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JOÃO MARCOS RIBEIRO DO CARMO

Invariant subspace problem and universal composition operators

Problema do subespaço invariante e operadores de composição

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João Marcos Ribeiro do Carmo

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Prof(a). Dr(a). SAHIBZADA WALEED NOOR

Prof(a). Dr(a). SERGIO ANTONIO TOZONI

Prof(a). Dr(a). LUCAS CATÃO DE FREITAS FERREIRA

Prof(a). Dr(a). VALENTIN MATACHE

Prof(a). Dr(a). ADI TCACIUC

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Resumo

O problema do subespaço invariante é um dos mais importantes problemas em aberto da área de teoria dos operadores, sabemos que o problema do subespaço invariante é equivalente a mostrar que todos subespaços minimais de um operador universal são unidimensionais. Sejam $\mathbb{D}$ e $\mathbb{C}_+$ o disco aberto unitário e o semiplano direito respectivamente. Neste trabalho caracterizaremos quais operadores de composição induzidos por símbolos lineares possuem uma translação universal nos espaços de Hardy $H^2(\mathbb{D})$ e $H^2(\mathbb{C}_+)$. Em ambos os casos conseguimos encontrar novos exemplos. O mais proeminente desses é o automorfismo afim de $\mathbb{D}$ definido por

$$C_{\phi_b}(z) = bz + 1 - b$$

para $0 < b < 1$. Neste trabalho também nos dedicamos na analise dos autovetores e subesbaços minimais de C_{ϕ_b} em $H^2(\mathbb{D})$.

Palavras-chave: operadores de composição. operador universal. problema do subespaço invariante.

Abstract

The invariant subspace problem is one of the most important open problems of operator theory. We know that the invariant subspace problem is equivalent to the statement that all minimal invariant subspaces for a universal operator are one dimensional. Let $\mathbb{D}$ and $\mathbb{C}_+$ be the open unit disk and right half-plane respectively. In this work we characterize which compositions operators induced by linear fractional symbols have universal translates on the Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{C}_+)$. In both cases new examples are discovered. The most prominent of these being the affine self-map of $\mathbb{D}$ defined by

$$\phi_b(z) = bz + 1 - b$$

for $0 < b < 1$. We also dedicate our attention to an analysis of the eigenvectors and minimal invariant subspaces of C_{ϕ_b} on $H^2(\mathbb{D})$.

Keywords: compositions operators. universal operator. invariant subspace problem.

List of symbols

$\mathbb{C}$	Set of complex numbers
$\mathbb{C}_+$	Right half-plane
$Re(s)$	Real part of complex number s
$Im(s)$	Imaginary part of complex number s
$\mathbb{D}$	Unit disc
$\mathbb{T}$	Unit circle in $\mathbb{C}$
$B(X)$	Space of bounded operators on the Banach space X
$R(T)$	Range of operator T
$Ker(T)$	Kernel of operator T
$\sigma(T)$	Spectrum of operator T
$\sigma_0(T)$	Point spectrum of the operator T
$r(T)$	Spectral radius of the operator T
T^*	Adjoint of operator T
$Lat(T)$	Set of invariant subspaces of operator T
$H^2(\mathbb{D})$	Hardy space of analytic functions on the open unit disc
$H^2(\mathbb{C}_+)$	Hardy space of analytic functions on the right half-plane
C_ϕ	Composition operator induced by analytic function ϕ

Contents

Introduction

The linear fractional transformations are the automorphisms of the Riemann sphere $\overline{\mathbb{C}}$ and the composition operators C_ϕ, where ϕ is a linear fractional transformation are the most studied class of composition operators on both $H^2(\mathbb{D})$ and on $H^2(\mathbb{C}_+)$.

The invariant subspace problem is the most important open problem in operator theory and it is not known who formulate it first. Apparently the problem was formulated after Beurling's publication on invariant subspaces of unilateral shift (BEURLING, 1949), or after the unpublished work of Von Neumann on compact operators.

In this thesis we will show a necessary and sufficient condition for operator $C_\phi - \lambda I$ to be universal, where ϕ is a linear fractional transformation, C_ϕ is a composition operator on $H^2(\mathbb{D})$ (or on $H^2(\mathbb{C}_+)$), and $0 \neq \lambda \in \sigma_0(C_\phi)$. Furthermore, we will show that the invariant subspace problem can be solved by analyzing the invariant subspaces of the operator C_{ϕ_b}, with $\phi_b : \mathbb{D} \to \mathbb{D}$ being a non-automorphic hyperbolic linear fractional transformation. The results of this work have been accepted for publication as:

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In the first chapter we will provide the basic results that will be utilized throughout the work. We will also define the linear fractional transformations, the concept of Hardy Hilbert space, the concept of composition operator and the proof of some properties of the compositions operators. The main reference for this chapter is (MARTÍNEZ-AVENDAÑO; ROSENTHAL, 2007).

In the first section of Chapter 2, a discussion will be made on the invariant subspace problem. In the first section we will cite the particular cases for which the problem has a solution. In the second section we will present the concept of universal operator and we will prove a result that relates the concept of universal operator to the invariant subspace problem. The main references for this chapter are (ENFLO, 1976), (LOMONOSOV, 1973), (CARADUS, 1969).

In Chapter 3, we will prove that the operator $C_\phi - \lambda I$ is universal on $H^2(\mathbb{C}_+)$, for $0 \neq \lambda$ in the interior of $\sigma(C_\phi)$, where ϕ is a certain type of hyperbolic linear fractional transformation. Furthermore, we will prove that this is the only type of linear fractional transformation on $\mathbb{C}_+$ such that $C_\phi - \lambda I$ is universal on $H^2(\mathbb{C}_+)$ for some $\lambda \in \mathbb{C}$. Also in this chapter, we will prove that if $\phi : \mathbb{D} \to \mathbb{D}$ is a non-automorphic hyperbolic linear fractional transformation without fix point in $\mathbb{D}$, the $C_\phi - \lambda I$ is universal for all $0 \neq \lambda \in \sigma(C_\phi)$. This result extends a classic result of (NORDGREN; ROSENTHAL; WINTROBE, 1987). Moreover, we also give a sufficient and necessary condition for $C_\phi - \lambda I$ to be universal, with $\phi : \mathbb{D} \to \mathbb{D}$ being a linear fractional transformation and $\lambda \in \mathbb{C}$.

In Chapter 4, we will work with the operator C_{ϕ_b}, where ϕ_b is a canonical form of the non-automorphic hyperbolic linear fractional transformation without fixed point in $\mathbb{D}$. In this chapter we will also prove various properties of eigenvectors of C_{ϕ_b} and will give results on minimal invariant subspaces of C_{ϕ_b} analogous to results by (MATACHE, 1993), (CHKLIAR, 1997) for the automorphic case.

1 Preliminaries

1.1 Linear Fractional Transformations

This section is based on (SHAPIRO, 1993).

A linear fractional transformation is a mapping of the form

$$T(z) = \frac{az+b}{cz+d}$$

with $ad - bc \neq 0$, which is a necessary and sufficient condition for T to be non-constant. We denote the set of all linear fractional transformations by $LFT(\overline{\mathbb{C}})$.

$LFT(\overline{\mathbb{C}})$ is a group under composition. Each of its members maps circles to other circles. Given any pair of circles, there are members of $LFT(\overline{\mathbb{C}})$ that maps one onto the other and the same is true for the set of triples of distinct points in $\overline{\mathbb{C}}$.

Each non-singular 2 x 2 complex matrix gives rise to a linear fractional transformation T_A. Clearly, $T_A = T_{\lambda A}$ for any $\lambda \in \mathbb{C}$, for this reason, it is convenient when working out general properties of $LFT(\overline{\mathbb{C}})$ to normalize the matrices to have determinant 1.

Definition 1. *If $ad - bc = 1$, we say T is in standard form.*

The utility of matrices in dealing with linear fractional transformations comes from the fact that $T_A \circ T_B = T_{AB}$. Clearly, the linear fractional transformation $\dfrac{az+b}{cz+d}$ fixes the point ∞ if and only if $c = 0$, in which case ∞ is the only fixed point if and only if $a = d$ and $b \neq 0$. Otherwise, the fixed point equation is a quadratic, with solutions

$$\alpha, \beta = \frac{(a-d) \pm [(a-d)^2 + 4bc]^{1/2}}{2c},$$

the trace, if $T(z) = \dfrac{az+b}{cz+d}$ is in standard form. It defines the trace of T to be $\chi(T) = \pm(a+d)$.

T has ∞ as its only fixed point on the sphere, if and only if $T(z) = z + b$, in which case $|\chi(T)| = 2$. If T has only finite fixed points, then the equations written above for these fixed points can be at least partially expressed in terms of the trace

$$\alpha, \beta = \frac{(a-d) \pm [\chi(T)^2 - 4]^{1/2}}{2c}.$$

This equation together with our previous remark about maps with unique fixed point at ∞ shows that:

$T \in LFT(\overline{\mathbb{C}})$ has an unique fixed point in $\mathbb{C}$ if and only if $|\chi(T)| = 2$.

A map $T \in LFT(\overline{\mathbb{C}})$ is called parabolic if it has a single fixed point in $\mathbb{C}$. Suppose T is parabolic and has its fixed point at $\alpha \in \mathbb{C}$. If $S \in LFT(\overline{\mathbb{C}})$ takes α to ∞, then $V = S \circ T \circ S^{-1}$ belongs to $LFT(\overline{\mathbb{C}})$ and fixes only the point ∞. Therefore $V(z) = z + \tau$ for some non-zero complex number τ. Thus every parabolic linear fractional map is conjugate to a translation.

If T is not parabolic, there are two fixed points $\alpha, \beta \in \mathbb{C}$. Let S be a linear fractional map that takes α to 0 and β to ∞. Then the map $V = S \circ T \circ S^{-1}$ belongs to $LFT(\overline{\mathbb{C}})$ and fixes both 0 and ∞, so it must have the form $V(z) = \lambda z$ for some complex number λ, which is called the multiplier for T. Thus

$$T(z) = S^{-1}(\lambda S(z)),$$

by the chain rule,

$$T'(\alpha) = \lambda \textit{ and } T'(\beta) = \frac{1}{\lambda}.$$

The equation above implies that, if $|\lambda| \neq 1$, then one of fixed points of T is attractive, this is, $T_n(z)$ tends for the attractive fixed point when n tends for $+\infty$. For example, if $|\lambda| < 1$, its attractive fixed point is α, and

$$T_n(z) \to \alpha.$$

Note that there is an ambiguity on the definition of multiplier. If the roles of α and β are interchanged, so that now S sends β to zero, then $1/\lambda$ is the multiplier of the transformation. The theorem on fixed points and derivatives, shows that

$$\lambda + \frac{1}{\lambda} = \chi(T)^2 - 2.$$

Definition 2. *Suppose $T \in LFT(\overline{\mathbb{C}})$ is neither parabolic nor the identity. Let $\lambda \neq 1$ be the multiplier of T. Then T is called*

1. *Elliptic if $|\lambda| = 1$*
2. *Hyperbolic if $\lambda > 0$*
3. *Loxodromic if T is neither eliptic nor parabolic*

1.2 Linear Fractional Transformations in $\mathbb{D}$

This section is based on (SHAPIRO, 1993).

Our interest here is in $LFT(\mathbb{D})$, the subgroup of $LFT(\mathbb{C})$ consisting of selfmaps of the unit disc $\mathbb{D}$. Those that take $\mathbb{D}$ onto itself are called (conformal) automorphisms. Consideration of normal forms quickly shows that

- Parabolic members of $LFT(\mathbb{D})$ have their fixed point on $\mathbb{T}$.
- Hyperbolic members of $LFT(\mathbb{D})$ must have attractive fixed point in $\overline{\mathbb{D}}$, with the other fixed point outside $\mathbb{D}$, and lying on $\mathbb{T}$ if and only if the map is an automorphism of $\mathbb{D}$.
- Loxodromic and elliptic members of $LFT(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside $\overline{\mathbb{D}}$. The elliptic ones are precisely the automorphisms in $LFT(\mathbb{D})$ with this fixed point configuration.

1.3 Hardy Spaces

One of the most familiar Hilbert spaces is l^2. This space consisted of the collection of all sequences of complexes numbers $(a_n)_{n=0}^{\infty}$, such that

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

In this space the vector addiction and vector multiplication by complex numbers are performed componentwise. The norm of the vector $(a_n)_{n=0}^{\infty}$ is

$$\|(a_n)_{n=0}^{\infty}\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2},$$

where the inner product of vectors $f = (a_n)_{n=0}^{\infty}$ and $g = (b_n)_{n=0}^{\infty}$ is

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

The space l^2 is separable, and all infinite dimensional separable complex Hilbert spaces are isomorphic to each other.

Definition 3. *The Hardy-Hilbert space, called $H^2(\mathbb{D})$, consists of all analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, this is*

$$H^2(\mathbb{D}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the inner product on $H^2(\mathbb{D})$ is defined by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

The norm of vector $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is

$$\|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.$$

The map $(a_n)_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} a_n z^n$ clearly is an isomorphism of l^2 to $H^2(\mathbb{D})$. Particularly $H^2(\mathbb{D})$ is a separable Hilbert space.

Note that if $|z_0| < 1$, then $\sum_{n=0}^{\infty} a_n z_0^n$ converges absolutely, because $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, thus $|a_n|$ is bounded, then $\limsup |a_n z_0^n|^{1/n} \leqslant |z_0| < 1$. Therefore all functions in $H^2(\mathbb{D})$ are analytic in the open unit disc $\mathbb{D}$.

Theorem 1. *Let f be analytic on $\mathbb{D}$. Then $f \in H^2(\mathbb{D})$ if and only if*

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

Moreover, for $f \in H^2(\mathbb{D})$,

$$\|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

Proof. Let f be an analytic function on $\mathbb{D}$ with power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then, for $0 < r < 1$,

$$|f(re^{i\theta})|^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{(n+m)} e^{i(n-m)\theta}.$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\theta}d\theta = \delta_{n,m},$$

integrating the expression above for $|f(re^{i\theta})|^2$ and dividing by 2π results in

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

If $f \in H^2(\mathbb{D})$, then $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leqslant \|f\|^2 < \infty$, for every $r \in [0,1)$. Thus

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leqslant \|f\|^2 < \infty.$$

Conversely, assume that the above supremum is finite. As shown above,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

$f \notin H^2(\mathbb{D})$, the right-hand side can be made arbitrarily large by taking r close to 1. This would contradict the assumption that the supremum on the left side of the equation is finite.

Note that, by the above considerations, it also follows that for $f \in H^2(\mathbb{D})$,

$$\|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta.$$

□

Some authors define the Hardy space $H^2(\mathbb{D})$ as the space of analytic maps f on $\mathbb{D}$, such that

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty$$

Definition 4. *Let $z_0 \in \mathbb{D}$. The function k_{z_0}, defined by*

$$k_{z_0}(z) = \sum_{n=0}^{\infty} \overline{z_0^n} z^n = \frac{1}{1-\overline{z_0}z}$$

is in $H^2(\mathbb{D})$ and is called reproducing kernel for z_0 in $H^2(\mathbb{D})$.

We have that for $z_0 \in \mathbb{D}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(\mathbb{D})$, $f(z_0) = \sum_{n=0}^{\infty} a_n z_0^n = \langle f, k_{z_0}\rangle$ and $\|k_{z_0}\| = \left(\sum_{n=0}^{\infty} |z_0|^{2n}\right)^{1/2} = \frac{1}{(1-|z_0|^2)^{1/2}}$, and hence, the map $f \mapsto f(z_0)$ is a bounded linear functional.

Definition 5. *Let f be analytic on $\mathbb{C}_+$, we say that f belongs to $H^2(\mathbb{C}_+)$, if*

$$\sup_{x>0} \frac{1}{\pi} \int_{-\infty}^{+\infty} |f(x+yi)|^2 dy < \infty$$

By (HOFFMAN, 1962), see also (PARTINGTON, 1988) we have that $H^2(\mathbb{C}_+)$ is a Hilbert space with norm

$$\|f\| = \left(\sup_{x>0} \frac{1}{\pi} \int_{-\infty}^{+\infty} |f(x+yi)|^2 dy \right)^{1/2}.$$

and

$$\begin{array}{rccl} \mathbb{W}: & H^2(\mathbb{D}) & \to & H^2(\mathbb{C}_+) \\ & f(w) & \mapsto & g(z) = \dfrac{f(\frac{1-z}{1+z})}{\sqrt{\pi}(1+z)} \end{array}$$

and

$$\begin{array}{rccl} \mathbb{W}^{-1}: & H^2(\mathbb{C}_+) & \to & H^2(\mathbb{D}) \\ & g(z) & \mapsto & f(w) = \dfrac{2\sqrt{\pi} g(\frac{1-w}{1+w})}{(1+w)} \end{array}$$

is an isometric isomorphism.

Another important isomorphism between the Hardy space $H^2(\mathbb{C}_+)$ and a better known Hilbert space is given by

Theorem 2. *(Paley-Wiener theorem)*

The map $P : L^2(\mathbb{R}_+) \to H^2(\mathbb{C}_+)$, defined by

$$\begin{array}{rccl} P: & L^2(\mathbb{R}_+) & \to & H^2(\mathbb{C}_+) \\ & F(t) & \mapsto & f(z) = \displaystyle\int_{\mathbb{R}_+} F(t)e^{-tz} dt. \end{array}$$

is an isometric isomorphism.

1.4 Composition Operators

Definition 6. *For each analytic function $\phi : \mathbb{D} \to \mathbb{D}$, we define the composition operator C_ϕ by*

$$(C_\phi f)(z) = f(\phi(z))$$

for all $f \in H^2(\mathbb{D})$.

For this definition we have that if $\phi : \mathbb{D} \to \mathbb{D}$ is analytic, then the composition operator C_ϕ is well defined and bounded on $H^2(\mathbb{D})$ with:

$$\|C_\phi\| \leqslant \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}.$$

Proposition 1. *If C_ϕ and C_ψ are composition operators then $C_\phi \circ C_\psi = C_{\psi\circ\phi}$.*

Proof. Let ϕ, ψ be analytic functions that map the unitary disc onto himself, and $f \in H^2(\mathbb{D})$, we have that $C_\phi C_\psi f = C_\phi f \circ \psi = f \circ \psi \circ \phi = C_{\psi\circ\phi} f$. □

The adjoint of a composition operator is in most cases, difficult to describe in simple terms, but its action on reproducing kernels is known:

Proposition 2. *If C_ϕ is a composition operator and k_λ is a reproducing kernel, then*

$$C_\phi^* k_\lambda = k_{\phi(\lambda)}.$$

Proof. Let $f \in H^2(\mathbb{D})$. We have that $\langle f, C_\phi^* k_\lambda \rangle = \langle C_\phi f, k_\lambda \rangle = f(\phi(\lambda))$, and also that $\langle f, k_{\phi(\lambda)} \rangle = f(\phi(\lambda))$. Thus, we have that $C_\phi^* k_\lambda = k_{\phi(\lambda)}$. □

Proposition 3. *If C_ϕ is a composition operator, then*

$$\frac{1}{\sqrt{1-|\phi(0)|^2}} \leqslant \|C_\phi\| \leqslant \frac{2}{\sqrt{1-|\phi(0)|^2}}.$$

Proof. By Proposition 2, we have that $C_\phi^* k_0 = k_{\phi(0)}$. Since $\|k_\lambda\|^2 = \dfrac{1}{1-|\lambda|^2}$, we have that $\|k_0\| = 1$ and $\|k_{\phi(0)}\| = \dfrac{1}{\sqrt{1-|\phi(0)|^2}}$. By $\|k_{\phi(0)}\| = \|C_\phi^* k_0\| \leqslant \|C_{\phi^*}\|\|k_0\|$, we have that

$$\frac{1}{\sqrt{1-|\phi(0)|^2}} \leqslant \|C_\phi^*\| = \|C_\phi\|.$$

We know that $\|C_\phi\| \leqslant \sqrt{\dfrac{1+|\phi(0)|}{1-|\phi(0)|}}$, and we note that for $0 \leqslant r < 1$, we have

$$\sqrt{\frac{1+r}{1-r}} = \sqrt{\frac{(1+r)^2}{1-r^2}} = \frac{1+r}{\sqrt{1-r^2}} \leqslant \frac{2}{\sqrt{1-r^2}}.$$

As a result

$$\|C_\phi\| \leqslant \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)}} \leqslant \frac{2}{\sqrt{1-|\phi(0)|^2}}.$$

□

Proposition 4. *The norm of a composition operator C_ϕ is 1 if and only if $\phi(0) = 0$.*

Proof. Assuming that $\phi(0) = 0$, since $\|C_\phi\| \leqslant \sqrt{\frac{1+|\phi(0)|}{1-|\phi(0)|}}$, we have $\|C_\phi\| \leqslant 1$. From Proposition 3, we have $\|C_\phi\| \geqslant 1$, thus $\|C_\phi\| = 1$.

Assuming that $\|C_\phi\| = 1$. Since $\frac{1}{\sqrt{1-|\phi(0)|^2}} \leqslant \|C_\phi\|$, we have $\frac{1}{1-|\phi(0)|^2} \leqslant 1$, then $\phi(0) = 0$.

□

Proposition 5. *If ϕ is a non-constant analytic function mapping the disk into itself and satisfying $\phi(a) = a$ for some $a \in \mathbb{D}$, and if there is a function f analytic on $\mathbb{D}$ that is not identically zero and satisfies the Schröder equation*

$$f(\phi(z)) = \lambda f(z)$$

for some λ, then either $\lambda = 1$ or there is a natural number k such that $\lambda = (\phi'(a))^k$.

Proof. The equations $\phi(a) = a$ and $f(\phi(z)) = \lambda f(z)$ yield $f(a) = \lambda f(a)$. If $f(a) \neq 0$, then clearly $\lambda = 1$ and the theorem is established in that case.

Suppose $f(a) = 0$. Since f is not identically zero, $f(z)$ has a power series expansion

$$f(z) = b_k(z-a)^k + b_{k+1}(z-a)^{k+1} + ...,$$

with $b_k \neq 0$ for some $k \geqslant 1$. It follows that

$$\frac{f(\phi(z))}{f(z)} = \left(\frac{\phi(z)-a}{z-a}\right)^k \left(\frac{b_k + b_{k+1}(\phi(z)-a) + b_{k+2}(\phi(z)-a)^2 + ...}{b_k + b_{k+1}(z-a) + b_{k+2}(z-a)^2 + ...}\right).$$

Since $\phi(a) = a$,

$$\lim_{z\to a} \frac{\phi(z)-a}{z-a} = \phi'(a).$$

Also, $\lim_{z\to a}(b_k + b_{k+1}(\phi(z)-a) + b_{k+2}(\phi(z)-a)^2 + ...) = b_k$ and $\lim_{z\to a}(b_k + b_{k+1}(z-a) + b_{k+2}(z-a)^2 + ...) = b_k$. Therefore

$$\begin{aligned} \lim_{z\to a} \frac{f(\phi(z))}{f(z)} &= \lim_{z\to a} \left(\frac{\phi(z)-a}{z-a}\right)^k \left(\frac{b_k + b_{k+1}(\phi(z)-a) + b_{k+2}(\phi(z)-a)^2 + ...}{b_k + b_{k+1}(z-a) + b_{k+2}(z-a)^2 + ...}\right) \\ &= (\phi'(a))^k \cdot 1 \\ &= (\phi'(a))^k. \end{aligned}$$

However, $\frac{f(\phi(z))}{f(z)} = \lambda$, so $\lambda = (\phi'(a))^k$.

□

Corollary 1. *If ϕ is a loxodromic or elliptic linear fractional transformation, then $\sigma_0(C_\phi)$ has empty interior.*

Unlike the case of $H^2(\mathbb{D})$, where all analytic $\phi : \mathbb{D} \to \mathbb{D}$ induce bounded compositions operators, the space $H^2(\mathbb{C}_+)$ has fewer bounded composition operators.

Theorem 3. *The operator C_ϕ is bounded if and only if $\phi(\infty) = \infty$ and $\phi'(\infty) < \infty$, in which case the following equalities hold*

$$\|C_\phi\| = r(C_\phi) = \|C_\phi\|_e = \sqrt{\phi'(\infty)}.$$

Proof. See Theorems 3.1 and 3.4 of (ELLIOTT; JURY, 2012). □

Corollary 2. *The only bounded composition operators on $H^2(\mathbb{C}_+)$ induced by linear fractional transformations are operators C_ϕ with symbol of form*

$$\phi(w) = aw + b$$

where $a > 0$ and $Re(b) \geqslant 0$

2 The Invariant subspace Problem for Hilbert Spaces

In the first section of this chapter a discussion will be made on the invariant subspace problem. We will cite for which particular cases the problem has a solution and in the second section we will present the concept of universal operator and we will proof a result that relates the concept of universal operator to the invariant subspace problem. This chapter is based principally in (CARMO, 2017) and the others mains references of this chapter are (ENFLO, 1976), (LOMONOSOV, 1973), (CARADUS, 1969).

2.1 The Invariant Subspace Problem

The invariant subspace problem can be stated as the following question: "Do all bounded linear operators T in a Banach space have a non-trivial invariant subspace?". The term invariant subspace means a closed subspace of H, such that the operator T maps the subspace into itself. The term non-trivial means different from $\{0\}$ and H. This problem is easily stated, but still is partly open and it is unknown whom was the first to state it.

For a complex Banach space the answer is negative, because Per H. Enflo proposed a counterexample in 1975 (ENFLO, 1976). The complete article was written in 1981 (ENFLO, 1980), but due to its complexity it was only published in 1987 (ENFLO, 1987).

Proposition 6. *Let T be a bounded linear operator on a Hilbert space H. If T has an eigenvalue λ, then T has a non-trivial invariant subspace.*

Proof. If $T = \lambda I$, we have that all closed subspace in H is invariant for T.

If $T \neq \lambda I$, then there is $x \in H$, such that $Tx \neq \lambda x$, thus $x \notin N(T - \lambda)$, therefore $N(T - \lambda)$ is a subspace different from H. The continuity of $T - \lambda$ implies that $N(T - \lambda) = (T - \lambda)^{-1}\{0\}$ is closed in H. If $x \in N(T - \lambda)$, then $Tx = \lambda x$, and $(T - \lambda)x = 0$. By λ to be an eigenvalue of T, we have that $N(T - \lambda) \neq \{0\}$, thus we have that $N(T - \lambda)$ is a non-trivial invariant subspace of T. □

Corollary 3. *If H is a finite dimensional complex space, then any linear operator T on H have a non-trivial invariant subspace.*

Proof. Since each linear operator T in a finite dimensional complex space with dimension n is similar to an operator M on $\mathbb{C}^n$, we have that T has an eigenvalue, and by Proposition 6, we have that T has a non-trivial invariant subspace. □

Proposition 7. *Any bounded linear operator T in a non-separable Hilbert space has a non-trivial invariant subspace.*

Proof. Let H be a non-separable Hilbert space, and let T be a bounded linear operator on H. We pick a non-zero vector x and we consider the closed subspace M generated by vectors $\{x, Tx, T^2x, ...\}$. Then M is invariant on T and $M \neq \{0\}$. Furthermore, $M \neq H$, because this would contradict the fact of H being non-separable. Thus, any operator T on a non-separable Hilbert space has a non-trivial invariant subspace. □

One of the oldest results on invariant subspace is the Aronszajn and Smith theorem, published in 1954. That theorem says that all compact operators have a non-trivial invariant subspace. A stronger result was proved by Lomonosov in 1973 (LOMONOSOV, 1973).

Definition 7. *Let X be a Banach space, $T \in B(X)$. A subspace $M \subset X$ it is said to be hyper-invariant for T, if M is invariant for all $S \in B(X)$, such that $ST = TS$.*

Lemma 1. *Let X be a Banach space and $\mathbb{A}$ a sub-algebra of $B(X)$, such that*

$$\bigcap_{A \in \mathbb{A}} Lat A = \{\{0\}, X\}.$$

Then, for every compact operator $K \in B(X) - \{0\}$, there is $A \in \mathbb{A}$ and $x_0 \in X - \{0\}$, such that $KAx_0 = x_0$.

Proof. For $y \in X$, we consider $B_1(y)$, the open ball with center y and radius 1. Given that $K \neq 0$, there is a $y \in X$, such that $C = \overline{B_1(y)}$ does not contain 0. Let $x \in X$, we consider $\mathbb{A}x = \{Sx : S \in \mathbb{A}\}$, since $\bigcap_{A \in \mathbb{A}} Lat A = \{\{0\}, X\}$, we have that $\overline{\mathbb{A}x} = X$, for all $x \neq 0$. Given that $0 \notin C$, we have that $\overline{\mathbb{A}x} = X$, for all $x \in C$, therefore $\mathbb{A}x \cap B_1(y)$, for all $x \in C$. Then

$$C \subset \bigcup_{A \in \mathbb{A}} A^{-1}(B_1(y)).$$

Given that C is compact, there are finite sets $\{A_1, A_2, ..., A_n\}$ of $\mathbb{A}$, such that

$$C \subset \bigcup_{i=1}^{n} A_i^{-1}(B_1(y)).$$

Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous map, such that $r^{-1}(\{0\}) = [1, +\infty)$. We define $f : C \to B_1(y)$ by

$$f(x) = \sum_{i=1}^{n} \frac{r(\|A_i x - y\|)}{\sum_{j=1}^{n} r(\|A_j x - y\|)} A_i x.$$

The function f is well defined, because for all $x \in C$, there is a i, with $0 \leqslant i \leqslant n$, such that $A_i x \in B_1(y)$, and thus $r(\|A_i x - y\|) > 0$. We consider $F = K \circ f$. By theorem 1.1.22 of (CHALENDAR; PARTINGTON, 2011), there is $x_0 \in C$, such that $F(x_0) = x_0$. We define $A \in \mathbb{A}$ by

$$Au = \sum_{i=1}^{n} \frac{r(\|A_i x_0 - y\|)}{\sum_{j=1}^{n} r(\|A_j x_0 - y\|)} A_i u.$$

We easily see that $Ax_0 = f(x_0)$ and that $KAx_0 = F(x_0) = x_0$, with $x_0 \neq 0$, because $x_0 \in C$.

□

Theorem 4. *(Lomonosov) Let $T \in B(X) - \mathbb{C}I$. If there exists a non-zero compact operator K, such that $TK = KT$, then T has a non-trivial hyper-invariant subspace.*

Proof. Arguing by contradiction, assume that the algebra $\mathbb{A} = \{A \in B(X) : AT = TA\}$ does not have a non-trivial invariant subspace. Then by Lemma there is $A \in \mathbb{A}$, such that

$$Ker(I - KA) \neq \{0\}.$$

Since KA is compact, the dimension of $Ker(I - KA)$ is finite, and for $KA \in \mathbb{A}$, we have that T has a non-trivial invariant subspace, thus T has an eigenvalue. Given that, all eigenspaces of T lie in $\bigcap_{A \in \mathbb{A}} LatA$, we have a contradiction. □

To the present moment it is not known that if all bounded linear operator on a infinite dimensional separable complex Hilbert space have a non-trivial invariant subspace, thus the invariant subspace problem can be stated as:

Invariant Subspace Problem - Every bounded linear operator T in a separable Hilbert space has a non-trivial invariant subspace.

2.2 Universal Operators

This section is based on (CARADUS, 1969).

Definition 8. *For any Banach space X, let $B(X)$ be the space of continuous endomorphisms of X. An operator $U \in B(X)$ is called universal if, for all $T \in B(X)$, a non-null multiple of T is similar to a part of U. That is, there is $\lambda \in \mathbb{C}$, $\lambda \neq 0$, a closed subspace $X_0 \subseteq X$, such that $UX_0 \subseteq X_0$, and a linear homeomorphism $\phi : X \to X_0$, such that $\lambda T = \phi^{-1}(U/X_0)\phi$.*

Theorem 5. *Let X be a separable Hilbert space and let $U \in B(X)$. If U has the following properties:*

- *The kernel of U has infinite dimension*
- *The range of U is the space X*

Then U is universal.

Proof. We start by building operators, V, W in $B(X)$, such that $UV = I$, $UW = 0$, $Ker(W) = \{0\}$, $R(W)$ is closed and $R(W) \perp R(V)$. We consider $\hat{U}$ the restriction of U in $Ker(U)^{\perp}$ and we define $V = \hat{U}^{-1}$. We pick an orthornormal basis $\{e'_n\}$ for $Ker(U)$ and we defined $We_n = e'_n$, where $\{e_n\}$ is an orthornormal basis of X. It is obvious that V and W have the properties required. Let $T \in B(X)$. We take $0 \neq \lambda \in \mathbb{C}$, such that $|\lambda|\|T\|\|V\| < 1$. We introduce $\phi = \sum_{k=0}^{\infty} \lambda^k V^k W T^k$. By choice of λ, this series converge in $B(X)$. It is also evident that

1. $U\phi = \lambda\phi T$
2. $\phi = \lambda V\phi T + W$

By (1), we have that $R(\phi)$ is invariant for U. To finish the proof it remains to show that ϕ is a homeomorphism. We suppose that $\phi(x) = 0$. Then, based on (2) and the relation $R(W) \perp R(V)$, we have that $V\phi Tx = W(x) = 0$. Since W is invertible, it follows that $x = 0$, thus $\phi : X \to R(\phi)$ is an isomorphism.

Let (x_n) be a sequence of elements of X, such that $\phi(x_n) \to y$. From (2), we have $\lambda V\phi Tx_n + Wx_n \to y$. Thus $Wx_n \to Py$, where P is the orthogonal projection onto $R(W)$. Since $R(W)$ is closed, there is x, such that $Wx_n \to Wx$. Then $x_n \to x$ and therefore $\phi(x_n) \to \phi(x) = y$. Thus $R(\phi)$ is a Hilbert space and by Open Map Theorem ϕ is an open map, therefore ϕ is a homeomorphism. □

The next theorem establishes a relation between the concept of universal operator and the invariant subspace problem.

Theorem 6. *Let X be a Hilbert space and let $U \in B(X)$ be a universal operator. Then the following are equivalent:*

1. *Every non-null $T \in B(X)$ has a non-trivial invariant subspace.*
2. *For all $M \subset X$, such that M is invariant for U, and isomorphic to X, the operator $U|_M$ has a non-trivial invariant subspace.*

Proof. Suppose (1) true. If $M \subset X$, is an invariant subspace for U, and isomorphic at X, we have, by (1) that $U|_M$ has a non-trivial invariant subspace, because X and M are isomorphic.

Suppose (2) true. If $T \in B(X)$ is non-null, then there is non-null λ and $M \subset X$, such that M is invariant for U, isomorphic at X and T is similar at $\lambda U|_M$, thus T has a non-trivial invariant subspace.

□

Proposition 8. *Let H be a Hilbert space, if $U \in B(H)$ is universal, then $\sigma_0(U)$ does not have empty interior.*

Proof. See (SCHRODERUS; TYLLI, 2018). □

Corollary 4. *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a linear fractional transformation. If for all $0 \neq \lambda \in \sigma_0(C_\phi)$, we have that $C_\phi - \lambda I$ is universal in $H^2(\mathbb{D})$, then ϕ is hyperbolic and ϕ has no fixed points on $\mathbb{D}$.*

Proof. If $\phi : \mathbb{D} \to \mathbb{D}$ is loxodromofic or elliptic, then ϕ has a fixed point in $\mathbb{D}$, therefore $\sigma_0(C_\phi)$ has empty interior, thus by Proposition 8, we have that $C_\phi - \lambda I$ is not universal for all $\lambda \in \mathbb{C}$.

If $\phi : \mathbb{D} \to \mathbb{D}$ is a parabolic non-automorphism, then by (COWEN, 1983) we have that $\sigma(C_\phi)$ has empty interior, thus by Proposition 8, we have that $C_\phi - \lambda I$ is not universal for all $\lambda \in \mathbb{C}$.

If $\phi : \mathbb{D} \to \mathbb{D}$ is a parabolic automorphism, then by Theorem 5.4.5 of (MARTÍNEZ-AVENDAÑO; ROSENTHAL, 2007) we have that $\sigma(C_\phi)$ has empty interior, thus by Proposition 8, we have that $C_\phi - \lambda I$ is not universal for all $\lambda \in \mathbb{C}$.

Therefore we have that if $C_\phi - \lambda I$ is universal for all $0 \neq \lambda \in \sigma_0(C_\phi)$, then ϕ is hyperbolic and ϕ has not fixed points on $\mathbb{D}$.

□

Corollary 5. *Let $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ be a linear fractional transformation. If for all $0 \neq \lambda \in \sigma_0(C_\phi)$, we have that $C_\phi - \lambda I$ is universal in $H^2(\mathbb{C}_+)$, then ϕ is hyperbolic and ϕ is not a automorphism of $\mathbb{C}_+$.*

In the next chapter, we will show a sufficient condition for translates of linear fractional composition operators to be universal.

2.3 Minimal Subspaces

Definition 9. *Let H be a Hilbert space and $T \in B(H)$. A closed subspace $M \subset H$ is minimal if M is invariant under T and $T|_M$ has not a non-trivial closed invariant subspace.*

Let H be a Hilbert space, $T \in B(H)$ and $x \in H$, we consider $K_x = \bigvee_{n=0}^{\infty} T^n x$, where $\bigvee_{n=0}^{\infty} T^n x = \overline{span\{T^n x : n \in \mathbb{N}\}}$. If T is invertible we can consider

$$\bigvee T^n x = \overline{span\{T^n x : n \in \mathbb{Z}\}}.$$

Proposition 9. *Let H be a Hilbert space, $T \in B(H)$ and M is a minimal subspace. Then, if $x \in M$, and $x \neq 0$, we have that $M = K_x$.*

Proof. Let $x \in M$, and $x \neq 0$, we have that K_x is non-trivial invariant subspace of M, since M is minimal, we have that $K_x = M$.

□

Proposition 10. *Let H be a Hilbert space, $T \in B(H)$ invertible and K_x is a minimal subspace. Then $T^{-1}K_x \subset K_x$, and*

$$K_x = \bigvee_{n=-\infty}^{n=+\infty} T^n x.$$

Proof. Since K_x is minimal, we have that $K_{Tx} = K_x$. Thus $K_x = \bigvee_{n=0}^{\infty} T^n x = T^{-1}(\bigvee_{n=1}^{\infty} T^n x) = T^{-1}K_x$. □

Proposition 11. *Let H be a Hilbert space and $T \in B(H)$. The following are equivalent:*

1. *If $M \subseteq H$ is a finite dimensional subspace and is invariant for T, then $T|_M$ has a non-trivial invariant subspace.*

2. *Let $u \in H \backslash \{0\}$. K_u is minimal if u is an eigenvector of T.*

Proof. Suppose that the item (1) is true. Let $u \in H \backslash \{0\}$, such that K_u is minimal, since K_u is invariant to T; by item (1) we have that K_u is finite dimensional, then we have that K_u has dimension 1 and for this reason u is an eigenvector of T.

Now suppose that the item (2) is true. Let $M \subseteq H$, a finite dimensional subspace that is an invariant to T, and let $u \in M \backslash \{0\}$, if $K_u \neq M$, we have that K_u is a non-trivial invariant subspace of $T|_M$, if $K_u = M$; we have that K_u does not have finite dimension, then u cannot be an eigenvector of T, then, by item (2) we have that $T|_{K_u} = T|_M$ has a non-trivial invariant subspace.

□

3 Universality on Hardy Spaces

The remaining two chapters are based on (CARMO; NOOR, 2021) and highlight the major new contributions of this thesis.

In this chapter we will prove a necessary and sufficient condition for the operator $C_\phi - \lambda I$ being universal, where $\phi : \mathbb{D} \to \mathbb{D}$ (or $\phi : \mathbb{C}_+ \to \mathbb{C}_+$) is a linear fractional transformation.

3.1 Bilateral Weighted Shift

This section is based on (PARTINGTON; POZZI, 2011).

Theorem 7. *Let $T : l^2(\mathbb{Z}, L^2(t_0, t_1)) \to l^2(\mathbb{Z}, L^2(t_0, t_1))$ be the weighted right bilateral shift given by*

$$T\left(\sum_{n\in\mathbb{Z}} x_n e_n\right) = \sum_{n\in\mathbb{Z}} k_n x_{n-1} e_n$$

where each k_n is a positive continuous function on $[t_0, t_1]$ such that $k_n \to b$ uniformly as $n \to -\infty$ and $k_n \to a$ uniformly as $n \to +\infty$. Then for any complex number $a < |\lambda| < b$, the operator $T - \lambda I$ is an universal operator.

Proof. Suppose that $a > 0$. Note that if $f = \sum\limits_{m\in\mathbb{Z}} g_m e_m$, where $\{e_m\}$ is the standard orthonormal basis of $l^2(\mathbb{Z})$, then

$$\|Tf\|_2 \leqslant \sup_{m\in\mathbb{Z}} \|k_m\|_\infty \|f\|_2.$$

Let $\epsilon > 0,\ m_0 \in \mathbb{Z},\ A_\epsilon \subseteq [t_0, t_1],\ \mu(A_\epsilon) > 0$ such that

$$\|k_{m_0}\|_\infty \geqslant \sup_{m\in\mathbb{Z}} \|k_m\|_\infty - \frac{\epsilon}{2},$$

and for $x \in A_\epsilon$,

$$|k_{m_0}(x)| > \|k_{m_0}\|_\infty - \frac{\epsilon}{2}.$$

Also, if we take $f = \chi_{A_\epsilon} e_{m_0}$, then we have

$$\|Tf\| > (\|k_{m_0}\|_\infty - \epsilon)\sqrt{\mu(A_\epsilon)}.$$

So, we have that $\|T\| = \sup\limits_{m\in\mathbb{Z}} \|k_m\|_\infty$. By an inductive argument, we obtain that for $n \in \mathbb{N}^*,\ \|T^n\| = \sup\limits_{m\in\mathbb{Z}} \|k_m k_{m+1} ... k_{m+n-1}\|_\infty$. Since k_n converges uniformly to b as n tends to $+\infty$, it follows that

$$\sup_{m\in\mathbb{Z}} \|k_m k_{m+1} ... k_{m+n-1}\|_\infty^{1/n} \to_{n\to\infty} b.$$

Since T^{-1} is an unitary equivalent to a bilateral right shift with weights $\overline{k}_n = \overline{k}_{-n}^{-1}$, in the same way, one can show that $\|T^{-n}\|^{1/n} \to_{n\to\infty} 1/a$. As 0 does not lie in the spectrum of $T, \sigma(T^{-1}) = \sigma(T)^{-1}$. For that reason, we have that $\sigma(T) \subseteq \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}$; since $\sigma(T^*) = \overline{\sigma(T)}$, we obtain

$$\sigma(T^*) \subseteq \{z \in \mathbb{C} : a \leqslant |Z| \leqslant b\}.$$

Note that if $a = 0$, then we have $\sigma(T^*) \subseteq \overline{\mathbb{D}}(0, b)$.

Let σ be with $a < |\sigma| < b$. If $f = \sum_{n\in\mathbb{Z}} g_n e_n$, in which all $n \in \mathbb{Z}$, g_n is in $L^2(t_0, t_1)$, the equation $T^* f = \lambda f$ gives

$$\lambda \sum_{n=-\infty}^{+\infty} g_n e_n = \sum_{n=-\infty}^{+\infty} g_n k_{n-1} e_{n-1} = \sum_{n=-\infty}^{+\infty} g_{n+1} k_n e_n$$

which implies that for all $n \in \mathbb{Z}$, $\lambda g_n = k_n g_{n+1}$. Setting g_0 to be any function of norm 1 in $L^2(t_0, t_1)$, and defining on (t_0, t_1),

$$g_n = \begin{cases} \lambda^n g_0/(k_0 k_1 ... k_{n-1}) \text{ for } n > 0 \\ \lambda^n g_0 k_n k_{n+1} ... k_{-1} \text{ for } n < 0 \end{cases}$$

we see easily that $\sum_{n\in\mathbb{Z}} \|g_n\|_2^2$ converges.

Hence, $f = \sum_{n=-\infty}^{+\infty} g_n e_n$ is an eigenvector of T^* and $\lambda \in \sigma_p(T^*)$ with infinite multiplicity. We conclude that

$$\sigma(T) = \sigma(T^*) = \{z \in \mathbb{C} : a \leqslant |z| \leqslant b\}.$$

It remains to check that for $\lambda \in (a, b)$, $T - \lambda I$ is bounded below. Suppose towards a contradiction that λ is an approximate eigenvalue of T. So, for each $i \in \mathbb{N}^*$, there is an unit vector $f(i) = \{f_j(i)\}_{j\in\mathbb{Z}}$ such that

$$\|Tf(i) - \lambda f(i)\| < 1/i.$$

Suppose first that $\liminf_{i\to\infty} \|f_0(i)\|_2 = 0$. So, for all $\epsilon > 0$, there is an index i such that $\|f_0(i)\|_2 < \epsilon$ and $\|Tf(i) - \lambda f(i)\| < \epsilon$. Denoting by $h(i) = f(i) - f_0(i)e_0$ (which is, setting to zero the component corresponding to $j = 0$), we have

$$\begin{aligned} \|Th(i) - \lambda h(i)\| &\leqslant \|Tf(i) - \lambda f(i)\| + \|Tf_0(i)e_0 - \lambda f_0(i)e_0\| \\ &< \epsilon + (\|T\| + |\lambda|)\epsilon \end{aligned}$$

and thus there is an approximate eigenvector $h(i)$ of norm 1 such that $h_0(i) = 0$ and $\|Th(i) - \lambda h(i)\| < \epsilon$.

We can write $h(i) = l(i) + r(i)$ where $l(i)$ is supported on the negative integers and $r(i)$ is supported on the positive integers. Since their supports are disjoint, $Th(i) - \lambda h(i)$ is the orthogonal sum of $Tl(i) - \lambda l(i)$ and $Tr(i) - \lambda r(i)$. Since $h(i)$ is of norm 1, one of $l(i)$ and $r(i)$ has norm bigger than 1/2 and thus, we may find a sequence of approximate eigenvectors supported entirely on either the positive or negative integers. We will denote it by $p(i)$, and may suppose without loss of generality that $\|p(i) = 1\|$ and $\|Tp(i) - \lambda p(i)\| < 1/i$. Now

$$T^n p(i) - \lambda^n p(i) = (T^{n-1} + \lambda T^{n-2} + ... + \lambda^{n-1} I)(Tp(i) - \lambda p(i))$$

and so $\|T^n p(i) - \lambda^n p(i)\| < C_n/i$, where C_n depends on λ and the weights but not on i. If $p(i) = \{p_j(i)\}$ is supported on the positive integers, then

$$\begin{aligned}
\|T^n p(i)\|_2 &= \left(\sum_{j=1}^{\infty} \|k_j k_{j+1} ... k_{j+n-1} p_j(i)\|_2^2\right)^{1/2} \\
&\geqslant \left(\inf_{j>0}\left(\min_{u\in[t_0,t_1]} k_j(u)k_{j+1}(u)...k_{j+n-1}(u)\right)^2 \sum_{j=1}^{\infty} \|p_j(i)\|_2^2\right)^{1/2} \\
&= \left[\inf_{j>0}\left(\min_{u\in[t_0,t_1]} k_j(u)k_{j+1}(u)...k_{j+n-1}(u)\right)\right].
\end{aligned}$$

Without loss of generality, we can suppose that $b > 1$. So, for n sufficiently large, we have

$$\inf_{j>0}\left(\min_{u\in[t_0,t_1]} k_j(u)k_{j+1}(u)...k_{j+n-1}(u)\right) > |\lambda^n| + 2.$$

Choosing i larger than C_n, we obtain a contradiction. Applying similar arguments to T^{-1}, we obtain a contraction when T has an approximate eigenvector supported on the negative integers.

Suppose that $\liminf_{i\to\infty} \|f_0(i)\|_2 = d > 0$. Since we have an approximate eigenvector $f(i) = \{f_n(i)\}$ of norm 1 such that $\|Tf(i) - \lambda f(i)\| < 1/i$, then a simple inductive argument shows that there are constants $\{D_n\}_{n\geqslant 0}$ independent of i such that

$$\|f_{n+1}(i) - \frac{k_0 k_1 ... k_n}{\lambda^{n+1}} f_0(i)\|_2 \leqslant \frac{D_n}{i}, \quad for\ n \in \mathbb{N}.$$

If $\|f_0(i)\|_2 \geqslant d/2$, then,

$$\begin{aligned}
\|f_{n+1}(i)\|_2 &\geqslant \|\frac{k_0 k_1 ... k_n}{\lambda^{n+1}} f_0(i)\|_2 - \frac{D_n}{i} \\
&\geqslant \frac{\min_{u\in[t_0,t_1]} k_j(u)k_{j+1}(u)...k_{j+n-1}(u)}{|\lambda^{n+1}|}\|f_0(i)\|_2 - \frac{D_n}{i}.
\end{aligned}$$

Since

$$\frac{\min_{u\in[t_0,t_1]} k_j(u)k_{j+1}(u)...k_{j+n-1}(u)}{|\lambda^{n+1}|} \to_{n\to+\infty} = +\infty,$$

we may find an index n such that $\|f_{n+1}(i)\|_2 \geqslant 2 - D_n/i$. But, as $f(i)$ is a vector of norm 1, if we choose i larger than D_n, we obtain a contradiction. □

3.2 Universality on $H^2(\mathbb{C}_+)$

Lemma 2. *Let $\mu \in (0,+\infty)$, $w \in \mathbb{C}_+$ and $\psi : \mathbb{C}_+ \to \mathbb{C}_+$ be the hyperbolic symbol $\psi(s) = \mu s + w$. Then the composition operator $C_\psi : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ is unitary equivalent to the operator $M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ defined by $(Mf)(t) = \dfrac{1}{\mu}e^{-tw/\mu}f(t/\mu)$.*

Proof. By the Paley-Wiener theorem, the map $P : L^2(\mathbb{R}_+) \to H^2(\mathbb{C}_+)$ defined by

$$(Pf)(s) = \int_{\mathbb{R}_+} f(t)e^{-ts}dt$$

is an isometric isomorphism. Let $f \in L^2(\mathbb{R}_+)$ and $F = P(f) \in H^2(\mathbb{C}_+)$. We get

$$\begin{aligned}(C_\psi Pf)(s) &= C_\psi F(s) = F(\mu s + w) = \int_{\mathbb{R}_+} f(t)e^{-t(\mu s+w)}dt \\ &= \int_{\mathbb{R}_+} f(t)e^{-tw}e^{-t\mu s}dt = \int_{\mathbb{R}_+} \frac{1}{\mu}f(t/\mu)e^{-tw/\mu}e^{-ts}dt \\ &= (PMf)(s).\end{aligned}$$

That is the reason for C_ψ on $H^2(\mathbb{C}_+)$ is unitary equivalent to M on $L^2(\mathbb{R}_+)$.

□

Lemma 3. *For $\mu \in (0,1) \cup (1,+\infty)$, the operator $M : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ defined by $(Mf)(t) = \dfrac{1}{\mu}e^{-tw/\mu}f(t/\mu)$ is unitary equivalent to the weighted left bilateral shift T defined by*

$$T\left(\sum_{n\in\mathbb{Z}} g_n e_n\right) = \sum_{n\in\mathbb{Z}} c_n g_{n+1} e_n$$

on $l^2(\mathbb{Z}, L^2[1,\mu])$ for $\mu > 1$ (resp. $l^2(\mathbb{Z}, L^2[\mu,1])$ for $\mu < 1$), and where

$$c_n(t) := \mu^{-1/2}e^{-tRe(w)\mu^{-n-1}}$$

are positive and continuous functions on $[1,\mu]$ (resp. $[\mu,1]$).

Proof. Let $f \in L^2(\mathbb{R}_+)$ and consider first the case $\mu > 1$. Then after a simple change of variables

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}_+)} &= \int_{\mathbb{R}_+} |f(t)|^2 dt = \sum_{n\in\mathbb{Z}} \int_{\mu^{-n}}^{\mu^{-n+1}} |f(t)|^2 dt = \sum_{n\in\mathbb{Z}} \int_1^{\mu} \mu^{-n} |f(t/\mu^n)|^2 dt \quad (3.1) \\ &= \sum_{n\in\mathbb{Z}} \int_1^{\mu} |\mu^{-n/2} f(t/\mu^n)|^2 dt \end{aligned}$$

define a sequence of unimodular functions by

$$a_n := \begin{cases} \prod_{k=0}^{n} e^{-itIm(w)\mu^{-k}} & \text{if } n \geqslant 0 \\ e^{-itIm(w)} \prod_{k=0}^{-n-1} e^{itIm(w)\mu^{k}} & \text{if } n < 0 \end{cases}$$

and note that $a_n/a_{n+1} = e^{itIm(w)\mu^{-n-1}}$ for all n. Therefore $\Psi : L^2(\mathbb{R}_+) \to l^2(\mathbb{Z}, L^2[1,\mu])$ defined by

$$\Psi(f) = \sum_{n\in\mathbb{Z}} h_n e_n$$

where $h_n(t) = \mu^{-n/2} a_n f(t/\mu^n)$ is an unitary operator. Indeed, the equation (3.1) and the unimodularity of the a_n gives

$$\|\Psi(f)\|^2_{l^2} = \sum_{n\in\mathbb{Z}} \|h_n\|^2_{L^2[1,\mu]} = \|f\|^2_{L^2(\mathbb{R}_+)}$$

for each $f \in L^2(\mathbb{R}_+)$. Hence $f \in L^2(\mathbb{R}_+)$, we get

$$\begin{aligned} (\Psi \circ M)f &= \Psi(\mu^{-1} e^{-tw/\mu} f(t/\mu)) = \sum_{n\in\mathbb{Z}} \mu^{-1} e^{-tw\mu^{-n-1}} \mu^{-n/2} a_n f(t/\mu^{n+1}) e_n \\ &= \sum_{n\in\mathbb{Z}} \mu^{-1/2} e^{-tw\mu^{-n-1}} \frac{a_n}{a_{n+1}} h_{n+1} e_n = \sum_{n\in\mathbb{Z}} \mu^{-1/2} e^{-tw\mu^{-n-1}} \frac{a_n}{a_{n+1}} h_{n+1} e_n \\ &= \sum_{n\in\mathbb{Z}} \mu^{-1/2} e^{-tRe(w)\mu^{-n-1}} h_{n+1} e_n = \sum_{n\in\mathbb{Z}} c_n h_{n+1} e_n \\ &= T(\sum_{n\in\mathbb{Z}} h_n e_n) = (T \circ \Psi) f. \end{aligned}$$

As a result M is unitary equivalent to T when $\mu > 1$. The case $0 < \mu < 1$ is analogous to the only change replacing $l^2(\mathbb{Z}, L^2[1,\mu])$ by $l^2(\mathbb{Z}, L^2[\mu,1])$ and equation (3.1) becoming

$$\begin{aligned} \|f\|^2_{L^2\mathbb{R}_+} &= \sum_{n\in\mathbb{Z}} \int_{\mu^{n+1}}^{\mu^n} |f(t)|^2 dt = \sum_{n\in\mathbb{Z}} \int_{\mu}^{1} \mu^n |f(t\mu^n)|^2 dt = \sum_{n\in\mathbb{Z}} \int_{\mu}^{1} |\mu^{n/2} f(t\mu^n)|^2 dt \\ &= \sum_{n\in\mathbb{Z}} \|h_n\|^2_{L^2[\mu,1]}. \end{aligned}$$

The constants a_n, functions h_n and operator $\Psi : L^2(\mathbb{R}_+) \to l^2(\mathbb{Z}, L^2[\mu, 1])$ are defined as the previous case and the rest of the proof follows verbatim.

□

Theorem 8. *Let $\mu \in (0,1) \cup (1,\infty)$, $w \in \mathbb{C}_+$ and the map $\psi : \mathbb{C}_+ \to \mathbb{C}_+$ be given by $\psi(s) = \mu s + w$. For $\mu > 1$, the operator $C_\psi - \lambda I$ is universal on $H^2(\mathbb{C}_+)$ for λ with $0 < |\lambda| < \mu^{-1/2}$. If $\mu < 1$ then $C_\psi^* - \lambda I$ is universal for all λ with $0 < |\lambda| < \mu^{-1/2}$. We have $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \mu^{-1/2}\} \subset \sigma_p(C_\psi)$ if $\mu > 1$ and $\sigma_p(C_\psi) = \varnothing$ if $\mu < 1$. In both cases $\sigma(C_\psi) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \mu^{-1/2}\}$.*

Proof. By Lemma 2 and Lemma 3 we see that C_ψ is unitary equivalent to a weighted left bilateral shift with weights

$$c_n(t) = \mu^{-1/2} e^{-tRe(w)\mu^{-n-1}}.$$

We first observe that any shift is unitary equivalent to the corresponding right shift with reversed weights

$$c_n'(t) := c_{-n}(t) = \mu^{-1/2} e^{-tRe(w)\mu^{n-1}},$$

and its adjoint is unitary equivalent to the right shift with the same weights $(c_n)_{n\in\mathbb{Z}}$. Hence when $\mu > 1$, we see that $c_n' \to 0$ as $n \to +\infty$ and $c_n' \to \mu^{-1/2}$ as $n \to -\infty$ uniformly on $[1, \mu]$. So $C_\psi - \lambda I$ is universal for $0 < |\lambda| < \mu^{-1/2}$ by Theorem 7. For the case $\mu < 1$, $c_n \to 0$ as $n \to +\infty$ and $c_n \to \mu^{-1/2}$ as $n \to -\infty$ uniformly on $[\mu, 1]$ implies that $C_\psi^* - \lambda I$ is universal for $0 < |\lambda| < \mu^{-1/2}$. The statement on the spectrum and point spectrum follow the Theorem 7.

□

Hence we get the following reformulation of the invariant subspace problem.

Corollary 6. *Let $\psi(s) = \mu s + w$ with $\mu > 1$ and $w \in \mathbb{C}_+$. Then every bounded linear operator on a separable complex Hilbert space has a proper invariant subspace if and only if the minimal non-trivial invariant subspaces of C_ψ in $H^2(\mathbb{C}_+)$ are all one dimensional.*

Proof. By Theorem 8, we have that $C_\psi - \lambda I$ is universal. Then by Proposition 11, we have that for $0 \neq \lambda \in \sigma(C_\psi)$ every bounded linear operator on a separable complex Hilbert space, has a proper invariant subspace if and only if the minimal non-trivial invariant subspaces of $C_\psi - \lambda$ in $H^2(\mathbb{C}_+)$ are all one dimensional.

Note that if M is an invariant subspace for C_ψ, then M is an invariant subspace for $C_\psi - \lambda I$. And if M is an invariant subspace for $C_\psi - \lambda I$ then M is an invariant subspace for C_ψ. Then we have that every bounded linear operator on a separable complex Hilbert space has a proper invariant subspace if and only if the minimal non-trivial invariant subspaces of C_ψ in $H^2(\mathbb{C}_+)$ are all one dimensional.

□

Proposition 12. *Let $\phi : \mathbb{C}_+ \to \mathbb{C}_+$ be a linear fractional transformation. Then $C_\phi - \lambda I : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ is universal for all $0 \neq \lambda \in \sigma(C_\phi)$, if and only if ϕ is a hyperbolic linear fractional transformation of form $\mu z + w$, with $\mu > 1$ and $Re(w) > 0$. Furthermore if ϕ is not of form $\mu z + w$, with $\mu < 1$ and $Re(w) > 0$, we have that $C_\phi - \lambda I$ is not universal for all $\lambda \in \mathbb{C}$.*

Proof. If ϕ is a parabolic linear fractional transformation, then $\sigma(C_\phi)$ has empty interior, thus $C_\phi - \lambda I$ is not universal, for all $\lambda \in \mathbb{C}$.

If $\phi(s) = \mu s + w$, where $\mu < 1$ and $Re(w) > 0$, then by Theorem 8 C_ϕ does not have eigenvectors, thus $C_\phi - \lambda I$ is not universal, for all $\lambda \in \mathbb{C}$.

If $\phi(s) = s + w$, where $Re(w) > 0$, we have that $\sigma(C_\phi)$ has empty interior, then C_ϕ does not have eigenvectors, thus $C_\phi - \lambda I$ is not universal, for all $\lambda \in \mathbb{C}$.

By Theorem 8, if $\phi(s) = \mu s + w$, with $\mu > 1$ and $Re(w) > 0$, then $C_\phi - \lambda I : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ is universal for all $0 \neq \lambda \in \sigma(C_\phi)$.

□

3.3 Universality in $H^2(\mathbb{D})$

Theorem 9. *Let ϕ be a hyperbolic linear fractional selfmap of $\mathbb{D}$. If $\lambda \neq 0$ is in the interior of $\sigma(C_\phi)$, then $C_\phi - \lambda I$ is universal on $H^2(\mathbb{D})$. In particular the ISP has a positive solution if and only if the minimal non-trivial invariant subspaces of C_ϕ are all one dimensional.*

This extends a thirty year old result of Nordgren, Rosenthal and Wintrobe (NORDGREN; ROSENTHAL; WINTROBE, 1987) where they proved this for hyperbolic automorphisms. Let ϕ be a non-automorphic hyperbolic self map of $\mathbb{D}$. Hence ϕ fixes one point $\chi \in \mathbb{T}$ and the other outside the closed unit disk $\overline{\mathbb{D}}$ (possibly ∞). It was shown by (HURST, 1997) that in this case C_ϕ is similar to C_{ϕ_b} where

$$\phi_b(z) = bz + 1 - b, \text{ with } b := \phi'(\chi) \in (0, 1).$$

It is known that a composition operator C_ψ on $H^2(\mathbb{C}_+)$ is unitary equivalent to the weighted composition operator W_Φ on $H^2(\mathbb{D})$ defined by

$$(W_\Phi f)(z) = \frac{1-\Phi(z)}{1-z} f(\Phi(z)) \qquad (3.2)$$

where $\Phi = \gamma^{-1} \circ \psi \circ \gamma : \mathbb{D} \to \mathbb{D}$ and $\gamma : \mathbb{D} \to \mathbb{C}_+$ is the conformal map $\gamma(z) = \dfrac{1+z}{1-z}$ and $\gamma^{-1}(z) = \dfrac{z-1}{z+1}$. Via this equivalence C_{ϕ_b} is equivalent to a scalar multiple of an universal composition operators on $H^2(\mathbb{C}_+)$.

Lemma 4. *Let $b \in (0,1)$ and ϕ_b be the self map of $\mathbb{D}$ given by $\phi_b(z) = bz + 1 - b$. Then C_{ϕ_b} on $H^2(\mathbb{D})$ is unitary equivalent to $b^{-1}C_{\psi_b}$ on $H^2(\mathbb{C}_+)$ where*

$$\psi_b(s) = b^{-1}s + (b^{-1} - 1).$$

Proof. We only have to determine (3.2) with $\Phi = \gamma^{-1} \circ \psi_b \circ \gamma$. We get

$$\begin{aligned}
(\gamma^{-1} \circ \psi_b \circ \gamma)(z) &= \gamma^{-1}(b^{-1}\gamma(z) + (b^{-1} - 1)) = \frac{b^{-1}\gamma(z) + (b^{-1}-1) - 1}{b^{-1}\gamma(z) + (b^{-1}-1) + 1} \\
&= \frac{b^{-1}\frac{1+z}{1-z} + b^{-1} - 2}{b^{-1}\frac{1+z}{1-z} + b^{-1}} = \frac{b^{-1}(1+z) + (b^{-1}-2)(1-z)}{b^{-1}(1+z) + b^{-1}(1-z)} \\
&= \frac{2b^{-1} - 2 + 2z}{2b^{-1}} = bz + (1-b) \\
&= \phi_b(z),
\end{aligned}$$

similarly

$$\frac{1-\Phi(z)}{1-z} = \frac{1-\phi_b(z)}{1-z} = \frac{b(1-z)}{1-z} = b.$$

As a result we see that C_{ϕ_b} is unitary equivalent to $W_\Phi = bC_{\phi_b}$.

For this reason, we are ready to treat the hyperbolic non-automorphism case which, together with the automorphism case, (see [(NORDGREN; ROSENTHAL; WINTROBE, 1987), Theorem 6.2]) proves Theorem 9.

□

Theorem 10. *Let ϕ be a hyperbolic linear fractional map with one fixed point $\chi \in \mathbb{T}$ and the other outside the closed unit disk $\overline{\mathbb{D}}$. If $b := \phi'(\chi) \in (0,1)$, then for each λ with $0 < |\lambda| < b^{-1/2}$ the operator $C_\phi - \lambda I$ is universal on $H^2(\mathbb{D})$.*

Proof. By Lemma 4 and the discussion before it we see that C_ϕ on $H^2(\mathbb{D})$ is similar to $b^{-1}C_{\psi_b}$ on $H^2(\mathbb{C}_+)$. Since $\psi_b(s) = b^{-1}s + (b^{-1} - 1)$ and $b^{-1} > 1$, it follows by Theorem 8 that $C_{\psi_b} - \lambda I$ is universal for $0 < |\lambda| < b^{1/2}$. Therefore $C_\phi - \lambda I$ must be universal for $0 < |\lambda| < b^{-1/2}$.

Notice that Theorem 10 implies that each point with $0 < |\lambda| < \phi'(\chi)^{-1/2}$ is an eigenvalue of C_ϕ of infinite multiplicity and that $\sigma(C_\phi) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \phi'(\chi)^{-1/2}\}$, hence arriving at a result of Hurst [(HURST, 1997), Theorem 8] with a different proof. □

Proposition 13. *Let $\phi : \mathbb{D} \to \mathbb{D}$ a linear fractional transformation, then $C_\phi - \lambda I$ is universal for all $0 \neq \lambda \in Int(\sigma(C_\phi))$ if and only if ϕ is hyperbolic and ϕ does not have a fixed point in $\mathbb{D}$. Furthermore if ϕ is not hyperbolic or has fixed point in $\mathbb{D}$, we have that $C_\phi - \lambda I$ is not universal for all $\lambda \in \mathbb{C}$.*

Proof. By Corollary 4 we have that if $\phi : \mathbb{D} \to \mathbb{D}$ is not hyperbolic or has fixed point on $\mathbb{D}$, then $C_\phi - \lambda I$ is not universal for all $\lambda \in \mathbb{C}$.

By (NORDGREN; ROSENTHAL; WINTROBE, 1987) we have that if $\phi : \mathbb{D} \to \mathbb{D}$ is a hyperbolic automorphism, then $C_\phi - \lambda I$ is universal for all $0 \neq \lambda \in Int(\sigma(C_\phi))$.

By theorem 9, if $\phi : \mathbb{D} \to \mathbb{D}$ is a hyperbolic non-automorphism without fixed point on $\mathbb{D}$, then $C_\phi - \lambda I$ is universal for all $0 \neq \lambda \in Int(\sigma(C_\phi))$. □

4 Minimal invariant subspaces of C_{ϕ_b}

In this chapter we consider the canonical hyperbolic non-automorphism

$$\phi_b(z) = bz + 1 - b,\ 0 < b < 1.$$

The operator C_{ϕ_b} on $H^2(\mathbb{D})$ was studied by (DEDDENS, 1972) where it was shown that the adjoint of C_{ϕ_b} is subnormal and where its spectrum was first determined. An easy inductive argument shows that the formula for the compositions iterates of ϕ_b is strikingly simple:

$$\phi_b^{[n]} = b^n z + 1 - b^n = \phi_{b^n}(z)$$

for each $n \in \mathbb{N}$. For this reason $\phi_b^{[n]}(z)$ is a convex combination between z and 1 for each $n \in \mathbb{N}$ and $\phi_b^{[n]} \to 1$ uniformly on $\mathbb{D}$ as $n \to \infty$. Also note that $C_{\phi_n}^n = C_{\phi_b^{[n]}} = C_{\phi_{b^n}}$. For each non-zero $f \in H^2(\mathbb{D})$, we denote by K_f the cyclic subspace defined by

$$K_f = \overline{span\{C_{\phi_b}^n f : n \geqslant 0\}}.$$

In addition to $a, b \in (0,1)$, we have that

$$\phi_a \circ \phi_b(z) = \phi_a(bz + 1 - b) = abz + a - ab + 1 - a = abz + 1 - ab = \phi_{ab}(z),$$

then $C_{\phi_a} \circ C_{\phi_b} = C_{\phi_b} \circ C_{\phi_a} = C_{\phi_{ab}}$. It is followed by $S = \{C_{\phi_b} : b \in (0,1)\}$ being a multiplicative semigroup of operator.

4.1 Eigenvectors of C_{ϕ_b}

Clearly each one of K_f is a closed invariant subspace for C_{ϕ_b}. If $K_f = H^2(\mathbb{D})$, then f is called a cyclic vector for C_{ϕ_b}. Furthermore K_f is a proper invariant subspace precisely when f is non-cyclic. Now if E is a closed invariant subspace of C_{ϕ_b} then obviously $K_f \subset E$ for all $f \in E$. But if E is also minimal invariant then in fact $K_f = E$ for all $f \in E$. Hence minimal invariant subspaces are necessarily cyclic. Now since $dim K_f = 1$ precisely when f is an eigenvector for C_{ϕ_b}, the ISP has a positive solution if and only if K_f is not minimal whenever f is not an eigenvector of C_{ϕ_b} (see Theorem 9). It suggests that the problem of eliminating as many functions as possible for which K_f is non-minimal. Since such functions must necessarily be not eigenvectors, it is interesting to know the behaviour of eigenvectors.

Example 1. *The main examples of eigenvectors for C_{ϕ_b} are the functions*

$$f_s(z) = (1-z)^s$$

for $s \in \mathbb{C}$. In fact for $z \in \mathbb{D}$ we have

$$(C_{\phi_b} f_s)(z) = (1-(bz+1-b))^s = b^s(1-z)^s = b^s f_s(z).$$

So $C_{\phi_b} f_s = b^s f_s$ for all $b \in (0,1)$ and in Lemma 7 of (HURST, 1997) it was showed that $f_s \in H^2$ if and only if $Re(s) > -1/2$.

For any non-zero $f \in H^2(\mathbb{D})$ we define the subset $A_f \subset (0,1)$ by

$$A_f = \{a \in (0,1) : f \text{ is an egeinvector of } C_{\phi_a}\}.$$

Theorem 11. *Let $f \in H^2(\mathbb{D})$ not be a scalar multiple of f_s for any $s \in \mathbb{C}$. Then either A_f is empty or $A_f = \{c^n : n \in \mathbb{N}\}$ for some $c \in (0,1)$.*

Proof. First we prove that A_f is closed in $(0,1)$ and has empty interior. Let

$$\Sigma = \{z \in \mathbb{D} : f(z) \neq 0\}$$

be the open subset of $\mathbb{D}$ where f is non-vanishing. Suppose A_f in non-empty and there is a sequence $(b_n)_{n\in\mathbb{N}}$ in A_f that converges to some $b \in (0,1)$ with $C_{\phi_{b_n}} f = \lambda_n f$. Then for $z \in \Sigma$ we have

$$\lambda := \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \frac{C_{\phi_{b_n}} f(z)}{f(z)} = \lim_{n\to\infty} \frac{f(b_n z + 1 - b_n)}{f(z)} = \frac{C_{\phi_b} f(z)}{f(z)}$$

which exists and for this reason $b \in A_f$. So A_f is closed in $(0,1)$. Now suppose A_f contains an open interval (s,t). Then let $\lambda : (s,t) \to \mathbb{C}$ be the function defined by

$$f(bz+1-b) = C_{\phi_b} f(z) = \lambda(b) f(z),$$

for $b \in (s,t)$. Then fixing $z \in \Sigma$ shows that λ is continuously differentiable on (s,t). Now differentiating with respect to b while fixing z gives

$$f'(bz+1-b) = \frac{\lambda'(b) f(z)}{z-1}$$

and doing the same with respect to z while fixing b gives

$$f'(bz+1-b) = \frac{\lambda(b) f'(z)}{b}.$$

Therefore $\dfrac{f'(z)}{f(z)} = \dfrac{-s(b)}{1-z}$ where $s(b) = \dfrac{b\lambda'(b)}{\lambda(b)}$ for all $z \in \Sigma$ and $b \in (s,t)$. This implies $f \neq 0$ in $\mathbb{D}$, otherwise f'/f would have a pole in $\mathbb{D}$. So f has a holomorphic logarithm g with $e^g = f$ in $\mathbb{D}$. Derivating the equation $fe^{-g} = 1$ gives $f'e^{-g} = g'fe^{-g} = g'$ or $g' = f'/f$. For that reason $g(z) = s(b)log(1-z) + C$ for a constant C and hence $f(z) = K(1-z)^{s(b)}$ for some constant K. This contradicts our assumption and as a result A_f has empty interior.

We now prove that $A_f = (c^n)_{n\in\mathbb{N}}$ for some $c \in (0,1)$ if $A_f \neq \emptyset$. Note that $a \in A_f$ implies $(a^n)_{n\in\mathbb{N}} \in A_f$. If $a, b \in A_f$ with $C_{\phi_a}f = \lambda_a f$ and $C_{\phi_b}f = \lambda_b f$, then $C_{\phi_{ab}}f = \lambda_a\lambda_b f$ and if $a < b$ we have that

$$C_{\phi_{a/b}}f = \frac{1}{\lambda_b}C_{\phi_{a/b}}C_{\phi_b}f = \frac{1}{\lambda_b}C_{\phi_a}f = \frac{\lambda_a}{\lambda_b}f,$$

where λ_b is non-zero since C_{ϕ_b} is injective. So $a, b \in A_f$ implies $ab \in A_f$ and also if $a < b$ then $a/b \in A_f$. Now, since the complement of A_f in $(0,1)$ is open and dense, there are $a, b \in A_f$ such that $(a,b) \cap A_f = \emptyset$. We define $c := a/b \in A_f$ and hence $(c^n)_{n\in\mathbb{N}} \subset A_f$. Since $c > a$ and $c \notin (a,b)$ imply $c \geqslant b$. We now claim that $a = c^{N+1}$ and $b = c^N$ for some $N \in \mathbb{N}$. Otherwise there is $n \in \mathbb{N}$ such that $c^{n+1} < a < b < c^n$ since $(a,b) \cap A_f = \emptyset$. But $b < c^n$ implies $a = cb < c^{n+1}$ which is a contradiction. So $a = c^{N+1}$. So $a = c^{N+1}$ and $b = c^N$ for some $N \in \mathbb{N}$. This implies that for all $n \in \mathbb{N}$

$$(c^{n+1}, c^n) \cap A_f = \emptyset$$

otherwise if $d \in (c^{n+1}, c^n) \cap A_f$ for some n then $c^{N-n}d \in (a,b) \cap A_f$. The only case that remains is if $d > c$ and $d \in A_f$. But this is also not possible since $c > dc > c^2$. Therefore $A_f = (c^n)_{n\in\mathbb{N}}$.

□

Example 2. *Let $h = f_s + f_{s+\frac{2\pi i}{\log b}}$ for some $b \in (0,1)$ and $R(s) > -1/2$. Then $C_{\phi_b}h = b^s f_s + b^{s+\frac{2\pi i}{\log b}} f_{s+\frac{2\pi i}{\log b}} = b^s h$ because $b^{\frac{2\pi i}{\log b}} = 1$ and hence $b \in A_h$. We will show that $A_h = (b^n)_{n\in\mathbb{N}}$. If some other $a \in A_h$ then*

$$C_{\phi_a}h = a^s f_s + a^{s+\frac{2\pi i}{\log b}} f_{s+\frac{2\pi i}{\log b}} = \lambda h$$

for some $\lambda \in \mathbb{C}$ if and only if $a^{\frac{2\pi i}{\log b}} = e^{2\pi i\frac{\log a}{\log b}} = 1$. So $\log a = n\log b = \log b^n$ for some $n \in \mathbb{Z}$. Hence $a = b^n$ for $n \geqslant 1$ since $a \in (0,1)$. For this reason $A_h = (b^n)_{n\in\mathbb{N}}$.

Let $b \in (0,1)$, the Example 2 leads to an interesting question:

$$\overline{span(\{f_{s+\frac{2k\pi i}{\log b}} : k \in \mathbb{Z})} = Ker(C_{\phi_b} - b^s I)?$$

For Theorem 11, we have an interesting question "let $b \in (0,1)$, and $f \in H^2(\mathbb{D})$ an eigenvector of C_{ϕ_b}, if f is not a scalar multiple of f_s, for some s, what can we say about $\overline{span(\{C_{\phi_a} f : a \in (0,1)\})}$?"

We know that if f is an eigenvector of C_{ϕ_b} with eigenvalue λ, and $a \in (0,1)$, then

$$C_{\phi_b}(C_{\phi_a} f) = C_{\phi_a}(C_{\phi_b} f) = \lambda C_{\phi_a} f,$$

this is, $C_{\phi_a} f$ is an eigenvector of C_{ϕ_b} with eigenvalue λ, then we have that

$$\overline{span(\{C_{\phi_a} f : a \in (0,1)\})},$$

is a closed subspace of $Ker(C_{\phi_b} - \lambda I)$, but we do not know under which condition $\overline{span(\{C_{\phi_a} f : a \in (0,1)\})}$ is finite dimensional, proper or equal to $Ker(C_{\phi_b} - \lambda I)$.

By Example 2 we have if $h = \alpha_1 f_{s+\frac{2k_1\pi i}{\log b}} + \alpha_2 f_{s+\frac{2k_2\pi i}{\log b}} + ... + \alpha_n f_{s+\frac{2k_n\pi i}{\log b}}$, with $\alpha_j \neq 0$, for $j = 1, 2, ..., n$, then $Dim(\overline{span(\{C_{\phi_a} h : a \in (0,1)\})}) = n$.

Proposition 14. *If $f \in C_{\phi_b}(H^2(\mathbb{D}))$, then there is a U open, such that $\overline{\mathbb{D}} - \{1\} \subset U$, and there is f_1 analytic on U, with $f_1(z) = f(z)$, for all $z \in \mathbb{D}$.*

Proof. Let $f \in C_{\phi_b}(H^2(\mathbb{D}))$, then there is $g \in H^2(\mathbb{D})$, such that $C_{\phi_b} g = f$.

Let $D_1 = \{z \in \mathbb{C} : |z - 1 + b^{-1}| < b^{-1}\}$, we have that $\phi_b(D_1) = \mathbb{D}$, then $g \circ \phi_b$ is an analytic extension to f and $\overline{D} - \{1\} \subset D_1$.

□

Note that

- If K_f is minimal, then $K_{C_{\phi_b} f} = K_f$ is minimal.
- If $K_{C_{\phi_b} f}$ is not minimal, then K_f is not minimal.

Then, when we work with C_{ϕ_b}-invariants spaces of form K_f, we can assume that f has an analytic extension to an open subset containing $\overline{D} - \{1\}$.

Theorem 12. *If f is an eigenvector of C_{ϕ_b}, then there is f_1 analytic on $\{z \in \mathbb{C} : Re(z) < 1\}$, such that $f_1(z) = f(z)$, for all $z \in \mathbb{D}$.*

Proof. Let $z_0 \in \mathbb{C}$. If $Re(z_0) < 1$, there is $n \in \mathbb{N}$, such that, $|z_0 - 1 + b^{-n}| < b^{-n}$, by proof of Proposition 14 there is f_n analitic on $\{z \in \mathbb{C} : |z - 1 + b^{-n}| < b^{-n}\}$, such that $f_n(z) = f(z)$, for all $z \in \mathbb{D}$.

□

Corollary 7. *Let f be an eigenvector of C_{ϕ_b} and g the analytic extension on $\{z \in \mathbb{C} : Re(z) < 1\}$. If $f(z_0) = 0$, for any $z_0 \in \mathbb{D}$. then $g(b^n z_0 + 1 - b^n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. For $n \geqslant 0$, we have that $g(b^n z_0 + 1 - b^n) = f(\phi_b(z_0)) = \lambda^n f(z_0) = 0$.

If $n < 0$, since $\lambda^n f \circ \phi_{b_{-n}}(z) = f(z)$, for all $z \in \mathbb{D}$, we have that $\lambda^n f \circ \phi_{b_{-n}}(z)$ is an analytic extension of f on $\{z \in \mathbb{C} : |z - 1 + b^n| < b^n\}$, thus $g(b^n z_0 + 1 - b^n) = \lambda^n f \circ \phi_{b_{-n}}(b^n z_0 + 1 - b^n) = \lambda^n f(z_0) = 0$.

□

Lemma 5. *Let f be an eigenvector of C_{ϕ_b}, with $C_{\phi_b} f = \lambda f$. If $f' \in H^2(\mathbb{D})$, then f' is an eigenvector of C_{ϕ_b} and $C_{\phi_b} f' = \dfrac{\lambda}{b} f'$. In particular, if the n-th derivative $f^{(n)} \in H^2(\mathbb{D})$ for all $n \in \mathbb{N}$ then f must be a polynomial.*

Proof. We have that $\dfrac{f(bz + 1 - b)}{f(z)} = \lambda$, then $\left(\dfrac{f(bz + 1 - b)}{f(z)}\right)' = 0$. Thus, we have that

$$\left(\frac{f(bz + 1 - b)}{f(z)}\right)' = \frac{bf'(bz + 1 - b)f(z) - f(bz + 1 - b)f'(z)}{f(z)^2} = 0,$$

then

$$\frac{f'(bz + 1 - b)}{f'(z)} = \frac{f(bz + 1 - b)}{bf(z)} = \frac{\lambda}{b}.$$

Therefore $C_{\phi_b} f'(z) = f'(bz + 1 - b) = \dfrac{\lambda}{b} f'(z)$.

□

Theorem 13. *If f is a non-zero C_{ϕ_b}-eigenvector that is analytic at the point 1, then $f(z) = K(1 - z)^n$ for some $n \in \mathbb{N}$ and scalar K.*

Proof. Since f is analytic at 1 and on $\{z \in \mathbb{C} : Re(z) < 1\}$ by Theorem 12, it is in particular analytic in a neighborhood of $\overline{\mathbb{D}}$. As a result all derivatives $f^{(n)}$ for $n \in \mathbb{N}$ are analytic in a neighborhood of $\overline{\mathbb{D}}$ and in particular belong to $H^2(\mathbb{D})$. Therefore f is a polynomial by Lemma 5. Clearly f has no zeros in $\mathbb{C}\backslash\{1\}$ by Corollary 7. If $f(1) \neq 0$ then f is a polynomial with no zeros in $\mathbb{C}$ and hence a constant. Otherwise if $f(1) = 0$ then $f(z) = K(1 - z)^n$ for some $n \in \mathbb{N}$ and scalar K. □

We have just seen that eigenvectors of C_{ϕ_b} are somehow determined by their behaviour at the boundary point 1. The next result shows how eigenvalues determine the radial limits at 1 of the corresponding eigenvectors.

Proposition 15. *Let $f \in H^2(\mathbb{D})$ be an eigenvector for C_{ϕ_b} with eigenvalue λ. Then*

$$f^*(1) := \lim_{r\to 1^-} f(r) = \begin{cases} 0, & \textit{if } |\lambda| < 1 \\ \infty, & \textit{if } |\lambda| > 1 \textit{and } f \textit{ does not have zeros in } (-1,1) \\ L, & \textit{if } f \textit{ is constant} \\ & \textit{does not exist otherwise.} \end{cases}$$

Proof. Let λ be in $\mathbb{C}$, with $|\lambda| < 1$ and $f \in H^2(\mathbb{D})$, such that $C_{\phi_b} f = \lambda f$. Fix $t_0 \in (-1,1)$. Consider $K = \sup_{z\in[t_0,\phi_b(t_0)]} (|f(z)|)$. Let $N_0 \in \mathbb{N}$, we know that if $\phi_{b^{N_0+1}}(t_0) < z < 1$, then there exists $z_0 \in [t_0, \phi_b(t_0)]$, with $\phi_{b^n}(z_0) = z$, for some $n > N_0$, that implies $|f(z)| = |f \circ \phi_{b^n}(z_0)| = |C_{\phi_{b^n}} f(z_0)| = |\lambda^n f(z_0)| < |\lambda|^{N_0} K$. Therefore $f^*(1) = 0$.

Let λ be in $\mathbb{C}$, with $|\lambda| > 1$ and $f \in H^2(\mathbb{D})$, such that $C_{\phi_b} f = \lambda f$ and f does not have zeros in $(-1,1)$. Fix $t_0 \in (-1,1)$. Consider $K = \inf_{z\in[t_0,\phi_b(t_0)]} (|f(z)|)$, by f does not have zeros in $(-1,1)$, we have that $K > 0$. Let $\epsilon > 0$, we know that there is $N_0 \in \mathbb{N}$, such that, if $z \in (-1,1)$ and $1 - z < \epsilon$, exists $n > N_0$, and $z_0 \in [t_0, \phi_b(t_0)]$, with $\phi_{b^n}(z_0) = z$, that implies $|f(z)| = |f \circ \phi_{b^n}(z_0)| = |C_{\phi_{b^n}} f(z_0)| = |\lambda^n f(z_0)| > |\lambda|^{N_0} K$. Therefore $f^*(1) = \infty$.

If f is constant, it is obvious that $f^*(1) = L \neq 0$.

Let $f \in H^2(\mathbb{D})$, such that f has zeros in $(-1,1)$ and $C_{\phi_b} f = \lambda f$ with $\lambda > 1$. Let $t_0 \in (-1,1)$ with $f(t_0) = 0$, and let $t_1 \in (-1,1)$ with $f(t_1) \neq 0$. We have that $f(\phi_{b^n}(t_0)) = 0$, for all $n \in \mathbb{N}$, and $f(\phi_{b^n}(t_1)) = \lambda^n f(t_1)$, for all $n \in \mathbb{N}$. By $\phi_{b^n}(t_0) \to 1$ and $\phi_{b^n}(t_1) \to 1$, when $n \to \infty$, we have that $\lim_{t\to 1^-} f(t)$ does not exist.

Let $f \in H^2(\mathbb{D})$, such that is not a constant function and $C_{\phi_b} f = f$. There are t_0 and t_1 in $(-1,1)$, with $f(t_0) \neq f(t_1)$, and we have $f(\phi_{b^n}(t_0)) = f(t_0)$ and $f(\phi_{b^n}(t_1)) = f(t_1)$ for all $n \in \mathbb{N}$. By $\phi_{b^n}(t_0) \to 1$ and $\phi_{b^n}(t_1) \to 1$, when $n \to \infty$, we have that $\lim_{t\to 1^-} f(t)$ does not exist.

Let $f \in H^2(\mathbb{D})$, with $C_{\phi_b} f = \lambda f$ and $|\lambda| = 1$, with $\lambda \neq 1$. Let $t \in (-1,1)$, $f(\phi_{b^n}(t)) = \lambda^n f(t)$, it does not converge, then we have that $f^*(1)$ cannot exist.

□

Example 3. *Recall that* $f_s = (1-z)^s = e^{s\log(1-z)}$ *where* $C_{\phi_b} f_s == b^s f_s$ *for all* $b \in (0,1)$ *and* $f_s \in H^2(\mathbb{D})$ *if and only if* $RE(s) > -1/2$. *Note also that* f_s *has no zeros in* $\mathbb{D}$. *If we write* $\lambda = b^s = e^{s\log b}$ *then* $|\lambda| < 1$ *precisely when* $Re(s) > 0$ *in which case clearly* $f_s^*(1) = 0$. *It is similar to* $|\lambda| > 1$ *precisely when* $Re(s) < 0$ *and in this case* $f_s^*(1) = \infty$. *Finally* $|\lambda| = 1$, *precisely when* $Re(s) = 0$ *in this case* $f_s^*(1) = \lim_{r\to 1^-} e^{iIm(s)\log(1-r)}$ *does not exist, unless* $Im(s) = 0$ *and in this case* $\lambda = 1$ *and* $f_0 \equiv 1$. *Also* f_s *is analytic at* 1 *if and* *only if $s \in \mathbb{N}$. If $s \notin \mathbb{N}$ and hence f_s is not a polynomial, then*

$$f_s^n = s(s-1)...(s-n+1)(1-z)^{(s-n)}$$

does not belong to $H^2(\mathbb{D})$ if $n \geqslant Re(s) + 1/2$ in accordance with Lemma 5. Finally we five an example of an eigenvector with zeros in $\{z \in \mathbb{C} : Re(z) < 1\}$. Let $h = f_s - f_{s+\frac{2\pi i}{\log b}}$. Then $C_{\phi_b} h = b^s h$ and $h(0) = 0$. Moreover $h(1-b^n) = b^{ns} - b^{ns+\frac{2\pi n i}{\log b}} = 0$ for all $n \in \mathbb{Z}$ in accordance with Corollary 7.

By Proposition 15, if $f^*(1) = L \neq 0$ and f is a non-constant function, then f cannot be an eigenvector for C_{ϕ_b}. In this case, the next result shows that the cyclic subspaces K_f are never minimal invariant.

Theorem 14. *Let $f \in H^2(\mathbb{D})$ with $f^*(1) = L \neq 0$. Then K_f is a minimal invariant subspace for C_{ϕ_b} if and only if f is the constant L.*

If ϕ is some holomorphic self map of $\mathbb{D}$, then the Nevanlinna counting function for ϕ is defined for all $w \in \mathbb{D}\backslash\{\phi(0)\}$ by

$$N_\phi(w) = \sum_{z \in \phi^{-1}\{w\}} log \frac{1}{|z|}$$

where $\phi^{-1}\{w\}$ is the sequence of ϕ-preimages of w repeated according to their multiplicities. If $w \notin \phi(\mathbb{D})$ then $N_\phi(w)$ is defined as 0. We shall need the following change of variables formula used by Shapiro in his seminal work on compact composition operators [(SHAPHIRO, 1987), Corollory 4.4]:

$$\|C_\phi f\|_2^2 = 2\int_{\mathbb{D}} |f'(w)|^2 N_\phi(w) dA(w) + |f(\phi(0))|^2$$

for any f holomorphic on $\mathbb{D}$ and where dA is the normalized area measure on $\mathbb{D}$.

Proof. Since $C_{\phi_b}^n = C_{\phi_b{}^n}$ and $\phi_{b^n}(0) = 1 - b^n$, we get

$$\begin{aligned} \|C_{\phi_b}^n f - L\|_2^2 &= \|C_{\phi_{b^n}}(f - L)\|_2^2 \\ &= 2\int_{\mathbb{D}} |f'(w)|^2 N_{\phi_{b^n}}(w) dA(w) + |f(1-b^n) - L|^2. \end{aligned}$$

We only need to prove that the integral on the right tends to 0 as $n \to \infty$. This is followed by a simple monotone convergence argument. Indeed, first notice that the images $\phi_{b^n}(\mathbb{D}) = b^n\mathbb{D} + (1-b^n)$ are open disks of decreasing radii b^n with centres $1-b^n$ tending to 1. Therefore for each $w \in \mathbb{D}$ we have $w \notin \phi_{b^n}(\mathbb{D})$ and hence $N_{\phi_{b^n}}(w) = 0$ for every sufficiently large n. So $N_{\phi_{b^n}}$ is a monotonically decreasing positive function on $\mathbb{D}$ with pointwise limit 0. Hence the integral above vanishes as claimed, and with $f(1-b^n) \to L$ as $n \to \infty$, this implies that $C^n_{\phi_b} f \to L$ in $H^2(\mathbb{D})$. Therefore the constant $L \in K_f$ and K_f is minimal if and only if $K_f = \mathbb{C}$, in this case f must be the constant L.

□

As a consequence, we have the following general results which include functions for which $f^*(1)$ is zero or does not exist.

Corollary 8. *Let $p \geqslant 0$, q any real number and suppose $f \in (1-z)^{p+iq} H^2(\mathbb{D})$ such that*

$$\lim_{r \to 1^-} \frac{f(r)}{(1-r)^{p+iq}} = L \neq 0. \qquad (4.1)$$

Then K_f contains the eigenvector $(1-z)^{p+iq}$ and therefore K_f is minimal invariant if and only if $f = L(1-z)^{p+iq}$.

Observe that $p = q = 0$ gives Theorem 14, while $p > 0$ implies $f^*(1) = 0$ and $p = 0$ but $q \neq 0$ implies $f^*(1)$ does not exist.

Proof. The hypothesis says that $f(z) = (1-z)^{p+iq} g(z)$ where $g \in H^2(\mathbb{D})$ and $g^*(1) = L \neq 0$. Hence $(1-z)^{p+iq} \in H^\infty(\mathbb{D})$ and Theorem 14 applied to g gives

$$\begin{aligned} \left\| \frac{C^n_{\phi_b} f}{b^{n(p+iq)}} - L(1-z)^{p+iq} \right\|_2^2 &= \|(1-z)^{p+iq}(C^n_{\phi_b} g - L)\|_2^2 \\ &\leqslant 2^p \|C^n_{\phi_b} g - L\|_2^2 \to 0 \end{aligned}$$

as $n \to \infty$. Therefore $C^n_{\phi_b} f / b^{n(p+iq)}$ tends to the eigenvector $L(1-z)^{p+iq}$ in $H^2(\mathbb{D})$ which implies $(1-z)^{p+iq} \in K_f$ and K_f is minimal if and only if $f = L(1-z)^{p+iq}$.

□

Corollary 9. *If $f \in H^2(\mathbb{D})$ is analytic on a neighborhood of 1 with $f(1) = 0$, then K_f is minimal if and only if f is an eigenvector.*

Proof. There must be an integer $k > 0$, a neighborhood U of 1 and a function g holomorphic on U with $g(1) = L \neq 0$ such that $f = (1-z)^k g$. Now

$$C_{\phi_{a^n}} = b^{nk}(1-z)^k (g \circ \phi_{b^n})$$

and if n is sufficiently large, say $n > n_0$ then $\phi_{b^n}(\mathbb{D}) = b^n \mathbb{D} + (1-b^n) \subset U$. Hence $g \circ \phi_{b^n}$ is a bounded holomorphic function on $\mathbb{D}$ for $n > n_0$ with $g \circ \phi_{b^n}(1) = L$. Now applying Corollary 8 with $h := C_{\phi_{b^n}} f / b^{nk}$ for some $n > n_0$ and $p + iq = k$ implies the eigenvector $(1-z)^k \in K_h \subset K_f$ which concludes the proof.

□

Finally we consider a class of functions for which condition (4.1) does not hold for any value of p or q. For $a > 0$, let E_a denote the singular shift-invariant subspace

$$E_a = e^{a\frac{z+1}{z-1}} H^2(\mathbb{D}).$$

It is clear that $E_a \subset E_{a'}$ if $a' < a$ because $e^{a'\frac{z+1}{z-1}}$ divides $e^{a\frac{z+1}{z-1}}$ as an inner function. Cowen and Wahl (see [(COWEN; WAHL, 2014), Theorem 5]) showed that if ϕ is any self-map of the disk with $\phi(1) = 1$ and $\phi'(1) \leqslant 1$, then each E_a is an invariant subspace for C_ϕ. In particular $C_{\phi_b} E_a \subset E_a$ for all $a > 0$. It is clear that $f^*(1) = 0$ for all $f \in E_a$ and that no such f satisfies (4.1) for any values of p or q. The next result shows that even in this case we get non-minimality of the corresponding cyclic subspace K_f.

Proposition 16. *For any $a > 0$ and $f \in E_a$ non-zero, the cyclic subspace K_f is not minimal invariant for C_{ϕ_b}.*

Proof. First note that

$$C_{\phi_b} e^{a\frac{z+1}{z-1}} = e^{\frac{a}{b}\frac{bz-a+2}{z-1}} = e^{\frac{a}{b}\frac{z+1}{z-1}} e^{\frac{a}{b}(b-1)} \in E_{a/b}$$

which implies that $C_{\phi_b}^n E_a \subset E_{a/b^n}$ for all $n \geqslant 1$ and clearly $E_{a/b^n} \subset E_a$. Then we prove that for each non-zero $f \in H^2(\mathbb{D})$ there is an integer N large enough (depending on f) such that $f \notin E_N$. Otherwise, the inner part of f would be divisible by each of the singular inner functions $I_n(z) := e^{n\frac{z+1}{z-1}}$ for $n \in N$. But this implies that

$$2\pi n = \mu_n(\{1\}) \leqslant \mu_f(\{1\})$$

for all $n \in \mathbb{N}$, where μ_n and μ_f are the singular measures on $\mathbb{T}$ corresponding to I_n and f respectively (see [(MARTÍNEZ-AVENDAÑO; ROSENTHAL, 2007), Theorem 2.6.7]). Hence $\mu_f(\{1\}) = \infty$ which is a contradiction. Now let $f \in E_a \backslash \{0\}$ for some $a > 0$ and suppose $f \notin E_N$ for some $N > a$. Then there is n_0 such that $C_{\phi_b}^n f \in E_{a/b^n} \subset E_N$ for all $n \geqslant n_0$. So if $g := C_{\phi_b}^n f$, then $K_g \subset E_N$ which implies that $f \notin K_g$. Therefore K_g is a proper closed invariant subspace of K_f under C_{ϕ_b} and hence K_f is not minimal.

□

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