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Affine Schur-Weyl duality and Weyl modules

Dualidade afim de Schur-Weyl e módulos de Weyl

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Resumo

Seja $\mathfrak{g} = \mathfrak{sl}_{n+1}$ a álgebra de Lie das matrizes $(n+1) \times (n+1)$ de traço nulo. A dualidade de Schur-Weyl é um resultado clássico que estabelece uma equivalência de categorias entre aquela dos módulos de dimensão finita do grupo simétrico S_ℓ e uma certa categoria de representações de $\mathfrak{g}$, se $\ell \leq n$. No artigo [17] de Chari e Pressley, uma versão análoga deste resultado foi estabelecida com a correspondente álgebra de Hecke afim $H_\ell(q^2)$ no lugar de S_ℓ e a álgebra afim quantizada $U_q(\hat{\mathfrak{g}}')$ no lugar de $\mathfrak{g}$, onde o parâmetro de quantização q não é uma raiz da unidade e $\hat{\mathfrak{g}}'$ denota a álgebra de Lie afim obtida da álgebra de laços de $\mathfrak{g}$ através da extensão central 1-dimensional. Em particular, os autores afirmaram que um esquema de demonstração análogo deveria prover uma outra versão da dualidade de Schur-Weyl, a qual denominamos aqui por dualidade afim de Schur-Weyl, com o grupo simétrico afim extendido $\tilde{\mathcal{S}}_\ell$ no lugar de $\mathcal{S}_\ell$ e a álgebra de Lie afim $\hat{\mathfrak{g}}'$ no lugar de $\mathfrak{g}$. Levar a cabo este esquema de maneira completa, obtendo assim todos os resultados análogos aos de [17] no contexto de $\tilde{\mathcal{S}}_\ell$-módulos e $\hat{\mathfrak{g}}'$-módulos, é o primeiro objetivo do presente trabalho. Indo além dos resultados de [17], iniciamos também uma investigação na direção de obter descrições dos $\tilde{\mathcal{S}}_\ell$-módulos correspondentes, por meio da dualidade afim de Schur-Weyl, aos chamados módulos de Weyl locais, objetos de alta relevância no estudo das representações de dimensão finita de $\hat{\mathfrak{g}}'$ que não haviam sido introduzidos na literatura ainda na época em que [17] foi publicado.

Palavras-chave: Álgebras de Kac-Moody, Grupo simétrico afim, Dualidade de Schur-Weyl, Módulos de Weyl.

Abstract

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ be the Lie algebra of traceless $(n+1) \times (n+1)$ matrices. The Schur-Weyl duality is a classical result that establishes an equivalence of categories between the category of finite-dimensional modules for the symmetric group $\mathcal{S}_\ell$ and a certain category of representations of $\mathfrak{g}$, if $\ell \leq n$. In Chari and Pressley's article [17], an analogous version of this result was established with the role of $\mathcal{S}_\ell$ being played by the affine Hecke algebra $\hat{H}_\ell(q^2)$, and the role of $\mathfrak{g}$ being played by the quantum affine algebra $U_q(\hat{\mathfrak{g}}')$, where the quantum parameter q is not a root of unity and $\hat{\mathfrak{g}}'$ is the affine Lie algebra obtained from the loop algebra of $\mathfrak{g}$ via the 1-dimensional central extension. In particular, the authors claimed that an analogous proof scheme should provide another version of the Schur-Weyl duality, which we here refer to as the affine Schur-Weyl duality, in which the role of $\mathcal{S}_\ell$ is played by the extended affine symmetric group $\tilde{\mathcal{S}}_\ell$ and the role of $\mathfrak{g}$ is played by the affine Lie algebra $\hat{\mathfrak{g}}'$. Carrying out this scheme thoroughly, thus obtaining in the setting of $\tilde{\mathcal{S}}_\ell$-modules and $\hat{\mathfrak{g}}'$-modules all the results analogous to those in [17], is the main objective of the present work. Going beyond the results of [17], we also set an investigation in the direction of describing the $\tilde{\mathcal{S}}_\ell$-modules corresponding, via the affine Schur-Weyl duality, to the so-called local Weyl modules, which are relevant objects in the setting of finite-dimensional representations of $\hat{\mathfrak{g}}'$ that were not yet introduced in the literature by the time that [17] was published.

Keywords: Kac-Moody algebras, Affine symmetric group, Schur-Weyl duality, Weyl modules.

Contents

Introduction

Given $\ell \in \mathbb{Z}_{\geq 0}$, the symmetric group $\mathcal{S}_\ell$ acts on the ℓ-th tensor power $\mathbb{V}^{\otimes \ell}$ by the usual left action of permuting factors, where $\mathbb{V} = \mathbb{C}^{n+1}$ is the $(n+1)$-dimensional left irreducible representation of the Lie group SL_{n+1} of $(n+1) \times (n+1)$ complex matrices with determinant 1. In fact, both the actions of SL_{n+1} and $\mathcal{S}_\ell$ on $\mathbb{V}^{\otimes \ell}$ commute with each other, and, given a right $\mathcal{S}_\ell$-module M, one can naturally associate with it the SL_{n+1}-module given by the tensor product over rings

$$\mathcal{F}_\ell(M) = M \otimes_{\mathbb{C}[\mathcal{S}_\ell]} \mathbb{V}^{\otimes \ell},$$

where $\mathbb{C}[\mathcal{S}_\ell]$ is the regular representation of $\mathcal{S}_\ell$ and the action of SL_{n+1} is the one induced from the action on $\mathbb{V}^{\otimes \ell}$. If $\ell \leq n$, it follows from the classical works of Weyl and Schur, extensively treated in the literature (see [31, 22, 9], for instance), that the irreducible summands of the SL_{n+1}-module $\mathbb{V}^{\otimes \ell}$ can be described in terms of $\mathcal{F}_\ell(\mathcal{S}(\xi))$, where $\mathcal{S}(\xi)$ is the irreducible $\mathcal{S}_\ell$-module generated by the Young symmetrizer associated to the partition ξ of the integer ℓ. In particular, the assignment $M \mapsto \mathcal{F}_\ell(M)$ establishes an equivalence between the categories of finite-dimensional $\mathcal{S}_\ell$-modules and the category of finite-dimensional SL_{n+1}-modules whose irreducible summands also occur in $\mathbb{V}^{\otimes \ell}$. Since representations of SL_{n+1} determine representations for its associated Lie algebra $\mathfrak{g} = \mathfrak{sl}_{n+1}$ of $(n+1) \times (n+1)$ traceless complex matrices, this equivalence of categories can also be interpreted from the point of view of the category of finite-dimensional $\mathcal{S}_\ell$-modules and the category of finite-dimensional $\mathfrak{g}$-modules whose simple summands occur in $\mathbb{V}^{\otimes \ell}$. In fact, this is one of the many consequences of a major classical result commonly known by the name of *Schur-Weyl duality*, which states that the image of $\mathbb{C}[\mathcal{S}_\ell]$ on $\mathrm{End}(\mathbb{V}^{\otimes \ell})$ and the image of $\mathbb{C}[GL_{n+1}]$ on $\mathrm{End}(\mathbb{V}^{\otimes \ell})$, both obtained as a result of their representations on $\mathbb{V}^{\otimes \ell}$, are each other's commutant, where GL_{n+1} is the Lie group of $(n+1) \times (n+1)$ invertible complex matrices and $\mathbb{C}[GL_{n+1}]$ is its group algebra.

Since the emergence of quantum groups in the mid 1980s independently introduced by the works of Drinfeld and Jimbo in the search for solutions to the so-called Yang-Baxter equations, many analogous versions of the aforementioned equivalence of categories have been developed in a wide range of different contexts. In [35], Jimbo proved a version of this equivalence in which $\mathfrak{g}$ is replaced by the quantum algebra $U_q(\mathfrak{g})$, and $\mathcal{S}_\ell$ is replaced by its Hecke algebra $H_\ell(q^2)$, where $q \in \mathbb{C}^\times$ is not a root of unity. In [21], Drinfeld obtained a version in which the role of $\mathfrak{g}$ is played by the Yangian $Y(\mathfrak{g})$ and the role of $\mathcal{S}_\ell$ is played by the degenerate affine Hecke algebra Λ_ℓ. In particular, in that same paper, Drinfeld conjectured that there should be another analogous version relating representations of the quantum affine algebra $U_q(\hat{\mathfrak{g}}')$ and the affine Hecke algebra $\hat{H}_\ell(q^2)$, where $\hat{\mathfrak{g}}'$ is the affine Lie algebra obtained from the loop algebra $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by the 1-dimensional central extension, and q is also not a root of unity. This latter equivalence of categories would only be proved years later in Chari and Pressley's article [17]. Over the years, with the introduction of quantum super algebras, super Yangians and quantum affine super algebras, all these equivalences have been further generalized to these contexts, as seen in the works of [46, 24, 40, 43, 38, 44], to name just a few. In particular, inspired by the original equivalence of categories obtained from the works of Weyl and Schur, it became a common practice in this more recent literature to refer to that very first equivalence as the *classical Schur-Weyl duality*, whilst the analogous versions that later unfolded are often accompanied by adjectives that identifies the concerning context, e.g. *quantum affine Schur-Weyl duality* in §1.5 of [38], *affine quantum super Schur-Weyl duality* in [24], *super Schur-Weyl duality* in §3.1 of [44].

In this present work, we study another Schur-Weyl duality type here referred to as the *affine Schur-Weyl duality*: the equivalence between the category of finite-dimensional $\hat{\mathfrak{g}}'$-modules whose simple components occur in $\mathbb{V}^{\otimes \ell}$ when regarded as $\mathfrak{g}$-modules and the category of finite-dimensional modules for the extended affine symmetric group $\tilde{\mathcal{S}}_\ell = \mathbb{Z}^\ell \rtimes \mathcal{S}_\ell$, where $\mathcal{S}_\ell$ acts on the additive group $\mathbb{Z}^\ell$ by permuting factors. This equivalence of categories first appeared also in Chari and Pressley's paper [17] and was claimed by the authors to follow if one adapts accordingly the proof scheme there originally developed in the context of finite-dimensional modules for $U_q(\hat{\mathfrak{g}}')$ and $\hat{H}_\ell(q^2)$. Indeed, assigning a right finite-dimensional $\tilde{\mathcal{S}}_\ell$-module M to the $\mathfrak{g}$-module $\mathcal{F}_\ell(M)$ as above, one derives an extension to an action of $\hat{\mathfrak{g}}'$ using the proper action of $\tilde{\mathcal{S}}_\ell$ in a suitable way. This assignment gives rise to the functor responsible for establishing the affine Schur-Weyl duality, if $\ell \leq n$. The main goal of this present work is to fully adapt the scheme of [17] and provide a precise characterization of the affine Schur-Weyl duality, as well as carry through other related results seen in that article to the setting of finite-dimensional $\hat{\mathfrak{g}}'$-modules and $\tilde{\mathcal{S}}_\ell$-modules. In particular, the analogous notion of Zelevisnky tensor products will be seen to behave suitably with respect to the functor of the affine Schur-Weyl duality, and this property will be instrumental in explicitly describing the $\tilde{\mathcal{S}}_\ell$-modules corresponding under this equivalence to the irreducible finite-dimensional $\hat{\mathfrak{g}}'$-modules.

One can carry out the study of finite-dimensional $\hat{\mathfrak{g}}'$-modules by concerning only with finite-dimensional $\tilde{\mathfrak{g}}$-modules, as the categories defined by these objects are equivalent (see [47]). Moreover, the category of finite-dimensional $\tilde{\mathfrak{g}}$-modules has been extensively discussed in a range of papers [16, 12, 50, 10], for instance. In particular, the simple objects were first classified in [10, 16] in terms of tensor products of evaluation representations along different scalars in $\mathbb{C}^\times$. Equivalently, this classification can also be obtained using the notions of ℓ-weights and highest-ℓ-weight modules, which can be thought of as "loop analogues" of the classical notions of weights and highest-weight modules. These concepts were inspired by Chari and Pressley's work [18] on finite-dimensional representations for the quantum affine algebra $U_q(\hat{\mathfrak{g}}')$. In particular, from the study of the classical ($q \to 1$) limits of simple $U_q(\hat{\mathfrak{g}}')$-modules, Chari and Pressley also introduced in [19] the notion of local Weyl modules, which are the type of finite-dimensional $\tilde{\mathfrak{g}}$-modules characterized by the property that any finite-dimensional highest-ℓ-weight module is a quotient of the corresponding local Weyl module of the same ℓ-weight. Since the publication of [19], the notion of local Weyl modules has also been generalized to a range of related contexts [5, 7, 28, 13, 8, 27], and further connections with the theory of Demazure modules and fusion products have also been established in [14, 26, 48].

By the time that the affine Schur-Weyl duality was first announced in [17], the notion of local Weyl modules was not yet introduced in the literature. In view of the significance of these modules, describing the $\tilde{\mathcal{S}}_\ell$-modules corresponding via the affine Schur-Weyl duality to the local Weyl modules (in which the central element acts trivially) is a problem of particular interest. In this work, we also set the very first steps in the direction of describing these modules. From the behavior of Zelevisnky tensor products with respect to the functor of the affine Schur-Weyl duality, it will be seen that it is sufficient to only describe the modules corresponding to local Weyl modules whose ℓ-weights have a unique associated root. By exploring the simplest examples of local Weyl modules at hand, it will be seen that these can be described as quotients of the group algebra $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ of $\tilde{\mathcal{S}}_\ell$ by ideals generated by symmetric polynomials. In particular, this approach will be generalized into a proof that describes the $\tilde{\mathcal{S}}_\ell$-modules corresponding to the local Weyl modules whose generator is, in particular, a vector of highest-weight $\ell\omega_1$, where ω_1 is the highest weight of $\mathbb{V}$. Furthermore, exploring the consequences of this proof, we also reobtain, in a independent manner, some results of [14, 26] regarding the dimension of these type of local Weyl modules.

This work is divided in three main chapters. In the first chapter, we establish some notational conventions and briefly give an overview of standard notions and results that are used to construct Chapters 2 and 3. In Chapter 2, we focus on the theory of finite-dimensional $\tilde{\mathfrak{g}}$-modules, and give a classification of finite-dimensional irreducible $\tilde{\mathfrak{g}}$-modules in terms of highest-ℓ-weight modules. Using the notion of highest ℓ-weight, we also characterize the local Weyl modules, and briefly study the structure of these modules. In the last chapter, we finally study both the classical and affine Schur-Weyl duality, and introduce the notion of Zelevinksy tensor product for $\tilde{\mathcal{S}}_\ell$. Using this notion, we describe all the $\tilde{\mathcal{S}}_\ell$-modules corresponding under the affine Schur-Weyl duality to the simple finite-dimensional $\hat{\mathfrak{g}}'$-modules, as well as set an investigation in the direction of also describing the modules corresponding to the local Weyl modules.

1 Preliminaries

The sole purpose of this chapter is to establish the notation and recall standard results that will be utilized throughout the whole text. The main references here used to construct the upcoming sections are [31, 34, 37, 47, 41]. Others will be specifically cited. The results whose proofs can be found in any of the references here employed are purposely omitted and only referenced.

1.1 Simple finite-dimensional complex Lie algebras

In this section we recall the basic structure of simple finite-dimensional Lie algebras with respect to their root systems.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank n. Fix a Cartan subalgebra $\mathfrak{h}$, and let Φ be the corresponding root system with associated root space decomposition (or Cartan decomposition):

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right), \qquad \text{where } \mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

The simple Lie algebra $\mathfrak{g}$ possesses a nondegenerate invariant symmetric bilinear form $\kappa : \mathfrak{g}\times\mathfrak{g} \to \mathbb{C}$, uniquely defined up to scalar multiple, whose restriction to $\mathfrak{h}$ is nondegenerate. Such bilinear form induces an isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ by assigning $\lambda \mapsto t_\lambda$, where $t_\lambda \in \mathfrak{h}$ is such that $(t_\lambda, h) = \lambda(h)$ for all $h \in \mathfrak{h}$, whence a nondegenerate symmetric bilinear form $(. , .) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ is also obtained by setting $(\alpha, \beta) = (t_\alpha, t_\beta)$ for $\alpha, \beta \in \mathfrak{h}^*$.

Set $I = \{1, \cdots , n\}$, and let $\{\alpha_i\}_{i\in I}$ be a set of **simple roots** in Φ with associated Cartan matrix $C = (c_{ij})_{i,j\,\in I}$. The Weyl group $\mathscr{W}$ is generated by the simple reflections $r_i : \mathfrak{h}^* \to \mathfrak{h}^*$, where

$$r_i(\lambda) = \lambda - 2\frac{(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}\alpha_i,\ i \in I.$$

Let $Q = \bigoplus_{i=1}^n \mathbb{Z}\,\alpha_i$ be the **root lattice** of $\mathfrak{g}$, and let $Q^+ = \bigoplus_{i=1}^n \mathbb{N}\,\alpha_i$ be the **associated monoid**. The set of **positive roots** Φ^+ is the intersection $\Phi \cap Q^+$. The lattice P of **integral weights** is the set of all $\lambda \in \mathfrak{h}^*$ such that $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi$. We denote by $P^+ \subseteq P$ the subset of integral weights λ such that $\lambda(h_i) \geq 0$ for all $i \in I$, and refer to its elements as **dominant integral weights**. The set P^+ is generated by the **fundamental weights** $\omega_j \in \mathfrak{h}^*$ uniquely defined by $\omega_j(h_i) = \delta_{ij}$ for all $i, j \in I$. In the set P, we consider the usual **partial order**

$$\lambda \leq \mu \iff \mu - \lambda \in Q^+,\ \lambda, \mu \in P. \tag{1.1.1}$$

In particular, Φ equipped with this same partial order has a unique maximal element θ.

For $\beta \in \Phi^+$, fix nonzero elements $x_\beta^\pm \in \mathfrak{g}_{\pm\beta}, h_\beta \in \mathfrak{h}$ satisfying

$$[x_\beta^+, x_\beta^-] = h_\beta,\ [h_\beta, x_\beta^\pm] = \pm 2x_\beta^\pm, \tag{1.1.2}$$

so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a **triangular decomposition** of $\mathfrak{g}$, where

$$\mathfrak{n}^\pm = \bigoplus_{\alpha\in\Phi^+} \mathbb{C}x_\alpha^\pm. \tag{1.1.3}$$

When $\beta = \alpha_i$, the elements $x_{\alpha_i}^\pm, h_{\alpha_i}$ are commonly referred to as the **Chevalley generators** of $\mathfrak{g}$, and we simplify notation by setting $x_i^\pm = x_{\alpha_i}^\pm, h_i = h_{\alpha_i}$. These generators give rise to the **Chevalley involution**, an involution $\omega : \mathfrak{g} \to \mathfrak{g}$ uniquely defined by

$$\omega(x_i^\pm) = -x_i^\mp, \omega(h_i) = -h_i \quad \text{for all} \quad i \in I.$$

The set $\{x_\beta^\pm : \beta \in \Phi^+\} \cup \{h_i : i \in I\}$ is a basis of $\mathfrak{g}$, with $\{x_\beta^\pm : \beta \in \Phi^+\}$ being a basis for $\mathfrak{n}^\pm$, and $\{h_i : i \in I\}$ being a basis for $\mathfrak{h}$. In particular, if $\beta \in \Phi^+$ is such that $\beta = \sum_i r_i\alpha_i$ for some $r_i \in \mathbb{Z}_{\geq 0}$, then

$$h_\beta = \sum_i \frac{r_j(\alpha_i, \alpha_i)}{(\beta, \beta)} h_i.$$

1.2 Affine Kac-Moody algebras of nontwisted type and loop algebras

In this section we recall an explicit realization of the extended nontwisted affine Kac-Moody algebra associated with a simple finite-dimensional Lie algebra.

Fix a simple finite-dimensional complex Lie algebra $\mathfrak{g}$ as in Section 1.1, and let $\mathbb{C}[t, t^{-1}]$ be the **Laurent polynomial algebra**. The **loop algebra** over $\mathfrak{g}$ is the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with the Lie bracket given by

$$[x \otimes p, y \otimes q] = [x, y] \otimes pq, \quad \text{where } x, y \in \mathfrak{g}, p, q \in \mathbb{C}[t, t^{-1}]. \tag{1.2.1}$$

The triangular decomposition of $\mathfrak{g}$ also induces a triangular decomposition of its loop algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$, where $\tilde{\mathfrak{n}}^\pm = \mathfrak{n}^\pm \otimes \mathbb{C}[t, t^{-1}]$, and $\tilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$.

Let $\frac{d}{dt} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ and $\mathrm{Res} : \mathbb{C}[t, t^{-1}] \to \mathbb{C}$ be the linear maps uniquely defined by

$$\mathrm{Res}(t^r) = \delta_{1,-r} \quad \text{and} \quad \frac{d}{dt}\Big(x \otimes p\Big) = x \otimes \frac{dp}{dt}, \tag{1.2.2}$$

where dp/dt is the formal derivative of polynomials. Fix a nondegenerate invariant symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, and define the $\mathbb{C}[t, t^{-1}]$-valued bilinear map $\kappa_t : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathbb{C}[t, t^{-1}]$:

$$\kappa_t(x \otimes p, y \otimes q) = \kappa(x, y)pq. \tag{1.2.3}$$

The maps (1.2.2),(1.2.3) give rise to the bilinear form $\nu : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \mathbb{C}$ defined by

$$\nu(f,g) = \mathrm{Res}\Bigg(\kappa_t\Bigg(\frac{d}{dt}\Big(f\Big), g\Bigg)\Bigg), \quad \text{for} \quad f, g \in \tilde{\mathfrak{g}}.$$

The **nontwisted affine Kac-Moody algebra of** $\mathfrak{g}$, or the **affine Lie algebra** of $\mathfrak{g}$, or the **affinization of** $\mathfrak{g}$, is the 1-dimensional extension $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{C}c$, where c is an additional formal central element, and the Lie bracket of $\hat{\mathfrak{g}}'$ is given by

$$[a + \lambda c, b + \mu c] = [a, b] + \nu(a, b)c, \quad \text{for} \quad a, b \in \tilde{\mathfrak{g}}, \lambda, \mu \in \mathbb{C}. \tag{1.2.4}$$

The **extended nontwisted affine Kac-Moody algebra** of $\mathfrak{g}$, or the **extended affine Lie algebra** of $\mathfrak{g}$, or the **extended affinization** of $\mathfrak{g}$, is the Lie algebra $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \mathbb{C}d$ obtained by adjoining a derivation d which acts on $\hat{\mathfrak{g}}'$ by

$$[d, x \otimes p] = x \otimes t\frac{dp}{dt} \;, \quad [d, c] = 0, \tag{1.2.5}$$

and extending linearly.

Remark. In the literature, it is more common to refer to the Lie algebra $\hat{\mathfrak{g}}$ as the affine Lie algebra associated with $\mathfrak{g}$. However, we chose not to follow this practice. In fact, we are following the nomenclature utilized in the reference [17] central to our work in Chapter 3.

Notice that $\hat{\mathfrak{g}}' = [\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$. The centralizer of d in $\hat{\mathfrak{g}}$ is $(\mathfrak{g} \otimes 1) \oplus \mathbb{C}c \oplus \mathbb{C}d$ and the 1-dimensional center of $\hat{\mathfrak{g}}$ is $\mathbb{C}c$. Also, since $\mathfrak{g} \otimes 1 \cong \mathfrak{g}$, we abuse notation by setting $\mathfrak{g} = \mathfrak{g} \otimes 1$, and maintain the notation $x_i^{\pm}, h_i$ for the elements $x_i^{\pm} \otimes 1, h_i \otimes 1, i \in I$, respectively. The affine analogue of the Cartan subalgebra in $\hat{\mathfrak{g}}$ is the $(n+2)$-dimensional subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$. We extend any functional $\lambda \in \mathfrak{h}^*$ to $\hat{\mathfrak{h}}^*$ by setting $\lambda(c) = \lambda(d) = 0$, and denote by $\delta \in \hat{\mathfrak{h}}^*$ the functional dual to the element d with respect to the direct sum of $\hat{\mathfrak{h}}$. The root system $\hat{\Phi}$ derived from the root space decomposition with respect to $\hat{\mathfrak{h}}$ is the set

$$\hat{\Phi} = \{\alpha + k\delta : k \in \mathbb{Z}, \alpha \in \Phi\} \cup \{k\delta : k \in \mathbb{Z}, k \neq 0\}, \tag{1.2.6}$$

and the root space decomposition itself is explicitly given by

$$\hat{\mathfrak{g}} = \hat{\mathfrak{h}} \oplus \Bigg(\bigoplus_{\beta \in \hat{\Phi}} \hat{\mathfrak{g}}_\beta\Bigg), \quad \text{where} \quad \hat{\mathfrak{g}}_{\alpha + k\delta} = \mathfrak{g}_\alpha \otimes t^k, \;\; \hat{\mathfrak{g}}_{k\delta} = \mathfrak{h} \otimes t^k, k \in \mathbb{Z}. \tag{1.2.7}$$

Using the Chevalley involution $\omega : \mathfrak{g} \to \mathfrak{g}$, let $E_0 \in \mathfrak{g}_\theta$ be such that $\kappa(E_0, \omega(E_0)) = -2/(\theta, \theta)$, and set $F_0 = -\omega(E_0)$. Now introduce the following elements in $\hat{\mathfrak{g}}$

$$x_0^+ = F_0 \otimes t, \;\; x_0^- = E_0 \otimes t^{-1}, \;\; h_0 = [x_0^+, x_0^-], \tag{1.2.8}$$

and notice that

$$h_0 = 2c/(\theta,\theta) - [E_0, F_0]. \tag{1.2.9}$$

Let $\hat{I} = \{0, 1, \cdots, n\}$. The elements $x_i^\pm, h_i, i \in \hat{I}$, are also referred to as **the Chevalley generators**, and regarding the lacing of these elements with polynomials in $\mathbb{C}[t, t^{-1}]$, we simplify notation by setting $x_{i,r}^\pm = x_i^\pm \otimes t^r$ and $h_{i,r} = h_i \otimes t^r$ for all $i \in \hat{I}, r \in \mathbb{Z}$, as well as $x_{\beta,r}^\pm = x_\beta^\pm \otimes t^r, \beta \in \Phi^+$.

Set $\alpha_0 = \delta - \theta$. Then, the matrix $\hat{C} = (c_{ij})_{i,\,j\,\in\,\hat{I}}$ labelled by $\hat{I}$ and with entries

$$c_{ij} = \alpha_j(h_i) \quad \text{for all } i, j \in \hat{I} \tag{1.2.10}$$

is said to be the **extended affine Cartan matrix** obtained from the Cartan matrix $C = (c_{i,j})_{i,j \in I}$ of $\mathfrak{g}$ by adding the 0-th row and 0-th column. The matrix $\hat{C}$ is a generalized **Cartan matrix of affine type**, i.e., $\hat{C}$ is a matrix of rank n with positive proper principal minors whose entries are such that

$$c_{ii} = 2,\ c_{ij} \in \mathbb{Z}_{\leq 0} \ \text{ if } \ i \neq j,\ c_{ij} = 0 \text{ if and only if } c_{ji} = 0,\ i, j \in \hat{I}. \tag{1.2.11}$$

In particular, there exists a unique vector $u^t = (c_0, \cdots, c_n)$ whose coordinates are positive integers with no common prime factors such that $u^t\hat{C} = 0$. Setting $h_0 = \sum_{i=0}^n c_i h_i$, it follows that $\alpha(h_0) = 0$ for all $\alpha \in \Phi$, so h_0 lies in the center of $\hat{\mathfrak{g}}'$, i.e., c is a multiple of h. In this sense, we normalize c by setting

$$c = \sum_{i=0}^{n} c_i h_i. \tag{1.2.12}$$

1.3 Representations and weight modules

This section has the sole purpose of establishing preliminaries pertinent to the context of representation theory of Lie algebras.

We first give a brief overview of the representation theory of $\mathfrak{g}$ as in Section 1.1. Let V be a nonzero $\mathfrak{g}$-module. Given $\lambda \in \mathfrak{h}^*$, the set

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \ \text{ for all } \ h \in \mathfrak{h}\}$$

is said to be the **weight space** of V of weight λ, and its nonzero elements (if any) are referred to as **weight vectors** of weight λ. If $V_\lambda \neq \{0\}$, λ is said to be a **weight** of V, and $\dim V_\lambda$ is said to be the **multiplicity** of λ. We denote by $\mathrm{wt}(V)$ the set of weights of V. Weight spaces corresponding to different linear functionals are linearly independent, and if V decomposes into the direct sum

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \tag{1.3.1}$$

then V is said to be a **weight module**. Submodules of weight modules are also weight modules, and all finite-dimensional $\mathfrak{g}$-modules are known to be weight modules.

Regarding the weights of a representation V, of particular interest are the weight vectors $v \in V_\lambda$ such that $\mathfrak{n}^+ v = 0$. These vectors are usually known as **highest-weight vectors** of weight λ, and if $V = U(\mathfrak{g})v$, then V is said to be a **highest-weight module** of highest weight λ. In this case, V is an indecomposable module such that its weights satisfy $\mu \leq \lambda$ for all $\mu \in \mathrm{wt}(V)$. Moreover, V possesses a unique maximal proper submodule, with a corresponding unique simple quotient. For every $\lambda \in \mathfrak{h}^*$, there exists a unique **simple highest-weight module** of weight λ up to isomorphism, which we denote by $V(\lambda)$. Such modules $V(\lambda)$ are known to be finite-dimensional if and only if $\lambda \in P^+$. In this way, since every finite-dimensional $\mathfrak{g}$-module V is written as a direct sum of simple modules by **Weyl's Theorem on complete reducibility**, then $V \cong \bigoplus_i V(\lambda_i)$, where the sum is finite and each $\lambda_i \in P^+$. Regarding this decomposition, we adopt the notation

$$[V : V(\lambda)] \in \mathbb{Z}_{\geq 0}$$

to sate that the amount of times V admits a simple summand isomorphic to $V(\lambda), \lambda \in P^+$, is equal to $[V : V(\lambda)]$.

We now recall a few facts on representation theory of $\mathfrak{sl}_2$. Let us denote by x^+, x^-, h the generators of $\mathfrak{sl}_2$ such that $[x^+, x^-] = h, [h, x^\pm] = \pm 2x^\pm$. In this case, the Cartan subalgebra is spanned by h, so the linear functionals of $(\mathbb{C}h)^*$ are identified with elements of $\mathbb{C}$. In this way, the simple highest-weight modules for $\mathfrak{sl}_2$ are more conveniently denoted by $V(m)$, where $m \in \mathbb{C}$.

Theorem 1.3.1. *([34], §7.2) Let $V(m)$ be a simple finite-dimensional highest-weight module for $\mathfrak{sl}_2$. Then, m is a nonnegative integer,* $\dim V(m) = m + 1, (x^-)^{m+1}v = 0$, *and the weights of $V(m)$ are exactly $\{-m, -(m-2), \cdots, m-2, m\}$, with each weight space being 1-dimensional.* □

Corollary 1.3.2. *([34], §7.2) Let V be a finite-dimensional $\mathfrak{sl}_2$-module. Then, the weights of h on V are all integers, and each occurs along with its negative with the same multiplicity.* □

Since the Lie algebras $\tilde{\mathfrak{g}}, \hat{\mathfrak{g}}'$ and $\hat{\mathfrak{g}}$ introduced in Section 1.2 have $\mathfrak{h}$ as a subalgebra, the concepts of weight modules, weight space, weight vectors and weights of a representation are naturally carried over to the context of representations of these Lie algebras.

Regarding the weights of a representation V of any of the Lie algebras here discussed, if they all lie in P, a range of results are available, as seen in the abstract theory of weights treated in [52, 34]. We finish this section with a few useful result on this theory. Given

$\lambda \in \mathfrak{h}^*$, we introduce the set

$$\mathrm{wt}(\lambda) = \{\sigma(\mu) : \sigma \in \mathscr{W}, \mu \in P^+, \mu \leq \lambda\}. \tag{1.3.2}$$

Theorem 1.3.3. *Let $\lambda \in P^+$.*

(i) $w(\lambda) \leq \lambda$ for all $w \in \mathscr{W}$. In particular, $\mu \leq \lambda$ for all $\mu \in \mathrm{wt}(\lambda)$.

(ii) The set $\mathrm{wt}(\lambda)$ is finite.

(iii) Any $\mu \in P$ satisfies $\mu = \sigma(\nu)$ for a unique $\nu \in P^+$. □

Corollary 1.3.4. *Let V be a weight module such that $\mu \in P$ for all $\mu \in \mathrm{wt}(V)$. If $\lambda \in P^+$ is such that $\mu \leq \lambda$ for all $\mu \in \mathrm{wt}(V)$, then $\mathrm{wt}(V) \subseteq \mathrm{wt}(\lambda)$, and $\mathrm{wt}(\lambda)$ is finite.* □

1.4 Universal enveloping algebras

In this section, we give a brief overview on the universal enveloping algebra of a given Lie algebra. In particular, we specialize some results and notations for the universal enveloping algebra of the simple finite-dimensional Lie algebra $\mathfrak{g}$ as in Section 1.1, as well as for its associated loop algebra $\tilde{\mathfrak{g}}$. Any given unital associative algebra A here considered is supposed to have a Lie algebra structure given by the usual commutator $[a, b] = ab - ba$ for any elements $a, b \in A$, and we utilize the notation $[A]$ to make clear that A is being regarded as a Lie algebra.

Let L be a Lie algebra. Recall that the **universal enveloping algebra** of L, denoted by $U(L)$, is the unital associative algebra together with an homomorphism of Lie algebras $i : L \to [U(L)]$ satisfying the following universal property: given any homomorphism of Lie algebras $f : L \to [A]$, where A is an unital associative algebra, there exists a unique homomorphism of unital associative algebras $\phi : U(L) \to A$ such that $\phi \circ i = f$. The map $i : L \to [U(L)]$, usually referred to as the **canonical map** of $U(L)$, is known to be injective, and it is usual to abuse notation by regarding $L \subseteq U(L)$.

It follows from the universal property of $U(L)$ and the canonical map $i : L \to [U(L)]$ that any L-module gives rise to a $U(L)$-module and vice-versa. When studying representations of $U(L)$ on tensor products, it will be useful to consider the k-th **comultiplication map** $\Delta^k : U(L) \to U(L)^{\otimes k}$ which is defined as the unique homomorphism of algebras extending the Lie algebra homomorphism $L \to U(L)^{\otimes k}$ given by

$$x \mapsto \sum_{j=1}^{k} \left(1^{\otimes(k-1)} \otimes x \otimes 1^{\otimes(k-j)}\right), \ x \in L. \tag{1.4.1}$$

We denote by 1 the identity element of $U(L)$ and set $x^0 = 1$ for any $x \in U(L)$. The **Poincaré–Birkhoff–Witt Theorem** below, shortly known as the **PBW Theorem**, gives a basis for $U(L)$.

Theorem 1.4.1. *([34], §17.2) Let L be a Lie algebra with an ordered basis $(x_i)_{i\in S}$ parametrized by a set S. Then, the set of elements*

$$(x_{i_1})^{r_1}(x_{i_2})^{r_2}\cdots(x_{i_m})^{r_m},$$

where $m \in \mathbb{Z}_{\geq 0}, i_j \in S$ are such that $i_1 < i_2 < \cdots < i_m$, and $r_j \in \mathbb{Z}_{\geq 0}$ for all $1 \leq j \leq m$, form a basis of $U(L)$. □

In case $L = \mathfrak{g}$ as in Section 1.1, we choose the ordering on the set of basis elements of $\mathfrak{g}$ to reflect the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ as follows. Let $\Phi^+ = \{\beta_1, \cdots, \beta_m\}$. Then, we define the ordering

$$\begin{aligned}
&x^{\pm}_{\beta_i} < x^{\pm}_{\beta_j} \ \text{ for } \ i < j,\ 1 \leq i, j \leq m \\
&h_k < h_\ell \ \text{ for } \ k < \ell,\ 1 \leq k, \ell \leq n, \\
&x^-_\beta < h_j < x^+_\alpha \ \text{ for all } \ \beta, \alpha \in \Phi^+, 1 \leq j \leq n.
\end{aligned}$$

Therefore, an arbitrary basis element of $U(\mathfrak{g})$ is given by:

$$\Big(\prod_{j=1}^{m}(x^-_{\beta_j})^{r^-_j}\Big)\Big(\prod_{i=1}^{n} h_i^{s_i}\Big)\Big(\prod_{j=1}^{m}(x^+_{\beta_j})^{r^+_j}\Big), \tag{1.4.2}$$

where $r^{\pm}_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq m$, $s_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq n$. In particular, this ordering when restricted to the basis of $\mathfrak{n}^{\pm}, \mathfrak{h}$ also assures that the elements of the form $\Big(\prod_{j=1}^{m}(x^{\pm}_{\beta_j})^{r^{\pm}_j}\Big)$ as above compose a basis of $U(\mathfrak{n}^{\pm})$, and similarly for $\Big(\prod_{i=1}^{n} h_i^{s_i}\Big)$ and $U(\mathfrak{h})$. Thus, regarding $U(\mathfrak{n}^{\pm}), U(\mathfrak{h})$ as subalgebras of $U(\mathfrak{g})$ via the universal property, it follows that the linear map $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+) \to U(\mathfrak{g})$ such that

$$y \otimes h \otimes x \mapsto yhx \in U(\mathfrak{g}) \ \text{ for all } \ y \in U(\mathfrak{n}^-), h \in U(\mathfrak{h}), x \in U(\mathfrak{n}^+)$$

is an isomoprhism of vector spaces.

From the triangular decomposition of the loop algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$, the set $\{x^{\pm}_{\beta,r} : r \in \mathbb{Z}, \beta \in \Phi^+\} \cup \{h_{i,r} : i \in I, r \in \mathbb{Z}\}$ is a basis of $\tilde{\mathfrak{g}}$ which can be ordered using the lexicographic order induced from the order of the basis of $\mathfrak{g}$ and the usual order of $\mathbb{Z}$. Therefore, one similarly obtains the isomorphism:

$$U(\tilde{\mathfrak{n}}^-) \otimes U(\tilde{\mathfrak{h}}) \otimes U(\tilde{\mathfrak{n}}^+) \cong U(\tilde{\mathfrak{g}}). \tag{1.4.3}$$

In $U(\tilde{\mathfrak{g}})$, it will be convenient to consider the elements $\Lambda_{\beta,r}$, $\beta \in \Phi^+, r \in \mathbb{Z}$, of $U(\tilde{\mathfrak{g}})$

defined by the following equality of power series in the variable u:

$$\Lambda_\beta^\pm(u) = \sum_{r=0}^{\infty} \Lambda_{\beta,\pm r}\, u^r = \exp\Bigg(- \sum_{s=1}^{\infty} \frac{h_{\beta,\pm s}}{s}\, u^s \Bigg). \tag{1.4.4}$$

If $\beta = \alpha_i,\ i \in I$, we simplify notation by setting $\Lambda_i^\pm(u) = \Lambda_{\alpha_i}^\pm(u)$ and $\Lambda_{i,r} = \Lambda_{\alpha_i,r}, r \in \mathbb{Z}$. It is clear that $\Lambda_{\beta,0} = 1$, and the remaining terms $\Lambda_{\beta,\pm r}$ for $r > 0$ are more explicitly written as follows. Given $\mathbf{s} = (s_1, \cdots, s_\ell) \in \mathbb{Z}_{>0}^\ell$, set $|\mathbf{s}| = \sum_j s_j$. Using this terminology, we have

$$\Lambda_{\beta,\pm r} = \sum_{\ell=1}^{r} \frac{1}{\ell!} \sum_{\substack{\mathbf{s} \in \mathbb{Z}_{>0}^\ell \\ |\mathbf{s}| = r}} \Bigg(\prod_{j=1}^{\ell} \frac{(-1)^\ell h_{\beta,\pm s_j}}{s_j} \Bigg). \tag{1.4.5}$$

In particular, in [32] it is proven that the comultiplication map $\Delta^k : U(\tilde{\mathfrak{g}}) \to U(\tilde{\mathfrak{g}})^{\otimes k}$ maps each $\Lambda_{\beta,\pm r}$ to

$$\Delta^k(\Lambda_{\beta,\pm r}) = \sum_{\substack{m_1, \cdots, m_k \geq 0 \\ m_1 + \cdots + m_k = r}} \Lambda_{\beta,\pm m_1} \otimes \cdots \otimes \Lambda_{\beta,\pm m_k}. \tag{1.4.6}$$

For $x \in U(\tilde{\mathfrak{g}}), i \in I,\ s, m \in \mathbb{Z}, m \geq 0$, we also introduce the notation

$$x^{(m)} = \frac{x^m}{m!}, \tag{1.4.7}$$

as well as the following power series in u

$$X_{i,s,\pm}^-(u) = \sum_{r=1}^{\infty} x_{i,\pm(r+s)}^-\, u^r \quad \text{and} \quad \Big(X_{i,s,\pm}^-(u)\Big)^{(m)} = \frac{1}{m!}\Big(X_{i,s,\pm}^-(u)\Big)^m. \tag{1.4.8}$$

We finish this section with the following result that will be useful in Chapters 2 and 3. A proof can be found in [32, 19].

Proposition 1.4.2. *For every $\beta \in \Phi^+$ and $s \in \mathbb{Z}$, we have*

$$(x_{\beta,\mp s}^+)^{(l)} (x_{\beta,\pm(s+1)}^-)^{(m)} = (-1)^l \Bigg(\Big(X_{\beta,s,\pm}^-(u) \Big)^{(m-l)} \Lambda_\beta^\pm(u) \Bigg)_m \quad \textit{mod}\ U(\tilde{\mathfrak{g}})U(\tilde{\mathfrak{n}}^+)^0,$$

where $(X_{i,s}^-(u))^{(m-l)}$ is understood to be zero if $m < l$, the subindex m on the right-hand side designates the coefficient of u^m in the given power series and $U(\tilde{\mathfrak{n}}^+)^0$ denotes the augmentation ideal of $U(\tilde{\mathfrak{n}}^+)$. □

1.5 Free Lie algebras and presentations by generators and relations

In this section we recall the concept of presentations of Lie algebras by generators and relations, and give such a presentation for the simple finite-dimensional Lie algebra $\mathfrak{g}$ of Section 1.1, its loop algebra $\tilde{\mathfrak{g}}$ and affinization $\hat{\mathfrak{g}}'$.

Let L be a Lie algebra and $S \subseteq L$ be a generating subset. We say that L **is free on** S if L satisfies the following universal property: given any map $f : S \to M$ into a Lie algebra M, there exists a unique homomorphism of Lie algebras $\phi : L \to M$ such that $\phi\Big|_S = f$. It is known that, given a set X, there exists a unique Lie algebra that is free on X, called **the Free lie algebra over** X and denoted by $FL(X)$. Given a subset $R \subseteq FL(X)$, let $\langle R\rangle$ be the ideal generated by R. Then, the quotient Lie algebra $L(X : R) = FL(X)/\langle R\rangle$ is said to be **the Lie algebra of formal generators** X **satisfying the defining relations** R. If a Lie algebra L is such that $L \cong L(X : R)$, then $L(X : R)$ is said to be a **presentation of** L **by generators and relations**. Whenever $x, y \in FL(X)$ are such that $x - y \in R$, it is usual to say that $L(X : R)$ satisfies the defining relation $x = y$.

We now give a presentation of the nontwisted affine Kac-Moody algebra $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{C}c$ by generators and relations using the extended affine Cartan matrix $\hat{C}$ as shown in §9 of [37]. Let $X = \{X_i^{\pm}, H_i : i \in \hat{I}\}$ be a set of formal generators indexed by the set $\hat{I}$, and let $FL(X)$ be the free Lie algebra over X. In $FL(X)$, consider the following set

$$R=\left\{[H_i,H_j],(1-\delta_{i,j})\,\mathrm{ad}(X_i^{\pm})^{1-c_{i,j}}(X_j^{\pm}),\ \pm c_{i,j}X_j^{\pm}-[H_i,X_j^{\pm}],[X_i^+,X_j^-]-\delta_{i,j}H_i : i,j\in\hat{I}\right\},$$

and let $\langle R\rangle$ be the ideal of $FL(X)$ generated by R. The Lie algebra $\hat{\mathfrak{g}}'$ is isomorphic to the quotient $FL(X)/\langle R\rangle$, i.e., $\hat{\mathfrak{g}}'$ is isomorphic to the Lie algebra with formal generators $X = \{X_i^{\pm}, H_i : i \in \hat{I}\}$ satisfying the defining relations

$$\begin{gathered}[H_i,H_j]=0;\ [X_i^+,X_j^-]=\delta_{i,j}H_i;\ [H_i,X_j^{\pm}]=\pm c_{i,j}X_j^{\pm}\\ (1-\delta_{i,j})\,\mathrm{ad}(X_i^{\pm})^{1-c_{i,j}}(X_j^{\pm})=0,\ \ i,j\in\hat{I}.\end{gathered}\tag{1.5.1}$$

In particular, restricting the set of generators to $X' = \{X_i^{\pm}, H_i : i \in I\}$ and the relations above to indexes belonging to I, we obtain a presentation of $\mathfrak{g}$ in terms of generators and relations by **Serre's Theorem** in §18 of [34].

Notice that under the isomorphism $\hat{\mathfrak{g}}' \cong FL(X)/\langle R\rangle$, the elements $x_i^{\pm}, h_i$, respectively, corresponds to $X_i^{\pm}, H_i^{\pm}, i \in \hat{I}$, so the 1-dimensional center of $FL(X)/\langle R\rangle$ is generated by $\sum_{i=0}^n c_i H_i$. Since $\tilde{\mathfrak{g}}$ is clearly isomorphic to the quotient of $\hat{\mathfrak{g}}'$ by its central element c, such presentation can be deduced from the corresponding presentation of $\hat{\mathfrak{g}}'$ by adjoining the expression of the central element itself to the set of defining relations. Thus, the set $R' = R \cup \left\{\sum_{i=0}^n c_i H_i\right\} \subseteq FL(X)$ is such that $\tilde{\mathfrak{g}} \cong FL(X)/\langle R'\rangle$, that is to say, the loop algebra $\tilde{\mathfrak{g}}$ is isomorphic to the Lie algebra with formal generators $X_i^{\pm}, H_i, i \in \hat{I}$, satisfying the defining relations

$$\begin{gathered}[H_i,H_j]=0,\ [X_i^+,X_j^-]=\delta_{i,j}H_i,\ [H_i,X_j^{\pm}]=\pm c_{i,j}X_j^{\pm}\quad\forall\, i,j\in\hat{I},\\ (1-\delta_{i,j})\,\mathrm{ad}(X_i^{\pm})^{1-c_{i,j}}(X_j^{\pm})=0,\ i\in\hat{I},\ \sum_{i=0}^n c_iH_i=0.\end{gathered}\tag{1.5.2}$$

1.6 Finite group theory and representations

In this section we give an overview of the representation theory of the symmetric group $\mathcal{S}_\ell$, as well as recall the concept of presentation of groups by generators and relations. We remark that all the group modules here considered are supposed to be finite-dimensional vector spaces over the field of complex numbers.

1.6.1 Presentation of groups by generators and relations

We now discuss the concept of presentation of groups by generators and relations as seen in [36]. Given a subset $S \subseteq G$ of a group G, then G is said to be **free over** S if G satisfies the following universal property: any map $t : S \to H$ of S into a group H uniquely extends to a homomorphism of groups $\theta : G \to H$. For any given set X, there exists a unique group that is free over X, called **the free group over** X and denoted by $FG(X)$. If $R \subseteq FG(X)$ is a subset, let $N \subseteq FG(X)$ be the smallest normal subgroup that contains R. Then, the quotient group $FG(X)/N$ is said to be the **group given by generators** X **satisfying the defining relations** R, and the notation $\langle X : R\rangle = FG(X)/N$ is usually adopted. If H is any group such that $H \cong \langle X : R\rangle$, then $\langle X : R\rangle$ is said to be a **presentation of** H **in terms of generators and relations**. Also, if $uv^{-1} \in R$ for some $u, v \in X$, we often say that these generators satisfy the defining relation $u = v$, and we abuse notation by setting $u = v$ as an element of R.

We finish this subsection by briefly discussing a material that will be useful in Section 3.2 of Chapter 3: presentations of semi-direct product of groups. Given groups G and H, let $H \times G \to G$ be a left action of H on G. The **semi-direct product of** G **and** H associated with this action, denoted by $S = G \rtimes H$, is the set $G \times H$ equipped with the group operation

$$(g, h) \cdot (g', h') = (g(h \cdot g'), hh')$$

for all $g, g' \in G, h, h' \in H$. Both G and H can be naturally seen as subgroups of $G \rtimes H$ by identification with the subsets $\{(g, 1) : g \in G\}$ and $\{(1, h) : h \in H\}$, respectively.

Proposition 1.6.1. *([36], Proposition 4.4) Let G, H be groups with presentations $G = \langle X : R\rangle$, $H = \langle Y : S\rangle$, and let $H \times G \to G$ be some left action of H on G. Then, the semi-direct product $G \rtimes H$ is isomorphic to the group of generators $X \cup Y$ satisfying the defining relations R and S along with*

$$yxy^{-1} = y \cdot x \quad \textit{for all} \;\; x \in X, y \in Y. \qquad \square$$

1.6.2 Symmetric group representation theory

Given $\ell \in \mathbb{Z}_{>0}$, let $\mathcal{S}_\ell$ be the **symmetric group** of order $\ell!$ defined over the set $\{1, \cdots, \ell\}$. It is well-known that this group is isomorphic to the group given by generators

$\sigma_i, 1 \leq i < \ell$, satisfying the defining relations

$$\sigma_i^2 = 1,\ 1 \leq i < \ell, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i < \ell - 1, \quad \text{and} \quad \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| > 1, \tag{1.6.1}$$

where $1 \in \mathcal{S}_\ell$ denotes the identity element and σ_i is identified with the transposition $(i, i+1), 1 \leq i < \ell$. Given $\sigma \in \mathcal{S}_\ell$, let $\operatorname{sgn}(\sigma)$ be the parity of σ obtained by the equality $\operatorname{sgn}(\sigma) = (-1)^m$, where m is the number of transpositions that appear on a given decomposition of σ in terms of transpositions.

Recall that the right modules of a finite group G are **completely reducible**, i.e., they decompose into a direct-sum of simple modules, and recall that a complete set of inequivalent (non-isomorphic) simple G-modules is in a bijective correspondence with the set of conjugacy classes of G. In case $G = \mathcal{S}_\ell$, it turns out that the set of conjugacy classes is in a bijective correspondence with the set of partitions of the integer ℓ and each partition gives rise to a simple $\mathcal{S}_\ell$-module, as we now present.

Recall that the **right regular representation** (or the **group algebra**) $\mathbb{C}[\mathcal{S}_\ell]$ of $\mathcal{S}_\ell$ is a right $\mathcal{S}_\ell$-module realized as the free vector space over $\mathcal{S}_\ell$ with an algebra structure inherited from the group multiplication and such that $e_\sigma \cdot e_\tau = e_{\sigma\tau}$ for all elements in the basis $(e_\sigma)_{\sigma\in\mathcal{S}_\ell}$. Setting $\mathcal{P}_\ell$ to be the **set of partitions** of ℓ, its elements are identified with eventually-zero sequences of integers $\xi = (\xi_j)_{j\geq 1}$ such that $\xi_j \geq \xi_{j+1} \geq 0$ for all $j \geq 1$ and $\sum_{j\geq 1} \xi_j = \ell$. The length of $\xi \in \mathcal{P}_\ell$ is defined as $|\xi| = \max\{j : \xi_j \neq 0\}$, and such a partition can also be denoted more explicitly as $\xi = (\xi_1, \cdots, \xi_m)$, where $m = |\xi|$. The irreducible representation associated to such a ξ is constructed as follows. Define the **Young diagram of** ξ as the collection of ℓ boxes arranged in m left-justified rows, with the j-th row length equal to ξ_j, $1 \leq j \leq m$. By a **standard tableau of** ξ we will mean the diagram obtained by labeling the Young diagram of ξ with the integers $1, \cdots, \ell$ in such a way that the labels increase in rows (from left to right) and increase in columns (from top to bottom). We denote by $\operatorname{STab}(\xi)$ the set of standard tableaux of ξ. Of particular interest is the **principal tableau of** ξ, denoted by $T(\xi) \in \operatorname{STab}(\xi)$, in which the labels increase by 1 in each row (see [30] for further details on Young tableaux). For instance, if $\xi = (3, 2) \in \mathcal{P}_5$, the following is an example of a standard tableau of ξ and the principal tableau of ξ, respectively:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\,, \qquad T(\xi) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}\,.$$

With respect to the principal tableu of $\xi \in \mathcal{P}_\ell$, let P_ξ, Q_ξ be the following subgroups of $\mathcal{S}_\ell$:

$$P_\xi = \{\sigma \in \mathcal{S}_\ell \colon \sigma \text{ preserves each row of } T(\xi)\},$$

and

$$Q_\xi = \{\sigma \in \mathcal{S}_\ell \colon \sigma \text{ preserves each column of } T(\xi)\}.$$

Using the basis $(e_\sigma)_{\sigma\in\mathcal{S}_\ell}$ of $\mathbb{C}[\mathcal{S}_\ell]$, set

$$a_\xi = \sum_{\sigma\in P_\xi} e_\sigma, \quad b_\xi = \sum_{\sigma\in Q_\xi} \operatorname{sgn}(\sigma) e_\sigma, \quad \text{and } c_\xi = a_\xi \cdot b_\xi.$$

The element c_ξ is called the **Young symmetrizer of** ξ, and the right **simple module attached to** c_ξ or the **Specht module of** ξ is the right cyclic $\mathcal{S}_\ell$-module:

$$\mathcal{S}(\xi) = c_\xi \cdot \mathbb{C}[\mathcal{S}_\ell]. \tag{1.6.2}$$

In particular, if $|\xi| = 1$, we have $c_\xi = \sum_{\sigma\in\mathcal{S}_\ell} \sigma$ and $\mathcal{S}(\xi)$ is the 1-dimensional **trivial representation**, whilst, if $|\xi| = \ell$, we have $c_\xi = \sum_{\sigma\in\mathcal{S}_\ell} \operatorname{sgn}(\sigma)\sigma$ and $\mathcal{S}(\xi)$ is the 1-dimensional **sign representation**. The Young symmetrizers are also seen to satisfy the following relations:

Proposition 1.6.2. *([31], Lemma 4.23)*

(i) If $\xi, \eta \in \mathcal{P}_\ell$ is such that $\xi \neq \eta$, then $c_\xi \cdot x \cdot c_\eta = 0$ for all $x \in \mathbb{C}[\mathcal{S}_\ell]$. In particular, $c_\xi \cdot c_\eta = 0$.

(ii) For all $x \in \mathbb{C}[\mathcal{S}_\ell]$, the element $c_\xi \cdot x \cdot c_\xi$ is a scalar multiple of c_ξ. In particular, we have $c_\xi \cdot c_\xi = \lambda c_\xi$ for some $\lambda \neq 0$. □

In fact, the above properties ensure that $\mathcal{S}(\xi)$ and $\mathcal{S}(\eta)$ are inequivalent whenever $\xi \neq \eta$, so the set $\{\mathcal{S}(\xi) : \xi \in \mathcal{P}_\ell\}$ gives all, up to isomorphism, inequivalent simple finite-dimensional representations of $\mathcal{S}_\ell$. Given a $\mathcal{S}_\ell$-module V, we denote by $[V : \mathcal{S}(\xi)]$ the multiplicity of $\mathcal{S}(\xi)$ as a simple summand of V, $\xi \in \mathcal{P}_\ell$.

We finish this subsection about the symmetric group $\mathcal{S}_\ell$ by recalling the direct-sum decomposition of the right regular representation $\mathbb{C}[\mathcal{S}_\ell]$ in terms of the simple summands $\mathcal{S}(\xi), \xi \in \mathcal{P}_\ell$. Let $d_\xi = [\mathbb{C}[\mathcal{S}_\ell] : \mathcal{S}(\xi)]$. Then, $d_\xi = \dim \mathcal{S}(\xi) = |\mathrm{STab}(\xi)|$, where $|\mathrm{STab}(\xi)|$ denotes the cardinality of the set of standard tableaux of ξ and we have

$$\mathbb{C}[\mathcal{S}_\ell] = \bigoplus_{\xi\in\mathcal{P}_\ell} \mathcal{S}(\xi)^{\oplus d_\xi}. \tag{1.6.3}$$

1.7 A few general results on $\mathfrak{sl}_{n+1}$ and $\hat{\mathfrak{sl}}_{n+1}$

We now specialize some of the contents seen in Subsections 1.1, 1.2 and 1.3 explicitly in terms of $\mathfrak{g} = \mathfrak{sl}_{n+1}$. These notions will be fundamental in the development of Chapter 3.

1.7.1 The basic structure of $\mathfrak{sl}_{n+1}$ and $\hat{\mathfrak{sl}}_{n+1}$

We start with the basic structure of $\mathfrak{g}$. Choose the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ to be the subalgebra of traceless $(n+1) \times (n+1)$ diagonal matrices. The root system thus obtained is the set

$$\Phi = \{\alpha_{i,j} : 1 \le i, j \le n+1,\, i \ne j\},$$

where $\alpha_{i,j} \in \mathfrak{h}^*$ is the unique element satisfying $\mathrm{diag}(h_1, \cdots, h_{n+1}) \mapsto h_i - h_j$. The associated Weyl group $\mathscr{W}$ is isomorphic to $\mathcal{S}_{n+1}$, and the root space decomposition is given by

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{1 \le i,j \le n} \mathbb{C}\, e_{i,j} \right),$$

where $e_{i,j}$ is the elementary $(n+1) \times (n+1)$ matrix with 1 in the i-th row and j-th column, and 0 elsewhere. The set $\{\alpha_i : i \in I\}$ of simple roots is given by $\alpha_i = \alpha_{i,i+1}$, and the associated Cartan matrix $C = (c_{i,j})_{i,j \in I}$ is the $n \times n$ matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1.7.1}$$

The set of positive roots Φ^+ coincides with the set $\{\alpha_{i,j} : 1 \le i \le j \le n\}$, with $\theta = \alpha_{1,n} = \sum_{i=1}^{n} \alpha_i$ being the maximal root. In particular, the elements in (1.1.2) with respect to $\beta \in \{\alpha_i : i \in I\} \cup \{\theta\}$ are as follows:

$$\begin{aligned} x_\theta^+ &= e_{1,n+1},\, x_\theta^- = e_{n+1,1},\, h_\theta = \sum_{i=1}^{n} h_i, \\ x_i^+ &= e_{i,i+1},\, x_i^- = e_{i+1,i},\, h_i = e_{i,i} - e_{i+1,i+1},\, i \in I. \end{aligned} \tag{1.7.2}$$

For $1 \le j \le n+1$, it is convenient to consider $\epsilon_j \in \mathfrak{h}^*$ given by $\mathrm{diag}(h_1, \cdots, h_{n+1}) \mapsto h_j$, so that each simple root and each fundamental weight can be written as

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad \omega_i = \epsilon_1 + \cdots + \epsilon_i,\; i \in I.$$

We now briefly discuss the structure of $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with Lie bracket given by (1.2.4),(1.2.5). Fix the standard Killing form κ as the nondegenerate invariant symmetric bilinear form of $\mathfrak{g}$, so that $(\theta, \theta) = 2$. In $\hat{\mathfrak{g}}$, notice that $E_0 = x_\theta^+$ is such that $\kappa(E_0, \omega(E_0)) = -1$ with respect to the Chevalley involution $\omega : \mathfrak{g} \to \mathfrak{g}$, so $F_0 = -\omega(E_0) = x_\theta^-$, and

$$x_0^+ = x_\theta^- \otimes t,\; x_0^- = x_\theta^+ \otimes t^{-1},\; h_0 = c - h_\theta. \tag{1.7.3}$$

In this way, using the simple root $\alpha_0 = \delta - \theta \in \hat{\mathfrak{h}}$, notice that

$$\alpha_0(h_0) = 2, \ \alpha_0(h_j) = -\delta_{1,j},$$

so the extended affine Cartan matrix $\hat{C} = (c_{i,j})_{i,j\in\hat{I}}$ obtained from (1.7.1) is as follows:

$$\hat{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \tag{1.7.4}$$

In particular, notice that the vector $u^t = (1, \cdots, 1) \in \mathbb{Z}^{n+1}$ is such that $u^t\hat{C} = 0$, so the central element c is explicitly given by

$$c = \sum_{i=0}^{n} h_i. \tag{1.7.5}$$

1.7.2 The $\mathfrak{sl}_{n+1}$-modules $V(\omega_i)$ and $V(\omega_1)^{\otimes\ell}$

In this subsection we provide a representative in the isomorphism class of each simple highest-weight module $V(\omega_i), i \in I$, of $\mathfrak{g}$ and study some properties of the ℓ-th tensor power $V(\omega_1)^{\otimes\ell}$.

Set $J = \{1, \cdots, n+1\}$, and let $(v_i)_{i\in J}$ be the canonical basis of the vector space $\mathbb{V} = \mathbb{C}^{n+1}$. We identify each $v \in \mathbb{V}$ with a column matrix, so the action of matrix multiplication $x \cdot v = xv$, $x \in \mathfrak{g}, v \in \mathbb{V}$, makes $\mathbb{V}$ trivially into a $\mathfrak{g}$-module. The following facts are easily calculated:

$$\begin{aligned} &[x_\theta^+, x_\theta^-] \cdot v_j = \delta_{1,j}v_j - \delta_{n,j-1}v_j, \\ &x_\theta^- v_j = \delta_{1,j}v_{n+1}, \ x_\theta^+ v_j = \delta_{n+1,j}v_1, \\ &x_i^- v_j = \delta_{i,j}v_{j+1}, \ x_i^+ v_j = \delta_{i,j-1}v_{j-1}, \\ &h_i \cdot v_j = \delta_{i,j}v_j - \delta_{i,j-1}v_j, \end{aligned} \tag{1.7.6}$$

where $i \in I, j \in J$, and $v_0 = v_{n+2} = 0$. Notice that each $\epsilon_j \in \mathfrak{h}^*$ is a weight of $\mathbb{V}$ such that

$$v_j \in \mathbb{V}_{\epsilon_j}, \ j \in J.$$

The modules $V(\omega_i), i \in I$, are obtained by taking the k-th exterior power of $\mathbb{V}$, with the $\mathfrak{g}$-action given by $x \cdot (w_1 \wedge \cdots \wedge w_k) = \sum_{i=1}^{k} w_1 \wedge \cdots \wedge x \cdot w_i \wedge \cdots \wedge w_k$, for $\omega_i \in \mathbb{V}$.

Theorem 1.7.1. *([3], §11.2)* $V(\omega_k) \cong \bigwedge^k(\mathbb{V})$ *for all* $k \in I$. □

We now devote the remainder of this section to study some basic results on the ℓ-th tensor power of $\mathbb{V}$ which will be utilized in Section 3.2. These results can be found in [17] with omitted proofs, so we give a proof of them here for the sake of completeness. Set $\rho : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$ to be the representation associated with the action of $\mathfrak{g}$ on $\mathbb{V}$, and let $1 \in \mathrm{End}(\mathbb{V})$ be the identity endomorphism. We first recall the action of $U(\mathfrak{g})$ on $\mathbb{V}^{\otimes \ell}$. By the universal property of universal enveloping algebras, let $\tilde{\rho} : U(\mathfrak{g}) \to \mathrm{End}(\mathbb{V})$ be the unique homomorphism of algebras extending ρ. Then, using the ℓ-comultiplication map seen on (1.4.1), we naturally obtain a representation of $U(\mathfrak{g})$ on $\mathbb{V}^{\otimes \ell}$ by the following composition of maps

$$U(\mathfrak{g}) \xrightarrow{\Delta^{\ell}} U(\mathfrak{g})^{\otimes \ell} \xrightarrow{\tilde{\rho}^{\otimes \ell}} \mathrm{End}(\mathbb{V})^{\otimes \ell} \xrightarrow{\cong} \mathrm{End}(\mathbb{V}^{\otimes \ell}). \tag{1.7.7}$$

In particular, every $x \in \mathfrak{g}$ is assigned to

$$x \mapsto \sum_{j=1}^{\ell} \Delta_j(x), \quad \text{where} \quad \Delta_j(x) = 1^{\otimes (j-1)} \otimes \rho(x) \otimes 1^{\otimes(\ell - j)} \in \mathrm{End}(\mathbb{V}^{\otimes \ell}), \tag{1.7.8}$$

so the usual action of $x \in \mathfrak{g}$ on $\mathbb{V}^{\otimes \ell}$ can be rewritten as

$$x \cdot v = \sum_{j=1}^{\ell} \Delta_j(x)(v), v \in \mathbb{V}^{\otimes \ell}, \tag{1.7.9}$$

Also, recall that $\mathbb{V}^{\otimes \ell}$ is a left $\mathcal{S}_\ell$-module by the action of permuting tensor factors

$$\sigma \cdot (w_1 \otimes \cdots \otimes w_\ell) = w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(\ell)}, \qquad \text{where } \sigma \in S_\ell, w_i \in \mathbb{V}, \tag{1.7.10}$$

and notice that the action of $\mathcal{S}_\ell$ and $\mathfrak{g}$ on $\mathbb{V}^{\otimes \ell}$ commute with each other.

Proposition 1.7.2. *If $\boldsymbol{v} = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathbb{V}^{\otimes \ell}$ with $i_1, \cdots, i_\ell \in J$ all distinct, then $\mathbb{V}^{\otimes \ell} = U(\mathfrak{g})\boldsymbol{v}$.*

Proof. For this proof, we may suppose without loss of generality that

$$i_1 < i_2 < \cdots < i_\ell. \tag{1.7.11}$$

If not, $\mathbf{v}$ satisfies $\sigma \cdot \mathbf{v} = v_{i'_1} \otimes \cdots \otimes v_{i'_\ell}$ with $i'_1 < i'_2 < \cdots < i'_\ell$ for some $\sigma \in S_\ell$. Assuming the result for such $\sigma \cdot \mathbf{v}$, given any $w \in \mathbb{V}^{\otimes \ell}$ we have $\sigma \cdot w = x \cdot (\sigma \cdot \mathbf{v})$ for some $x \in U(\mathfrak{g})$. Since the action of σ and $\mathfrak{g}$ commute, it follows that $\sigma \cdot w = \sigma \cdot (x \cdot \mathbf{v})$, whence $w = x \cdot \mathbf{v}$.

Under the condition (1.7.11), it suffices to show that any $\mathbf{w} = v_{j_1} \otimes \cdots \otimes v_{j_\ell}$ lies in $U(\mathfrak{g})\mathbf{v}$. We begin by proving the result for $\mathbf{w}$ with $j_1 \leq j_2 \leq \cdots \leq j_\ell$ via induction on s, where

$$s = \#\{k : j_k < i_k\}.$$

If $s = 0$, then $i_r \le j_r$ for all $1 \le r \le \ell$. Setting

$$x^-_{(r)} = \begin{cases} x^-_{j_r-1} \cdots x^-_{i_r+1} x^-_{i_r}, & \text{if} \quad i_r < j_r \\ 1 \in U(\mathfrak{g}), & \text{if} \quad i_r = j_r \end{cases},$$

then the conditions $j_r \le j_k, i_r \le j_k$ for all k such that $k > r$ ensures that

$$\left(x^-_{(1)} \cdots x^-_{(\ell-1)} x^-_{(\ell)}\right) \cdot \mathbf{v} = \mathbf{w}.$$

Now, assuming the result for $s > 0$, let k_0 be the smallest element of $\{k : j_k < i_k\}$. Thus, any $r < k_0$ must satisfy $j_r \ge i_r$, whence $i_r \le j_r \le j_{k_0} < i_{k_0}$ for all such r. Setting $\mathbf{v}' = \left(x^+_{j_{k_0}-1} \cdots x^+_{i_{k_0}-1}\right) \cdot \mathbf{v}$, we see that $\mathbf{v}' = v_{i'_1} \otimes \cdots \otimes v_{i'_\ell}$ with $i'_{k_0} = j_{k_0}$, $i'_k = i_k$ for $k \ne k_0$. In particular, the inductive hyphotesis applies to $\mathbf{v}'$, so there exists $x \in U(\mathfrak{g})$ such that $\mathbf{w} = x \cdot \mathbf{v}' = \left(x x^+_{j_{k_0}-1} \cdots x^+_{i_{k_0}-1}\right) \cdot \mathbf{v}$.

It remains to consider $\mathbf{w} = v_{j_1} \otimes \cdots \otimes v_{j_\ell}$ for arbitrary indexes $j_1, \cdots, j_\ell$ satisfying no particular order. Let $k_1, \cdots, k_\ell \in \{1, \cdots, \ell\}$ be such that

$$j_{k_1} \le j_{k_2} \le \cdots \le j_{k_\ell},$$

so there exists $x \in U(\mathfrak{g})$ such that $x \cdot \mathbf{v} = v_{j_{k_1}} \otimes \cdots \otimes v_{j_{k_\ell}}$ by the previous paragraph. Given indexes $1 \le i, j \le n+1$ with $i \ne j$, let $x^i_j \in \mathfrak{g}$ be the unique element satisfying

$$x^i_j(v_k) = \begin{cases} v_j, & \text{if } k = i \\ v_i, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}$$

Given $\tau \in S_\ell$, let $\tau = (s_1, s_1+1) \cdots (s_p, s_p+1)$ be any decomposition of τ in terms of transpositions, where $s_i \in \{1, \cdots, \ell-1\}$ and p is the length of τ. If we prove that

$$\left(x^{j_{s_1}}_{j_{s_1+1}} x^{j_{s_1}}_{j_{s_1+1}} \cdots x^{j_{s_p}}_{j_{s_p+1}} x^{j_{s_p}}_{j_{s_p+1}}\right) x \cdot \mathbf{v} = x' \cdot \mathbf{v} + a\tau \cdot (x \cdot \mathbf{v}) \quad \text{for some } x' \in U(\mathfrak{g}),\ a \ne 0, \tag{1.7.12}$$

choosing $\tau \in S_\ell$ such that $\tau \cdot (x \cdot \mathbf{v}) = \mathbf{w}$, then (1.7.12) clearly completes the proof of the proposition.

We prove (1.7.12) by induction on p. If $p = 1$, then $x^{j_{s_1}}_{j_{s_1+1}}(x \cdot \mathbf{v}) = \mathbf{v}' + \mathbf{v}''$, where $\mathbf{v}'$ is the vector obtained from $x \cdot \mathbf{v}$ by replacing $v_{j_{s_1+1}}$ at the $(s_1{+}1)$-th tensor factor by $v_{j_{s_1}}$ and $\mathbf{v}''$ is obtained by replacing $v_{j_{s_1}}$ at the s_1-th tensor factor by $v_{j_{s_1+1}}$. Now, one checks that $x^{j_{s_1}}_{j_{s_1+1}} \mathbf{v}' = x^{j_{s_1}}_{j_{s_1+1}} \mathbf{v}'' = x \cdot \mathbf{v} + (s_1, s_1+1) \cdot (x \cdot \mathbf{v})$, whence

$$x^{j_{s_1}}_{j_{s_1+1}} x^{j_{s_1}}_{j_{s_1+1}} (x \cdot \mathbf{v}) = 2x \cdot \mathbf{v} + 2(s_1, s_1+1) \cdot (x \cdot \mathbf{v}) \tag{1.7.13}$$

shows that (1.7.12) holds for $p = 1$. If $p > 1$, applying the inductive hyphotesis on

$\tau' = (s_1, s_1+1)\tau$ of length $p-1$ yields

$$\left(x^{j_{s_2}}_{j_{s_2+1}} x^{j_{s_2}}_{j_{s_2+1}} \cdots x^{j_{s_p}}_{j_{s_p+1}} x^{j_{s_p}}_{j_{s_p+1}}\right) x \cdot \mathbf{v} = x' \cdot \mathbf{v} + a\tau' \cdot (x \cdot \mathbf{v}),\ a \neq 0. \tag{1.7.14}$$

In particular, this shows that $\tau' \cdot (x \cdot \mathbf{v}) \in U(\mathfrak{g})\mathbf{v}$. Similarly as in (1.7.13), we obtain

$$x^{j_{s_1}}_{j_{s_1+1}} x^{j_{s_1}}_{j_{s_1+1}} \tau' \cdot (x \cdot \mathbf{v}) = 2\tau' \cdot (x \cdot \mathbf{v}) + 2\tau \cdot (x \cdot \mathbf{v}).$$

Thus, applying $x^{j_{s_1}}_{j_{s_1+1}} x^{j_{s_1}}_{j_{s_1+1}}$ on both sides of (1.7.14) we see that

$$\left(x^{j_{s_1}}_{j_{s_1+1}} x^{j_{s_1}}_{j_{s_1+1}} \cdots x^{j_{s_p}}_{j_{s_p+1}} x^{j_{s_p}}_{j_{s_p+1}}\right) x \cdot \mathbf{v} = x'' \cdot \mathbf{v} + a'\tau \cdot (x \cdot \mathbf{v}) \text{ for some } x'' \in U(\mathfrak{g}),\ a' \neq 0,$$

so (1.7.12) follows by induction. □

Proposition 1.7.3. *Let $\ell \leq n$.*

(i) The set $\{\sigma \cdot (v_1 \otimes \cdots \otimes v_\ell) : \sigma \in S_\ell\}$ is a basis of the weight space $\mathbb{V}^{\otimes \ell}_{\omega_\ell}$.

(ii) The set $\{\sigma \cdot (v_{n+1} \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_\ell) : \sigma \in S_\ell\}$ is a basis of $\mathbb{V}^{\otimes \ell}_{\omega_\ell - \theta}$.

Proof. For $\mathbf{i} = (i_1, \cdots, i_\ell) \in J^{\times \ell}$, write $v_\mathbf{i} = v_{i_1} \otimes \cdots \otimes v_{i_\ell}$. From $v_j \in \mathbb{V}_{\epsilon_j}, j \in J$, notice that $v_\mathbf{i}$ is a weight vector of weight $\sum_{k=1}^{\ell} \epsilon_{i_k}$. Since any $v \in \mathbb{V}^{\otimes \ell}$ is written uniquely as a finite sum $v = \sum_{\mathbf{i} \in J^{\times \ell}} a_\mathbf{i} v_\mathbf{i}$, if v has weight μ, then the linear independence of the weight spaces implies that

$$a_\mathbf{i} \neq 0 \quad \text{only if} \quad \sum_{k=1}^{\ell} \epsilon_{i_k} = \mu. \tag{1.7.15}$$

Now, using the relation $\omega_1 = \epsilon_1, \alpha_i = \epsilon_i - \epsilon_{i+1}, i \in I$, one easily checks that

$$\epsilon_j = \omega_1 - (\alpha_1 + \cdots + \alpha_{j-1}),\ j \in J. \tag{1.7.16}$$

In particular, we have $\mu = \sum_{k=1}^{\ell} \epsilon_{i_k}$ equal to

$$\mu = \sum_{j=1}^{\ell} \left(\omega_1 - \alpha_1 - \cdots - \alpha_{i_j - 1}\right) = \ell\omega_1 - \sum_{k=1}^{n} c_k \alpha_k, \text{ where } c_k = \#\{j : i_j > k\}. \tag{1.7.17}$$

(i) Since $\omega_\ell = \sum_{j=1}^{\ell} \epsilon_j$, then $\omega_\ell = \ell\omega_1 - (\ell-1)\alpha_1 - (\ell-2)\alpha_2 - \cdots - \alpha_{\ell-1}$ by (1.7.16). Setting $\mu = \omega_\ell$, from (1.7.17) we have

$$\ell\omega_1 - \sum_{k=1}^{n} c_k \alpha_k = \ell\omega_1 - (\ell-1)\alpha_1 - (\ell-2)\alpha_2 - \cdots - \alpha_{\ell-1}.$$

Using the linear independence of the simple roots to compare the coefficients of every α_k in the above equality, one checks that it must be $\{i_1, \cdots, i_\ell\} = \{1, \cdots, \ell\}$. Therefore, $v_\mathbf{i} = \sigma \cdot v_1 \otimes \cdots \otimes v_\ell$ for some $\sigma \in S_\ell$.

(ii) We have $\omega_\ell - \theta = \epsilon_2 + \cdots + \epsilon_\ell + \epsilon_{n+1}$, whence (1.7.16) gives us

$$\omega_\ell - \theta = \ell\omega_1 - \ell\alpha_1 - (\ell-1)\alpha_2 - (\ell-2)\alpha_3 - \cdots - 2\alpha_{\ell-1} - \alpha_\ell - \cdots - \alpha_n.$$

Taking $\mu = \omega_\ell - \theta$, (1.7.17) analogously implies that $\{i_1, \cdots, i_\ell\} = \{n+1, 2, 3, \cdots, \ell\}$, so $v_{\mathbf{i}} = \sigma \cdot v_{n+1} \otimes v_2 \otimes \cdots \otimes v_\ell$ for some $\sigma \in S_\ell$. □

1.8 Complete homogeneous and elementary symmetric polynomials

In this section we introduce the elementary and complete homogeneous symmetric polynomials which are relevant to the development of Section 3.4.

The symmetric group $\mathcal{S}_\ell$ acts on the ring of multivariate polynomials $\mathbb{C}[x_1, \cdots, x_\ell]$ in the obvious way:

$$\sigma \cdot f(x_1, \cdots, x_\ell) = f(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(\ell)}), \text{ for } f \in X, \sigma \in \mathcal{S}_\ell. \tag{1.8.1}$$

The set of **symmetric polynomials** in the variables x_j, denoted by $\mathbb{C}[x_1, \cdots, x_n]^{\mathcal{S}_\ell}$, is defined by the elements invariant under the action of $\mathcal{S}_\ell$, i.e., all the polynomials $f \in X$ such that $\sigma \cdot f = f$ for all $\sigma \in \mathcal{S}_\ell$. It is well-known that

$$\mathbb{C}[x_1, \cdots, x_n]^{\mathcal{S}_\ell} = \mathbb{C}[h_1, \cdots, h_\ell] = \mathbb{C}[e_1, \cdots, e_\ell], \tag{1.8.2}$$

where the $e_j = e_j(x_1, \cdots, x_n)$ and $h_j = h_j(x_1, \cdots, x_n)$ are, respectively, the so-called elementary and complete homogeneous symmetric polynomials defined as follows.

The **elementary symmetric polynomials** $e_k(x_1, \cdots, x_\ell)$ in the variables $x_1, \cdots, x_\ell$ are defined by the expression

$$e_k(x_1, \cdots, x_\ell) = \sum_{1 \le i_1 < \cdots < i_k \le \ell} x_{i_1} \cdots x_{i_k}, \ 1 \le k \le \ell, \tag{1.8.3}$$

or, equivalently, as the coefficients of the expansion of the following polynomial in the variable u:

$$\prod_{j=1}^{\ell} (1 - x_j u) = \sum_{j=0}^{\ell} (-1)^j e_j(x_1, \ldots, x_\ell) u^j = E(x_1, \ldots, x_\ell; u). \tag{1.8.4}$$

The **complete homogeneous symmetric polynomials** $h_k(x_1, \cdots, x_\ell)$ in the variables $x_1, \cdots, x_\ell$ are defined by

$$h_k(x_1, \cdots, x_\ell) = \sum_{1 \le i_1 \le \cdots \le i_k \le \ell} x_{i_1} \cdots x_{i_k}, \ 1 \le k \le \ell, \tag{1.8.5}$$

or, equivalently, as the coefficients of the expansion in the variable u of the polynomial:

$$\prod_{j=1}^{\ell}(1-x_j u)^{-1} = \sum_{j\geq 0} h_j(x_1,\dots,x_\ell)u^j = H(x_1,\dots,x_\ell;u). \tag{1.8.6}$$

It is clear that (1.8.4) and (1.8.6) satisfy $E(x_1,\cdots,x_\ell : u)H(x_1,\cdots,x_\ell;u) = 1$, whence we see that the elementary and complete homogeneous symmetric polynomials are related by the following equality:

$$\sum_{j=0}^{k}(-1)^j e_j(x_1,\cdots,x_\ell)h_{k-j}(x_1,\cdots,x_\ell) = 0,\ k\geq 1. \tag{1.8.7}$$

1.9 Notes on Category Theory

In this section we briefly present some results on Category theory and fix notation that will be relevant to Chapter 3.

Let $\mathscr{C}$ be a category. We denote its **objects** by $\operatorname{Ob}\mathscr{C}$, and by $\operatorname{Hom}_{\mathscr{C}}(A,B)$ the collection of **morphisms** for each $A,B\in\operatorname{Ob}\mathscr{C}$, with **identity morphisms** 1_A for each object A. A **functor** between categories $\mathcal{F}:\mathscr{C}\to\mathscr{D}$, or simply a functor $\mathcal{F}$, will be denoted by the assignments

$$A\mapsto\mathcal{F}(A),\ f\mapsto\mathcal{F}(f),$$

where $A\in\operatorname{Ob}\mathscr{C},\mathcal{F}(A)\in\operatorname{Ob}\mathscr{D}, f\in\operatorname{Hom}_{\mathscr{C}}(A,B)$, and $\mathcal{F}(f)\in\operatorname{Hom}_{\mathscr{D}}(\mathcal{F}(A),\mathcal{F}(B))$. The functor $\mathcal{F}$ is said to be **faithful** (resp. **full**) on morphisms if for every $A,B\in\operatorname{Ob}\mathscr{C}$ the assignment $\operatorname{Hom}_{\mathscr{C}}(A,B)\to\operatorname{Hom}_{\mathscr{D}}(\mathcal{F}(A),\mathcal{F}(B))$ is injective (resp. surjective), and it is said to be **essentially surjective** on objects if, for every $B\in\operatorname{Ob}\mathscr{D}$, there exists $A\in\operatorname{Ob}\mathscr{C}$ such that $B\cong\mathcal{F}(A)$, i.e., if $f\circ g = 1_B$ and $g\circ f = 1_{\mathcal{F}(A)}$ for some $f\in\operatorname{Hom}_{\mathscr{D}}(\mathcal{F}(A),B), g\in\operatorname{Hom}_{\mathscr{D}}(B,\mathcal{F}(A))$. We say that $\mathcal{F}$ **establishes an equivalence**, or that the categories $\mathscr{C}$ and $\mathscr{D}$ are **equivalent**, if $\mathcal{F}$ is full, faithful and essentially surjective.

We are mainly interested in abelian categories $\mathscr{C}$ whose objects are left or right R-modules, where R is an associative ring with identity. Here, the zero object is denoted by $0_{\mathscr{C}}$ (or simply 0), the biproduct of a finite collection $A_1,\cdots,A_n$ of objects is denoted by $A_1\oplus\cdots\oplus A_n$, while kernels and cokernels of morphisms $f\in\operatorname{Hom}_{\mathscr{C}}(A,B)$ are denoted by $\operatorname{Ker}(f)$ and $B/\operatorname{Im}(f)$, respectively. In the context of such categories of R-modules, it is convenient to introduce the notation $\operatorname{length}_{\mathscr{C}}(A)$ to denote the length of any composition series of a R-module A. We shall refer to irreducible modules or simple modules in categories of R-modules by **simple objects**. In case $\mathscr{C}$ and $\mathscr{D}$, respectively, are **abelian** categories whose objects, respectively, are composed of R-modules and R'-modules, any equivalence $\mathcal{F}:\mathscr{C}\to\mathscr{D}$ is known to satisfy the following essential properties:

(i) $\mathcal{F}$ is an **additive functor**, i.e, given $A\oplus B\in\mathscr{C}$, then $\mathcal{F}(A\oplus B)\cong\mathcal{F}(A)\oplus\mathcal{F}(B)$

in $\mathscr{D}$.

(ii) $\mathcal{F}$ is an **exact functor**, that is to say, given a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R-modules in $\mathscr{C}$, then the sequence $0 \to \mathcal{F}(M_1) \to \mathcal{F}(M_2) \to \mathcal{F}(M_3) \to 0$ of R'-modules in $\mathscr{D}$ is also short exact.

(iii) $\mathcal{F}$ **reflects exactness**, in the sense that if $0 \to B_1 \to B_2 \to B_3 \to 0$ in $\mathscr{D}$ is short exact, then the sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ is short exact in $\mathscr{C}$, where $A_j \in \operatorname{Ob}\mathscr{C}$ are such that $\mathcal{F}(A_j) \cong B_j$ for $1 \leq j \leq 3$.

In particular, $\mathcal{F}$ maps simple objects of $\mathscr{C}$ into simple objects of $\mathscr{D}$, and if $\mathcal{F}(A)$ is simple, then $A \in \operatorname{Ob}\mathscr{C}$ is simple. In the context of exact sequences,

In some contexts, it will be useful to consider a **full subcategory** $\mathscr{C}'$ of a given abelian category $\mathscr{C}$ of R-modules, which is defined by a subclass $\operatorname{Ob}\mathscr{C}'$ of the class of objects $\operatorname{Ob}\mathscr{C}$ with the additional requirement that $\operatorname{Hom}_{\mathscr{C}'}(A, B) = \operatorname{Hom}_{\mathscr{C}}(A, B)$ for all pair of objects. However, it is known that such full subcategories may not be abelian. The following result is an useful tool with respect to this matter.

Lemma 1.9.1. *([51] Proposition 5.92) Let $\mathscr{C}'$ be a full subcategory of an abelian category $\mathscr{C}$ composed of R-modules. If, for all $A, B \in Ob\mathscr{C}'$ and $f \in Hom_{\mathscr{C}'}(A, B)$, we have:*

(i) $0 \in Ob\mathscr{C}'$,

(ii) For all $A, B \in Ob\mathscr{C}'$, the direct-sum of R-modules $A \oplus B \in Ob\mathscr{C}'$,

(iii) $\operatorname{Ker}(f)$, $B/Im(f) \in Ob\mathscr{C}'$,

then $\mathscr{C}'$ is an abelian category. □

Since representations of a Lie algebra are equivalent to representations of its universal enveloping algebra, the category of representations of a Lie algebra is a category of R-modules, where R is the universal enveloping algebra itself. In particular, all the discussion above applies to the context of representations of Lie algebras.

2 Finite-dimensional Representation Theory of Affine Lie Algebras

In this chapter, unless explicitly stated otherwise, we set $\mathfrak{g}$ to be an arbitrary complex simple finite-dimensional Lie algebra as in Section 1.1, with associated loop algebra $\tilde{\mathfrak{g}}$ and affinization $\hat{\mathfrak{g}}'$ as in Section 1.2. The goal of this chapter is to study noteworthy representations in the category of finite-dimensional $\hat{\mathfrak{g}}'$-modules: the irreducible representations and the local Weyl modules. In fact, it will be seen that this category is equivalent to the category of finite-dimensional $\tilde{\mathfrak{g}}$-modules, and, in view of this equivalence, the study of these representations will be carried out in the setting of $\tilde{\mathfrak{g}}$-modules with the action extended to an action of $\hat{\mathfrak{g}}'$ by setting the central element c to act trivially. The finite-dimensional simple $\tilde{\mathfrak{g}}$-modules were classified in terms of tensor products of evaluation representations in [16, 12]. Following [47], we shall introduce the notion of highest-ℓ-weight modules and rewrite this classification in terms of this concept. Along the way, we shall also briefly discuss the category of integrable representations, in order to introduce general results that will be instrumental in the characterization of the local Weyl modules, which is the culminating point of this chapter.

In the sections that follow, we remark that [47] is the main reference of Sections 2.1 and 2.2, while [47, 19, 13, 14] are the main references of Sections 2.3 and 2.4. In this sections, all the proofs that can be found in the corresponding main references of each section are purposely omitted.

2.1 Finite-dimensional modules for the loop and affine Lie algebras

Let $\hat{\mathscr{G}}'$ and $\tilde{\mathscr{G}}$, respectively, be the abelian **categories** of finite-dimensional left $\hat{\mathfrak{g}}'$-modules and left $\tilde{\mathfrak{g}}$-modules, and let $V \in \mathrm{Ob}\,\tilde{\mathscr{G}}$. Regarding the central element c and the direct sum $\hat{\mathfrak{g}}' = \tilde{\mathfrak{g}} \oplus \mathbb{C}c$, one can extend the action of $\tilde{\mathfrak{g}}$ on V to an action of $\hat{\mathfrak{g}}'$ by setting $c \cdot v = 0$ for all $v \in V$. Denoting by $\hat{V}$ the $\hat{\mathfrak{g}}'$-module obtained through this extension, one defines a functor $\mathcal{F} : \tilde{\mathscr{G}} \to \hat{\mathscr{G}}'$ mapping each object V and each morphism f by

$$V \mapsto \mathcal{F}(V) = \hat{V}, \ \ f \mapsto \mathcal{F}(f) = f.$$

It is clear that $\mathcal{F}$ satisfies the defining conditions of a functor, and that $\mathcal{F}$ is fully and faithful with respect to morphisms. The next theorem ensures that this functor is also essentially surjective on objects.

Theorem 2.1.1. *([47], Proposition 2.1.1) If V is a finite-dimensional $\hat{\mathfrak{g}}'$-module, then the* *central element $c \in \hat{\mathfrak{g}}'$ is such that $c \cdot v = 0$ for all $v \in V$.* □

Indeed, any finite-dimensional $\hat{\mathfrak{g}}'$-module V can be regarded as a $\tilde{\mathfrak{g}}$-module by restriction, whence it follows from the above theorem that $\hat{V} \cong V$ as $\hat{\mathfrak{g}}'$-modules. Therefore, $\mathcal{F}$ establishes an equivalence between the categories $\hat{\mathscr{G}}'$ and $\tilde{\mathscr{G}}$.

Corollary 2.1.2. *The categories $\hat{\mathscr{G}}'$ and $\tilde{\mathscr{G}}$ are equivalent.* □

Since the study of objects in $\hat{\mathscr{G}}'$ is equivalent to the study of objects in $\tilde{\mathscr{G}}$, we now focus on the category $\tilde{\mathscr{G}}$ only. In the next section, we introduce the concepts of ℓ-weights and highest-ℓ-weight modules, and classify all the simple objects in $\tilde{\mathscr{G}}$ in these terms.

2.2 The classification of simple finite-dimensional $\tilde{\mathfrak{g}}$-modules

We begin by introducing the concept of evaluation modules. Given $a \in \mathbb{C}^\times$, let $\mathrm{ev}_a : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the **evaluation map** at a given by $x \otimes g(t) \mapsto g(a)x$, which is a homomorphism of Lie algebras. For any $\mathfrak{g}$-module V, denote by $V(a)$ the $\tilde{\mathfrak{g}}$-module obtained by pulling back the action of $\mathfrak{g}$ along the homomorphism ev_a, i.e., $V(a)$ is the vector space V endowed with the action

$$(x \otimes g(t)) \cdot v = g(a)x \cdot v, \quad x \in \mathfrak{g}, g(t) \in \mathbb{C}[t, t^{-1}], v \in V. \tag{2.2.1}$$

In particular, notice that $V(a)$ is simple if and only if V is simple. Also, if $f : V \to W$ is any homomorphism of $\mathfrak{g}$-modules, then $f : V(a) \to W(a)$ is also a homomorphism of $\tilde{\mathfrak{g}}$-modules, since, for all $v \in V, x \in \mathfrak{g}, r \in \mathbb{Z}$, we have

$$f((x \otimes t^r) \cdot v) = a^r x \cdot f(v) = (x \otimes t^r) \cdot f(v). \tag{2.2.2}$$

Modules of the form $V(a)$ are usually called **evaluation modules**. In the case $V = V(\lambda)$ for some $\lambda \in P^+$, we use the more convenient notation $V(\lambda, a)$ instead of $V(\lambda)(a)$ to denote the evaluation module obtained from $V(\lambda)$.

We now state the well-known theorem classifying simple objects of $\tilde{\mathscr{G}}$ in terms of tensor product of evaluations modules along different scalars:

Theorem 2.2.1. *([12], Theorem 4.1) Every simple finite-dimensional $\tilde{\mathfrak{g}}$-module is isomorphic to some tensor product of evaluation modules of the form*

$$V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m), \quad \text{where} \;\; \lambda_1, \cdots, \lambda_m \in P^+ \setminus \{0\}, \; a_i \neq a_j \;\; \text{for all} \;\; i \neq j. \tag{2.2.3}$$

Conversely, any such tensor product is a simple finite-dimensional $\tilde{\mathfrak{g}}$-module. □

Our goal in this section is to rewrite this result using the language of ℓ-weights. We shall see that evaluation modules naturally give rise to the concept of ℓ-weights, from which the concepts of ℓ-weight modules and highest-ℓ-weight modules also emerge. The simple finite-dimensional $\tilde{\mathfrak{g}}$-modules will be characterized as a particular case of the latter.

By the universal property of universal enveloping algebras, the homomorphism of Lie algebras ev_a is uniquely extended to a homomorphism of unital associative algebras $U(\tilde{\mathfrak{g}}) \to U(\mathfrak{g})$, which we still denote by ev_a. In particular, using (1.4.5), one may check that the elements $\Lambda_{i,r}, i \in I, r \in \mathbb{Z}$, are mapped as follows:

$$\mathrm{ev}_a(\Lambda_{i,\pm r}) = (-a)^{\pm r}\binom{h_i}{r}, \quad \text{where} \quad \binom{h_i}{r} = \frac{h_i(h_i-1)\cdots(h_i-(r-1))}{r!} \in U(\tilde{\mathfrak{g}}). \tag{2.2.4}$$

Moreover, if $v \in V(a)_\mu$ is any weight vector of the evaluation module $V(a)$ of a given $\mathfrak{g}$-module V, then (2.2.4) implies that

$$\Lambda_{i,\pm r} \cdot v = (-a)^{\pm r}\tbinom{\mu(h_i)}{r} v, \tag{2.2.5}$$

where $\binom{\mu(h_i)}{r} \in \mathbb{C}$ is the generalized binomial coefficient. In fact, this equality can also be expressed in terms of power series in the following way. Given $f(u) \in \mathbb{C}[u]$ of degree n with $f(0) = 1$, write $f(u) = 1 + c_1 u + \cdots + c_n u^n$, and let

$$f^-(u) = c_n^{-1} u^n f(u^{-1}). \tag{2.2.6}$$

Writing $f^-(u) = 1 + c_{-1}u + \cdots + c_{-n}u^n$, the coefficients of $f(u)$ and $f^-(u)$ are related by

$$c_n c_{-r} = c_{n-r}, \ \text{ for } \ 0 \le r \le n. \tag{2.2.7}$$

In particular, if $f(u) = \prod_{r=1}^m (1 - a_r u)$ is the factorization of $f(u)$, then the factorization of $f^-(u)$ is given by $f^-(u) = \prod_{r=1}^m (1 - a_r^{-1} u)$. Since any rational function $q(u) \in \mathbb{C}(u)$ such that $q(0) = 1$ can be written as a quotient $f(u)/g(u)$ of elements in $\mathbb{C}[u]$ with $f(0) = g(0) = 1$, the assignment $f \mapsto f^-$ can be extended to all rational functions of this type as well. Of particular interest is the I-tuple $\boldsymbol{\omega}_{\mu,a} \in \mathbb{C}(u)^I$ for $\mu \in P, a \in \mathbb{C}^\times$, whose i-th component is the rational function

$$(\boldsymbol{\omega}_{\mu,a})_i(u) = (1 - au)^{\mu(h_i)},\ i \in I,$$

satisfying $(\boldsymbol{\omega}_{\mu,a})_i(0) = 1$. Using the assignment $(\boldsymbol{\omega}_{\mu,a})_i \mapsto (\boldsymbol{\omega}_{\mu,a})_i^-$, let also $\boldsymbol{\omega}_{\mu,a}^- \in \mathbb{C}(u)^I$ be the I-tuple given by $(\boldsymbol{\omega}_{\mu,a}^-)_i(u) = (\boldsymbol{\omega}_{\mu,a})_i^-(u) = (1 - a^{-1}u)^{\mu(h_i)}$ for all $i \in I$. Now, let us identify any polynomial in the variable u with a power series in the obvious way, and any rational function of the form $(1 - au)^{-1}$ with the formal geometric series $\sum_{r \ge 0} a^r u^r \in \mathbb{C}[[u]], a \in \mathbb{C}^\times$. Then, any rational function $q(u) \in \mathbb{C}(u)$ such that $q(0) = 1$ can be identified with a unique element of $\mathbb{C}[[u]]$ by taking formal multiplication and division of power series, if necessary. In particular, we regard both $\boldsymbol{\omega}_{\mu,a}$ and $\boldsymbol{\omega}^-_{\mu,a}$ as elements of $\mathbb{C}[[u]]^I$. In this way, (2.2.5) implies the following identity of formal power series in the variable u with coefficients in $V(a)$:

$$\Lambda_i^{\pm}(u) \cdot v = (\boldsymbol{\omega}^{\pm}_{\mu,a})_i(u) \cdot v, i \in I, \tag{2.2.8}$$

where $\boldsymbol{\omega}^+_{\mu,a} = \boldsymbol{\omega}_{\mu,a}$, and each power series above acts on v component-wise.

The set $\mathbb{C}[[u]]^I$ of I-tuples of power series is a ring under coordinate-wise addition and multiplication. Let $\mathscr{P} \subseteq \mathbb{C}[[u]]^I$ be the multiplicative subgroup generated by all such $\boldsymbol{\omega}_{\mu,a}$, and let $\mathscr{P}^+$ be the submonoid generated by all $\boldsymbol{\omega}_{\mu,a}$ with $\mu \in P^+$. The abelian group $\mathscr{P}$ is called the **ℓ-weight lattice of $\tilde{\mathfrak{g}}$** and its elements are called **ℓ-weights**, while the elements of $\mathscr{P}^+$ are usually known as **dominant ℓ-weights** or as **Drinfeld polynomials**, and the ℓ-weights of the form $\boldsymbol{\omega}_{\omega_i,a}, i \in I$, are more specifically referred to as **fundamental ℓ-weights**. For notational convenience, set $\boldsymbol{\omega}_{i,a} = \boldsymbol{\omega}_{\omega_i,a}$ and, given $\boldsymbol{\mu} = \prod_i \boldsymbol{\omega}_{\mu_i,a_i} \in \mathscr{P}$, set $\boldsymbol{\mu}^- = \prod_i \boldsymbol{\omega}^-_{\mu_i,a_i}$, $\boldsymbol{\mu}^+ = \boldsymbol{\mu}$. Notice that $\mathscr{P}$ coincides with the subgroup generated by the fundamental ℓ-weights. Also, one checks that any $\boldsymbol{\mu} \in \mathscr{P}^+$ uniquely decomposes as

$$\boldsymbol{\mu} = \prod_{i=1}^{m} \boldsymbol{\omega}_{\lambda_i, a_i} \;\text{ for some }\; \lambda_i \in P^+ \setminus \{0\},\, m \geq 0, \;\text{ and }\; a_i \in \mathbb{C}^\times \text{ with } a_i \neq a_j \text{ for } i \neq j. \tag{2.2.9}$$

Indeed, writing each $\lambda_i = k_1\omega_1 + \cdots + k_n\omega_n$, then $k_j \in \mathbb{Z}_{\geq 0}$ is the multiplicity of $(1 - a_i u)$ in the factorization of the j-th component of $\boldsymbol{\mu}$.

Let $\mathrm{wt} : \mathscr{P} \to P$ be the unique group homomorphism satisfying

$$\mathrm{wt}(\boldsymbol{\omega}_{\mu,a}) = \mu \;\text{ for all }\; \mu \in P, a \in \mathbb{C}^\times. \tag{2.2.10}$$

Given $\boldsymbol{\mu} \in \mathscr{P}$, the element $\mathrm{wt}(\boldsymbol{\mu})$ is usually referred to as the **underlying weight** of $\boldsymbol{\mu}$. The group $\mathscr{P}$ can be identified with a subset of $U(\tilde{\mathfrak{h}})^*$ as follows. Given $\boldsymbol{\mu} \in \mathscr{P}$, denote by $\boldsymbol{\mu}_i^{\pm}(u) = \sum_{r \geq 0} \boldsymbol{\mu}^{\pm}_{i,r}\, u^r$ the i-th power series of $\boldsymbol{\mu}^{\pm}$. Since $U(\tilde{\mathfrak{h}})$ is the commutative unital associative algebra freely generated by the elements $h_i, \Lambda_{i,r}, i \in I, r \in \mathbb{Z}, r \neq 0$, there exists a unique algebra map $U(\tilde{\mathfrak{h}}) \to \mathbb{C}$ such that

$$h_i \mapsto \mathrm{wt}(\boldsymbol{\mu})(h_i) \quad \text{and} \quad \Lambda_{i,\pm r} \mapsto \boldsymbol{\mu}^{\pm}_{i,r} \quad \text{for all } \; i \in I, r \in \mathbb{Z}. \tag{2.2.11}$$

We abuse notation and still denote the above map by $\boldsymbol{\mu} : U(\tilde{\mathfrak{h}}) \to \mathbb{C}$. Under this identification, we shall continue referring to these functionals as ℓ-weights $\boldsymbol{\mu} \in \mathscr{P}$, and the nomenclatures established in the previous paragraph will also be carried over. In this way, given a $\tilde{\mathfrak{g}}$-module W, set the **ℓ-weight space** associated with $\boldsymbol{\mu}$ to be

$$W_{\boldsymbol{\mu}} = \{w \in W : \text{for all } x \in U(\tilde{\mathfrak{h}}),\, \exists\, n \in \mathbb{Z}_{>0} \text{ such that } (x - \boldsymbol{\mu}(x))^n\, w = 0\}. \tag{2.2.12}$$

If $W_{\boldsymbol{\mu}} \neq 0$, then $\boldsymbol{\mu}$ is said to be an **ℓ-weight of W** and its nonzero elements are called

ℓ-weight vectors.

In any $\tilde{\mathfrak{g}}$-module W, the ℓ-weight vector which is a common eigenvector for the action of $\tilde{\mathfrak{h}}$ and which is annihilated by $\tilde{\mathfrak{n}}^+$ is referred to as a **highest-ℓ-weight vector**, and the associated ℓ-weight is referred to as a **highest ℓ-weight**. We also refer to the $\tilde{\mathfrak{g}}$-module generated by a highest-ℓ-weight vector as a **highest-ℓ-weight module**. The next proposition gives some fundamental results on highest-ℓ-weight modules. In [47], its proof is omitted, so we shall present it here.

Proposition 2.2.2. *([47], Proposition 2.1.9)*

(i) Every highest-ℓ-weight module V is a weight module. Moreover, if $\boldsymbol{\mu}$ is the corresponding highest ℓ-weight with $\mu = \mathrm{wt}(\boldsymbol{\mu})$, then $\dim V_\mu = 1$ and $V_\nu \neq 0$ only if $\nu \leq \mu$.

(ii) Every highest-ℓ-weight module has a unique proper maximal submodule and, hence, a unique irreducible quotient. In particular, every highest-ℓ-weight module is an indecomposable module.

(iii) Two highest-ℓ-weight modules are isomorphic only if they have the same highest-ℓ-weight.

Proof. (i) Let $V = U(\tilde{\mathfrak{g}})v$ be a highest-ℓ-weight module generated by a highest-ℓ-weight vector v of ℓ-weight $\boldsymbol{\mu} \in U(\tilde{\mathfrak{h}})^*$. Write $\mu = \mathrm{wt}(\boldsymbol{\mu}) \in P$. From the decomposition (1.4.3) of $U(\tilde{\mathfrak{g}})$, we see that V is spanned by the following elements:

$$(x^-_{\beta_1,r_1})^{m_1} \cdots (x^-_{\beta_p,r_p})^{m_p}\, v, \quad \text{where} \quad \beta_j \in \Phi^+, m_j \in \mathbb{Z}_{\geq 0}, r_j \in \mathbb{Z} \text{ for all } 1 \leq j \leq p. \tag{2.2.13}$$

As $v \in V_\mu$, notice that the relations $[h, x^-_{\beta,r}] = -\alpha(h)x^-_{\beta,r}$ for all $h \in \mathfrak{h}, \beta \in \Phi^+, r \in \mathbb{Z}$ imply that $x^-_{\beta,r}V_\mu \subseteq V_{\mu-\beta}$. Therefore, the vector $(x^-_{\beta_1,r_1})^{m_1} \cdots (x^-_{\beta_p,r_p})^{m_p}\, v$ has weight ν

$$\nu = \mu - \sum_{k=1}^{p} m_k \beta_{i_k}. \tag{2.2.14}$$

In other words, all vectors of the form (2.2.13) have weight $\nu \leq \mu$. Thus, since (2.2.13) is a generating set for V, and since different weights are linearly independent, then V is a weight module whose weights ν all satisfy $\nu \leq \mu$. Furthermore, (2.2.14) implies that the only way to write an element of V_μ as a linear combination of vectors (2.2.13) is to have all exponents $m_1 = \cdots = m_p = 0$, whence $\dim V_\mu = 1$.

(ii) Since V is a weight module, any submodule $W \subseteq V$ of a weight module is also a weight module. If W is proper, the condition $V = U(\tilde{\mathfrak{g}})v$ implies $V_\mu \nsubseteq W$ and W lies in the sum of weight spaces other than V_μ, so any sum of proper submodules of V is still proper. In particular, V is indecomposable, and has a maximal proper submodule.

(iii) Let $W = U(\tilde{\mathfrak{g}})w$ be another highest-ℓ-weight module generated by w of ℓ-weight $\boldsymbol{\lambda} \in U(\tilde{\mathfrak{h}})^*$ with $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$ and such that $V \cong W$. Fix any $f : V \to W$ isomorphism of $\tilde{\mathfrak{g}}$-modules. Since f, f^{-1} commute with the action of $\mathfrak{h}$, notice that $f(v)$ and $f^{-1}(w)$ have weight μ and λ, respectively. In particular, $\mu \leq \lambda$ and $\lambda \leq \mu$ shows that $\mu = \lambda$, so $\dim V_\mu = \dim V_\lambda = 1$ implies that $f(v) = aw$ for some scalar $a \neq 0$. Up to rescaling, let us suppose that $a = 1$. Thus,

$$\boldsymbol{\lambda}_{i,r}^{\pm} f(v) = \Lambda_{i,\pm r} f(v) = f(\Lambda_{i,\pm r}\, v) = \boldsymbol{\mu}_{i,r}^{\pm} f(v) \tag{2.2.15}$$

gives us $\boldsymbol{\lambda}_{i,r}^{\pm} = \boldsymbol{\mu}_{i,r}^{\pm}$ for all $i \in I, r \in \mathbb{Z}$, and the result follows from (2.2.11). □

For $\boldsymbol{\mu} \in \mathscr{P}$, let $M(\boldsymbol{\mu})$ be the quotient of $U(\tilde{\mathfrak{g}})$ by the ideal generated by $\tilde{\mathfrak{n}}^+$ and all $y - \boldsymbol{\mu}(y), y \in U(\tilde{\mathfrak{h}})$. The decomposition seen in (1.4.3) ensures that $M(\boldsymbol{\mu})$ is isomorphic to the free rank one $U(\tilde{\mathfrak{n}}^-)$-module, so it is nonzero. In particular, it is clear that every highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\mu}$ is isomorphic to a quotient of $M(\boldsymbol{\mu})$. Setting $V(\boldsymbol{\mu})$ to be the unique irreducible quotient of $M(\boldsymbol{\mu})$, then $V(\boldsymbol{\mu})$ is the unique **simple highest-ℓ-weight module** of highest ℓ-weight $\boldsymbol{\mu}$, up to isomorphism.

A natural source of highest-ℓ-weight vectors are in fact the objects of $\tilde{\mathscr{G}}$. Indeed, regarding the action of $\mathfrak{h}$ on any given $V \in \mathrm{Ob}\,\tilde{\mathscr{G}}$, V is a weight module as in (1.3.1), and the relations $[h, x_{i,r}^{\pm}] = \pm\alpha_i(h)x_{i,r}^{\pm}$ for all $h \in \mathfrak{h}, i \in I, r \in \mathbb{Z}$ imply the inclusion of sets

$$x_{i,r}^{\pm} V_\mu \subseteq V_{\mu \pm \alpha_i} \text{ for all } \mu \in \mathrm{wt}(V). \tag{2.2.16}$$

In particular, the finite dimension of V implies that there exists $\lambda \in \mathrm{wt}(V)$ maximal, so there exists $v \in V_\lambda$ such that $\tilde{\mathfrak{n}}^+ v = 0$. Also, since $\tilde{\mathfrak{h}}$ is abelian, then

$$y\, V_\lambda \subseteq V_\lambda \;\text{ for all } y \in U(\tilde{\mathfrak{h}}) \tag{2.2.17}$$

shows that there exists a nonzero $v \in V$ that is a common eigenvector for the action of $\tilde{\mathfrak{h}}$ and such that $\tilde{\mathfrak{n}}^+ v = 0$. It turns out that such a v is a highest-ℓ-weight vector, as seen in the following theorem.

Theorem 2.2.3. *([47], Theorem 2.1.13) Let V be a finite-dimensional $\tilde{\mathfrak{g}}$-module, and let $v \in V$ be an eigenvector for the action of $\tilde{\mathfrak{h}}$ such that $\tilde{\mathfrak{n}}^+ v = 0$. Then, v is a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\mu} \in \mathscr{P}^+$. In particular, if V is simple, then $V \cong V(\boldsymbol{\mu})$.* □

We are now interested in proving the converse of the last assertion of Theorem 2.2.3, i.e., showing that the modules of the form $V(\boldsymbol{\mu}), \boldsymbol{\mu} \in \mathscr{P}^+$, are finite-dimensional. The next result easily follows from the presentation given in Section 2.1 of [47]. However, regarding the way we chose to develop this section, an explicit proof is seen as necessary.

Proposition 2.2.4. *([47], Corollary 2.1.10) Given* $\boldsymbol{\mu} \in \mathscr{P}^+$ *as in* (2.2.9)*, then* $V(\boldsymbol{\mu}) \cong V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m)$. *In particular,* $V(\boldsymbol{\mu})$ *is finite-dimensional.*

Proof. Write $V = V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m)$. From (2.2.8), it is clear that

$$V(\lambda_j, a_j)_{\lambda_j} = V(a)_{\boldsymbol{\omega}_{\lambda_j, a_j}},\ 1 \leq j \leq m.$$

In particular, since each $V(\lambda_j, a_j)$ is simple, then $V(\lambda_j, a_j) \cong V(\boldsymbol{\omega}_{\lambda_j, a_j})$. Now, using the comultiplication formula (1.4.6) for the elements $\Lambda_{i,\pm r}, i \in I, r \in \mathbb{Z}$, one may check that

$$V(\lambda_1, a_1)_{\boldsymbol{\omega}_{\lambda_1, a_1}} \otimes \cdots \otimes V(\lambda_m, a_m)_{\boldsymbol{\omega}_{\lambda_m, a_m}} \subseteq V_{\boldsymbol{\mu}}.$$

Since each vector in $V(\lambda_j, a_j)_{\boldsymbol{\omega}_{\lambda_j, a_j}}$ is a highest-ℓ-weight vector, it follows that V possesses a nonzero highest-ℓ-weight vector of weight $\boldsymbol{\mu} \in \mathscr{P}^+$. Therefore, $V \cong V(\boldsymbol{\mu})$ follows by the fact that V is simple, as seen in Theorem 2.2.1. □

We finally see that Proposition 2.2.4 and Theorem 2.2.3 reestablish the result seen in Theorem 2.2.1 in the following way: every simple finite-dimensional $\tilde{\mathfrak{g}}$-module is of the form $V(\boldsymbol{\mu})$ for some $\boldsymbol{\mu} \in \mathscr{P}^+$. Conversely, every $V(\boldsymbol{\mu})$ with $\boldsymbol{\mu} \in \mathscr{P}^+$ is a simple finite-dimensional $\tilde{\mathfrak{g}}$-module. In fact, recalling that $V(\boldsymbol{\mu})$ and $V(\boldsymbol{\lambda})$ are inequivalent whenever $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$, one can also state that the isomorphism classes of simple finite-dimensional $\tilde{\mathfrak{g}}$-modules are in bijective correspondence with the set of Drinfeld polynomials.

2.3 Integrable modules

In this section, we briefly set aside the study of objects in the category $\tilde{\mathscr{G}}$ to focus on the broader category of integrable $\tilde{\mathfrak{g}}$-modules. This change of focus is seen as a technical prerequisite in order to study a few standard results that will be fundamental in the characterization of the local Weyl modules as being finite-dimensional in the next session.

Given a $\tilde{\mathfrak{g}}$-module V, an element $x \in \tilde{\mathfrak{g}}$ is said to act **locally nilpotently** on V if, for every $v \in V$, there exists some positive integer $k = k(v)$ such that $x^k v = 0$. We say that V is **integrable** if V is a weight-module in the sense of (1.3.1), and if the Chevalley generators $x_i^{\pm}, i \in I$, act locally nilpotently on V. Let $\mathscr{I}$ denote the **category** of all $\tilde{\mathfrak{g}}$-modules which are integrable. Since finite-dimensional $\tilde{\mathfrak{g}}$-modules are weight modules, it is clear from (2.2.16) that these modules are integrable.

The following lemma is adapted from Lemma 3.4 of [37], and is a useful tool in characterizing elements of a Lie algebra that act locally nilpotently on cyclic representations.

Lemma 2.3.1. *([37], Lemma 3.4) Let* $\{x_j : j \in \mathbb{N}\}$ *be a system of generators of a Lie algebra* L*, and let* $V = U(L) \cdot v$ *be a cyclic* L*-module. If* $x \in L$ *is such that* $(ad\, x)^{N_j} x_j = 0$

for some positive integers $N_j, j \in \mathbb{N}$, and if $x^m v = 0$ for some positive integer m, then x acts locally nilpotently on V. □

We now study a few standard properties of integrable modules. The proofs of (2.3.2)(2.3.3), (2.3.5) below are omitted in [47, 14], so we present them here using the results seen in Section 1.3. Let $V \in \mathrm{Ob}\,\mathscr{I}$. For $\lambda \in \mathrm{wt}(V)$, set

$$V_\lambda^+ = \{v \in V_\lambda : \tilde{\mathfrak{n}}^+ v = 0\}. \tag{2.3.1}$$

A nonzero vector in V_λ^+ (if any) is also referred to as a **highest-weight vector** of weight λ. For every $i \in I$, let $\mathfrak{g}_{(i)}$ be the Lie algebra generated by x_i^+, x_i^-, which is isomorphic to $\mathfrak{sl}_2$ in the obvious way by (1.1.2). Let $0 \neq v \in V_\lambda$ and set U to be the $\mathfrak{g}_{(i)}$-module $U = U(\mathfrak{g}_{(i)})v \subseteq V$. Since x_i^+, x_i^- both act locally nilpotently on V and v is a weight vector, it is clear that U is finite-dimensional by the PBW Theorem applied for $U(\mathfrak{g}_{(i)})$. In particular, from Corollary (1.3.2), we see that $\lambda(h_i) \in \mathbb{Z}$. Therefore,

$$\lambda \in \mathrm{wt}(V) \Rightarrow \lambda \in P. \tag{2.3.2}$$

Moreover, if $v \in V_\lambda^+$, then U is a simple highest-weight $\mathfrak{sl}_2$-module, and the following facts are consequences of Theorem 1.3.1:

$$\lambda \in P^+, \ \ (x_i^-)^{\lambda(h_i)+1} v = 0 \ \text{ for all } v \in V_\lambda^+,\ i \in I. \tag{2.3.3}$$

Furthermore, if $V = U(\tilde{\mathfrak{g}})V_\lambda^+$, then all the integral weights $\mu \in \mathrm{wt}(V)$ are such that $\mu \leq \lambda$, whence it follows that

$$\mathrm{wt}(V) \subseteq \mathrm{wt}(\lambda), \tag{2.3.4}$$

with $\mathrm{wt}(\lambda)$ being a finite set by Corollary 1.3.4. Also, since the vectors of the form $(x_{i,r_1}^-) \cdots (x_{i,r_{\lambda(h_i)+1}}^-) \cdot v$ have weight $\mu = \lambda - (\lambda(h_i)+1)\alpha_i$ with $\mu \notin \mathrm{wt}(\lambda)$, then

$$(x_{i,r_1}^-) \cdots (x_{i,r_{\lambda(h_i)+1}}^-) \cdot v = 0 \ \text{ for all } \ i \in I, r_j \in \mathbb{Z}, 1 \leq j \leq \lambda(h_i)+1. \tag{2.3.5}$$

We finish this section with some useful facts about V_λ^+.

Proposition 2.3.2. *([19], Proposition 1.1) Let $V \in Ob\,\mathscr{I}$ and suppose that $V_\lambda^+ \neq 0$.*

(i) $\Lambda_{i,\pm r} V_\lambda^+ = 0$ *for all* $i \in I, r > \lambda(h_i)$.

(ii) $\Lambda_{i,\pm\lambda(h_i)} V_\lambda^+ \subseteq V_\lambda^+ \setminus \{0\}$.

(iii) $(\Lambda_{i,\lambda(h_i)}\Lambda_{i,-s} - \Lambda_{i,\lambda(h_i)-s})V_\lambda^+ = 0$ *for all* $i \in I$, $0 \leq s \leq \lambda(h_i)$. □

2.4 The universal finite-dimensional highest-ℓ-weight modules

In this section, we introduce two relevant objects in the category $\mathscr{I}$: the global Weyl module (resp. local Weyl module) corresponding to a dominant integral weight (resp. Drinfeld polynomial). These objects will be seen to carry with them some notion of universality in the category $\mathscr{I}$, in the sense that any integrable module generated by a highest-weight vector (resp. finite-dimensional module generated by a highest-ℓ-weight vector) is isomorphic to a quotient of the global Weyl module (resp. local Weyl module) of the same weight (resp. ℓ-weight). In fact, the local Weyl modules will be constructed as quotients of global Weyl modules of the same highest-weight, by requiring the highest-weight space to be 1-dimensional.

Let $\lambda \in P^+$. The **global Weyl module** of highest-weight λ, denoted by $W(\lambda)$, is the quotient of $U(\tilde{\mathfrak{g}})$ by the left ideal $I(\lambda)$ generated by the elements

$$\tilde{\mathfrak{n}}^+, \quad (x_i^-)^{\lambda(h_i)+1} \text{ and } \quad h - \lambda(h), \quad \text{where } i \in I, h \in \mathfrak{h}. \tag{2.4.1}$$

Let us denote by w_λ the image of $1 \in U(\tilde{\mathfrak{g}})$ in $W(\lambda)$, so that $W(\lambda) = U(\tilde{\mathfrak{g}})w_\lambda$. If V is any integrable $\tilde{\mathfrak{g}}$-module generated by $v \in V_\lambda^+$, it is clear from (2.3.3) that v is annihilated by the generators in (2.4.1), so V is isomorphic to a quotient of $W(\lambda)$. In particular, if $\boldsymbol{\lambda} \in \mathscr{P}^+$ is such that $\lambda = \mathrm{wt}(\boldsymbol{\lambda})$, then the finite-dimensional irreducible $\tilde{\mathfrak{g}}$-module $V(\boldsymbol{\lambda})$ is isomorphic to a quotient of $W(\lambda)$, so $W(\lambda) \neq \{0\}$. It follows from the defining relations (1.5.2) of $\tilde{\mathfrak{g}}$ and Proposition 2.3.1 that the Chevalley generators act locally nilpotently on $W(\lambda)$. Also, it is clear from the PBW Theorem that $W(\lambda)$ is a weight module, so $W(\lambda)$ is an integrable $\tilde{\mathfrak{g}}$-module.

We regard $W(\lambda)$ as a right $U(\tilde{\mathfrak{h}})$-module via the action

$$(u\, w_\lambda) \cdot y = (uy) \cdot w_\lambda \quad \text{for all } u \in U(\tilde{\mathfrak{g}}), y \in U(\tilde{\mathfrak{h}}). \tag{2.4.2}$$

Proposition 2.4.1. *([13], Section 3.4) The above $U(\tilde{\mathfrak{h}})$-action on $W(\lambda)$ is well-defined.* □

The next theorem was proved in [19, 13]. The proof we present is adapted from [47] and was originally developed in the context of Weyl modules for quantum affine algebras.

Theorem 2.4.2. *The global Weyl module $W(\lambda)$ is finitely generated as a right $U(\tilde{\mathfrak{h}})$-module.*

Proof. Set $W = W(\lambda)$. Since W is integrable with $W = U(\tilde{\mathfrak{g}})W_\lambda^+$, then (2.3.4) holds. Therefore, it suffices to show that all the weight spaces are finitely generated as $U(\tilde{\mathfrak{h}})$-modules. More precisely, given $\mu \in \mathrm{wt}(W)$ such that $\sum_{i=1}^m \alpha_{i_j} = \lambda - \mu$, we show by induction on $m \geq 1$ that W_μ is generated by elements of the form

$$(x_{i_m,r_m}^-) \cdots (x_{i_1,r_1}^-) yv, \quad \text{where } v \in W_\lambda^+,\ y \in U(\tilde{\mathfrak{h}}),\ 0 \leq r_j < \lambda(h_{i_j}) + j - 1. \tag{2.4.3}$$

Let $m = 1, i = i_1$. In this case, a general element of W_μ is of the form $x^-_{i,r}\, yv$, where $y \in U(\tilde{\mathfrak{h}}), r \in \mathbb{Z}$. Since $(x^-_{i,s})^{\lambda(h_i)+1} v = 0$ for all $s \in \mathbb{Z}$, from Proposition 1.4.2 we derive the equality $0 = \left(X^-_{i,s-1,+}(u)\Lambda^+_i(u)\right)_{\lambda(h_i)+1} v = \sum_{t=0}^{\lambda(h_i)} x^-_{i,\lambda(h_i)+s-t}\Lambda_{i,t}v$. Isolating the terms $x^-_{i,\,\lambda(h_i)+s}v$, $x^-_{i,s}\Lambda_{i,\lambda(h_i)}v$, and using that $\left(\Lambda_{i,\lambda(h_i)}v\right)\Lambda_{i,-\lambda(h_i)} = v$, we have

$$x^-_{i,\lambda(h_i)+s}\, yv = -\sum_{t=1}^{\lambda(h_i)} x^-_{i,\lambda(h_i)+s-t}\Lambda_{i,t}yv \tag{2.4.4}$$

$$x^-_{i,s}\, yv = -\sum_{t=0}^{\lambda(h_i)-1} x^-_{i,\lambda(h_i)+s-t}\Lambda_{i,t}\Lambda_{i,-\lambda(h_i)}yv. \tag{2.4.5}$$

If $r \geq \lambda(h_i)$, we set $s = r - \lambda(h_i)$ in the first equality and proceed by induction on r to obtain the result, with the right-hand side of (2.4.4) being a sum of elements of the form (2.4.3) in the inductive step. If $r < 0$, we set $s = r$ in (2.4.5) and similarly proceed by induction on $|r|$.

Let $i = i_{m+1}, r = r_{m+1}$. Using the inductive hyphotesis on m, a general vector of W_μ is assumed to be of the form

$$x^-_{i,r}x^-_{i_m,r_m}\cdots x^-_{i_1,r_1}\, yv, \text{ where } 0 \leq r_j < \lambda(h_{i_j}) + j - 1,\ 1 \leq j \leq m. \tag{2.4.6}$$

Suppose first that $r \geq \lambda(h_i) + m$. From the equality

$$x^-_{i,r-1}x^-_{i_m,r_m+1} - x^-_{i_m,r_m+1}x^-_{i,r-1} = [x^-_i, x^-_{i_m}] \otimes t^{r+r_m} = x^-_{i,r}x^-_{i_m,r_m} - x^-_{i_m,r_m}x^-_{i,r},$$

we isolate the term $x^-_{i,r}x^-_{i_m,r_m}$ and substitute it on (2.4.6) to obtain:

$$x^-_{i,r}x^-_{i_m,r_m}\cdots x^-_{i_1,r_1}\, yv = \underbrace{x^-_{i,r-1}x^-_{i_m,r_m+1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\, yv}_{a} + \underbrace{x^-_{i_m,r_m}x^-_{i,r}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\, yv}_{b} - \underbrace{x^-_{i_m,r_m+1}x^-_{i,r-1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\, yv}_{c}.$$

On account of the above equality, it suffices to show that a, b and c lie in the span of vectors of the form (2.4.3). By the inductive hyphotesis on m, the vector $x^-_{i,r}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\, yv$ lies in the span of vectors of this form. Since $0 \leq r_m < \lambda(h_{i_m}) + m - 1 < \lambda(h_{i_m}) + m$, then b is shown to be written as a sum of vectors in the desired form. The same conclusion holds for c by a similar argument, since $0 \leq r_m + 1 < \lambda(h_{i_m}) + m$. As for a, the vector $x^-_{i_m,r_m+1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\, yv$ lies in the span of vectors of type (2.4.3), whence a is written as a sum of vectors of the form $x^-_{i,r-1}x^-_{i'_m,s_m}\cdots x^-_{i'_1,s_1}\, yv$, where $0 \leq s_j < \lambda(h_{i'_j}) + j - 1$, $i'_j \in \{i_1, \cdots, i_m\}$ for $1 \leq j \leq m$. Now, an inductive argument on $r \geq \lambda(h_i) + m$ establishes that these vectors lie in the span of vectors satisfying (2.4.3), so a is seen to be in that desired form as well.

As for the case $r < 0$, from the equality $x^-_{i,r}x^-_{i_m,r_m} = x^-_{i,r+1}x^-_{i_m,r_m-1} + x^-_{i_m,r_m}x^-_{i,r} -$

$x^-_{i_m,r_m-1}x^-_{i,r+1}$ we get

$$x^-_{i,r}x^-_{i_m,r_m}\cdots x^-_{i_1,r_1}y\,v = \underbrace{x^-_{i,r+1}x^-_{i_m,r_m-1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\,yv}_{a} + \underbrace{x^-_{i_m,r_m}x^-_{i,r}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\,yv}_{b} - \underbrace{x^-_{i_m,r_m}x^-_{i,r+1}x^-_{i_{m-1},r_{m-1}}\cdots x^-_{i_1,r_1}\,yv}_{c}.$$

Proceeding as in the previous case, the summands b and c can be written in the desired form by the inductive hyphotesis on m, and a further inductive argument on $|r|$ establishes the same conclusion for a. □

A noteworthy subalgebra of $\tilde{\mathfrak{g}}$ is $\mathfrak{g}[t] = \mathfrak{g}\otimes\mathbb{C}[t]$, which is generally known as the **current algebra** over $\mathfrak{g}$. Notice that the proof of the above theorem shows that $W(\lambda) = U(\mathfrak{g}[t])W^+_\lambda$. The following corollary follows from (2.2.17) and (2.4.3) seen in the proof above.

Corollary 2.4.3. *Let V be a quotient of $W(\lambda)$ such that V^+_λ is finite-dimensional. Then, V is a finite-dimensional $\tilde{\mathfrak{g}}$-module.* □

The result above gives the central idea behind the definition of the local Weyl modules which we now present. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ be a Drinfeld polynomial, and let $\lambda = \mathrm{wt}(\boldsymbol{\lambda}) \in P^+$ be its underlying weight. The **local Weyl module** of highest-ℓ-weight $\boldsymbol{\lambda}$, denoted by $W(\boldsymbol{\lambda})$, is the quotient of the global module $W(\lambda)$ by the ideal generated by the elements $y - \boldsymbol{\lambda}(y), y \in U(\tilde{\mathfrak{h}})$. More precisely, $W(\boldsymbol{\lambda})$ is set to be the quotient of $U(\tilde{\mathfrak{g}})$ by the left ideal $I(\boldsymbol{\lambda})$ generated by

$$\tilde{\mathfrak{n}}^+,\ \ y - \boldsymbol{\lambda}(y),\ \ (x^-_i)^{\lambda(h_i)+1} \quad \text{for all} \quad y \in U(\tilde{\mathfrak{h}}), i \in I. \tag{2.4.7}$$

By (2.4.7), it is clear that $W(\boldsymbol{\lambda})$ is a highest-ℓ-weight module with $W(\boldsymbol{\lambda})_\lambda$ being finite-dimensional and spanned by the image of $1 \in U(\tilde{\mathfrak{g}})$ in $W(\boldsymbol{\lambda})$, which we denote by $w_{\boldsymbol{\lambda}}$. In particular, $W(\boldsymbol{\lambda})$ is finite-dimensional by Corollary 2.4.3. In fact, $W(\boldsymbol{\lambda})$ is universal among the finite-dimensional highest-ℓ-weight modules of highest ℓ-weight $\boldsymbol{\lambda}$, as they are clearly isomorphic to a quotient of $W(\boldsymbol{\lambda})$ by (2.3.3). Furthermore, since $V(\boldsymbol{\lambda})$ is finite-dimensional by Proposition 2.2.4, we see that $V(\boldsymbol{\lambda})$ is actually the unique irreducible quotient of $W(\boldsymbol{\lambda})$. In this sense, one could also introduce the finite-dimensional highest-ℓ-weight modules $V(\boldsymbol{\lambda})$ as the irreducible quotients of the local Weyl module $W(\boldsymbol{\lambda})$ of the same ℓ-weight.

We utilize the $U(\tilde{\mathfrak{h}})$-action (2.4.2) on the global Weyl module $W(\lambda)$ to give an alternative characterization of the local Weyl module $W(\boldsymbol{\lambda})$ as seen in [47, 13], which is equivalent to the characterization seen here. Let $\mathscr{A}$ be the subalgebra of $U(\tilde{\mathfrak{h}})$ generated by the elements $\Lambda_{i,r}, i \in I, r = 1, \cdots, \lambda(h_i)$, so that $W(\lambda)_\lambda$ is generated by the action of $\mathscr{A}$ by Proposition 2.3.2. We denote by ann_λ the kernel of this representation, which is proper since $\Lambda_{i,\lambda(h_i)} \notin \mathrm{ann}_\lambda$, and denote by $\mathscr{A}_\lambda$ the quotient $\mathscr{A}/\mathrm{ann}_\lambda$ with associated canonical homomorphism $\pi : \mathscr{A} \to \mathscr{A}_\lambda$. Since $\mathscr{A}_\lambda$ is a finitely generated commutative algebra over an algebraically closed field, its maximal ideals in $\mathrm{specm}(\mathscr{A}_\lambda)$ are generated by elements of the form $\pi(\Lambda_{i,r}) - a_{i,r}$ for some $a_{i,r} \in \mathbb{C}$. In particular, we may consider the maximal ideal $\mathfrak{m} \subseteq \mathscr{A}_\lambda$ generated by $\pi(\Lambda_{i,r}) - \boldsymbol{\lambda}(\Lambda_{i,r})$. Since $W(\lambda)_\lambda$ is an $\mathscr{A}_\lambda$-module whose action is the one inherited by $\mathscr{A}$, let M be the submodule generated by $\mathfrak{m}W(\lambda)_\lambda$, and set $W(\mathfrak{m}) = W(\lambda)/M$.

Proposition 2.4.4. *([13]) $W(\mathfrak{m})$ and $W(\boldsymbol{\lambda})$ are isomorphic as $\tilde{\mathfrak{g}}$-modules.* □

For $\lambda \in P^+, a \in \mathbb{C}^\times$, let us refer to the ℓ-weights of the form $\boldsymbol{\omega}_{\lambda,a}$ as **Drinfeld polynomials with a unique root.** The following theorem shows that the decomposition (2.2.9) of $\boldsymbol{\lambda} \in \mathscr{P}^+$ translates into a decomposition of $W(\boldsymbol{\lambda})$ in terms of tensor products of local Weyl modules whose ℓ-weights are Drinfeld polynomials with a unique root.

Theorem 2.4.5. *([19], Theorem 2) If $\boldsymbol{\lambda} \in \mathscr{P}^+$ decomposes as in* (2.2.9)*, then*

$$W(\boldsymbol{\lambda}) \cong W(\boldsymbol{\omega}_{\lambda_1,a_1}) \otimes \cdots \otimes W(\boldsymbol{\omega}_{\lambda_m,a_m}).$$ □

Since in the next chapter we shall be concerned exclusively with local Weyl modules for the Lie algebra $\mathfrak{sl}_{n+1}$, we now further characterize the local Weyl modules in the particular context of $\mathfrak{sl}_{n+1}$. Henceforth, we drop the hyphotesis of $\mathfrak{g}$ being a general simple finite-dimensional Lie algebra, and set $\mathfrak{g} = \mathfrak{sl}_{n+1}$.

We first develop necessary and sufficient criteria for a given local Weyl module to be irreducible, i.e., to satisfy $W(\boldsymbol{\lambda}) \cong V(\boldsymbol{\lambda})$. Given $\beta \in \Phi^+, \boldsymbol{\lambda} \in \mathscr{P}^+$, define $\pi_\beta^{\boldsymbol{\lambda}} \in \mathbb{C}[[u]]$ by

$$\Lambda_\beta^+(u) \cdot w_{\boldsymbol{\lambda}} = \pi_\beta^{\boldsymbol{\lambda}}(u) \cdot w_{\boldsymbol{\lambda}}.$$

Since $W(\boldsymbol{\lambda})$ is finite-dimensional, $\pi_\beta^{\boldsymbol{\lambda}}$ is in fact a polynomial. Now, if $\boldsymbol{\lambda} = \prod_{i=1}^m \boldsymbol{\omega}_{\lambda_i, a_i}$ is the decomposition of $\boldsymbol{\lambda}$ as in (2.2.9), define also the polynomials

$$\pi_j = \prod_{i=1}^m \left(\boldsymbol{\omega}_{\lambda_i(h_j)\omega_j, a_i}\right)_j \in \mathbb{C}[u], \text{ for } 1 \leq j \leq n. \tag{2.4.8}$$

Lemma 2.4.6. *([19], Lemma 3.1) Let $\beta = \sum_{i=1}^n r_i\alpha_i \in \Phi^+$. Then,*

$$\pi_\beta^{\boldsymbol{\lambda}}(u) = \prod_{i=1}^n \pi_i^{r_i(\alpha_i,\alpha_i)/(\beta,\beta)}(u).$$ □

It turns out that the roots of $\pi_\theta^{\boldsymbol{\lambda}}$ give the intended irreducibility criteria for $W(\boldsymbol{\lambda})$.

Theorem 2.4.7. *([19], Theorem 3) Given $\boldsymbol{\lambda} \in \mathscr{P}^+$, the local Weyl module $W(\boldsymbol{\lambda})$ is irreducible if and only if $\pi_\theta^{\boldsymbol{\lambda}}$ has simple roots. In particular, $W(\boldsymbol{\omega}_{i,a}) \cong V(\omega_i, a)$ for $i \in I, a \in \mathbb{C}^\times$.* □

We further specialize Theorem 2.4.7 with a description of $W(\boldsymbol{\lambda})$ in terms of tensor products of evaluation modules. Let $\boldsymbol{\lambda} \in \mathscr{P}^+$ and suppose that $\pi_\theta^{\boldsymbol{\lambda}}$ has simple roots. Since $\theta = \sum_{i=1}^n \alpha_i$ is the maximal root of $\mathfrak{g}$, notice that Lemma 2.4.6 gives us

$$\pi_\theta^{\boldsymbol{\lambda}}(u) = \prod_{i=1}^n \pi_i(u).$$

By our supposition on $\pi_\theta^{\boldsymbol{\lambda}}$, notice that π_j has simple roots for all $1 \leq j \leq n$, and that π_i is relatively prime to π_j for all $i \neq j$. Thus, regarding (2.4.8), any λ_i must satisfy $0 \leq \lambda_i(h_j) \leq 1$ for all $1 \leq j \leq n$, and, if $\lambda_i(h_{j_0}) \neq 0$ for some j_0, then $\lambda_i(h_k) = 0$ for all $k \neq j_0$. Therefore, each λ_i must be a fundamental weight, and $\boldsymbol{\lambda} \in \mathscr{P}^+$ is of the form

$$\boldsymbol{\lambda} = \prod_{k=1}^m \boldsymbol{\omega}_{i_k,a_k}, \quad \text{where } i_k \in \{1, \cdots, n\}, 1 \leq k \leq m, \text{ and } a_i \neq a_j \text{ for } i \neq j. \tag{2.4.9}$$

In particular, since $W(\boldsymbol{\omega}_{i_k,a_k}) \cong V(\omega_{i_k}, a_k)$ for all $1 \leq k \leq m$, it follows from Theorem 2.4.5 that, if $W(\boldsymbol{\lambda})$ is simple, $W(\boldsymbol{\lambda})$ decomposes as the tensor product

$$W(\boldsymbol{\lambda}) \cong \bigotimes_{k=1}^m V(\omega_{i_k}, a_k). \tag{2.4.10}$$

Conversely, any such tensor product is isomorphic to a simple local Weyl module whose corresponding ℓ-weight is of the form (2.4.9) with π_θ^λ having distinct roots.

We finish this section with a decomposition theorem for the local Weyl modules when regarded as $\mathfrak{g}$-modules. In face of Theorem 2.4.5, it will be sufficient to study how this decomposition unfolds with respect to the local Weyl modules whose corresponding ℓ-weight is a Drinfeld polynomial with a unique root. Such result will be obtained as a consequence of studying the Weyl modules for the current algebra $\mathfrak{g}[t]$ to be briefly discussed now.

Let $W(\boldsymbol{\omega}_{\lambda,a})$ be a local Weyl module with generator $w_{\lambda,a}$ and regard it as a module for $\mathfrak{g}[t]$. From (1.4.5), notice that each $h_{i,r}, i \in I, r \in \mathbb{Z}_{>0}$, can be written as a linear combination of $\Lambda_{i,r}$ and products of terms $h_{i,s}$ with $0 < s < r$. A straightforward inductive argument on r then shows that $h_{i,r} \cdot w_{\lambda,a} = \lambda(h_i) a^r w_{\lambda,a}$, which is equivalent to

$$\Big(h_i \otimes (t-a)^r\Big) \cdot w_{\lambda,a} = 0 \text{ for all } r \geq 1. \tag{2.4.11}$$

Setting $\varphi_a : \mathfrak{g}[t] \to \mathfrak{g}[t]$ to be the homomorphism of Lie algebras

$$x \otimes t^k \mapsto x \otimes (t-a)^k, k \geq 0,$$

let $\varphi_a^*(W(\boldsymbol{\omega}_{\lambda,a}))$ be the pullback of $W(\boldsymbol{\omega}_{\lambda,a})$ along φ_a, so $\varphi_a^*(W(\boldsymbol{\omega}_{\lambda,a}))$ is the $\mathfrak{g}[t]$-module with the action

$$\left(x \otimes t^k\right) \cdot u = (x \otimes (t-a)^k) \cdot u \quad \text{for all} \quad u \in W(\boldsymbol{\omega}_{\lambda,a}), k \geq 0.$$

Following [14], the **Weyl module for the current algebra** $\mathfrak{g}[t]$ of highest-weight λ, denoted by $W_t(\lambda)$, is the quotient of $U(\mathfrak{g}[t])$ by the ideal generated by the elements

$$\mathfrak{n}^+ \otimes \mathbb{C}[t], \quad h \otimes t^k - \delta_{0,k}\lambda(h), k \in \mathbb{Z}_{\geq 0}, \quad (x_i^-)^{\lambda(h_i)+1}. \tag{2.4.12}$$

From (2.4.11) and the generators in (2.4.7), it is clear that $w_{\lambda,a}$ is annihilated by the generators in (2.4.12), so $\varphi_a^*(W(\boldsymbol{\omega}_{\lambda,a}))$ is isomorphic to a quotient of $W_t(\lambda)$. The Weyl module $W_t(\lambda)$ has been the object of attention in a series of papers, [14, 26, 48] for instance. In particular, it follows from the work seen in those articles that these modules are in fact isomorphic. Furthermore, in [14] a further connection between $W_t(\lambda)$ and the so-called Demazure modules has also been established, with the latter having an explicit known decomposition as a $\mathfrak{g}$-module in terms of tensor products. We summarize both of these fundamental results in the next theorem.

Theorem 2.4.8. *([14, 26])*

(i) As $\mathfrak{g}[t]$-modules, $W_t(\lambda) \cong \varphi_a^*((W(\boldsymbol{\omega}_{\lambda,a}))$ for all $a \in \mathbb{C}^\times$.

(ii) As $\mathfrak{g}$-modules, $W_t(\lambda) \cong V(\omega_1)^{\otimes\lambda(h_1)} \otimes \cdots \otimes V(\omega_n)^{\otimes\lambda(h_n)}$. □

The structure of the local Weyl module $W(\boldsymbol{\omega}_{\lambda,a})$ as a $\mathfrak{g}$-module is explicitly obtained as a immediate consequence of the above theorem. In fact, regarding (2.2.9), we more generally state:

Corollary 2.4.9. *Let $\boldsymbol{\lambda} \in \mathscr{P}^+$, and set $\lambda = wt(\boldsymbol{\lambda})$ to be the underlying weight. As $\mathfrak{g}$-modules,*

$$W(\boldsymbol{\lambda}) \cong V(\omega_1)^{\otimes\lambda(h_1)} \otimes \cdots \otimes V(\omega_n)^{\otimes\lambda(h_n)}.$$

In particular, the structure of $W(\boldsymbol{\lambda})$ as a $\mathfrak{g}$-module is solely dependent on $\lambda = wt(\boldsymbol{\lambda})$. □

3 The classical and affine Schur-Weyl Duality

In this chapter, we restrict the symbols $\mathfrak{g}, \tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ to denote, respectively, the Lie algebra $\mathfrak{sl}_{n+1}$, its associated loop algebra and affinization. All the notation developed in Section 1.7 will be followed closely. In the upcoming sections, we shall give a detailed proof of both of the equivalences of categories mentioned in the Introduction: the classical Schur-Weyl duality and the affine Schur-Weyl duality. Moreover, following [17], we also introduce the analogue of Zelevinsky tensor products of affine Hecke algebras in the setting of modules for the extended affine symmetric group. It will be seen that this tool is instrumental not only for a complete description of the simple objects assigned by the affine Schur-Weyl duality, but also for an investigation of the modules corresponding via this functor to the local Weyl modules. Unless explicitly stated otherwise, all the local Weyl modules here considered are supposed to be $\tilde{\mathfrak{g}}$-modules.

The problem of writing a precise characterization of the affine Schur-Weyl duality and describing the modules corresponding to local Weyl modules under this equivalence was originally addressed as part of the Postdoctoral project [4]. A draft containing the progress made in the context of that project was the base of our initial steps, serving mostly as a reading guide to the existing literature. Example 3.4.4 below is the only result here that is more heavily supported by the original material contained in the mentioned draft.

The main references utilized in this chapter are [31, 17]. In Sections 3.1, 3.2 and 3.3 below, except for Theorem 3.1.3, all results here presented either have their proofs omitted in these references or they were originally developed in the context of quantum affine algebras and Hecke algebras. Since one of the objectives of this chapter is to give a very precise version of both the classical and affine Schur-Weyl duality, these results will be accompanied by a proof.

3.1 The classical Schur-Weyl duality

In this section we give a proof of the classical Schur-Weyl duality. As discussed in the Introduction of this work, although the Schur-Weyl duality was originally presented as a result stating that the image of $\mathbb{C}[\mathcal{S}_\ell]$ and $\mathbb{C}[GL_{n+1}]$ on $\mathrm{End}(\mathbb{V}^{\otimes \ell})$ are each other's commutant, we shall concern ourselves exclusively with one of the consequences of this result which establishes an equivalence between certain categories of finite-dimensional modules for the symmetric group $\mathcal{S}_\ell$ and the Lie algebra $\mathfrak{g}$.

One of the categories of our interest is the abelian **category** $\mathscr{S}_\ell$ of finite-dimensional right $\mathcal{S}_\ell$-modules. The other category of our interest will be defined as a full subcategory of the abelian **category** $\mathscr{G}$ of finite-dimensional left $\mathfrak{g}$-modules as follows. Given $\lambda \in \mathfrak{h}^*$, set $\mathrm{lv}(\lambda)$ to be the **level** of λ defined by the scalar

$$\mathrm{lv}(\lambda) = \sum_{j=1}^{n} j\lambda(h_j).$$

Setting $P_\ell^+ = \{\lambda \in P^+ : \mathrm{lv}(\lambda) = \ell\}$, let $\mathscr{G}_\ell$ be the **full subcategory** of $\mathscr{G}$ given by all finite-dimensional left $\mathfrak{g}$-modules V such that

$$[V : V(\lambda)] \neq 0 \Rightarrow \lambda \in P_\ell^+.$$

Since $\{0\} \in \mathscr{G}_\ell$ vacuously, it follows from Weyl's theorem on complete reducibility and Lemma 1.9.1 that $\mathscr{G}_\ell$ is also an abelian category of modules. Moreover, Weyl's theorem also ensures that the objects of $\mathscr{G}_\ell$ can be equivalently characterized as the finite-dimensional $\mathfrak{g}$-modules whose irreducible summands also occur in $\mathbb{V}^{\otimes \ell}$, as seen in the next result.

Proposition 3.1.1. *If $\ell \leq n$, the weight $\lambda \in P_\ell^+$ if and only if $[\mathbb{V}^{\otimes \ell} : V(\lambda)] \neq 0$.*

Proof. Write $\lambda = \sum_{k=1}^n c_k \omega_k$ so that $\sum_k kc_k = \ell$. Since $V(\omega_k) \cong \Lambda^k(\mathbb{V})$ by Theorem 1.7.1, we choose any highest-weight vector $w_k \in \Lambda^k(\mathbb{V})$ of weight ω_k, $1 \leq k \leq n$. Notice that $w_k^{\otimes c_k} \in \left(\Lambda^k(\mathbb{V})\right)^{\otimes c_k}$ is then a highest-weight vector of weight $c_k\omega_k$, whence

$$w = w_1^{\otimes c_1} \otimes \cdots \otimes w_n^{\otimes c_n} \in \bigotimes_{k=1}^{n} \left(\Lambda^k(\mathbb{V})\right)^{\otimes c_k} \text{ is a highest-weight vector of weight } \lambda.$$

Using the canonical inclusion $\alpha_k : \Lambda^k(\mathbb{V}) \hookrightarrow V^{\otimes k}$, $1 \leq k \leq n$, the tensor product of linear transformations $\alpha_k^{\otimes c_k} : \left(\Lambda^k(\mathbb{V})\right)^{\otimes c_k} \to \mathbb{V}^{\otimes kc_k}$ maps $w_k^{\otimes c_k}$ into a non-zero vector of $\mathbb{V}^{\otimes kc_k}$, so $\alpha = \alpha_1^{\otimes c_1} \otimes \cdots \otimes \alpha_n^{\otimes c_n}$ maps w into a non-zero vector of $\mathbb{V}^{\otimes \ell}$. Since each α_k is a homomorphism of $\mathfrak{g}$-modules, in particular, the map α is also a homomorphism of $\mathfrak{g}$-modules, hence $v = \alpha(w)$ is a highest-weight vector of weight λ in $\mathbb{V}^{\otimes \ell}$. Thus, the submodule $U(\mathfrak{g})v$ generated by this vector, being indecomposable and finite-dimensional, implies that $U(\mathfrak{g})v \cong V(\lambda)$, whence $[\mathbb{V}^{\otimes \ell} : V(\lambda)] \neq 0$.

Now, suppose that $[\mathbb{V}^{\otimes \ell} : V(\lambda)] \neq 0$. Then, there exists $\beta \in Q^+$ such that $\lambda = \ell\omega_1 - \beta$ by (1.7.17). Noticing that $\mathrm{lv}(\alpha_i) = 0$ for $i < n$ and $\mathrm{lv}(\alpha_n) = n+1$, then $\mathrm{lv}(\beta) = k(n+1)$ for some $k \in \mathbb{Z}_{\geq 0}$. Since $\mathrm{lv}(\lambda) = \ell - \mathrm{lv}(\beta)$, if $k > 0$, then $\ell \leq n$ implies the contradiction $0 < \mathrm{lv}(\lambda) = \ell - \mathrm{lv}(\beta) = \ell - k(n+1) \leq 0$. Therefore, $\mathrm{lv}(\lambda) = \ell$. □

We now state the classical Schur-Weyl duality.

Theorem 3.1.2. *Under the restriction $\ell \leq n$, the categories $\mathscr{S}_\ell$ and $\mathscr{G}_\ell$ are equivalent.*

Henceforth, we assume $\ell \leq n$. Let $\mathbb{V} = \mathbb{C}^{n+1}$ and let $\{v_1, \cdots, v_{n+1}\}$ be its canonical basis. Since the ℓ-th tensor power $\mathbb{V}^{\otimes \ell}$ is a left $\mathcal{S}_\ell$-module by the action (1.7.10) of permuting factors, given a right $\mathcal{S}_\ell$-module $M \in \mathscr{S}_\ell$, one can consider the tensor product of modules over the ring $\mathbb{C}[\mathcal{S}_\ell]$:

$$\mathcal{F}_\ell(M) = M \underset{\mathbb{C}[\mathcal{S}_\ell]}{\otimes} \mathbb{V}^{\otimes \ell}, \tag{3.1.1}$$

which is a $\mathfrak{g}$-module via the induced action of $\mathfrak{g}$ given by

$$x\cdot(m \otimes v) = m \otimes x\cdot v \quad \text{for } m \in M, v \in \mathbb{V}^{\otimes \ell}, x \in \mathfrak{g}. \tag{3.1.2}$$

In particular, we record the following isomorphisms of $\mathfrak{g}$-modules

$$\mathcal{F}_\ell(\mathbb{C}[\mathcal{S}_\ell]) \cong \mathbb{V}^{\otimes \ell} \quad \text{and} \quad \mathcal{F}_\ell(\mathcal{S}(\xi)) \cong c_\xi \cdot \mathbb{V}^{\otimes \ell} \tag{3.1.3}$$

given by $m \otimes v \mapsto m \cdot v$. The construction (3.1.1) gives rise to the **functor** $\mathcal{F}_\ell : \mathscr{S}_\ell \to \mathscr{G}_\ell$ mapping each object and each morphism $f : M \to N$ of $\mathcal{S}_\ell$-modules as

$$M \mapsto \mathcal{F}_\ell(M), \; f \mapsto \mathcal{F}_\ell(f) = f \otimes 1,$$

where 1 denotes the identity map of $\mathrm{End}(\mathbb{V}^{\otimes \ell})$. It is clear from the properties of tensor products that $\mathcal{F}_\ell$ is an additive functor. Moreover, since the $\mathfrak{g}$-structure of $\mathcal{F}_\ell(M)$ is the one induced by $\mathbb{V}^{\otimes \ell}$, it follows from Proposition 3.1.1 that $[\mathcal{F}_\ell(M) : V(\lambda)] \neq 0 \Rightarrow \lambda \in P_\ell^+$, showing that indeed $\mathcal{F}_\ell(M) \in \mathscr{G}_\ell$.

Let $\mathrm{wt} : \mathcal{P}_\ell \to P^+$ be the map defined on the set of partitions of ℓ by

$$\xi = (\xi_j)_{j\geq 1} \mapsto \sum_{j=1}^{n} (\xi_j - \xi_{j+1})\omega_j. \tag{3.1.4}$$

We have $P_\ell^+ = \mathrm{Im}\,\mathrm{wt}$. Indeed, given $\xi \in \mathcal{P}_\ell$, then $|\xi| \leq n$ implies that

$$\mathrm{lv}(\mathrm{wt}(\xi)) = \sum_{j=1}^{n} j(\xi_j - \xi_{j+1}) = \sum_{j=1}^{n} \xi_j = \ell, \tag{3.1.5}$$

whence $\mathrm{Im}\,\mathrm{wt} \subseteq P_\ell^+$ follows. Conversely, given $\lambda \in P_\ell^+$, set ξ_λ to be the partition

$$\xi_\lambda = \left(\sum_{i=1}^{n} \lambda(h_i), \sum_{i=2}^{n} \lambda(h_i), \; \cdots \; , \lambda(h_n) \right) \tag{3.1.6}$$

such that $\sum_j (\xi_\lambda)_j = \mathrm{lv}(\lambda) = \ell$ and $\lambda(h_j) = (\xi_\lambda)_j - (\xi_\lambda)_{j+1}$. Thus, $\mathrm{wt}(\xi_\lambda) = \lambda$ shows the remaining inclusion $P_\ell^+ \subseteq \mathrm{Im}\,\mathrm{wt}$.

The following theorem plays an essential role in the proof of Theorem 3.1.2.

Theorem 3.1.3. *([31], Proposition 15.15) Let $M \cong \mathcal{S}(\xi)$ be an irreducible $\mathcal{S}_\ell$-module for $\xi \in \mathcal{P}_\ell$. Then, $\mathcal{F}_\ell(M) \cong V(wt(\xi))$ as $\mathfrak{g}$-modules. In particular, $\mathcal{F}_\ell(M) \in \mathscr{G}_\ell$.* □

We finally prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Since we are dealing with the categories $\mathscr{G}_\ell$ and $\mathscr{S}_\ell$ in which every object is completely reducible, and $\mathcal{F}_\ell : \mathscr{S}_\ell \to \mathscr{G}_\ell$ is additive, it suffices to show that $\mathcal{F}_\ell$ is essentially surjective, full and faithful for irreducible objects in the corresponding categories as in the steps below.

1. Given V in $\mathscr{G}_\ell$ irreducible, there exists $\xi \in \mathcal{P}_\ell$ such that $\mathcal{F}_\ell(\mathcal{S}(\xi)) \cong V$.

Since V is irreducible, it follows that $V \cong V(\lambda)$ for some $\lambda \in P_\ell^+$. Using the fact that wt is surjective on P_ℓ^+, we have $\mathrm{wt}(\xi_\lambda) = \lambda$, and Theorem 3.1.3 ensures $\mathcal{F}_\ell(\mathcal{S}(\xi_\lambda)) \cong V(\lambda)$.

2. Given $M, N \in \mathscr{S}_\ell$ simple, set $V = \mathcal{F}_\ell(M), W = \mathcal{F}_\ell(N)$. Then, $\mathcal{F}_\ell : \mathrm{Hom}_{\mathscr{S}_\ell}(M,N) \to \mathrm{Hom}_{\mathscr{G}_\ell}(V,W)$, defined by $f \mapsto f \otimes 1$, is an isomorphism.

Let $M \cong \mathcal{S}(\xi)$ and $N \cong \mathcal{S}(\chi)$, where $\xi, \chi \in \mathcal{P}_\ell$. By Theorem 3.1.3, it follows that $V \cong V(\lambda)$ and $W \cong V(\mu)$, where $\lambda = \mathrm{wt}(\xi)$ and $\mu = \mathrm{wt}(\chi)$. By irreducibility, notice that $\mathrm{Hom}_{\mathscr{G}_\ell}(V,W) \neq \{0\}$ if and only if V and W are isomorphic. Since both are highest-weight modules, it turns out that $\mathrm{Hom}_{\mathscr{G}_\ell}(V,W) \neq \{0\}$ if and only if $\lambda = \mu$. Similarly, by irreducibility, we have $\mathrm{Hom}_{\mathscr{S}_\ell}(M,N) \neq \{0\}$ if and only if $\xi = \chi$. Therefore, $\mathrm{Hom}_{\mathscr{S}_\ell}(M,N) \neq \{0\}$ if and only if $\mathrm{Hom}_{\mathscr{G}_\ell}(V,W) \neq \{0\}$. Now, if $\mathrm{Hom}_{\mathscr{G}_\ell}(V,W) = \{0\}$, the map $\mathcal{F}_\ell$ is clearly surjective. Assuming $\mathrm{Hom}_{\mathscr{G}_\ell}(V,W) \neq \{0\}$, then $\mathrm{Hom}_{\mathscr{S}_\ell}(M,N) \neq \{0\}$. Since $\dim \mathrm{Hom}_{\mathscr{G}_\ell}(V,W) = 1$, it suffices to exhibit $f \in \mathrm{Hom}_{\mathscr{S}_\ell}(M,N)$ such that $0 \neq f \otimes 1$ in $\mathrm{Hom}_{\mathscr{G}_\ell}(V,W)$. Choosing any $0 \neq f \in \mathrm{Hom}_{\mathscr{S}_\ell}(M,N)$, it is necessarily an isomorphism of $\mathcal{S}_\ell$-modules and, since $(f \otimes 1) \circ (f^{-1} \otimes 1) = 1_V, (f^{-1} \otimes 1) \circ (f \otimes 1) = 1_W$, it follows that $f \otimes 1 \neq 0$, so $\mathcal{F}_\ell$ must be surjective. For the injectivity, assume that $f \otimes 1 = g \otimes 1$. Hence, $(f-g) \otimes 1 = 0$ and $f - g = 0$ follows, otherwise $f - g \neq 0$ is an isomorphism and $(f-g) \otimes 1 \neq 0$ by the previous argument. ☐

We finalize this section with the next two results in [17] which will be relevant to the study of the affine Schur-Weyl duality in the next section.

Proposition 3.1.4. *Let $M \in \mathscr{S}_\ell$ and write $\mathcal{P}_\ell(M) = \{\xi \in \mathcal{P}_\ell : [M : \mathcal{S}(\xi)] \neq 0\}$. Suppose that $v_0 \in \mathbb{V}^{\otimes \ell}$ satisfies $[U(\mathfrak{g})v_0 : V(wt(\xi))] \neq 0$ for all $\xi \in \mathcal{P}_\ell(M)$. Then, the linear map $M \to \mathcal{F}_\ell(M)$, $m \mapsto m \otimes v_0$, is injective.*

Proof. First, let us suppose that M is simple, say $M \cong \mathcal{S}(\eta) = c_\eta \cdot \mathbb{C}[\mathcal{S}_\ell]$, where $\eta \in \mathscr{S}_\ell$ and $c_\eta \in \mathbb{C}[\mathcal{S}_\ell]$ is the corresponding Young symmetrizer. Fix any isomorphism $f : M \to \mathcal{S}(\eta)$ and notice that $\mathcal{F}_\ell(M) \cong c_\eta \cdot \mathbb{V}^{\otimes \ell}$ via the isomorphism $m \otimes v \mapsto f(m) \cdot v$. If $0 \neq m \in M$ is such that $m \otimes v_0 = 0$, then $f(m) \cdot v_0 = 0$ implies that $\mathrm{ann}\, v_0 = \{s \in \mathcal{S}(\eta) : s \cdot v_0 = 0\}$ is a non-zero submodule of $\mathcal{S}(\eta)$, so that $\mathcal{S}(\eta) = \mathrm{ann}\, v_0$ by irreducibility. In particular, this implies that $c_\eta \cdot v_0 = 0$, thus, in order to show the injectivity of $m \mapsto m \otimes v_0$, it suffices to show that $c_\eta \cdot v_0 \neq 0$.

Since $\mathbb{C}[\mathcal{S}_\ell]$ decomposes as in (1.6.3), and $\mathcal{F}_\ell(\mathcal{S}(\xi)) \cong c_\xi \cdot \mathbb{V}^{\otimes \ell}$ by (3.1.3), we have

$$\bigoplus_{\xi \in \mathcal{P}_\ell} \mathcal{F}_\ell(\mathcal{S}(\xi))^{\oplus d_\xi} \cong \bigoplus_{\xi \in \mathcal{P}_\ell} \left(c_\xi \cdot \mathbb{V}^{\otimes \ell}\right)^{\oplus d_\xi} = \mathbb{V}^{\otimes \ell}.$$

We claim that the projection v' of v_0 in $\left(c_\eta \cdot \mathbb{V}^{\otimes \ell}\right)^{\oplus d_\eta}$ is not zero. Indeed, given any $\xi \in \mathcal{P}_\ell$, by Theorem 3.1.3 and (3.1.3) we have $V(\mathrm{wt}(\xi)) \cong c_\xi \cdot \mathbb{V}^{\otimes \ell}$, so $[c_\xi \cdot \mathbb{V}^{\otimes \ell} : V(\mathrm{wt}(\eta))] = 0$ for all $\xi \neq \eta$. In particular, we have $\left[\left(c_\xi \cdot \mathbb{V}^{\otimes \ell}\right) \cap U(\mathfrak{g})v_0 : V(\mathrm{wt}(\eta))\right] = 0$. Since by hyphotesis we have $[U(\mathfrak{g})v_0 : V(\mathrm{wt}(\eta))] \neq 0$, it follows that $v' \neq 0$.

By Proposition 1.6.2, recall that $c_\eta^2 = \lambda c_\eta$ for some scalar $\lambda \neq 0$. Writing $v' = c_\eta \cdot v''$ with $v'' \in \left(\mathbb{V}^{\otimes \ell}\right)^{\oplus d_\eta} \setminus \{0\}$, we have

$$c_\eta \cdot v' = \lambda c_\eta \cdot v'' = \lambda v' \neq 0,$$

whence $c_\eta \cdot v_0 \neq 0$. This proves the injectivity of $m \mapsto m \otimes v_0$ in the case that M is simple.

Now, for the general case, we have $M \cong \bigoplus_j \mathcal{S}(\xi_j)$ and $\mathcal{F}_\ell(M) \cong \bigoplus_j \mathcal{F}_\ell(\mathcal{S}(\xi_j))$. Since $\mathcal{S}(\xi_j) \to \mathcal{F}_\ell(\mathcal{S}(\xi_j))$ is injective, if $x = x_1 + \cdots + x_n \in M$ satisfies $x \otimes v_0 = 0$, as the sum is direct, we have $x_1 \otimes v_0 = \cdots = x_n \otimes v_0 = 0$, then $x_1 = \cdots = x_n = 0$. □

Corollary 3.1.5. *If $\boldsymbol{v} = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathbb{V}^{\otimes \ell}$ with $i_j \neq i_k$ for $j \neq k$, then the linear map $M \to \mathcal{F}_\ell(M), m \mapsto m \otimes \boldsymbol{v}$ is injective.*

Proof. Since the factors composing $\mathbf{v}$ are all distinct, it follows from Proposition 1.7.2 that $\mathbb{V}^{\otimes \ell} = U(\mathfrak{g})\mathbf{v}$. Given $\xi \in \mathcal{P}_\ell$ such that $[M : \mathcal{S}(\xi)] \neq 0$, since $\mathrm{wt}(\xi) \in \mathcal{P}_\ell^+$ by (3.1.5), then $[\mathbb{V}^{\otimes \ell} : V(\mathrm{wt}(\xi))] \neq 0$ follows from Proposition 3.1.1. Thus, $\mathbb{V}^{\otimes \ell} = U(\mathfrak{g})\mathbf{v}$ guarantees the hyphotesis of Proposition 3.1.4 for $v_0 = \mathbf{v}$, whence $m \mapsto m \otimes \mathbf{v}, m \in M$, is injective. □

3.2 The affine Schur-Weyl duality

In this section we give a detailed proof of the affine Schur-Weyl duality. As mentioned in the Introduction, this equivalence between certain categories of finite-dimensional modules for the extended affine symmetric group $\tilde{\mathcal{S}}_\ell$ and the affine Lie algebra $\hat{\mathfrak{g}}'$ was claimed in Chari and Pressley's article [17] to follow from an analogous proof scheme that was originally developed in the context of categories of finite-dimensional modules for the quantum affine algebra $U_q(\hat{\mathfrak{g}}')$ and the affine Hecke algebra $\hat{H}_\ell(q^2)$. The proof here presented is in fact a straightforward adaptation of the proof scheme developed in that article. Since the categories of finite-dimensional modules for $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}'$ are equivalent, for simplicity we rather chose to state this equivalence of categories in terms of categories of finite-dimensional modules for $\tilde{\mathcal{S}}_\ell$ and the loop algebra $\tilde{\mathfrak{g}}$.

The **extended affine symmetric group** $\tilde{\mathcal{S}}_\ell$ is the semi-direct product $\tilde{\mathcal{S}}_\ell = \mathbb{Z}^\ell \rtimes \mathcal{S}_\ell$ obtained by the action (1.7.10) of $\mathcal{S}_\ell$ on $\mathbb{Z}^\ell$ of permuting factors. By Proposition 1.6.1, the group $\tilde{\mathcal{S}}_\ell$ is isomorphic to the group given by generators σ_i, $1 \leq i \leq \ell-1$, $y_j^{\pm 1}$, $1 \leq j \leq \ell$, satisfying the defining relations (1.6.1) of $\mathcal{S}_\ell$ along with

$$\begin{gathered} y_i^{-1} y_i = y_i y_i^{-1} = 1, \quad y_i y_j = y_j y_i \\ y_j \sigma_i = \sigma_i y_j,\ j \notin \{i,\, i+1\}, \quad \sigma_i y_i = y_{i+1}\sigma_i \ \text{ for all } \ i, j, \end{gathered} \tag{3.2.1}$$

where the first and second line above refer, respectively, to the defining relations of $\mathbb{Z}^\ell$ and how the relations of $\mathcal{S}_\ell$ and $\mathbb{Z}^\ell$ behave with respect to the semi-direct product. Here, the elements σ_i are identified with the transpositions $(0, (i, i+1))$ and $y_j^{\pm 1}$ are identified with $(\pm \boldsymbol{e}_j, 1)$, where $\boldsymbol{e}_j$ denotes the j-th vector of the canonical basis of $\mathbb{Z}^\ell$. In what follows, for convenience, we shall identify $\tilde{\mathcal{S}}_\ell$ with its presentation given by generators and relations so that we can refer directly to the variables y_j, σ_i as being elements in $\tilde{\mathcal{S}}_\ell$. In this sense, $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ is seen to contain the ring $\mathbb{C}[y_1^{\pm 1}, \cdots, y_\ell^{\pm 1}]$, and the subgroup generated by σ_i is identified with the symmetric group $\mathcal{S}_\ell$.

We now define the categories which will be shown to be equivalent. Let $\tilde{\mathscr{S}}_\ell$ be the abelian **category** of right finite-dimensional $\tilde{\mathcal{S}}_\ell$-modules, and let $\tilde{\mathscr{G}}_\ell$ be the **full subcategory** of the abelian category of left finite-dimensional $\tilde{\mathfrak{g}}$-modules whose objects lie in $\mathscr{G}_\ell$ when regarded as $\mathfrak{g}$-modules. Notice that the $\mathfrak{g}$-structure of the objects in $\tilde{\mathscr{G}}_\ell$ ensures that this full subcategory is also abelian by Lemma 1.9.1. Henceforth, we construct the desired functor between these two categories, as well as develop all the necessary tools to give a detailed proof of their equivalence.

Given $M \in \tilde{\mathscr{S}}_\ell$, by the canonical inclusion $\mathcal{S}_\ell \to \tilde{\mathcal{S}}_\ell$, one can consider the $\mathfrak{g}$-module $\mathcal{F}_\ell(M) = M \otimes_{\mathbb{C}[\mathcal{S}_\ell]} \mathbb{V}^{\otimes \ell} \in \mathscr{G}_\ell$ studied in the context of the classical Schur-Weyl duality, with the corresponding representation of $\mathfrak{g}$ induced from (3.1.2) being denoted here by $\rho_M : \mathfrak{g} \to \mathrm{End}(\mathcal{F}_\ell(M))$. Recalling (1.7.8), (1.7.9), notice that

$$\rho_M(x)(m \otimes v) = \sum_{j=1}^{\ell} m \otimes \Delta_j(x)(v) \ \text{ for all } \ m \in M, v \in \mathbb{V}^{\otimes \ell}. \tag{3.2.2}$$

We aim to extend the above representation of $\mathfrak{g}$ to a representation $\tilde{\rho}_M : \tilde{\mathfrak{g}} \to \mathfrak{gl}(\mathcal{F}_\ell(M))$ in a way that the corresponding endomorphism of $x_0^{\pm} \in \tilde{\mathfrak{g}}$ satisfies

$$\tilde{\rho}_M(x_0^{\pm})(m \otimes v) = \sum_{j=1}^{\ell} m \cdot y_j^{\pm 1} \otimes \Delta_j(x_\theta^{\mp})(v)\,. \tag{3.2.3}$$

Once this extension is acquired, the $\tilde{\mathfrak{g}}$-module $\tilde{\mathcal{F}}_\ell(M)$ thus obtained will establish the intended functor between the categories $\tilde{\mathscr{S}}_\ell$ and $\tilde{\mathscr{G}}_\ell$ by assigning $M \mapsto \tilde{\mathcal{F}}_\ell(M)$.

Lemma 3.2.1. *There exist unique $\tilde{\rho}_M(x_0^{\pm}) \in End(\mathcal{F}_\ell(M))$ satisfying (3.2.3).*

Proof. We only show the result for $\tilde{\rho}_M(x_0^+)$, since the existence and uniqueness of $\tilde{\rho}_M(x_0^-)$ can be proved analogously.

Set $\rho_0^+ : M \times \mathbb{V}^{\otimes \ell} \to \mathcal{F}_\ell(M)$ by $(m, v) \mapsto \sum_{j=1}^{\ell} m \cdot y_j \otimes \Delta_j(x_\theta^-)(v)$ for $m \in M, v \in \mathbb{V}^{\otimes \ell}$. We have to verify that ρ_0^+ is $\mathbb{C}[\mathcal{S}_\ell]$-bilinear, since the universal property of tensor products over rings ensures the existence of a unique $\tilde{\rho}_M(x_0^+) \in \mathrm{End}(\mathcal{F}_\ell(M))$ satisfying (3.2.3). Since ρ_0^+ is clearly $\mathbb{C}$-linear in each of its entries, we are left to show that $\rho_0^+(m\cdot\sigma_i, v) = \rho_0^+(m, \sigma_i \cdot v)$ holds for arbitrary $\sigma_i \in \mathcal{S}_\ell, m \in M$ and $v \in \mathbb{V}^{\otimes \ell}$. In explicit terms, we have to show that

$$\sum_{j=1}^{\ell} m \cdot \sigma_i y_j \otimes \Delta_j(x_\theta^-)(v) = \sum_{j=1}^{\ell} m \cdot y_j \otimes \Delta_j(x_\theta^-)(\sigma_i \cdot v). \tag{3.2.4}$$

Given index $j \notin \{i, i+1\}, 1 \leq j \leq \ell$, we have $\sigma_i \cdot \Delta_j(x_\theta^-)(v) = \Delta_j(x_\theta^-)(\sigma_i \cdot v)$. Since in this case $\sigma_i y_j = y_j \sigma_i$ by (3.2.1), then

$$m\cdot\sigma_i y_j \otimes \Delta_j(x_\theta^-)(v) = m\cdot y_j \sigma_i \otimes \Delta_j(x_\theta^-)(v) = m\cdot y_j \otimes \sigma_i \cdot \Delta_j(x_\theta^-)(v) = m\cdot y_j \otimes \Delta_j(x_\theta^-)(\sigma_i\cdot v).$$

Therefore, (3.2.4) reduces to verifying that

$$m\cdot\sigma_i y_i \otimes \Delta_i(x_\theta^-)(v) + m\cdot\sigma_i y_{i+1} \otimes \Delta_{i+1}(x_\theta^-)(v) = m\cdot y_{i+1} \otimes \Delta_{i+1}(x_\theta^-)(\sigma_i\cdot v) + m\cdot y_i \otimes \Delta_i(x_\theta^-)(\sigma_i\cdot v). \tag{3.2.5}$$

Now, one checks that $\sigma_i \cdot \Delta_i(x_\theta^-)(v) = \Delta_{i+1}(x_\theta^-)(\sigma_i \cdot v)$ and that $\sigma_i \cdot \Delta_{i+1}(x_\theta^-)(v) = \Delta_i(x_\theta^-)(\sigma_i \cdot v)$. Thus, from the relations $\sigma_i y_i = y_{i+1}\sigma_i$, $\sigma_i y_{i+1} = y_i \sigma_i$ in (3.2.1), we obtain

$$\begin{aligned} m \cdot \sigma_i y_i \otimes \Delta_i(x_\theta^-)(v) &= m \cdot y_{i+1} \otimes \Delta_{i+1}(x_\theta^-)(\sigma_i \cdot v) \\ m \cdot \sigma_i y_{i+1} \otimes \Delta_{i+1}(x_\theta^-)(v) &= m \cdot y_i \otimes \Delta_i(x_\theta^-)(\sigma_i \cdot v), \end{aligned}$$

whence (3.2.5) immediately follows. □

In order to utilize the endomorphisms $\tilde{\rho}_M(x_0^{\pm})$ given by the previous lemma and the endomorphisms $\rho_M(x_i), \rho_M(h_i), i \in I$, given by the representation ρ_M of $\mathfrak{g}$ on $\mathcal{F}_\ell(M)$ to construct the intended representation of $\tilde{\mathfrak{g}}$ on $\mathcal{F}_\ell(M)$, we shall work extensively with the presentation of $\tilde{\mathfrak{g}}$ in terms of generators and relations seen in Section 1.5. Since the central element of $\hat{\mathfrak{g}}$ is the one seen in (1.7.5), by (1.5.2) the loop algebra $\tilde{\mathfrak{g}}$ is isomorphic to the Lie algebra $FL(X)/\langle R' \rangle$, where $FL(X)$ is the free Lie algebra over the set of formal generators $X = \{X_i^{\pm}, H_i^{\pm} : i \in \hat{I}\}$, and R' is the set

$$\left\{ [H_i, H_j], (1-\delta_{i,j})\mathrm{ad}(X_i^{\pm})^{1-c_{i,j}}(X_j^{\pm}), \pm c_{i,j} X_j^{\pm} - [H_i, X_j^{\pm}], [X_i^+, X_j^-] - \delta_{i,j} H_i, \sum_{i=0}^{n} H_i : i, j \in \hat{I} \right\}$$

with the integers c_{ij} above being the entries of the Cartan matrix of affine type (1.7.4).

We now define a map $r_X : X \to \mathfrak{gl}(\mathcal{F}_\ell(M))$ by assigning

$$\begin{aligned} X_i^{\pm} \mapsto \rho_M(x_i^{\pm}), H_i \mapsto \rho_M(h_i),\ i \in I, \\ X_0^{\pm} \mapsto \tilde{\rho}_M(x_0^{\pm}),\ H_0 \mapsto \Big[\tilde{\rho}_M(x_0^{+}), \tilde{\rho}_M(x_0^{-})\Big]. \end{aligned} \tag{3.2.6}$$

By the universal property of free Lie algebras, this map gives rise to a unique representation $\varrho : FL(X) \to \mathfrak{gl}(\mathcal{F}_\ell(M))$ such that $\varrho\Big|_X = r_X$. In the next theorem, we show that $R' \subseteq \operatorname{Ker} \varrho$, so that we obtain an induced homomorphism of Lie algebras

$$\tilde{\varrho} : FL(X)/\langle R' \rangle \to \mathfrak{gl}(\mathcal{F}_\ell(M)),$$

i.e., a representation of $FL(X)/\langle R' \rangle$ on $\mathcal{F}_\ell(M)$. We abuse notation and also denote by X the image of X under the canonical homomorphism $FL(X) \to FL(X)/\langle R' \rangle$, as well as keep the notation $X_i^{\pm}, H_i, i \in \hat{I}$, for its elements in this quotient. In particular, since $\tilde{\mathfrak{g}} \cong FL(X)/\langle R' \rangle$ and $\tilde{\varrho}\Big|_X = r_X$, the representation $\tilde{\varrho}$ ensures that there exists $\tilde{\rho}_M : \tilde{\mathfrak{g}} \to \mathfrak{gl}(\mathcal{F}_\ell(M))$ extending ρ_M as desired.

Theorem 3.2.2. *We have $R' \subseteq \operatorname{Ker} \varrho$. In particular, there exists a representation $\tilde{\rho}_M : \tilde{\mathfrak{g}} \to \mathfrak{gl}(\mathcal{F}_\ell(M))$ extending ρ_M and such that the action of the elements $x_0^{\pm}$ satisfies (3.2.3).*

Proof. By Serre's Theorem on semi-simple Lie algebras, the Lie algebra $\mathfrak{g}$ is isomorphic to the subalgebra of $F(X)/\langle R' \rangle$ generated by $X_i^{\pm}, H_i^{\pm}, i \in I$, satisfying the defining relations (1.5.2) above restricted to indexes belonging to I. In this way, since $\varrho(X_i^{\pm}) = \rho_M(x_i^{\pm}), \varrho(H_i) = \rho_M(h_i), i \in I$, and ρ_M is a proper representation of $\mathfrak{g}$, all the elements of R' not involving $X_0^{\pm}, H_0$ are, thus, already in the kernel of ϱ. Therefore, we are left to verify that the following elements are vanished by ϱ:

$$\begin{gathered} [H_0, H_j]\,,\ \pm c_{0,j}\, X_j^{\pm} - [H_0, X_j^{\pm}]\,,\ \pm c_{j,0}\, X_0^{\pm} - [H_j, X_0^{\pm}]\,, \\ \pm 2X_0^{\pm} - [H_0, X_0^{\pm}],\ [X_0^{+}, X_j^{-}]\,,\ \operatorname{ad}(X_0^{\pm})^{1-c_{0,j}}(X_j^{\pm})\,, \sum_{i=0}^{n} H_i\,,\ j \neq 0. \end{gathered}$$

This verification is performed in the steps below. For the particular elements $\pm c_{0,j}\, X_j^{\pm} - [H_0, X_j^{\pm}],\ \pm c_{j,0}\, X_0^{\pm} - [H_j, X_0^{\pm}],\ 2X_0^{\pm} - [H_0, X_0^{\pm}]$ and $\operatorname{ad}(X_0^{\pm})^{1-c_{0,j}}(X_j^{\pm})$, we remark that we restricted ourselves to just verifying that $X_j^{+} - [H_0, X_j^{+}],\ c_{j,0}\, X_0^{+} - [H_j, X_0^{+}]$, *etc* lie in $\operatorname{Ker} \varrho$, as the remaining ones can be verified analogously. Also, since $y \in \operatorname{Ker} \varrho$ if and only if $\varrho(y)(m \otimes v) = 0$ for every pure tensor of the form $m \otimes v$, all the calculations in this proof were performed on an arbitrary $m \otimes v \in \mathcal{F}_\ell(M)$.

1. $\sum_{i=0}^{n} H_i \in \operatorname{Ker} \varrho$.

We begin by expressing the action of H_0 on $\mathcal{F}_\ell(M)$ more explicitly. From the definition

(3.2.3) of $\rho_M(x_0^{\pm})$, we have

$$\begin{aligned}\tilde{\rho}_M(x_0^+)(\tilde{\rho}_M(x_0^-)(m\otimes v)) &= \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} m y_j^{-1} y_k \otimes \Delta_k(x_\theta^-)(\Delta_j(x_\theta^+)(v)),\\ \tilde{\rho}_M(x_0^-)(\tilde{\rho}_M(x_0^+)(m\otimes v)) &= \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} m y_j y_k^{-1} \otimes \Delta_k(x_\theta^+)(\Delta_j(x_\theta^-)(v)).\end{aligned} \tag{3.2.7}$$

For $k \neq j$, one checks that $\Delta_k(x_\theta^-)(\Delta_j(x_\theta^+)(v)) = \Delta_j(x_\theta^+)(\Delta_k(x_\theta^-)(v))$, so (3.2.7) implies

$$\begin{aligned}\varrho(H_0)(m\otimes v) = &\sum_{k=1}^{\ell} m \otimes \Big(\Delta_k(x_\theta^-)\Delta_k(x_\theta^+) - \Delta_k(x_\theta^+)\Delta_k(x_\theta^-)\Big)(v) + \\ &+ \sum_{j=1}^{\ell}\sum_{k\neq j}^{\ell} m(y_j^{-1}y_k - y_j y_k^{-1}) \otimes \Delta_k(x_\theta^-)(\Delta_j(x_\theta^+)(v)).\end{aligned} \tag{3.2.8}$$

Since $y_j^{-1}y_k = y_k y_j^{-1}$ by the group relations (3.2.1), then

$$\sum_{j=1}^{\ell}\sum_{k\neq j}^{\ell} m(y_j^{-1}y_k - y_j y_k^{-1}) = m\Bigg(\underbrace{\sum_{j<k} y_j^{-1}y_k - \sum_{k<j} y_j y_k^{-1}}_{=0} + \underbrace{\sum_{k<j} y_j^{-1}y_k - \sum_{j<k} y_j y_k^{-1}}_{=0}\Bigg) = 0. \tag{3.2.9}$$

Plugging this relation back into (3.2.8), it then reduces to

$$\varrho(H_0)(m\otimes v) = \sum_{k=1}^{\ell} m \otimes \Big(\Delta_k(x_\theta^-)\Delta_k(x_\theta^+) - \Delta_k(x_\theta^+)\Delta_k(x_\theta^-)\Big)(v) = \sum_{k=1}^{\ell} m \otimes \Delta_k([x_\theta^-, x_\theta^+])(v),$$

showing that $\varrho(H_0) = -\rho_M([x_\theta^+, x_\theta^-])$. Since $[x_\theta^+, x_\theta^-] = \sum_{i=1}^n h_i$ holds in $\mathfrak{g}$ by (1.7.2), then $\varrho\Big(\sum_{i=1}^n H_i\Big) = \rho_M\Big([x_\theta^+, x_\theta^-]\Big)$ follows, whence $\varrho\Big(\sum_{i=0}^n H_i\Big) = 0$.

2. $[H_0, H_j], [X_0^+, X_j^-] \in \operatorname{Ker}\varrho$.

Since $\varrho(H_0) = -\varrho\Big(\sum_{i=1}^n H_i\Big) = -\sum_{i=1}^n \rho_M(h_i)$ by the previous step, then

$$\varrho([H_0, H_j]) = [\varrho(H_0), \varrho(H_j)] = -\sum_{i=1}^{n}[\rho_M(h_i), \rho_M(h_j)] = -\sum_{i=1}^{n}\underbrace{\rho_M([h_i, h_j])}_{=0} = 0,$$

since we have $[h_i, h_j] = 0$ in $\mathfrak{g}$ for all $i, j \in I$, and ρ_M is a representation of $\mathfrak{g}$.

Noticing that $\Delta_k(x_\theta^-)(\Delta_s(x_j^-)(v)) = \Delta_s(x_j^-)(\Delta_k(x_\theta^-)(v))$ for $k \neq s$, from (3.2.2) and

(3.2.3) we obtain

$$\varrho\Big([X_0^+, X_j^-]\Big)(m \otimes v) = [\tilde{\rho}_M(x_0^+), \rho_M(x_j^-)](m \otimes v) = \sum_{k=1}^{\ell} my_k \otimes \Delta_k([x_\theta^-, x_j^-])(v) + \\ + \sum_{s=1}^{\ell}\sum_{k\neq s}^{\ell} my_s \otimes \underbrace{(\Delta_s(x_\theta^-)\Delta_k(x_j^-) - \Delta_k(x_j^-)\Delta_s(x_\theta^-))(v)}_{=0} = \sum_{k=1}^{\ell} my_k \otimes \Delta_k([x_\theta^-, x_j^-])(v). \tag{3.2.10}$$

Since $[x_\theta^-, x_j^-] = 0$ in $\mathfrak{g}$, then $\Delta_k([x_\theta^-, x_j^-])(v) = 0$ for all $1 \leq k \leq \ell$, whence $\varrho([X_0^+, X_j^-]) = 0$ follows as well.

3. $\mathrm{ad}(X_0^+)^{1-c_{0,j}}(X_j^+) \in \operatorname{Ker}\varrho$.

From the entries $c_{0,j}$ of (1.7.4), we only have to verify that the following terms lie in $\operatorname{Ker}\varrho$:

$$[X_0^+, [X_0^+, X_1^+]],\ [X_0^+, [X_0^+, X_n^+]] \ \text{ and } \ [X_0^+, X_j^+] = 0 \text{ for } j \notin \{1, n\}.$$

We begin with $[X_0^+, X_j^+] = 0$, $j \notin \{1, n\}$. Since $[x_\theta^-, x_j^+] = 0$ holds in $\mathfrak{g}$ for $j \notin \{1, n\}$, then $\Delta_k(x_\theta^-)(\Delta_s(x_j^+)(v)) = \Delta_s(x_j^+)(\Delta_k(x_\theta^-)(v))$ for all $1 \leq k, s \leq \ell$. Hence,

$$\begin{aligned}\tilde{\rho}_M(x_0^+)(\rho_M(x_j^-)(m \otimes v)) &= \sum_{s=1}^{\ell}\sum_{k=1}^{\ell} my_s \otimes \Delta_s(x_\theta^-)(\Delta_k(x_j^-)(v)) \\ &= \sum_{k=1}^{\ell}\sum_{s=1}^{\ell} my_s \otimes \Delta_k(x_j^-)(\Delta_s(x_\theta^-)(v)) = \rho_M(x_j^-)(\tilde{\rho}_M(x_0^+)(m \otimes v))\end{aligned}$$

shows that $\varrho([X_0^+, X_j^+]) = [\tilde{\rho}_M(x_0^+), \rho_M(x_j^-)] = 0$.

We now show that $[X_0^+, [X_0^+, X_1^+]] \in \operatorname{Ker}\varrho$. By similar reasoning which led to (3.2.10), we have

$$[\tilde{\rho}_M(x_0^+), \rho_M(x_1^+)](m \otimes v) = \sum_{k=1}^{\ell} my_k \otimes \Delta_k([x_\theta^-, x_1^+])(v).$$

Since $\Delta_k(x_\theta^-)(\Delta_j([x_\theta^-, x_1^+])(v)) = \Delta_j(x_\theta^-)(\Delta_k([x_\theta^-, x_1^+])(v))$ for $k \neq j$, taking the bracket with $\tilde{\rho}_M(x_0^+)$ on both sides yields

$$[\tilde{\rho}_M(x_0^+), [\tilde{\rho}_M(x_0^+), \rho_M(x_1^+)]](m \otimes v) = \sum_{k=1}^{\ell} my_k \otimes \Delta_k([x_\theta^-, [x_\theta^-, x_1^+]])v + \\ + \sum_{j=1}^{\ell}\sum_{k\neq j}^{\ell} m(y_j^{-1}y_k - y_j y_k^{-1}) \otimes \Delta_k(x_\theta^-)\Delta_j([x_\theta^-, x_1^+])v. \tag{3.2.11}$$

Since $[x_\theta^-, [x_\theta^-, x_1^+]] = 0$ in $\mathfrak{g}$, then $\Delta_k([x_\theta^-, [x_\theta^-, x_1^+]]) = 0$ for all $1 \leq k \leq \ell$, which combined with (3.2.9) ensures that (3.2.11) above vanishes, showing that $\varrho([X_0^+, [X_0^+, X_1^+]]) = 0$. An analogous argument also shows that $[X_0^+, [X_0^+, X_n^+]] \in \operatorname{Ker} \varrho$.

4. $[H_j, X_0^+] - c_{j,0}X_0^+ \in \operatorname{Ker} \varrho$.

We first suppose $j \notin \{1, n\}$. Since $\varrho(H_j) = \rho_M(h_j) = [\rho_M(x_j^+), \rho_M(x_j^-)] = \varrho([X_j^+, X_j^-])$, from the Jacobi Identity we notice that

$$\varrho([H_j, X_0^+]) = \varrho([X_j^+, [X_j^-, X_0^+]]) - \varrho([X_j^-, [X_j^+, X_0^+]]).$$

By steps 2 and 3 above, we have $\varrho([X_j^-, X_0^+]) = \varrho([X_j^+, X_0^+]) = 0$, so $\varrho([H_j, X_0^+]) = 0$. Since $c_{j,0} = 0$ for $j \notin \{1, n\}$, it follows that $[H_j, X_0^+] - c_{j,0}X_j^+ \in \operatorname{Ker} \varrho$.

It now remains to show that $[H_1, X_0^+]+X_0^+, [H_n, X_0^+]+X_0^+ \in \operatorname{Ker} \varrho$, as $c_{1,0} = c_{1,n} = -1$. We provide the details only for $[H_1, X_0^+] + X_0^+$. Once more, applying similar reasoning which led to (3.2.10), we obtain that

$$[\rho_M(h_1), \tilde{\rho}_M(x_0^+)](m \otimes v) = \sum_{k=1}^{\ell} my_k \otimes \Delta_k([h_1, x_\theta^-])(v).$$

Since $[h_1, x_\theta^-] = -x_\theta^-$ in $\mathfrak{g}$, the above implies that $[\rho_M(h_1), \tilde{\rho}_M(x_0^+)] + \tilde{\rho}_M(x_0^+) = 0$ by (3.2.3), which is equivalent to $\varrho([H_1, X_0^+] + X_0^+) = 0$.

5. $[H_0, X_j^+] - c_{0,j}X_j^+, [H_0, X_0^+] - 2X_0^+ \in \operatorname{Ker} \varrho$.

Since $\varrho(H_0) = -\sum_{i=1}^n \varrho(H_i)$ by step 1, we obtain

$$\varrho([H_0, X_0^+]) = -\sum_{i=1}^{n} \varrho([H_i, X_0^+]) = -\sum_{i=1}^{n} c_{j,0}\, \varrho(X_0^+) = 2\varrho(X_0^+),$$

where in the second identity we used the previous step. Similarly, since $\varrho([H_i, X_j^+]) = \rho_M[h_i, x_j^+] = c_{i,j}\, \varrho(X_j^+)$, we also have

$$\varrho([H_0, X_j^+]) = -\sum_{i=1}^{n} \varrho([H_i, X_j^+]) = -\sum_{i=1}^{n} c_{i,j}\, \varrho(X_j^+) = c_{0,j}\, \varrho(X_j^+),$$

and step 5 follows. □

Using the $\tilde{\mathfrak{g}}$-module $\tilde{\mathcal{F}}_\ell(M)$ given by the extension $\tilde{\rho}_M$ of the previous theorem, we define a **functor** $\tilde{\mathcal{F}}_\ell : \tilde{\mathscr{S}}_\ell \to \tilde{\mathscr{G}}_\ell$ mapping each object and each morphism $f : M \to N$ of $\tilde{\mathcal{S}}_\ell$-modules by

$$M \mapsto \tilde{\mathcal{F}}_\ell(M),\ f \mapsto \tilde{\mathcal{F}}_\ell(f) = f \otimes 1.$$

Our next goal is to prove that under the restriction $\ell \leq n$ this functor establishes the affine Schur-Weyl duality.

Theorem 3.2.3. *When $\ell \leq n$, the functor $\tilde{\mathcal{F}}_\ell : \tilde{\mathscr{S}}_\ell \to \tilde{\mathscr{G}}_\ell$ establishes an equivalence of categories.*

From now on, suppose that $\ell \leq n$. Using the canonical basis $\{v_1, \cdots, v_{n+1}\}$ of $\mathbb{V} = \mathbb{C}^{n+1}$, we inductively set

$$\begin{aligned} &\mathbf{v}^{(1)} = v_{n+1} \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_\ell, \quad \mathbf{v}^{(j)} = (j-1, j) \cdot \mathbf{v}^{(j-1)} \text{ for } 2 \leq j \leq \ell, \\ &\mathbf{w}^{(1)} = v_1 \otimes v_2 \otimes \cdots \otimes v_\ell, \text{ and } \mathbf{w}^{(j)} = (j-1, j) \cdot \mathbf{w}^{(j-1)} \text{ for } 2 \leq j \leq \ell. \end{aligned} \tag{3.2.12}$$

Regarding the decomposition of $\mathbb{V}^{\otimes \ell}$ as a weight module, notice that

$$\mathbf{v}^{(j)} \in \mathbb{V}^{\otimes \ell}_{\omega_\ell - \theta} \text{ and } \mathbf{w}^{(j)} \in \mathbb{V}^{\otimes \ell}_{\omega_\ell}, \ 1 \leq j \leq \ell. \tag{3.2.13}$$

Fix an arbitrary $W \in \tilde{\mathscr{G}}_\ell$ and let $M \in \mathscr{S}_\ell$ be such that $\mathcal{F}_\ell(M) \cong W$ as $\mathfrak{g}$-modules by the classical Schur-Weyl duality. We want to define an action of $\tilde{\mathcal{S}}_\ell$ on M so that the $\tilde{\mathcal{S}}_\ell$-module $\tilde{M}$ thus obtained is such that $\tilde{\mathcal{F}}_\ell(\tilde{M}) \cong W$ as $\tilde{\mathfrak{g}}$-modules. Fix an isomorphism of $\mathfrak{g}$-modules $f\colon \mathcal{F}_\ell(M) \to W$ and define an action of $\tilde{\mathfrak{g}}$ on $\mathcal{F}_\ell(M)$ by

$$x \cdot u = f^{-1}(x \cdot f(u)), \ x \in \tilde{\mathfrak{g}}, \ u \in \mathcal{F}_\ell(M). \tag{3.2.14}$$

In particular, notice that if $x \in \mathfrak{g}$, then (3.2.14) is equal to the standard action (3.2.2) of x on $\mathcal{F}_\ell(M)$.

Lemma 3.2.4. *There exist unique $A_j^\pm \in End(M)$, up to a choice of f, such that*

$$x_0^-(m \otimes \boldsymbol{v}^{(j)}) = A_j^-(m) \otimes \boldsymbol{w}^{(j)} \ \textit{and} \ x_0^+(m \otimes \boldsymbol{w}^{(j)}) = A_j^+(m) \otimes \boldsymbol{v}^{(j)}, \ 1 \leq j \leq \ell.$$

Proof. Recall that the decomposition of $\mathcal{F}_\ell(M)$ as a weight module is the one induced by the decomposition of $\mathbb{V}^{\otimes \ell}$ itself. In particular, we have $m \otimes \mathbf{v}^{(j)} \in \mathcal{F}_\ell(M)_{\omega_\ell - \theta}$ by (3.2.13). Moreover, since $x_0^\pm \in \mathfrak{g}_{\mp\theta} \otimes t^{\pm 1}$, then $x_0^-(m \otimes \mathbf{v}^{(j)}) \in \mathcal{F}_\ell(M)_{\omega_\ell}$ follows by (2.2.16). Now, notice that $\{\sigma \cdot \mathbf{w}^{(j)} : \sigma \in \mathcal{S}_\ell\}$ is a basis of $\mathbb{V}^{\otimes \ell}_{\omega_\ell}$ by Proposition 1.7.3, so there exist unique $m_{\sigma,j} \in M$ such that

$$x_0^-(m \otimes \mathbf{v}^{(j)}) = \sum_{\sigma \in \mathcal{S}_\ell} m_{\sigma,j} \otimes \sigma \cdot \mathbf{w}^{(j)} = \sum_{\sigma \in \mathcal{S}_\ell} m_{\sigma,j} \cdot \sigma \otimes \mathbf{w}^{(j)} = \left(\sum_{\sigma \in \mathcal{S}_\ell} m_{\sigma,j} \cdot \sigma \right) \otimes \mathbf{w}^{(j)}.$$

In other words,

$$x_0^-(m \otimes \mathbf{v}^{(j)}) = m_j^- \otimes \mathbf{w}^{(j)}, \text{ where } m_j^- = \sum_{\sigma \in \mathcal{S}_\ell} m_{\sigma,j} \cdot \sigma.$$

Let $A_j^- \in \mathrm{End}(M)$ be the linear transformation $m \mapsto m_j^-$. It remains to check its uniqueness. Since the factors composing $\mathbf{w}^{(j)}$ are all distinct, it follows from Corollary 3.1.5 that the linear map $m \mapsto m \otimes \mathbf{w}^{(j)}$, $m \in M$, is injective. If $A \in \mathrm{End}(M)$ satisfies

$x_0^-(m\otimes\mathbf{v}^{(j)}) = A(m)\otimes\mathbf{w}^{(j)}$ for all $m \in M$, then $A(m)\otimes\mathbf{w}^{(j)} = x_0^-(m\otimes\mathbf{v}^{(j)}) = A_j^-(m)\otimes\mathbf{w}^{(j)}$ implies that $A_j^-(m) = A(m)$, so A_j^- is unique.

Analogously, since $x_0^+\cdot\mathcal{F}_\ell(M)_{\omega_\ell} \subseteq \mathcal{F}_\ell(M)_{\omega_\ell-\theta}$ and $\{\sigma\cdot\mathbf{v}^{(j)} : \sigma \in \mathcal{S}_\ell\}$ is a basis of the weight space $\mathbb{V}^{\otimes\ell}_{\omega_\ell-\theta}$, we can write

$$x_0^+(m \otimes \mathbf{w}^{(j)}) = m_j^+ \otimes \mathbf{v}^{(j)}$$

and define a unique $A_j^+ \in \mathrm{End}(M)$ by $m \mapsto m_j^+$. □

The linear operators $A_j^\pm$ relate to the action (3.2.14) of $x_0^\pm$ on $\mathcal{F}_\ell(M)$ as follows:

Lemma 3.2.5. *For every $u \in \mathcal{F}_\ell(M)$, the action of $x_0^\pm$ given by* (3.2.14) *satisfies*

$$x_0^\pm \cdot u = \sum_{j=1}^{\ell}\Big(A_j^\pm(m) \otimes \Delta_j(x_\theta^\mp)\Big)(u). \tag{3.2.15}$$

Proof. It suffices to consider u of the form $u = m\otimes\mathbf{v}$, where $m \in M$ and $\mathbf{v} = v_{i_1}\otimes\cdots\otimes v_{i_\ell}$ is a basis element of $\mathbb{V}^{\otimes\ell}$, $\{i_1, \cdots, i_\ell\} \subseteq \{1, \cdots, n+1\}$. We write the details only for x_0^-, since the proof for x_0^+ is analogous. In this case, (3.2.15) can be rewritten as

$$x_0^-(m \otimes \mathbf{v}) = \sum_{k:i_k=n+1} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}). \tag{3.2.16}$$

Define the set

$$\mathrm{K}_\mathbf{v} = \{k : 1 < i_k < n+1\}$$

and let $r = \#\{k : i_k = 1\},\ s = \#\{k : i_k = n+1\}$. For convenience, denote by $\mathfrak{a}$ the Lie subalgebra generated by $x_i^\pm, 1 < i < n$, and notice that $[x_0^-, \mathfrak{a}] = 0$ by (1.7.4). The fact that x_0^- lies in the center of $\mathfrak{a}$ will be used multiple times in this proof without any further mention.

If $s = 0$, the right-hand side of (3.2.16) is clearly zero. Since $\mathbf{v}$ has weight $\mu = \ell\omega_1 - \sum_k c_k\alpha_k$ with $c_k \geq 0$ as in (1.7.17), the assumption $s = 0$ also implies that $c_n = 0$. If $x_0^-(m \otimes \mathbf{v}) \neq 0$, then $x_0^-(m \otimes \mathbf{v}) \in \mathcal{F}_\ell(M)_{\mu+\theta}$ with $\mu + \theta \in \mathrm{wt}\Big(\mathbb{V}^{\otimes\ell}\Big)$. In particular, since $c_n = 0$ and $\theta = \sum_{k=1}^n \alpha_k$, then

$$\mu + \theta = \ell\omega_1 - \tilde{\mu} \ \text{ for some } \ \tilde{\mu} \notin Q^+,$$

which contradicts the fact, given by (1.7.17), that any weight of $\mathbb{V}^{\otimes\ell}$ is lower than $\ell\omega_1$. In this way, we can assume $s \geq 1$ and $1 \leq r+s \leq \ell$. Since $\ell \leq n$, then $\#\mathrm{K}_\mathbf{v} = \ell-r-s \leq n-1$, so the vector $\mathbf{v}$ may satisfy the condition

$$i_j \neq i_k \quad \text{for every} \quad j, k \in \mathrm{K}_\mathbf{v}, j \neq k. \tag{3.2.17}$$

For now, assume that we have proved (3.2.16) for the particular case of vectors satisfying this condition and let us deduce the general case from it. Given $\mathbf{v}$ that does not satisfy (3.2.17), choose any $\mathbf{v}' = v_{i'_1} \otimes \cdots \otimes v_{i'_\ell}$ with $K_{\mathbf{v}'} = K_{\mathbf{v}}$ that satisfies (3.2.17) as well as

$$i'_k = i_k \text{ for all } \quad k \notin K_{\mathbf{v}'}. \tag{3.2.18}$$

Notice that $\mathbf{v} = x \cdot \mathbf{v}'$ for some $x \in U(\mathfrak{a})$ by Proposition 1.7.2. Applying (3.2.16) to $\mathbf{v}'$ yields the following

$$x_0^-(m \otimes \mathbf{v}) = x x_0^-(m \otimes \mathbf{v}') = x \cdot \left(\sum_{k: i_k = n+1} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}') \right). \tag{3.2.19}$$

Since $[x_\theta^+, x] = 0$, it then follows that

$$x_0^-(m \otimes \mathbf{v}) = \sum_{k: i_k = n+1} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}),$$

obtaining (3.2.16) as desired. Thus, it remains to prove the lemma for $\mathbf{v}$ satisfying (3.2.17). We proceed by induction on s with subinduction on r as follows.

We begin by first proving the inductive argument on r for all s, instead of the base case $r = 0$ by induction on s, which is more complex. So let us assume the inductive hyphotesis on $r > 0$ and choose any k_0 such that $i_{k_0} = 1$. First, suppose that $2 \notin \{i_k : k \in \mathrm{K}_{\mathbf{v}}\}$. In this case, let $\mathbf{v}'$ be the vector obtained from $\mathbf{v}$ by replacing v_1 at the k_0-th tensor factor by v_2 so that $\mathbf{v} = x_1^+ \cdot \mathbf{v}'$. In particular, notice that the induction hypothesis on r applies to $\mathbf{v}'$. Since $[x_0^-, x_1^+] = [x_1^+, x_\theta^+] = 0$, an application of the computation (3.2.19) with $x = x_1^+$ shows that the result holds in this case. Now, suppose that $2 \in \{i_k : k \in \mathrm{K}_{\mathbf{v}}\}$. Since $r + s \geq 2$, notice that $\{2, \cdots, n\} \not\subseteq \{i_k : k \in \mathrm{K}_{\mathbf{v}}\}$, so there exists $\mathbf{v}' = v_{i'_1} \otimes \cdots \otimes v_{i'_\ell}$ satisfying (3.2.17), (3.2.18), and such that $2 \notin \{i'_k : k \in \mathrm{K}_{\mathbf{v}'}\}$. We have just shown that (3.2.15) holds for such a $\mathbf{v}'$, while Proposition 1.7.2 ensures that $\mathbf{v} = x \cdot \mathbf{v}'$ for some $x \in U(\mathfrak{a})$. Another application of (3.2.19) thus completes the induction step on r.

We are left to treat the case $r = 0$ by induction on s. If $s = 1$, let k be the unique element such that $i_k = n + 1$ and let $x \in U(\mathfrak{a})$ be such that $\mathbf{v} = x \cdot \mathbf{v}^{(k)}$ by Proposition 1.7.2. Then, Lemma 3.2.4 says that (3.2.16) holds for $\mathbf{v}^{(k)}$, so (3.2.19) with $\mathbf{v}' = \mathbf{v}^{(k)}$ shows that the result holds for this case as well. Suppose that $s > 1$ and assume the inductive hyphotesis on s. The condition $s > 1$ also implies $\{2, \cdots, n\} \not\subseteq \{i_k : k \in \mathrm{K}_{\mathbf{v}}\}$, so there exists basis vectors satisfying (3.2.17), (3.2.18), and such that none of its tensor factors equals to v_n. It then suffices to prove that the result holds for this type of vectors, since this case combined with another application of Proposition 1.7.2 and (3.2.19) ensure that the result holds for a general $\mathbf{v}$ with $s > 1$ and $r = 0$. So suppose that $n \notin \{i_k : k \in \mathrm{K}_{\mathbf{v}}\}$. Let $k_0 \neq k'_0$ be such that $i_{k_0} = i_{k'_0} = n + 1$ and let $\mathbf{v}'$ be the vector obtained from $\mathbf{v}$ by replacing v_{n+1} at the k_0, k'_0-th tensor factor by v_n. From the Serre relation $[[x_0^-, x_n^-]x_n^-] = 0$

and the construction of $\mathbf{v}'$, we have the following identities

$$x_n^- x_n^- \cdot \mathbf{v}' = 2\mathbf{v} \quad \text{and} \quad x_0^- x_n^- x_n^-(m \otimes \mathbf{v}') = 2x_n^- x_0^- x_n^-(m \otimes \mathbf{v}') - x_n^- x_n^- x_0^-(m \otimes \mathbf{v}'),$$

which combine into

$$x_0^-(m \otimes \mathbf{v}) = x_n^- x_0^- x_n^-(m \otimes \mathbf{v}') - \frac{1}{2} x_n^- x_n^- x_0^-(m \otimes \mathbf{v}'). \tag{3.2.20}$$

In the following, we calculate each of the summands of the right-hand side of (3.2.20) in order to show that their sum verifies (3.2.16). We begin with $x_n^- x_0^- x_n^-(m \otimes \mathbf{v}')$. Denote by $\mathbf{v}''$ ($\mathbf{v}'''$) the vector obtained from $\mathbf{v}'$ by replacing v_n at the k_0-th (k_0'-th) tensor factor by v_{n+1} and notice that

$$x_n^-(m \otimes \mathbf{v}') = m \otimes \mathbf{v}'' + m \otimes \mathbf{v}'''.$$

Since both $\mathbf{v}'', \mathbf{v}'''$ satisfy the induction hypothesis on s, applying x_0^- to both sides of the above equality yields

$$x_0^- x_n^-(m \otimes \mathbf{v}') = \sum_{\substack{k: i_k = n+1 \\ k \neq k_0'}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}'') + \sum_{\substack{k: i_k = n+1 \\ k \neq k_0}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}''').$$

Now, $\mathbf{v}''$ ($\mathbf{v}'''$) has v_n only at the k_0'-th (k_0-th) tensor factor by the fact that $n \notin \{i_k : k \in K_{\mathbf{v}}\}$, whence $x_n^- \cdot \mathbf{v}'' = x_n^- \cdot \mathbf{v}''' = \mathbf{v}$. Since $[x_\theta^+, x_n^-] = 0$, applying x_n^- to each one of the summands of the right-hand side of the above equality provides

$$x_n^- \left(\sum_{\substack{k: i_k = n+1 \\ k \neq k_0'}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}'') \right) = \sum_{\substack{k: i_k = n+1 \\ k \neq k_0'}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v})$$

$$x_n^- \left(\sum_{\substack{k: i_k = n+1 \\ k \neq k_0}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}''') \right) = \sum_{\substack{k: i_k = n+1 \\ k \neq k_0}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}),$$

so the expression for $x_n^- x_0^- x_n^-(m \otimes \mathbf{v}')$ becomes

$$x_n^- x_0^- x_n^-(m \otimes \mathbf{v}') = \sum_{k: i_k = n+1} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}) \; + \sum_{\substack{k: i_k = n+1 \\ k \neq k_0, k_0'}} A_k^-(m) \otimes \Delta_k(x_\theta^+)(\mathbf{v}). \tag{3.2.21}$$

We now calculate $x_n^- x_n^- x_0^-(m \otimes \mathbf{v}')$. Note that if $s = 2$, then $\mathbf{v}'$ has no tensor factor equal to v_{n+1}, so $x_n^- x_n^- \underbrace{x_0^-(m \otimes \mathbf{v}')}_{=0} = 0$ reduces (3.2.20) to

$$x_0^-(m \otimes \mathbf{v}) = x_n^- x_0^- x_n^-(m \otimes \mathbf{v}').$$

Furthermore, in this case (3.2.21) reduces to

$$x_n^- x_0^- x_n^-(m \otimes \mathbf{v}') = A_{k_0'}^-(m) \otimes \Delta_{k_0'}(x_\theta^+)(\mathbf{v}) + A_{k_0}^-(m) \otimes \Delta_{k_0}(x_\theta^+)(\mathbf{v}),$$

whence (3.2.16) is verified for $\mathbf{v}$ with $s=2$. Hence, we can assume $s>2$. This condition ensures that $\{2,\cdots,n{-}1\}\nsubseteq\{i_k : k\in \mathrm{K}_{\mathbf{v}}\}$ and, once more, it suffices to suppose that $n{-}1\notin\{i_k : k\in\mathrm{K}_{\mathbf{v}}\}$ by the same type of argument involving Proposition 1.7.2 and (3.2.19). In particular, notice that $n{-}1\notin\{i_k : k\in\mathrm{K}_{\mathbf{v}'}\}$. Denote by $\mathbf{v}_0$ the vector obtained from $\mathbf{v}'$ by replacing v_n at the k_0-th tensor factor by v_{n-1} so that $x_{n-1}^-\cdot\mathbf{v}_0=\mathbf{v}'$. The induction hypothesis on the vector $\mathbf{v}_0$ gives us

$$x_0^-(m\otimes\mathbf{v}_0)=\sum_{\substack{k:i_k=n+1\\ k\neq k_0,k_0'}}A_k^-(m)\otimes\Delta_k(x_\theta^+)(\mathbf{v}_0).$$

Since $x_n^-x_n^-\cdot\mathbf{v}'=2\mathbf{v}$, it follows that $x_n^-x_n^-x_{n-1}^-\cdot\mathbf{v}_0=2\mathbf{v}$. Also, using that $[x_\theta^+,x_{n-1}^-]=[x_n^-,x_\theta^+]=0$, then

$$\begin{aligned}\frac{1}{2}\,x_n^-x_n^-x_0^-(m\otimes\mathbf{v}')&=\frac{1}{2}\,x_n^-x_n^-x_{n-1}^-x_0^-(m\otimes\mathbf{v}_0)\\ &=\frac{1}{2}x_n^-x_n^-x_{n-1}^-\Bigg(\sum_{\substack{k:i_k=n+1\\ k\neq k_0,k_0'}}A_k^-(m)\otimes\Delta_k(x_\theta^+)(\mathbf{v}_0)\Bigg)\\ &=\sum_{\substack{k:i_k=n+1\\ k\neq k_0,k_0'}}A_k^-(m)\otimes\Delta_k(x_\theta^+)(\mathbf{v}).\end{aligned}\tag{3.2.22}$$

One may check that plugging (3.2.22) and (3.2.21) back into (3.2.20) finally give the result (3.2.16), finishing the base case of induction for r and the proof. □

We now use the operators $A_j^\pm$ to extend the given representation $\eta:\mathcal{S}_\ell\to\mathrm{GL}(M)$ of $\mathcal{S}_\ell$ to a representation $\tilde\eta$ of $\tilde{\mathcal{S}}_\ell$ on $M\in\mathscr{S}_\ell$. Similarly as in the discussion preceding Theorem 3.2.2, there exists a unique linear map defined on the free group of generators $\sigma_i,y_j^{\pm1}$ such that the corresponding automorphism for σ_i equals $\eta(\sigma_i)$ and for $y_j^{\pm1}$ equals $A_j^\pm$, so we are left to verify that these automorphisms satisfy the same defining relations as their group generators counterparts.

Theorem 3.2.6. *There exists $\tilde\eta:\tilde{\mathcal{S}}_\ell\to GL(M)$ a representation of $\tilde{\mathcal{S}}_\ell$ extending η and such that $\tilde\eta(y_j^{\pm1})=A_j^\pm$ for all $1\le j\le\ell$.*

Proof. Since η is a representation of $\mathcal{S}_\ell$, the endomorphisms $\eta(\sigma_i)$ already satisfy the defining relations associated with (1.6.1). Thus, from the group relations (3.2.1), we only have to verify the following equalities

$$A_r^-A_r^+=A_r^+A_r^-=1,\ \ A_r^+A_s^+=A_s^+A_r^+,\ r\neq s,$$
$$A_{r+1}^+=\eta(\sigma_r)A_r^+\eta(\sigma_r),\ \ A_r^+\eta(\sigma_s)=\eta(\sigma_s)A_r^+,r\notin\{s,s+1\}$$

as in the steps below.

1. $A_r^- A_r^+ = A_r^+ A_r^- = 1.$

Let $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in \mathbb{V}^{\otimes \ell}$ be any basis element, and let $m \in M$. By a similar reasoning which led to (3.2.8), notice that Lemma 3.2.5 shows that $[x_0^+, x_0^-](m \otimes \mathbf{v})$ is equal to

$$I_1 = \sum_{k=1}^{\ell} \left(A_k^+(A_k^-(m)) \otimes \Delta_k(x_\theta^-)(\Delta_k(x_\theta^+)(\mathbf{v})) - A_k^-(A_k^+(m)) \otimes \Delta_k(x_\theta^+)(\Delta_k(x_\theta^-)(\mathbf{v})) \right) + \\ + \sum_{j=1}^{\ell} \sum_{k \neq j}^{\ell} \left(A_j^+ A_k^- - A_k^- A_j^+ \right)(m) \otimes \Delta_k(x_\theta^+)(\Delta_j(x_\theta^-)(\mathbf{v})).$$

Since $[x_0^+, x_0^-] = -h_\theta$ in $\tilde{\mathfrak{g}}$, then $[x_0^+, x_0^-](m \otimes \mathbf{v})$ is also equal to

$$I_2 = \sum_{k=1}^{\ell} m \otimes \left(\Delta_k(x_\theta^-)\Delta_k(x_\theta^+) - \Delta_k(x_\theta^+)\Delta_k(x_\theta^-) \right)(\mathbf{v}),$$

whence we obtain the equality $I_1 = I_2$. Choose $\mathbf{v}$ with $i_r = n+1$ and the remaining $i_k \in \{2, \cdots, n\}$ with $i_k \neq i_j$ for $k \neq j$. In this case, we have $\Delta_k(x_\theta^-)(\mathbf{v}) = 0$ for all k and $\Delta_k(x_\theta^+)(\mathbf{v}) = 0$ for all $k \neq r$. Thus, setting such $\mathbf{v}$ in $I_1 = I_2$, we obtain that

$$A_r^+(A_r^-(m)) \otimes \Delta_r(x_\theta^-)(\Delta_r(x_\theta^+)(\mathbf{v})) = m \otimes \Delta_r(x_\theta^-)(\Delta_r(x_\theta^+)(\mathbf{v})). \tag{3.2.23}$$

Since the linear map $m \mapsto m \otimes \mathbf{v}$ is injective by Corollary 3.1.5, and $\Delta_r(x_\theta^-)(\Delta_r(x_\theta^+)(\mathbf{v})) = \mathbf{v}$, (3.2.23) shows that $A_r^+ A_r^- = 1$. Similarly, one can prove the relation $A_r^- A_r^+ = 1$ as well.

2. $A_r^+ A_s^+ = A_s^+ A_r^+$, $r \neq s$.

Since $A_k^- A_k^+ = A_k^+ A_k^- = 1$ for $k \in \{r, s\}$ by the previous step, notice that it suffices to show $A_r^+ A_s^- = A_s^- A_r^+$. Furthermore, step 1 also implies that

$$A_k^-(A_k^+(m)) \otimes \Delta_k(x_\theta^-)(\Delta_k(x_\theta^+)(\mathbf{v})) - A_k^+(A_k^-(m)) \otimes \Delta_k(x_\theta^+)(\Delta_k(x_\theta^-)(\mathbf{v})) \\ = m \otimes \left(\Delta_k(x_\theta^-)\Delta_k(x_\theta^+) - \Delta_k(x_\theta^+)\Delta_k(x_\theta^-) \right)(\mathbf{v}),$$

so the equality $I_1 = I_2$ above reduces to

$$\sum_{j=1}^{\ell} \sum_{k \neq j}^{\ell} \left(A_j^+ A_k^- - A_k^- A_j^+ \right)(m) \otimes \Delta_k(x_\theta^+)(\Delta_j(x_\theta^-)(\mathbf{v})) = 0. \tag{3.2.24}$$

We now choose any $\mathbf{v}$ with $i_r = 1, i_s = n+1$, and the remaining $i_k \in \{2, \cdots, n\}$ all different. Then, we have $\Delta_k(x_\theta^+)(\Delta_j(x_\theta^-)(\mathbf{v})) = 0$ whenever $(j,k) \neq (r,s)$, and (3.2.24) reduces to the equality

$$\left(A_r^+ A_s^- - A_s^- A_r^+ \right)(m) \otimes \Delta_s(x_\theta^+)(\Delta_r(x_\theta^-)(\mathbf{v})) = 0.$$

Since $\mathbf{v}' = \Delta_s(x_\theta^+)(\Delta_r(x_\theta^-)(\mathbf{v}))$ has different composing tensor factors, the map $m \mapsto m \otimes \mathbf{v}'$ is injective, so the above equality shows $A_r^+ A_s^- = A_s^- A_r^+$.

3. $A_{r+1}^+ = \eta(\sigma_r) A_r^+ \eta(\sigma_r)$.

Using Lemma 3.2.4 we obtain

$$A_r^+(m \cdot \sigma_r) \otimes \mathbf{v}^{(r)} = x_0^+(m \cdot \sigma_r \otimes \mathbf{w}^{(r)}), \quad A_{r+1}^+(m) \otimes \mathbf{v}^{(r+1)} = x_0^+(m \otimes \mathbf{w}^{(r+1)}).$$

Since $\sigma_r \cdot \mathbf{w}^{(r)} = \mathbf{w}^{(r+1)}$ and $\sigma_r \cdot \mathbf{v}^{(r)} = \mathbf{v}^{(r+1)}$, then $x_0^+(m \cdot \sigma_r \otimes \mathbf{w}^{(r)}) = x_0^+(m \otimes \mathbf{w}^{(r+1)})$, whence it follows that $A_r^+(m \cdot \sigma_r) \otimes \mathbf{v}^{(r)} = A_{r+1}^+(m) \otimes \mathbf{v}^{(r+1)} = A_{r+1}^+(m) \cdot \sigma_r \otimes \mathbf{v}^{(r)}$. Thus, $A_r^+ \eta(\sigma_r) = \eta(\sigma_r) A_{r+1}^+$ follows by the fact that $m \mapsto m \otimes \mathbf{v}^{(r)}$ is injective.

4. $A_r^+ \eta(\sigma_s) = \eta(\sigma_s) A_r^+, r \notin \{s, s+1\}$.

By Lemma 3.2.5, we have

$$x_0^+(m \otimes \sigma_s \cdot \mathbf{w}^{(r)}) = \sum_{j=1}^{\ell} A_j^+(m) \otimes \Delta_j(x_\theta^-)(\sigma_s \cdot \mathbf{w}^{(r)}) = A_r^+(m) \otimes \Delta_r(x_\theta^-)(\sigma_s \cdot \mathbf{w}^{(r)}),$$

since $r \notin \{s, s+1\}$ ensures that $\sigma_s \cdot \mathbf{w}^{(r)}$ has a unique tensor factor equal to v_1 at the r-th position. Also, notice that $\Delta_r(x_\theta^-)(\sigma_s \cdot \mathbf{w}^{(r)}) = \sigma_s \cdot \Delta_r(x_\theta^-)(\mathbf{w}^{(r)}) = \sigma_s \cdot \mathbf{v}^{(r)}$, so the above equality reduces to $x_0^+(m \otimes \sigma_s \cdot \mathbf{w}^{(r)}) = A_r^+(m) \otimes \sigma_s \cdot \mathbf{v}^{(r)} = A_r^+(m) \cdot \sigma_s \otimes \mathbf{v}^{(r)}$. On the other hand, from Lemma 3.2.4 we have $A_r^+(m \cdot \sigma_s) \otimes \mathbf{v}^{(r)} = x_0^+(m \cdot \sigma_s \otimes \mathbf{w}^{(r)}) = x_0^+(m \otimes \sigma_s \cdot \mathbf{w}^{(r)})$, whence the equality $A_r^+(m \cdot \sigma_s) \otimes \mathbf{v}^{(r)} = A_r^+(m) \cdot \sigma_s \otimes \mathbf{v}^{(r)}$ follows. Once more, the map $m \mapsto m \otimes \mathbf{v}^{(r)}$ being injective gives us the result. □

We are finally able to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. We begin by proving that $\tilde{\mathcal{F}}_\ell : \tilde{\mathscr{S}}_\ell \to \tilde{\mathscr{G}}_\ell$ is essentially surjective on objects. More explicitly, we use the $\tilde{\mathcal{S}}_\ell$-module $\tilde{M}$ obtained from the previous theorem and consider $\tilde{\mathcal{F}}_\ell(\tilde{M})$ given by Theorem 3.2.2 to prove that $\tilde{\mathcal{F}}_\ell(\tilde{M}) \cong W$ as $\tilde{\mathfrak{g}}$-modules, where $W \in \tilde{\mathscr{G}}_\ell$ is the arbitrary $\tilde{\mathfrak{g}}$-module we have fixed earlier in this section and which satisfies $\mathcal{F}_\ell(M) \cong W$ via the isomorphism of $\mathfrak{g}$-modules $f : \mathcal{F}_\ell(M) \to W$. The isomorphism between W and $\tilde{\mathcal{F}}_\ell(\tilde{M})$ will be established by f itself, once we ensure that f commutes with the action of $\tilde{\mathfrak{g}}$. In order to check it, it suffices to only verify that f commutes with the action of $x_0^{\pm}$. Indeed, for all $m \otimes v \in \tilde{\mathcal{F}}_\ell(\tilde{M})$ we have

$$\begin{aligned} f(x_0^{\pm}(m \otimes v)) &\overset{\text{Theorem 3.2.2}}{=} f\left(\sum_{j=1}^{\ell} m\cdot y_j^{\pm 1} \otimes \Delta_j(x_\theta^{\mp})(v)\right) \\ &\overset{\text{Theorem 3.2.6}}{=} f\left(\sum_{j=1}^{\ell} A_j^{\pm}(m) \otimes \Delta_j(x_\theta^{\mp})(v)\right) \\ &\overset{(3.2.15),(3.2.14)}{=} f(f^{-1}(x_0^{\pm}\cdot f(m \otimes v))) = x_0^{\pm}(f(m \otimes v)). \end{aligned}$$

It remains to prove that $\tilde{\mathcal{F}}_\ell$ is full and faithful. Given any $M, N \in \tilde{\mathscr{S}}_\ell$, set $V = \tilde{\mathcal{F}}_\ell(M), W = \tilde{\mathcal{F}}_\ell(N)$ and consider $\mathrm{Hom}_{\tilde{\mathscr{S}}_\ell}(M, N) \to \mathrm{Hom}_{\tilde{\mathscr{G}}_\ell}(V, W)$, $f \mapsto f \otimes 1$. If $f \otimes 1 = g \otimes 1$, in particular f, g are morphisms of $\mathcal{S}_\ell$-modules with $M, N \in \mathscr{S}_\ell$, whence $f = g$ by the equivalence of categories $\mathcal{F}_\ell : \mathscr{S}_\ell \to \mathscr{G}_\ell$ of the classical Schur-Weyl duality.

Now, given any morphism of $\tilde{\mathfrak{g}}$-modules $F : V \to W$, since F is in particular a morphism of $\mathfrak{g}$-modules with $V, W \in \mathscr{G}_\ell$, we have $F = f \otimes 1$ for some morphism of $\mathcal{S}_\ell$-modules $f : M \to N$ given by the equivalence of Theorem 3.1.2. We show that f commutes with the action of $\tilde{\mathcal{S}}_\ell$ on M, N given by Theorem 3.2.6 by showing that it commutes with the action of each $y_j^{\pm 1}$, $1 \le j \le \ell$. Since $x_0^{+}(F(m \otimes \mathbf{v})) = F(x_0^{+}(m \otimes \mathbf{v}))$ for any $m \otimes \mathbf{v} \in V$ with $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_\ell}$, we have that

$$\sum_{k=1}^{\ell} f(m \cdot y_k) \otimes \Delta_k(x_\theta^{-})(\mathbf{v}) = \sum_{k=1}^{\ell} f(m) \cdot y_k \otimes \Delta_k(x_\theta^{-})(\mathbf{v}). \tag{3.2.25}$$

Thus, given any $1 \le j \le \ell$, choose $\mathbf{v}$ with $i_j = 1$ and the remaining $i_k \in \{2, \cdots, n+1\}$ all different, so that (3.2.25) reduces to

$$f(m\cdot y_j) \otimes \Delta_j(x_\theta^{-})(\mathbf{v}) = f(m)\cdot y_j \otimes \Delta_j(x_\theta^{-})(\mathbf{v}),$$

which shows that $f(m \cdot y_j) = f(m) \cdot y_j$ by Corollary 3.1.5. Since $y_j^{-1} y_j = 1$, then

$$\Big(f(m\cdot y_j^{-1}) - f(m)\cdot y_j^{-1}\Big)\cdot y_j = f(m \cdot y_j^{-1} y_j) - f(m) = 0,$$

whence $f(m)\cdot y_j^{-1} = f(m\cdot y_j^{-1})$ follows as well. Hence, the map $f : M \to N$ is shown to be an homomorphism of $\tilde{\mathcal{S}}_\ell$-modules. □

Remark. Since $\tilde{\mathcal{F}}_\ell : \tilde{\mathscr{S}}_\ell \to \tilde{\mathscr{G}}_\ell$ is an equivalence between abelian categories, then $\tilde{\mathcal{F}}_\ell$ is exact and reflect exactness. In particular, $\tilde{\mathcal{F}}_\ell(M)$ is a simple object if and only if M is a simple object.

3.3 Zelevinsky tensor products and evaluation representations

In Chari and Pressley's article [17], the notion of Zelevinsky tensor product was introduced in the context of modules for the affine Hecke algebra $\hat{H}_\ell(q^2)$. In particular, it was shown that this notion was well-behaved with respect to the functor that established the equivalence between the categories of $U_q(\hat{\mathfrak{g}}')$-modules and $\hat{H}_\ell(q^2)$-modules there discussed. In this section, we shall develop in the context of the $\tilde{\mathcal{S}}_\ell$-modules an analogous concept to the Zelevinsky tensor product. Similarly to the setting of $\hat{H}_\ell(q^2)$-modules, this concept will be seen to behave in a special way with respect to the functor $\tilde{\mathcal{F}}_\ell$ of the affine Schur-Weyl duality. This property, along with the notion of evaluation representations for $\tilde{\mathcal{S}}_\ell$-modules, will be instrumental in the process of describing, up to isomorphism, the simple $\tilde{\mathcal{S}}_\ell$-modules corresponding via $\tilde{\mathcal{F}}_\ell$ to the simple $\tilde{\mathfrak{g}}$-modules.

We first develop the concept of Zelevinsky tensor products in the category $\mathscr{S}_\ell$ as follows. For $\ell_1, \ell_2 \in \mathbb{Z}_{>0}$, let $\iota_{\ell_1,\ell_2} : \mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}] \to \mathbb{C}[\mathcal{S}_{\ell_1+\ell_2}]$ be the unique homomorphism of algebras such that we have

$$\begin{aligned} \iota_{\ell_1,\ell_2}(\sigma_i \otimes 1) &= \sigma_i, \ i = 0, \cdots, \ell_1 - 1, \\ \iota_{\ell_1,\ell_2}(1 \otimes \sigma_i) &= \sigma_{i+\ell_1}, \ i = 0, \cdots, \ell_2 - 1, \end{aligned} \tag{3.3.1}$$

where we set $\sigma_0 = \sigma_{\ell_1} = 1$. Given right S_{ℓ_i}-modules $M_i \in \mathscr{S}_{\ell_i}, i = 1, 2$, recall that the tensor product $M_1 \otimes M_2$ is a right $\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]$-module such that

$$(m_1 \otimes m_2) \cdot (s_1 \otimes s_2) = (m_1 \cdot s_1) \otimes (m_2 \cdot s_2) \quad \text{for } m_i \in M_i, s_i \in \mathbb{C}[\mathcal{S}_{\ell_i}], \ i = 1, 2.$$

Set $\ell = \ell_1 + \ell_2$. Since $\mathbb{C}[\mathcal{S}_\ell]$ can be regarded as a left $\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]$-module by the pullback along ι_{ℓ_1,ℓ_2}, we define the **Zelevisnky tensor product** $M_1 \boxtimes M_2$ as

$$M_1 \boxtimes M_2 = \mathrm{Ind}_{\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]}^{\mathbb{C}[\mathcal{S}_\ell]} (M_1 \otimes M_2) = (M_1 \otimes M_2) \bigotimes_{\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]} \mathbb{C}[\mathcal{S}_\ell]. \tag{3.3.2}$$

In particular, $M_1 \boxtimes M_2$ is naturally a right $\mathbb{C}[S_\ell]$-module, so we may consider $\mathcal{F}_\ell(M_1 \boxtimes M_2)$ given by (3.1.1). The following lemma is essential to investigate how the functor $\mathcal{F}_\ell$ behaves with respect to the Zelevinsky tensor product just defined.

Lemma 3.3.1. *Let A_j be a unital associative complex algebra, M_j be a right A_j-module and V_j be a left A_j-module, $j = 1, 2$. As complex vector spaces, we have*

$$(M_1 \otimes_{A_1} V_1) \otimes (M_2 \otimes_{A_2} V_2) \cong (M_1 \otimes M_2) \bigotimes_{A_1 \otimes A_2} (V_1 \otimes V_2).$$

Proof. To shorten notation, set $M = (M_1 \otimes M_2) \bigotimes_{A_1 \otimes A_2} (V_1 \otimes V_2)$ and $M' = (M_1 \otimes_{A_1} V_1) \otimes (M_2 \otimes_{A_2} V_2)$.

For $(m_1, v_1) \in M_1 \times V_1$, let $\varphi_{m_1,v_1} : M_2 \times V_2 \to M$ be given by

$$\varphi_{m_1,v_1}(m_2, v_2) = (m_1 \otimes m_2) \otimes (v_1 \otimes v_2).$$

Using the action of $(1 \otimes a_2) \in A_1 \otimes A_2, a_2 \in A_2$, on $M_1 \otimes M_2$ and $V_1 \otimes V_2$, we have

$$\begin{aligned} \varphi_{m_1,v_1}(m_2 \cdot a_2, v_2) &= \Big((m_1 \otimes m_2) \cdot (1 \otimes a_2)\Big) \otimes (v_1 \otimes v_2) \\ &= (m_1 \otimes m_2) \otimes \Big((1 \otimes a_2) \cdot (v_1 \otimes v_2)\Big) = \varphi_{m_1,v_1}(m_2, a_2 \cdot v_2), \end{aligned}$$

so the map φ_{m_1,v_1} is A_2-bilinear and there exists a unique $\mathbb{C}$-linear map $\tilde{\varphi}_{m_1,v_1} : M_2 \otimes_{A_2} V_2 \to M$ satisfying $\tilde{\varphi}_{m_1,v_1}(m_2 \otimes v_2) = (m_1 \otimes v_1) \otimes (m_2 \otimes v_2)$ by the universal property of tensor products of modules over rings.

Furthermore, by verifying on generators, we have the following equality of maps:

$$\begin{aligned} \tilde{\varphi}_{m_1 \cdot a_1, v_1} &= \tilde{\varphi}_{m_1,\, a_1 \cdot v_1} \\ \tilde{\varphi}_{m_1 + m_1', v_1} &= \tilde{\varphi}_{m_1,v_1} + \tilde{\varphi}_{m_1',v_1} \\ \tilde{\varphi}_{m_1, v_1 + v_1'} &= \tilde{\varphi}_{m_1,v_1} + \tilde{\varphi}_{m_1,v_1'} \end{aligned}$$

for all $a_1 \in A_1, m_1, m_1' \in M_1$ and $v_1, v_1' \in V_1$. In this way, $\psi_\eta : M_1 \times V_1 \to M$ depending on $\eta \in M_2 \otimes_{A_2} V_2$ and given by

$$\psi_\eta(m_1, v_1) = \tilde{\varphi}_{m_1,v_1}(\eta)$$

is seen to be A_1-bilinear, whence a $\mathbb{C}$-linear map $\tilde{\psi}_\eta : M_1 \otimes V_1 \to M$ such that $\tilde{\psi}_\eta(m_1 \otimes v_1) = \tilde{\varphi}_{m_1,v_1}(\eta)$ is obtained.

Finally, define $\phi : \Big(M_1 \otimes_{A_1} V_1\Big) \times \Big(M_2 \otimes_{A_2} V_2\Big) \to M$ by

$$\phi(\xi, \eta) = \tilde{\psi}_\eta(\xi).$$

This map is clearly $\mathbb{C}$-linear in its first entry. Since

$$\tilde{\psi}_{\eta+\eta'}(m_1 \otimes v_1) = \tilde{\varphi}_{m_1,v_1}(\eta + \eta') = \Big(\tilde{\psi}_\eta + \tilde{\psi}_{\eta'}\Big)(m_1 \otimes v_1)$$

holds for all $m_1 \otimes v_1 \in M_1 \otimes_{A_1} V_1$, we conclude that ϕ is also $\mathbb{C}$-linear in its second entry. Therefore, the existence of a unique $\mathbb{C}$-linear map $\tilde{\phi} : M' \to M$ such that $\tilde{\phi}(\xi \otimes \eta) = \tilde{\psi}_\eta(\xi)$ is ensured. In particular, we notice that

$$\tilde{\phi}\Big((m_1 \otimes v_1) \otimes (m_2 \otimes v_2)\Big) = (m_1 \otimes m_2) \otimes (v_1 \otimes v_2), \quad m_i \in M_i, v_i \in V_i,\ i = 1, 2. \tag{3.3.3}$$

The inverse of $\tilde{\phi}$ can be described analogously. Given $(m_1, m_2) \in M_1 \times M_2, v \in V_1 \otimes V_2$, using the universal property of $\mathbb{C}$-bilinear maps, one checks that the following $\mathbb{C}$-linear maps $\tilde{f}_{m_1,m_2} : V_1 \otimes V_2 \to M'$ and $\tilde{g}_v : M_1 \otimes M_2 \to M'$ such that

$$\tilde{f}_{m_1,m_2}(v_1 \otimes v_2) = (m_1 \otimes v_1) \otimes (m_2 \otimes v_2) \quad \text{for all } \ v_i \in V_i,$$
$$\tilde{g}_v(m_1 \otimes m_2) = \tilde{f}_{m_1,m_2}(v) \quad \text{for all } \ m_i \in M_i, \ \ i = 1, 2,$$

are well-defined. We then set $h : (M_1 \otimes M_2) \times (V_1 \otimes V_2) \to M'$ by

$$h(m, v) = \tilde{g}_v(m).$$

By analogous procedures previously done for the map ϕ above, one checks that h is linear in each entry. Moreover, since

$$\tilde{g}_{v_1 \otimes v_2}\Big((m_1 \otimes m_2) \cdot (a_1 \otimes a_2)\Big) = \tilde{g}_{(a_1 \otimes a_2)\cdot(v_1 \otimes v_2)}(m_1 \otimes m_2)$$

holds for all $m_i \in M_i, v_i \in V_i, a_i \in A_i, i = 1, 2$, we see that h is $A_1 \otimes A_2$-bilinear, so a $\mathbb{C}$-linear map $\tilde{h} : M \to M'$ is obtained accordingly. In particular, it satisfies

$$\tilde{h}\Big((m_1 \otimes m_2) \otimes (v_1 \otimes v_2)\Big) = (m_1 \otimes v_1) \otimes (m_2 \otimes v_2), \tag{3.3.4}$$

which combined with (3.3.3) shows that they are mutual inverses. □

Proposition 3.3.2. *$\mathcal{F}_\ell(M_1 \boxtimes M_2) \cong \mathcal{F}_{\ell_1}(M_1) \otimes \mathcal{F}_{\ell_2}(M_2)$ as $\mathfrak{g}$-modules.*

Proof. Since $\mathbb{C}[\mathcal{S}_\ell]$ can be regarded both as a right $\mathbb{C}[\mathcal{S}_\ell]$-module and as a $\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]$-bimodule by the pullback along (3.3.1), by standard results on commutative algebra we obtain that

$$\left(\Big(M_1 \otimes M_2\Big) \bigotimes_{\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]} \mathbb{C}[\mathcal{S}_\ell]\right) \otimes_{\mathbb{C}[\mathcal{S}_\ell]} \mathbb{V}^{\otimes \ell} \cong \Big(M_1 \otimes M_2\Big) \bigotimes_{\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]} \left(\mathbb{C}[\mathcal{S}_\ell] \otimes_{\mathbb{C}[\mathcal{S}_\ell]} \mathbb{V}^{\otimes \ell}\right). \tag{3.3.5}$$

Since $\mathbb{C}[\mathcal{S}_\ell] \otimes_{\mathbb{C}[\mathcal{S}_\ell]} \mathbb{V}^{\otimes \ell} \cong \mathbb{V}^{\otimes \ell}$ as left $\mathbb{C}[\mathcal{S}_\ell]$-modules, the pullback along (3.3.1) also ensures that this isomorphism holds as left $\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]$-modules. Moreover, one checks that the isomorphism $f : \mathbb{V}^{\otimes \ell} \to \mathbb{V}^{\otimes \ell_1} \otimes \mathbb{V}^{\otimes \ell_2}$ mapping

$$(w_1 \otimes \cdots \otimes w_\ell) \mapsto (w_1 \otimes \cdots \otimes w_{\ell_1}) \otimes (w_{\ell_1+1} \otimes \cdots \otimes w_\ell), w_i \in \mathbb{V},$$

commutes with the action of $\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]$, whence it follows that $\mathbb{V}^{\otimes \ell} \cong \mathbb{V}^{\otimes \ell_1} \otimes \mathbb{V}^{\otimes \ell_2}$ as left $\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]$-modules. Thus, (3.3.5) reduces to the isomorphism

$$\mathcal{F}_\ell(M_1 \boxtimes M_2) \cong (M_1 \otimes M_2) \bigotimes_{\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \mathbb{C}[\mathcal{S}_{\ell_2}]} \mathbb{V}^{\otimes \ell_1} \otimes \mathbb{V}^{\otimes \ell_2}. \tag{3.3.6}$$

In particular, the above isomorphism assigns $\Big(m \otimes s\Big) \otimes v \in \mathcal{F}_\ell(M_1 \boxtimes M_2)$ as follows

$$\Big(m \otimes s\Big) \otimes v \mapsto m \otimes f(s \cdot v),$$

where $m \in M_1 \otimes M_2, s \in \mathbb{C}[\mathcal{S}_\ell]$ and $v \in \mathbb{V}^{\otimes \ell}$. Since f and $\mathbb{C}[\mathcal{S}_\ell]$ commute with the action of $\mathfrak{g}$, (3.3.6) is in particular an isomorphism of $\mathfrak{g}$-modules. Now, the right-hand side of (3.3.6) is isomorphic to $\mathcal{F}_{\ell_1}(M_1) \otimes \mathcal{F}_{\ell_2}(M_2)$ as a vector space by Lemma 3.3.1. By (3.3.3) and (3.1.2), we see that this isomorphism maps

$$x \cdot \Big((m_1 \otimes m_2) \otimes (w_1 \otimes w_2)\Big) \mapsto (m_1 \otimes x \cdot w_1) \otimes (m_2 \otimes w_2) + (m_1 \otimes w_1) \otimes (m_2 \otimes x \cdot w_2)$$

for all $m_i \in M_i, w_i \in \mathbb{V}^{\otimes \ell_i}$, $i = 1, 2, x \in \mathfrak{g}$, showing that it commutes with the action of $\mathfrak{g}$ as well. Composing this isomorphism with (3.3.6), then $\mathcal{F}_\ell(M_1 \boxtimes M_2) \cong \mathcal{F}_{\ell_1}(M_1) \otimes \mathcal{F}_{\ell_2}(M_2)$ as $\mathfrak{g}$-modules. □

We denote by $\mathbb{C}[\tilde{\mathcal{S}}_k]$ the right **regular representation** (or the **group algebra**) of $\tilde{\mathcal{S}}_k, k \in \mathbb{Z}_{>0}$. One can also define the homomorphism of algebras $\tilde{\iota}_{\ell_1,\ell_2} : \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}] \to \mathbb{C}[\tilde{\mathcal{S}}_\ell]$ mapping each generator of the form $\sigma_i \otimes 1$ and $1 \otimes \sigma_i$ as in (3.3.1), along with

$$\begin{aligned} \tilde{\iota}_{\ell_1,\ell_2}(y_j \otimes 1) &= y_j, \ j = 1, \cdots, \ell_1, \\ \tilde{\iota}_{\ell_1,\ell_2}(1 \otimes y_j) &= y_{j+\ell_1}, \ j = 1, \cdots, \ell_2. \end{aligned} \tag{3.3.7}$$

In particular, notice that there exists a unique $y' \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]$ such that $\tilde{\iota}_{\ell_1,\ell_2}(y') = y$ for any given $y \in \mathbb{C}[y_1, \cdots, y_\ell] \cap \mathbb{C}[\tilde{\mathcal{S}}_\ell]$.

The group algebra $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ is a left $\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]$-module by the pullback along $\tilde{\iota}_{\ell_1,\ell_2}$, and $M_1 \otimes M_2$ is a right $\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]$-module in the obvious way, $M_i \in \tilde{\mathscr{S}}_{\ell_i}, i = 1, 2$. Thus, we set the **affine Zelevisnky tensor product** $M_1 \tilde{\boxtimes} M_2$ to be

$$M_1 \tilde{\boxtimes} M_2 = \mathrm{Ind}_{\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]}^{\mathbb{C}[\tilde{\mathcal{S}}_\ell]} (M_1 \otimes M_2). \tag{3.3.8}$$

We introduce the notation $M\Big|_{\mathcal{S}_k}$ for $M \in \tilde{\mathscr{S}}_k, k \in \mathbb{Z}_{>0}$, to make explicit that M is being regarded as an $\mathcal{S}_k$-module.

Proposition 3.3.3. $\Big(M_1 \tilde{\boxtimes} M_2\Big)\Big|_{\mathcal{S}_\ell} \cong M_1\Big|_{\mathcal{S}_{\ell_1}} \boxtimes M_2\Big|_{\mathcal{S}_{\ell_2}}$ *as $\mathcal{S}_\ell$-modules.*

Proof. Set $M = M_1\Big|_{\mathcal{S}_{\ell_1}} \boxtimes M_2\Big|_{\mathcal{S}_{\ell_2}}$ and $\tilde{M} = \Big(M_1 \tilde{\boxtimes} M_2\Big)\Big|_{\mathcal{S}_\ell}$.

Given $(m_1, m_2) \in M_1 \times M_2$, let $\varphi_{m_1,m_2} : \mathbb{C}[\tilde{\mathcal{S}}_\ell] \to M$ be the unique homomorphism of $\mathcal{S}_\ell$-modules such that

$$\begin{aligned} \sigma &\mapsto (m_1 \otimes m_2) \otimes \sigma, \ \sigma \in \mathcal{S}_\ell, \\ y &\mapsto \Big((m_1 \otimes m_2) \cdot \tilde{\iota}^{-1}_{\ell_1,\ell_2}(y)\Big) \otimes 1, \ y \in \mathbb{C}[y_1, \cdots, y_\ell]. \end{aligned} \tag{3.3.9}$$

In particular, we notice that

$$\begin{aligned} \varphi_{m_1,m_2}(y_j^{\pm 1}) &= (m_1 \cdot y_j^{\pm 1} \otimes m_2) \otimes 1, \\ \varphi_{m_1,m_2}(y_k^{\pm 1}) &= (m_1 \otimes m_2 \cdot y_k^{\pm 1}) \otimes 1 \end{aligned} \tag{3.3.10}$$

for $1 \le j \le \ell_1, 1 \le k \le \ell_2$. One checks that $\psi_s : M_1 \times M_2 \to M$ depending on $s \in \mathbb{C}[\tilde{\mathcal{S}}_\ell]$ and given by

$$\psi_s(m_1, m_2) = \varphi_{m_1,m_2}(s)$$

is $\mathbb{C}$-bilinear, whence $\tilde{\psi}_s : M_1 \otimes M_2 \to M$ such that $\tilde{\psi}_s(m_1 \otimes m_2) = \varphi_{m_1,m_2}(s)$ is well-defined by the universal property of tensor products. Now, set $\phi : (M_1 \otimes M_2) \times \mathbb{C}[\tilde{\mathcal{S}}_\ell] \to M$ by

$$\phi(m, s) = \tilde{\psi}_s(m).$$

Since any element of $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ is a sum of elements of the form $y\sigma$ for $y \in \mathbb{C}[y_1, \cdots, y_\ell], \sigma \in \mathbb{C}[\mathcal{S}_\ell]$, it follows from (3.3.7) and (3.3.9) that

$$\phi(m_1 \otimes m_2 \cdot (s_1 \otimes s_2), y\sigma) = \varphi_{m_1 \cdot s_1, m_2 \cdot s_2}(y) \cdot \sigma = \Big((m_1 \otimes m_2) \cdot (s_1 \otimes s_2)\tilde{\iota}_{\ell_1,\ell_2}^{-1}(y)\Big) \otimes \sigma$$

holds for all $m_i \in M_i, s_i \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell_i}], i = 1, 2$, which shows that ϕ is $\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]$-bilinear, and the existence of a unique linear map $\tilde{\phi} : \tilde{M} \to M$ satisfying $\tilde{\phi}(m \otimes s) = \phi(m, s)$ is esnured.

The existence of a unique $\mathbb{C}$-linear map $f : M \to \tilde{M}$ such that $f(m \otimes r) = m \otimes r$ holds for all $m \in (M_1 \otimes M_2), r \in \mathbb{C}[\mathcal{S}_\ell]$ can be proved analogously. In particular, for every $m_i \in M_i$ we have

$$\begin{aligned} (m_1 \cdot y_j^{\pm 1} \otimes m_2) \otimes 1 &\overset{f}{\longmapsto} \Big((m_1 \otimes m_2) \cdot (y_j^{\pm 1} \otimes 1)\Big) \otimes 1 = (m_1 \otimes m_2) \otimes y_j^{\pm 1},\ j = 1, \cdots, \ell_1 \\ (m_1 \otimes m_2 \cdot y_j^{\pm 1}) \otimes 1 &\overset{f}{\longmapsto} \Big((m_1 \otimes m_2) \cdot (1 \otimes y_j^{\pm 1})\Big) \otimes 1 = (m_1 \otimes m_2) \otimes y_j^{\pm 1},\ j = 1, \cdots, \ell_2, \end{aligned} \tag{3.3.11}$$

which, combined with (3.3.10), shows that $\tilde{\phi}$ and f are mutual inverses. Since $\tilde{\phi}$ commutes with the right action of $\mathbb{C}[\mathcal{S}_\ell]$ trivially, the isomorphism follows. □

The next proposition shows that the functor $\tilde{\mathcal{F}}_\ell$ also behaves well with respect to the affine Zelevinsky tensor product.

Proposition 3.3.4. *$\tilde{\mathcal{F}}_\ell(M_1 \tilde{\boxtimes} M_2) \cong \tilde{\mathcal{F}}_{\ell_1}(M_1) \otimes \tilde{\mathcal{F}}_{\ell_2}(M_2)$ as $\tilde{\mathfrak{g}}$-modules.*

Proof. Given $M \in \tilde{\mathscr{S}}_\ell$, the identity map establishes the isomorphism $\tilde{\mathcal{F}}_\ell(M) \cong_{\mathfrak{g}} \mathcal{F}_\ell(M\big|_{\mathcal{S}_\ell})$. Hence, it follows from Proposition 3.3.2 and Proposition 3.3.3 that

$$\tilde{\mathcal{F}}_\ell(M_1 \tilde{\boxtimes} M_2) \cong_{\mathfrak{g}} \mathcal{F}_\ell\Big(M_1\big|_{\mathcal{S}_{\ell_1}} \boxtimes M_2\big|_{\mathcal{S}_{\ell_2}}\Big) \cong_{\mathfrak{g}} \mathcal{F}_{\ell_1}(M_1\big|_{\mathcal{S}_{\ell_1}}) \otimes \mathcal{F}_{\ell_2}(M_2\big|_{\mathcal{S}_{\ell_2}}) \cong_{\mathfrak{g}} \tilde{\mathcal{F}}_{\ell_1}(M_1) \otimes \tilde{\mathcal{F}}_{\ell_2}(M_2).$$

Denoting this isomorphism by $f : \tilde{\mathcal{F}}_{\ell_1}(M_1) \otimes \tilde{\mathcal{F}}_{\ell_2}(M_2) \to \tilde{\mathcal{F}}_{\ell}(M_1 \tilde{\boxtimes} M_2)$, notice that

$$(m_1 \otimes v_1) \otimes (m_2 \otimes v_2) \xmapsto{f} ((m_1 \otimes m_2) \otimes 1) \otimes (w_1 \otimes w_2), \text{ for } m_i \in M_i, w_i \in \mathbb{V}^{\otimes \ell_i}, i = 1, 2. \tag{3.3.12}$$

In order to prove the claim, we are left to check that the action of $x_0^{\pm}$ on these pure tensors commutes with f. Using the action of $x_0^{\pm}$ on $\tilde{\mathcal{F}}_{\ell_i}(M_i)$ given by (3.2.3), we have $x_0^{\pm} \cdot ((m_1 \otimes w_1) \otimes (m_2 \otimes w_2)) = S_1 + S_2$, where

$$\begin{aligned} S_1 &= \sum_{j=1}^{\ell_1} \Big(m_1 \cdot y_j^{\pm 1} \otimes \Delta(x_\theta^{\mp})(w_1) \Big) \otimes (m_2 \otimes w_2), \\ S_2 &= \sum_{j=1}^{\ell_2} (m_1 \otimes w_1) \otimes \big(m_2 \cdot y_j^{\pm 1} \otimes \Delta(x_\theta^{\mp})(w_2)\big). \end{aligned} \tag{3.3.13}$$

Now, using (3.3.12) and the action of $\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]$ on $M_1 \otimes M_2$, we see that

$$\begin{aligned} f(S_1) &= \sum_{j=1}^{\ell_1} \Big((m_1 \otimes m_2) \cdot (y_j^{\pm 1} \otimes 1) \Big) \otimes \Big(\Delta(x_\theta^{\mp})(w_1) \otimes w_2 \Big), \\ f(S_2) &= \sum_{j=1}^{\ell_2} \Big((m_1 \otimes m_2) \cdot (1 \otimes y_j^{\pm 1}) \Big) \otimes \Big(w_1 \otimes \Delta(x_\theta^{\mp})(w_2) \Big). \end{aligned}$$

Using the action $\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}]$ on $\mathbb{C}[\tilde{\mathcal{S}}_{\ell}]$ established by (3.3.7), notice that

$$f(S_1) + f(S_2) = \sum_{j=1}^{\ell_1+\ell_2} ((m_1 \otimes m_2) \otimes 1\Big) \cdot y_j^{\pm 1} \otimes \Delta_j(x_\theta^{\mp})(w_1 \otimes w_2). \tag{3.3.14}$$

Now, the right-hand side of (3.3.14) is exactly the term $x_0^{\pm}(f(m_1 \otimes v_1) \otimes (m_2 \otimes v_2))$, which shows that f commutes with the action of $x_0^{\pm}$. □

Given $\ell_1, \cdots, \ell_k \in \mathbb{Z}_{>0}$, now set $\ell = \sum_{j=1}^{k} \ell_j$. One defines the homomorphism of algebras $\iota_{\ell_1, \cdots, \ell_k} : \mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \cdots \otimes \mathbb{C}[\mathcal{S}_{\ell_k}] \to \mathbb{C}[\mathcal{S}_{\ell}]$ in a similar fashion as in (3.3.1), and sets the Zelevinsky tensor product over right modules $M_j \in \mathscr{S}_{\ell_j}$ by

$$M_1 \boxtimes \cdots \boxtimes M_k = \operatorname{Ind}_{\mathbb{C}[\mathcal{S}_{\ell_1}] \otimes \cdots \otimes \mathbb{C}[\mathcal{S}_{\ell_k}]}^{\mathbb{C}[\mathcal{S}_{\ell}]} (M_1 \otimes \cdots \otimes M_k).$$

Analogously, one also defines the homomorphism of algebras $\tilde{\iota}_{\ell_1, \cdots, \ell_k}$ and the affine Zelevinsky tensor product $M_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} M_k$ over right modules $M_j \in \tilde{\mathscr{S}}_{\ell_j}$.

With this definition, we see that the affine Zelevinsky tensor product is associative:

Proposition 3.3.5. *Let $M_i \in \tilde{\mathscr{S}}_{\ell_j}$, $1 \le j \le 3$. Then, as $\tilde{\mathcal{S}}_{\ell}$-modules,*

$$M_1 \tilde{\boxtimes} M_2 \tilde{\boxtimes} M_3 \cong M_1 \tilde{\boxtimes} (M_2 \tilde{\boxtimes} M_3) \cong (M_1 \tilde{\boxtimes} M_2) \tilde{\boxtimes} M_3.$$

Proof. We only show that $M_1 \tilde{\boxtimes} M_2 \tilde{\boxtimes} M_3 \cong M_1 \tilde{\boxtimes} (M_2 \tilde{\boxtimes} M_3)$, as the case $M_1 \tilde{\boxtimes} M_2 \tilde{\boxtimes} M_3 \cong$

$(M_1 \tilde{\boxtimes} M_2)\tilde{\boxtimes} M_3$ can be shown analogously. Given the homomorphism of algebras $\tilde{\iota}_{\ell_2,\ell_3}$, set

$$\tilde{\iota} = Id_{\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}]} \otimes \tilde{\iota}_{\ell_2,\ell_3} : \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \Big(\mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_3}]\Big) \to \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2+\ell_3}],$$

so the composition

$$h = \tilde{\iota}_{\ell_1,\ell_2+\ell_3} \circ \tilde{\iota} : \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \Big(\mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_3}]\Big) \to \mathbb{C}[\tilde{\mathcal{S}}_{\ell}]$$

is also a homomorphism of algebras. Thus, $\mathbb{C}[\tilde{\mathcal{S}}_{\ell}]$ is a $\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \Big(\mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_3}]\Big)$-module by the pullback along h. In particular, one easily checks that

$$\begin{aligned}
&h(\sigma_i \otimes (1 \otimes 1)) = \tilde{\iota}_{\ell_1,\ell_2,\ell_3}(\sigma_i \otimes 1 \otimes 1),\ i = 1, \cdots \ell_1 - 1\\
&h(1 \otimes (\sigma_j \otimes 1) = \tilde{\iota}_{\ell_1,\ell_2,\ell_3}(1 \otimes \sigma_j \otimes 1),\ j = 1, \cdots, \ell_2 - 1\\
&h(1 \otimes (1 \otimes \sigma_k)) = \tilde{\iota}_{\ell_1,\ell_2,\ell_3}(1 \otimes 1 \otimes \sigma_k), k = 1, \cdots, \ell_3 - 1\\
&h(y_r \otimes (1 \otimes 1)) = \tilde{\iota}_{\ell_1,\ell_2,\ell_3}(y_r \otimes 1 \otimes 1), r = 1, \cdots, \ell_1\\
&h(1 \otimes (y_s \otimes 1)) = \tilde{\iota}_{\ell_1,\ell_2,\ell_3}(1 \otimes y_s \otimes 1), s = 1, \cdots, \ell_2\\
&h(1 \otimes (1 \otimes y_t)) = \tilde{\iota}_{\ell_1,\ell_2,\ell_3}(1 \otimes 1 \otimes y_t), t = 1, \cdots, \ell_3.
\end{aligned}$$

Since the action of $\mathbb{C}[S_{\ell_1}] \otimes \mathbb{C}[S_{\ell_2}] \otimes \mathbb{C}[S_{\ell_3}]$ and $\mathbb{C}[S_{\ell_1}] \otimes \Big(\mathbb{C}[S_{\ell_2}] \otimes \mathbb{C}[S_{\ell_3}]\Big)$ on $\mathbb{C}[S_\ell]$ are the same up to associativity and $M_1 \otimes M_2 \otimes M_3 \cong M_1 \otimes (M_2 \otimes M_3)$, we have

$$M_1 \otimes M_2 \otimes M_3 \bigotimes_{\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_3}]} \mathbb{C}[\tilde{\mathcal{S}}_{\ell}] \cong M_1 \otimes \Big(M_2 \otimes M_3\Big) \bigotimes_{\mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes (\mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_3}])} \mathbb{C}[\tilde{\mathcal{S}}_{\ell}]. \tag{3.3.15}$$

Denote by M the right-side of the isomorphism above and set $M_0 = M_1 \boxtimes (M_2 \boxtimes M_3)$. Given $s \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell}]$, by the universal property of tensor products, one checks that there exists a unique linear map $\tilde{g}_s : M_1 \otimes (M_2 \otimes M_3) \to M_0$ such that

$$\tilde{g}_s(m \otimes m') = (m \otimes (m' \otimes 1)) \otimes s \ \text{ for all } \ m \otimes m' \in M_1 \otimes (M_2 \otimes M_3).$$

Using these maps, set $\tilde{\phi} : M_1 \otimes (M_2 \otimes M_3) \times \mathbb{C}[\tilde{\mathcal{S}}_{\ell}] \to M_0$ by

$$\tilde{\phi}(x, s) = \tilde{g}_s(x),\ x \in M_1 \otimes (M_2 \otimes M_3), s \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell}].$$

For $m \in M_1, m' \in M_2 \otimes M_3, r \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}], r' \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell_1}] \otimes \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2}], s \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell}]$, notice that

$$\tilde{\phi}((m \otimes m') \cdot (r \otimes r'), s) = (m \cdot r \otimes ((m' \cdot r') \otimes 1)) \otimes s$$

and also

$$\tilde{\phi}(m \otimes m', (r \otimes r') \cdot s) = (m \otimes (m' \otimes 1)) \otimes h(r \otimes r') \cdot s.$$

Manipulating the term $\tilde{\phi}((m \otimes m') \cdot (r \otimes r'), s) \in M_0$, we obtain that

$$\begin{aligned}(m \cdot r \otimes ((m' \cdot r') \otimes 1)) \otimes s &= (m \cdot r \otimes (m' \otimes \tilde{\iota}_{\ell_2,\ell_3}(r')) \otimes s \\ &= ((m \otimes (m' \otimes 1)) \cdot (r \otimes \tilde{\iota}_{\ell_2,\ell_3}(r'))) \otimes s \\ &= (m \otimes (m' \otimes 1)) \otimes h(r \otimes r') \cdot s,\end{aligned}$$

which shows that $\tilde{\phi}(m \otimes m' \cdot (r \otimes r'), s) = \tilde{\phi}(m \otimes m', (r \otimes r') \cdot s)$. One can also check that ϕ is bilinear, so by the universality of tensor products there exists $\phi : M \to M_0$ such that

$$\phi((m \otimes m') \otimes s) = \tilde{\phi}(m \otimes m', s) = (m \otimes (m' \otimes 1)) \otimes s. \tag{3.3.16}$$

The inverse can be described in a similar fashion. In fact, for every $m \in M_1$, there exist $f_m : M_2 \tilde{\boxtimes} M_3 \to M$ such that

$$m' \otimes r \mapsto (m \otimes m') \otimes \tilde{\iota}_{\ell_1,\ell_2+\ell_3}(1 \otimes r) \ , m' \in M_2 \otimes M_3, r \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2+\ell_3}],$$

and also $g : M_1 \otimes (M_2 \tilde{\boxtimes} M_3) \to M$ such that

$$g(m \otimes u) \mapsto f_m(u), \ u \in M_2 \tilde{\boxtimes} M_3, m \in M_1.$$

In this way, $\psi : M_0 \to M$ such that $\psi(v \otimes s) = g(v) \cdot s$ is well-defined, for every $v \in M_1 \otimes (M_2 \tilde{\boxtimes} M_3), s \in \mathbb{C}[\tilde{\mathcal{S}}_\ell]$. In particular, we have

$$\psi((m \otimes (m' \otimes r)) \otimes s) = (m \otimes m') \otimes \tilde{\iota}_{\ell_1,\ell_2+\ell_3}(1 \otimes r) \cdot s \tag{3.3.17}$$

for all $m \in M_1, m' \in M_2 \otimes M_3, r \in \mathbb{C}[\tilde{\mathcal{S}}_{\ell_2+\ell_3}], s \in \mathbb{C}[\tilde{\mathcal{S}}_\ell]$. It follows from (3.3.17),(3.3.16) that ψ, ϕ are mutual inverses and the isomorphism $M \cong M_0$ follows. Therefore, the isomorphism (3.3.15) implies that $M_1 \tilde{\boxtimes} M_2 \tilde{\boxtimes} M_3 \cong M_1 \tilde{\boxtimes} (M_2 \tilde{\boxtimes} M_3)$. □

A direct adaptation of the above proof for the simpler case of $\mathcal{S}_\ell$-modules gives us that

$$M_1 \boxtimes M_2 \boxtimes M_3 \cong (M_1 \boxtimes M_2) \boxtimes M_3 \cong M_1 \boxtimes (M_2 \boxtimes M_3) \text{ for any } M_j \in \mathscr{S}_{\ell_j}, 1 \leq j \leq 3,$$

so the Zelevinsky tensor product is also associative. Moreover, an inductive argument along with the associativity of Zelevinsky tensor products and Proposition 3.3.2 show

$$\mathcal{F}_\ell(M_1 \boxtimes \cdots \boxtimes M_k) \cong \mathcal{F}_{\ell_1}(M_1) \otimes \cdots \otimes \mathcal{F}_{\ell_k}(M_k) \tag{3.3.18}$$

for any $M_i \in \mathscr{S}_{\ell_i}, 1 \leq i \leq k, \ell = \sum_i \ell_i$. Analogously, the associativity of the affine Zelevinsky tensor product and Proposition 3.3.4 gives us that

$$\tilde{\mathcal{F}}_\ell(M_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} M_k) \cong \tilde{\mathcal{F}}_{\ell_1}(M_1) \otimes \cdots \otimes \tilde{\mathcal{F}}_{\ell_k}(M_k) \tag{3.3.19}$$

for any $M_i \in \tilde{\mathscr{S}}_{\ell_i}, \ell = \sum_i \ell_i$.

We now define the analogous concept of evaluation modules in the setting of $\tilde{\mathcal{S}}_\ell$-modules. For $a \in \mathbb{C}^\times$, let $\mathrm{ev}'_a : \mathbb{C}[\tilde{\mathcal{S}}_\ell] \to \mathbb{C}[\mathcal{S}_\ell]$ be the homomorphism of unital associative algebras mapping each σ_i to itself and $y_j^{\pm 1}$ to $a^{\pm 1}$ for all $1 \leq j \leq \ell$. Given $M \in \mathscr{S}_\ell$, let $M(a)$ be the $\tilde{\mathcal{S}}_\ell$-module obtained by the pullback along ev'_a, and set $\mathcal{S}(\xi, a) = \mathcal{S}(\xi)(a)$ for $\xi \in \mathcal{P}_\ell$. If $\xi = (\xi_1, \cdots, \xi_m), m = |\,\xi\,|$, we shall sometimes utilize the more explicit notation $\mathcal{S}(\xi_1, \cdots, \xi_m, a)$ to denote $\mathcal{S}(\xi)(a)$.

Lemma 3.3.6. *For $a \in \mathbb{C}^\times$ and $M \in \mathscr{S}_\ell$, we have $\tilde{\mathcal{F}}_\ell(M(a)) \cong \mathcal{F}_\ell(M)(a)$ as $\tilde{\mathfrak{g}}$-modules. In particular, $\tilde{\mathcal{F}}_\ell(\mathcal{S}(\xi, a)) \cong V(wt(\xi), a)$, $\xi \in \mathcal{P}_\ell$.*

Proof. Since both $\tilde{\mathcal{F}}_\ell(M(a))$ and $\mathcal{F}_\ell(M)(a)$ are the same underlying set, we show that the identity map $i : \tilde{\mathcal{F}}_\ell(M(a)) \to \mathcal{F}_\ell(M)(a)$ establishes the isomorphism. In fact, we only have to verify that the action of $x_0^\pm = x_\theta^\mp \otimes t^{\pm 1}$ commutes with i on the generators $m \otimes v$:

$$\begin{aligned} i(x_0^\pm(m \otimes v)) &\overset{(3.2.3)}{=} i\left(\sum_{j=1}^{\ell} m \cdot y_j^{\pm 1} \otimes \Delta_j(x_\theta^\mp)(v)\right) \\ &\overset{\mathrm{ev}'_a(y_j^{\pm 1}) = a^{\pm 1}}{=} i\left(\sum_{j=1}^{\ell} a^{\pm 1} m \otimes \Delta_j(x_\theta^\mp)(v)\right) \\ &= a^{\pm 1}\left(\sum_{j=1}^{\ell} m \otimes \Delta_j(x_\theta^\mp)(v)\right) \overset{(2.2.1)}{=} x_0^\pm(i(m \otimes (v))). \end{aligned}$$

Finally, the assertion $\tilde{\mathcal{F}}_\ell(\mathcal{S}(\xi, a)) \cong V(\mathrm{wt}(\xi), a)$ immediately follows from Theorem 3.1.3. □

We now suppose $\ell \leq n$ so that $\tilde{\mathcal{F}}_\ell : \tilde{\mathscr{S}}_\ell \to \tilde{\mathscr{G}}_\ell$ is an equivalence of categories. With all the tools here developed so far, we are finally able to describe up to isomorphism all the simple $\tilde{\mathcal{S}}_\ell$-modules that corresponds via the functor to the simple $\tilde{\mathfrak{g}}$-modules.

Theorem 3.3.7. *Let $W \in \tilde{\mathscr{G}}_\ell$ be a simple module, and let $M \in \tilde{\mathscr{S}}_\ell$ be such that $\tilde{\mathcal{F}}_\ell(M) \cong W$. Then, M is isomorphic to an affine Zelevinsky tensor product of the form*

$$\mathcal{S}(\xi_1, a_1) \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{S}(\xi_m, a_m), \quad \textit{where} \ \ \xi_j \in \mathcal{P}_{\ell_j}, \ \sum_{j=1}^{m} \ell_j = \ell, \ \textit{and} \ a_j \in \mathbb{C}^\times \ \textit{with} \ a_i \neq a_j, \ i \neq j.$$

Proof. By Theorem 2.2.1, we have

$$W \cong V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m), \ \text{ where } \lambda_j \in P^+ \setminus \{0\} \text{ and } a_i \neq a_j \text{ for } i \neq j. \quad (3.3.20)$$

Set $\mu = \sum_{j=1}^m \lambda_j$, $\ell_j = \mathrm{lv}(\lambda_j)$, and let $w_j \in V(\lambda_j, a_j)$ be any highest-weight vector of weight $\lambda_j, 1 \leq j \leq m$. Then, considering the $\mathfrak{g}$-submodule $U(\mathfrak{g})w$, where w is the highest-weight vector $w = w_1 \otimes \cdots \otimes w_m$, we see that $[V : V(\mu)] \neq 0$. In this way, since $\tilde{\mathcal{F}}_\ell(M) \in \tilde{\mathscr{G}}_\ell$, then $\ell = \sum_j \ell_j$. For $1 \le j \le m$, let $\xi_j \in \mathcal{P}_{\ell_j}$ be such that $\mathrm{wt}(\xi_j) = \lambda_j$ by (3.1.6), so that $\tilde{\mathcal{F}}_{\ell_j}(\mathcal{S}(\xi_j, a_j) \cong V(\lambda_j, a_j)$ by Lemma 3.3.6. Now, from (3.3.19) we have

$$\tilde{\mathcal{F}}_\ell(\mathcal{S}(\xi_1, a_1)\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathcal{S}(\xi_m, a_m)) \cong V(\lambda_1, a_1) \otimes \cdots \otimes V(\lambda_m, a_m), \tag{3.3.21}$$

whence the result follows. □

In particular, since $\tilde{\mathcal{F}}_\ell$ reflects exactness, we see that the theorem above also describes up to isomorphism all the simple objects in the category $\tilde{\mathscr{S}}_\ell$. For $\mathbf{a} = (a_1, \cdots, a_\ell) \in (\mathbb{C}^\times)^\ell$, denote by $M_{\mathbf{a}}$ the quotient of $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ by the right ideal generated by $y_j - a_j$, $1 \le j \le \ell$. We finish this section establishing a few properties of $M_{\mathbf{a}}$, as well as equivalently characterizing simple objects in $\tilde{\mathscr{S}}_\ell$ in terms of $M_{\mathbf{a}}$.

Theorem 3.3.8.

(i) $M_{\mathbf{a}} \cong \mathbb{C}[\mathcal{S}_\ell]$ *as right* $\mathcal{S}_\ell$*-modules. In particular,* $M_{\mathbf{a}} \cong \bigoplus_{\xi\in\mathcal{P}_\ell} \mathcal{S}(\xi, a)^{\oplus d_\xi}$ *as right* $\tilde{\mathcal{S}}_\ell$*-modules if* $a_j = a$ *for all* $1 \le j \le \ell$.

(ii) $\tilde{\mathcal{F}}_\ell(M_{\mathbf{a}}) \cong \mathbb{V}(a_1) \otimes \cdots \otimes \mathbb{V}(a_\ell)$ *as* $\tilde{\mathfrak{g}}$*-modules.*

(iii) $M_{\mathbf{a}}$ *is irreducible if and only if* $a_i \neq a_j$ *for all* $i \neq j$.

(iv) Every irreducible $\mathbb{C}\tilde{\mathcal{S}}_\ell$*-module is isomorphic to a quotient of some* $M_{\mathbf{a}}$.

Proof. (i) Notice that the ideal generated by $y_j - a_j, 1 \le j \le \ell$, is contained in the kernel of the surjective homomorphism of $\mathcal{S}_\ell$-modules $\varphi_{\mathbf{a}} : \mathbb{C}[\tilde{\mathcal{S}}_\ell] \to \mathbb{C}[\mathcal{S}_\ell]$ given by $\sigma_i \mapsto \sigma_i$ and $y_j^{\pm 1} \mapsto a_j^{\pm 1}$. Still denoting by $\varphi_{\mathbf{a}} : M_{\mathbf{a}} \to \mathbb{C}[\mathcal{S}_\ell]$ the induced homomorphism obtained from $\varphi_{\mathbf{a}}$, since $M_{\mathbf{a}}$ and $\mathbb{C}[\mathcal{S}_\ell]$ have the same dimension, this homomorphism is thus an isomorphism. Now, given $M_1, \cdots, M_k \in \tilde{\mathscr{S}}_\ell$, notice that the identity map $\bigoplus_{i=1}^k M_i \to \bigoplus_{i=1}^k M_i$ gives the isomorphism of $\tilde{\mathcal{S}}_\ell$-modules $\Big(\bigoplus_{i=1}^k M_i\Big)(a) \cong \bigoplus_{i=1}^k M_i(a)$. Therefore, if $a_j = a$ for all j, the isomorphism $M_{\mathbf{a}} \cong \bigoplus_\xi \mathcal{S}(\xi, a)^{\oplus d_\xi}$ follows immediately from (1.6.3).

(ii) Clearly, $\mathbb{V}^{\otimes\ell} \cong_{\mathfrak{g}} \mathbb{V}(a_1) \otimes \cdots \otimes \mathbb{V}(a_\ell)$ by the identity map. Since $\varphi_{\mathbf{a}} : M_{\mathbf{a}} \to \mathbb{C}[\mathcal{S}_\ell]$ establishes an isomorphism of $\mathcal{S}_\ell$-modules and $\mathcal{F}_\ell(\mathbb{C}[\mathcal{S}_\ell]) \cong \mathbb{V}^{\otimes\ell}$ by (3.1.3), it follows that the map $\phi : \mathcal{F}_\ell(M_{\mathbf{a}}) \to \mathbb{V}(a_1) \otimes \cdots \otimes \mathbb{V}(a_\ell)$ such that

$$m \otimes v \mapsto \varphi_{\mathbf{a}}(m) \cdot v, \ \text{ for } m \in M_{\mathbf{a}},\, v \in \mathbb{V}^{\otimes\ell}, \tag{3.3.22}$$

is an isomorphism of $\mathfrak{g}$-modules. In order to verify that ϕ is also an isomorphism of $\tilde{\mathfrak{g}}$-modules, it suffices to verify that the action of $x_0^{\pm} = x_\theta^{\mp} \otimes t^{\pm 1} \in \tilde{\mathfrak{g}}$ commutes with ϕ. For every $v \in \mathbb{V}^{\otimes\ell}$, we have

$$x_0^{\pm}(m \otimes v) \overset{\phi}{\mapsto} \sum_{j=1}^{\ell} \varphi_{\mathbf{a}}(m \cdot y_j^{\pm 1}) \cdot \Big(\Delta_j(x_\theta^{\mp})(v)\Big) = \varphi_{\mathbf{a}}(m) \cdot \Big(\sum_{j=1} a_j^{\pm 1} \Delta_j(x_\theta^{\mp})(v)\Big) = x_0^{\pm}(\varphi_{\mathbf{a}}(m) \cdot v),$$

showing that $\phi(x_0^{\pm}(m \otimes \mathbf{v})) = x_0^{\pm}\, \phi(m \otimes \mathbf{v})$.

(iii) By Theorem 2.2.1, $V(\omega_1, a_1) \otimes \cdots \otimes V(\omega_1, a_\ell)$ is simple if and only if $a_i \neq a_j$ for $i \neq j$. Since $\tilde{\mathcal{F}}_\ell(M_{\mathbf{a}}) \cong V(\omega_1, a_1) \otimes \cdots \otimes V(\omega_1, a_\ell)$ by item (ii), then $M_{\mathbf{a}}$ is simple if and only if $a_i \neq a_j$ for $i \neq j$.

(iv) Let $M \cong \mathcal{S}(\xi_1, a_1)\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathcal{S}(\xi_m, a_m) \in \tilde{\mathscr{S}}_\ell$ be a simple module, so $\tilde{\mathcal{F}}_\ell(M)$ is isomorphic to $V = \bigotimes_{j=1}^m V(\lambda_j, a_j)$, where $\lambda_j = \mathrm{wt}(\xi_j)$. Set $\mu = \sum_{j=1}^m \lambda_j$, $\ell_j = \mathrm{lv}(\lambda_j)$, so that $\ell = \mathrm{lv}(\mu) = \sum_j \ell_j$. By Proposition 3.1.1 and Weyl's theorem on complete reducibility, for every $1 \leq j \leq m$, we have a short exact sequence of $\mathfrak{g}$-modules

$$0 \to W_j' \to \mathbb{V}^{\otimes \ell_j} \to V(\lambda_j) \to 0$$

for some submodule $W_j' \subseteq \mathbb{V}^{\otimes \ell_j}$. In particular, since a homomorphism of $\mathfrak{g}$-modules induces a homomorphism between the corresponding evaluation modules as shown in (2.2.2), we also obtain the short exact sequence

$$0 \to W_j'(a_j) \to \mathbb{V}^{\otimes \ell_j}(a_j) \cong \mathbb{V}(a_j)^{\otimes \ell_j} \xrightarrow{\pi_j} V(\lambda_j, a_j) \to 0,\ \ 1 \leq j \leq m.$$

Setting $\pi = \pi_1 \otimes \cdots \otimes \pi_m : \mathbb{V}(a_1)^{\otimes \ell_1} \otimes \cdots \otimes \mathbb{V}(a_m)^{\otimes \ell_m} \to V$, then by considering the kernel of π we see that V is isomorphic to a quotient of $\mathbb{V}(a_1)^{\otimes \ell_1} \otimes \cdots \otimes \mathbb{V}(a_m)^{\otimes \ell_m}$. Thus, setting $b_j = a_i$ for $\ell_{i-1} \leq j \leq \ell_i,\ 1 \leq i \leq m$ and $\ell_0 = 1$, it follows that $V \cong \tilde{\mathcal{F}}_\ell(M)$ is isomorphic to a quotient of $\mathbb{V}(b_1) \otimes \cdots \otimes \mathbb{V}(b_\ell)$ as desired. Since $\mathbb{V}(b_1) \otimes \cdots \otimes \mathbb{V}(b_\ell) \cong \tilde{\mathcal{F}}_\ell(M_{\mathbf{a}})$, for $\mathbf{a} = (b_1, \cdots, b_\ell)$, M is then isomorphic to a quotient of $M_{\mathbf{a}}$.

□

3.4 Modules corresponding to the local Weyl modules

Following the same spirit of the last section, we now direct our efforts towards investigating the problem of explicitly describing the $\tilde{\mathcal{S}}_\ell$-modules corresponding to the local Weyl modules via the equivalence of categories established by the functor $\tilde{\mathcal{F}}_\ell$ of the affine Schur-Weyl duality, for $\ell \leq n$. From the decomposition theorem of Weyl modules into tensor products and the behavior of $\tilde{\mathcal{F}}_\ell$ with respect to affine Zelevinsky tensor products, it will be seen that this problem is equivalent to classifying the $\tilde{\mathcal{S}}_\ell$-modules corresponding to the local Weyl modules whose highest ℓ-weight are Drinfeld polynomials with a unique root. However, the task of classifying these type of local Weyl modules seemingly less complex is still a difficult task. In fact, using the proof that $\tilde{\mathcal{F}}_\ell$ is essentially surjective on objects, we first approach this problem from the most straightforward manner as possible with respect to the simplest examples of local Weyl modules at hand. Further exploring these very first examples, we shall develop a different approach by studying quotients of $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$. In particular, this approach will originate a theorem characterizing the modules corresponding to the local Weyl modules $W(\boldsymbol{\lambda})$ with $\mathrm{wt}(\boldsymbol{\lambda}) = \ell\omega_1$ as being the cyclic modules whose generator satisfies certain relations that can be described in terms of symmetric polynomials.

3.4.1 First examples

We begin by first specifying the result of Theorem 3.3.7 for the simple $\tilde{\mathcal{S}}_\ell$-modules corresponding to the simple local Weyl modules. Let $W(\boldsymbol{\lambda}) \in \tilde{\mathscr{G}}_\ell$ be simple with $\boldsymbol{\lambda} \in \mathscr{P}^+$ of the form (2.4.9) and $W(\boldsymbol{\lambda})$ isomorphic to (2.4.10), as seen in Section 2.3. For $1 \le k \le m$, consider the partition of $i_k \in \{1, \cdots, n\}$ of maximal length obtained from (3.1.6):

$$\xi_{\omega_{i_k}} = (1, \cdots, 1) \in \mathcal{P}_{i_k}, \;\; |\xi_{\omega_{i_k}}| = i_k,$$

so that $\mathrm{wt}(\xi_{\omega_{i_k}}) = \omega_{i_k}$ and $\mathcal{S}(\xi_{\omega_{i_k}}) \in \mathscr{S}_{i_k}$ is the 1-dimensional sign representation. Thus, applying (3.3.19) and Proposition 3.3.6 successively, we have

$$\tilde{\mathcal{F}}_\ell\Big(\tilde{\boxtimes}_{k=1}^m \mathcal{S}(\xi_{\omega_{i_k}}, a_k)\Big) \cong W(\boldsymbol{\lambda}), \tag{3.4.1}$$

showing that $\tilde{\boxtimes}_{k=1}^m \mathcal{S}(\xi_{\omega_{i_k}}, a_k)$ corresponds to the simple local Weyl module $W(\boldsymbol{\lambda})$ via the functor $\tilde{\mathcal{F}}_\ell$. In other words, the affine Zelevinsky tensor product of evaluation modules of sign representations are the $\tilde{\mathcal{S}}_\ell$-modules corresponding via the functor $\tilde{\mathcal{F}}_\ell$ to the simple local Weyl modules.

In fact, one notices that the above description (3.4.1) essentially depended only on two factors: the tensor product decomposition (2.4.10) and the fact that the modules $\mathcal{S}(\xi_{\omega_{i_k}, a_k})$ are such that $\tilde{\mathcal{F}}_{i_k}(\mathcal{S}(\xi_{\omega_{i_k}}, a_k)) \cong W(\boldsymbol{\omega}_{i_k, a_k}), 1 \le k \le m$. This observation generalize as follows: let $W(\boldsymbol{\lambda}) \in \tilde{\mathscr{G}}_\ell$. If $\boldsymbol{\lambda} = \prod_{j=1}^m \boldsymbol{\omega}_{\lambda_j, a_j}$ is the decomposition of $\boldsymbol{\lambda}$ as in (2.2.9), then $W(\boldsymbol{\lambda}) \cong \bigotimes_{j=1}^m W(\boldsymbol{\omega}_{\lambda_j, a_j})$ by Theorem 2.4.5. By the affine Schur-Weyl duality, let $M_j \in \tilde{\mathscr{S}}_{\ell_j}$ be such that $\tilde{\mathcal{F}}_{\ell_j}(M_j) \cong W(\boldsymbol{\omega}_{\lambda_j, a_j})$, where $\ell_j = \mathrm{lv}(\lambda_j)$ for $1 \le j \le m$ and $\ell = \sum_j \ell_j$. Then, from (3.3.19) we see that

$$\tilde{\mathcal{F}}_\ell(M_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} M_m) \cong \bigotimes_{j=1}^m W(\boldsymbol{\omega}_{\lambda_j, a_j}),$$

which shows that the problem of describing the module corresponding to $W(\boldsymbol{\lambda})$ reduces to describing each of the modules M_j corresponding to the local Weyl modules $W(\boldsymbol{\omega}_{\lambda_j, a_j}) \in \tilde{\mathscr{G}}_{\ell_j}$ whose highest ℓ-weight has a unique root a_j, $1 \le j \le m$.

However, so far we have not developed any systematic method of explicitly describing modules corresponding to local Weyl modules of the form $W(\boldsymbol{\omega}_{\lambda,a})$ not necessarily irreducible. An immediate approach in this direction is to consider the proof seen in Section 3.2 of $\tilde{\mathcal{F}}_\ell$ being essentially surjective, which gives a construction of the $\tilde{\mathcal{S}}_\ell$-module corresponding to $W(\boldsymbol{\omega}_{\lambda,a})$ in the following way: choose $M \in \mathscr{S}_\ell$ such that $\mathcal{F}_\ell(M) \cong W(\omega_{\lambda,a})$

as $\mathfrak{g}$-modules and fix a particular isomorphism of $\mathfrak{g}$-modules $f : \mathcal{F}_\ell(M) \to W(\boldsymbol{\omega}_{\lambda,a})$ so that we can use (3.2.14) to calculate the automorphisms $A_j^{\pm}, 1 \le j \le \ell$, given by Lemma 3.2.5 in a very explicit manner. Thus, the $\tilde{\mathcal{S}}_\ell$-module $\tilde{M}$ obtained from M via the extension to an action of $\tilde{\mathcal{S}}_\ell$ given by Theorem 3.2.6 is such that $\tilde{\mathcal{F}}_\ell(\tilde{M}) \cong W(\boldsymbol{\omega}_{\lambda,a})$. If such approach could give formulas for the automorphisms $A_j^{\pm}$ that follow a certain pattern dependent on ℓ and on the weight λ, one could expect that our description problem would then reduce to deducing a general formula for these automorphisms.

In the following example, we shall put this approach into practice for $\ell = 2$ regardless if the given ℓ-weight of the local Weyl module has a unique root or not, so that we can get the clearest possible first impression out of this method.

Example 3.4.1. Let $\ell = 2$ and let $W(\boldsymbol{\lambda}) \in \tilde{\mathscr{G}}_2$. Since $[W(\boldsymbol{\lambda}) : V(\mathrm{wt}(\boldsymbol{\lambda}))] \neq 0$, the ℓ-weight $\boldsymbol{\lambda} \in \mathscr{P}^+$ is one of the following type:

$$\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b},\ \boldsymbol{\omega}_{2\omega_1,a},\ \boldsymbol{\omega}_{2,a},\quad \text{where } a, b \in \mathbb{C}^\times, a \neq b.$$

We first study the cases $\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b}$ and $\boldsymbol{\omega}_{2\omega_1,a}$ together.

Case 1. Since $\mathrm{wt}(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b}) = 2\omega_1$, then $W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b}) \cong_{\mathfrak{g}} \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ by Theorem 2.4.9. In fact, as a result of the decomposition $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} = V(2\omega_1) \oplus V(\omega_2)$, an isomorphism $g : \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \to W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$ is solely dependent on how any fixed highest-weight vectors of weight $2\omega_1, \omega_2$ are assigned. Denoting by $\mathbf{w}$ the generator of $W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$, the vectors $x_1^-\mathbf{w}$ and $x_{1,1}^-\mathbf{w}$ generate the weight space of weight ω_2 by (2.4.3), so a highest-weight vector of weight ω_2 is of the form $k'x_1^-\mathbf{w} + kx_{1,1}^-\mathbf{w}$, where $k, k' \in \mathbb{C}$. Applying x_1^+ on this vector, we have

$$0 = k'h_1\mathbf{w} - k\Lambda_{1,1}\mathbf{w} = 2k'\mathbf{w} + k(a+b)\mathbf{w} \Rightarrow k' = -\frac{k(a+b)}{2}, \tag{3.4.2}$$

whence $kx_{1,1}^-\mathbf{w} - 2^{-1}k(a+b)x_1^-\mathbf{w}$ is a highest-weight vector of weight ω_2 for all $k \in \mathbb{C}$. In this way, since $v_1 \otimes v_1$ and $v_1 \otimes v_2 - v_2 \otimes v_1$ are, respectively, highest-weight vectors of weight $2\omega_1$ and ω_2 in $\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$, we choose $g : \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \to W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$ given by

$$v_1 \otimes v_1 \mapsto \mathbf{w}, \quad v_1 \otimes v_2 - v_2 \otimes v_1 \mapsto kx_{1,1}^-\mathbf{w} - \frac{k(a+b)}{2}x_1^-\mathbf{w}. \tag{3.4.3}$$

For $\ell = 2$, notice that the vectors in (3.2.12) can be written as follows:

$$\begin{aligned}
\mathbf{w}^{(1)} &= v_1 \otimes v_2 = \frac{1}{2}\Big(x_1^-(v_1 \otimes v_1) + (v_1 \otimes v_2 - v_2 \otimes v_1)\Big)\\
\mathbf{w}^{(2)} &= v_2 \otimes v_1 = \frac{1}{2}\Big(x_1^-(v_1 \otimes v_1) - (v_1 \otimes v_2 - v_2 \otimes v_1)\Big)\\
\mathbf{v}^{(1)} &= v_{n+1} \otimes v_2 = \frac{1}{2}\Big(x_\theta^- x_1^-(v_1 \otimes v_1) + x_\theta^-(v_1 \otimes v_2 - v_2 \otimes v_1)\Big)\\
\mathbf{v}^{(2)} &= v_2 \otimes v_{n+1} = \frac{1}{2}\Big(x_\theta^- x_1^-(v_1 \otimes v_1) - x_\theta^-(v_1 \otimes v_2 - v_2 \otimes v_1)\Big).
\end{aligned}$$

In particular, g maps these elements by

$$\begin{aligned} g(v_1 \otimes v_2) &= \frac{2-k(a+b)}{4} x_1^- \mathbf{w} + \frac{k}{2} x_{1,1}^- \mathbf{w}, \\ g(v_2 \otimes v_1) &= \frac{2+k(a+b)}{4} x_1^- \mathbf{w} - \frac{k}{2} x_{1,1}^- \mathbf{w}, \\ g(v_{n+1} \otimes v_2) &= \frac{k}{2} x_\theta^- x_{1,1}^- \mathbf{w} + \frac{2-k(a+b)}{4} x_\theta^- x_1^- \mathbf{w}, \\ g(v_2 \otimes v_{n+1}) &= \frac{2+k(a+b)}{4} x_\theta^- x_1^- \mathbf{w} - \frac{k}{2} x_\theta^- x_{1,1}^- \mathbf{w}. \end{aligned} \tag{3.4.4}$$

Set $M = \mathbb{C}[\mathcal{S}_1] \boxtimes \mathbb{C}[\mathcal{S}_1]$, and notice that $\mathcal{F}_2(M) \cong \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ by Proposition 3.3.2. Therefore, we obtain an isomorphism of $\mathfrak{g}$-modules $f : \mathcal{F}_2(M) \to W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$ by composing the isomorphism $h : \mathcal{F}_2(M) \to \mathcal{F}_1(\mathbb{C}[\mathcal{S}_1]) \otimes \mathcal{F}_1(\mathbb{C}[\mathcal{S}_1]) \to \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1}$ with g, i.e., $f = g \circ h$. In particular, f maps any pure tensor of the form $((1 \otimes 1) \otimes s) \otimes (w_1 \otimes w_2) \in \mathcal{F}_2(M)$, where $s \in \mathcal{S}_2, w_i \in \mathbb{C}^{n+1}, i = 1, 2$, as follows:

$$((1 \otimes 1) \otimes s) \otimes (w_1 \otimes w_2) \overset{f}{\longmapsto} g(s \cdot (w_1 \otimes w_2)) \in W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b}). \tag{3.4.5}$$

Since $\mathrm{wt}(\boldsymbol{\omega}_{2\omega_1,a}) = 2\omega_1$ as well, then (3.4.3) also gives an isomorphism $g' : \mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1} \to W(\boldsymbol{\omega}_{2\omega_1,a})$ by setting $a = b$ and $\mathbf{w}$ to be the generator of $W(\boldsymbol{\omega}_{2\omega_1,a})$, so an isomorphism $f' : \mathcal{F}_2(M) \to W(\boldsymbol{\omega}_{2\omega_1,a})$ is similarly obtained by composing $f' = g' \circ h$. In this sense, since g and g' are virtually the "same" map, it suffices to only calculate the endomorphism $A_j^{\pm}$ regarding the ℓ-weight $\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b}$, with the respective endomorphisms for the case $\boldsymbol{\omega}_{2\omega_1,a}$ being obtained by just setting $a = b$ in the calculated formulas of $A_j^{\pm}$.

Since A_j^- is set to be the inverse of A_j^+, we first concern ourselves exclusively with calculating the image of A_j^+ on the basis $m_1 = (1 \otimes 1) \otimes 1$, $m_2 = (1 \otimes 1) \otimes \sigma$ of M, where $\sigma = (1, 2) \in \mathcal{S}_2$. Using (3.2.14) and Lemma 3.2.4, the vectors $A_j^+(m_i)$ are derived from the equality

$$x_0^+(m_i \otimes \mathbf{w}^{(j)}) = A_j^+(m_i) \otimes \mathbf{v}^{(j)} \text{ for } i, j = 1, 2 \tag{3.4.6}$$

as in the steps below.

1. $A_1^+(m_2)$. Since $\sigma \cdot \mathbf{w}^{(1)} = v_2 \otimes v_1$, then $f(m_2 \otimes \mathbf{w}^{(1)}) = g(v_2 \otimes v_1)$ by (3.4.5). Since $v_2 \otimes v_1$ has weight ω_2, then $x_0^+ g(v_2 \otimes v_1)$ lies in the weight space of weight $\omega_2 - \theta = \epsilon_2 + \epsilon_{n+1}$, so there exists scalars λ_1, λ_2 such that $\lambda_1 g(v_{n+1} \otimes v_2) + \lambda_2 g(v_2 \otimes v_{n+1}) = x_0^+ g(v_2 \otimes v_1)$. Applying (3.4.4) to this equality yields

$$\begin{aligned} &\frac{2(\lambda_1+\lambda_2)+(\lambda_2-\lambda_1)k(a+b)}{4} x_\theta^- x_1^- \mathbf{w} + \frac{(\lambda_1-\lambda_2)k}{2} x_\theta^- x_{1,1}^- \mathbf{w} \\ &= \frac{2+k(a+b)}{4} x_0^+ x_1^- \mathbf{w} - \frac{k}{2} x_0^+ x_{1,1}^- \mathbf{w}. \end{aligned} \tag{3.4.7}$$

We aim to apply x_θ^+ on (3.4.7) in order to deduce λ_1, λ_2. From Proposition 1.4.2, we have

$$0 = (x_{1,1}^-)^2(x_1^-)^3\mathbf{w} = (-1)^2\bigg(X_{1,-1,+}^-(u)\Lambda_1^+(u)\bigg)_3\mathbf{w} = x_1^-\Lambda_{1,2}\mathbf{w} + x_{1,1}^-\Lambda_{1,1}\mathbf{w} + x_{1,2}^-\mathbf{w},$$

so $x_{1,2}^-\mathbf{w} = (a+b)x_{1,1}^- - abx_1^-\mathbf{w}$. Thus, we obtain

$$x_\theta^+x_0^+x_{1,1}^-\mathbf{w} = h_{\theta,1}x_{1,1}^-\mathbf{w} = (a+b)x_{1,1}^-\mathbf{w} - x_{1,2}^-\mathbf{w} = abx_1^-\mathbf{w}. \tag{3.4.8}$$

Similarly, we also have

$$\begin{aligned} &x_\theta^+x_0^+x_1^-\mathbf{w} = h_{\theta,1}x_1^-\mathbf{w} = (a+b)x_1^-\mathbf{w} - x_{1,1}^-\mathbf{w} \\ &x_\theta^+x_\theta^-x_{1,1}^-\mathbf{w} = h_\theta x_{1,1}^-\mathbf{w} = x_{1,1}^-\mathbf{w} \\ &x_\theta^+x_\theta^-x_1^-\mathbf{w} = h_\theta x_1^-\mathbf{w} + x_\theta^-[x_\theta^+, x_1^-]\mathbf{w} = h_\theta x_1^-\mathbf{w} = -\alpha_1(\theta)x_1^-\mathbf{w} + 2\omega_1(\theta)x_1^-\mathbf{w} = x_1^-\mathbf{w}. \end{aligned} \tag{3.4.9}$$

Therefore, applying x_0^+ to (3.4.7), and using the above equalities just derived, we obtain

$$\begin{aligned} &\frac{2(\lambda_1+\lambda_2)+(\lambda_2-\lambda_1)k(a+b)}{4}x_1^-\mathbf{w} + \frac{(\lambda_1-\lambda_2)k}{2}x_{1,1}^-\mathbf{w} \\ &= \frac{2(a+b)+ka^2+kb^2}{4}x_1^-\mathbf{w} - \frac{2+k(a+b)}{4}x_{1,1}^-\mathbf{w}, \end{aligned} \tag{3.4.10}$$

which implies that

$$\lambda_1 = \frac{k^2(a-b)^2-4}{8k},\ \lambda_2 = \frac{4+4k(a+b)+k^2(a-b)^2}{8k}.$$

Now, noticing that $x_0^+(m_2 \otimes \mathbf{w}^{(1)}) = f^{-1}(x_0^+g(v_2 \otimes v_1)) = (\lambda_1 m_1 + \lambda_2 m_2) \otimes \mathbf{v}^{(1)}$, it follows from (3.4.6) that

$$A_1^+(m_2) = \frac{k^2(a-b)^2-4}{8k}m_1 + \frac{4+4k(a+b)+k^2(a-b)^2}{8k}m_2.$$

2. $A_1^+(m_1)$. We have $f(m_1 \otimes \mathbf{w}^{(1)}) = g(v_1 \otimes v_2)$, and $x_0^+g(v_1 \otimes v_2)$ lies in the weight space of weight $\epsilon_2 + \epsilon_{n+1}$. Writing $\mu_1 g(v_{n+1} \otimes v_2) + \mu_2 g(v_2 \otimes v_{n+1}) = x_0^+g(v_1 \otimes v_2)$, then

$$\begin{aligned} &\frac{2(\mu_1+\mu_2)+(\mu_2-\mu_1)k(a+b)}{4}x_\theta^-x_1^-\mathbf{w} + \frac{(\mu_1-\mu_2)k}{2}x_\theta^-x_{1,1}^-\mathbf{w} \\ &= \frac{2-k(a+b)}{4}x_0^+x_1^-\mathbf{w} + \frac{k}{2}x_0^+x_{1,1}^-\mathbf{w}. \end{aligned}$$

Applying x_θ^+ to the equality above, (3.4.8) and (3.4.9) gives us

$$\begin{aligned} &\frac{2(\mu_1+\mu_2)+(\mu_2-\mu_1)k(a+b)}{4}x_\theta^-x_1^-\mathbf{w} + \frac{(\mu_1-\mu_2)k}{2}x_\theta^-x_{1,1}^-\mathbf{w} \\ &= \frac{2(a+b)-ka^2-kb^2}{4}x_1^-\mathbf{w} - \frac{2-k(a+b)}{4}x_{1,1}^-\mathbf{w}, \end{aligned}$$

whence

$$\mu_1 = \frac{4k(a+b)-4-k^2(a-b)^2}{8k},\ \mu_2 = \frac{4-k^2(a-b)^2}{8k}. \tag{3.4.11}$$

Therefore, since $f^{-1}((x_0^+ g(v_1 \otimes v_2)) = (\mu_1 m_1 + \mu_2 m_2) \otimes \mathbf{v}^{(1)}$, we conclude that

$$A_1^+(m_1) = \frac{4k(a+b)-4-k^2(a-b)^2}{8k} m_1 + \frac{4-k^2(a-b)^2}{8k} m_2.$$

3. $A_2^+(m_2)$. Since $f(m_2 \otimes \mathbf{w}^{(2)}) = g(v_1 \otimes v_2)$, we can use the calculations made for $A_1^+(m_1)$ above in order to deduce $A_2^+(m_2)$. In fact, since $f^{-1}(x_0^+ f(m_2 \otimes \mathbf{w}^{(2)})) = (\mu_1 m_2 + \mu_2 m_1) \otimes \mathbf{v}^{(2)}$, then

$$A_2^+(m_2) = \frac{4k(a+b)-4-k^2(a-b)^2}{8k} m_2 + \frac{4-k^2(a-b)^2}{8k} m_1.$$

4. $A_2^+(m_1)$. Since $f(m_1 \otimes \mathbf{w}^{(2)}) = g(v_2 \otimes v_1)$, from the calculations made for $A_1^+(m_2)$, we have

$$A_2^+(m_1) = \frac{k^2(a-b)^2-4}{8k} m_2 + \frac{4+4k(a+b)+k^2(a-b)^2}{8k} m_1.$$

Using the relations $A_j^- A_j^+ = 1$ and the vectors $A_j^+(m_i)$ just calculated, one can deduce the vectors $A_j^-(m_i)$ for $i, j = 1, 2$ by solving linear systems. Therefore, the $\tilde{\mathcal{S}}_2$-module $M^{a,b}$ obtained from M through the extension $m \cdot y_j^{\pm 1} = A_j^{\pm}(m)$ is such that $\tilde{\mathcal{F}}_2(M^{a,b}) \cong W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$. Let $A_j'^{\pm}$ be the automorphisms of M obtained by setting $a = b$ in the formulas of $A_j^{\pm}$ above. The module M^a obtained by means of the extension $m \cdot y_j^{\pm 1} = A_j'^{\pm}(m)$ is such that $\tilde{\mathcal{F}}_2(M^a) \cong W(\boldsymbol{\omega}_{2\omega_1,a})$.

It is clear that our construction of g given by (3.4.3) and the automorphism $A_j^{\pm}$ hence calculated are all dependent on the particular choice of the scalar k. In particular, if one takes $k = 2$, then (3.4.3) reduces to the assignment

$$v_1 \otimes v_1 \mapsto \mathbf{w},\ v_1 \otimes v_2 - v_2 \otimes v_1 \mapsto 2x_{1,1}^- \mathbf{w} - (a+b)x_1^- \mathbf{w},$$

with the vectors $A_j^{\pm}(m_i)$ simplifying to the following clearer expressions:

$$\begin{aligned}
A_1^+(m_2) &= \frac{(a-b)^2-1}{4} m_1 + \frac{(a-b)^2+1+2(a+b)}{4} m_2,\\
A_1^+(m_1) &= \frac{2(a+b)-(a-b)^2-1}{4} m_1 + \frac{1-(a-b)^2}{4} m_2,\\
A_2^+(m_2) &= \frac{2(a+b)-(a-b)^2-1}{4} m_2 + \frac{1-(a-b)^2}{4} m_2,\\
A_2^+(m_1) &= \frac{(a-b)^2-1}{4} m_2 + \frac{(a-b)^2+1+2(a+b)}{4} m_1.
\end{aligned} \tag{3.4.12}$$

Case 2. We have $W(\boldsymbol{\omega}_{2,a}) \cong_{\mathfrak{g}} V(\omega_2) = \Lambda^2(\mathbb{C}^{n+1})$ by the isomorphism $g : \Lambda^2(\mathbb{C}^{n+1}) \to$

$W(\boldsymbol{\omega}_{2,a})$ which maps

$$v_1 \wedge v_2 \mapsto \mathbf{w},$$

where $\mathbf{w}$ is the generator of $W(\boldsymbol{\omega}_{2,a})$. In particular, $g(v_{n+1} \wedge v_2) = x_\theta^- \mathbf{w}$. Regarding the partition $(1,1) \in \mathcal{P}_2$, its associated Young generator is $c = 1 - \sigma$. Therefore, since $c \otimes (v_i \otimes v_j) \mapsto 2v_i \wedge v_j$ gives the isomorphism $\mathcal{F}_2(\mathcal{S}(1,1)) \cong \Lambda^2(\mathbb{C}^{n+1})$, we obtain an isomorphism $f : \mathcal{F}_2(\mathcal{S}(1,1)) \to W(\boldsymbol{\omega}_{2,a})$ by composing these maps appropriately.

In order to obtain the automorphisms A_j^+, we only have to calculate $f^{-1}(x_0^+ f(c \otimes \mathbf{w}^{(j)}))$, as c is basis of $\mathcal{S}(1,1)$. In fact, we have $x_0^+ f(c \otimes \mathbf{w}^{(1)}) = 2x_0^+ \mathbf{w}$, and since $x_0^+ \mathbf{w}$ lies in the weight space of weight $\epsilon_2 + \epsilon_{n+1}$, we write $2x_0^+ \mathbf{w} = \lambda g(v_{n+1} \wedge v_2) = \lambda x_\theta^- \mathbf{w}$. Applying x_θ^+ to both sides of this equality, and noticing that $x_\theta^+ x_0^+ \mathbf{w} = a\mathbf{w}$, $x_\theta^+ x_\theta^- \mathbf{w} = \mathbf{w}$, then $\lambda = 2a$. Because $f^{-1}(x_0^+ f(c \otimes \mathbf{w}^{(1)})) = ac \otimes \mathbf{v}^{(1)}$, it follows that $A_1^+(c) = ac$, showing that $A_1^+ = a\, Id_{\mathcal{S}(1,1)}$. Similarly, one checks that $A_2^+ = a\, Id_{\mathcal{S}(1,1)}$ as well. In this way, the module obtained from $\mathcal{S}(1,1)$ by means of the extension $m \cdot y_j^{+1} = A_j^+(m) = am$ is exactly $\mathcal{S}(1,1,a)$, which we knew beforehand to be such that $\tilde{\mathcal{F}}_2(\mathcal{S}(1,1,a)) \cong W(\boldsymbol{\omega}_{2,a})$. $\diamond$

In the example above, even though our approach worked in a very straightforward manner with respect to Case 2, a few drawbacks in Case 1 stood out as we now discuss. As expected from Lemma 3.2.4, our approach used in Case 1 was seen to heavily rely on the choice of the isomorphism g, in the sense that, depending on the highest-weight vectors chosen to be assigned by g, we may end up with entirely different automorphisms $A_j^\pm$. In addition to that, the complexity of the calculations and formulas that later unfolded is also directly dependent on how g was chosen in the first place. In fact, what essentially differs Case 1 from Case 2 is the fact that the local Weyl module considered in the latter is a simple $\mathfrak{g}$-module, which ultimately simplify the choice of the isomorphism g. In this sense, since the vast majority of the local Weyl modules in the context of $\tilde{\mathscr{G}}_\ell$, $\ell \geq 2$, are reducible $\mathfrak{g}$-modules, including the ones whose ℓ-weight has a unique root, such drawbacks associated with the choice of this isomorphism of $\mathfrak{g}$-modules strongly discourage expectations of deducing similarities between the formulas of $A_j^\pm$ obtained in each of these contexts. With that in mind, we shall revisit the $\tilde{\mathcal{S}}_2$-modules studied in Case 1 of Example 3.4.1 looking for a different, but equivalent, way of characterizing them.

Example 3.4.2. We now revisit the $\tilde{\mathcal{S}}_2$-modules $M^{a,b}$ and M^a seen in Example 3.4.1. Here, we suppose that $M^{a,b}$ is obtained from $\mathbb{C}[\mathcal{S}_1] \boxtimes \mathbb{C}[\mathcal{S}_1]$ through the extension given by the automorphisms $A_j^\pm$ satisfying (3.4.12), while M^a is the extension through the automorphisms $A_j'^\pm$ obtained from (3.4.12) by setting $a = b$.

Let us begin with M^a by first proving that it is cyclic. This will be shown using a certain short exact sequence involving M^a that does not split. Since $V(2\omega_1, a)$ is the irreducible quotient of $W(\boldsymbol{\omega}_{2\omega_1,a})$, we obtain a short exact sequence for some submodule

$W \subseteq W(\boldsymbol{\omega}_{2\omega_1,a})$:

$$0 \to W \to W(\boldsymbol{\omega}_{2\omega_1,a}) \to V(2\omega_1, a) \to 0. \tag{3.4.13}$$

As a $\mathfrak{g}$-module, we have the following decomposition into simple summands

$$W(\boldsymbol{\omega}_{2\omega_1,a}) \cong (\mathbb{C}^{n+1})^{\otimes 2} = V(2\omega_1) \oplus V(\omega_2),$$

and, since the ℓ-weight space of weight $\boldsymbol{\omega}_{2\omega_1,a}$ is 1-dimensional, any highest-weight vector of weight $2\omega_1$ is also a highest-ℓ-weight vector of weight $\boldsymbol{\omega}_{2\omega_1,a}$. Therefore, the proper module W must satisfy $W \cong_{\mathfrak{g}} V(\omega_2)$. In particular, W is a simple $\tilde{\mathfrak{g}}$-module, so we have $W \cong V(\omega_2, a)$. Now, the equivalence of categories $\tilde{\mathcal{F}}_2$ induces from Proposition 3.3.4 a short exact sequence of $\tilde{\mathcal{S}}_2$-modules

$$0 \to \mathcal{S}(1,1,a) \to M^a \xrightarrow{\pi} \mathcal{S}(2,a) \to 0. \tag{3.4.14}$$

In particular, since both $\mathcal{S}(1,1,a)$ and $\mathcal{S}(2,a)$ are irreducible, this short exact sequence shows that any submodule of M^a has length at the most equal to 2. From the matrices of A'^{+}_j in the basis m_1, m_2 given by

$$\left[A'^{+}_1\right] = \begin{pmatrix} \frac{4a-1}{4} & -\frac{1}{4} \\ \\ \frac{1}{4} & \frac{4a+1}{4} \end{pmatrix} \quad \text{and} \quad \left[A'^{+}_2\right] = \begin{pmatrix} \frac{4a+1}{4} & \frac{1}{4} \\ \\ -\frac{1}{4} & \frac{4a-1}{4} \end{pmatrix}, \tag{3.4.15}$$

one can check that a is the unique eigenvalue of A'^{+}_j, with $q' = m_1 - m_2$ being the unique, up to scalar, common eigenvector of eigenvalue a. In particular, it follows that (3.4.14) does not split, as it would require the existence of a common eigenvector for A'^{+}_j linearly independent with q'. Choosing $m \in M^a$ such that $\pi(m) \neq 0$, then $\pi(m)$ generates $\mathcal{S}(2,a)$ by the fact that $\mathcal{S}(2,a)$ is irreducible, so $\mathcal{S}(2,a)$ is isomorphic to a quotient of $m \cdot \mathbb{C}[\tilde{\mathcal{S}}_2]$ by the restriction of π to this submodule. In particular, $m \cdot \mathbb{C}[\tilde{\mathcal{S}}_2]$ is not irreducible, otherwise we would have $m \cdot \mathbb{C}[\tilde{\mathcal{S}}_2] \cong \mathcal{S}(2,a)$ and (3.4.14) would split. Therefore, we have $1 < \text{length}_{\tilde{\mathscr{S}}_2}\left(m \cdot \mathbb{C}[\tilde{\mathcal{S}}_2]\right) \leq 2$, whence $m \cdot \mathbb{C}[\tilde{\mathcal{S}}_2] = M^a$. Therefore, M^a is cyclic, with a generator being any vector m such that $\pi(m) \neq 0$.

Set $q = m_1 + m_2$, so $\mathbb{C}q$ is isomorphic to the trivial representation of $\mathcal{S}_2$ and we have $\mathcal{S}(2,a) \cong_{\mathcal{S}_2} \mathbb{C}q$. In view of this, we conclude that $\pi(q) \neq 0$ and we have $M^a = q \cdot \mathbb{C}[\tilde{\mathcal{S}}_2]$ by the argument above. In particular, using (3.4.15), one checks that

$$q \cdot y_1 = a.q - \frac{1}{2}q' \ \text{ and } q \cdot y_2 = a.q + \frac{1}{2}q',$$

so q satisfies the following relations

$$\begin{aligned} q \cdot (y_1 - a)(y_1 - a) &= 0 \\ q \cdot (y_1 + y_2 - 2a) &= 0 \\ q \cdot \sigma &= q. \end{aligned} \tag{3.4.16}$$

Let I be right ideal of $\mathbb{C}[\tilde{\mathcal{S}}_2]$ generated by the elements

$$\sigma - 1,\ y_1 + y_2 - 2a, \text{ and } (y_1 - a)(y_1 - a). \tag{3.4.17}$$

From (3.4.16), we see that M^a is isomorphic to a quotient of $\mathbb{C}[\tilde{\mathcal{S}}_2]/I$, so it suffices to show that $\mathbb{C}[\tilde{\mathcal{S}}_2]/I$ is a 2-dimensional module in order to conclude that $M^a \cong \mathbb{C}[\tilde{\mathcal{S}}_2]/I$. Instead of performing these computations here, we refer to the results seen in Subsection 3.4.2 for the general case. In particular, it will follow from those results that $v, v \cdot y_1$ form linearly independent set in $\mathbb{C}[\tilde{\mathcal{S}}_2]/I$ and that their image under the action of the inverse elements y_1^{-1}, y_2^{-1} lie in their span, where v denote the image of $1 \in \mathbb{C}[\tilde{\mathcal{S}}_2]$ in the quotient $\mathbb{C}[\tilde{\mathcal{S}}_2]/I$.

Finally, we study how the relations (3.4.17) modify when considering the module $M^{a,b}$. Since the local Weyl module $W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$ is irreducible with $\tilde{\mathcal{F}}_2(M^{a,b}) \cong W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$, from (3.4.1) we see that $M^{a,b} \cong \mathbb{C}[\mathcal{S}_1](a)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](b)$, so it suffices to derive relations for $\mathbb{C}[\mathcal{S}_1](a)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](b)$ only. Notice that $m_1 = (1 \otimes 1) \otimes 1$ and $m_2 = (1 \otimes 1) \otimes \sigma$ are also basis elements of this module which satisfy

$$\begin{aligned} m_1 y_1 &= a m_1 \qquad & m_1 y_2 &= b m_1 \\ m_2 y_1 &= b m_2 \qquad & m_2 y_2 &= a m_2. \end{aligned}$$

Because $\mathbb{C}[\mathcal{S}_1](a)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](b)$ is irreducible, every vector generates it. In particular, for $q = m_1 + m_2$, we see that

$$q \cdot y_1 = bq + (a-b)m_1 \text{ and } q \cdot y_2 = aq + (b-a)m_1,$$

so q satisfies the relations

$$\begin{aligned} q \cdot (y_1 - b)(y_1 - a) &= 0 \\ q \cdot (y_1 + y_2 - (a+b)) &= 0 \\ q \cdot \sigma &= q. \end{aligned} \tag{3.4.18}$$

Similarly as in the previous case, we have $\mathbb{C}[\mathcal{S}_1](a)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](b) \cong \mathbb{C}[\tilde{\mathcal{S}}_2]/I'$, where I' is the right ideal generated by $\sigma - 1, y_1 + y_2 - (a+b)$ and $(y_1 - b)(y_1 - a)$. ◇

The relations seen in the last example satisfied by the chosen generator of both M^a and $\mathbb{C}[\mathcal{S}_1](a)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](b)$ are strikingly similar. Indeed, not only both modules are generated by the same generator $m_1 + m_1 \cdot \sigma$, but also the relations satisfied by this generator in the context of M^a can be acquired from the relations derived in the context of $\mathbb{C}[\mathcal{S}_1](a)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](b)$ by just setting $a = b$. Given the fact that both of these modules corresponds via the functor to $W(\boldsymbol{\omega}_{2\omega_1,a})$ and $W(\boldsymbol{\omega}_{1,a}\boldsymbol{\omega}_{1,b})$, respectively, these similarities suggest that, if one wants to characterize the modules corresponding to the local Weyl modules $W(\boldsymbol{\omega}) \in \tilde{\mathscr{G}}_\ell$ whose underlying weight are such that $\mathrm{wt}(\boldsymbol{\omega}) = \ell\omega_1$ for ℓ other than $\ell = 2$, it is essential to study the relations satisfied by the generator $m = \sum_{\sigma\in\mathcal{S}_\ell} m_1 \cdot \sigma$ of the simple module $M = \mathbb{C}[\mathcal{S}_1](a_1)\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](a_\ell) \in \mathscr{S}_\ell$, where $m_1 = (1^{\otimes \ell}) \otimes 1 \in M, a_i \in \mathbb{C}^\times$ with $a_i \neq a_j$ for all $i \neq j$.

In the next example, we further explore these relations for $\ell = 3$ and establish some connections with the symmetric polynomials.

Example 3.4.3. Let $\ell = 3$. The module $\mathbb{C}[\mathcal{S}_1](a_1)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](a_2)\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](a_3)$ is 6-dimensional with a basis given by

$$m_1 = (1 \otimes 1 \otimes 1) \otimes 1, \quad m_2 = (1 \otimes 1 \otimes 1 \otimes)\sigma_1, \quad m_3 = (1 \otimes 1 \otimes 1) \otimes \sigma_2$$
$$m_4 = (1 \otimes 1 \otimes 1) \otimes \sigma_1\sigma_2\sigma_1, \quad m_5 = 1 \otimes 1 \otimes 1 \otimes \sigma_1\sigma_2, \quad m_6 = (1 \otimes 1 \otimes 1 \otimes 1) \otimes \sigma_2\sigma_1.$$

In this case, the generator m is equal to $\sum_{j=1}^{6} m_j$, and it is clear that m is stabilized by $\mathcal{S}_3$. From the defining relations (3.2.1), we have

$$y_3\sigma_1 = \sigma_1 y_3, \quad y_1\sigma_2 = \sigma_2 y_1$$
$$\sigma_1 y_1 = y_2\sigma_1, \quad \sigma_2 y_2 = y_3\sigma_2,$$

whence we obtain that

$$\begin{aligned}
m_1y_1 &= a_1m_1, & m_1y_2 &= a_2m_1, & m_1y_3 &= a_3m_1,\\
m_2y_1 &= a_2m_2, & m_2y_2 &= a_1m_2, & m_2y_3 &= a_3m_2,\\
m_3y_1 &= a_1m_3, & m_3y_2 &= a_3m_3, & m_3y_3 &= a_2m_3,\\
m_4y_1 &= a_3m_4, & m_4y_2 &= a_2m_4, & m_4y_3 &= a_1m_4,\\
m_5y_1 &= a_2m_5, & m_5y_2 &= a_3m_5, & m_5y_3 &= a_1m_5,\\
m_6y_1 &= a_3m_6, & m_6y_2 &= a_1m_6, & m_6y_3 &= a_2m_6.
\end{aligned}$$

We shall only deduce relations involving the elements $y_j, 1 \leq j \leq \ell$. From the fact that $m = \sum_{j=1}^{6} m_j$, one checks that the above relations imply

$$\begin{aligned}
&m \cdot (y_1 - a_1)(y_1 - a_2)(y_1 - a_3) = 0\\
&m \cdot (y_2 - a_1)(y_2 - a_2)(y_2 - a_3) = 0 \qquad (3.4.19)\\
&m \cdot (y_1 + y_2 + y_3 - (a + b + c)) = 0.
\end{aligned}$$

Looking for linear dependence between the vectors $m, m \cdot y_j, m \cdot y_j^2, m \cdot y_iy_j$ for $i, j = 1, 2,$

we deduce that the following one also holds:

$$(a_1a_2+a_1a_3+a_2a_3)m-(a_1+a_2+a_3)m\cdot(y_1+y_2)+m\cdot(y_1^2+y_2^2)+m\cdot y_1y_2=0. \quad (3.4.20)$$

It is worth noticing that (3.4.20) above can be rewritten in terms of complete homogeneous and elementary symmetric polynomials as follows:

$$m\cdot\left(\sum_{j=0}^{2}(-1)^j e_j(a_1,a_2,a_3)h_{2-j}(y_1,y_2)\right)=0.$$

Similarly, (3.4.19) also rewrites as

$$m\cdot\left(\sum_{j=0}^{1}(-1)^j e_j(a_1,a_2,a_3)h_{1-j}(y_1,y_2,y_3)\right)=0$$
$$m\cdot\left(\sum_{j=0}^{3}(-1)^j e_j(a_1,a_2,a_3)h_{3-j}(y_i)\right)=0,\ \ 1\leq i\leq 2.$$

Revisiting Example 3.4.2 from this perspective as well, we see that the first two relations in (3.4.18) rewrite as

$$q\cdot\left(\sum_{j=0}^{k}e_j(a,b)h_{k-j}(y_1,y_2)\right)=0,\ 1\leq k\leq 2.$$

◇

The considerations seen above generalize as follows. Set $\mathbf{a}=(a_1,\cdots,a_\ell)\in(\mathbb{C}^\times)^\ell$, and write $e_j(\mathbf{a})=e_j(a_1,\cdots,a_\ell), h_j(\mathbf{a})=h_j(a_1,\cdots,a_\ell)$ for all $0\leq j\leq\ell$. Then, the vector $m=\sum_{\sigma\in\mathcal{S}_\ell}m_1\cdot\sigma\in M=\mathbb{C}[\mathcal{S}_1](a_1)\tilde{\boxtimes}\cdots\tilde{\boxtimes}\mathbb{C}[\mathcal{S}_1](a_\ell)$ satisfies the following relations

$$\begin{aligned}&m\cdot(\sigma-1)=0\ \text{ for all }\ \sigma\in\mathcal{S}_\ell,\\&m\cdot\left(\sum_{j=0}^{r}(-1)^j e_j(\mathbf{a})h_{r-j}(y_1,\cdots,y_\ell)\right)=0,\ 1\leq r\leq\ell.\end{aligned} \quad (3.4.21)$$

It is clear that $m\cdot\sigma=m$ holds for every $\sigma\in\mathcal{S}_\ell$. As for the remaining relations in (3.4.21), set $(y_1)_j=(1\otimes\cdots\otimes y_1\otimes\cdots\otimes 1)\in\mathbb{C}[\tilde{\mathcal{S}}_1]\otimes\cdots\otimes\mathbb{C}[\tilde{\mathcal{S}}_1]$ with y_1 in the j-th tensor factor, $1\leq j\leq\ell$. Thus, recalling the usual action of $\mathbb{C}[\tilde{\mathcal{S}}_1]\otimes\cdots\otimes\mathbb{C}[\tilde{\mathcal{S}}_1]$ on $\mathbb{C}[\mathcal{S}_1](a_1)\otimes\cdots\otimes\mathbb{C}[\mathcal{S}_1](a_\ell)$, notice that $(1^{\otimes\ell})\cdot(y_1)_j=a_j1^{\otimes\ell}$. In this way, since we have $m_1\cdot y_j=((1^{\otimes\ell})\cdot(y_1)_j)\otimes 1$, then

$$m_1\cdot y_j=a_j\,m_1,\ 1\leq j\leq\ell.$$

In particular, it follows that $m_1\cdot h_j(y_1,\cdots,y_\ell)=h_j(\mathbf{a})m_1$ for every complete homogeneous symmetric polynomial $h_j(y_1,\cdots,y_\ell)$, $0\leq j\leq\ell$. Moreover, from (3.2.1) and the fact that

$h_j(y_1, \cdots, y_\ell)$ is a symmetric polynomial, notice that

$$\sigma h_j(y_1, \cdots, y_\ell) = h_j(y_1, \cdots, y_\ell)\sigma$$

for every $\sigma \in \mathcal{S}_\ell$. Therefore, we obtain

$$(m_1\sigma) \cdot \left(\sum_{j=0}^{r} (-1)^j e_j(\mathbf{a}) h_{r-j}(y_1, \cdots, y_\ell) \right) = \left(\sum_{j=0}^{r} (-1)^j e_j(\mathbf{a}) h_{r-j}(\mathbf{a}) \right) m_1\sigma.$$

Now, using (1.8.7) for $x_j = a_j$, from the above equality we see that

$$(m_1\sigma) \cdot \left(\sum_{j=0}^{r} (-1)^j e_j(\mathbf{a}) h_{r-j}(y_1, \cdots, y_\ell) \right) = 0,$$

whence we finally deduce that (3.4.21) holds. This inspires the main theorem of the next subsection.

We finish this subsection with an example related to a local Weyl module whose highest-weight is not of the form $\ell\omega_1$ as in the previous examples.

Example 3.4.4. Set $\ell = 3, \boldsymbol{\omega} = \boldsymbol{\omega}_{\omega_1+\omega_2, a}$, and consider the Weyl module $W = W(\boldsymbol{\omega}) \in \tilde{\mathscr{G}}_3$. The irreducible quotient of W is $V(\omega_1 + \omega_2, a)$, so we obtain the following short exact sequence for some proper submodule W':

$$0 \to W' \to W \to V(\omega_1 + \omega_2, a) \to 0. \tag{3.4.22}$$

As a $\mathfrak{g}$-module, we have the following decomposition into simple summands

$$W \cong_{\mathfrak{g}} V(\omega_1) \otimes V(\omega_2) = V(\omega_1 + \omega_2) \oplus V(\omega_3).$$

Since $\dim W_{\boldsymbol{\omega}} = 1$, any highest-weight vector of weight $\omega_1 + \omega_2$ of W is also a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$, so the proper submodule W' must satisfy $W' \cong_{\mathfrak{g}} V(\omega_3)$. In particular, W' is a simple $\tilde{\mathfrak{g}}$-module. Therefore, we have $W' \cong V(\omega_3, a)$.

The Weyl module W being indecomposable ensures that the exact sequence (3.4.22) does not split. We now show that any non-splitting exact sequence

$$0 \to V(\omega_3, a) \to X \xrightarrow{\pi} V(\omega_1 + \omega_2, a) \to 0 \tag{3.4.23}$$

implies $W \cong X$, where $X \in \tilde{\mathscr{G}}_3$. By the surjectivity of π, let $u \in X$ be a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}_{1+2,a}$, so that $V(\omega_1 + \omega_2, a)$ is isomorphic to a quotient of $U(\tilde{\mathfrak{g}}){\cdot}u$ by the restriction of π to this submodule. In particular, $U(\tilde{\mathfrak{g}}){\cdot}u$ is not irreducible, otherwise $U(\tilde{\mathfrak{g}}){\cdot}u \cong V(\omega_1 + \omega_2, a)$ would imply that (3.4.23) splits. Since the length of X is 2 by (3.4.23), we have $1 < \text{length}_{\tilde{\mathscr{G}}}(U(\tilde{\mathfrak{g}}){\cdot}u) \leq 2$, whence $X = U(\tilde{\mathfrak{g}}){\cdot}u$. Therefore, by the universal property of Weyl modules, X is a quotient of W. Noticing that both have the same dimension by (3.4.23) and (3.4.22), it follows that $X \cong W$.

From the equivalence of categories $\tilde{\mathcal{F}}_3$ and (3.4.22), we obtain a non-splitting short exact sequence of $\tilde{\mathcal{S}}_3$-modules:

$$0 \to \mathcal{S}(1,1,1,a) \to M \to \mathcal{S}(2,1,a) \to 0,$$

where $\tilde{\mathcal{F}}_3(M) \cong W$. Moreover, since any non-splitting short exact sequence as in (3.4.23) implies $X \cong W$, then any non-splitting short exact sequence $0 \to \mathcal{S}(1,1,1,a) \to M' \to \mathcal{S}(2,1,a) \to 0$ implies $M' \cong M$ by the equivalence of categories. We shall use this property to construct M' suitably as follows. Namely, setting M' to be the external direct sum of vector spaces $M' = \mathcal{S}(1,1,1,a) \times \mathcal{S}(2,1,a)$, we aim to define an action of $\tilde{\mathcal{S}}_3$ on M' and construct a non-trivial short exact sequence $0 \to \mathcal{S}(1,1,1,a) \to M' \to \mathcal{S}(2,1,a) \to 0$ so that $M' \cong M$ as $\tilde{\mathcal{S}}_3$-modules.

Let $\rho : \mathcal{S}_3 \to \mathrm{GL}(M')$ be the representation associated with the action of $\mathcal{S}_3$ on M':

$$(v \cdot x, w \cdot x) = \rho(x)(v,w), \; x \in \mathcal{S}_3, (v,w) \in M'.$$

We extend this representation to a representation of $\tilde{\mathcal{S}}_3$ by defining automorphisms $\tilde{\rho}(y_j)$ which, along with $\rho(\sigma_i)$, satisfy the defining relations of $\tilde{\mathcal{S}}_3$. As a $\mathcal{S}_3$-module, we identify $\{0\} \times \mathcal{S}(2,1,a)$ with $\mathcal{S}(2,1)$ generated by the Young symmetrizer:

$$c = (1 + (1,2))(1 - (1,3)) = 1 - (1,3) + (1,2) - (1,3,2),$$

and with basis $w_1 = c$ and $w_2 = c \cdot (1,2)$. Set $S(1,1,1,a) = \mathbb{C}v$. We shall construct $\tilde{\rho}(y_j) \in \mathrm{GL}(M')$ as the unique automorphisms defined by

$$\tilde{\rho}(y_j)(v,0) = (av,0), \quad \tilde{\rho}(y_j)(0,w_k) = (a_{j,k}v, aw_k) \tag{3.4.24}$$

for some nontrivial choice of $a_{j,k} \in \mathbb{C}, 1 \le j \le 3, 1 \le k \le 2$. In face of (3.2.1), we want these scalars $a_{j,k}$ to be such that we also have

$$\rho(\sigma_1)\tilde{\rho}(y_3)\rho(\sigma_1) = \tilde{\rho}(y_3) \tag{3.4.25}$$

$$\rho(\sigma_1)\tilde{\rho}(y_1)\rho(\sigma_1) = \tilde{\rho}(y_2) \tag{3.4.26}$$

$$\rho(\sigma_2)\tilde{\rho}(y_1)\rho(\sigma_2) = \tilde{\rho}(y_1) \tag{3.4.27}$$

$$\rho(\sigma_2)\tilde{\rho}(y_2)\rho(\sigma_2) = \tilde{\rho}(y_3) \tag{3.4.28}$$

$$\tilde{\rho}(y_i)\tilde{\rho}(y_j) = \tilde{\rho}(y_j)\tilde{\rho}(y_i), \; 1 \le i,j \le 3. \tag{3.4.29}$$

One easily notices that the above identities are satisfied when evaluated at $(v,0) \in M'$, so we only have to work with vectors of the form $(0,w_k) \in M'$.

1. For all $(0,w_k) \in M'$, we have that $\tilde{\rho}(y_i)\tilde{\rho}(y_j)(0,w_k) = \tilde{\rho}(y_i)(a_{j,k}v, aw_k) = (aa_{i,k}v +$

$aa_{j,k}v, a^2w_k)$. Similarly, we obtain $\tilde{\rho}(y_j)\tilde{\rho}(y_i)(0,w_k) = (aa_{j,k}v + aa_{i,k}v, a^2w_k)$, whence (3.4.29) is satisfied for any choice of $a_{j,k}$.

2. Notice that $(\rho(\sigma_1)\tilde{\rho}(y_3)\rho(\sigma_1))(0,w_1) = (-a_{3,2}v, aw_1)$. Since $\tilde{\rho}(y_3)(0,w_1) = (a_{3,1}, aw_1)$, in order to satisfy (3.4.25), we must have

$$a_{3,1} + a_{3,2} = 0.$$

Similarly calculating on the vector $(0,w_2)$, we once more derive the equality $a_{3,1} + a_{3,2} = 0$.

3. We have $((\rho(\sigma_1)\tilde{\rho}(y_1)\rho(\sigma_1))(0,w_1) = (-a_{1,2}v, aw_1)$. In order to satisfy (3.4.26) on $(0,w_1)$, we must have

$$a_{2,1} + a_{1,2} = 0.$$

Now, calculating on the vector $(0,w_2)$, we derive that

$$a_{2,2} + a_{1,1} = 0.$$

4. Notice that $w_1 \cdot (2,3) = w_1 - w_2$ and that $w_2 \cdot (2,3) = -w_2$. In this way, we obtain that $((\rho(\sigma_2)\tilde{\rho}(y_1)\rho(\sigma_2))(0,w_1) = ((a_{1,2} - a_{1,1})v, aw_1)$, whence from (3.4.27) we derive

$$a_{1,2} - 2a_{1,1} = 0.$$

We also have $((\rho(\sigma_2)\tilde{\rho}(y_1)\rho(\sigma_2))(0,w_2) = (a_{1,2}v, aw_2)$, which is exactly $\tilde{\rho}(y_1)(0,w_2)$ for any choice of $a_{1,2}$.

5. Note that $((\rho(\sigma_2)\tilde{\rho}(y_2)\rho(\sigma_2))(0,w_1) = ((a_{2,2} - a_{2,1})v, aw_1)$, whence it follows from (3.4.28) that

$$a_{2,2} - a_{2,1} - a_{3,1} = 0.$$

Analogously, we have $((\rho(\sigma_2)\tilde{\rho}(y_2)\rho(\sigma_2))(0,w_2) = (a_{2,2}v, aw_2)$, so we obtain

$$a_{2,2} - a_{3,2} = 0.$$

One can check that the associated linear system obtained from the variables $a_{j,k}$ and the equations above has a nontrivial solution for all $a_{1,2} \neq 0$. In this way, by our construction of these automorphisms based on the choice of the scalars $a_{j,k}$, we obtain a representation $\tilde{\rho} : \tilde{\mathcal{S}}_3 \to GL(M')$ extending $\rho : \mathcal{S}_3 \to GL(M')$. In particular, it is clear that $\mathcal{S}(1,1,1,a) \times \{0\} \cong \mathcal{S}(1,1,1,a)$ as $\tilde{\mathcal{S}}_3$-modules. Setting $\phi : M' \to \mathcal{S}(2,1,a)$ to be the epimorphism of $\tilde{\mathcal{S}}_3$-modules $(v,w) \mapsto w$, then $\operatorname{Ker}\phi = \mathcal{S}(1,1,1,a) \times \{0\}$ gives us a short exact sequence

$$0 \to \mathcal{S}(1,1,1,a) \to M' \xrightarrow{\phi} \mathcal{S}(2,1,a) \to 0.$$

Supposing by absurd that this exact sequence splits, it follows that M' has a submodule isomorphic to $\mathcal{S}(2,1,a)$. As an $\mathcal{S}_3$-module, M' has a unique submodule isomorphic to

$\mathcal{S}(2,1)$, which is the submodule generated by the vectors $(0, w_k), 1 \le k \le 2$. Therefore, by uniqueness, the submodule of M' isomorphic to $S(2,1,a)$ must coincide with the submodule generated by the vectors $(0, w_k)$. However, by our construction of the scalars $a_{j,k}$ above, this set is not an $\tilde{\mathcal{S}}_3$-module, obtaining a contradiction. Therefore, the sequence above does not split, and M' with the action of $\tilde{\mathcal{S}}_3$ explicitly given by $\tilde{\rho}$ is the desired module such that $\tilde{\mathcal{F}}_3(M') \cong W$. $\diamond$

3.4.2 Modules corresponding to Weyl modules of highest-weight $\ell\omega_1$

Let $\mathbf{a} = (a_1, \ldots, a_\ell) \in (\mathbb{C}^\times)^\ell$ and consider the following elements of $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$:

$$F_k = F_k(\mathbf{a}) = \sum_{j=0}^{k} (-1)^j e_j(\mathbf{a}) h_{r-j}(y_1, \cdots, y_\ell),\ 1 \le k \le \ell. \tag{3.4.30}$$

Let $R_{\mathbf{a}}$ be the quotient of $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ by the right ideal generated by $F_k, 1 \le k \le \ell$, together with $\sigma - 1, \sigma \in \mathcal{S}_\ell$. Now, let $\boldsymbol{\omega} \in \mathscr{P}^+$ be such that $\mathrm{wt}(\boldsymbol{\omega}) = \ell\omega_1$ and suppose that

$$\boldsymbol{\omega} = \prod_{j=1}^{\ell} \boldsymbol{\omega}_{1,a_j}. \tag{3.4.31}$$

Relations (3.4.21) inspire the following theorem.

Theorem 3.4.5. *In the above notation, we have $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}}) \cong W(\boldsymbol{\omega})$ as $\tilde{\mathfrak{g}}$-modules.*

In particular, since $W(\boldsymbol{\omega}) \cong \mathbb{V}^{\otimes \ell} \cong \mathcal{F}_\ell(\mathbb{C}[S_\ell])$ as $\mathfrak{g}$-modules, it follows from the fact that $\mathcal{F}_\ell$ is an equivalence of categories that we have the isomorphisms of right $\mathcal{S}_\ell$-modules

$$R_{\mathbf{a}} \cong \mathbb{C}[\mathcal{S}_\ell] \quad \text{for all} \quad \mathbf{a} \in (\mathbb{C}^\times)^\ell. \tag{3.4.32}$$

In the following, we show that (3.4.32) holds independently of Theorem 3.4.5.

It will be convenient to identify $R_{\mathbf{a}}$ with a quotient of the polynomial ring $\mathbb{C}[y_1, \cdots, y_\ell]$ as follows. Consider the right action of $\mathcal{S}_\ell$ on $\mathbb{C}[y_1, \cdots, y_\ell]$ of permuting the variables $y_j, 1 \le j \le \ell$. Regarding the elements in (3.4.30) as elements of $\mathbb{C}[y_1, \cdots, y_\ell]$, let $I_{\mathbf{a}}$ be the ideal generated by $F_k(\mathbf{a}), 1 \le k \le \ell$. In particular, we have $f \cdot \sigma \in I_{\mathbf{a}}$ for all $f \in I_{\mathbf{a}}$ and $\sigma \in \mathcal{S}_\ell$, so there is a natural induced right action of $\mathcal{S}_\ell$ on the quotient ring $\mathbb{C}[y_1, \cdots, y_\ell]/I_{\mathbf{a}}$. Let us denote this right $\mathcal{S}_\ell$-module by $\mathcal{P}_{\mathbf{a}}$.

Let $S \subseteq \mathbb{C}[y_1, \cdots, y_\ell]$ be the multiplicative set generated by the variables $y_j, 1 \le j \le \ell$. Thus, the localization $S^{-1}\mathbb{C}[y_1, \cdots, y_\ell]$ is the Laurent polynomial ring $\mathbb{C}[y_1^{\pm 1}, \cdots, y_\ell^{\pm 1}]$. Consider also the localized module $S^{-1}\mathcal{P}_{\mathbf{a}}$. Since $I_{\mathbf{a}} = \mathrm{Ann}(\mathcal{P}_{\mathbf{a}})$ by definition and $\mathbb{C}[y_1, \cdots, y_\ell]$ is a finitely generated $\mathbb{C}[y_1, \cdots, y_\ell]$-module, we have

$$\mathrm{Ann}(S^{-1}\mathcal{P}_{\mathbf{a}}) = S^{-1} I_{\mathbf{a}},$$

where $S^{-1}I_{\mathbf{a}}$ is the ideal of $\mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]$ generated by the image of $I_{\mathbf{a}}$ under the localization map $\mathbb{C}[y_1, \cdots, y_\ell] \to \mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]$. Let $\rho : \mathbb{C}[y_1, \cdots, y_\ell] \to \mathrm{End}(\mathcal{P}_{\mathbf{a}})$ be the associated representation. At the end of this section, we shall provide a proof that $\rho(S) \subseteq \mathrm{Aut}(\mathcal{P}_{\mathbf{a}})$. Assuming for now that $\rho(S) \subseteq \mathrm{Aut}(\mathcal{P}_{\mathbf{a}})$ holds, the universal property of localization then implies that there exists a unique extension of representation $\tilde{\rho} : \mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}] \to \mathrm{End}(\mathcal{P}_{\mathbf{a}})$ and, furthermore, the map

$$S^{-1}\mathcal{P}_{\mathbf{a}} \to \mathcal{P}_{\mathbf{a}}, \ \ m/s \mapsto \tilde{\rho}(s)^{-1}m,$$

is an isomorphism of $\mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]$-modules. In other words, we have

$$\mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]/S^{-1}I_{\mathbf{a}} \cong \mathcal{P}_{\mathbf{a}} \tag{3.4.33}$$

as $\mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]$-modules. The action of $\mathcal{S}_\ell$ on $\mathbb{C}[y_1, \cdots, y_\ell]$ extends naturally to an action on $\mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]$. Therefore, regarding $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$ as $\mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}] \rtimes \mathbb{C}[\mathcal{S}_\ell]$ in view of the isomorphism of algebras $\mathbb{C}[\mathbb{Z}^n] \cong \mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]$, then (3.4.33) is an isomorphism of $\mathbb{C}[\tilde{\mathcal{S}}_\ell]$-modules.

Let v be the image of 1 in the $\tilde{\mathcal{S}}_\ell$-module $R_{\mathbf{a}}$. Since $v \cdot \mathcal{S}_\ell = \{v\}$ by definition, it follows from (3.2.1) that $R_{\mathbf{a}} \cong \mathbb{C}[y_1^{\pm1}, \cdots, y_\ell^{\pm1}]/S^{-1}I_{\mathbf{a}}$ as $\tilde{\mathcal{S}}_\ell$-modules, whence (3.4.33) implies the isomorphism of $\tilde{\mathcal{S}}_\ell$-modules:

$$R_{\mathbf{a}} \cong \mathcal{P}_{\mathbf{a}}. \tag{3.4.34}$$

Therefore, in order to show (3.4.32), it suffices to show that $\mathcal{P}_{\mathbf{a}} \cong \mathbb{C}[\mathcal{S}_\ell]$ as $\mathcal{S}_\ell$-modules.

Let us denote by $Y = \mathbb{C}[y_1, \cdots, y_\ell]$ and by $Y^{\mathcal{S}_\ell}$ the symmetric polynomials in the variables $y_j, 1 \leq j \leq \ell$. Let $I_{\mathbf{a}}^{\mathcal{S}_\ell}$ be the ideal of $Y^{\mathcal{S}_\ell}$ generated by the symmetric polynomials $F_k(\mathbf{a})$ in (3.4.30), $1 \leq k \leq \ell$. Thus, considering the quotient $Y^{\mathcal{S}_\ell}/I_{\mathbf{a}}^{\mathcal{S}_\ell}$, by the very definition of the elements $F_k(\mathbf{a})$ one easily checks that the image of each $h_k = h_k(y_1, \cdots, y_\ell)$ in the quotient $Y^{\mathcal{S}_\ell}/I_{\mathbf{a}}^{\mathcal{S}_\ell}$ is a scalar, whence by (1.8.2) it follows that $I_{\mathbf{a}}^{\mathcal{S}_\ell}$ is a maximal ideal of $Y^{\mathcal{S}_\ell}$. In particular, there exists $\mathbf{z} = (z_1, \cdots, z_\ell) \in \mathbb{C}^\ell$ such that $I_{\mathbf{a}}^{\mathcal{S}_\ell}$ equals the ideal $J(\mathbf{z})$ of $Y^{\mathcal{S}_\ell}$ generated by the elements $h_k - z_k, 1 \leq k \leq \ell$. Setting $I(\mathbf{z})$ to be the ideal of Y generated by the elements $h_j - z_j, 1 \leq j \leq \ell$, one sees that $I_{\mathbf{a}} = I(\mathbf{z})$ as ideals of Y. We thus work equivalently with the quotient $\mathcal{P}(\mathbf{z}) = Y/I(\mathbf{z})$ instead of $\mathcal{P}_{\mathbf{a}}$.

Set $\mathcal{P}(\mathbf{0})$ to be the quotient of Y by the ideal generated by $h_k, 1 \leq k \leq \ell$. This quotient in the literature is referred to as the **ring of coinvariants** and the fact that $\mathcal{P}(\mathbf{0}) \cong \mathbb{C}[\mathcal{S}_\ell]$ as $\mathcal{S}_\ell$-modules traces back to Chevalley and a proof can be found in [42, Proposition 3.32], for instance. A more constructive approach to $P(\mathbf{0})$ will be utilized in order to characterize $\mathcal{P}(\mathbf{z})$ as follows. Recall the set $\mathrm{STab}(\xi)$ of standard tableaux of the partition $\xi \in \mathcal{P}_\ell$ seen in Subsection 1.6.2 and set

$$\mathcal{T} = \bigcup_{\xi \in \mathcal{P}_\ell} \Big(\mathrm{STab}(\xi) \times \mathrm{STab}(\xi)\Big).$$

The proof of the following theorem was sketched in [53] and a detailed proof of a more general statement was given in [2, Theorem 2] and [1] (see also [33] for further generalizations).

Theorem 3.4.6. *There exists a set $\mathscr{F} = \{F_T^S \in Y : (T,S) \in \mathcal{T}\}$ satisfying the following properties:*

(i) $\mathscr{F}$ is basis for Y as a $Y^{\mathcal{S}_\ell}$-module.

(ii) The image of $\mathscr{F}$ in $\mathcal{P}(\mathbf{0})$ is a $\mathbb{C}$-basis.

(iii) For each $\xi \in \mathcal{P}_\ell$ and $S \in \mathrm{STab}(\xi)$, the $\mathbb{C}$-span of the set $\mathscr{F}_\xi^S := \{F_T^S : T \in \mathrm{STab}(\xi)\}$ is a simple $\mathcal{S}_\ell$-submodule of Y isomorphic to $S(\xi)$. □

The polynomials F_T^S are explicitly described combinatorially and are known as **higher Specht polynomials**. Denoting by V_ξ^S the simple submodule mentioned in part (iii) of Theorem 3.4.6, it follows from $(i), (iii)$ that

$$Y = \bigoplus_{\xi \in \mathcal{P}_\ell} \bigoplus_{S \in \mathrm{Stab}(\xi)} V_\xi^S Y^{\mathcal{S}_\ell}. \tag{3.4.35}$$

The following lemma will be fundamental in characterizing $\mathcal{P}(\mathbf{z})$.

Lemma 3.4.7. *Let A be a commutative associative ring with 1 and B a subring (with 1) of A. Let J be an ideal of B and let I be the ideal of A generated by J (the extension of J in A). Suppose further that A is a free (right) B-module with basis $(a_k)_{k\in K}$ for some index set K. Then, $I = \oplus_k a_k J$. In particular, if A is a domain, $(a_k + I)_{k\in K}$ is a basis of the B/J-module A/I.*

Proof. Since I is a B-submodule of A such that $IJ \subseteq I$, then A/I is naturally a B/J-module. By hyphotesis, we have

$$A = \bigoplus_{k\in K} a_k B. \tag{3.4.36}$$

Since $a_k J \subseteq a_k B$, (3.4.36) implies that the sum $\sum_k a_k J$ is direct and, since $a_k J \subseteq I$ by the definition of I, we have $\oplus_k a_k J \subseteq I$. Conversely, given $i \in I$, we can write i as the finite sum $i = \sum_s c_s j_s$ for some $c_s \in A, j_s \in J$. From (3.4.36), we also write $c_s = \sum_k a_k b_{k,s}$ for some $b_{k,s} \in B$ to get

$$i = \sum_s \Big(\sum_k a_k b_{k,s}\Big) j_s = \sum_k a_k \Big(\sum_s b_{k,s} j_s\Big) \in \sum_k a_k J,$$

thus proving that $I \subseteq \oplus_k a_k J$.

Now, set $B_k = a_k B$ and $J_k = a_k J = B_k J, k \in K$, so J_k is a B-submodule of the B-module B_k. Let $\pi : A \to \bigoplus_k B_k/J_k$ be the epimorphism of B-modules given by

$$a = \sum_s b_s \mapsto \sum_s \Big(b_s + J_s\Big),$$

where $b_s \in B_s$ and $\sum_s \Big(b_s + J_s\Big) \in \oplus_k B_k/J_k$ is the corresponding sequence such that the s-th coordinate belongs to B_s/J_s. Noticing that π has kernel $I = \oplus_k J_k$, then $A/I \cong \bigoplus_k (B_k/J_k)$ as B-modules. Let $f_k : B \to B_k/J_k$ be the epimorphism of B-modules given by $b \mapsto a_k b + J_k$. If $f_k(b) = J_k$, then $a_k b = a_k j$ for some $j \in J$. Since A is a domain by assumption, then $b = j \in J$, showing that $\operatorname{Ker} f = J$. Therefore, $B_k/J_k \cong B/J$ as B-modules and, hence, also as B/J-modules by the fact that $J_k = B_k J$. In particular, B_k/J_k is a cyclic B/J-module generated by $a_k + J_k$. Therefore, in view of the isomorphism $A/I \cong \bigoplus_k (B_k/J_k)$ it follows that $(a_k + I)_{k \in K}$ is linearly independent over B/J and generates A/I as a B/J-module. □

In light of (3.4.35) and using Lemma 3.4.7 with $A = Y, B = Y_\ell^{\mathcal{S}_\ell}, J = J(\mathbf{z}), I = I(\mathbf{z}), K = \mathcal{T}$, $a_k = F_T^S, k = (T, S)$, we get

$$\mathcal{P}(\mathbf{z}) = Y/I(\mathbf{z}) = \bigoplus_{\xi \in \mathcal{P}_\ell} \bigoplus_{S \in \mathrm{Stab}(\xi)} (V_\xi^S + I(\mathbf{z})), \tag{3.4.37}$$

which implies $\mathcal{P}_{\mathbf{a}} = \mathcal{P}(\mathbf{z}) \cong \mathbb{C}[\mathcal{S}_\ell]$ since $V_\xi^S + I(\mathbf{z}) \cong S(\xi)$ and the cardinality of $\mathrm{STab}(\xi)$ equals $\dim V(\xi)$ (see [53]).

We now return to Theorem 3.4.5. In the next lemma, we identify vectors in $\tilde{\mathcal{F}}_\ell(R_{\boldsymbol{a}})$ which are highest-ℓ-weight vectors of ℓ-weight $\boldsymbol{\omega}$.

Lemma 3.4.8. *Set $\boldsymbol{v} = v_1^{\otimes \ell} \in \mathbb{V}^{\otimes \ell}$. Any nonzero pure tensor of the form $u \otimes \boldsymbol{v} \in \tilde{\mathcal{F}}_\ell(R_{\boldsymbol{a}})$ is a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$.*

Proof. The vector $u \otimes \mathbf{v}$ has weight $\ell\omega_1$. Since there can be no weight of $\mathbb{V}^{\otimes \ell}$ higher than $\ell\omega_1$, then $\tilde{\mathfrak{n}}^+(u \otimes \mathbf{v}) = 0$. Moreover, since $x_i^- \mathbf{v} = 0$ for $i \neq 1$, then $h_{i,r}(u \otimes \mathbf{v}) = [x_{i,r}^+, x_i^-](u \otimes \mathbf{v}) = 0$ follows as well, whence we obtain $\Lambda_{i,r}(u \otimes \mathbf{v}) = 0$, $r \in \mathbb{Z}_{>0}$. Thus, in order to show that $u \otimes \mathbf{v}$ is a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$, we are left to verify that $u \otimes \mathbf{v}$ is an eigenvector for the action of $\Lambda_{1,r}$ of eigenvalue equal to the coefficient of t^r in the polynomial $\prod_{i=1}^{\ell}(1 - a_i t)$. In fact, it suffices to calculate the eigenvalue of $\Lambda_{\theta,r}$ on $(u \otimes \mathbf{v})$, since $\Lambda_{\theta,r} = \prod_{i \in I} \Lambda_{i,r}$ and $\Lambda_{i,r}(u \otimes \mathbf{v}) = 0$ for $i \neq 1$. Also, one notices from (1.8.4) that the coefficient of t^r in $\prod_{i=1}^{\ell}(1 - a_i t)$ equals $(-1)^r e_r(a_1, \cdots, a_\ell)$, so we have to show

$$\Lambda_{\theta,r}(u \otimes \mathbf{v}) = (-1)^r e_r(a_1, \cdots, a_\ell)(u \otimes \mathbf{v}), r \in \mathbb{Z}_{\geq 0}. \tag{3.4.38}$$

Setting $\beta = \theta$ and $s = 0$ in Proposition 1.4.2, it follows that

$$(x_\theta^+)^{(r)}(x_0^+)^{(r)}(u \otimes \mathbf{v}) = (-1)^r \Lambda_{\theta,r}(u \otimes \mathbf{v}), r \in \mathbb{Z}_{\geq 0}, \tag{3.4.39}$$

so we only have to handle the left-hand side of (3.4.39). Iterating (3.2.3), we get

$$(x_0^+)^{(r)}(u \otimes \mathbf{v}) = \frac{1}{r!} \sum_{\boldsymbol{j} \in J_r} u \cdot y_{\boldsymbol{j}} \otimes \Delta_{\boldsymbol{j}}(x_\theta^-)(\mathbf{v}), \tag{3.4.40}$$

where $J_r = \{\boldsymbol{j} = (j_1, j_2, \ldots, j_r) : 1 \leq j_i \leq \ell, j_i \neq j_s \text{ for } i \neq s\}$ and

$$\begin{aligned} y_{\boldsymbol{j}} &= y_{j_1} \cdots y_{j_r}, \\ \Delta_{\boldsymbol{j}}(x_\theta^-) &= \Delta_{j_1}(x_\theta^-) \cdots \Delta_{j_r}(x_\theta^-),\ \boldsymbol{j} \in J_r. \end{aligned}$$

The condition $j_i \neq j_s$ follows from $\Delta_k(x_\theta^-)^m(\mathbf{v}) = 0$ for $m > 1$. Furthermore, noticing that $\Delta_{\boldsymbol{j}}(x_\theta^-)(\mathbf{v}) = 0$ for $\boldsymbol{j} \in J_r$ with $r > \ell$, then $(x_0^+)^{(r)}(u \otimes \mathbf{v}) = 0$ shows that (3.4.38) holds for $r > \ell$, so we move to the case $0 \leq r \leq \ell$. Noticing that in this case $(x_\theta^+)^r \cdot \Delta_{\boldsymbol{j}}(x_\theta^-)(\mathbf{v}) = r!\mathbf{v}, \boldsymbol{j} \in J_r$, then

$$(x_\theta^+)^{(r)}(x_0^+)^{(r)}(u \otimes \mathbf{v}) = \frac{1}{r!} \sum_{\boldsymbol{j} \in J_r} \Big(u \cdot y_{\boldsymbol{j}} \otimes \mathbf{v}\Big).$$

Using the fact that $y_i y_j = y_j y_i$, we collect all the $r!$ repeating terms of the form $y_{j_1} \cdots y_{j_r}$ with $1 \leq j_1 < \cdots < j_r \leq \ell$ to finally obtain

$$(x_\theta^+)^{(r)}(x_0^+)^{(r)}(u \otimes \mathbf{v}) = u \cdot e_r(y_1, \cdots, y_\ell) \otimes \mathbf{v}, \tag{3.4.41}$$

since $e_r(y_1, \cdots, y_\ell)$ satisfies (1.8.3). We now proceed inductively to show that

$$u \cdot e_r(y_1, \cdots, y_\ell) = e_r(a_1, \cdots, a_\ell)u,\ 0 \leq r \leq \ell. \tag{3.4.42}$$

It is clear that (3.4.42) holds for $r = 0$. Let $0 < r \leq \ell$ and and assume that it also holds for all $0 < s < r$. Using (1.8.7) and the inductive hyphotesis, we get

$$u \cdot e_r(y_1, \cdots, y_\ell) = (-1)^{r+1} u \cdot h_r(y_1, \cdots, y_\ell) + \sum_{j=1}^{r-1} (-1)^{j+r+1} e_j(a_1, \cdots, a_\ell)\Big(u \cdot h_{r-j}(y_1, \cdots, y_\ell)\Big). \tag{3.4.43}$$

Now, since $v\, F_r = 0$ in $R_{\boldsymbol{a}}$, where v the image of 1, the following relation holds in $R_{\boldsymbol{a}}$ as well:

$$(-1)^{r+1} u \cdot h_r(y_1, \cdots, y_\ell) = \sum_{j=1}^{r} (-1)^{j+r+2} e_j(a_1, \cdots, a_\ell)\Big(u \cdot h_{r-j}(y_1, \cdots, y_\ell)\Big).$$

Plugging this relation back in (3.4.43), then $u \cdot e_r(y_1, \cdots, y_\ell) = e_r(a_1, \cdots, a_\ell)u$ follows, finishing the induction on r. Thus, (3.4.42) and (3.4.41) combined with (3.4.39) shows that (3.4.38) holds, completing the proof. □

Proof of Theorem 3.4.5. Notice that (3.4.32) guarantees the existence of $u \in R_{\mathbf{a}}$ such that $u \otimes \mathbf{v} \neq 0$. Namely, any u such that the image of u under the isomorphism (3.4.32) has nonzero projection on the submodule $S(\xi_0) \subseteq \mathbb{C}[\mathcal{S}_\ell]$, where $S(\xi_0)$ is the 1-dimensional trivial representation associated with the Young symmetrizer $c_0 = \sum_{\sigma \in \mathcal{S}_\ell} \sigma$ of the partition $\xi_0 = (\ell)$ of ℓ. In this way, the submodule $V_{\mathbf{a}}$ of $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}})$ generated by $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}})_{\ell\omega_1}$ is nonzero by the previous lemma. In particular, any vector in $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}})_{\ell\omega_1}$ is also a highest-ℓ-weight vector of ℓ-weight $\boldsymbol{\omega}$. We now show that

$$V_{\mathbf{a}} = \tilde{\mathcal{F}}_\ell(R_{\mathbf{a}}). \tag{3.4.44}$$

If that were not the case, we would have a short exact sequence

$$0 \to V_{\mathbf{a}} \to \tilde{\mathcal{F}}_\ell(R_{\boldsymbol{a}}) \to W \to 0 \tag{3.4.45}$$

for some nonzero module W. Moreover, since $V_{\mathbf{a}}$ is generated by $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}})_{\ell\omega_1}$, it follows that $\ell\omega_1$ is not a weight of W. From the equivalence established by $\tilde{\mathcal{F}}_\ell$, there would also exists a short exact sequence of $\tilde{\mathcal{S}}_\ell$-modules

$$0 \to K \to R_{\boldsymbol{a}} \to L \to 0 \tag{3.4.46}$$

such that $\tilde{\mathcal{F}}_\ell(K) = V_{\mathbf{a}}$ and $\tilde{\mathcal{F}}_\ell(L) = W$. Considering a splitting of this sequence in the category of $\mathcal{S}_\ell$-modules, say $R_{\mathbf{a}} = K \oplus L'$, it follows that

$$u \otimes \mathbf{v} = 0 \quad \text{for all} \quad u \in L', \tag{3.4.47}$$

otherwise W would admit a nonzero vector of weight $\ell\omega_1$ by the previous lemma. Now, notice that v, the image of 1 in $R_{\mathbf{a}}$, is such that $\mathbb{C}v \cong S(\xi_0)$ as $\mathcal{S}_\ell$-modules by definition of v. In particular, this shows that $v \otimes \mathbf{v} \neq 0$. Moreover, it follows from (1.6.3), and the isomorphism $R_{\mathbf{a}} \cong \mathbb{C}[\mathcal{S}_\ell]$, that $\mathbb{C}v$ is the unique submodule of $R_{\mathbf{a}}$ isomorphic to $S(\xi_0)$. Writing $v = k_0 + l_0$ for some $k_0 \in K, l_0 \in L'$, then $v \cdot \sigma = v$ for $\sigma \in \mathcal{S}_\ell$ and $K \cap L' = \{0\}$ shows that $k_0\sigma = k_0, l_0 \cdot \sigma = l_0$ for all $\sigma \in \mathcal{S}_\ell$, i.e., we have $\mathbb{C}k_0 \cong S(\xi_0)$ and $\mathbb{C}l_0 \cong S(\xi_0)$ if both $k_0, l_0 \neq 0$. Therefore, by the uniqueness of $\mathbb{C}v$ as the submodule isomorphic to $S(\xi_0)$, then either $k_0 = 0$ or $l_0 = 0$. Now, $v \otimes \mathbf{v} \neq 0$ combined with (3.4.47) implies that $k_0 \neq 0$, so $v = k_0 \in K$ and it follows that $R_{\mathbf{a}} = K$, as v is the generator of $R_{\mathbf{a}}$. Consequently, regarding (3.4.45) and (3.4.46), the conclusion (3.4.44) follows.

Now, it is clear from the definition of $V_{\mathbf{a}}$ and the equality $V_{\mathbf{a}} = \tilde{\mathcal{F}}_\ell(R_{\mathbf{a}})$ shown above that $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}})$ is a quotient of the local Weyl module $W(\boldsymbol{\omega})$. Since $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}}) \cong \mathbb{V}^{\otimes \ell} \cong W(\boldsymbol{\omega})$ as $\mathfrak{g}$-modules by (3.4.32) and Corollary 2.4.9, it then follows that $\tilde{\mathcal{F}}_\ell(R_{\mathbf{a}}) \cong W(\boldsymbol{\omega})$ as $\tilde{\mathfrak{g}}$-modules, proving Theorem 3.4.5. □

We finish this section with the promised proof mentioned in the beginning of this section that $\rho(S) \subseteq \mathrm{Aut}(\mathcal{P}_{\mathbf{a}})$. For this proof, we will heavily utilize properties of symmetric polynomials that we first deduce as follows. For convenience, let us denote by $X_n = \mathbb{C}[x_1, \cdots, x_n], n \in \mathbb{Z}_{>0}$. Since the symmetric polynomials are generated by the elementary $e_k = e_k(x_1, \cdots, x_n)$ and complete homogeneous symmetric polynomials $h_k = h_k(x_1, \cdots, x_n)$, for each $h_k, 1 \le k \le n$, there exists a unique $q_k \in X_k$ such that

$$h_k = q_k(e_1, \cdots, e_k). \tag{3.4.48}$$

For instance, $q_1 = x_1, q_2 = x_1^2 - x_2$, and $q_3 = x_1^3 - 2x_1x_2 + x_3$. It follows from the classical relation (1.8.7) that

$$q_k = \sum_{j=1}^{k} (-1)^{j+1} x_j q_{k-j}. \tag{3.4.49}$$

Conversely, let $\tilde{q}_k \in X_k$ be such that $e_k = \tilde{q}_k(h_1, \cdots, h_k)$. One can also write (1.8.7) in the form $0 = \sum_{j=0}^{k} (-1)^j e_{k-j} h_j$, whence

$$\tilde{q}_k = \sum_{j=1}^{k} (-1)^{j+1} x_j \tilde{q}_{k-j}, \tag{3.4.50}$$

showing that $\tilde{q}_k = q_k$ and also that

$$x_k = \sum_{j=1}^{k} (-1)^{j+1} x_{k-j} q_j, \quad 1 \le k \le \ell, \tag{3.4.51}$$

when considering (3.4.48). Let us check that

$$q_k(q_1, \ldots, q_k) = x_k \quad \text{for all} \quad 1 \le k \le \ell. \tag{3.4.52}$$

Proceeding by induction, since $q_1 = x_1$, we readily see that the induction starts. Assume $k > 1$ and that (3.4.52) holds for $q_j, j < k$. By (3.4.49), we then get

$$q_k(q_1, \ldots, q_k) = \sum_{j=1}^{k} (-1)^{j+1} q_j q_{k-j}(q_1, \ldots, q_{k-j}) \overset{\text{Ind. Hyp.}}{=} \sum_{j=1}^{k} (-1)^{j+1} q_j x_{k-j} \overset{(3.4.51)}{=} x_k,$$

completing the induction.

Now, let $q_{k,\ell} \in X_\ell$ be the unique element such that

$$h_k = q_{k,\ell}(h_1, \ldots, h_\ell) \quad \text{for all} \quad k \ge 0. \tag{3.4.53}$$

For instance, $q_{k,\ell} = t_k$ if $k \le \ell$. Since $e_k = 0$ for $k > \ell$, (1.8.7) and $\tilde{q}_k = q_k$ imply

$$q_{\ell+1,\ell} = \sum_{j=1}^{\ell} (-1)^{j+1} x_{\ell+1-j} q_j. \tag{3.4.54}$$

Comparing with (3.4.51), we get

$$q_\ell = (-1)^\ell (q_{\ell,\ell-1} - x_\ell). \tag{3.4.55}$$

Combining with (3.4.52), we get

$$x_\ell = q_\ell(q_1, \ldots, q_\ell) = (-1)^\ell \left(q_{\ell,\ell-1}(q_1, \ldots, q_{\ell-1}) - q_\ell\right). \tag{3.4.56}$$

We finally use these properties to deduce our desired result.

Lemma 3.4.9. *For $\boldsymbol{a} \in (\mathbb{C}^\times)^\ell$, we have $\rho(S) \subseteq Aut(\mathcal{P}_{\boldsymbol{a}})$*

Proof. Set $Y_k = \mathbb{C}[y_1, \cdots, y_k], k \in \mathbb{Z}_{>0}$. We need to show that $\rho(y_j) \in \mathrm{Aut}(\mathcal{P}_{\mathbf{a}})$ for all $1 \leq j \leq \ell$. Since the argument is symmetric in j, we may assume that $j = \ell$ for simplicity. Thus, we show that $y_\ell r = 0$ for some $r \in \mathcal{P}_{\mathbf{a}}$ implies $r = 0$. Let R be the submodule of $\mathcal{P}_{\mathbf{a}}$ generated by such r. In what follows, we prove that $R = 0$.

Recall the elements $F_k(\mathbf{a}) \in I_{\mathbf{a}}$. Since $F_k(\mathbf{a})r = 0$ by the very definition of $\mathcal{P}_{\mathbf{a}}$, and since $y_\ell r = 0$, it then follows that R is a cyclic $Y_{\ell-1}$-module generated by a vector r which satisfies (at least) the relations

$$F'_k(\mathbf{a})r = 0, 1 \leq i \leq \ell, \tag{3.4.57}$$

where

$$F'_k(\mathbf{a}) = \sum_{j=0}^{k} (-1)^j e_j(a_1, \cdots, a_\ell) h_{k-j}(y_1, \cdots, y_{\ell-1}), 1 \leq k \leq \ell. \tag{3.4.58}$$

Let $\mathcal{P}'_{\mathbf{a}}$ be the quotient of $Y_{\ell-1}$ by the ideal generated by $F'_k(\mathbf{a}), 1 \leq k \leq \ell$, so that R is isomorphic to a quotient of $\mathcal{P}'_{\mathbf{a}}$. Let r' be the image of 1 in $\mathcal{P}'_{\mathbf{a}}$. In order to conclude that $R = 0$, it suffices to show $\mathcal{P}'_{\mathbf{a}} = 0$ or, equivalently, $r' = 0$.

Henceforth, to shorten notation, set $c_j = e_j(a_1, \cdots, a_\ell)$, $F'_j = F'_j(\mathbf{a})$ as well as $H_i = h_j(y_1, \cdots, y_{\ell-1}), 1 \leq j \leq \ell$. We suppose that $r' \neq 0$ and derive a contradiction. Let us assume that we know that r' is a common eigenvector for $H_i, 1 \leq i \leq \ell$, more precisely:

$$H_i r' = q_i(c_1, c_2, \ldots, c_i) r', \tag{3.4.59}$$

where q_i is given by (3.4.48). By (3.4.53) we have

$$H_\ell r' = q_{\ell,\ell-1}(H_1, \ldots, H_{\ell-1}) r', \tag{3.4.60}$$

and, hence,

$$q_{\ell,\ell-1}(q_1, \ldots, q_{\ell-1}) = q_\ell(c_1, c_2, \ldots, c_\ell), \tag{3.4.61}$$

where, for short, we wrote q_i in place of $q_i(c_1, c_2, \ldots, c_i)$ on the left hand side. However, (3.4.56) implies

$$q_{\ell,\ell-1}(q_1, \ldots, q_{\ell-1}) - q_\ell(c_1, c_2, \ldots, c_\ell) = (-1)^\ell c_\ell = (-1)^\ell a_1 a_2 \cdots a_\ell, \tag{3.4.62}$$

yielding a contradiction since $\mathbf{a} \in (\mathbb{C}^\times)^\ell$.

It remains to prove (3.4.59), which we shall do by induction on $1 \leq i \leq \ell$. The case $i = 1$ is clear from (3.4.57) and $q_1 = x_1$. Assume $i > 1$ and use the induction hypothesis on (3.4.57) to get

$$0 = F_i' r' = H_i r' + \sum_{j=1}^{i} (-1)^j c_j H_{i-j} r' = H_i r' + \left(\sum_{j=1}^{i} (-1)^j c_j q_{i-j}(c_1, \ldots, c_{i-j}) \right) r'.$$

An application of (3.4.49) completes the proof. □

3.4.3 Discussions and further directions

We now discuss how the results obtained in the last subsection relate to some other results previously established in the literature about local Weyl modules.

The proof of Theorem 3.4.5 here presented was conducted in two parts: the first part in which we show that $\tilde{\mathcal{F}}_\ell(R_\mathbf{a})$ is isomorphic to a quotient of $W(\boldsymbol{\omega})$ and the second part in which we conclude that $W(\boldsymbol{\omega}) \cong \tilde{\mathcal{F}}_\ell(R_\mathbf{a})$, with the latter being heavily dependent on Corollary 2.4.9 obtained from the results of [14, 26]. In particular, from the first part and from the isomorphism of $\mathcal{S}_\ell$-modules $R_\mathbf{a} \cong \mathbb{C}[\mathcal{S}_\ell]$, it follows that

$$\dim W(\boldsymbol{\omega}) \geq \dim \mathbb{V}^{\otimes \ell}, \tag{3.4.63}$$

with the proof of this inequality here given being independent from the fact derived in [14, 26] that $W(\boldsymbol{\omega}) \cong \mathbb{V}^{\otimes \ell}$ as $\mathfrak{g}$-modules. However, a few concerns arise from the context in which we derived (3.4.63), as we now discuss.

Since we are dealing with the affine Schur-Weyl duality, at first (3.4.63) is only valid for values $\ell \leq n$, since the proof that $\tilde{\mathcal{F}}_\ell(R_\mathbf{a}) \cong W(\boldsymbol{\omega})$ of the previous subsection is heavily dependent on the fact that $\tilde{\mathcal{F}}_\ell$ is an equivalence of categories. However, this hyphotesis was unnecessary in Corollary 1.3.4 of [14] which gives the following inequality with respect to the dimension of an arbitrary local Weyl module $W(\boldsymbol{\lambda})$:

$$\dim W(\boldsymbol{\lambda}) \geq \prod_{i=1}^{n} \binom{n+1}{i}^{\mathrm{wt}(\boldsymbol{\lambda})(h_i)}, \tag{3.4.64}$$

and from which (3.4.63) follows as a particular case. Nevertheless, if one considers the broader category of finite-dimensional $\tilde{\mathfrak{g}}$-modules, the functor $\tilde{\mathcal{F}}_\ell$ is still well-defined even for values $\ell > n$. In particular, one can ask whether the local Weyl modules $W(\boldsymbol{\omega})$ in this broader setting are still such that there exists finite-dimensional $\tilde{\mathcal{S}}_\ell$-modules M with $\tilde{\mathcal{F}}_\ell(M) \cong W(\boldsymbol{\omega})$. If that is the case, and the functor still behaves suitably with respect to exact sequences, one can hope to give a proof of (3.4.63) independently of [14, 26] and which is not dependent on $\ell \leq n$. We are still investigating this.

The general inequality (3.4.64) which gives (3.4.63) was originally pointed out in [15] to follow from the works seen in [19, 54, 11] involving the dimension of local Weyl modules in the setting of finite-dimensional representations of $U_q(\hat{\mathfrak{g}}')$ and the classical limits ($q \to 1$) of these modules. In [14], (3.4.64) was reobtained using the notion of fusion products, and, in [26], this result was developed using connections with the notion of Demazure modules. In fact, these external connections was also used to obtain the reverse direction of this inequality, and in both articles the bases of local Weyl modules for the affine Lie algebra of $\mathfrak{sl}_2$, first developed in [19] using the quantum setting, was assumed to construct the general argument. A construction of bases for the local Weyl modules, including bases for the local Weyl modules for the affine Lie algebra of $\mathfrak{sl}_2$ independently from the quantum setting construction in [19], was given in [6, 20, 39, 49] in different approaches.

It is interesting to notice that, in our context, even with the concerns above, we managed to obtain (3.4.63) only using the theory of $\mathcal{S}_\ell$ and $\tilde{\mathcal{S}}_\ell$-modules and the affine Schur-Weyl duality established by the functor $\tilde{\mathcal{F}}_\ell$, without relying on the notions of fusion products, Demazure modules or any known construction of bases of local Weyl modules for the affine Lie algebra of $\mathfrak{sl}_2$. Now, if one wants to obtain the reverse inequality without external connections, it is natural to raise the following question: In the same way the isomorphism of $\mathcal{S}_\ell$-modules $R_{\mathbf{a}} \cong \mathbb{C}[\mathcal{S}_\ell]$ was fundamental to show (3.4.63), is it possible to obtain the reverse inequality by working exclusively in the setting of $\mathcal{S}_\ell$-modules or $\tilde{\mathcal{S}}_\ell$-modules? For example, the local Weyl modules are characterized by the universal property that any finite-dimensional highest-ℓ-weight $\tilde{\mathfrak{g}}$-module is isomorphic to a quotient of the corresponding local Weyl module. Regarding the $\tilde{\mathcal{S}}_\ell$-modules corresponding to the local Weyl modules, can one expect that they can also be characterized by some universal property regarding quotients? If that is the case, is then $R_{\mathbf{a}}$ such that it satisfies exactly the universal property of that module? We still leave this questions unanswered. In particular, if one is successful in such characterization, the reverse inequality of (3.4.63) would follow independently from the works of [14, 26], although probably still dependent on the hyphotesis $\ell \leq n$. In fact, if properly generalized, it leads to a direction of further exploring the modules corresponding to the local Weyl modules whose ℓ-weights $\boldsymbol{\lambda}$ are not necessarily such that $\mathrm{wt}(\boldsymbol{\lambda}) = \ell\omega_1$, by characterizing these modules in terms of some universal property.

We end this work by doing a comparison with [25] whose content has a considerable intersection with Sections 3.2 and 3.3 above. The author of the present work initiated the study of the affine Schur-Weyl duality as an undergraduate research project in the context of Fapesp IC Grant [29]. In particular, Section 3.2 above is a significantly improved version of Section 3 of the final report concerning that grant, submitted to Fapesp in January 2020. In March 2020, we were told by V. Chari, who was aware of the efforts in this direction previously made in [4], that Y. Flicker had posted a related preprint on his own website. That preprint, at that time, culminated exactly with the result established by Theorem 3.2.3 and did not address any of the topics discussed in Sections 3.3 and 3.4 which we were working on at that moment. The present work is supported by Fapesp Grant 2019/23380-0. In February 2021, we reported to Fapesp that we had made a study of that preliminary version of [25] which contributed to a further improvement of what became Section 3.2, which is now indeed more detailed than the corresponding part in the present published version of [25]. Unfortunately, at the time we wrote that report, the preprint had been withdrawn from the author's webpage, so we could not include a link to it in the report to Fapesp.

On February 2nd 2022, we visited the author's webpage again and discovered that Flicker's initial preprint had been substantially expanded, growing from 8 to 15 sections, and published in *Journal of Lie Theory* in Setember of 2021. One interesting addition to the published version of [25], which is related to Section 3.2, is the proof that the functor in Theorem 3.2.3 is not an equivalence of categories when $\ell = n + 1$. This is achieved in Section 15 by showing that, in that case, the evaluation modules of the sign representation for $\mathcal{S}_\ell$ admit non-trivial self-extensions, while their image under the functor, the trivial representation for the loop algebra, are known not to admit nontrivial self-extensions (Section 15 of [25] contains other interesting points). The material covered in Section 3.3 above was also added to the published version of [25] as Section 8. Although we have developed this material independently of [25] - given that it was not present in the early version of the preprint we had studied -, since both works are heavily based on the quantum version originally shown in [17], it turns out that the approaches are unavoidably very similar. Section 3.3 contains more details than [25, Section 8]. The terminology "Zelevinsky tensor product", which we use following [17], is not used in [25]. We regard the fact that this master's project unintentionally acquired a feel of competition with [25], whose author is a senior and highly accomplished mathematician, as a strong indication of the relevance of this topic.

The material of Section 3.4, which was the main goal of our project since the early stages of [4] and [29], is not addressed in [25]. However, another addition to the published version of [25] was a citation to the paper [23] which, among other results, describes a version of the Schur-Weyl duality of $\mathbb{Z}$-graded modules for the current algebra $\mathfrak{g}[t]$ on the one hand and the degenerate affine Hecke algebra $\mathcal{H}_\ell = \mathbb{C}[\mathcal{S}_\ell] \ltimes \mathbb{C}[y_1, \ldots, y_\ell]$ on the other. In particular, since the Weyl module $W_t(\lambda)$ for the current algebra $\mathfrak{g}[t]$ defined in Section 2.4 is isomorphic to the the pull-back $\varphi_a^*(W(\boldsymbol{\omega}_{\lambda,a}))$ along the homomorphism

$t \overset{\varphi_a}{\longmapsto} t-a, a \in \mathbb{C}^\times$, which in turn is isomorphic to a level-1 Demazure module (see [14]), the Weyl module $W_t(\lambda)$ is then a $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module and one may consider the $\mathcal{H}_\ell$-module corresponding to $W_t(\lambda)$. By exploring the theory of highest-weight categories, the authors of [23] show that the module corresponding to $W_t(\lambda)$ under their version of the Schur-Weyl duality is a certain module referred to as the local Kato module, which we shall denote here by $K(\lambda)$. It will be convenient to think of λ both as an element of P^+ as well as the corresponding partition of ℓ by the correspondence (3.1.6). It is then proved in [23] that the graded $\mathcal{S}_\ell$-character of $K(\lambda)$ is given by the so-called q-Whitacker function and that we have an isomorphism of $\mathcal{S}_\ell$-modules

$$K(\lambda) \cong \mathrm{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_\ell} S,$$

where $\mathcal{S}_\lambda$ is the subgroup $\mathcal{S}_{\xi_1} \times \mathcal{S}_{\xi_2} \times \cdots \mathcal{S}_{\xi_m}$ associated to the partition $\xi = (\xi_1, \ldots, \xi_m)$ given by the transpose of the partition associated to λ, and S is the sign representation of $\mathcal{S}_\ell$ regarded as an $\mathcal{S}_\lambda$-module by restriction. However, the description of the action of the elements y_j on $K(\lambda)$ in [23] is not as transparent as one might desire. For instance, following the path we trailed in Subsection 3.4.2, for $\lambda = \ell\omega_1$ one might expect that $K(\lambda)$ is isomorphic to the quotient R_0 of $\mathcal{H}_\ell$ by the right ideal generated by $\sigma - 1, \sigma \in \mathcal{S}_\ell$, together with the elements $F_k(\mathbf{a})$ with $\mathbf{a} = (0, 0, \ldots, 0)$, since $h \otimes t^k, k \geq 1$, annihilates the generator of $W_t(\ell\omega_1)$. As of this moment, we do not see how to show this from the very abstract description of $K(\lambda)$ given in [23]. A far as we know, the question of studying the modules corresponding to local Weyl modules has not been addressed elsewhere beside [23] and our work. In particular, a description of the answer in terms of generators and relation as we gave in Section 3.4.2 for the case $\lambda = \ell\omega_1$ has not been given before ours. At the moment, we are focusing on unraveling the action of the elements y_j on $K(\lambda)$ for general λ hoping to be able to extend the results of Section 3.4.2.

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