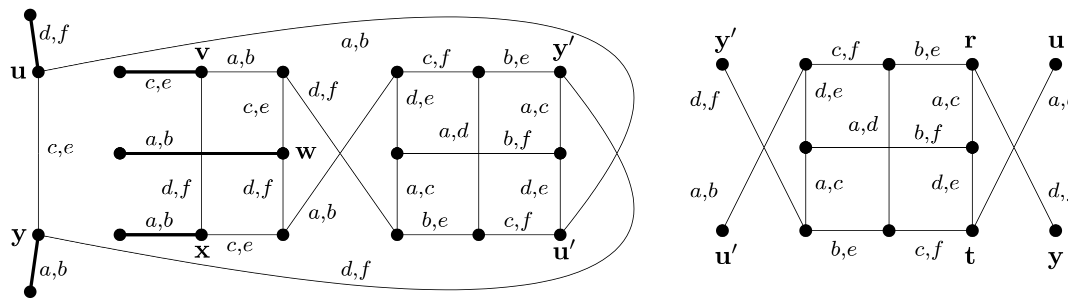
Proof. The proof is by induction on k. Consider Fulkerson functions $\Lambda^j_5 \colon E(B^{J+}_5) \to L^{(2)}$, with $1 \leq j \leq 4$, exhibited in Figure 4.21. By inspecting the figure, we conclude that Λ^j_∂ is a restriction of Λ^j_5. Note that, in each function Λ^1_5, Λ^3_5, and Λ^4_5, the labels of edges r_1r_2, s_1s_2, and t_1t_2 are, respectively, ce, ab, and df. Also, note that in function Λ^2_5, the labels of these same edges are, respectively, cd, cf, and de.

Let k odd, $k \geq 5$. By the induction hypothesis, there exist Fulkerson functions $\Lambda^j_k \colon E(B^{J+}_k) \to L^{(2)}$, with $1 \leq j \leq 4$, such that:

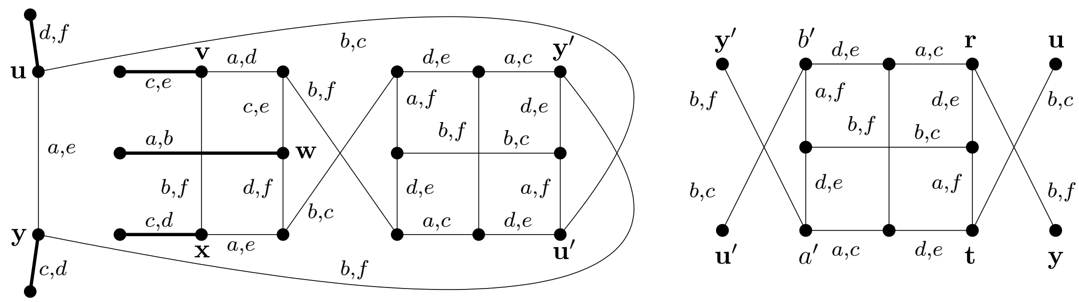
(i) Λ^j_∂ is a restriction of Λ^j_k;

(ii) in B^{J+}_k, for each function Λ^1_k, Λ^3_k, and Λ^4_k, the labels of edges r_1r_2, s_1s_2, and t_1t_2 are, respectively, ce, ab, and df;

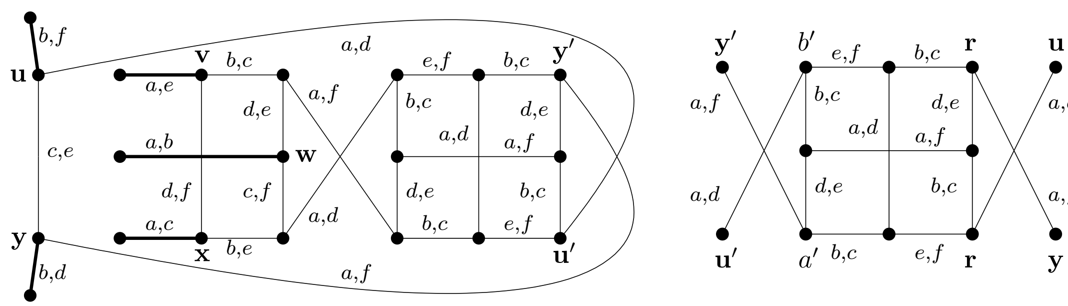
(iii) in B^{J+}_k, the values of Λ^2_k for edges r_1r_2, s_1s_2, and t_1t_2 are, respectively, cd, cf, and de.



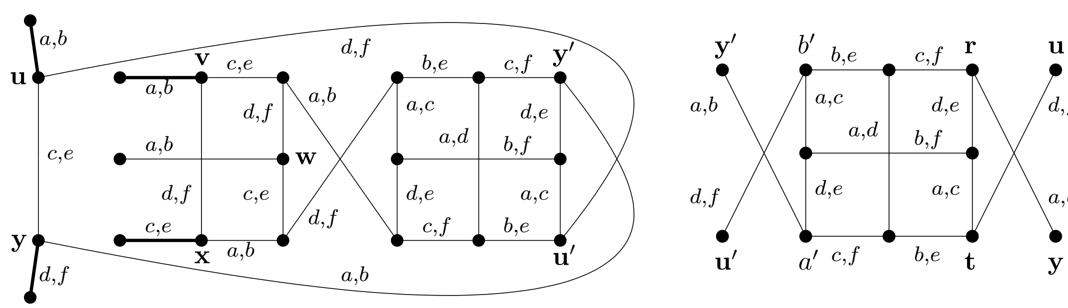
(a) Fulkerson functions Λ_1^1 for B_1 and Λ_L^1 for B_L.



(b) Fulkerson functions Λ_1^2 for B_1 and Λ_L^2 for B_L.

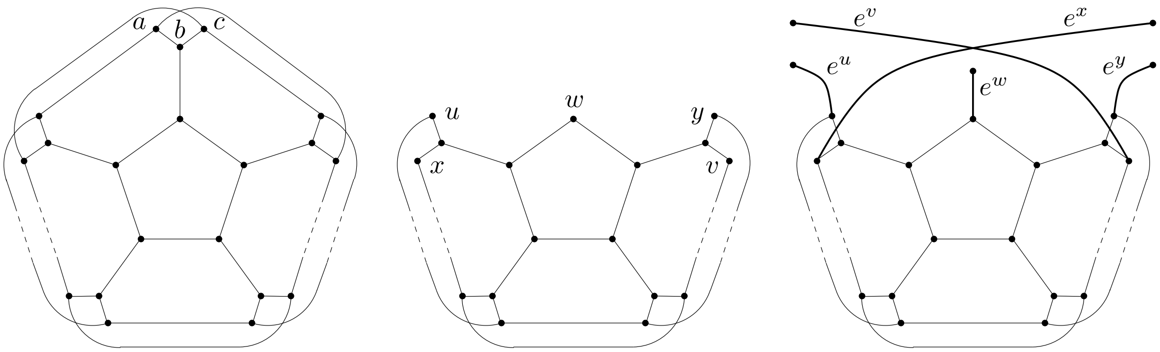


(c) Fulkerson functions Λ_1^3 for B_1 and Λ_L^3 for B_L.



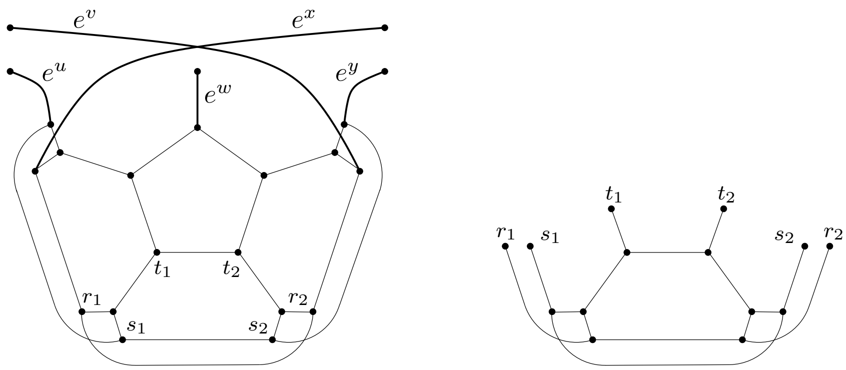
(d) Fulkerson functions Λ_1^4 for B_1 and Λ_L^4 for B_L.

Figure 4.18: Models Λ_i^1, Λ_i^2, Λ_i^3, and Λ_i^4 for the blocks of $\mathcal{B}_{Blanuša}$. Models for B_1^+ and B_L are provided.



(a) Flower Snark J_k. (b) Block B_k^J. (c) Extended block B_k^{J+}.

Figure 4.19: Flower Snark J_k (a), block B_k^J (b), and extended block B_k^J (c).

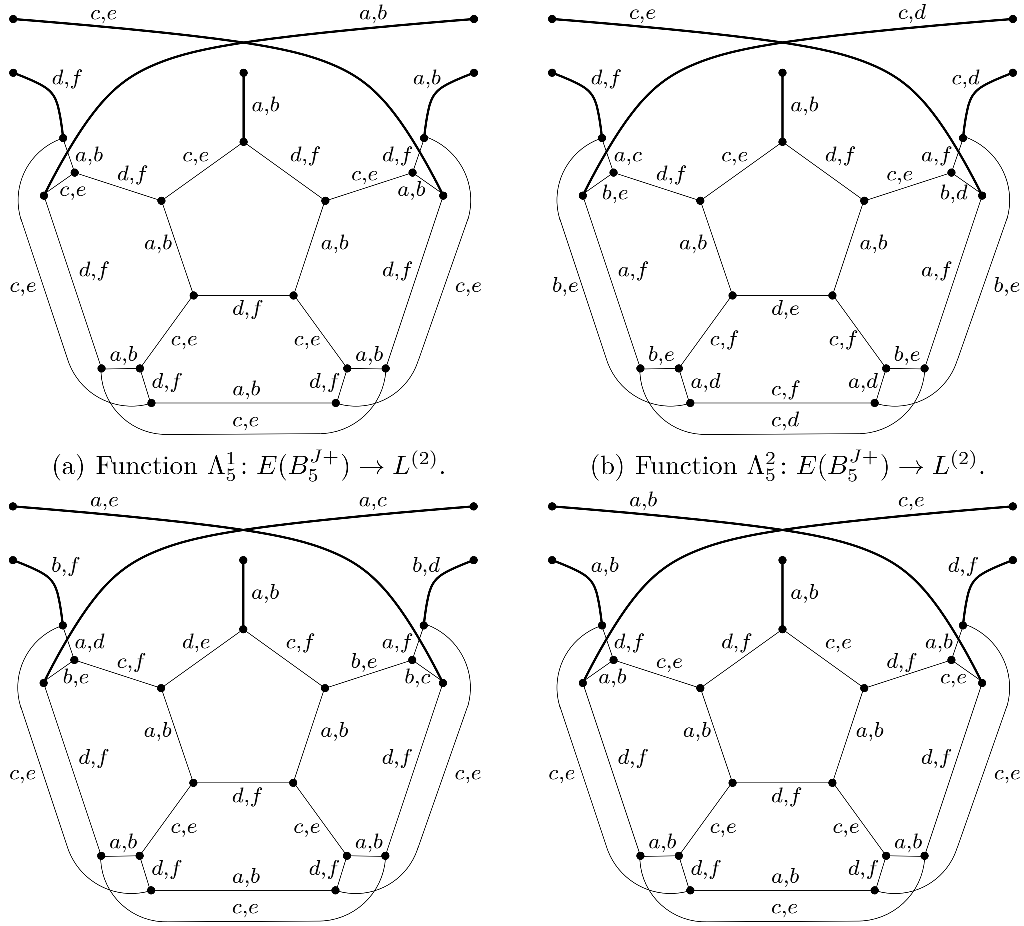


(a) (b)

Figure 4.20: Extended block B_5^{J+} (a) and link graph J_L (b).

In order to construct B_{k+2}^{J+}, join J_L to B_k^{J+}: remove edges r_1r_2, s_1s_2, and t_1t_2 from B_k^{J+}, and identify each vertex $r_1, s_1, t_1, r_2, s_2, t_2$ of J_L with vertex of same name of B_k^{J+}. To construct Λ_{k+2}^j, where $j \in \{1, 3, 4\}$, use Fulkerson function Λ_k^j of B_k^{J+} and the Fulkerson function of J_L exhibited in Figure 4.22(a). Similarly, to construct Λ_{k+2}^2, use Fulkerson function Λ_k^2 and the Fulkerson function of J_L exhibited in Figure 4.22(b). Observe that in all cases each edge e removed from B_k^{J+} has the same label as the new edges of J_L incident with the ends of e. Thus, Λ_{k+2}^1, Λ_{k+2}^2, Λ_{k+2}^3, and Λ_{k+2}^4 are Fulkerson functions. Finally, we adjust the names of vertices of B_k^{J+} such that conditions (ii) and (iii) are satisfied for $k+2$. □

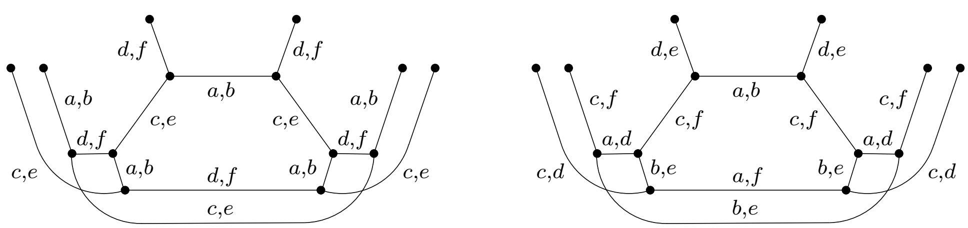
Theorem 4.17. *Every Loupekine snark such that each of its blocks is isomorphic to one of the blocks of $\mathcal{B}_{Flower} \cup \mathcal{B}_{Blanuša} \cup \{B(P)\}$ and its central subgraph is 3-edge-colourable admits a Fulkerson colouring.* □



(a) Function $\Lambda_5^1\colon E(B_5^{J+}) \to L^{(2)}$. (b) Function $\Lambda_5^2\colon E(B_5^{J+}) \to L^{(2)}$.

(c) Function $\Lambda_5^3\colon E(B_5^{J+}) \to L^{(2)}$. (d) Function $\Lambda_5^4\colon E(B_5^{J+}) \to L^{(2)}$.

Figure 4.21: Fulkerson functions Λ_5^1, Λ_5^2, Λ_5^3, and Λ_5^4 for B_5^{J+}.



(a) (b)

Figure 4.22: Fulkerson functions of link graph J_L.

Chapter 5

Conclusions

Fulkerson's Conjecture has been an open problem for more than 40 years. Although partial results have been published, a complete solution for this problem seems to be far from being achieved. The conjecture is restricted to snarks, increasing the importance of this class of graphs, which is related to important problems in Graph Theory.

In this work, we present developments of Fulkerson's Conjecture since it was stated. We selected some remarkable results, emphasizing their reach and connections with other conjectures. We also present equivalences between Fulkerson's Conjecture and other conjectures, expecting to impart the generality of the former. The intent of the first part of this dissertation is to give a full picture of the progress of Fulkerson's Conjecture.

In the second part of this work, we verify Fulkerson's Conjecture for selected infinite families of snarks. Initially, we prove the result for the two families of generalised Blanuša Snarks[1]. Then, we show that the family of LP_0-snarks, constructed with Loupekine's method, also satisfies Fulkerson's Conjecture. Using a different technique, we generalise this result by proving that Fulkerson's Conjecture holds for every LP_1-snark. We extend these results to an even broader class of snarks constructed with Loupekine's method. More specifically, let $\mathcal{B}$ be the infinite set of blocks comprising the block derived from the Petersen Graph, a set of blocks derived from the first family of generalised Blanuša Snarks, $\mathcal{B}_{Blanuša}$, and a set of blocks derived from the Flower Snarks, $\mathcal{B}_{Flower}$, as defined in Section 4.2.1. We show that every Loupekine Snark constructed with blocks of $\mathcal{B}$ (a snark may contain different blocks) and with a 3-edge-colourable central subgraph satisfies Fulkerson's Conjecture. In fact, this result can be extended for any set of blocks provided that for each block there exist four Fulkerson colourings satisfying certain conditions (see Section 4.2.1, Figure 4.16). Table 5.1 presents a non-exhaustive list of classes of snarks known to satisfy Fulkerson's Conjecture.

We conclude this dissertation by mentioning a few conjectures and problems.

[1]Although this fact was already known, we prove it here using a different technique

Author	**Class**	**Year**
Celmins [16]	Flower Snarks; Double Star Snark; dot product successively applied on the Petersen Graph, Flower Snarks and Double Star Snark (includes generalised Blanuša Snarks and the Szekeres Snark)	1979
Hao et al. [38]	Flower Snarks; Goldberg Snarks	2009
Fouquet, Vanherpe [28]	Flower Snarks; generalised Blanuša Snarks; Szekeres Snark; dot product successively performed under certain conditions on the Petersen Graph and Flower Snarks	2011
Karam, Campos [53]	Loupekine Snarks: LP_0-snarks	2012
Brinkmann et al. [9]	Snarks of order $n \leq 36$	2013
Karam, Campos [54, 55]	Loupekine Snarks: LP_1-snarks and $L\mathcal{B}$-snarks with a 3-edge-colourable central subgraph	2013
Karam, Sasaki [56]	Subfamilies of two families of snarks constructed by Jonas Hägglund [37]	2013

Table 5.1: Classes of snarks known to satisfy Fulkerson's Conjecture.

If a cubic graph G is 3-edge-colourable, then G is bridgeless (or, equivalently, G is a 3-graph). Since a 3-edge-colouring of G is a 1-cover by perfect matchings, the statement of Fulkerson's Conjecture can be interpreted as a claim that every bridgeless cubic graph is very close to be 3-edge-colourable, if it is not so. In the same spirit, the Generalised Berge-Fulkerson Conjecture can be seen as a claim that every r-graph, $r \geq 3$, is very close to be r-edge-colourable. The problem of determining the chromatic index of an arbitrary graph, and in particular of a cubic graph, is NP-complete [41]. This fact probably eliminates the existence of a polynomially checkable property implying that a cubic graph is 3-edge-colourable. Problem 5.1 was proposed by Seymour [81], restricting the edge-colouring problem to r-graphs. We wonder whether this restriction makes the problem any easier.

Problem 5.1. *What is a necessary and sufficient condition for an r-graph to be r-edge-colourable?*

A natural continuation of the work presented in this dissertation is the investigation of Fulkerson's Conjecture in the light of other constructions of snarks, such as Goldberg's [32] and Kochol's [63] methods. This approach, however, achieves limited progress, since it often requires the use of ad hoc techniques. On the other hand, Fulkerson's Conjecture can be approached in many other ways. We now suggest a few.

Máčajová and Škoviera [68, 69] showed that every bridgeless cubic graph with oddness at most two has three simultaneously disjoint perfect matchings. This is a special case of the Fan-Raspaud Conjecture. Inspired by this result, it is natural to consider the following special case of Fulkerson's Conjecture.

Conjecture 5.2. *Every bridgeless cubic graph with oddness at most two satisfies Fulkerson's Conjecture.*

Conjecture 5.2 could be studied considering the fact that the Cycle Double Cover Conjecture is true for cubic graphs with oddness at most four [96]. Furthermore, it can be interesting to generalise Conjecture 5.2 to other oddness values.

Fouquet and Vanherpe [28] showed that a bridgeless cubic graph satisfies Fulkerson's Conjecture if and only if it has two compatible FR-triples (see Section 3.4). Motivated by Máčajová and Škoviera's result, Conjecture 5.2 could be approached in the following form.

Conjecture 5.3. *Every bridgeless cubic graph with oddness at most two has two compatible FR-triples.*

Jaeger and Swart [47] conjectured that every snark has cyclic edge-connectivity at most six (Conjecture 2.33). In case this conjecture is true, a possible approach to Fulkerson's Conjecture would be to show that a minimal counterexample is cyclically 7-edge-connected.

Conjecture 5.4. *A minimal counterexample to Fulkerson's Conjecture is cyclically* 7*-edge-connected.*

We conclude by stating the problem of generalising Seymour's Theorem 3.12. This problem is related to the Generalised Berge-Fulkerson Conjecture (Conjecture 3.6).

Problem 5.5. *Is $\mathscr{P}(G)$ eventually periodic, with period at most two, for every r-graph G, $r \geq 3$?*

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