Invariantes de Planaridade

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Dissertação de Mestrado

Instituto de Computação
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Prefácio

O *splitting number* de um grafo G consiste no número mínimo de operações de quebra de vértice que devem ser realizadas em G para produzir um grafo planar, onde uma operação de quebra de vértice em um determinado vértice u significa substituir algumas das arestas (u, v) por arestas (u', v), onde u' é um novo vértice. O *skewness* de G é o número mínimo de arestas que devem ser removidas de G para torná-lo planar. O *vertex deletion number* de G é o menor inteiro k tal que existe um subgrafo induzido planar de G obtido através da remoção de k vértices de G.

Neste trabalho, apresentamos valores exatos para o *splitting number*, o *skewness* e o *vertex deletion number* dos grafos $C_n \times C_m$, onde C_n é o circuito simples com n vértices, e para o *splitting number* e o *vertex deletion number* de uma triangulação dos grafos $C_n \times C_m$.

Abstract

The *splitting number* of a graph G is the minimum number of splitting steps needed to turn G into a planar graph; where each step replaces some of the edges (u, v) incident to a selected vertex u by edges (u', v), where u' is a new vertex. The *skewness* of G is the minimum number of edges that need to be deleted from G to produce a planar graph. The *vertex deletion number* of G is the smallest integer k such that there is a planar induced subgraph of G obtained by the removal of k vertices of G.

In this work, we show exact values for the *splitting number*, *skewness* and *vertex deletion number* of the graphs $C_n \times C_m$, where C_n is the simple circuit on n vertices, and for the *splitting number* and *vertex deletion number* of a triangulation of $C_n \times C_m$.

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Capítulo 1

Introdução

No tempo presente é muito comum encontrarmos uma vasta formação de padrões de formas diagramáticas para representar de maneira pictórica a informação. Citamos como exemplos: fluxogramas, diagramas hierárquicos, diagramas de Sistemas de Informação ER (Entity Relationship) [7, 35]. Contudo, a representação da informação por meio de uma forma diagramática não é uma condição suficiente para a compreensão da informação pelo usuário, pois um diagrama pode ser mostrado de maneira totalmente ilegível.

A área de pesquisa dentro de Teoria dos Grafos chamada Desenho de Grafos investiga formas legíveis de visualização de dados por meio de grafos. Nesta área de pesquisa procuramos, no espaço das possíveis representações de um dado grafo, uma maneira pictórica que seja legível e ao mesmo tempo geral e simples.

Há vários critérios que devem ser respeitados para que de algum modo se traduza o que o usuário quer dizer por um "bom" desenho. Contudo, criar um desenho com características visuais boas é um problema de caráter subjetivo relativo a cada usuário. Entretanto, podemos estabelecer critérios gerais que podem ser de grande interesse para a maioria dos usuários. Estes critérios gerais incluem, entre outros, a planaridade de grafos. Outra razão do interesse neste critério em particular segue do fato que desenhos planares são muito mais fáceis de serem entendidos pelo usuário, e em alguns casos é a única representação possível. Quando a não planaridade é inevitável, cruzamentos devem ser tratados com cuidado e de maneira clara para o observador. Este interesse levou à produção de um imenso acervo de algoritmos que tratam de grafos planares [3].

Uma característica importante do sistema visual humano é a interpretação do conteúdo de um volume pela visão da superfície. Citamos como exemplo deste fato a infinidade de pacotes gráficos que tratam somente a superfície de objetos. Uma vez que as representações mais comuns nos dias de hoje são bidimensionais, podemos então afirmar que um dos critérios mais relevantes à elaboração de um bom desenho, segundo o aspecto de visualização, é a planaridade.

A seguir apresentamos os conceitos básicos sobre grafos e a terminologia que adotamos ao longo da tese. Sugerimos ao leitor familiarizado que prossiga a partir da Seção 1.2.

1.1 Definições e Terminologia

Um *grafo* G é uma tripla $G = (V, E, \psi(G))$ tal que, V é um conjunto de vértices, E é um conjunto de arestas, e $\psi(G)$ é uma função de incidência que associa a cada aresta e um par não ordenado de vértices distintos u e v, chamados de *extremos* de e (se a função de incidência admite que $u = v$ então dizemos que o grafo contém *laços*). Neste caso dizemos que o vértice v é *vizinho* de u ou que u e v são *adjacentes*. Quando duas arestas distintas possuem um mesmo extremo u elas são ditas *adjacentes a* ou *incidentes em u*.

Um grafo com arestas *múltiplas* é um grafo que admite que arestas diferentes compartilhem o mesmo par de extremos. Quando um grafo não admite arestas múltiplas e nem laços dizemos que este grafo é um grafo *simples*. O *grau* de um vértice u, denotado por $d(u)$, é o número de arestas incidentes em u, laços contados duas vezes. A *vizinhança* de um vértice u, denotada por $N(u)$, é o conjunto dos vértices adjacentes a u. Note que se o grafo for simples então $d(u) = |N(u)|$.

Este trabalho considera somente grafos simples, portanto omitiremos a palavra simples. Omitiremos também a função de incidência escrevendo somente que um grafo G é um par $G = (V, E)$.

Se todo os vértices de um grafo G são dois a dois adjacentes, então G é dito *completo*. Um grafo K_n é um grafo completo com n vértices. Um grafo bipartido G é um grafo cujos vértices são particionados em dois conjuntos S_1 e S_2 de tal maneira que para todo vértice $u \in S_1$, $N(u) \subseteq S_2$ e para todo vértice $v \in S_2$, $N(v) \subseteq S_1$. Dado um grafo G bipartido, se para todo vértice $u \in S_1$, $N(u) = S_2$ e para todo vértice $v \in S_2$, $N(v) = S_1$ então G é dito *completo bipartido* e é representado por $K_{n,m}$, onde $n = |S_1|$ e $m = |S_2|$.

Um grafo $G' = (V', E')$ é chamado de *subgrafo* de um grafo $G = (V, E)$ se V' é subconjunto de V e E' é subconjunto de E.

A *subdivisão de uma aresta* $e = uv$ em um grafo G consiste em substituir a aresta e por um novo vértice n_e e duas arestas un_e e n_ev. Um grafo G é dito conter uma *subdivisão* de um grafo H se existe um subgrafo de G isomorfo a um grafo obtido por meio de subdivisão de arestas de H.

Um *desenho simples* de um grafo G é um desenho no plano tal que: nenhuma aresta cruza a si mesma, arestas adjacentes não se cruzam, duas arestas se cruzam no máximo uma vez, arestas não interceptam vértices exceto seus extremos, e no máximo duas arestas se cruzam em um mesmo ponto. Neste trabalho, todos os desenhos são simples.

Um grafo é *planar* quando ele possui um desenho simples no plano sem cruzamentos de arestas. Uma outra maneira de definir um grafo planar, usada com freqüência em nosso trabalho, é a caracterização de Kuratowski [28]: um grafo é planar se e somente se ele não contém uma subdivisão do K_5 nem uma subdivisão do $K_{3,3}$.

Um grafo é dito *toroidal* se ele pode ser desenhado no toro sem que arestas se cruzem e sem que haja sobreposição de arestas e vértices. O *toro* é uma superfície topologicamente igual a uma esfera com uma alça, como mostra a Figura 1.1.

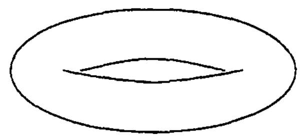

Figura 1.1: Desenho do toro

Um *ciclo* ou *circuito* C_n é um grafo com n vértices $\{v_0, v_1, ..., v_{n-1}\}$ tal que todo vértice v_i é adjacente exatamente a dois vértices $v_{(i-1) \bmod n}$ e $v_{(i+1) \bmod n}$ em C_n.

O produto cartesiano $C_n \times C_m$ dos ciclos C_n e C_m é o grafo contendo nm vértices $\{v_{i,j}\}$ e $2nm$ arestas $\{v_{i,j}, v_{(i+1) \bmod n, j}\}$ e $\{v_{i,j}, v_{i,(j+1) \bmod m}\}$, para $0 \leq i < n$ e $0 \leq j < m$. Considerando que cada vértice do $C_n \times C_m$ é representado por um ponto no plano com coordenadas (i, j), chamamos as duas famílias de arestas acima de *horizontal* e *vertical*, respectivamente. Um ciclo do grafo $C_n \times C_m$ é chamado *meridiano* se ele usa apenas arestas verticais e *paralelo* se ele usa apenas arestas horizontais. Então, o grafo $C_n \times C_m$ possui n meridianos isomorfos ao C_m e m paralelos isomorfos ao C_n. Como este trabalho se concentra nos grafos não planares e os grafos $C_n \times C_m$ com $\min\{m, n\} \leq 2$ são planares, suporemos $m, n \geq 3$.

Um *biciclo* do grafo $C_n \times C_m$ é a união de um meridiano e um paralelo. Observe que um meridiano e um paralelo têm precisamente um vértice em comum.

Uma *triangulação de* $C_n \times C_m$, denotada por $\mathcal{T}_{C_n \times C_m}$, é o grafo obtido após acrescentarmos as arestas $\{v_{i,j}, v_{(i+1) \bmod n, (j+1) \bmod m}\}$ a cada vértice do $C_n \times C_m$.

Um *n-cubo* é um grafo cujos 2^n vértices são as ênuplas de 0's e 1's e dois vértices $u = (u_1, u_2, ..., u_n)$ e $v = (v_1, v_2, ..., v_n)$ são adjacentes se e somente se $u_i \neq v_i$ para exatamente um único i, $1 \leq i \leq n$. Denotamos um n-cubo por Q_n.

Dois grafos G e H são *isomorfos* se existe uma bijeção $\psi : G \rightarrow H$ tal que dois vértices distintos x e y de G são adjacentes se e somente se os vértices $\psi(x)$ e $\psi(y)$ são adjacentes em H. Tal função ψ é chamada de *isomorfismo* de G para H. Um *automorfismo* é um isomorfismo em que $G = H$. É fácil notar que $C_n \times C_m$ é isomorfo a $C_m \times C_n$ e que $\mathcal{T}_{C_n \times C_m}$ é isomorfo a $\mathcal{T}_{C_m \times C_n}$.

Seja $\mathcal{F}$ uma família de subgrafos de G. Dizemos que G é *$\mathcal{F}$-transitivo* se, para quaisquer dois elementos F, H de $\mathcal{F}$, existe um automorfismo de G cuja restrição a $V(F)$ é isomorfismo de F em H.

Note que $C_n \times C_m$ é vértice-transitivo, meridiano-transitivo, paralelo-transitivo e biciclo-transitivo.

1.2 Invariantes de Planaridade

Dada a importância do critério de planaridade no desenho de grafos, estamos interessados em fazer desenhos que sejam o mais planar possível, isto é, contendo um número mínimo de cruzamentos de arestas. Este número é chamado *crossing number* de G e o denotaremos aqui por $cr(G)$.

Quando um grafo G não pode ser desenhado no plano sem cruzamentos de arestas, podemos efetuar operações de "planarização" em G, ou seja, alterá-lo de forma que o grafo resultante seja planar. Dentre as operações possíveis destacamos: quebra de vértice, remoção de vértice e remoção de aresta.

O *splitting number* de um grafo G consiste no número mínimo de operações de quebra de vértice que devem ser realizadas em G para produzir um grafo planar, onde uma operação de quebra de vértice em um determinado vértice u significa substituir algumas das arestas (u, v) por arestas (u', v), onde u' é um novo vértice.

O *vertex deletion number* de um grafo G, que denotamos por $vd(G)$, é o número mínimo de vértices que precisam ser removidos de G para produzir um grafo planar. Note que remover um vértice implica em remover também todas as suas arestas incidentes.

O *skewness* de um grafo G, $sk(G)$, consiste no número mínimo de arestas que devem ser removidas de G para produzir um grafo planar.

O *splitting number*, o *vertex deletion number* e o *skewness* também são invariantes de planaridade e a seguinte relação: $vd(G) \leq sp(G) \leq sk(G) \leq cr(G)$, cuja demonstração é apresentada no Capítulo 3, revela a interdependência entre as quatro importantes invariantes citadas.

Além do *crossing number*, *skewness*, *splitting number* e *vertex deletion number*, outras invariantes de planaridade também têm sido estudadas. Para maiores informações sobre invariantes e operações de planarização, aconselhamos a coletânea escrita recentemente por Liebers [30].

1.2.1 Valores Conhecidos

Pouco se sabe a respeito do *crossing number*, *splitting number*, *skewness* e *vertex deletion number* para classes específicas de grafos. Os problemas de decisão correspondentes para grafos gerais são todos NP-completos [18, 15, 19, 43]. Contudo, para um valor fixo k esses problemas se tornam polinomiais [18, 38].

Os valores das invariantes de planaridade conhecidos são restritos a poucas classes de grafos, em geral grafos simétricos ou com características regulares, tais como: K_n, $K_{n,m}$, Q_n e $C_n \times C_m$.

Um limite superior para o *crossing number* dos grafos completos K_n foi apresentado em [20, 21] e o *skewness* pode ser facilmente extraído a partir da fórmula de Euler como visto em [30]. O valor exato do *splitting number* dos grafos completos também foi determinado [24]. O *vertex deletion number* é $n-4$ para $n > 4$.

Com relação aos grafos completos bipartidos, foi estabelecido um limite superior para o *crossing number* [6] cuja igualdade é satisfeita se $\min\{n,m\} \leq 6$. Esta igualdade foi estendida em [42] para os grafos $K_{7,7}, K_{7,8}, K_{7,9}$ e $K_{7,10}$. O *splitting number* [26] e o *skewness* [30] do $K_{n,m}$ são conhecidos. O *vertex deletion number* é $\min\{n,m\}-2$ para $n,m \geq 3$.

Os valores das invariantes determinados para o K_n e o $K_{n,m}$ são mostrados na Tabela 1.1.

	cr	sk	sp	vd
K_n	$\leq \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$	$\frac{(n-3)(n-4)}{2}$	$\lceil\frac{(n-3)(n-4)}{6}\rceil;\ n \neq 6,7,9$ $\lceil\frac{(n-3)(n-4)}{6}\rceil+1;\ n = 6,7,9$	$n-4;\ n>4$
$K_{n,m}$	$\leq \lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor$	$(n-2)(m-2)$	$\lceil\frac{(m-2)(n-2)}{2}\rceil;\ m,n \geq 2$	$\min\{n,m\}-2;\ n,m \geq 3$

Tabela 1.1: Invariantes do K_n e do $K_{n,m}$

Há vários resultados parciais a respeito do *crossing number* do cubo Q_n [12, 13, 20, 21, 31, 41] e somente o Q_4 teve seu valor exato estabelecido [9]. O *skewness* do cubo Q_n foi estabelecido [8]. Em [16] foram apresentados o *splitting number* do Q_4 e um limite inferior para o *splitting number* do Q_n.

Existe uma conjetura para o *crossing number* dos grafos $C_n \times C_m$ e um valor exato no caso do $C_3 \times C_3$ como mostrado em [23]. Este resultado foi usado para estabelecer em [37] o *crossing number* dos grafos $C_3 \times C_n$. A conjetura foi provada para o $C_4 \times C_4$ em [9], para o $C_5 \times C_5$ em [36] e para os grafos $C_5 \times C_n$ em [27]. O *crossing number* também foi provado para o $C_6 \times C_6$ [1] e para o $C_7 \times C_7$ [2]. Recentemente a conjetura foi reforçada [39]. Schaffer estabeleceu o *splitting number* dos grafos $C_n \times C_m$ em 1986 [40], contudo este resultado nunca foi publicado.

Os valores das invariantes determinados para Q_n e $C_n \times C_m$ são mostrados na Tabela 1.2.

	cr	sk	sp
Q_n	8; se $n = 4$ $\leq \frac{4^n}{6} - n^2 2^{n-3} + 2^{n-4}3 + \frac{(-2)^n}{48}$	$2^n(n-2) - n2^{n-1} + 4$	4; se $n = 4$ $\geq 2^{n-2}$
$C_n \times C_m$	$\leq (m-2)n$; se $3 \leq m \leq n$		$\begin{cases} 1 & \text{se } n+m=6 \\ 2 & \text{se } n+m=7 \\ \min\{n,m\} & \text{c.c.} \end{cases}$

Tabela 1.2: Invariantes do Q_n e do $C_n \times C_m$

1.2.2 Nossa Contribuição

Neste trabalho determinamos o *splitting number*, o *skewness* e o *vertex deletion number* dos grafos $C_n \times C_m$. Determinamos também o *vertex deletion number* e o *splitting number* de $\mathcal{T}_{C_n \times C_m}$.

Veja os resultados na Tabela 1.3, onde $\xi_{i,j}(k_1, k_2)$ é o número de condições verdadeiras dentre: (i) $k_1 = k_2 \leq i$ e (ii) $k_1 + k_2 \leq j$.

	sk	sp	vd
$C_n \times C_m$	$\min\{n,m\} - \xi_{2,7}(n,m)$	$\min\{n,m\} - \xi_{3,7}(n,m)$	$\min\{n,m\} - \xi_{5,9}(n,m)$
$\mathcal{T}_{C_n \times C_m}$		$\min\{n,m\}$	$\min\{n,m\}$

Tabela 1.3: Invariantes determinadas para $C_n \times C_m$ e $\mathcal{T}_{C_n \times C_m}$

A Tabela 1.4 mostra os valores de $sp(C_n \times C_m)$, $sk(C_n \times C_m)$ e $vd(C_n \times C_m)$ para valores pequenos de n e m.

sk	3	4	5	6	7
3	2	2	3	3	3
4	2	4	4	4	4
5	3	4	5	5	5
6	3	4	5	6	6
7	3	4	5	6	7

sp	3	4	5	6	7
3	1	2	3	3	3
4	2	4	4	4	4
5	3	4	5	5	5
6	3	4	5	6	6
7	3	4	5	6	7

vd	3	4	5	6	7
3	1	2	2	2	3
4	2	2	3	4	4
5	2	3	4	5	5
6	2	4	5	6	6
7	3	4	5	6	6

Tabela 1.4: *sk*, *sp* e *vd* de $C_n \times C_m$ para n e m pequenos

1.3 Organização da Tese

Neste primeiro capítulo procuramos situar o leitor no contexto em que se insere o nosso trabalho. Além de darmos os conceitos básicos da área, falamos um pouco a respeito de invariantes de planaridade, apresentando os resultados mais importantes e revelando a nossa contribuição com este trabalho.

Nos Capítulos de 2 a 4 apresentamos os resultados alcançados, todos eles na forma de artigos, em inglês, que foram submetidos a revista e congresso internacionais. O primeiro deles apresenta o *splitting number* e o *skewness* do $C_n \times C_m$. Este artigo foi submetido ao *Journal of Graph Theory*. O segundo artigo, submetido ao *Latin'2000*, mostra o *vertex deletion number* do $C_n \times C_m$, cuja demonstração completa gerou um relatório técnico que apresentamos no Apêndice A. O Capítulo 4 é outro artigo submetido ao *Journal of Graph Theory*, no qual mostramos o *vertex deletion number* e o *splitting number* dos grafos $\mathcal{T}_{C_n \times C_m}$.

No quinto e último capítulo, apresentamos a conclusão e os trabalhos futuros.

Capítulo 2

The Splitting Number and Skewness of $C_n \times C_m$

Prólogo

Este capítulo contém a réplica do artigo que submetemos ao *Journal of Graph Theory*, no qual apresentamos o *splitting number* e o *skewness* dos grafos $C_n \times C_m$. Mais especificamente, mostramos uma nova demonstração do resultado de Schaffer [40] e apresentamos uma fórmula exata que determina o *skewness* para esta classe de grafos.

The Splitting Number and Skewness of $C_n \times C_m$ [1]

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Abstract: The skewness of a graph G is the minimum number of edges that need to be deleted from G to produce a planar graph. The splitting number of a graph G is the minimum number of splitting steps needed to turn G into a planar graph; where each step replaces some of the edges $\{u, v\}$ incident to a selected vertex u by edges $\{u', v\}$, where u' is a new vertex. We show that the splitting number of the toroidal grid graph $C_n \times C_m$ is $\min\{n, m\} - \xi_{3,7}(n, m)$ and its skewness is $\min\{n, m\} - \xi_{2,7}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$.

Keywords: topological graph theory, graph drawing, toroidal mesh, planarity.

2.1 Introduction

The skewness $sk(G)$ and splitting number $sp(G)$, defined below, are two natural measures of the non-planarity of a graph G. These topological invariants play important roles in automatic graph drawing and circuit design [10, 32, 11, 29, 22, 25].

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[2]Research done while author was working at Instituto de Computação, UNICAMP, SP, Brazil.

In this paper, we determine exact values for the skewness and splitting number of the graphs $C_n \times C_m$, where C_n is the chordless cycle on n vertices. These graphs can be drawn as regular latitude-longitude grids on the torus, and thus are also known as 'toroidal rectangular grids' or similar names. They occur often as interconnection diagrams of multiprocessor computers and cellular automata [22, 25], and so our results are relevant to the physical design of such machines.

It turns out that the obvious upper bound $\min\{n,m\}$ is always tight except for $C_3 \times C_3$. Specifically, we show that

$$sp(C_n \times C_m) = \min\{n,m\} - \xi_{3,7}(n,m) \quad (2.1)$$

$$sk(C_n \times C_m) = \min\{n,m\} - \xi_{2,7}(n,m) \quad (2.2)$$

where $\xi_{i,j}(k_1,k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$.

Table 2.1 shows these bounds explicitly for small values of n and m.

sp					
	3	4	5	6	7
3	1	2	3	3	3
4	2	4	4	4	4
5	3	4	5	5	5
6	3	4	5	6	6
7	3	4	5	6	7

sk					
	3	4	5	6	7
3	2	2	3	3	3
4	2	4	4	4	4
5	3	4	5	5	5
6	3	4	5	6	6
7	3	4	5	6	7

Tabela 2.1: Values of *sp* and *sk* for small values of n and m

Our strategy to prove these results is as follows. In section 2.4, we prove that $sp(G) \leq sk(G)$, for any graph G. In section 2.5, we show that formula (2.1) is a lower bound for the splitting number *sp*, and in section 2.6, we prove that formula (2.2) is an upper bound for the skewness *sk*. It follows that the two invariants coincide except for $C_3 \times C_3$. To complete the proof we show in sections 2.5 and 2.6 that $sp(C_3 \times C_3) = 1$ and $sk(C_3 \times C_3) = 2$, respectively.

2.2 Notation and Definitions

For basic concepts—*graph, path, cycle, complete graph*, etc.—we borrow the definitions and nomenclature from Bondy and Murty [5].

Two graphs G and H are said to be *isomorphic* if there is a bijection $\alpha\colon V(G) \to V(H)$, such that $\{u,v\} \in E(G)$ if and only if $\{\alpha(u),\alpha(v)\} \in E(H)$. The bijection α is called an *isomorphism* from G to H. An *automorphism* of a graph is an isomorphism from the graph to itself.

Additionally, we define an *open arc* as a bounded subset of the plane $\mathbb{R}^2$ homeomorphic to the real line $\mathbb{R}$ in the standard topology. A *drawing* of a graph G is a mapping φ of the vertices of G to points of the plane, and of the edges of G to open arcs - the *vertices* and *edges* of the drawing, respectively - such that (1) the vertices of the drawing are pairwise distinct, and disjoint from all its edges; (2) any two edges of the drawing are either disjoint, or cross at a single point; (3) for every edge $e = \{u, v\}$ of G, the external frontier of $vd(e)$ is $\{vd(u), vd(v)\}$; and (4) no three edges of the drawing go through the same point.

We say that a graph is *planar* if it has a drawing without crossing edges. We denote by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph between m vertices and n vertices. In our proofs, we rely heavily on Kuratowski's theorem [28], which says that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph. In fact, we always prove that a graph is not planar by showing that it contains a subdivision of $K_{3,3}$, shown in Figure 2.1.

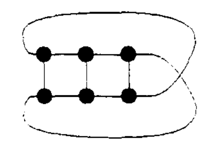


Figura 2.1: $K_{3,3}$

We also make use of the fact that a planar graph remains planar if an edge is deleted or contracted.

The *skewness* $sk(G)$ is the minimum number of edges that must be removed from G to produce a planar graph.

A *vertex splitting operation*, or *splitting* for short, consists in replacing some of the edges $\{u, v\}$ incident to a selected vertex u by edges $\{u', v\}$, where u' is a single new vertex. The *(vertex) splitting number* $sp(G)$ is the minimum number of splittings needed to turn G into a planar graph.

Note that, for any sequence of splittings, there is a sequence of the same length that produces the same graph, and is such that the vertex u affected by each splitting is always an original vertex of G, not one of the vertices introduced by previous steps.

For $n \geq 3$, we denote by C_n the chordless cycle with n vertices and n edges. The $n \times m$ *toroidal grid* $C_n \times C_m$ is the graph-theoretic product of C_n and C_m; that is, the graph with nm vertices $\{v_{ij} : 0 \leq i < n,\ 0 \leq j < m\}$, and $2nm$ edges $\{\{v_{ij}, v_{(i+1) \bmod n, j}\}, \{v_{ij}, v_{i,(j+1) \bmod m}\} : 0 \leq i < n,\ 0 \leq j < m\}$.

In our drawings of $C_n \times C_m$, vertex v_{ij} is represented by a point on the plane with coordinates (i, j). Based on this convention, we call the two families of edges above *horizontal* and *vertical*, respectively.

A cycle of $C_n \times C_m$ is called a *meridian* if it uses only vertical edges, and a *parallel* if it uses only horizontal ones. Thus the $n \times m$ toroidal grid has n meridians isomorphic to C_m, and m parallels isomorphic to C_n.

Let $\mathcal{F}$ be a family of isomorphic subgraphs of a graph G. We say that G is *$\mathcal{F}$-transitive* if for any two elements F and H of $\mathcal{F}$ there is an automorphism of G that takes F to H. Note that $C_n \times C_m$ is meridian-transitive, and parallel-transitive.

2.3 Previous Results

The problems of verifying and computing the invariants *sk* and *sp* for general graphs have been shown to be respectively NP-complete [19, 15] and MAX SNP-hard [14, 17], even for cubic graphs. However, it can be checked in polynomial time whether the skewness *sk* is equal to a fixed k [18]. We have shown [17] that the same holds for the splitting number *sp*, by the results of Robertson and Seymour [38].

The difficulty in computing the invariants *sk* and *sp* for general graphs justifies their analysis for special families of graphs. Exact explicit formulas have been found for the splitting number of complete graphs and complete bipartite graphs [24, 26], and for the skewness of the n-cube Q_n [8].

For the toroidal grid $C_n \times C_m$, in particular, there are only a few partial results concerning these invariants. The upper bounds $sk(C_n \times C_m) \leq \min\{n, m\}$ and $sp(C_n \times C_m) \leq \min\{n, m\}$ are fairly obvious, too (see lemma 2.19).

The splitting number $sp(C_n \times C_m)$ was determined exactly by Schaffer in his 1986 thesis [40], but not published elsewhere. The special case of $C_4 \times C_4$, which is isomorphic to the 4-cube Q_4, was proved by Faria et al. [16]. In this article we give a new proof of Schaffer's result, and also an exact formula for the skewness $sk(C_n \times C_m)$.

There are many partial results about the *crossing number* $cr(G)$ (the minimum number of edge crossings in any drawing of G) for $G = C_n \times C_m$. Harary et al. [23] conjectured that $cr(C_n \times C_m) = (n-2)m$, for all n, m satisfying $3 \leq n \leq m$. This has been proved only for n, m satisfying $m \geq n$, and $n \leq 5$ [37, 9, 4, 36, 27], and for the special cases $n = m = 6$ [1], and $n = m = 7$ [2]. A recent result [39] based on the asymptotic behaviour of the minimum crossing numbers of wide classes of drawings for $C_n \times C_m$ also supports the conjecture. The general conjecture $cr(C_n \times C_m) = (n-2)m$ remains open for all but a finite number of values of n. It can be shown that $cr(G)$ is always an upper bound for $sk(G)$ and $sp(G)$ [16]. However, for $C_n \times C_m$ this bound is not tight, and so the results above cited are not directly useful for our problem.

2.4 Skewness versus Splitting

The following general properties of skewness and splitting numbers are easily proved:

Lemma 2.1 *If H is a subgraph of G, then* $\mathrm{sp}(H) \leq \mathrm{sp}(G)$ *and* $\mathrm{sk}(H) \leq \mathrm{sk}(G)$.

Lemma 2.2 *If H is a subdivision of G, then* $\mathrm{sp}(H) = \mathrm{sp}(G)$ *and* $\mathrm{sk}(H) = \mathrm{sk}(G)$.

Lemma 2.3 *If a vertex v of a graph G has at most one neighbor, then* $\mathrm{sp}(G) = \mathrm{sp}(G-v)$.

Proof. Consider a minimum sequence of splittings that turns $G' = G - v$ into a planar graph H'. Since these splittings do not affect the edge $\{u, v\}$, if we apply the same splittings to G, then we will get a graph H equal to H' with the extra vertex v and extra edge $\{u, v\}$; which is obviously planar like H'. Thus $sp(G) \leq sp(G - v)$. The claim then follows by lemma 2.1. □

We also need the following inequality between the invariants:

Lemma 2.4 *For every graph G, we have* $\mathrm{sp}(G) \leq \mathrm{sk}(G)$.

Proof. We prove the lemma by induction on $sk(G)$. If $sk(G) = 0$, then G is planar and therefore $sp(G) = 0$. Otherwise, there is some edge $e = \{u, v\}$ such that $sk(G - e) = sk(G) - 1$. Now let H be the result of adding a vertex u' to G and replacing the edge e by $e' = \{u', v\}$, as shown in Figure 2.2. This is a splitting step, so $sp(G) \leq sp(H) + 1$. By lemma 2.3, $sp(H) = sp(H - u') = sp(G - e)$. Since $sp(G - e) \leq sk(G - e)$ by the induction hypothesis, we conclude that $sp(G) \leq (sk(G) - 1) + 1 = sk(G)$. □

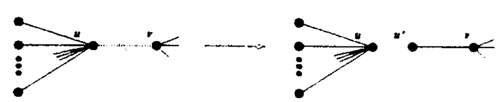


Figura 2.2: $sp(G) \leq sk(G)$

2.5 A Lower Bound for the Splitting Number

Lemma 2.5 *The splitting number of $C_3 \times C_3$ is* 1.

Proof. The graph $C_3 \times C_3$ has a subdivision of the $K_{3,3}$ as shown in Figure 2.3(a), where the edges belonging to the subdivision of the $K_{3,3}$ are thicker and vertices are emphasized. It follows that $sp(C_3 \times C_3) \geq 1$. On the other hand, we can obtain a planar graph from $C_3 \times C_3$ with a single splitting as shown in Figure 2.3(b) which implies that $sp(C_3 \times C_3) \leq 1$. Therefore, $sp(C_3 \times C_3) = 1$. □

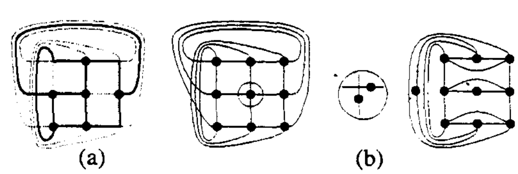


Figura 2.3: $sp(C_3 \times C_3) \leq 1$

Lemma 2.6 *The splitting number of $C_3 \times C_4$ is at least* 2.

Proof. Let H be the graph obtained from $C_3 \times C_4$ by a single vertex splitting. Without loss of generality, we may assume that the split vertex is $v_{2,0}$ (indicated by $\times$ in Figure 2.4). That splitting leaves untouched the subdivision of $K_{3,3}$ shown in Figure 2.4. It follows that $sp(C_3 \times C_4) \geq 2$. □

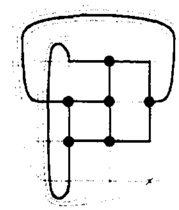


Figura 2.4: $sp(C_3 \times C_4) \geq 2$

Lemma 2.7 *If G can be obtained from $C_3 \times C_5$ by two splittings on the same vertex u, then* $\mathrm{sp}(G) \geq 1$.

Proof. As in the proof of lemma 2.6, the two splittings in the vertex $v_{2,0}$ will not destroy the copy of $K_{3,3}$ shown in Figure 2.5. Therefore G is not planar, and $sp(G) \geq 1$. □

Lemma 2.8 *If G can be obtained from $C_3 \times C_5$ by two splittings on distinct vertices of $C_3 \times C_5$, which belong to the same parallel or to adjacent parallels, then* $\mathrm{sp}(G) \geq 1$.

Proof. If the two vertices are on the same parallel (C_3), then without loss of generality we may assume that they are $v_{1,0}$ and $v_{2,0}$. In that case the copy of $K_{3,3}$ shown in Figure 2.5 is not affected by the splittings. The same is true if u and v belong to consecutive parallels: we can always map them by an automorphism to two of the vertices marked $\times$ in Figure 2.5, which can be split without destroying the $K_{3,3}$. Therefore G is not planar, and $sp(G) \geq 1$. □

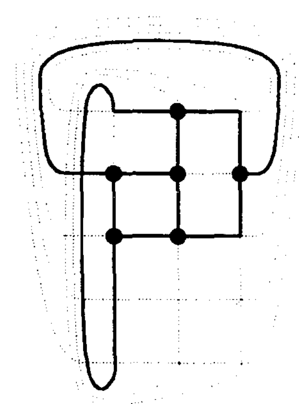


Figura 2.5: $sp(G) \geq 1$

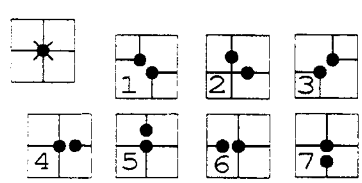


Figura 2.6: Possible ways to split a vertex of $C_n \times C_m$

As shown in Figure 2.6, there are at most seven different ways to split a vertex of $C_n \times C_m$ (assuming we do not care which of the two resulting vertices is the new one). We need this fact to prove the next two lemmas.

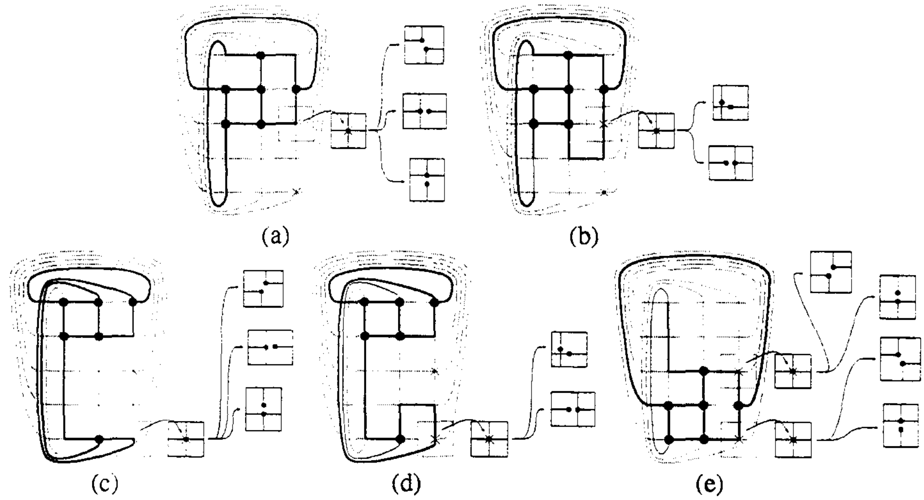


Figura 2.7: $sp(G) \geq 1$

Lemma 2.9 *If G is obtained from $C_3 \times C_5$ by splitting two non-adjacent vertices on the same meridian of $C_3 \times C_5$, then* $\mathrm{sp}(G) \geq 1$.

Proof. Without loss of generality, we may assume that the two vertices are $v_{2,0}$ and $v_{2,2}$. Figure 2.7 shows all $7 \times 7 = 49$ possible ways to split these two vertices, grouped into five cases. In each case there is a subdivision of $K_{3,3}$, shown in Figure 2.7, that is contained in $C_3 \times C_5$ and is not destroyed by the splits. Therefore G is not planar, and $sp(G) \geq 1$. □

Lemma 2.10 *If G is the result of splitting two vertices of $C_3 \times C_5$ that lie at distance 3 from each other, then* $\mathrm{sp}(G) \geq 1$.

Proof. Without loss of generality, we may assume that one of the vertices is $v_{1,2}$. There are four vertices at distance 3 from $v_{1,2}$, namely $v_{0,0}$, $v_{2,0}$, $v_{0,4}$, and $v_{2,4}$. Without loss of generality, we may assume the other split vertex is $v_{2,0}$. Figure 2.8 shows all $7 \times 7 = 49$ possible ways to split these two vertices, grouped into five cases. In each case there is a subdivision of $K_{3,3}$ contained in $C_3 \times C_5$ that is not destroyed by the splittings. Therefore G is not planar, and $sp(G) \geq 1$. □

Lemma 2.11 *The splitting number of $C_3 \times C_5$ is at least* 3.

Proof. Consider a sequence of splittings that turns $C_3 \times C_5$ into a planar graph. We may assume that all splittings are applied to vertices of $C_3 \times C_5$. By lemma 2.6, the sequence has at least two steps; let u and v be the affected vertices, and d their distance

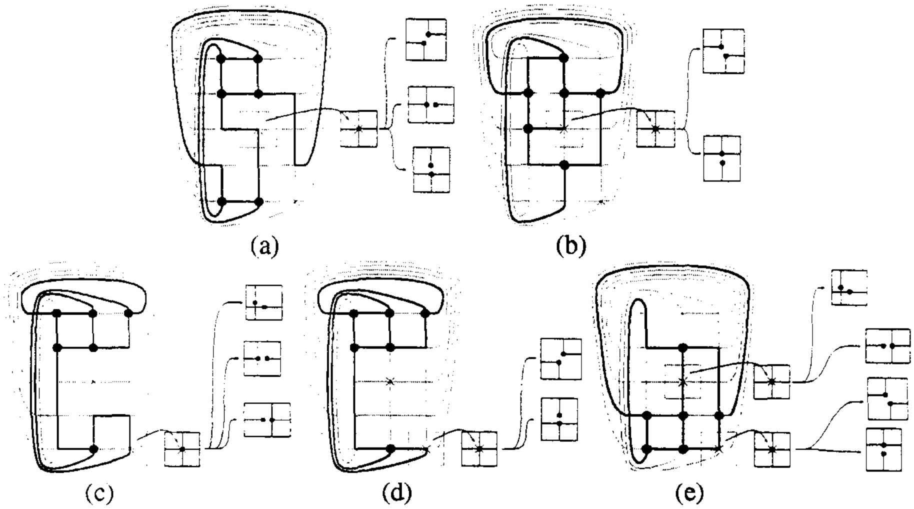


Figura 2.8: $sp(G) \geq 1$

in $C_3 \times C_5$. If $d = 0$, then $u = v$, and lemma 2.7 applies. If $d = 1$, then u and v lie on the same parallel or on adjacent parallels, and lemma 2.8 applies. If $d = 2$, then they either lie on adjacent parallels, or are non-adjacent vertices of the same meridian, and either lemma 2.8 or lemma 2.9 applies. Finally, if $d = 3$, then lemma 2.10 applies. Since there are no pairs of vertices with $d > 3$, we conclude that two splittings are not enough to turn $C_3 \times C_5$ into a planar graph. □

Lemma 2.12 *The splitting number of $C_3 \times C_m$, for $m \geq 5$, is at least* 3.

Proof. This result follows from lemmas 2.1, 2.2 and 2.11, since $C_3 \times C_m$ contains a subgraph that is isomorphic to a subdivision of $C_3 \times C_5$. □

Lemma 2.13 *The splitting number of $C_4 \times C_4$ is* 4.

Proof. The graph $C_4 \times C_4$ is isomorphic to the 4-cube Q_4; the result $sp(Q_4) = 4$ was proved by Faria, Figueiredo and Mendonça [16]. □

Lemma 2.14 *The splitting number of $C_k \times C_k$, for $k \geq 4$, is at least k.*

Proof. We prove this assertion by induction on k. The induction basis is the case $k = 4$, proved by lemma 2.13.

Now let k be greater than 4, and let Z be any sequence of splittings that turns $G = C_k \times C_k$ into a planar graph H. We may assume that all splittings in Z are applied to vertices of G. Let v be one of the vertices split by Z, and let G' be the graph $G - v$. It is easy to see that the graph G' contains a subgraph that is isomorphic to a subdivision of $C_{k-1} \times C_{k-1}$; hence, by induction, $sp(G') \geq k - 1$. It follows that the sequence Z has at least $k - 1 + 1 = k$ steps. □

Lemma 2.15 *The splitting number of $C_n \times C_m$, for $n, m \geq 4$, is at least* $\min\{n, m\}$.

Proof. Without loss of generality suppose that $n \leq m$. The assertion follows from the fact that $C_n \times C_m$ contains a subgraph that is isomorphic to a subdivision of $C_n \times C_n$, which has splitting number at least n. □

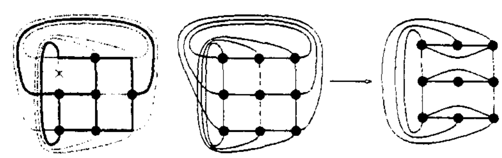
Lemma 2.16 *The splitting number of $C_n \times C_m$ is at least $min\{n, m\} - \xi_{3,7}(n, m)$,*

Proof. The assertion follows from lemmas 2.5–2.15. □

2.6 An Upper Bound for the Skewness

Lemma 2.17 *The skewness of $C_3 \times C_3$ is 2.*

Proof. Let e be any edge of $C_3 \times C_3$; without loss of generality, we may assume that e is the vertical edge $\{v_{0,1}, v_{0,2}\}$, marked with $\times$ in Figure 2.9(a). Deleting e from $C_3 \times C_3$ does not affect the subdivision of $K_{3,3}$ indicated in the figure; therefore $C_3 \times C_3 - e$ is not planar, and $sk(C_3 \times C_3) > 1$.



(a) (b) (c)

Figura 2.9: $sk(C_3 \times C_3) \leq 2$

On the other hand, the removal of the two edges marked $\times$ in Figure 2.9(b) results in a planar graph, as shown in Figure 2.9(c). Therefore $sk(C_3 \times C_3) = 2$. □

Lemma 2.18 *The skewness of $C_3 \times C_4$ is at most* 2.

Proof. Figure 2.10 exhibits two edges of $C_3 \times C_4$ whose removal results in a planar graph. □

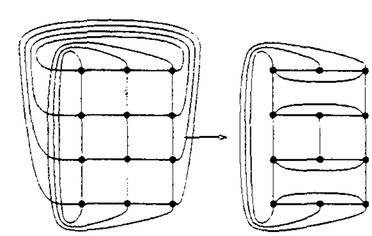


Figura 2.10: $sk(C_3 \times C_4) \leq 2$

Lemma 2.19 *The skewness of $C_n \times C_m$ is at most* $\min\{n, m\}$.

Proof. Suppose without loss of generality that $n \leq m$. Figure 2.11 exhibits a set of n edges of $C_n \times C_m$ whose removal obtains a planar graph. □

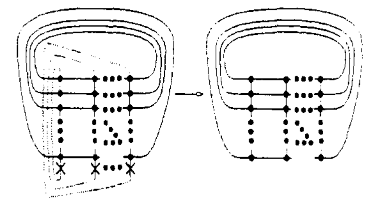


Figura 2.11: $sk(C_n \times C_m) \leq \min\{n, m\}$

Theorem 2.20 *The splitting number and the skewness of $C_n \times C_m$ are:*

$$\mathrm{sp}(C_n \times C_m) = \min\{n, m\} - \xi_{3,7}(n, m) \quad (2.3)$$

$$\mathrm{sk}(C_n \times C_m) = \min\{n, m\} - \xi_{2,7}(n, m) \quad (2.4)$$

Proof. For all cases except $n = m = 3$, formulas (2.3) and (2.4) follow from the inequality $sp(G) \leq sk(G)$ (lemma 2.4) and from the fact that the lower bound for sp (lemma 2.16) equals the upper bound for sk (lemma 2.19).

For the case $n = m = 3$, the formulas are shown valid by lemmas 2.5 and 2.17. □

Capítulo 3

The Vertex Deletion Number of $C_n \times C_m$

Prólogo

Neste capítulo apresentamos o artigo submetido ao *Latin'2000* no qual determinamos o *vertex deletion number* dos grafos $C_n \times C_m$. A demonstração completa deste resultado é apresentada no Apêndice A.

The Vertex Deletion Number of $C_n \times C_m$ [1]

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Abstract: The vertex deletion number of a graph G is the smallest integer $k \geq 0$ such that there is a planar induced subgraph of G obtained by the removal of k vertices of G. The toroidal grid graphs $C_n \times C_m$ have distinguished place in Computer Science. Several authors have devoted articles to proving the minimum number of crossings in optimum drawings and other planarity invariants such as skewness and splitting number. In this work we give a proof that the vertex deletion number of $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$.

3.1 Introduction

Graph Drawing applications for visualization or VLSI projects require layout techniques of nonplanar graphs. However, the wealth of layout algorithms are limited to a special class of graphs, particularly to planar graphs. These algorithms are useless for nonplanar graphs. One possible approach to handle nonplanarity in graph drawing algorithms is to consider topological invariants of the graph which are used as measures of nonplanarity. The vertex deletion number, defined below, is a natural measure of the non-planarity of a graph G. Research on topological properties of the $C_n \times C_m$ graphs is important for applications such as parallel processing.

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[1]Partially supported by CAPES, CNPq, FAPERJ, FAPESP and Araucária Foundation.
[2]Research done while author was working at Instituto de Computação, UNICAMP, SP, Brazil.

In this paper, we determine exact values for the vertex deletion number of the graphs $C_n \times C_m$, where C_n is the chordless cycle on n vertices. These graphs can be drawn as regular latitude-longitude grids on the torus, and thus are also known as 'toroidal rectangular grids' or similar names. They occur often as interconnection diagrams of multiprocessor computers and cellular automata [22, 25], and so our results are relevant to the physical design of such machines. In this article we prove that the vertex deletion of $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$.

A *simple drawing* of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. A graph is *planar* when there is a simple drawing for this graph in the plane such that no edges cross. In what follows, all drawings are assumed to be simple. We denote by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph between m vertices and n vertices. In our proofs, we rely heavily on the following characterization by Kuratowski [28]: a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph. In fact, we always prove that a graph is not planar by showing that it contains a subdivision of $K_{3,3}$, shown in Figure 3.1.

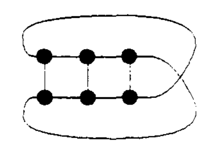

Figura 3.1: $K_{3,3}$

A drawing of a graph G is optimum when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $cr(G)$.

The *skewness* $sk(G)$ is the smallest integer $k \geq 0$ such that the removal of k edges from G yields a planar graph.

The *vertex deletion number* $vd(G)$ is the smallest integer $k \geq 0$ such that the removal of k vertices from G yields a planar graph.

The *splitting number* $sp(G)$ of a graph is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A *vertex splitting operation*, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 e P_2 and adds to $G \backslash v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a sequence of k splittings, we say that H is the *resulting graph* of this set of k splittings in G.

Some aspects of the study of splitting numbers have been considered by Eades and Mendonça [11, 10]: they successfully used splitting numbers in layout algorithm design.

Very little is known about vertex deletion number, splitting numbers, skewness or crossing numbers for specific classes of graphs. The corresponding decision problems for general graphs are all NP-complete [18, 15, 19]. However, it can be checked in polynomial time whether the skewness *sk* or the crossing number *cr* is equal to a fixed k [18]. We have shown [17] that the same holds for the splitting number *sp*, by the results of Robertson and Seymour [38]. The difficulty of finding the values of these invariants can justify entire articles in which just one type of graph is considered. For instance, the crossing numbers for the graphs $C_3 \times C_3$, $C_4 \times C_4$, $C_6 \times C_6$ and $C_7 \times C_7$ were recently established [23, 9, 1, 2], the splitting number for the graph Q_4 was established in [16]. The knowledge of the smallest nonplanar element in a class of graphs can help to find the values or bounds for this invariant for every element in the class. For instance, the crossing number of the $C_3 \times C_n$ was established in [37] using the crossing number of the $C_3 \times C_3$. Also the splitting number of the Q_4, which is isomorphic to $C_4 \times C_4$, was used in [33] to determine the lowerbound for the graphs $C_n \times C_m$ for $n, m \geq 4$.

The splitting number has been computed for complete graphs [24], for complete bipartite graphs [26] and for the $C_n \times C_m$ graphs [33]. The skewness has been computed for the n-cube graphs Q_n [8] and for the $C_n \times C_m$ graphs [33]. The crossing number has been computed for $C_n \times C_m$ graphs [39]. Bounds for the crossing number have been computed for complete graphs [21] for the complete bipartite graphs [6] and for n-cubes [12, 31, 41].

Note that the vertex deletion number is trivial for the complete graphs K_n (which is $n-4$, if $n > 4$) and for the complete bipartite graphs $K_{n,m}$ (which is $\min\{n, m\} - 2$, if $\min\{n, m\} > 2$). However, we show in this work that for the $C_n \times C_m$ graphs this number is not trivial, and except for a few values of n and m it is the same as the vertex splitting number and skewness [33].

Research on vertex deletion number can be also justified by the interdependency of the crossing number, skewness, splitting number and vertex deletion number. The following three lemmas show that for every graph G, $cr(G) \geq sk(G) \geq sp(G) \geq vd(G)$.

Lemma 3.1 *For every graph G,* $\mathrm{cr}(G) \geq \mathrm{sk}(G)$.

Proof. Consider an optimum drawing of a graph G with $cr(G)$ crossings, now for each pair of crossing edges remove one of the edges. The removal of this set of edges of size at most $cr(G)$ produces a planar graph from G which implies that $cr(G) \geq sk(G)$. □

Lemma 3.2 *For every graph G,* $\mathrm{sk}(G) \geq \mathrm{sp}(G)$.

Proof. Let H be a subgraph of G obtained by the removal of $k = sk(G)$ edges of G. For each edge $e_i = u_i v_i$ $(i = 1, 2, ..., k)$ removed from G to build H, build a splitting operation in u_i such that the new vertices u_i' and u_i'' have neighborhood $N(u_i') = N(u_i) \setminus \{v_i\}$ and $N(u_i'') = \{v_i\}$. □

Lemma 3.3 *For every graph G, $\mathrm{sp}(G) \geq \mathrm{vd}(G)$.*

Proof. Delete vertices instead of splitting them. □

For $n \geq 3$, we denote by C_n the chordless cycle with n vertices and n edges. The $n \times m$ *toroidal grid* $C_n \times C_m$ is the graph-theoretic product of C_n and C_m; that is, the graph with nm vertices $\{v_{ij} : 0 \leq i < n,\ 0 \leq j < m\}$, and $2nm$ edges $\{v_{ij}v_{(i+1) \bmod n, j}, v_{ij}v_{i,(j+1) \bmod m} : 0 \leq i < n,\ 0 \leq j < m\}$.

Two graphs G and H are *isomorphic* if there is a bijection $\psi : VG \to VH$ such that two distinct vertices x and y are adjacent in G if and only if the vertices $\psi(x)$ and $\psi(y)$ are adjacent in H. Such a function is called an *isomorphism* from G to H. It is obvious that $C_n \times C_m$ is isomorphic to $C_m \times C_n$.

An *automorphism* of a graph G is an isomorphism between G and itself. We observe that $C_n \times C_m$ has $4nm$ automorphisms if $n \neq m$, and $8nm$ if $n = m$.

Let $\mathcal{F}$ be a family of isomorphic subgraphs of a graph G. We say that G is *$\mathcal{F}$-transitive* if for any two elements F and H of $\mathcal{F}$ there is an automorphism of G that takes F to H. Note that the graph $C_n \times C_m$ is vertex-transitive. Therefore, a particular vertex may be chosen without loss of generality.

We define the *i, j-small-values-detection function* $\xi_{i,j} : N^2 \to \{0, 1, 2\}$ so that $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following:

(i) $k_1 = k_2 \leq i$, and

(ii) $k_1 + k_2 \leq j$.

Our strategy in this work is as follows. In section 3.2 we show that the upperbound of the vertex deletion number of $C_n \times C_m$ is at most $\min\{n, m\} - \xi_{5,9}(n, m)$. In section 3.3 we show that the lowerbound of the vertex deletion number of $C_n \times C_m$ is at least $\min\{n, m\} - \xi_{5,9}(n, m)$.

3.2 Upperbounds for $vd(C_n \times C_m)$

Theorem 3.4 *The vertex deletion number of the $C_n \times C_m$ graphs is at most $\min\{n, m\} - \xi_{5,9}(n, m)$.*

Proof. Figure 3.2(a) displays the $C_3 \times C_3$ graph and a planar drawing of the induced graph obtained by the removal of one vertex indicated by ✖. Dotted lines indicate isomorphism. Thus, $vd(C_3 \times C_3) \leq 1$. Figure 3.2(b), (c), (d) and (e) display the graphs $C_3 \times C_4$, $C_3 \times C_5$, $C_3 \times C_6$ and $C_4 \times C_4$, respectively, and planar drawings of the induced subgraphs after the removal of two vertices. Thus, $vd(C_3 \times C_4) \leq 2$, $vd(C_3 \times C_5) \leq 2$, $vd(C_3 \times C_6) \leq 2$, $vd(C_4 \times C_4) \leq 2$. Figure 3.2(f) displays the $C_4 \times C_5$ and a planar drawing after the removal of three vertices. Thus, $vd(C_4 \times C_5) \leq 3$. Figure 3.2(g) displays the $C_5 \times C_5$ and a planar drawing of the induced subgraph after the removal of four vertices. Thus, $vd(C_5 \times C_5) \leq 4$. Finally, Figure 3.2(h) displays the $C_n \times C_m$ graph and a planar drawing of the induced subgraph after the removal of $\min\{n, m\}$ vertices. Thus, $vd(C_n \times C_m) \leq \min\{n, m\}$. All these results can be summarized by the inequality $vd(C_n \times C_m) \leq \min\{n, m\} - \xi_{5,9}(n, m)$. □

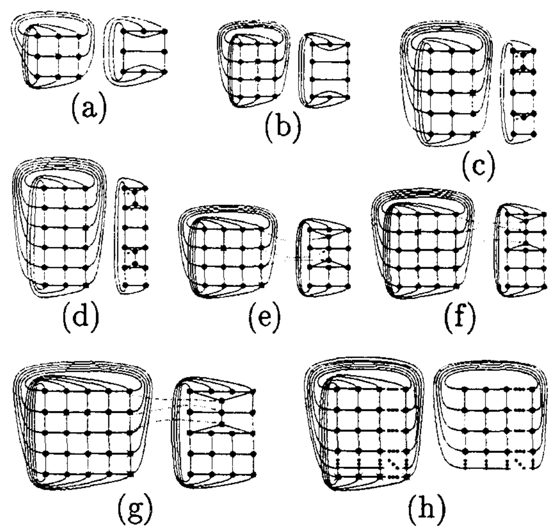

Figura 3.2: $vd(C_n \times C_m) \leq \min\{n, m\} - \xi_{5,9}(n, m)$

3.3 Lowerbounds for $vd(C_n \times C_m)$

To prove our claim we must show that some induced subgraphs of the $C_n \times C_m$ contain a subdivision of the $K_{3,3}$. These graphs have in some cases a surprisingly enormous number of analogous cases. Therefore, we will represent such subdivisions of the $K_{3,3}$ and induced subgraphs as follows:

- the vertices $v_{i,j}$ are represented by the integer points $\{(i, j), 0 \leq i < n, 0 \leq j < m\}$;
- deleted vertices are drawn with the symbol $\times$;
- edges used by the subdivision of the $K_{3,3}$ are drawn solid and vertices of the $K_{3,3}$ are drawn as large circles with degree 3;
- each horizontal half-edge drawn along the left side of the grid connects to the half-edge at the right side, on the same row; and analogously for vertical half-edges;
- all other vertices are drawn as small dots, and all other edges are omitted.

(a) (b)

Figura 3.3: (a) $vd(C_3 \times C_3) \geq 1$. (b) $vd(C_3 \times C_4) \geq 2$

To reduce the amount of work we wrote two simple combinatorics programs: **Find-Analogous** and **Find-$K_{3,3}$**. The former generates all the non-analogous subgraphs of $C_n \times C_m$ that result from the deletion of k vertices, for given n, m and k. We say that two subgraphs of $C_n \times C_m$ are *analogous* if they are isomorphic by an automorphism of $C_n \times C_m$. Note that two non-analogous subgraphs may be isomorphic. Due to the automorphisms of $C_n \times C_m$ we need to generate only subgraphs with the top left corner vertex deleted. This reduces the number of subgraphs that need to be considered from $\binom{nm}{k}$ to $\binom{nm-1}{k-1}$. Furthermore when deciding whether a subgraph is analogous to a previously generated one, we need to consider only $4k$ or $8k$ automorphisms of $C_n \times C_m$, instead of $4nm$ or $8nm$.

The second program **Find-$K_{3,3}$** checks whether each subgraph of $C_n \times C_m$ generated by **Find-Analogous** contains a subdivision of $K_{3,3}$. The subdivisions of the $K_{3,3}$ are added to a list by the user. If **Find-$K_{3,3}$** fails to find a subdivision of $K_{3,3}$ it stops printing the subgraph. If **Find-$K_{3,3}$** finds a subdivision of $K_{3,3}$ for each subgraph of $C_n \times C_m$ generated by **Find-Analogous** it prints all solutions.

Lemma 3.5 *The vertex deletion number of $C_3 \times C_3$ is at least* 1.

Proof. The graph $C_3 \times C_3$ contains a subdivision of $K_{3,3}$ as shown in Figure 3.3(a). Therefore, it is not planar which implies that $vd(C_3 \times C_3) \geq 1$. □

Lemma 3.6 *The vertex deletion number of $C_3 \times C_4$ is at least* 2.

Proof. Let G be the subgraph induced by all vertices of the $C_3 \times C_4$ minus one vertex. Without loss of generality we suppose that the deleted vertex is at the top left corner as shown in Figure 3.3(b). This graph contains a subdivision of $K_{3,3}$. Therefore, it is not planar which implies that $vd(C_3 \times C_4) \geq 2$. □

Corollary 3.7 *The vertex deletion number of $C_3 \times C_5$, $C_3 \times C_6$ and $C_4 \times C_4$ are at least* 2.

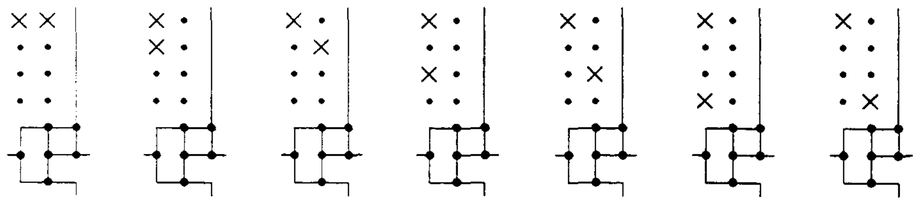

Figura 3.4: $vd(C_3 \times C_7) \geq 3$

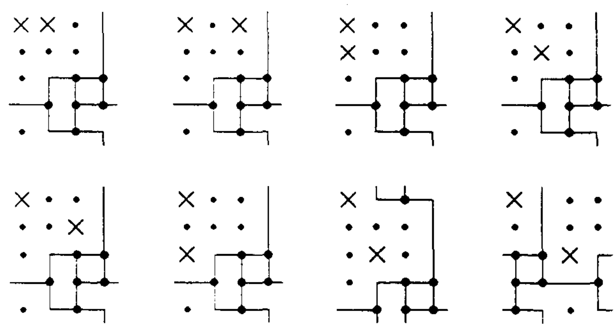

Figura 3.5: $vd(C_4 \times C_5) \geq 3$

Proof. All of these graphs contain a subdivision of $C_3 \times C_4$. □

Lemma 3.8 *The vertex deletion number of* $C_3 \times C_7$ *is at least* 3.

Proof. Figure 3.4 displays the 7 non-analogous possible ways to delete two vertices from $C_3 \times C_7$ generating different induced subgraphs. Figure 3.4 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_3 \times C_7) \geq 3$. □

Corollary 3.9 *The vertex deletion number of* $C_3 \times C_m$*, for* $m \geq 7$*, is at least* 3.

Proof. The graph $C_3 \times C_m$ contains a subdivision of $C_3 \times C_7$ which has vertex deletion number at least 3. □

Lemma 3.10 *The vertex deletion number of* $C_4 \times C_5$ *is at least* 3.

Proof. Figure 3.5 displays the 8 non-analogous subgraphs obtained by deleting two vertices from $C_4 \times C_5$. Figure 3.5 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar, which implies that $vd(C_4 \times C_5) \geq 3$. □

Lemma 3.11 *The vertex deletion number of* $C_4 \times C_6$ *is at least* 4.

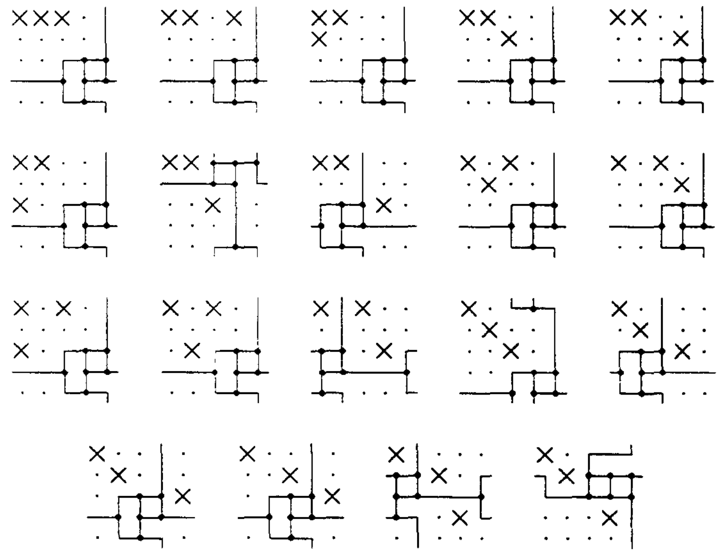

Figura 3.6: $vd(C_5 \times C_5) \geq 4$

Proof. There are 34 non-analogous ways to delete 3 vertices from $C_4 \times C_6$. We omitted this figure to conserve space. In [34] we show that there is a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_4 \times C_6) \geq 4$. □

Corollary 3.12 *The vertex deletion number of* $C_4 \times C_m$*, for* $m \geq 6$*, is at least* 4.

Proof. The graph $C_4 \times C_m$ contains a subdivision of $C_4 \times C_6$ which has vertex deletion number at least 4. □

Lemma 3.13 *The vertex deletion number of* $C_5 \times C_5$ *is at least* 4.

Proof. Figure 3.6 displays the 19 non-analogous ways to delete three vertices from $C_5 \times C_5$. Figure 3.6 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_5 \times C_5) \geq 4$. □

Lemma 3.14 *The vertex deletion number of* $C_5 \times C_6$ *is at least* 5.

Proof. There are 291 non-analogous ways to delete four vertices from $C_5 \times C_6$. We omitted this figure to conserve space. In [34] we show that there is a subdivision $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_5 \times C_6) \geq 5$. □

Corollary 3.15 *The vertex deletion number of* $C_5 \times C_m$*, for* $m \geq 6$*, is at least* 5.

Proof. The graph $C_5 \times C_m$ contains a subdivision of $C_5 \times C_6$ which has vertex deletion number at least 5. □

Lemma 3.16 *The vertex deletion number of* $C_6 \times C_6$ *is at least* 6.

Proof. There are 1455 non-analogous ways to delete five vertices from $C_6 \times C_6$. We omitted this figure to conserve space. In [34] we show that there are a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_6 \times C_6) \geq 6$. □

Lemma 3.17 *The vertex deletion number of* $C_k \times C_k$*, for* $k \geq 6$*, is at least* k.

Proof. We prove this assertion by induction in k. The induction basis is the graph $C_6 \times C_6$. The induction hypothesis is that for all graphs $C_l \times C_l$ the vertex deletion number is at least l, where $6 \leq l < k$. Without loss of generality we may suppose that the vertex s at the top left corner is deleted. The remaining graph has a subdivision of $C_{k-1} \times C_{k-1}$. It follows from the induction hypothesis that the $C_k \times C_k \backslash s$ has vertex deletion number at least $k-1$ and therefore, the graph $C_k \times C_k$ has vertex deletion number at least k. □

Corollary 3.18 *The vertex deletion number of* $C_n \times C_m$*, for* $n, m \geq 6$*, is at least* $\min\{n, m\}$.

Proof. The graph $C_n \times C_m$ contains a subdivision of $C_n \times C_n$ which has vertex deletion number at least n. □

Now our Theorem 3.19 follows from Lemma 3.5, Lemma 3.6, Corollary 3.7, Lemma 3.8, Corollary 3.9, Lemma 3.10, Lemma 3.11, Corollary 3.12, Lemma 3.13, Lemma 3.14, Corollary 3.15, Lemma 3.16, Lemma 3.17 and Corollary 3.18.

Theorem 3.19 *The vertex deletion number of* $C_n \times C_m$ *is at least* $\min\{n, m\} - \xi_{5,9}(n, m)$.

Theorem 3.20 *The vertex deletion number of* $C_n \times C_m$ *is* $\min\{n, m\} - \xi_{5,9}(n, m)$.

Proof. The assertion follows from Theorem 3.4 and Theorem 3.19. □

As is well-known, a map defines implicitly a topological embedding of the graph G_M in some 2D manifold. The orbits of φ_M are the faces of the embedding.

A (*two-dimensional*) *toroidal mesh* is a map M whose underlying manifold is a torus, with two authomorphisms τ and ς such that (1) $\varsigma\tau = \tau\varsigma$, (2) $u\tau$ and $u\varsigma$ are neighbors of u, for any u, and (3) the set of vertices $\{u\tau^m\varsigma^n : m, n \in Z\}$ covers the whole graph G.

Because of their symmetry and regularity, toroidal meshes are popular topologies for the connection networks of SIMD parallel machines. The automorphisms τ and ς represent the basic "parallel data shifting" operations whereby each node passes some datum to a specific neighbor in the network. One important information are the topological invariants such as vertex deletion number, splitting number, skewness and crossing number as a measure of nonplanarity of a graph G_M. There are several applications [29] which make use of this information such as Graph Drawing applications and VLSI design.

One of the most popular of the regular toroidal meshes is the $C_n \times C_m$ graphs for which entire articles were dedicated to proving the minimum number of crossings in optimum drawings [23, 37, 9, 1, 2, 39], and other planarity invariants such as skewness and splitting number [33, 16, 34, 40]. In this work we give a proof that the vertex deletion number and the splitting number of $\mathcal{T}_{C_n \times C_m}$ is $\min\{n, m\}$. The graph $\mathcal{T}_{C_n \times C_m}$ consists of a regular triangulation of the torus formed by adding the edges $v_{i,j}v_{(i+1) \bmod n,(j+1) \bmod m}$ to each vertex of $C_n \times C_m$.

A *simple drawing* of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. A graph is *planar* when there is a simple drawing for this graph in the plane such that no edges cross. In what follows, all drawings are assumed to be simple.

In our proofs we depend heavily on the following characterization by Kuratowski[28]: a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see Figure 4.1) as a subgraph.

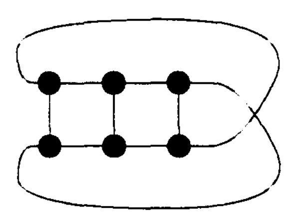

Figura 4.1: $K_{3,3}$

A drawing of a graph G is optimum when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $cr(G)$.

The *skewness* $sk(G)$ is the smallest integer $k \geq 0$ such that the removal of k edges from G yields a planar graph.

The *vertex deletion number* $vd(G)$ is the smallest integer $k \geq 0$ such that the removal of k vertices from G yields a planar graph.

The *splitting number* of a graph G, $sp(G)$, is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A ***vertex splitting operation***, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 e P_2 and adds to $G \backslash v$ two new and nonadjacent vertices v_1

and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a sequence of k splittings, we say that H is the *resulting graph* of this set of k splittings in G.

Some aspects of the study of splitting number have been considered by Eades and Mendonça [11, 10]: they successfully used splitting numbers in layout algorithm design.

Very little is known about vertex deletion number, splitting numbers, skewness or crossing numbers for specific classes of graphs. The corresponding decision problems are all NP-complete [18, 15, 19]. For a fixed k, *crossing number* turns to be polynomial [18], recently Robertson and Seymour [38] have shown *vertex deletion number*, *splitting number* and *skewness* also turn to be polynomial. The difficulty of finding the values of these invariants can justify entire articles in which just one type of graph is considered. For instance, the crossing numbers for the graphs $C_3 \times C_3$, $C_4 \times C_4$, $C_6 \times C_6$ and $C_7 \times C_7$ were recently established [23, 9, 1, 2], the splitting number for the graph Q_4 was established in [16]. The knowledge of the smallest nonplanar element in a class of graphs can help to find the values or bounds for this invariant for every element in the class. For instance, the crossing number of the $C_3 \times C_n$ was established in [37] using the crossing number of the $C_3 \times C_3$. Also the splitting number of the Q_4 which is isomorphic to $C_4 \times C_4$ was used in [33] to determine the lowerbound for the graphs $C_n \times C_m$ where $n, m \geq 4$.

The vertex deletion number has been computed for $C_n \times C_m$. This number is (except for a few values of n and m) the same as the vertex splitting number and skewness [34]. The splitting number has been computed for complete graphs [24], for complete bipartite graphs [26] and for $C_n \times C_m$ graphs [33]. The skewness has been computed for Q_n cubes [8] and for $C_n \times C_m$ graphs [33]. The crossing number has been computed for $C_n \times C_m$ graphs [39]. Bounds for the crossing number have been computed for complete graphs [21], for the complete bipartite graphs [6] and for n-cubes [12, 31, 41].

Note that the vertex deletion number is trivial for the complete graphs K_n (which is $n-4$ if $n > 4$) and for the complete bipartite graphs $K_{n,m}$ (which is $min\{n, m\} - 2$ if $min\{n, m\} > 2$).

Research on vertex deletion number can be also justified by the interdependency of the crossing number, skewness, splitting number and vertex deletion number. The following three Lemmas show that for any graph G, $cr(G) \geq sk(G) \geq sp(G) \geq vd(G)$.

Lemma 4.1 *For all graph G, $cr(G) \geq sk(G)$,*

Proof. Consider an optimum drawing of a graph G with $cr(G)$ crossings, now for each pair of edges that cross remove one of the edges. The removal of this set of edges of size at most $cr(G)$ produces a planar graph from G which implies that $cr(G) \geq sk(G)$. □

Lemma 4.2 *For all graph G, $sk(G) \geq sp(G)$.*

Proof. Let H be a planar subgraph of G obtained by the removal of $k = sk(G)$ edges of G. For each edge $e_i = u_i v_i$ $(i = 1, 2, ..., k)$ removed from G to build H, build a splitting operation in u_i such that the new vertices u_i' and u_i'' have neighborhood $N(u_i') = N(u_i) \setminus \{v_i\}$ and $N(u_i'') = \{v_i\}$. □

Lemma 4.3 *For all graph G, $sp(G) \geq vd(G)$.*

Proof. Delete vertices instead of splitting them. □

A *chordless circuit* or simply *circuit* C_k, $k \geq 3$ of a graph G is a set of vertices $C_k = \{v_0, v_1, ..., v_{k-1}\}$ where each vertex v_i has exactly two neighbors $v_{(i-1) \bmod k}$ and $v_{(i+1) \bmod k}$ in C_k. We say that a circuit C is a *k-circuit* if it is a circuit of k vertices.

Let q and r be the maximum common divisor and minimum common multiple of n and m, respectively. A *triangulation of $C_n \times C_m$*, denoted by $\mathcal{T}_{C_n \times C_m}$, is a graph with nm vertices where each vertex $v_{i,j}$ $(i = 0, 1, ..., n-1$ and $j = 0, 1, ..., m-1)$ has exactly six neighbors $v_{(i-1) \bmod n, j}$, $v_{(i+1) \bmod n, j}$, $v_{i,(j-1) \bmod m}$, $v_{i,(j+1) \bmod m}$ $v_{(i-1) \bmod n,(j-1) \bmod m}$ and $v_{(i+1) \bmod n,(j+1) \bmod m}$. Let a *row n-circuit* be the m n-circuits $R_n^j = \{v_{0,j}, v_{1,j}, ..., v_{n-1,j}\}$ (for $j = 0, 1, ..., m-1$), a *column m-circuit* be the n m-circuits $C_m^i = \{v_{i,0}, v_{i,1}, ..., v_{i,m-1}\}$ (for $i = 0, 1, ..., n-1$), and a *diagonal r-circuit* be the q r-circuits $C_r^k = \{v_{k,0}, v_{(k+1) \bmod n, 1}, ..., v_{(k+r-1) \bmod n,(r-1) \bmod m}\}$ (for $k = 0, 1, ..., q-1$). Note that this triangulation does not cover all regular triangulation of the torus.

Two graphs G and H are *isomorphic* if there is a bijection $\psi : VG \to VH$ such that two distinct vertices x and y of G are adjacent if and only if the vertices $\psi(x)$ and $\psi(y)$ are adjacent in H. Such a function is called an *isomorphism* from G to H. It is obvious that $\mathcal{T}_{C_n \times C_m}$ is isomorphic to $\mathcal{T}_{C_m \times C_n}$.

An *automorphism* of a graph G is an isomorphism between G and itself. We observe that $C_n \times C_m$ has $4nm$ automorphisms if $n \neq m$, and $8nm$ if $n = m$.

Given a graph G and a subgraph S of G, we say that G is *S-transitive* if for each pair F, H subgraphs of G, where F and H are isomorphic to S, there is an automorphism α of G such that if $v \in V(F)$, then $\alpha(v) \in V(H)$.

It is an easy exercise to show that the graph $\mathcal{T}_{C_n \times C_m}$ is vertex-transitive. Therefore, a particular vertex may be chosen without loss of generality.

Our strategy in this work is as follows. In section 4.2 we show that the upperbound of the splitting number of $\mathcal{T}_{C_n \times C_m}$ is at most $\min\{n, m\}$. In section 4.3 we show that the lowerbound of the vertex deletion number of $\mathcal{T}_{C_n \times C_m}$ is at least $\min\{n, m\}$.

4.2 Upperbounds for $sp(\mathcal{T}_{C_n \times C_m})$

Lemma 4.4 *The splitting number of $\mathcal{T}_{C_n \times C_m}$ is at most* $\min\{n, m\}$.

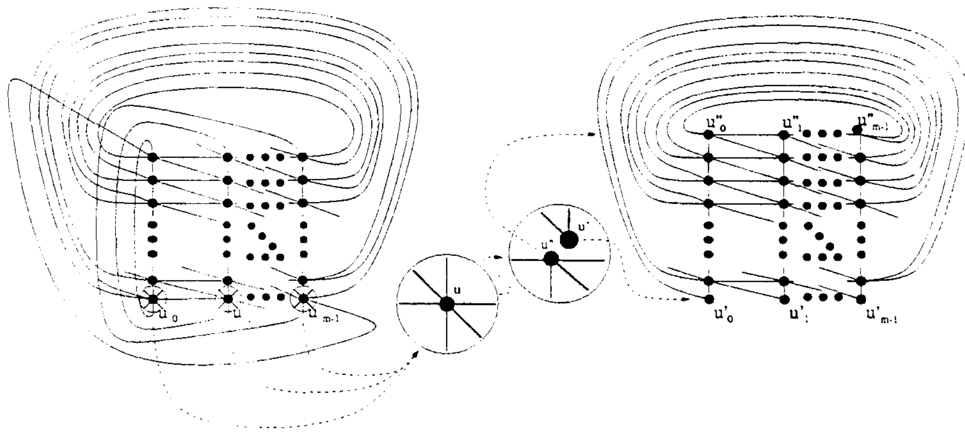

Figura 4.2: $sp(\mathcal{T}_{C_n\times C_m}) \leq \min\{n, m\}$

Proof. Without loss of generality we may suppose that $m \leq n$. Figure 4.2 displays a planar drawing of the graph obtained after $\min\{n, m\} = m$ splitting operations of the $\mathcal{T}_{C_n\times C_m}$. Therefore, $sp(\mathcal{T}_{C_n\times C_m}) \leq \min\{n, m\} = m$. □

4.3 Lowerbounds for $vd(\mathcal{T}_{C_n\times C_m})$

Lemma 4.5 *The vertex deletion number of $\mathcal{T}_{C_2\times C_3}$ is at least* 2.

Proof. The graph $\mathcal{T}_{C_2\times C_3}$ contains a subgraph isomorphic to K_6. Therefore, $vd(\mathcal{T}_{C_2\times C_3}) \geq 2$. □

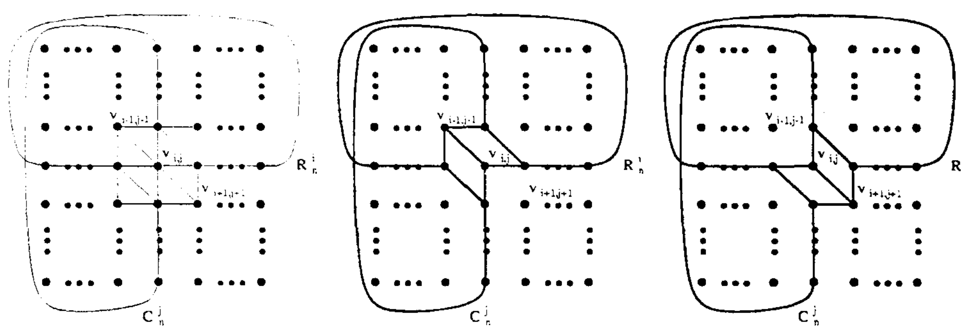

Figura 4.3: If $v_{(i-1) \bmod n,(j-1) \bmod n}$ or $v_{(i+1) \bmod n,(j+1) \bmod n}$ do not belong to D then H contains a subdivision of $K_{3,3}$

Lemma 4.6 *The vertex deletion number of $\mathcal{T}_{C_n\times C_n}$ is at least n.*

Proof. Let $G = (V, E)$ be the graph $\mathcal{T}_{C_n \times C_n}$ and D be a subset of the vertices of G such that $|D| = k = n - 1 \geq 2$. Let H be the subgraph of G induced by $V \setminus D$. Let s be number of different rows n-circuits that intersects D. We prove the assertion by induction in s.

Since $k < n$, H contains a column n-circuit C_n^j.

Base: there are two cases.

case 1: n is odd and $s < \frac{n}{2}$. In this case by the pigeon hole principle there are at least two consecutive rows n-circuits, lets say R_n^i and $R_n^{(i+1) \bmod n}$. Therefore, $C_n^j \cup R_n^i \cup \{v_{(i+1) \bmod n,(j+1) \bmod n}\} \subset H$ contains a subdivision of $K_{3,3}$ (see Figure 4.3).

case 2: n is even and $s = \frac{n}{2}$. In this case, if at least 2 rows n-circuits are consecutive we have a subdivision of $K_{3,3}$ as in the previous case. Otherwise (there are not 2 consecutives rows n-circuits) there is at least one vertex $w_l = v_{(l-1) \bmod n,(j-1) \bmod n}$ or $w_l = v_{(l+1) \bmod n,(j+1) \bmod n}$ (for each row n-circuit R_n^l) that does not belong to D. Therefore, $C_n^j \cup R_n^i \cup \{w\} \subset H$ contains a subdivision of $K_{3,3}$ as shown in Figure 4.3.

Hippothesis: If $s < k$ then H contains a subdivision of $K_{3,3}$.

Thesis: $s = k$. In this case, h contains at least 1 row n-circuit $R_n^i = \{v_{i,0}, v_{i,1}, ..., v_{i,n-1}\}$ such that $D \cap R_n^i = \emptyset$. If at least one of the vertices $v_{(i-1) \bmod n,(j-1) \bmod n}$ or $v_{(i+1) \bmod n,(j+1) \bmod n}$ does not belong to D then H contains a subdivision of $K_{3,3}$ (see Figure 4.3). Conversely, if both vertices belong to D consider the automorphism φ of H where $\varphi(v_{t,u}) = v'_{t,u} = v_{(t-u+j) \bmod n,(2j-u) \bmod n}$. Note that $\varphi(H)$ keeps the vertex $v_{i,j}$ in the same position. Furthermore, the row n-circuit $R_n'^i$ contains both vertices $v'_{i,(j-1) \bmod n} = v_{(i+1) \bmod n,(j+1) \bmod n}$ and $v'_{i,(j+1) \bmod n} = v_{(i-1) \bmod n,(j-1) \bmod n}$. Therefore, the number of intersections between D and the n rows n-circuits of $\varphi(H)$ is at most $s - 1$ which implies by the induction hippothesis it contains a subdivision of $K_{3,3}$.

The subdivision of $K_{3,3}$ found in H implies that it is not planar. Therefore, $vd(G) \geq n$. □

Corollary 4.7 *The vertex deletion number of* $\mathcal{T}_{C_n \times C_m}$ *is at least* $\min\{n, m\}$.

Proof. Whitout loss of generality suppose that $m \geq n$. Now contract the edges belonging to the comlumn m-circuits between the rows n-circuits R_n^0 and R_n^1 $m-n$ times, if $n, m \geq 3$, otherwise contract only $m - n + 1$ times. Next, remove all multiple edges. The remaining graph is a $\mathcal{T}_{C_n \times C_n}$ when $n, m \geq 3$ and a $\mathcal{T}_{C_2 \times C_3}$, otherwise. It is a well known result that both operations (edges contraction an edge deletion) do not increase the vertex deletion number. Therefore, in this case, it follows from this fact and from Lemma 4.6 and Lemma 4.5 that $vd(\mathcal{T}_{C_n \times C_m}) \geq n = \min\{n, m\}$. □

Theorem 4.8 *The vertex deletion number and splitting number of* $\mathcal{T}_{C_n \times C_m}$ *is* $\min\{n, m\}$.

Proof. The assertion follows from Theorem 4.4 and Corollary 4.7. □

Capítulo 5

Conclusão

Dada a complexidade de se determinar os valores das invariantes de planaridade para classes gerais de grafos, é de grande importância a descoberta destes valores para classes específicas de grafos. Prova disso está na quantidade de trabalhos publicados apresentando resultados deste tipo.

Sendo assim, concentramos nosso trabalho no estudo das invariantes de planaridade de duas classes especiais de grafos: $C_n \times C_m$ e $\mathcal{T}_{C_n \times C_m}$. Para os grafos $C_n \times C_m$, conseguimos determinar o *skewness* e o *vertex deletion number*, além de apresentarmos uma nova demonstração para o *splitting number*. E para os grafos $\mathcal{T}_{C_n \times C_m}$, estabelecemos o *splitting number* e o *vertex deletion number*.

Embora a relação $vd(G) \leq sp(G) \leq sk(G)$ tenha se mostrado "estreita" neste trabalho, isto é, os valores das invariante que determinamos para $C_n \times C_m$ e $\mathcal{T}_{C_n \times C_m}$ foram iguais ou bem próximos, este comportamento não é uma regra e tais invariantes podem estar tão distantes quanto se queira, dependendo da classe de grafos estudada. Por exemplo, a Figura 5.1 mostra um grafo G com $sp(G) = n$ e $vd(G) = 1$, e um outro grafo H com $sk(H) = n$ e $sp(H) = 1$.

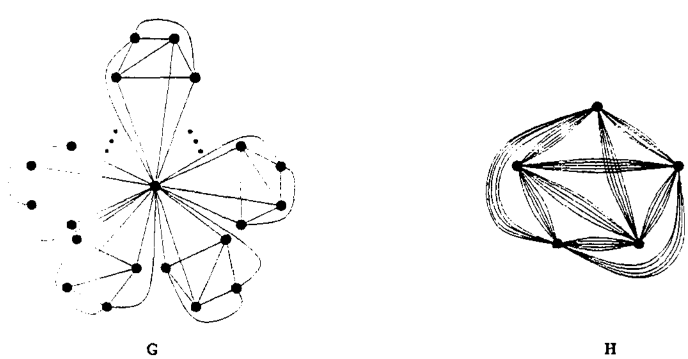

Figura 5.1: Exemplos em que *sk, sp* e *vd* estão bem distantes

Como trabalhos futuros, podemos citar o estudo das invariantes de planaridade de outros *tilings regulares do toro*, isto é, famílias de grafos que subdividem o toro em formas geométricas regulares. Os dois grafos que estudamos fazem parte deste conjunto; o $C_n \times C_m$ é um *tiling retangular ortogonal* e o $\mathcal{T}_{C_n \times C_m}$ é um *tiling triangular*. Contudo, existem vários outros *tilings*.

Veja por exemplo o *tiling hexagonal*. Este grafo pode ser obtido de duas maneiras: tomando-se o dual no toro de uma triangulação $\mathcal{T}_{C_n \times C_m}$, como mostrado na Figura 5.2; ou a partir do $C_n \times C_m$, alterando-se cada vértice como mostrado na Figura 5.3.

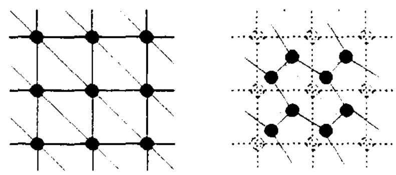

Figura 5.2: Dual de $\mathcal{T}_{C_n \times C_m}$

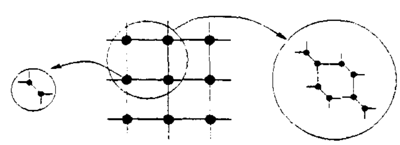

Figura 5.3: Quebrando os vértices do $C_n \times C_m$

Um outro resultado que fica em aberto e que desperta ainda mais interesse após este trabalho é o *skewness* de $\mathcal{T}_{C_n \times C_m}$. A nossa conjetura é que $sk(\mathcal{T}_{C_n \times C_m}) = 2\min\{n, m\}$.

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Apêndice A

The Vertex Deletion Number of $C_n \times C_m$

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Abstract: The vertex deletion number of a graph G is the smallest integer $k \geq 0$ such that there is an planar induced subgraph of G obtained by the removal of k vertices of G. The $C_n \times C_m$ graphs has distinguished place in Computer Science. Several authors have devoted articles to proving the minimum number of crossings in optimum drawings [23, 37, 9, 1, 2, 39], and other planarity invariants such as skewness and splitting number [33, 16, 40]. In this work we give a proof that the vertex deletion number of the $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$.

[1]Research done while author was working at UNICAMP, Brazil.

Keywords: vertex deletion number, vertex splitting number, skewness, planarity invariants.

A.1 Introduction

Graph Drawing applications for visualization or VLSI projects require layout techniques of nonplanar graphs. However, the wealth of layout algorithms are limited to a special class of graphs, particularly to planar graphs. These algorithms are useless for nonplanar graphs. One possible approache to handling nonplanarity in graph drawing algorithms is to consider topological invariants of the graph such as the vertex deletion number which are used as measure of nonplanarity. Research on topological properties of the $C_n \times C_m$ graphs is important for applications such as parallel processing. In this article we prove that the vertex deletion of $C_n \times C_m$ is $\min\{n, m\} - \xi_{5,9}(n, m)$, where $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following: (i) $k_1 = k_2 \leq i$ and (ii) $k_1 + k_2 \leq j$ (see Figure A.2).

A *simple drawing* of a graph G is a drawing of G on the plane such that no edge crosses itself, adjacent edges do not cross, crossing edges do so only once, edges do not cross vertices, and no more than two edges cross at a common point. A graph is *planar* when there is a simple drawing for this graph in the plane such that no edges cross. In what follows, all drawings are assumed to be simple. In our proofs we depend heavily on the following characterization by Kuratowski[28]: a graph is planar if and only if it does not contains a subdivision of K_5 or $K_{3,3}$ as a subgraph. In fact we only use the nonplanarity of the subdivision of $K_{3,3}$ (see Figure A.1).

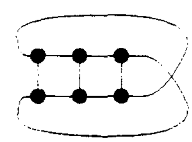


Figura A.1: $K_{3,3}$

A drawing of a graph G is optimum when it has the minimum number of crossings among all drawings of G. This number is called the *crossing number* of G and is denoted by $cr(G)$.

The *skewness* $sk(G)$ is the smallest integer $k \geq 0$ such that the removal of k edges from G yields a planar graph.

The *vertex deletion number* $vd(G)$ is the smallest integer $k \geq 0$ such that the removal of k vertices from G yields a planar graph.

The *splitting number* $sp(G)$ of a graph is the smallest integer $k \geq 0$ such that a planar graph can be obtained from G by k vertex splitting operations. A ***vertex splitting operation***, or simply *splitting*, of a vertex $v \in V(G)$ partitions the set of neighbors of v into two nonempty sets P_1 e P_2 and adds to $G \backslash v$ two new and nonadjacent vertices v_1 and v_2, such that P_1 is the set of neighbors of v_1 and P_2 is the set of neighbors of v_2. If a graph H is obtained from G by a sequence of k splittings, we say that H is the ***resulting graph*** of this set of k splittings in G.

Some aspects of the study of splitting number have been considered by Eades and Mendonça [11, 10]: they successfully used splitting numbers in layout algorithm design.

Very little is known about vertex deletion number, splitting numbers, skewness or crossing numbers for specific classes of graphs. The corresponding decision problems for general graphs are all NP-complete [18, 15, 19]. For a fixed k, CROSSING NUMBER turns to be polynomial [18], recently Robertson and Seymour [38] have shown VERTEX DELETION NUMBER, SPLITTING NUMBER and SKEWNESS also turn to be polynomial. The difficulty of finding the values of these invariants can justify entire articles in which just one type of graph is considered. For instance, the crossing numbers for the graphs $C_3 \times C_3$, $C_4 \times C_4$, $C_6 \times C_6$ and $C_7 \times C_7$ were recently established [23, 9, 1, 2], the splitting number for the graph Q_4 was established in [16]. The knowledge of the smallest nonplanar element in a class of graphs can help to find the values or bounds for this invariant for every element in the class. For instance, the crossing number of the $C_3 \times C_n$ was established in [37] using the crossing number of the $C_3 \times C_3$. Also the splitting number of the Q_4 which is isomorphic to $C_4 \times C_4$ was used in [33] to determine the lowerbound for the graphs $C_n \times C_m$ where $n, m \geq 4$.

The splitting number has been computed for complete graphs [24], for complete bipartite graphs [26] and for the $C_n \times C_m$ graphs [33]. The skewness has been computed for the n-cube graphs Q_n [8] and for the $C_n \times C_m$ graphs [33]. The crossing number has been computed for $C_n \times C_m$ graphs [39]. Bound for the crossing number have been computed for complete graphs [21] for the complete bipartite graphs [6] and for n-cubes [12, 31, 41].

Note that the vertex deletion number is trivial for the complete graphs K_n (which is $n - 4$ if $n > 4$) and for the complete bipartite graphs $K_{n,m}$ (which is $\min\{n, m\} - 2$ if $\min\{n, m\} > 2$). However, we show in this work that for the $C_n \times C_m$ this number is not trivial, and except for a few values of n and m it is the same as the vertex splitting number and skewness [33].

Research on vertex deletion number can be also justified by the interdependency of the crossing number, skewness, splitting number and vertex deletion number. The following three Lemmas show that for any graph G, $cr(G) \geq sk(G) \geq sp(G) \geq vd(G)$.

Lemma A.1 *For all graph G,* $\mathrm{cr}(G) \geq \mathrm{sk}(G)$,

Proof. Consider an optimum drawing of a graph G with $cr(G)$ crossings, now for each pair of edges that cross remove one of the edges. The removal of this set of edges of size at most $cr(G)$ produces a planar graph from G which implies that $cr(G) \geq sk(G)$. □

Lemma A.2 *For all graph G,* $\mathrm{sk}(G) \geq \mathrm{sp}(G)$.

Proof. Let H be a subgraph of G obtained by the removal of $k = sk(G)$ edges of G. For each edge $e_i = u_i v_i$ $(i = 1, 2, ..., k)$ removed from G to build H, build a splitting operation in u_i such that the new vertices u_i' and u_i'' have neighborhood $N(u_i') = N(u_i) \setminus \{v_i\}$ and $N(u_i'') = \{v_i\}$. □

Lemma A.3 *For all graph G,* $\mathrm{sp}(G) \geq \mathrm{vd}(G)$.

Proof. Delete vertices instead of splitting them. □

A *chordless circuit* or simply *circuit* C_k, $k \geq 3$ of a graph G is a set of vertices $C_k = \{v_0, v_1, ..., v_{k-1}\}$ where each vertex v_i has exactly two neighbors $v_{(i-1) \bmod k}$ and $v_{(i+1) \bmod k}$ in C_k.

A $C_n \times C_m$ *graph* is a graph with nm vertices where each vertex $v_{i,j}$ $(i = 0, 1, ..., n-1$ and $j = 0, 1, ..., m-1)$ has exactly four neighbors $v_{(i-1) \bmod n, j}$, $v_{(i+1) \bmod n, j}$, $v_{i,(j-1) \bmod m}$ and $v_{i,(j+1) \bmod m}$. It is an easy exercise to show that $C_n^j = \{v_{0,j}, v_{1,j}, ..., v_{n-1,j}\}$ is a circuit C_n in $C_n \times C_m$ for $j = 0, 1, ..., m-1$ and that $C_m^i = \{v_{i,0}, v_{i,1}, ..., v_{i,m-1}\}$ is a circuit C_m in $C_n \times C_m$ for $i = 0, 1, ..., n-1$.

Two graphs G and H are *isomorphic* if there is a bijection $\psi : VG \to VH$ such that two distinct vertices x and y of G are adjacent if and only if the vertices $\psi(x)$ and $\psi(y)$ are adjacent in H. Such a function is called an *isomorphism* from G to H. It is obvious that $C_n \times C_m$ is isomorphic to $C_m \times C_n$.

An *automorphism* of a graph G is an isomorphism between G and itself. We observe that $C_n \times C_m$ has $4nm$ automorphisms if $n \neq m$, and $8nm$ if $n = m$.

We define the i,j-*small-values-detection function* $\xi_{i,j} : N^2 \to \{0, 1, 2\}$ so that $\xi_{i,j}(k_1, k_2)$ is the number of true conditions among the following:

(i) $k_1 = k_2 \leq i$, and

(ii) $k_1 + k_2 \leq j$.

$\xi_{5,9}$	3	4	5	6	7
3	2	1	1	1	0
4	1	2	1	0	0
5	1	1	1	0	0
6	1	0	0	0	0
7	0	0	0	0	0

Figura A.2: Values of $\xi_{5,9}(n, m)$

Given a graph G and a subgraph S of G, we say that G is S-*transitive* if for each pair F, H subgraphs of G, where F and H are isomorphic to S, there is an automorphism α of G such that if $v \in V(F)$, then $\alpha(v) \in V(H)$.

It is an easy exercise to show that the graph $C_n \times C_m$ is vertex-transitive. Therefore, a particular vertex may be chosen without loss of generality.

Our strategy in this work is as follows. In section A.2 we show that the upperbound of the vertex deletion number of $C_n \times C_m$ is at most $\min\{n, m\} - \xi_{5,9}(n, m)$. In section A.3 we show that the lowerbound of the vertex deletion number of $C_n \times C_m$ is at least $\min\{n, m\} - \xi_{5,9}(n, m)$.

A.2 Upperbounds for $vd(C_n \times C_m)$

Theorem A.4 *The vertex deletion number of the $C_n \times C_m$ graphs is at most* $\min\{n, m\} - \xi_{5,9}(n, m)$.

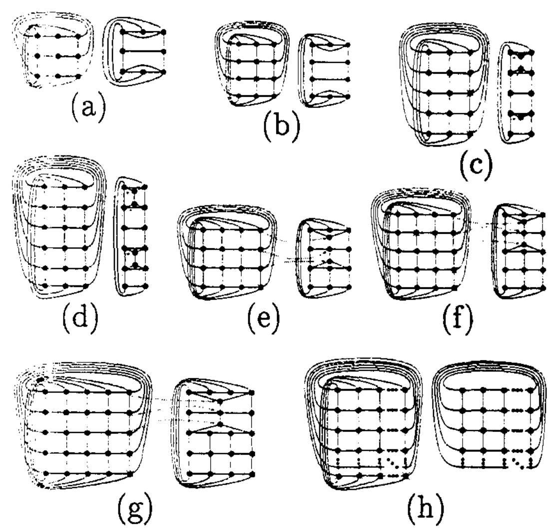


Figura A.3: $vd(C_n \times C_m) \leq \min\{n, m\} - \xi_{5,9}(n, m)$

Proof. Figure A.3 (a) displays the $C_3 \times C_3$ graph and a planar drawing of the induced graph after the removal of one vertex indicated by ∗. Dotted lines indicate isomorphism. Thus, $vd(C_3 \times C_3) \leq 1$. Figure A.3 (b), (c), (d) and (e) display the graphs $C_3 \times C_4$, $C_3 \times C_5$, $C_3 \times C_6$ and $C_4 \times C_4$, respectively, and planar drawings of the induced subgraphs after the removal of two vertices. Thus, $vd(C_3 \times C_4) \leq vd(C_3 \times C_5) \leq vd(C_3 \times C_6) \leq vd(C_4 \times C_4) \leq 2$. Figure A.3 (f) displays the $C_4 \times C_5$ and a planar drawing after the removal of three vertices. Thus, $vd(C_4 \times C_5) \leq 3$. Figure A.3 (g) displays the $C_5 \times C_5$ and a planar drawing of the induced subgraph after the removal of four vertices. Thus, $vd(C_5 \times C_5) \leq 4$ Finally, Figure A.3 (h) displays the $C_n \times C_m$ graph and a planar drawing of the induced subgraph after the removal of $\min\{n, m\}$. Thus, $vd(C_n \times C_m) \leq \min\{n, m\}$ vertices. All these results can be summarized by the inequation $vd(C_n \times C_m) \leq \min\{n, m\} - \xi_{5,9}(n, m)$. □

A.3 Lowerbounds for $vd(C_n \times C_m)$

To prove our claim we must show that some induced subgraphs of the $C_n \times C_m$ contains a subdivision of the $K_{3,3}$. These graphs have in some cases a surprisingly enormous number of analogous cases. Therefore, we will represent such subdivisions of the $K_{3,3}$ and induced subgraphs as follows:

- the vertices $v_{i,j}$ are represented by the integer points $\{(i,j), 0 \leq i < n, 0 \leq j < m\}$,
- deleted vertices are drawn with the symbol $\times$,
- edges used by the subdivision of the $K_{3,3}$ are drawn solid and vertices of the $K_{3,3}$ are drawn as large circles with degree 3,
- each horizontal half-edge drawn along the left edge side of the grid connects to the half-edge at the right side, on the same row; and analogously for vertical half-edges,
- all other vertices are drawn as small dots, and all other edges are omitted.

To reduce the amount of work we wrote two simple combinatorics programs: **Find-Analogous** and **Find-$K_{3,3}$**. The former generates all the non-analogous subgraphs of $C_n \times C_m$ that result from the deletion of k vertices, for given n, m and k. We say that two subgraphs of $C_n \times C_m$ are *analogous* if they are isomorphic by an automorphism of $C_n \times C_m$. Note that two non-analogous subgraphs may be isomorphic. Due to the automorphisms of $C_n \times C_m$ we need to generate only subgraphs with the top left corner vertex deleted. This reduces the number of subgraphs that needs to be considered from $\binom{nm}{k}$ to $\binom{nm-1}{k-1}$. Furthermore when deciding whether a subgraph is analogous to a previously generated one, we need to consider only $4k$ or $8k$ automorphisms of $C_n \times C_m$, instead of $4nm$ or $8nm$.

The second program **Find-$K_{3,3}$** checks whether each subgraph of $C_n \times C_m$ generated by **Find-Analogous** contains a subdivision of $K_{3,3}$. The subdivisions of the $K_{3,3}$ are added to a list by the user. If **Find-$K_{3,3}$** fails to find a subdivision of $K_{3,3}$ it stops printing the subgraph. If **Find-$K_{3,3}$** finds a subdivision of $K_{3,3}$ for each subgraph of $C_n \times C_m$ generated by **Find-Analogous** it prints all solutions.

Lemma A.5 *The vertex deletion number of $C_3 \times C_3$ is at least* 1.

Proof. The graph $C_3 \times C_3$ contains a subdivision of $K_{3,3}$ as shown in Figure A.4. Therefore, it is not planar which implies that $vd(C_3 \times C_3) \geq 1$. □

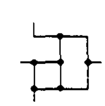


Figura A.4: $vd(C_3 \times C_3) \geq 1$

Lemma A.6 *The vertex deletion number of $C_3 \times C_4$ is at least* 2.

Proof. Let G be the subgraph induced by all vertices of the $C_3 \times C_4$ minus one vertex. Without loss of generality we suppose that the deleted vertex is at the top left corner as

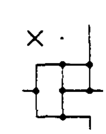


Figura A.5: $vd(C_3 \times C_4) \geq 2$

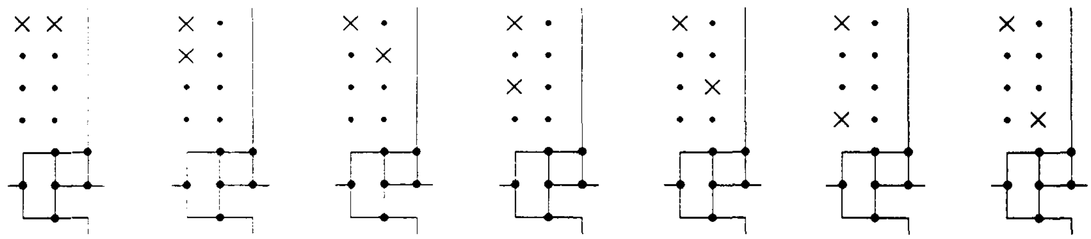


Figura A.6: $vd(C_3 \times C_7) \geq 3$

shown in Figure A.5. This graph contains a subdivision of $K_{3,3}$. Therefore, it is not planar which implies that $vd(C_3 \times C_4) \geq 2$. □

Corollary A.7 *The vertex deletion number of $C_3 \times C_5$, $C_3 \times C_6$ and $C_4 \times C_4$ are at least* 2.

Proof. All of these graphs contain a subdivision of $C_3 \times C_4$. □

Lemma A.8 *The vertex deletion number of $C_3 \times C_7$ is at least* 3.

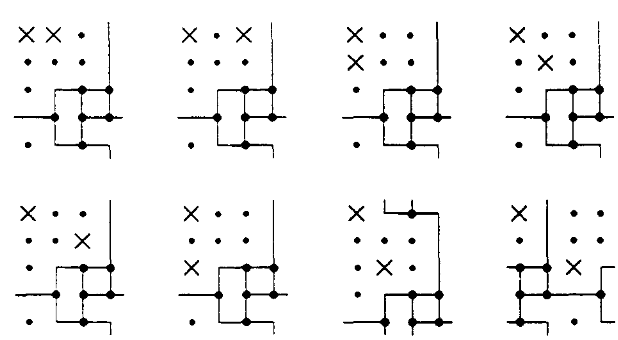
Proof. Figure A.6 displays the 7 non-analogous possible ways to delete 2 vertices from $C_3 \times C_7$ generating different induced subgraphs. Figure A.6 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_3 \times C_7) \geq 3$. □

Corollary A.9 *The vertex deletion number of $C_3 \times C_m$, where $m \geq 7$ is at least* 3.

Proof. The graph $C_3 \times C_m$ contains a subdivision of $C_3 \times C_7$ which has vertex deletion number at least 3. □

Lemma A.10 *The vertex deletion number of $C_4 \times C_5$ is at least* 3.

Proof. Figure A.7 displays the 8 non-analogous subgraphs obtained by deleting 2 vertices from $C_4 \times C_5$. Figure A.7 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_4 \times C_5) \geq 3$. □



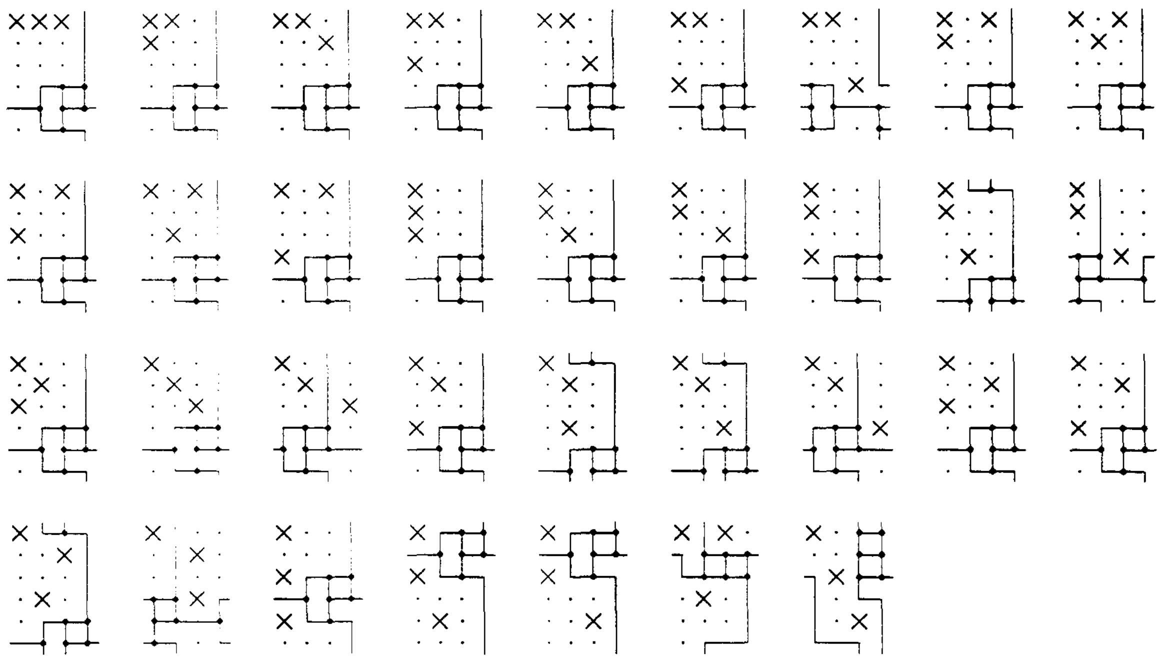


Figura A.8: $vd(C_4 \times C_6) \geq 4$

Lemma A.11 *The vertex deletion number of* $C_4 \times C_6$ *is at least* 4.

Proof. Figure A.8 displays the 34 non-analogous subgraphs obtained by deleting 3 vertices from $C_4 \times C_6$. Figure A.8 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_4 \times C_6) \geq 4$. □

Corollary A.12 *The vertex deletion number of* $C_4 \times C_m$*, where* $m \geq 6$ *is at least* 4.

Proof. The graph $C_4 \times C_m$ contains a subdivision of $C_4 \times C_6$ which has vertex deletion number at least 4. □

Lemma A.13 *The vertex deletion number of* $C_5 \times C_5$ *is at least* 4.

Proof. Figure A.9 displays the 19 non-analogous ways to delete 3 vertices from $C_5 \times C_5$. Figure A.9 also displays a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_5 \times C_5) \geq 4$. □

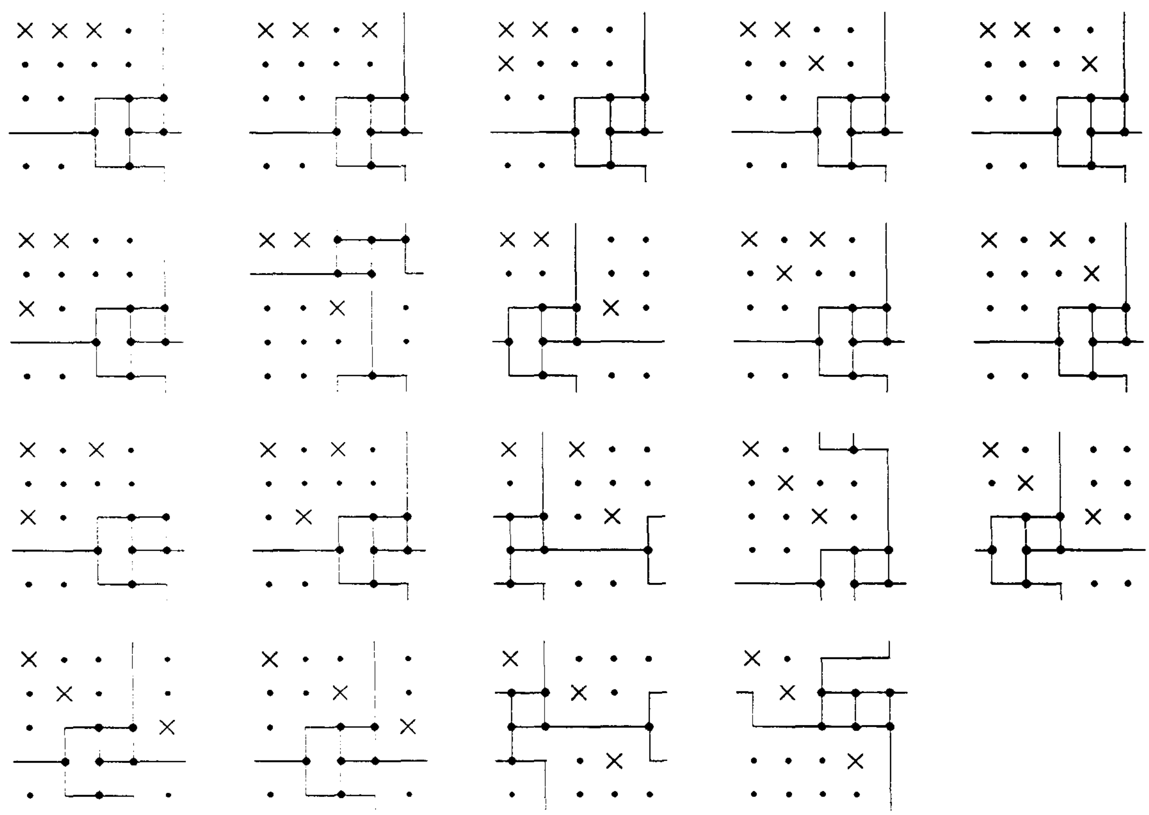


Figura A.9: $vd(C_5 \times C_5) \geq 4$

Lemma A.14 *The vertex deletion number of* $C_5 \times C_6$ *is at least* 5.

Proof. Figures A.10 to A.14 display the 291 non-analogous subgraphs obtained by deleting 4 vertices from $C_5 \times C_6$. Figure A.10 to A.14 also display a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_5 \times C_6) \geq 5$. □

Corollary A.15 *The vertex deletion number of* $C_5 \times C_m$*, where* $m \geq 6$ *is at least* 5.

Proof. The graph $C_5 \times C_m$ contains a subdivision of $C_5 \times C_6$ which has vertex deletion number at least 5. □

Lemma A.16 *The vertex deletion number of* $C_6 \times C_6$ *is at least* 6.

Proof. Figures A.10 to A.38 display the 1455 non-analogous subgraphs obtained by deleting 5 vertices from $C_6 \times C_6$. Figures A.15 to A.38 also display a subdivision of $K_{3,3}$ in each subgraph. Therefore, none of them are planar which implies that $vd(C_5 \times C_6) \geq 5$. □

Lemma A.17 *The vertex deletion number of* $C_k \times C_k$ *for an integer* $k \geq 6$ *is at least* k.

Proof. We will prove this assertion by induction in k. The induction basis is the graph $C_6 \times C_6$. The induction hypothesis is that for all graphs $C_l \times C_l$ the vertex deletion number is at least l, where $6 \leq l < k$. Without loss of generality we may suppose that the vertex s at the top left corner is deleted. The remaining graph has a subdivision of $C_{k-1} \times C_{k-1}$. It follows from the induction hypothesis the $C_k \times C_k \backslash s$ has vertex deletion number at least $k-1$ and therefore, the graph $C_k \times C_k$ has vertex deletion number at least k. □

Corollary A.18 *The vertex deletion number of* $C_n \times C_m$*, where* $n, m \geq 6$ *is at least* $\min\{n, m\}$.

Proof. The graph $C_n \times C_m$ contains a subdivision of $C_n \times C_n$ which has vertex deletion number at least n. □

Theorem A.19 *The vertex deletion number of* $C_n \times C_m$ *is at least* $\min\{n, m\} - \xi_{5,9}(n, m)$.

Proof. The assertion follows from Lemma A.5, Lemma A.6, Corollary A.7, Lemma A.8, Corollary A.9, Lemma A.10, Lemma A.11, Corollary A.12, Lemma A.13, Lemma A.14, Corollary A.15, Lemma A.16, Lemma A.17 and Corollary A.18. □

Theorem A.20 *The vertex deletion number of* $C_n \times C_m$ *is* $\min\{n, m\} - \xi_{5,9}(n, m)$.

Proof. The assertion follows from Theorem A.4 and Theorem A.19. □

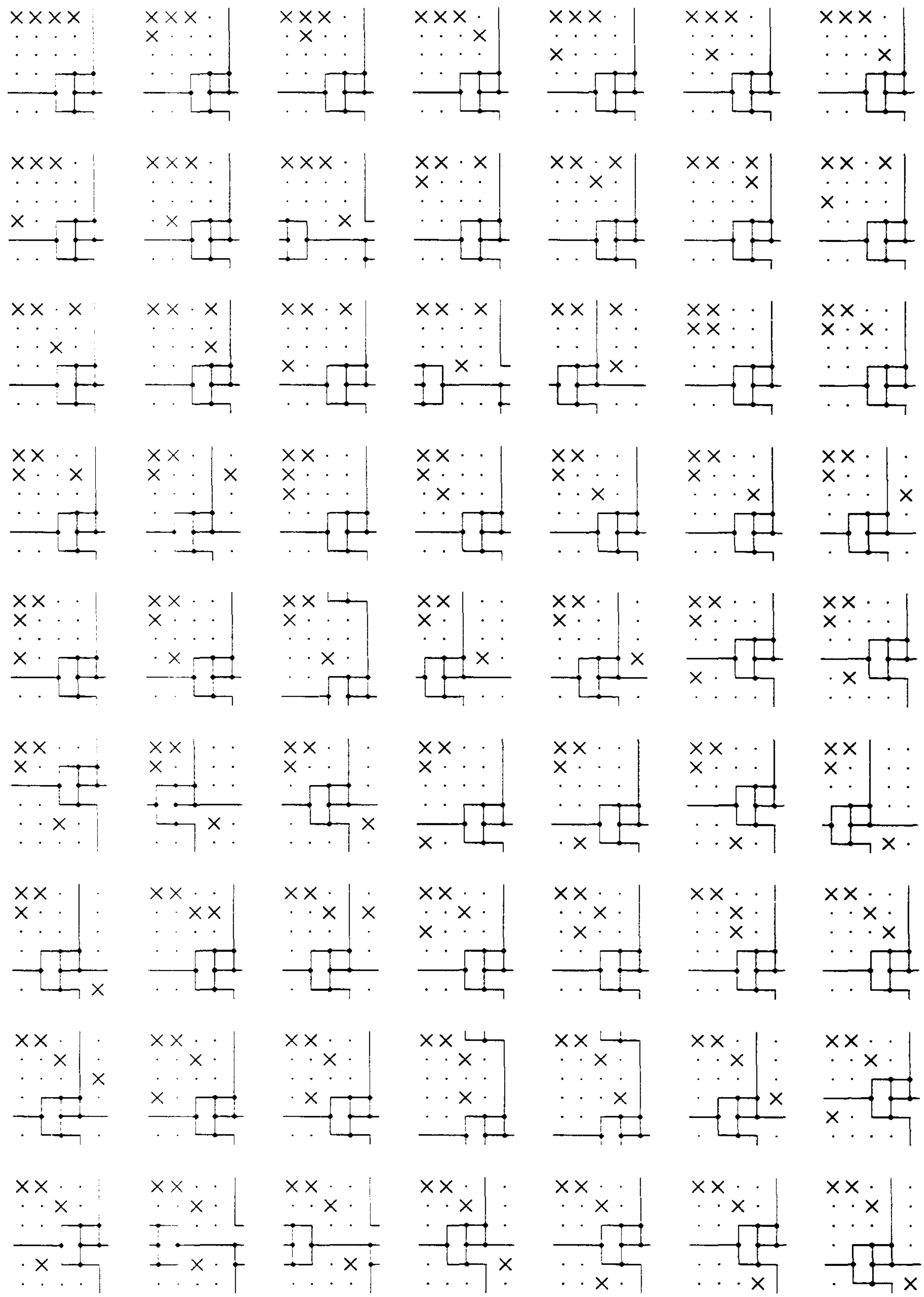


Figura A.10: $vd(C_5 \times C_6) \geq 5$.

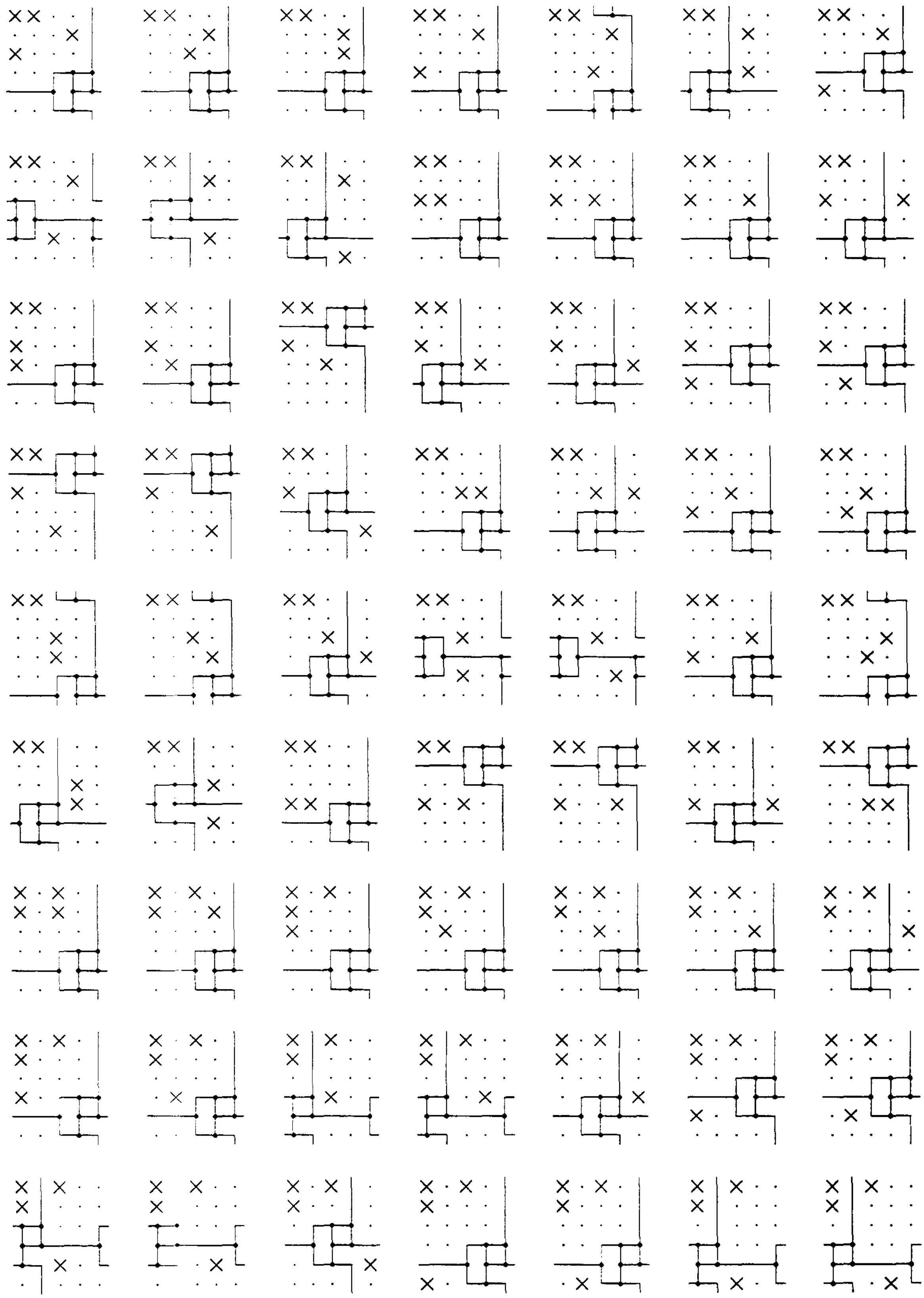


Figura A.11: $vd(C_5 \times C_6) \geq 5$.

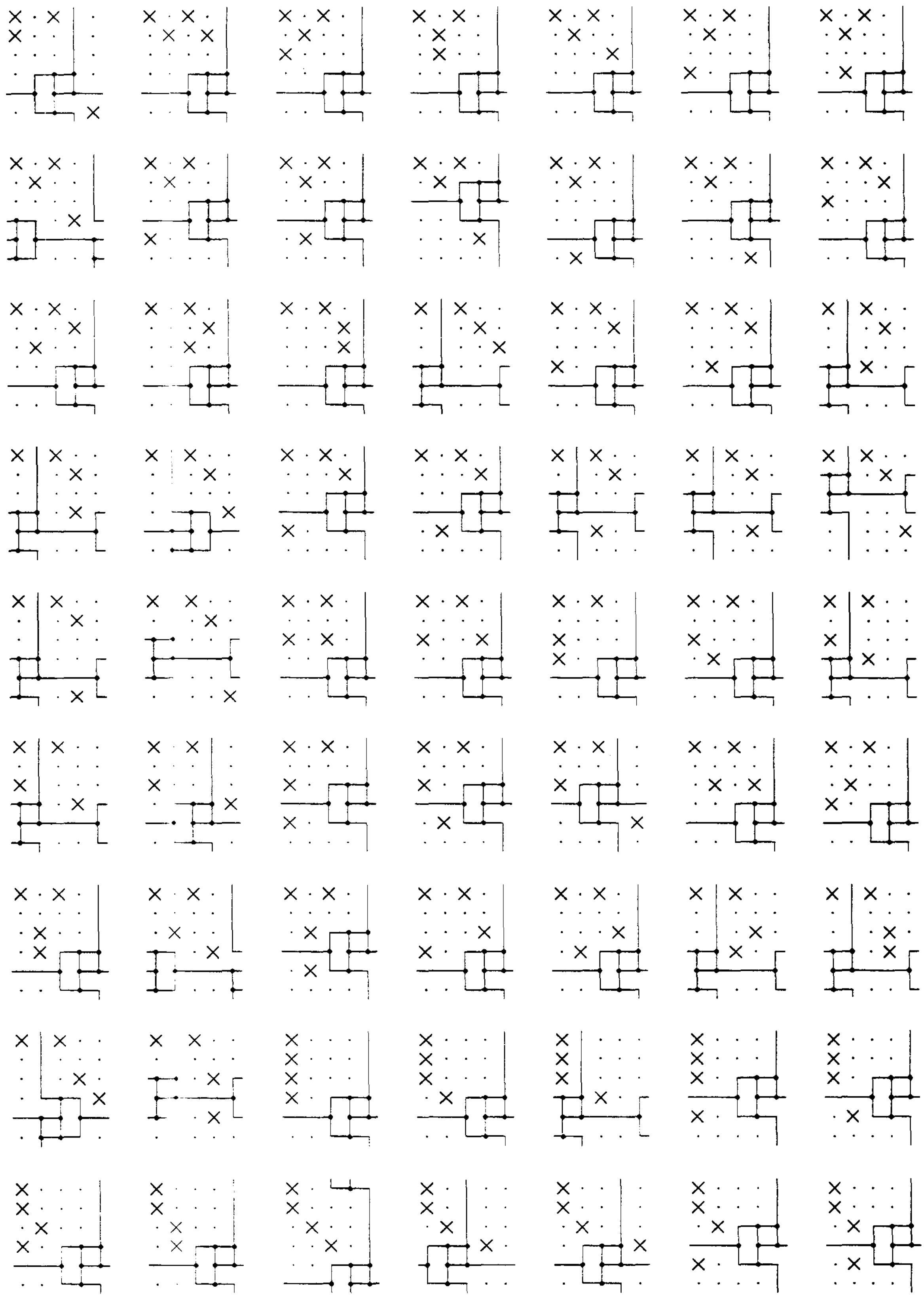


Figura A.12: $vd(C_5 \times C_6) \geq 5$.

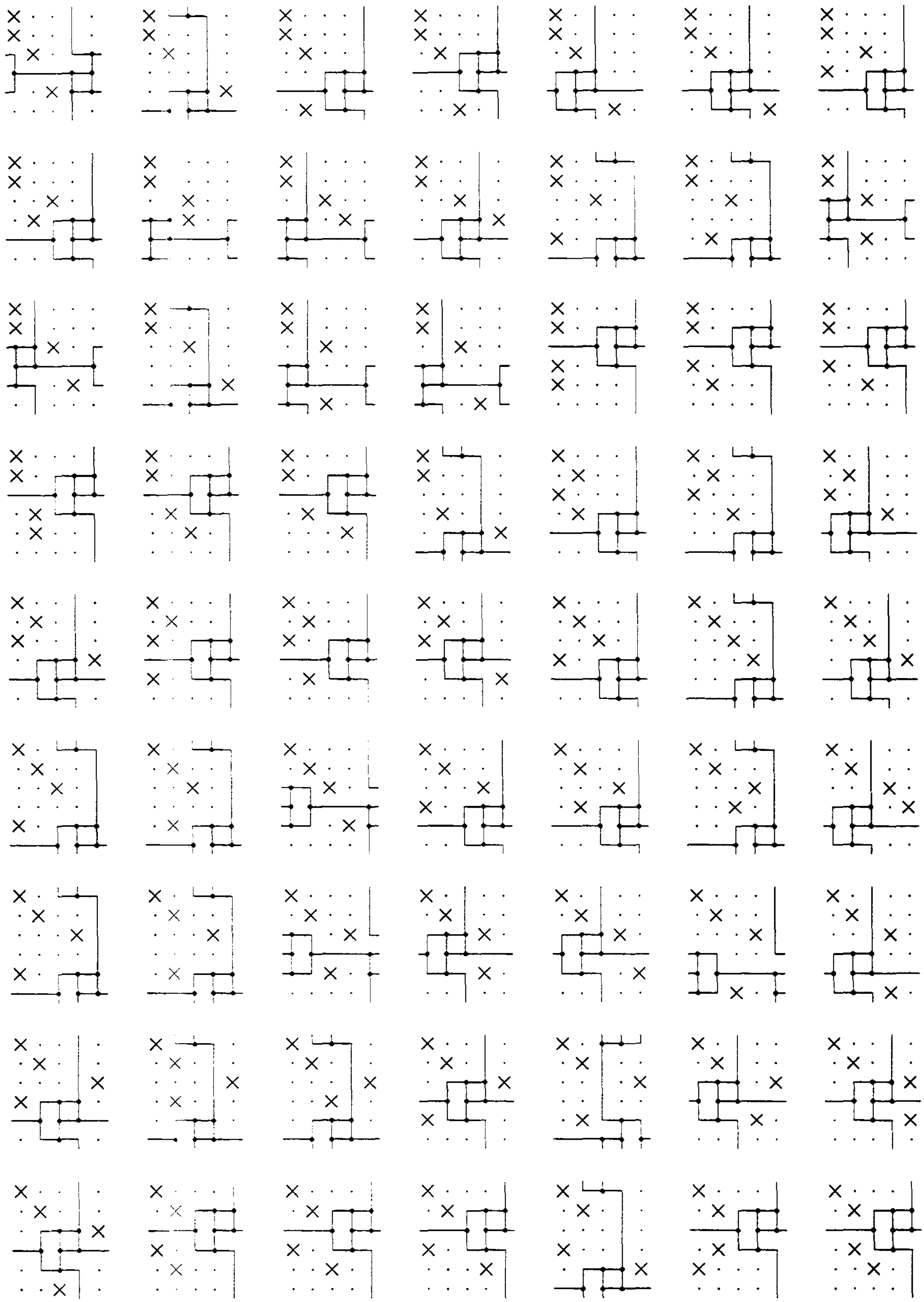


Figura A.13: $vd(C_5 \times C_6) \geq 5$.

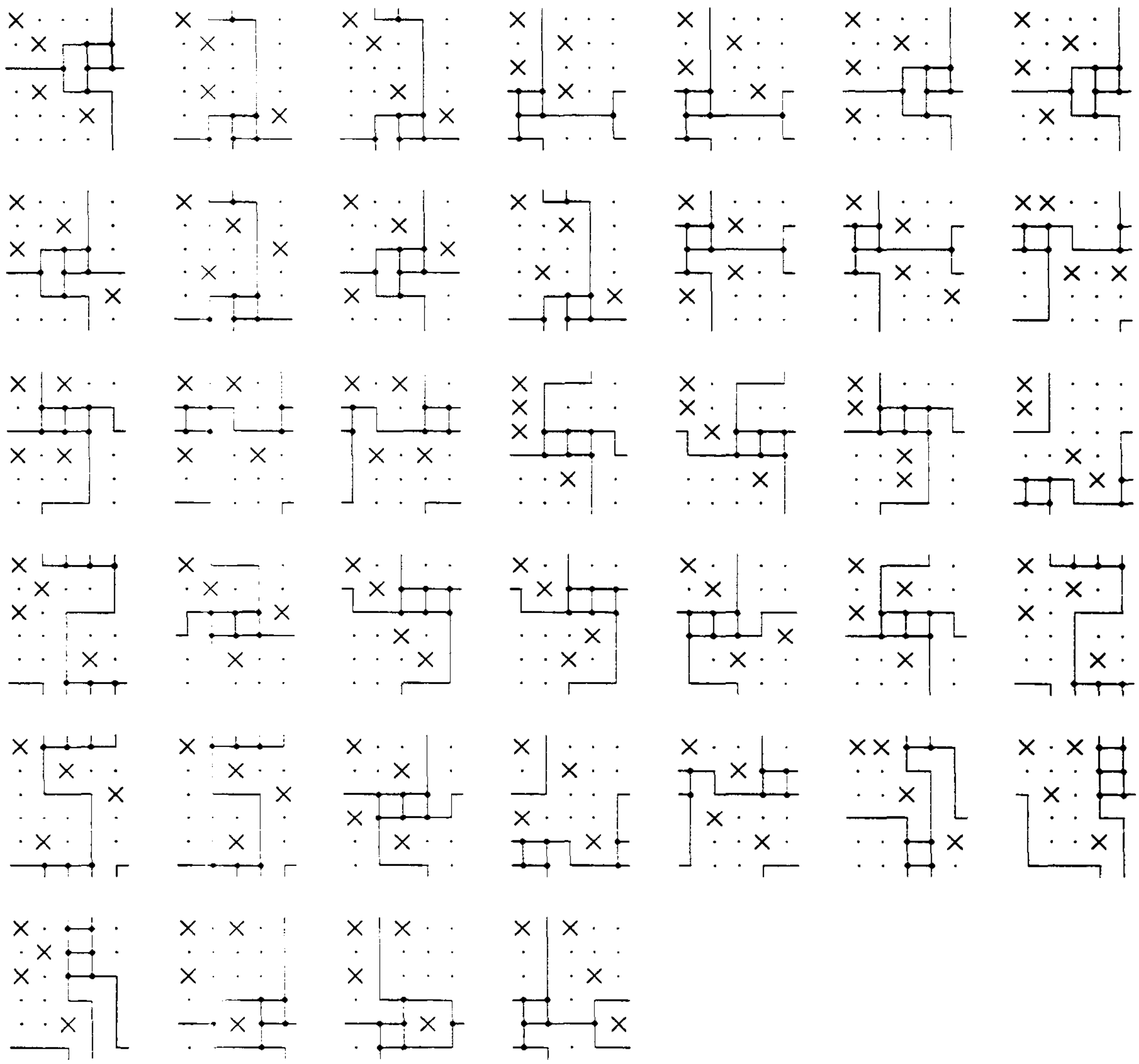


Figura A.14: $vd(C_5 \times C_6) \geq 5$.

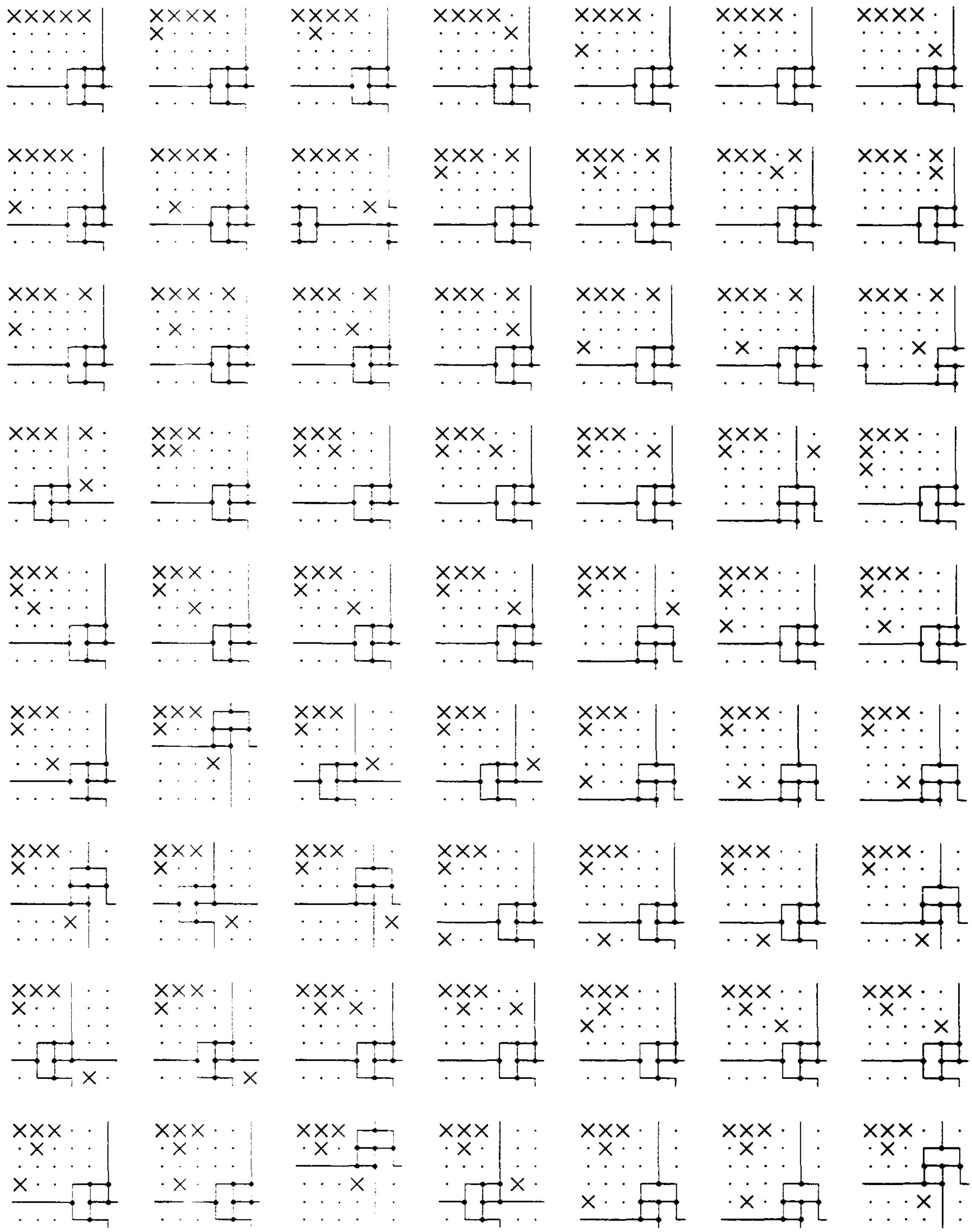


Figura A.15: $vd(C_6 \times C_6) \geq 6$.

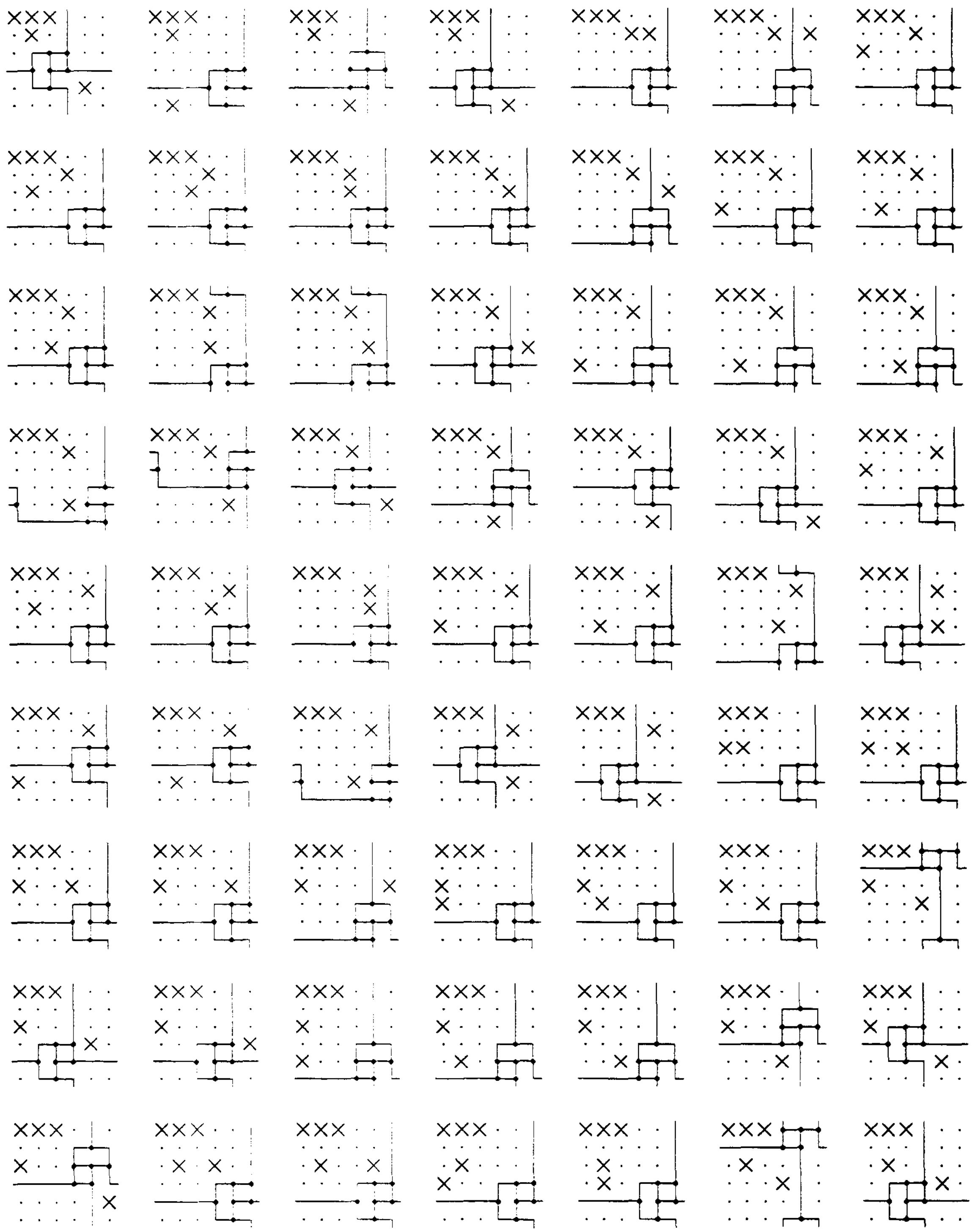


Figura A.16: $vd(C_6 \times C_6) \geq 6$.

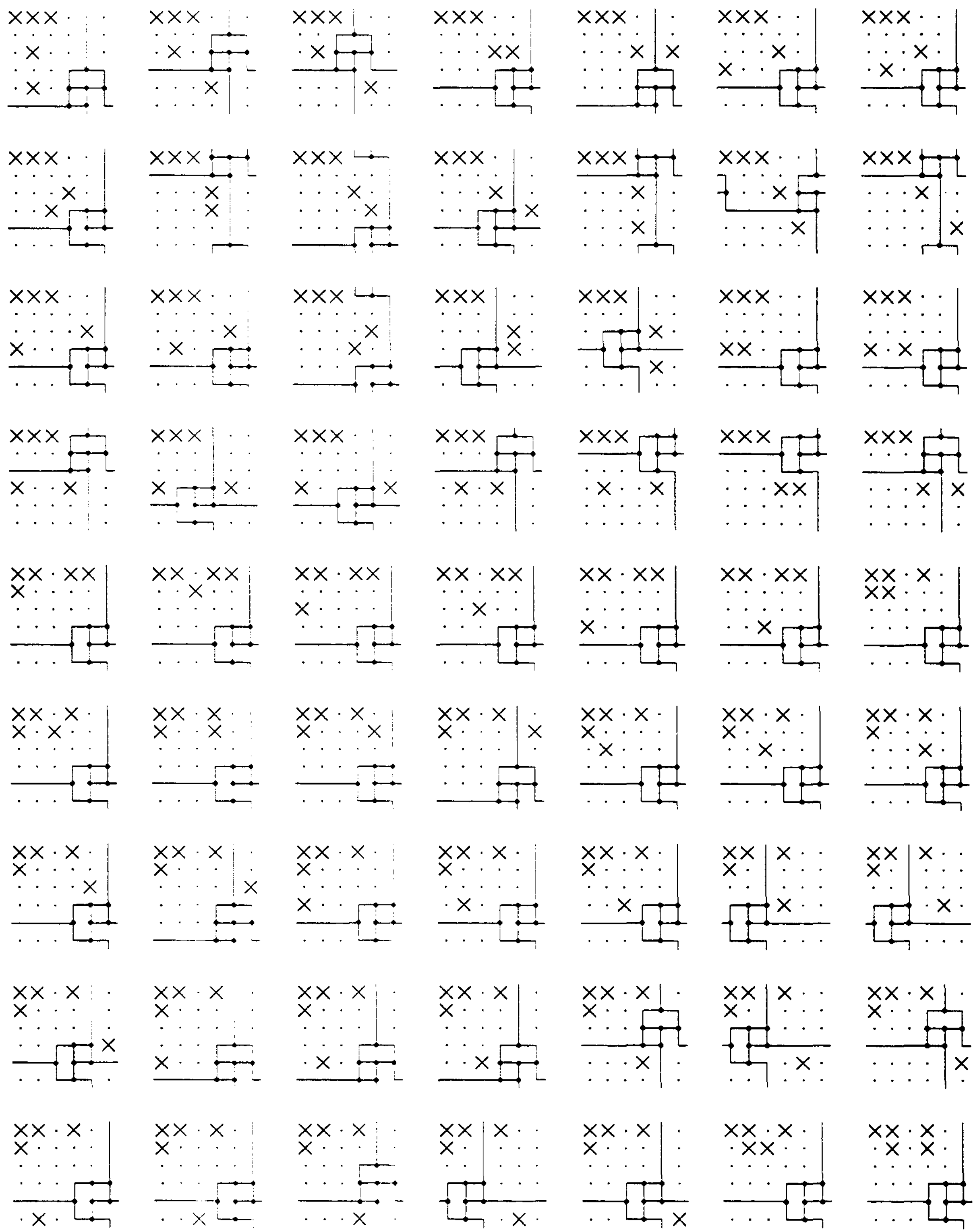


Figura A.17: $vd(C_6 \times C_6) \geq 6$.

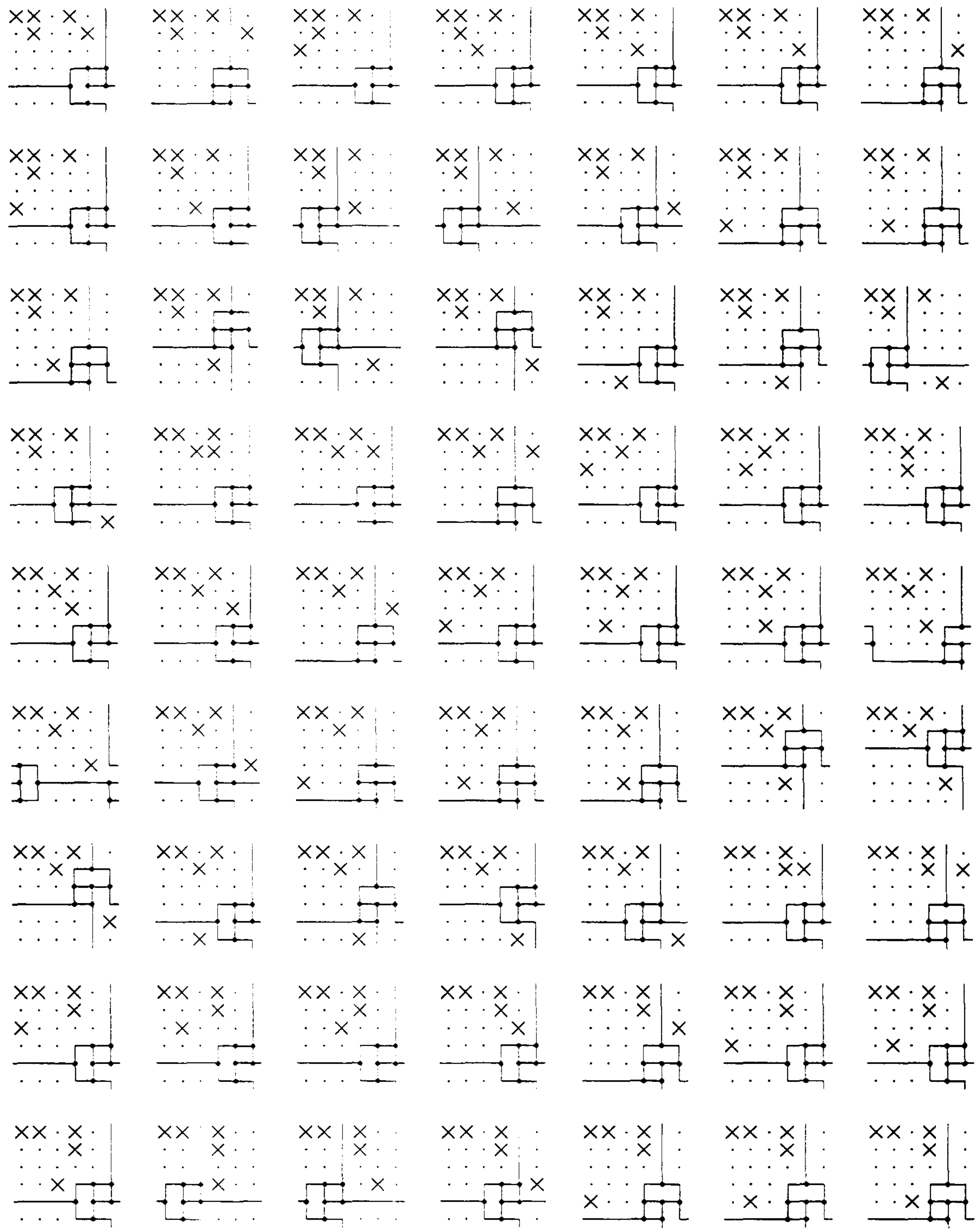


Figura A.18: $vd(C_6 \times C_6) \geq 6$.

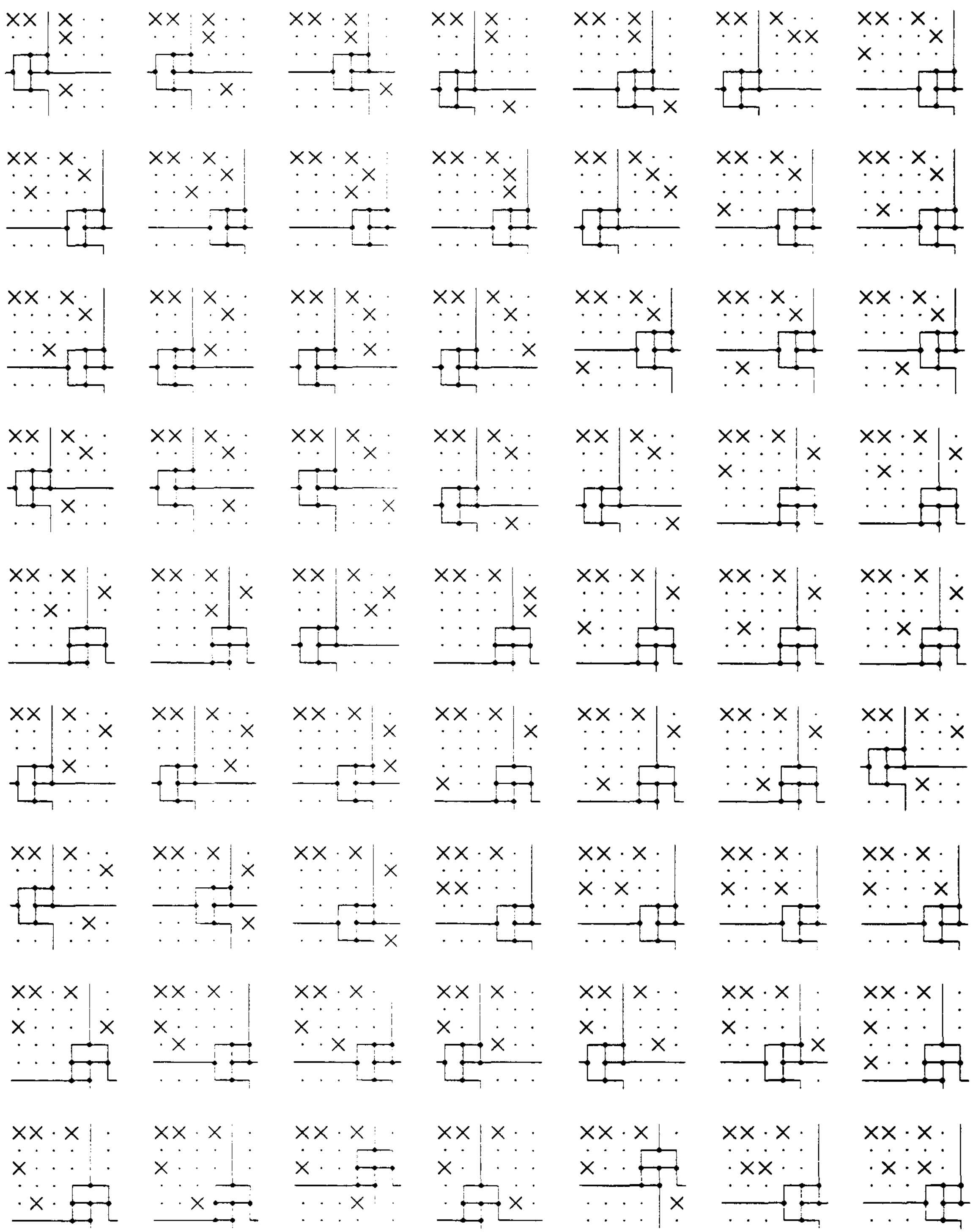


Figura A.19: $vd(C_6 \times C_6) \geq 6$.

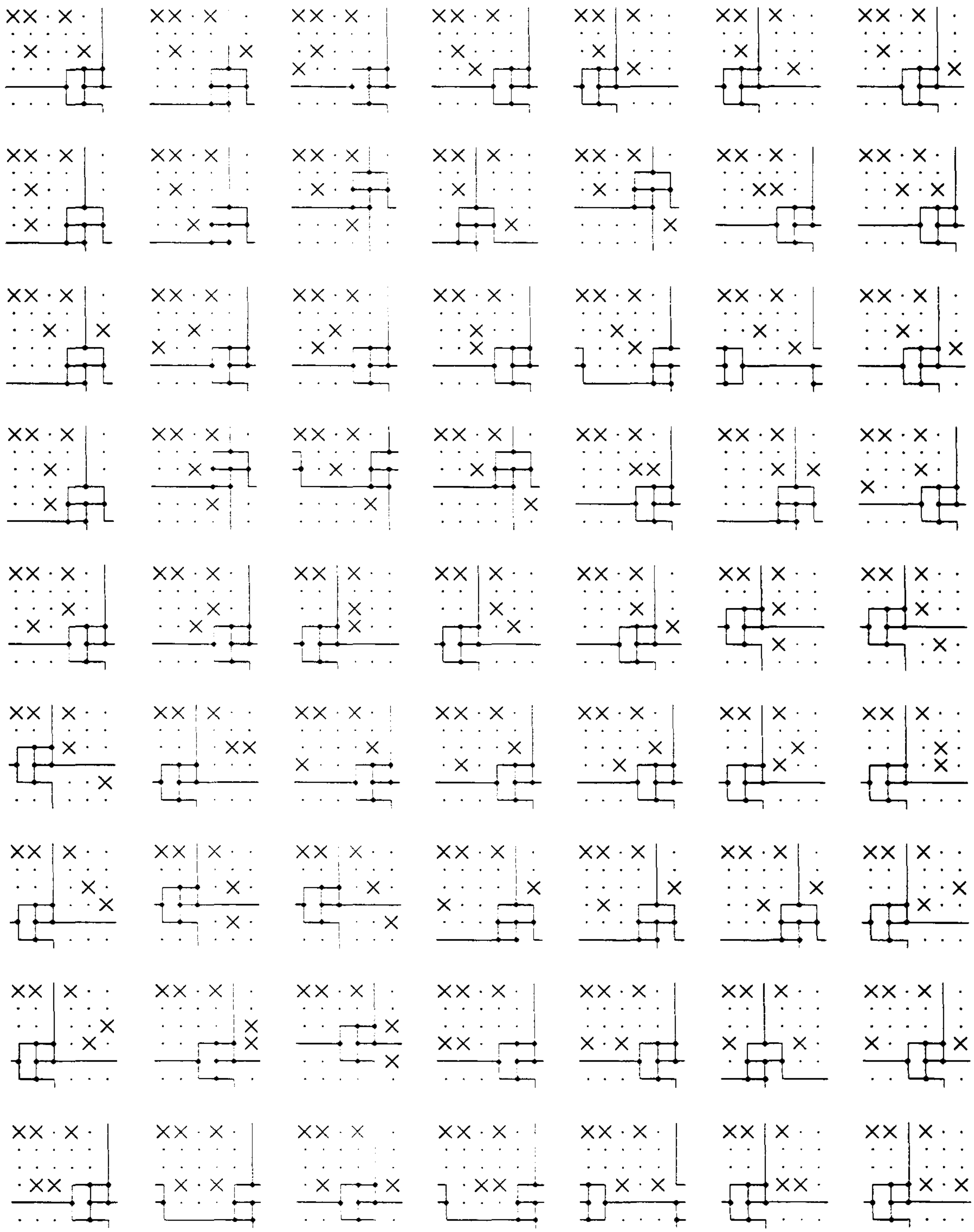


Figura A.20: $vd(C_6 \times C_6) \geq 6$.

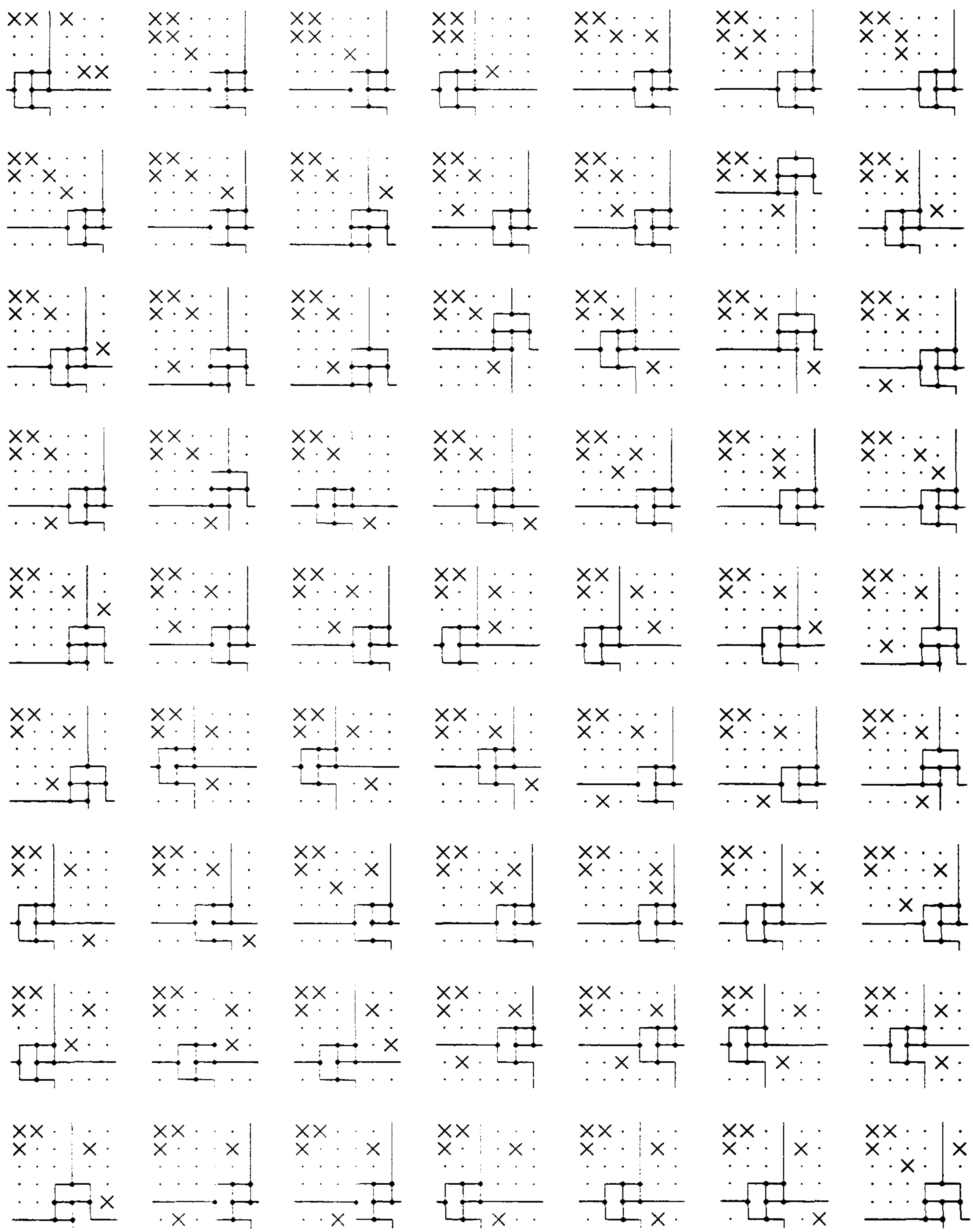


Figura A.21: $vd(C_6 \times C_6) \geq 6$.

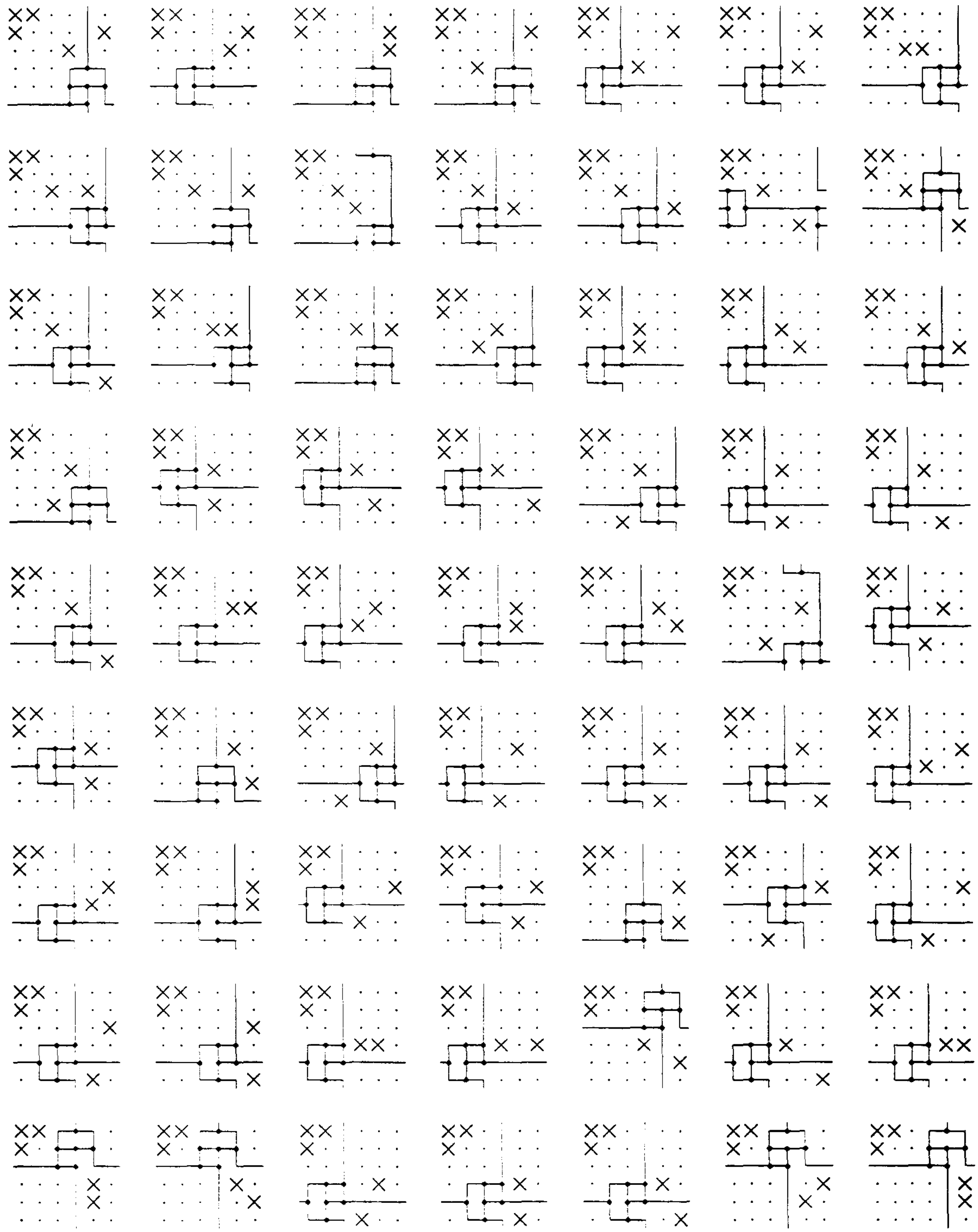


Figura A.22: $vd(C_6 \times C_6) \geq 6$.

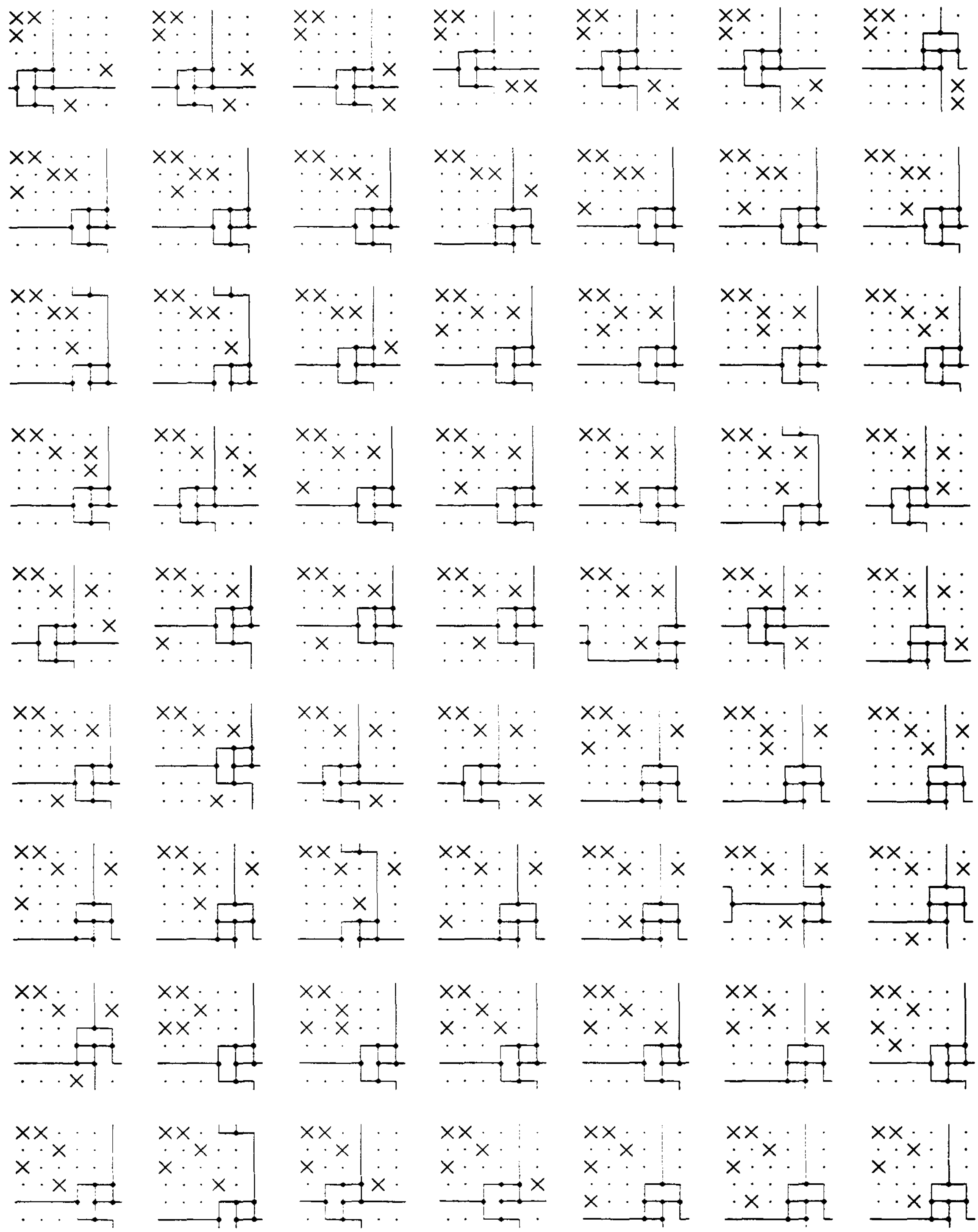


Figura A.23: $vd(C_6 \times C_6) \geq 6$.

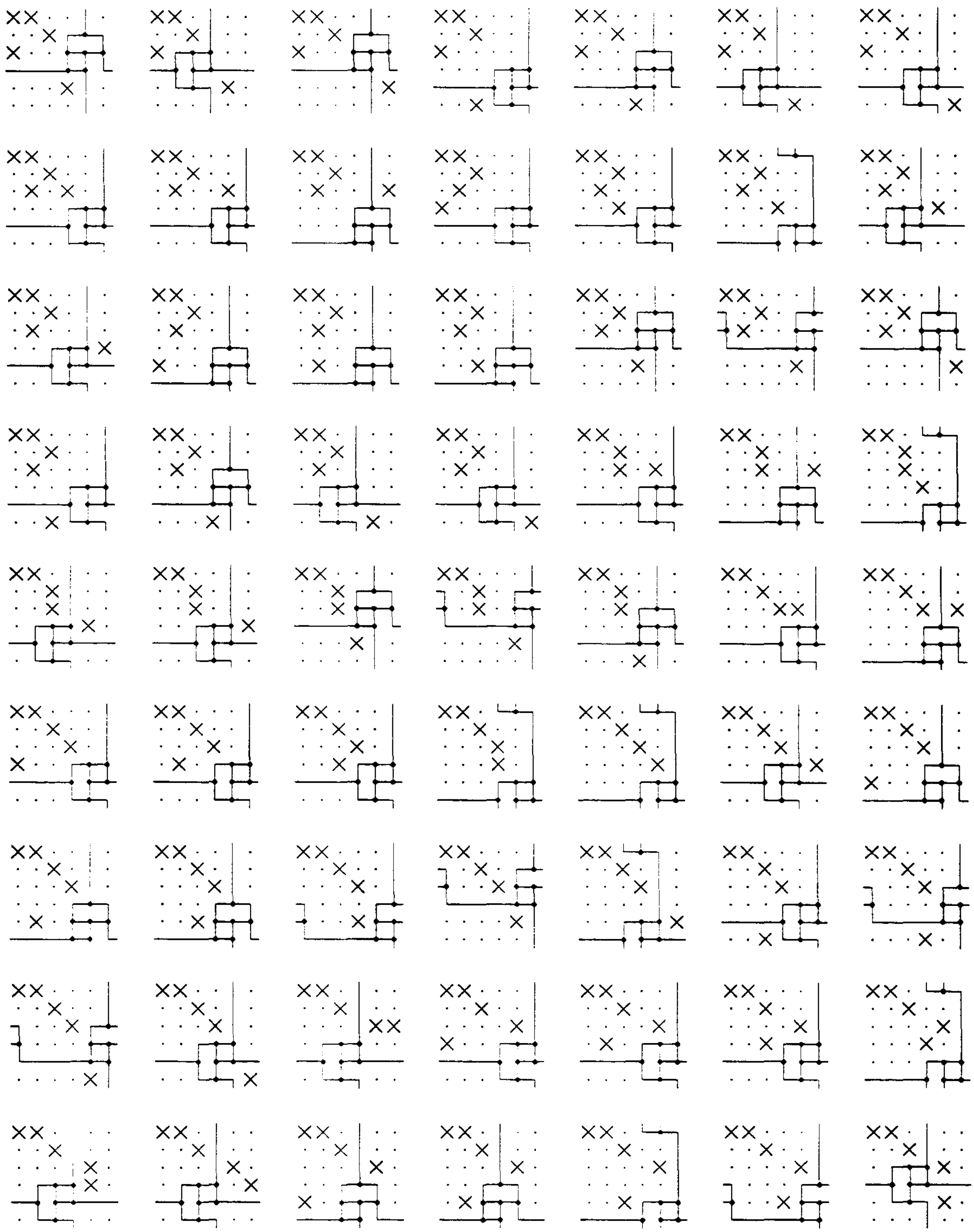


Figura A.24: $vd(C_6 \times C_6) \geq 6$.

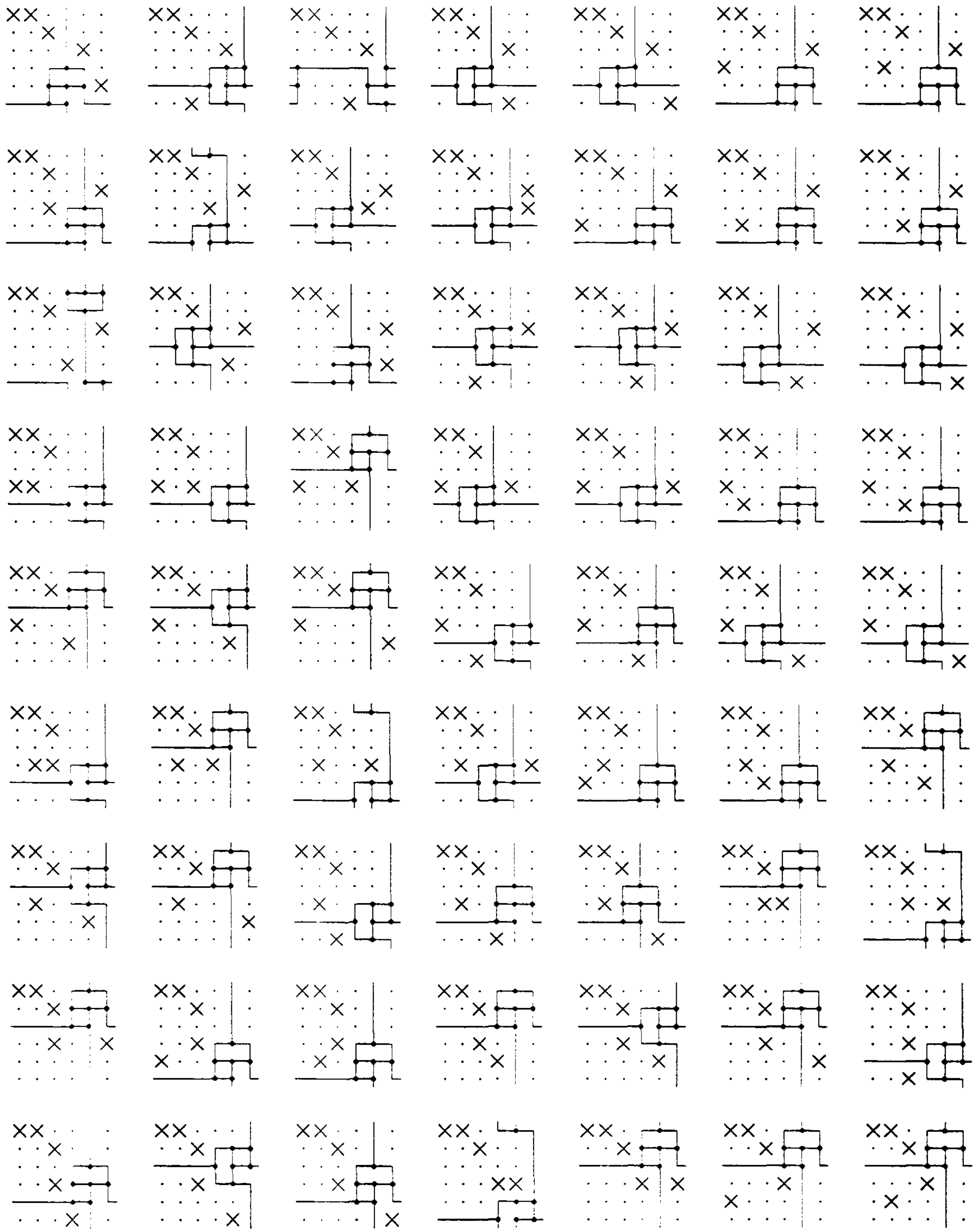


Figura A.25: $vd(C_6 \times C_6) \geq 6$.

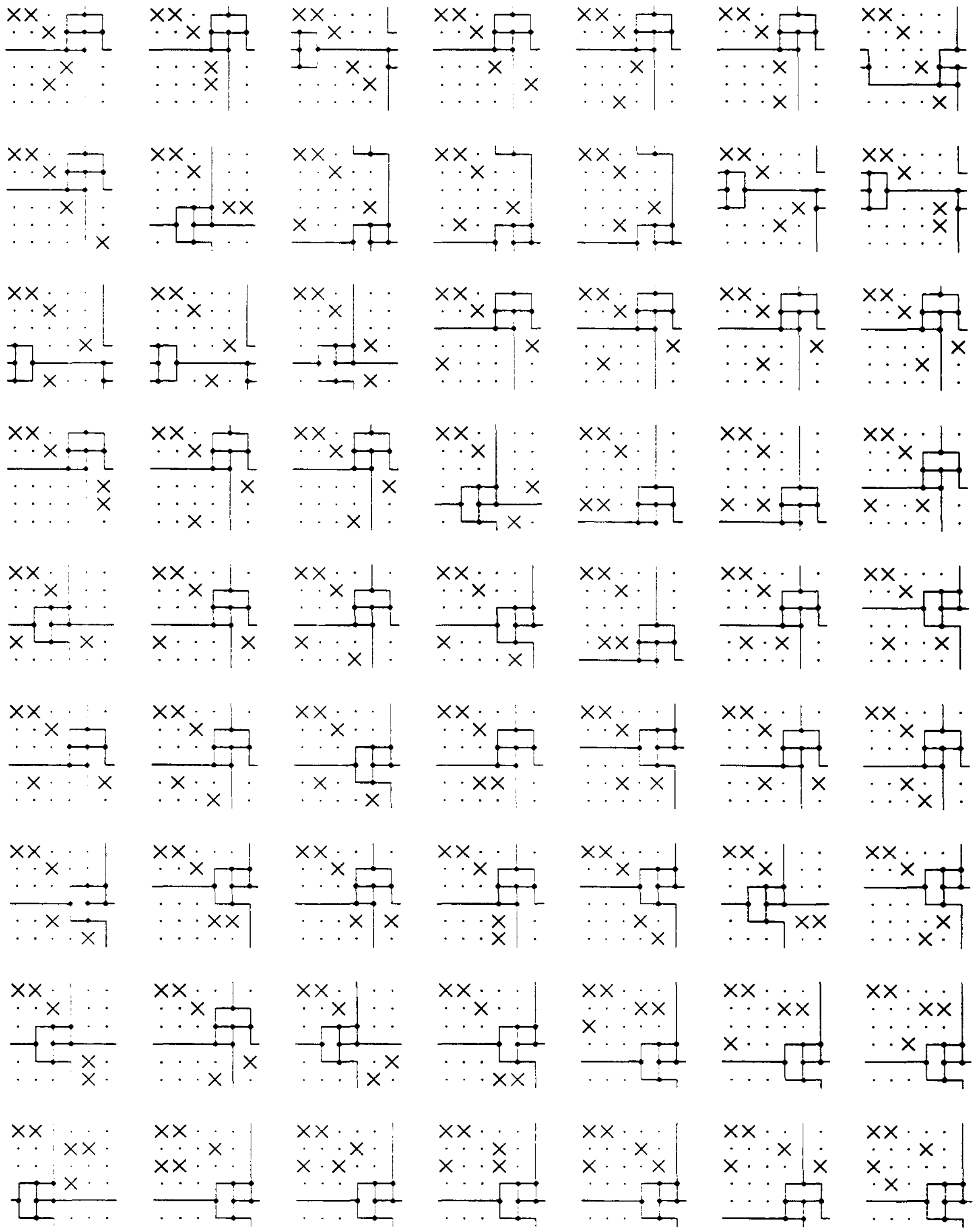


Figura A.26: $vd(C_6 \times C_6) \geq 6$.

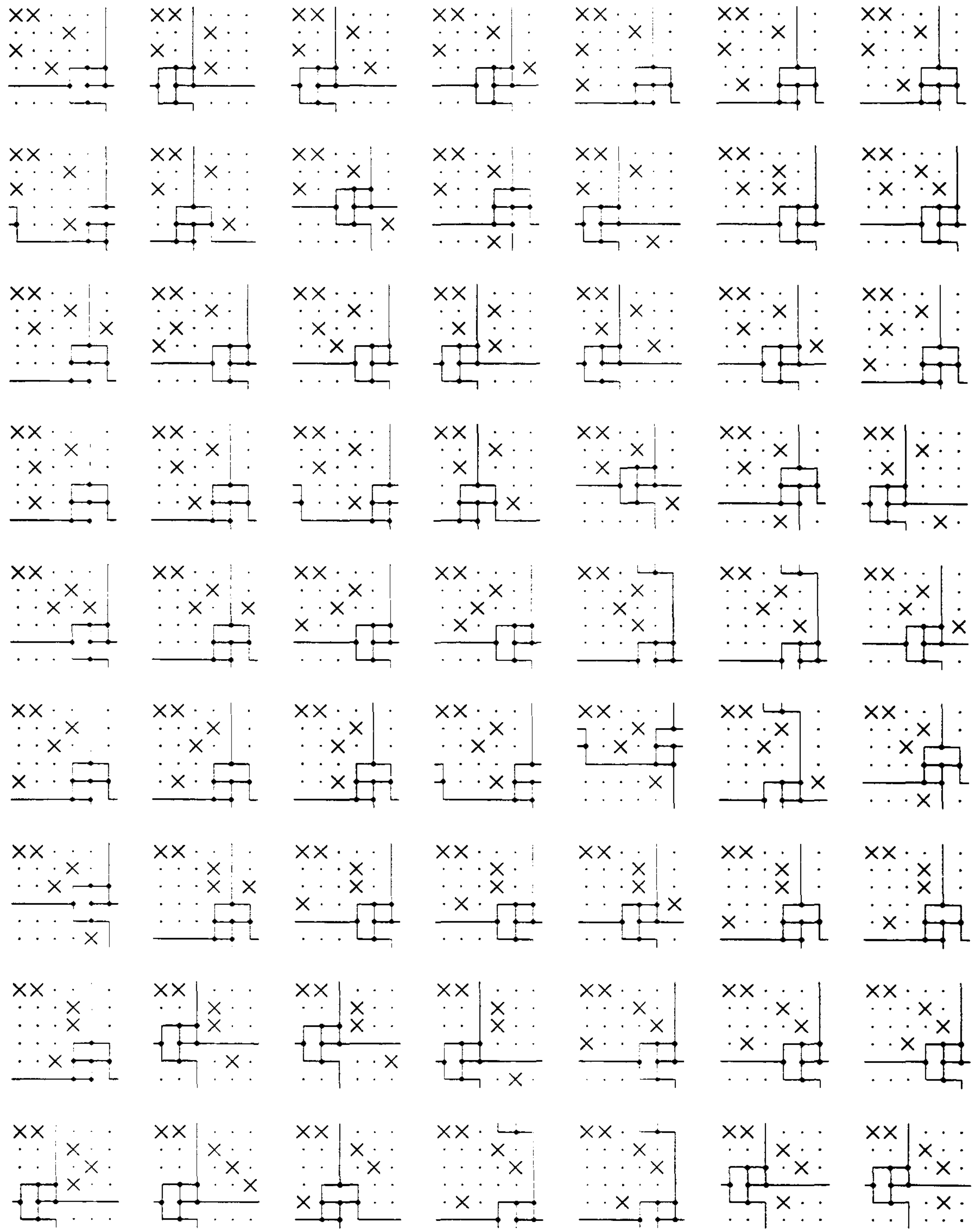


Figura A.27: $vd(C_6 \times C_6) \geq 6$.

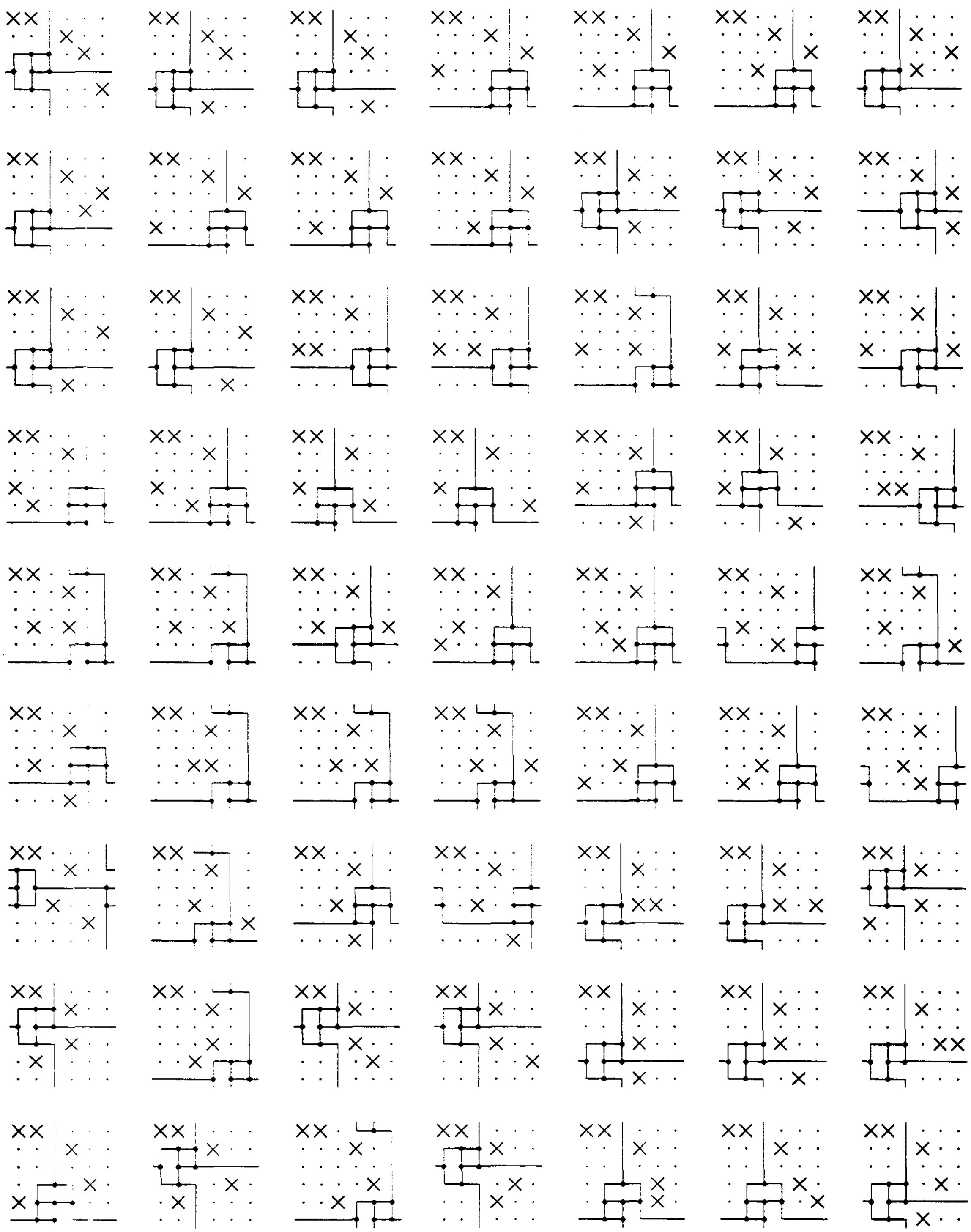


Figura A.28: $vd(C_6 \times C_6) \geq 6$.

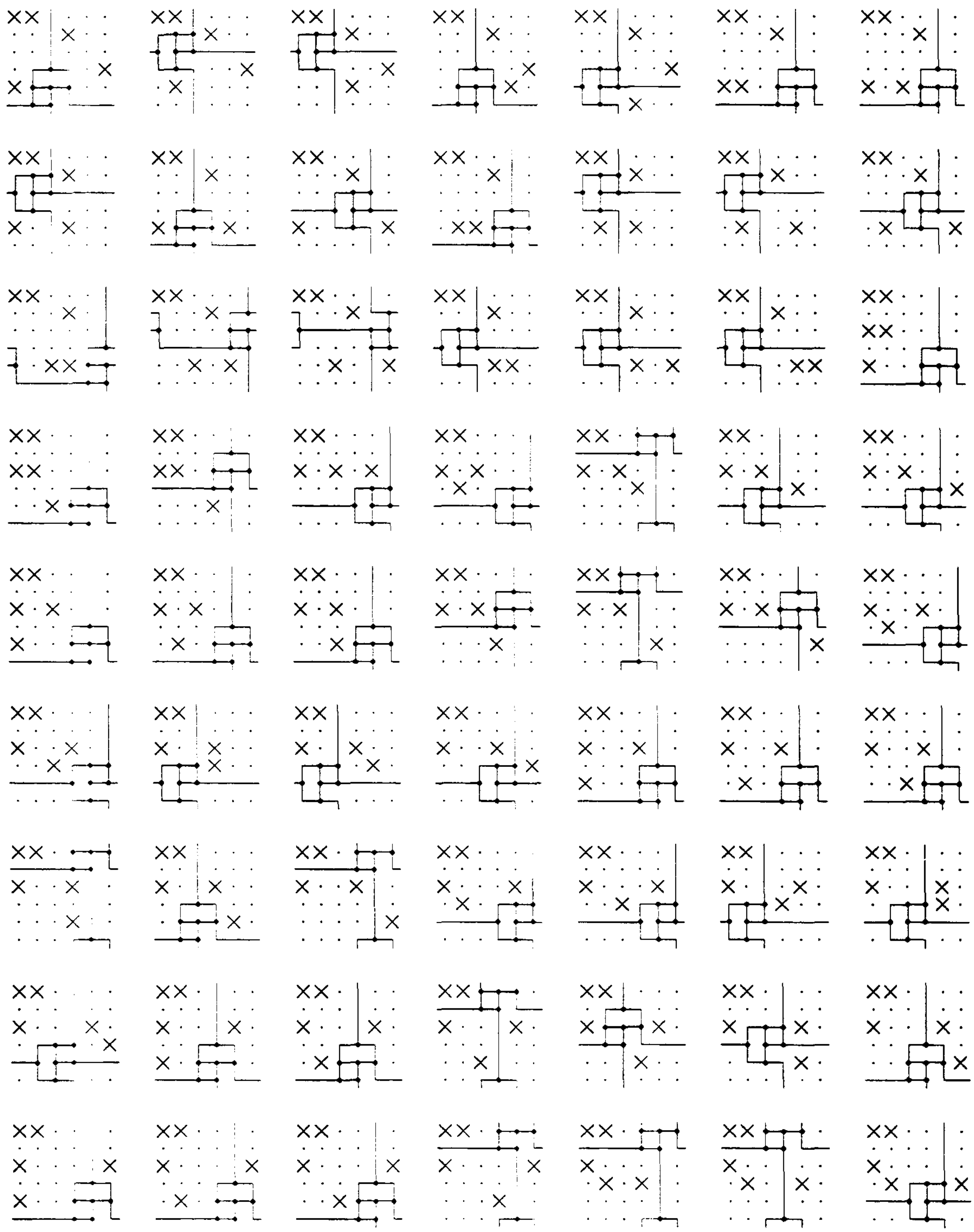


Figura A.29: $vd(C_6 \times C_6) \geq 6$.

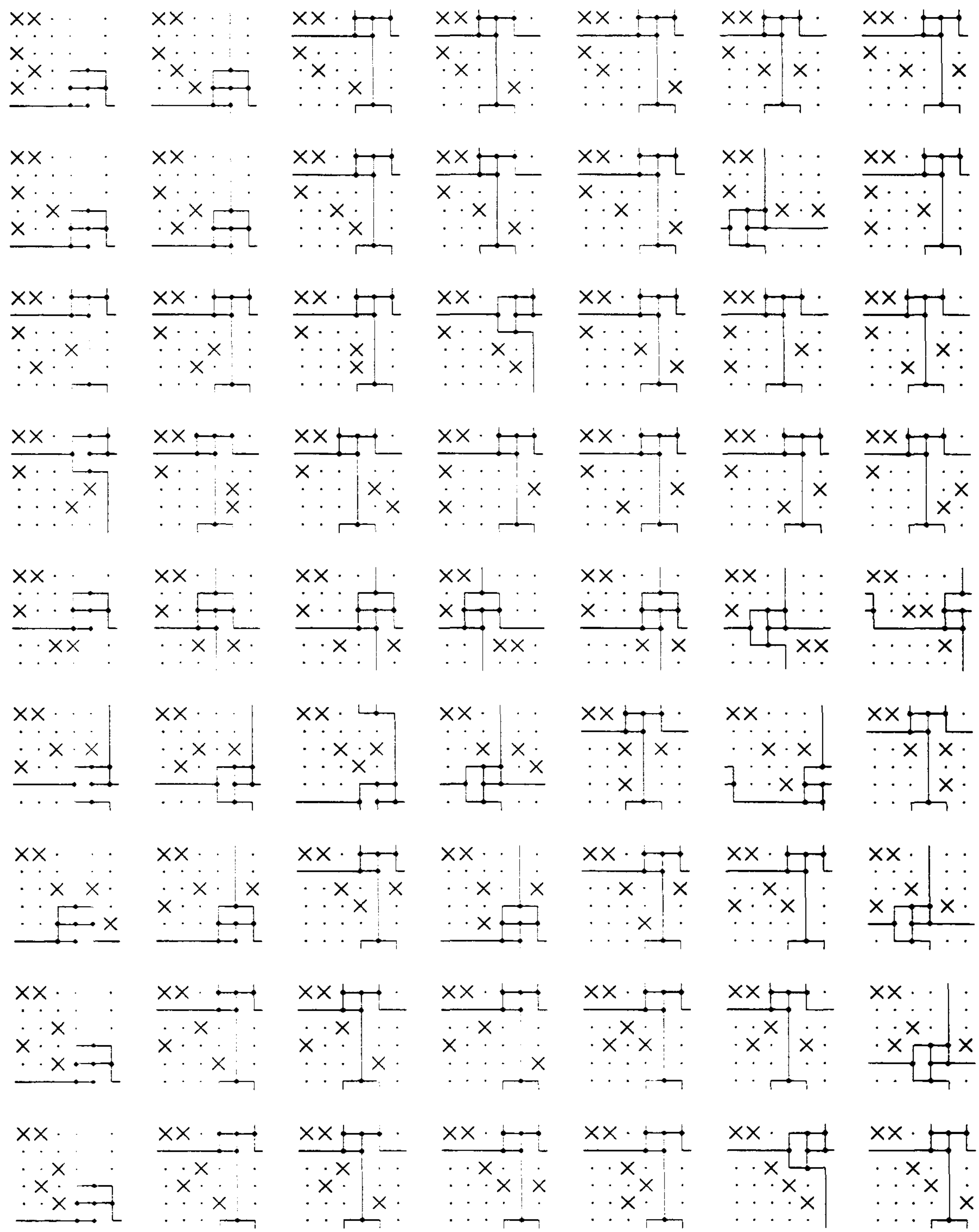


Figura A.30: $vd(C_6 \times C_6) \geq 6$.

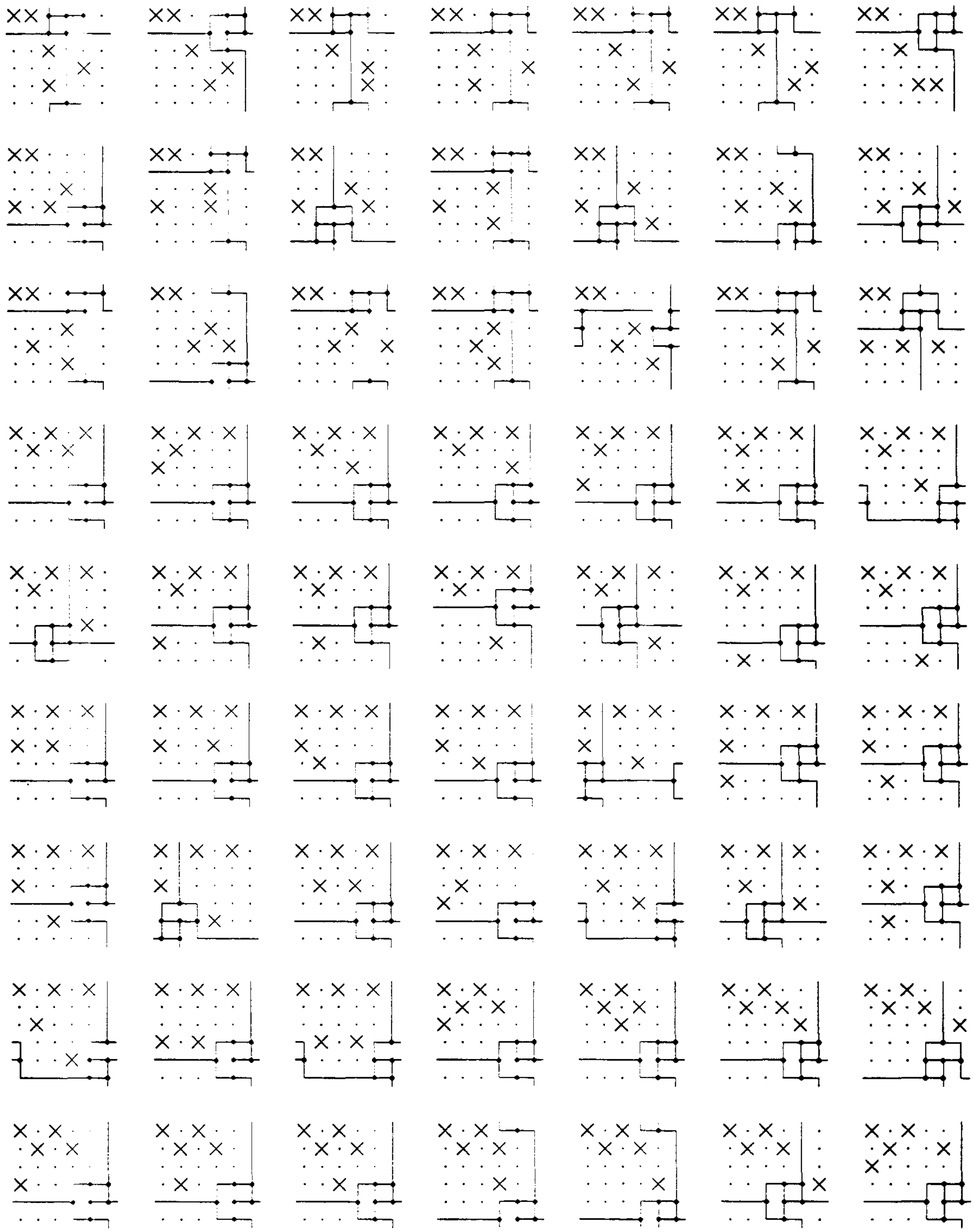


Figura A.31: $vd(C_6 \times C_6) \geq 6$.

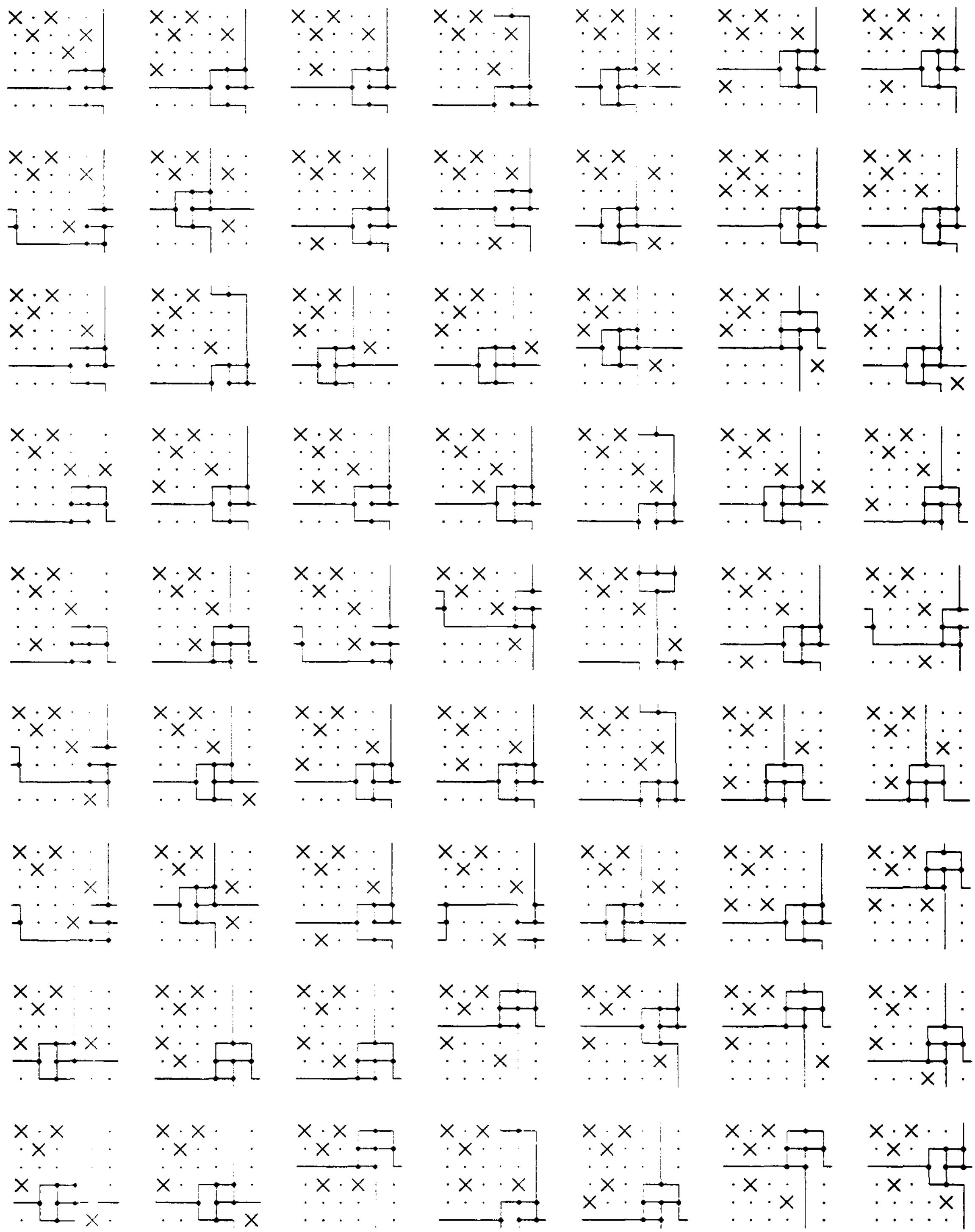


Figura A.32: $vd(C_6 \times C_6) \geq 6$.

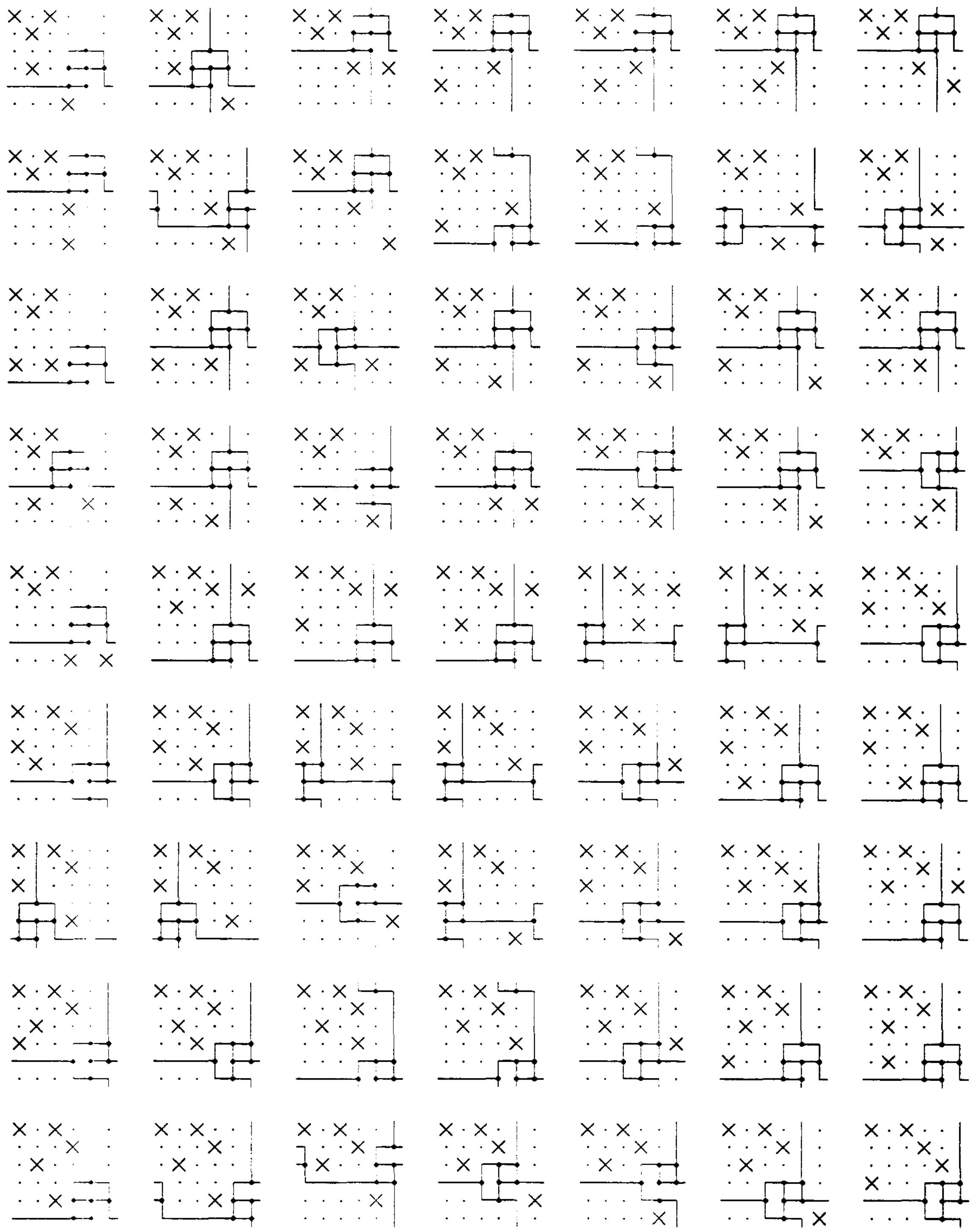


Figura A.33: $vd(C_6 \times C_6) \geq 6$.

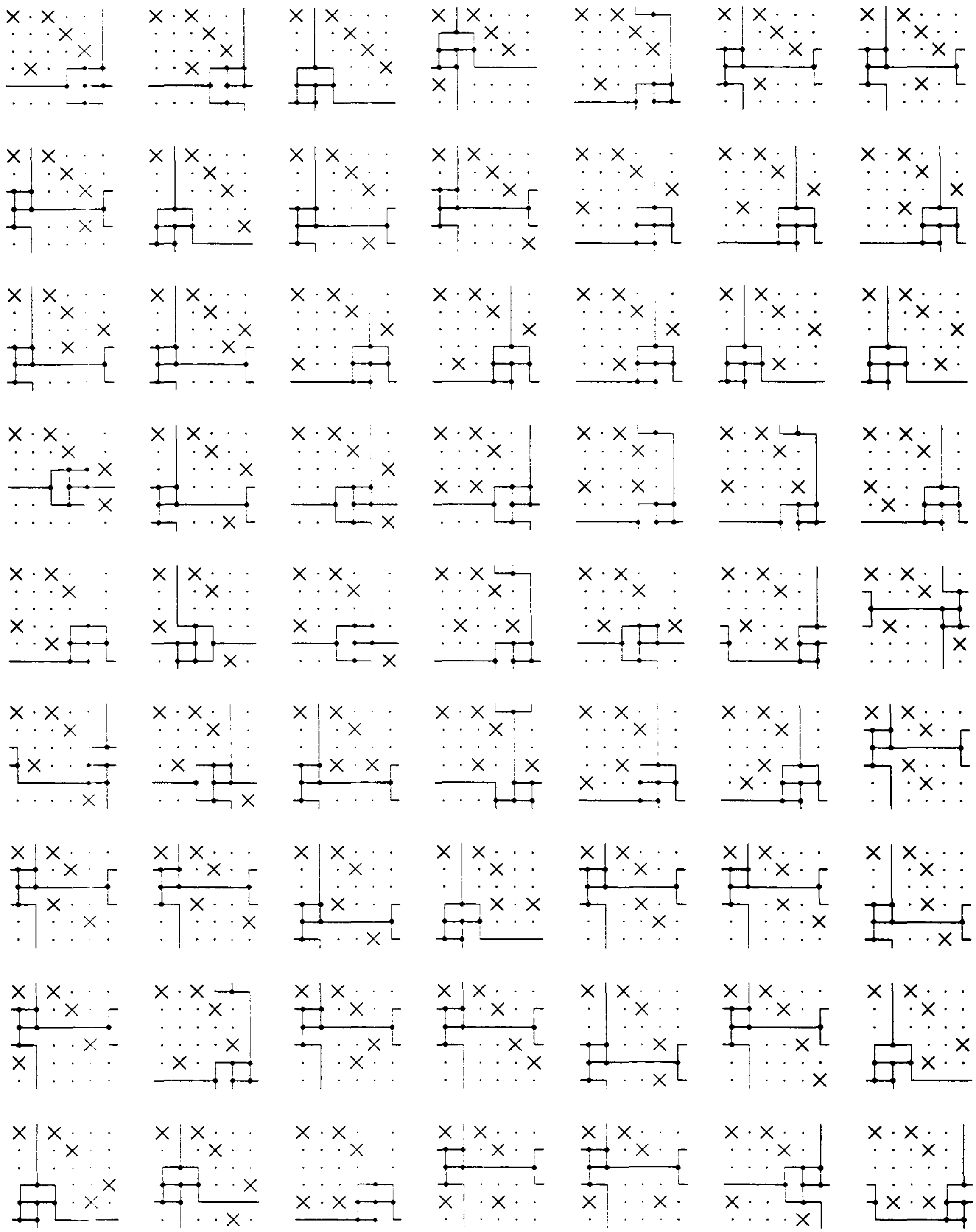


Figura A.34: $vd(C_6 \times C_6) \geq 6$.

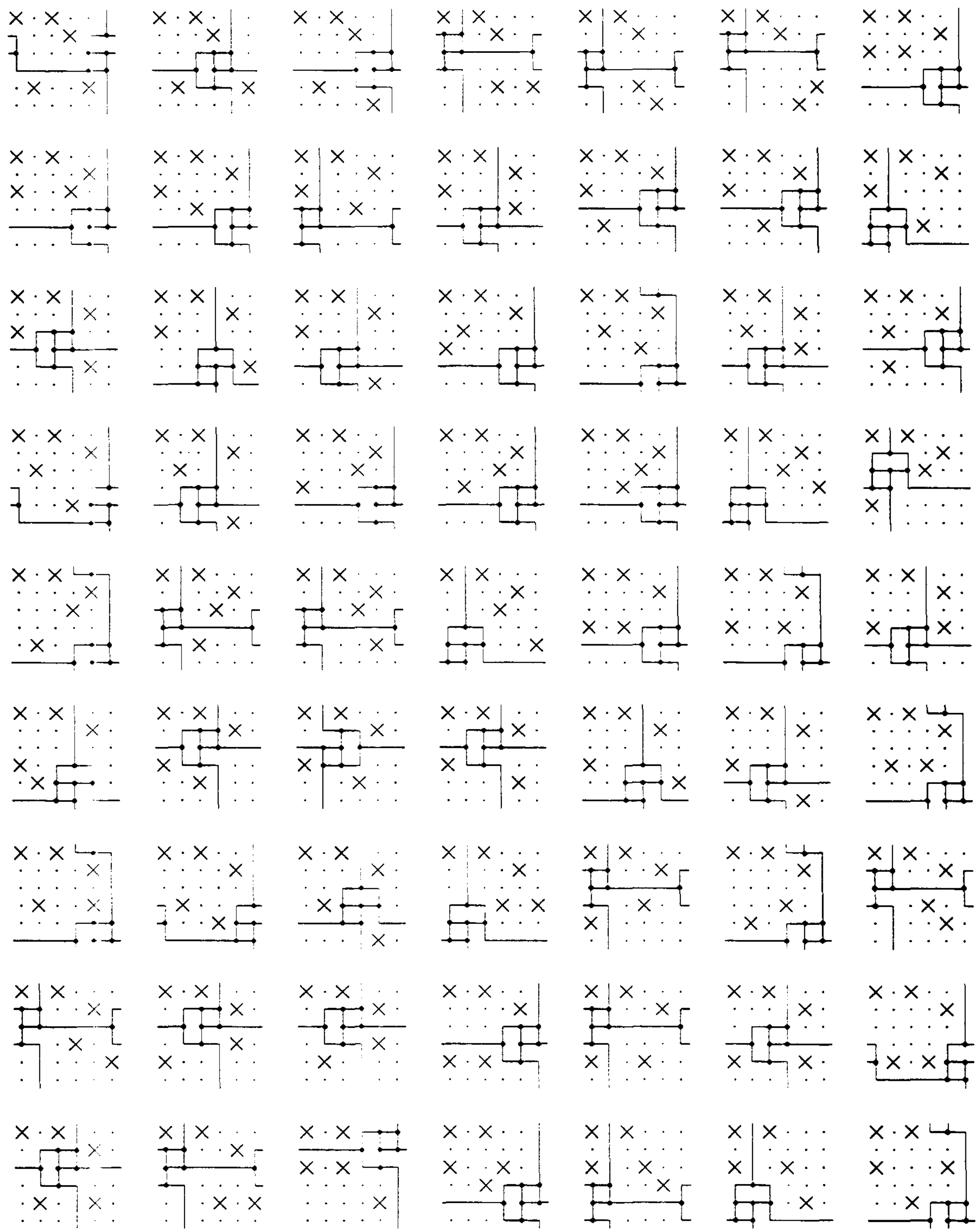


Figura A.35: $vd(C_6 \times C_6) \geq 6$.

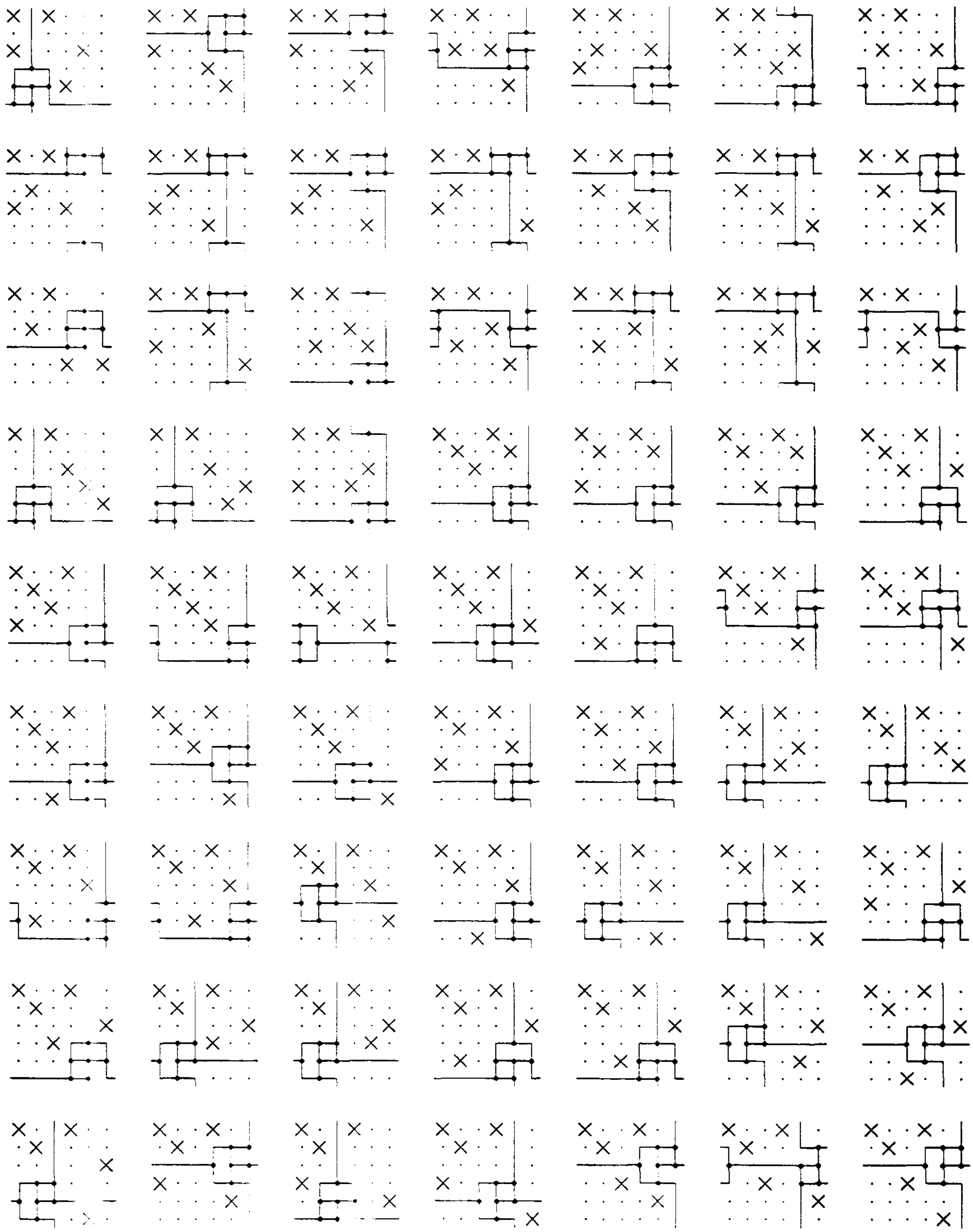


Figura A.36: $vd(C_6 \times C_6) \geq 6$.

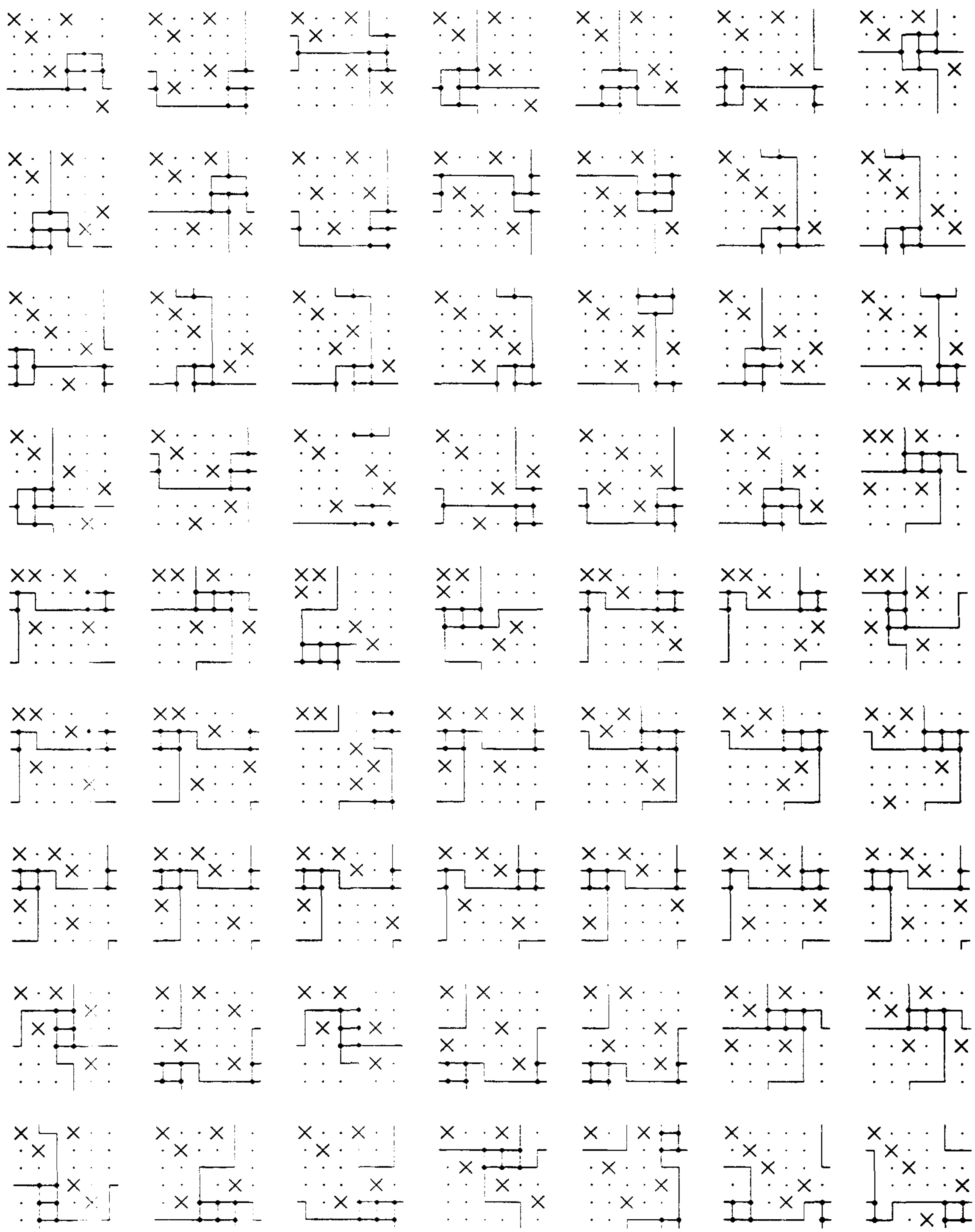


Figura A.37: $vd(C_6 \times C_6) \geq 6$.

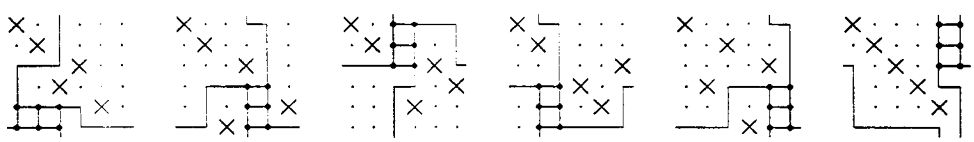


Figura A.38: $vd(C_6 \times C_6) \geq 6$.