DANIELA MOURA PRATA DOS SANTOS

REPRESENTATIONS OF QUIVERS AND VECTOR BUNDLES OVER PROJECTIVE SPACES

REPRESENTAÇÕES DE QUIVERS E FIBRADOS VETORIAIS SOBRE ESPAÇOS PROJETIVOS

CAMPINAS
2012

Processo FAPESP número 07/07469-3

UNIVERSIDADE ESTADUAL DE CAMPINAS
INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E COMPUTAÇÃO CIENTÍFICA

DANIELA MOURA PRATA DOS SANTOS

REPRESENTATIONS OF QUIVERS AND VECTOR BUNDLES OVER PROJECTIVE SPACES

REPRESENTAÇÕES DE QUIVERS E FIBRADOS VETORIAIS SOBRE ESPAÇOS PROJETIVOS

Doctorate Thesis presented at the Institute of Mathematics, Statistics and Computer Science of the State University of Campinas to obtain the Ph.D grade in Mathematics

Tese de Doutorado apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas para obtenção do título de Doutor em Matemática

SUPERVISOR: PROFESSOR MARCOS BENEVENUTO JARDIM

ORIENTADOR: PROF. DR. MARCOS BENEVENUTO JARDIM

ESTE EXEMPLAR CORRESPONDE À VERSÃO FINAL DA TESE DEFENDIDA PELA ALUNA DANIELA MOURA PRATA DOS SANTOS E ORIENTADA PELO PROF. DR. MARCOS BENEVENUTO JARDIM.

Marcos Jard

Prof. Dr. Marcos Benevenuto Jardim

CAMPINAS, 2012

FICHA CATALOGRÁFICA ELABORADA POR ANA REGINA MACHADO – CRB8/5467
BIBLIOTECA DO INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E
COMPUTAÇÃO CIENTÍFICA – UNICAMP

P887r

Prata, Daniela Moura, 1984-
Representations of quivers and vector bundles over projective spaces = Representações de quivers e fibrados vetoriais sobre espaços projetivos / Daniela Moura Prata dos Santos. – Campinas, SP : [s.n.], 2012.

Orientador: Marcos Benevenuto Jardim.
Tese (doutorado) - Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Representações de Quivers (Matemática). 2. Fibrados vetoriais. 3. Resoluções livres (Álgebra). 4. Koszul, Complexo de. 5. Geometria algébrica. I. Jardim, Marcos Benevenuto,1973-. II. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. III. Título.

Informações para Biblioteca Digital

Título em inglês: Representations of quivers and vector bundles over projective spaces
Palavras-chave em inglês:
Representations of quivers (Mathematics)
Vector bundles
Free resolutions (Algebra)
Koszul complex
Algebraic geometry
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Marcos Benevenuto Jardim [Orientador]
Renato Vidal da Silva Martins
Daniel Levcovitz
Rosa Maria Miró-Roig
Elizabeth Terezinha Gasparim
Data da defesa: 23-03-2012
Programa de Pós-Graduação: Matemática

Tese de Doutorado defendida em 23 de março de 2012 e aprovada

Pela Banca Examinadora composta pelos Profs. Drs.

Prof.(a). Dr(a). MARCOS BENEVENUTO JARDIM

Prof.(a). Dr(a). ROSA MARIA MIRÓ-ROIG

Prof.(a). Dr(a). RENATO VIDAL DA SILVA MARTINS

Prof.(a). Dr(a). DANIEL LEVCOVITZ

Prof.(a). Dr(a). ELIZABETH TEREZINHA GASPARIM

“Buscai, pois, em primeiro lugar, o reino de Deus e a sua justiça, e todas estas coisas vos serão acrescentadas.” (Mt 6:33)

Agradecimentos

Agradeço a Deus em primeiro lugar, que me fez chegar até aqui, e me abençoou em todas as coisas.

Gostaria de agradecer ao Marcos, meu orientador, pela confiança em meu trabalho, pela paciência, pelas idéias e discussões matemáticas e também pela amizade.

Agradeço a todos os professores e funcionários do IMECC, que direta ou indiretamente contribuíram para minha formação. Agradeço a cada um dos meus colegas de turma, com os quais estudei boa parte desses anos e que estavam sempre prontos a me ajudar nas disciplinas e exercícios. Também aos colegas de pesquisa, Patrícia, Paula, Vitor com quem convivi mais nos últimos dois anos.

Agradeço a Prof. Rosa Maria, da Universidade de Barcelona, por sua hospitalidade, pelas várias idéias e pela oportunidade. Também aos professores Laura Costa, Joan Elias e Juan Llopis, pela ajuda matemática e ótima companhia. Agradeço a Helena Soares pelas discussões e exemplos.

Agradeço aos professores da banca pelas sugestões e comentários, que ajudaram a melhorar este trabalho.

Agradeço aos irmãos de Campinas, a cada um com quem convivi. Obrigada pelas orações e pelo carinho.

Agradeço aos meus pais, Décio e Luzia, minha avó Romilde, e meus irmãos pelo amor e carinho. Também a minha família em Itu, Percival, Lena, Ju e Carlão, Laura e Carlos, pelo incentivo, apoio, pelo carinho e pelas orações.

Agradeço ao meu marido, Marcos, por seu amor e compreensão. Pelos momentos que me incentivou a prosseguir no trabalho e nunca me deixou desanimar, e pelos momentos em que me ajudou a descansar.

Finalmente, agradeço à FAPESP pelo apoio financeiro ao longo desses quatro anos.

Resumo

Neste trabalho relacionamos algumas classes de fibrados vetoriais em $\mathbb{P}^n$ com certos *quivers*. Estudamos os fibrados conúcleo, introduzidos por Brambilla [9], e mostramos que a categoria de fibrados conúcleo em $\mathbb{P}^n$ é equivalente a categoria de representações globalmente injetivas do *quiver* de Kronecker. Com isso, usando resultados de Kac [27, 28] sobre decomponibilidade de representações de quivers, demos uma nova demonstração ao teorema de Brambilla. Provamos também que um fibrado conúcleo é decomponível, se a representação associada for decomponível.

Estudamos os fibrados *syzygy*, e mostramos que estão relacionados com representações globalmente sobrejetivas de um certo *quiver*. Mostramos que o fibrado *syzygy* é decomponível se a representação globalmente sobrejetiva associada for decomponível.

Um de nossos principais resultados, mostra como construir fibrados simples de posto n em $\mathbb{P}^n$ com qualquer dimensão homológica. Também provamos que, tomando a feixificação do complexo de Koszul de certa álgebra, os fibrados que aparecem nas sequências exatas curtas deste complexo são simples. Mostramos a existência de um fibrado de posto $n-1$ em $\mathbb{P}^n$, com dimensão homológica dois. Tal fibrado é um caso particular dos fibrados de Tango com pesos, construídos por Cascini [11].

Por fim, feixificando a resolução minimal de uma *compressed Gorenstein Artinian graded algebra*, provamos que os fibrados vetoriais que aparecem nas sequências exatas curtas desta resolução são simples.

Abstract

In this work we relate some classes of vector bundles on $\mathbb{P}^n$ with certain quivers. We study the cokernel bundles introduced by Brambilla [9], and show that the category of cokernel bundles is equivalent to the category of globally injective representations of the Kronecker quiver. With this equivalence and using Kac's results [27, 28] of decomposability of quiver representations, we are able to provide a new proof to Brambilla's theorem. We also prove that a cokernel bundle is decomposable if the associated representation is decomposable.

We study syzygy bundles and relate them to globally surjective representations of a certain quiver. We prove that a syzygy bundle is decomposable if the associated representation is decomposable.

One of our main results shows how to construct simple rank n vector bundles on $\mathbb{P}^n$, with any homological dimension. We prove that taking the sheafification of the Koszul complex of a certain algebra, the bundles that appear in the short exact sequences of the complex are simple. We also prove the existence of a rank $n-1$ vector bundle on $\mathbb{P}^n$ with homological dimension 2. This is a particular case of the weighted Tango bundles constructed by Cascini in [11]. However, our approach is different from Cascini's.

Finally, sheafifying the minimal free resolution of a compressed Gorenstein Artinian graded algebra, we prove that the bundles that appear in the short exact sequence of this resolution are simple.

Contents

Introdução

Fibrados vetoriais sobre variedades algébricas são objetos centrais em geometria algébrica, com importantes aplicações em geometria complexa, teoria de representações e física matemática.

O estudo de fibrados sobre espaços projetivos merece particular destaque. Grothendieck [19], mostrou que todo fibrado em $\mathbb{P}^1$ se decompõe como soma de fibrados de linha. Para $n \geq 2$, conhecemos fibrados indecomponíveis em $\mathbb{P}^n$ de posto maior que 1, como por exemplo o fibrado tangente, de posto n.

Entretanto, em característica zero, são poucos os exemplos conhecidos de fibrados vetoriais em $\mathbb{P}^n$ de posto r, com $2 \leq r \leq n-1$. São eles o fibrado de Horrocks-Mumford, de posto 2 em $\mathbb{P}^4$ [23], também o *parent bundle* de Horrocks, de posto 3 em $\mathbb{P}^5$ [22], os fibrados de correlação nula generalizados, de posto 2 em $\mathbb{P}^3$ [15], os fibrados tipo instanton, de posto $2n$ em $\mathbb{P}^{2n+1}$ [3], o fibrado de posto 3 em $\mathbb{P}^4$ de Sasakura [1], os fibrados de Tango com pesos de posto $n-1$ em $\mathbb{P}^n$ [11], que generalizam o fibrado de Tango [41], e mais recentemente os fibrados de posto 3 em $\mathbb{P}^4$, dados por Kumar, Peterson e Rao [30]. Além disso, Hartshorne em [21] conjecturou o seguinte.

Conjectura. Para $n \geq 7$, não existem fibrados indecomponíveis de posto 2 em $\mathbb{P}^n$.

Nossa motivação vem do problema de encontrar novos exemplos de fibrados indecomponíveis com posto baixo, ou seja, entre 2 e $n-1$.

Seja E um fibrado em $\mathbb{P}^n$. Uma *resolução* de E é uma sequência exata da forma

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

onde cada F_i se quebra como soma direta de fibrados de linha. O menor d das resoluções acima é chamado *dimensão homológica* e denotado por $\mathrm{hd}(E)$. Bohnhorst e Spindler mostraram que todo fibrado em $\mathbb{P}^n$ admite resolução. Eles também mostraram em [8, Corolário 1.7] o seguinte.

Seja E fibrado vetorial em $\mathbb{P}^n$, $n \geq 2$, que não se quebra como soma de fibrados de linha. Então

$$\mathrm{rk}(E) \geq n + 1 - \mathrm{hd}(E).$$

Vemos que se for possível construir fibrados indecomponíveis em $\mathbb{P}^n$ com dimensão homológica alta, então talvez possamos produzir um novo exemplo de fibrado indecomponível em $\mathbb{P}^n$ com posto entre 2 e $n-1$.

Um dos objetivos deste trabalho é também estudar propriedades de alguns tipos de fibrados, como decomponibilidade, usando sua relação com representações de quivers. O primeiro caso abordado é o dos fibrados conúcleo. Fibrados conúcleo foram definidos pela primeira vez por Brambilla em [9]. Um fibrado C é dito *conúcleo* se é dado pela sequência exata curta

$$0 \longrightarrow E^a \longrightarrow F^b \longrightarrow C \longrightarrow 0$$

satisfazendo as seguintes condições

(1) E e F são simples;

(2) $\mathrm{Hom}(F, E) = 0$;

(3) $\mathrm{Ext}^1(F, E) = 0$;

(4) $E^* \otimes F$ é globalmente gerado;

(5) $\dim \mathrm{Hom}(E, F) \geq 3$.

Quando $E = \mathcal{O}_{\mathbb{P}^n}(-1)$ e $F = \mathcal{O}_{\mathbb{P}^n}$ temos exatamente os fibrados Steiner introduzidos por Dolgachev e Kapranov [14], e estudados por muitos autores como Ancona e Ottaviani [4], Cascini [12], Brambilla [10], Jardim e Martins [26]. Soares também estudou fibrados Steiner sobre hiperquádricas em [37], e Miró e Soares introduziram em [35] uma generalização para Steiner bundles, e deram uma caracterização cohomológica para esses fibrados. Essa definição é mais forte que a de fibrados conúcleo, no sentido de que os fibrados Steiner definidos por Miró-Roig e Soares são caso particular dos fibrados conúcleo, mas não o contrário.

No nosso trabalho, provamos uma equivalência entre a categoria de fibrados conúcleo em $\mathbb{P}^n$ e a categoria de representações globalmente injetivas do quiver de Kronecker. Usando essa equivalência e alguns resultados de Kac [27, 29] sobre quivers, fornecemos uma nova demonstração para o seguinte resultado de Brambilla.

Teorema 1 *Seja C um fibrado conúcleo genérico em $\mathbb{P}^n_{\mathbb{C}}$ dado por resolução*

$$0 \longrightarrow E^a \longrightarrow F^b \longrightarrow C \longrightarrow 0\,. \tag{1}$$

Então são equivalentes:

(*i*) *C é simples;*

(*ii*) $a^2 + b^2 - \dim \mathrm{Hom}(E, F)ab \leq 1$.

Para relacionar fibrados conúcleo e representações do quiver de Kronecker, precisamos da seguinte definição. Considere um fibrado conúcleo genérico dado por resolução (1). Fixe uma base $\sigma = \{\sigma_1, \cdots, \sigma_w\}$ de $\mathrm{Hom}(E, F)$ onde $w = \dim \mathrm{Hom}(E, F)$. Seja K_w o quiver de Kronecker com w flechas. Dizemos que uma representação $A = (\{\mathbb{C}^a, \mathbb{C}^b\}, \{A_i\}_{i=1}^w)$ de K_w é $(E, F, \sigma)-$*globalmente injetiva*, se o mapa $\alpha(P) = \sum_{i=1}^w A_i \otimes \sigma_i(P)$ for injetivo para todo $P \in \mathbb{P}^n_{\mathbb{C}}$.

Um dos principais resultados deste trabalho é o seguinte teorema.

Teorema 2 *A categoria de fibrados conúcleo em $\mathbb{P}^n$ dados por resolução* (1) *é equivalente a categoria de representações $(E, F, \sigma)-$globalmente injetivas de K_w.*

Um exemplo de fibrados conúcleo são os fibrados Steiner, introduzidos por Dolgachev e Kapranov em [14], e estudados por Jardim e Martins em [26], Brambilla em [10], entre outros. Usando a caracterização dada por Jardim em Martins [26, Teorema 4.2], e a equivalência entre categorias citada acima, provamos o seguinte.

Proposição 3 *Seja S um fibrado Steiner em $\mathbb{P}^n$ e R a representação do quiver de Kronecker K_{n+1} associada. Então S é indecomponível se e somente se R é indecomponível.*

Usando então a equivalência entre categoria de fibrados conúcleo em $\mathbb{P}^n$ e a categoria de representações globalmente injetivas do quiver de Kronecker, podemos mostrar alguns resultados de decomponibilidade de fibrados usando propriedades de quivers.

Para tentar encontrar mais exemplos, aplicamos esta mesma idéia a outra classe de fibrados, os fibrados syzygy.

Seja $R = k[x_0, \cdots, x_n]$, k corpo algebricamente fechado de característica 0. Seja $\mathfrak{m} = (x_0, \cdots x_n)$ e $\mathbb{P}^n = \mathrm{Proj}R$. Um *fibrado syzygy* [32], é o fibrado $S = \ker \alpha$ dado pela seguinte sequência exata curta

$$0 \longrightarrow S \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_m)^{a_m} \xrightarrow{\alpha} \mathcal{O}^c_{\mathbb{P}^n} \longrightarrow 0 \tag{2}$$

onde $\alpha = (\alpha_1, \cdots, \alpha_m)$ é sobrejetivo para todos os pontos de $\mathbb{P}^n$, α_i é polinômio homogêneo de grau i.

Desta vez, mostramos que existe uma correspondência (não um a um) entre os fibrados syzygy em $\mathbb{P}^n$ e as representações globalmente sobrejetivas do quiver Q_S

$$\begin{tikzcd}
 & \bullet & \\
\bullet \arrow[ur, shift left=2, "l_1^1"] \arrow[ur, shift right=2, "l_{k_1}^1"'] & \cdots & \bullet \arrow[ul, shift right=2, "l_1^m"'] \arrow[ul, shift left=2, "l_{k_m}^m"]
\end{tikzcd}$$

onde $k_j = \binom{n+d_j}{d_j}$, $j = 1, \cdots, m$.

Construímos um funtor G entre a categoria das representações globalmente sobrejetivas de Q_S e a categoria de fibrados syzygy dados por resolução (2). Provamos o seguinte.

Proposição 4 *O funtor G é fiel e essencialmente sobrejetivo.*

Proposição 5 *Seja S um fibrado syzygy dado por resolução* (2)*, e R a representação globalmente sobrejetiva de Q_S associada. Então S é decomponível se R é decomponível.*

Com relação aos fibrados syzygy, esses são os resultados que conseguimos provar até agora. Na tentativa de encontrar os exemplos de fibrados com posto baixo em $\mathbb{P}^n$ com qualquer dimensão homológica, provamos um dos principais teoremas deste trabalho.

Teorema 6 *Seja $n \in \mathbb{Z}, n \geq 4$. Para cada $1 \leq l \leq n-1$, existe um fibrado simples E em $\mathbb{P}^n$, com posto n e* $\mathrm{hd}(E) = l$*, dado por resolução*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}(d_2)^{2n} \xrightarrow{\alpha_2} \cdots$$

$$\cdots \xrightarrow{\alpha_{d_{l-1}}} \mathcal{O}_{\mathbb{P}^n}(d_l)^{2n} \xrightarrow{\alpha_l} E \longrightarrow 0$$

para inteiros $d_0, d_1, \cdots, d_l$ com $d_1 = 0 < d_2 < \cdots < d_l$, $d_0 > 0$.

A prova deste teorema é construtiva e feita por indução. Partimos de um fibrado de dimensão homológica um, e a cada passo aumentamos a dimensão homológica. Usando o resultado de Brambilla e de Bohnhorst e Spindler, garantimos que o fibrado é simples e tem exatamente a dimensão homológica desejada.

Durante estágio na Universidade de Barcelona, trabalhando com a professora Rosa Maria Miró-Roig, aplicamos os resultados sobre fibrados conúcleo a outra resolução. Mostramos o seguinte.

Seja $R = k[x_0, \cdots, x_n]$ o anel de polinômios em $n+1$ variáveis, e $\{f_1, \cdots, f_{n+1}\}$ uma sequência regular genérica de formas de grau d. Seja $I = (f_1, \cdots, f_{n+1})$ o ideal gerado pelas formas e $\mathbb{P}^n = \mathrm{Proj}(R)$. Feixificando o complexo de Koszul temos a seguinte resolução minimal livre

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \xrightarrow{\alpha^t} \mathcal{O}_{\mathbb{P}^n}(-nd)^{n+1} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0 \quad (3)$$

onde $\alpha : \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \to \mathcal{O}_{\mathbb{P}^n}$ é o mapa dado pelas formas.

Teorema 7 *Considere as sequências exatas curtas no complexo* (3).

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-nd)^{\binom{n+1}{n}} \longrightarrow F_1 \longrightarrow 0$$

$$0 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n-1)d)^{\binom{n+1}{n-1}} \longrightarrow F_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-2} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d)^{\binom{n+1}{2}} \longrightarrow F_{n-1} \longrightarrow 0$$

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Então F_i é simples de posto $\mathrm{rk}(F_i) = \binom{n}{i}$, $1 \leq i \leq n-1$.

Usando o mesmo complexo (3), conseguimos construir de maneira diferente um caso particular dos fibrados de Tango com pesos, introduzidos por Cascini [11].

Teorema 8 *Dados $n \geq 3$ e $d \geq 1$, existe um fibrado simples em $\mathbb{P}^n$ com* $\mathrm{rk}(E) = n-1$ *dado por resolução*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{2n-1} \longrightarrow E \longrightarrow 0\,.$$

Por fim, aplicando as mesmas idéias em um outro tipo de resolução, temos o seguinte.

Novamente $R = k[x_0, \cdots, x_n]$, $\mathbb{P}^n = \mathrm{Proj}(R)$. Seja $I = (f_1, \cdots, f_{\alpha_1})$ um ideal gerado por α_1 formas de grau $t+1$, tal que R/I é uma *compressed Gorenstein Artinian graded algebra* de dimensão de mergulho $n+1$ e grau *socle* $2t$. Usando [33, Proposição 3.2], feixificando a resolução da álgebra R/I, temos

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n)^{\alpha_n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n+1)^{\alpha_{n-1}} \longrightarrow \cdots \quad (4)$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-p)^{\alpha_p} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-2)^{\alpha_2} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

onde β é o mapa dado pelas α_1 formas de grau $t+1$ e

$$\alpha_i = \binom{t+i-1}{i-1}\binom{t+n+1}{n+1-i} - \binom{t+n-i}{n+1-i}\binom{t+n}{i-1}, \; para\, i = 1, \cdots, n.$$

Teorema 9 *Considere as sequências exatas curtas em* (4).

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n)^{\alpha_n} \longrightarrow F_1 \longrightarrow 0 \quad (5)$$

$$0 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n+1)^{\alpha_{n-1}} \longrightarrow F_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-p} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-p)^{\alpha_p} \longrightarrow F_{n-(p-1)} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Então para cada $1 \le i \le n-1$, F_i é simples.

Com isso enunciamos os principais resultados de nosso trabalho, que está dividido da seguinte forma.

Capítulo 1 Neste capítulo, damos algumas definições básicas para a leitura dos capítulos subsequentes. Na primeira seção, recordamos as definições de feixe globalmente gerado e regularidade de Castelnuovo-Mumford de feixes. Relembramos também alguns resultados sobre grupo Ext e vemos propriedades das classes de Chern. As referências são [2, 20]. Na seção 1.2 vemos definições e resultados de álgebra comutativa, como resoluções livres de módulos, complexo de Koszul, sequências regulares e *mapping cone procedure*, com referências em [16, 31, 42]. Já na última seção, tratamos sobre quivers e representações e vemos alguns resultados de Kac, que serão úteis em demonstrações posteriores. Temos como referência [6, 27, 28, 29].

Capítulo 2 No segundo capítulo, estudamos a relação entre os fibrados tipo conúcleo e certos tipos de representações do quiver de Kronecker. Vemos que existe uma correspondência um a um entre estes objetos, e estudamos propriedades das categorias relacionadas e do funtor equivalência entre elas. Fibrados conúcleo foram introduzidos por Brambilla em [9]. Eles generalizam alguns tipos de fibrados como os Steiner. Na seção 2.1 definimos representações globalmente injetivas do quiver de Kronecker, que são a chave para mostrarmos a equivalencia entre a categoria de representações globalmente injetivas desse quiver e a categoria de fibrados conúcleo, Teorema 2 (cf. Teorema 2.1.1). Nas seções 2.2 e 2.3 estudamos as propriedades da categoria das representações globalmente injetivas de K_w e propriedades do funtor equivalência, respectivamente. Mostramos em particular que a categoria é exata, e com relação ao funtor, provamos que ele preserva decomponibilidade dos objetos, Lema 2.3.1. Na seção 2.3 ainda fornecemos uma nova prova do Teorema de Brambilla [9, Teorema 4.3], com base na equivalência de categorias e nos resultados de Kac [27, 28]. Esse é o Teorema 1 (cf. Teorema 2.3.4). Por fim, na seção 2.4 estudamos os fibrados Steiner em $\mathbb{P}^n$, caso particular dos fibrados conúcleo. Todos os resultados da seção anterior valem para esses fibrados, mas como a categoria de fibrados Steiner é exata, podemos provar algo mais. Provamos que um fibrado Steiner em $\mathbb{P}^n$ é indecomponível se e somente se, a representação globalmente injetiva de K_n associada é indecomponível, Proposição 3 (cf. Proposição 2.4.5).

Capítulo 3 Neste capítulo, relacionamos os fibrados syzygy com certo tipo de representação de quiver. A idéia é encontrar uma equivalência de categorias como no capítulo anterior. Definimos representação globalmente sobrejetiva do quiver relacionado ao fibrado syzygy. Na seção 3.1, mostramos que o funtor correspondência entre representações globalmente sobrejetivas e fibrados syzygy é fiel e essencialmente sobrejetivo, Proposição 4 (cf. Lemas 3.1.2 e 3.1.3). Também provamos que a categoria das representações globalmente sobrejetivas do quiver associado é fechada por quocientes, e se a representação é decomponível, então o fibrado syzygy correspondente é decomponível, Proposição 5 (cf. Proposição 3.1.8).

Capítulo 4 Finalmente, neste último capítulo, aplicamos os resultados dos capítulos anteriores para provar os principais teoremas. Relembramos a definição de resolução e dimensão homológica, e vemos que todo fibrado em $\mathbb{P}^n$ admite tal resolução. Recordamos resultados de Jardim e Martins [26], e Bohnhorst e Spindler [8], e vemos alguns exemplos de fibrados e suas dimensões homológicas. Na seção 4.1 provamos um dos principais resultados, que nos dá uma maneira de construir fibrados simples de posto n em $\mathbb{P}^n$ com qualquer dimensão homológica, Teorema 6 (ver Teorema 4.1.1). Já na seção 4.2, tomando sequências exatas curtas de um complexo de Koszul feixificado, provamos que os fibrados que aparecem nas sequências exatas curtas deste complexo são simples, Teorema 7 (cf. Teorema 4.2.3). Na seção 4.3, usando esse mesmo complexo, mostramos a exitência de um fibrado simples e de posto $n-1$ em $\mathbb{P}^n$. Com isso fornecemos uma nova prova para um caso particular dos fibrados de Tango com pesos [11], Teorema 8 (cf. Teorema 4.3.1). Por fim, na seção 4.4, partimos de uma resolução de uma *compressed Gorenstein Artinian graded algebra* [33], e tomando as sequências exatas curtas que aparecem nesta resolução, mostramos que os fibrados são simples, usando a relação entre fibrados conúcleo e representações globalmente injetivas do quiver de Kronecker, Teorema 9 (cf. Teorema 4.4.2).

Chapter 1

Preliminaries

In this chapter we give some basic facts about coherent sheaves, commutative algebra and some notions of quiver representation.

In section 1.1 we introduce some definitions about coherent sheaves, in particular the Castelnuovo-Mumford regularity, some properties of cohomology of sheaves and we define simplicity and exceptionality of vector bundles. The main references for this section are [2, 18, 20, 40].

In section 1.2 we give some definitions of commutative algebra, like regular sequences, Koszul complex and mapping cone procedure, that will be very important for some proofs. The main references for this section are [16, 42].

Finally in section 1.3 we define quivers, quiver representations, and the Tits form of a quiver. We also give Kac's results, that will play an important role proving a result about decomposability of cokernel bundles. As references we have [6, 17, 27, 28, 29].

1.1 Coherent sheaves on algebraic varieties

In this section, X will be a nonsingular projective variety over an algebraically closed field k.

Regularity of sheaves Let E be a coherent sheaf on X. We say that E is *globally generated* (or generated by global sections) if the canonical morphism of sheaves

$$\varphi : H^0(X, E) \otimes_k \mathcal{O}_X \to E$$

is surjective, where $\varphi_x(s \otimes h) = hs_x$, for each $x \in X$. Then we have

$$\mathcal{O}_X^n \longrightarrow E \longrightarrow 0$$

for some $n \geq 1$. See [20, Chapter II, Section 5].

Definition 1.1.1 (Castelnuovo-Mumford). *Let E be a coherent sheaf on a projective variety X. We say that E is $n-$regular if*

$$H^i(\mathbb{P}^n, E(n-i)) = 0 \textit{ for all } i > 0.$$

We write "regular" for " $0-$regular"

Note that we always have such n, since $\mathcal{O}_X(1)$ is very ample. Cf. [20, Theorem 5.17].

The Lemma below can be found in [2].

Lemma 1.1.2. *If E is $n-$regular then*

(1) *E is $m-$regular for all $m \geq n$;*

(2) *$H^i(E(n)) = 0$ for all $i > 0$, hence $\dim H^0(E(n)) = P(E, n)$;*

(3) *$E(n)$ is globally generated, meaning that the natural evaluation map $\varepsilon_n : H^0(E(n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-n) \to E$ is surjective;*

(4) *the multiplication maps $H^0(E(n)) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(m-n)) \to H^0(E(m))$ are surjective for all $m \geq n$.*

Here $P(E)$ is the Hilbert polynomial that is given by

$$P(E, l) = \chi(E(l)) = \sum_{i=0}^{\infty} (-1)^i h^i(E(l))$$

where $h^i(F) = \dim H^i(F)$.

Example 1.1.3. *The line bundles $\mathcal{O}_{\mathbb{P}^n}(a)$ are $m-$regular for $m \geq -a$.*

Indeed,

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a + m - i)) = 0, \ 0 < i < n, \ i > n,$$

and

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a + m - n)) = 0, \ m \geq -a.$$

In particular, by Lemma 1.1.2, $\mathcal{O}_{\mathbb{P}^n}(a)$ are globally generated for $a \geq 0$.

Ext groups Let E, F and G be sheaves of $\mathcal{O}_X-$modules. The propositions below can be found in [20, Chapter III, Section 6].

Proposition 1.1.4 (Proposition 6.3,(c))**.** *For any F an $\mathcal{O}_X-$module,*

$$\operatorname{Ext}^i(\mathcal{O}_X, F) \cong H^i(X, F), \text{ for all } i \geq 0.$$

Proposition 1.1.5 (Proposition 6.4)**.** *If* $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ *is a short exact sequence, then for any $\mathcal{O}_X-$module H we have a long exact sequence*

$$0 \longrightarrow \operatorname{Hom}(G, H) \longrightarrow \operatorname{Hom}(F, H) \longrightarrow \operatorname{Hom}(E, H) \longrightarrow \operatorname{Ext}^1(G, H) \longrightarrow \cdots$$

Definition 1.1.6. *Let E be a vector bundle on X. We say E is* simple *if*

$$\operatorname{Hom}(E, E) \simeq k.$$

When E is simple and

$$\operatorname{Ext}^i(E, E) = 0, \ i > 0$$

we say E is exceptional*.*

Chern classes Let $\mathcal{E}$ be a locally free sheaf of rank r. We can find the definition of the Chern classes of $\mathcal{E}$ in Hartshorne's book, [20, Apendix A]. The *total Chern class* of $\mathcal{E}$ is

$$c(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E}) + \cdots + c_r(\mathcal{E}).$$

The *Chern polynomial* is given by

$$c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + \cdots + c_r(\mathcal{E})t^r.$$

They satisfy the following.

(*i*) $c_0(\mathcal{E}) = 1$;

(*ii*) If $\mathcal{E}$ is a line bundle, $\mathcal{E} \simeq \mathcal{L}(D)$ for a divisor D, then

$$c_t(\mathcal{E}) = 1 + Dt;$$

(*iii*) If $f : Y \to X$ is a morphism, then for each i

$$c_i(f^*\mathcal{E}) = f^*c_i(\mathcal{E});$$

(iv) if we have a short exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

of locally free sheaves on X, then

$$c_t(\mathcal{E}) = c_t(\mathcal{E}').c_t(\mathcal{E}'').$$

Genericity Let Y be an algebraic variety. A property P is *generic* if there is some non-empty open subset $U \subset Y$ such that every point $x \in U$ satisfies P. The point x is said to be *generic* for the property P.

Now let E, F be coherent sheaves on $\mathbb{P}^n$. Consider a morphism $\varphi : E^a \to F^b$. Giving a morphism φ is equivalent to give a matrix m of size $b \times a$ with entries in the affine space $\mathrm{Hom}(E, F)$.

Consider the vector bundle C over X given by exact sequence

$$0 \longrightarrow E^a \xrightarrow{m} F^b \longrightarrow C \longrightarrow 0\,.$$

We say that C is *generic* if m is generic for certain property.

1.2 Commutative algebra

The definitions below can be found in [16, 31, 34].

Let R be a commutative ring, and let M be a graded $R-$module.

Definition 1.2.1. *An $R-$module $N \neq 0$ is said to be a $k-$*syzygy *of M if there is an exact sequence of graded $R-$modules*

$$0 \longrightarrow N \longrightarrow F_k \xrightarrow{\varphi_k} F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} M \longrightarrow 0 \tag{1.1}$$

where the modules F_i are free $R-$modules.

Definition 1.2.2. *A complex of $R-$modules is a sequence of modules F_i*

$$\cdots \longrightarrow F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots$$

such that $\varphi_i \circ \varphi_{i+1} = 0$. The homology *of the complex at F_i is the module*

$$\ker \varphi_i / \mathrm{im}\ \varphi_{i+1}.$$

A free resolution *of an $R-$module M is a complex*

$$\mathcal{F} : \cdots \longrightarrow F_n \xrightarrow{\varphi_n} \cdots \longrightarrow F_1 \longrightarrow F_0$$

of free $R-$modules such that $\operatorname{coker}\varphi_1 = M$ *and $\mathcal{F}$ is exact. We say that*

$$\mathcal{F} : \cdots \longrightarrow F_n \xrightarrow{\varphi_n} \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a resolution *of M.*

Now let m be a maximal ideal of R and let $\varphi : F \to M$ be a morphism of $R-$modules where F is free. Then φ is said to be a *minimal homomorphism* if $\varphi \otimes \mathrm{id}_{R/m} : F/mF \to M/mM$ is the zero map when M is free, and an isomorphism when φ is surjective.

Consider the resolution (1.1).

$$0 \longrightarrow N \longrightarrow F_k \xrightarrow{\varphi_k} F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} M \longrightarrow 0$$

When each morphism $\varphi_i, i = 1, \cdots, k$, is a minimal homomorphism, N is called a *minimal $k-$syzygy* of M. If in addition N is free, then the exact sequence (1.1) is called a *minimal free resolution* of M.

Let M be an $R-$module. A sequence of elements $x_1, \cdots, x_n \in R$ is called a *regular sequence* on M if

1. $(x_1, \cdots, x_n)M \neq M$, and
2. for $i = 1, \cdots, n$, x_i is a nonzerodivisor on $M/(x_1, \cdots, x_{i-1})M$.

Let $x_1, \cdots, x_n \in R$. The *Koszul complex* $K(x) = K(x_1, \cdots, x_r)$ is the complex

$$K(x) : \qquad 0 \longrightarrow K_r(x) \longrightarrow \cdots \longrightarrow K_p(x) \longrightarrow \cdots \longrightarrow K_1(x) \longrightarrow K_0(x) \longrightarrow 0$$

where

$K_0(x) = R$;

$K_1(x) = E$, a free module with basis $\{e_1, \cdots, e_r\}$;

$K_p(x) = \wedge^p E$, a free module wit basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}$, $i_1 < \cdots < i_{i_p}$;

$K_r(x) = \wedge^r E$, a free module of rank 1 with basis $e_1 \wedge \cdots \wedge e_r$.

From [31, Theorem 4.6 (a)] we have the following.

Lemma 1.2.3. *Let $R = k[x_0, \cdots, x_n]$ be the ring of polynomials in $n+1$ variables. Let $\{\alpha_1, \cdots, \alpha_{n+1}\}$ be a regular sequence of forms of degree d on R and $I = (\alpha_1, \cdots, \alpha_{n+1})$ the ideal generated by the forms. The Koszul complex*

$$0 \longrightarrow \wedge^{n+1}(R(-d)^{n+1}) \longrightarrow \wedge^{n}(R(-d)^{n+1}) \longrightarrow \cdots \longrightarrow \wedge^{1}(R(-d)^{n+1}) \longrightarrow$$

$$\longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is a free resolution of R/I.

The following result will be very useful in the last chapter. A reference is [42].

Lemma 1.2.4 (Mapping cone procedure)**.** *Consider the short exact sequence*

$$0 \longrightarrow M \xrightarrow{\alpha} N \longrightarrow P \longrightarrow 0$$

of finitely generated $R-$modules, and let us consider free resolutions

$$e_\bullet : 0 \longrightarrow G_{n+1} \xrightarrow{e_{n+1}} G_n \longrightarrow \cdots \xrightarrow{e_1} G_0 \xrightarrow{e_0} M \longrightarrow 0$$

and

$$d_\bullet : 0 \longrightarrow F_{n+1} \xrightarrow{f_{n+1}} F_n \longrightarrow \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \longrightarrow 0\ .$$

Then the morphism α lifts to a morphism between the resolutions $\alpha_\bullet : e_\bullet \to d_\bullet$ and a (non necessarily minimal) free resolution for P is

$$0 \longrightarrow G_{n+1} \xrightarrow{c_{n+2}} G_n \oplus F_{n+1} \xrightarrow{c_{n+1}} \cdots \xrightarrow{c_3} G_1 \oplus F_2 \xrightarrow{c_2} G_0 \oplus F_1 \xrightarrow{c_1} F_0 \xrightarrow{c_0} P \longrightarrow 0$$

where

$$c_{i+1} = \begin{pmatrix} -e_1 & 0 \\ \alpha_i & d_{i+1} \end{pmatrix}.$$

1.3 Quivers and representations

In this section we define quivers, representations of quivers and we give some results proved by Kac. See [6, 28, 29].

Let k be an algebraically closed field with characteristic zero.

Definition 1.3.1. *A* quiver Q *consists on a pair* (Q_0, Q_1) *of sets where* Q_0 *is the set of vertices and* Q_1 *is the set of arrows and a pair of maps* $t, h : Q_1 \to Q_0$ *the tail and head maps.*

An example is the Kronecker quiver K_n with 2 vertices and n arrows.

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "n"'] & \bullet \end{tikzcd} \tag{1.2}$$

Given a quiver we can associate a graph that has the same vertices of the quiver and the arrows without orientation. A subgraph Γ' of a graph Γ is such that the vertices $\Gamma'_0 \subseteq \Gamma$ and the set of edges Γ'_1 is the set of edges of Γ that are not linked to the set of vertices of $\Gamma \backslash \Gamma'$.

Definition 1.3.2. *Let* Γ *be the graph associated to the quiver* Q. *We say* Q *is* hyperbolic *if every conexed subgraph of* $\Gamma' \neq \Gamma$ *is a Dynkin diagram of type ADE or ADE extended. For references see* [27].

For instance the Kronecker K_n is hyperbolic, but the quiver $A_{n,m}$ given by

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "n"'] & \bullet & \bullet \arrow[l, shift right=2, "1"'] \arrow[l, phantom, "\vdots"] \arrow[l, shift left=2, "m"] \end{tikzcd}$$

is not hyperbolic if $m \geq 3$ or $n \geq 3$ because

$$\begin{tikzcd} \bullet \arrow[r, shift left=2, "1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "n"'] & \bullet \end{tikzcd}$$

is not ADE extended type.

Definition 1.3.3. *A* representation *of* Q, $A = (\{V_i\}, \{A_a\})$ *consists of a collection of* $k-$*finite dimensional vector spaces*

$$\{V_i; i \in Q_0\}$$

together with a collection of linear maps

$$\{A_a : V_{t(a)} \to V_{h(a)}; a \in Q_1\}.$$

Given a representation A *of* Q *we associate a dimension vector* $\alpha \in \mathbb{Z}^{Q_0}$ *whose entries are* $\alpha_i = \dim V_i$.

We say $A' = (\{V'_i\}, \{A'_a\})$ is a *subrepresentation* of $(\{V_i\}, \{A_a\})$ if $V'_i \subset V_i,\ \forall\, i \in Q_0$, and $A_{a|V'_{t(a)}} = A'_a,\ \forall\, a \in Q_1$. The representation A such that $V_i = 0,\ \forall\, i \in Q_0$, and $A_a = 0,\ \forall\, a \in Q_1$ is called *zero representation.* A representation A is *simple* if the only subrepresentations are the zero representation and itself.

A *morphism* f between two representations $A = (\{V_i\}, \{A_a\})$ and $B = (\{W_i\}, \{B_a\})$ is a collection of linear maps $\{f_i\}$ such that for each $a \in Q_1$ the diagram bellow is commutative

$$\begin{tikzcd} V_{t(a)} \arrow[r, "A_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\ W_{t(a)} \arrow[r, "B_a"'] & W_{h(a)} \end{tikzcd}$$

The representations of Q form a category $Rep(Q)$, whose objects are representations and the morphisms are the morphisms between representations.

Given a dimension vector we associate a bilinear form, the *Euler form* on $\mathbb{Z}^{Q_0}$, given by

$$< \alpha, \beta >= \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

The *Tits form* is given by

$$q(\alpha) =< \alpha, \alpha > .$$

For instance, if $Q = K_n$ and $\alpha = (a, b) \in \mathbb{Z}^2$ then we have $q(a, b) = a^2 + b^2 - nab$. For the quiver $A_{n,m}$

$$1 \;\; \begin{matrix} \xrightarrow{1} \\ \vdots \\ \xrightarrow[n]{} \end{matrix} \;\; 2 \;\; \begin{matrix} \xleftarrow{1} \\ \vdots \\ \xleftarrow[m]{} \end{matrix} \;\; 3$$

we have

$$q(a, b, c) = a^2 + b^2 + c^2 - (na + mb)c$$

where c corresponds to the vertex 2.

Definition 1.3.4. *(Theorem) Given an hyperbolic quiver Q and $\alpha \in \mathbb{Z}^{Q_0}$ we say that α is a root if $q(\alpha) \leq 1$.*

Definition 1.3.5. *We say that $\alpha \in \mathbb{Z}^{Q_0}$ is Schur when* $\operatorname{Hom}(R, R) = k$ *for R a generic representation with dimension vector α.*

A reference for generic representations and Schur roots is [39]. For more informations about roots and root systems see [28]. Kac related quiver representations with roots.

Proposition 1.3.6. *Let Q be an hyperbolic quiver and $\alpha \in \mathbb{Z}^{Q_0}$ a dimension vector of Q. If $q(\alpha) \leq 1$ then there is an indecomposable representation with dimension vector α. For any quiver Q such that $q(\alpha) > 1$, every representation with dimension vector α is decomposable.*

Proposition 1.3.7. *Let $Q = K_n$ be the Kronecker quiver with $n \geq 3$ and $q(\alpha) \leq 1$. Then α is Schur.*

With these results we can give a criteria in the next chapter, to check when certain bundles are indecomposable.

Chapter 2

Cokernel Bundles

In this chapter we are going to relate a class of bundles to representations of the Kronecker quiver. We give the definition of cokernel bundles that was introduced by Brambilla in [9].

In section 2.1 we prove a one to one correspondence between the cokernel bundles on $\mathbb{P}^n$ and certain representations of the Kronecker quiver. To do so, we introduce the notion of *globally injective representations* of the Kronecker quiver, necessary to prove the equivalence of categories.

In section 2.2 we establish properties of the category of globally injective representations of the Kronecker quiver. The main property is that the category is exact.

In section 2.3 we prove properties of the functor that establishes the equivalence between the category of cokernel bundles and the category of globally injective representations. In particular, this functor is additive and preserves exact sequences. The main property is that the functor preserves decomposability of objects. This allow us to prove that if a quiver representation is decomposable, then the correspondent cokernel bundle is decomposable. We also provide a new proof for a theorem due to Brambilla [9, Theorem 4.3], that characterizes when the generic cokernel bundle is simple.

Finally in the last section, as an example of cokernel bundles, we have the Steiner bundles, that were studied, for instance, by Bohnhorst and Spindler [8], Jardim and Martins [26] among many others. We rewrite some results of the previous section for this case. Moreover, using a particular property of the category of Steiner bundles, we can show that these bundles are decomposable, if and only if, the associated globally injective representations of the Kronecker quiver is decomposable.

Fix $k = \mathbb{C}$, E and F different vector bundles on $\mathbb{P}^n$, $n \geq 2$, such that

(1) E and F are simple, that is, $\mathrm{Hom(E,E)} = \mathrm{Hom(F,F)} = \mathbb{C}$;

(2) $\mathrm{Hom}(F, E) = 0$;

(3) $\mathrm{Ext}^1(F, E) = 0$;

(4) the sheaf $E^* \otimes F$ is globally generated;

(5) $W = \mathrm{Hom}(E, F)$ has dimension $w \geq 3$.

Definition 2.0.8. *A cokernel bundle C on $\mathbb{P}^n$ is the bundle with resolution*

$$0 \longrightarrow E^{\oplus a} \xrightarrow{\alpha} F^{\oplus b} \longrightarrow C \longrightarrow 0 \tag{2.1}$$

$E^{\oplus a} = \mathbb{C}^a \otimes E$, $F^{\oplus b} = \mathbb{C}^b \otimes F$, $b\,\mathrm{rk}F - a\,\mathrm{rk}E \geq n$, $a, b \in \mathbb{N}$ and E, F satisfies the conditions above.

To simplify the notation we will write E^a instead of $E^{\oplus a}$ and so on. The cokernel bundles form a full subcategory of the category of coherent sheaves on $\mathbb{P}^n$ that we denote by $\mathcal{C}(\mathbb{P}^n)$.

Below we have some examples of cokernel bundles.

Example 2.0.9. *1. The bundle given by exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow C \longrightarrow 0$$

with $d \geq 1$. The map α is given by a matrix of forms of degree d.

2. The bundle

$$0 \longrightarrow \Omega^p(p)^c \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^b \longrightarrow C \longrightarrow 0$$

where $0 < p \leq n$ and α is a matrix given by $p-$forms.

Now we show how cokernel bundles are related to quivers.

Fix $\sigma = \{\sigma_1, \cdots, \sigma_w\}$ basis of $\mathrm{Hom}(E, F)$.

Definition 2.0.10. *Let $A = (\{\mathbb{C}^a, \mathbb{C}^b\}, \{A_i\}_{i=1}^w)$ be a representation of K_w. We say that A is $(E, F, \sigma)-$globally injective when the map $\alpha(P) = \sum_{i=1}^w A_i \otimes \sigma_i(P)$ is injective for every $P \in \mathbb{P}^n$.*

The category of $(E, F, \sigma)-$globally injective representations of K_w is a full subcategory of the category of representation of K_w, $Rep(K_w)$, and we denote by $Rep(K_w)^{g.i}$. From now on, since (E, F, σ) are fixed, we will just say globally injective representations. The next result relates the category of globally injective representations of K_w to the category of cokernel bundles.

2.1 Equivalence of Categories $\mathcal{C}(\mathbb{P}^n)$ and $Rep(K_w)^{g.i}$

For this section and the next ones we have fixed a basis σ of $\mathrm{Hom}(E, F)$. Our first main result is the following.

Theorem 2.1.1. *There is an equivalence between the category of globally injective representations of K_w $Rep(K_w)^{g.i}$ and the category of cokernel bundles $\mathcal{C}(\mathbb{P}^n)$ given by resolution* (2.1).

Proof. Let us construct a functor $F : Rep(K_w)^{g.i} \to \mathcal{C}(\mathbb{P}^n)$. Let $A = (\{\mathbb{C}^a, \mathbb{C}^b\}, \{A_i\}_{i=1}^w)$ be a globally injective representation of K_w. We define a map $\alpha : E^a \to F^b$ given by $\alpha = A_1 \otimes \sigma_1 + \cdots + A_w \otimes \sigma_w$. Since A is globally injective, $\alpha(P)$ in injective for every $P \in \mathbb{P}^n$. Then α is injective and $\dim \operatorname{coker} \alpha(\mathrm{P}) = \mathrm{b}\,\mathrm{rkF} - \mathrm{a}\,\mathrm{rkE}$ for each $P \in \mathbb{P}^n$. Therefore $C : \{P \to \operatorname{coker} \alpha(\mathrm{P})\}$ is a bundle that is a cokernel bundle.

Now given two globally injective representations $A = (\{\mathbb{C}^a, \mathbb{C}^b\}, \{A_i\}_{i=1}^w)$ and $B = (\{\mathbb{C}^c, \mathbb{C}^d\}, \{B_i\}_{i=1}^w)$ and $f = (f_1, f_2)$ a morphism between them, let $F(A) = C_1, F(B) = C_2$ be the cokernel bundles and α_1, α_2 the maps associated to A and B respectively. We want a morphism $F(f) : C_1 \to C_2$.

Since we have $f_1 : \mathbb{C}^a \to \mathbb{C}^c, f_2 : \mathbb{C}^b \to \mathbb{C}^d$, we have maps $f_1' = f_1 \otimes 1_E \in \mathrm{Hom}(E^a, E^c)$ and $f_2' = f_2 \otimes 1_F \in \mathrm{Hom}(F^b, F^d)$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & E^a \arrow[r, "\alpha_1"] \arrow[d, "f_1'"'] & F^b \arrow[r, "\pi_1"] \arrow[d, "f_2'"'] & C_1 \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & E^c \arrow[r, "\alpha_2"] & F^d \arrow[r, "\pi_2"] & C_2 \arrow[r] & 0
\end{tikzcd} \tag{2.2}$$

where π_1, π_2 are the projections. Since

$$\alpha_1 = \sum_{i=1}^{w} A_i \otimes \sigma_i \text{ e } \alpha_2 = \sum_{i=1}^{w} B_i \otimes \sigma_i$$

then

$$\alpha_2 f_1^{'} = (\sum_{i=1}^{w} B_i \otimes \sigma_i)(f_1 \otimes 1_E) = \sum_{i=1}^{w} B_i f_1 \otimes (\sigma_i 1_E)$$

$$= \sum_{i=1}^{w} f_2 A_i \otimes (1_F \sigma_i) = (f_2 \otimes 1_F)(\sum_{i=1}^{w} A_i \otimes \sigma_i) = f_2^{'} \alpha_1$$

and the left square commutes.

We want a morphism $\phi : C_1 \to C_2$. Applying the left exact contravariant functor $\mathrm{Hom}(-, C_2)$ to the upper sequence on (2.2) we have

$$0 \longrightarrow \mathrm{Hom}(C_1, C_2) \xrightarrow{\delta_1} \mathrm{Hom}(F^b, C_2) \xrightarrow{\delta_2} \mathrm{Hom}(E^a, C_2) \longrightarrow \cdots$$

Note that $\delta_2(\pi_2 f_2^{'}) = \pi_2 f_2^{'} \alpha_1 = \pi_2 \alpha_2 f_1^{'} = 0$. Then there is a map $\phi \in \mathrm{Hom}(C_1, C_2)$ such that

$$\delta_1(\phi) = \phi \pi_1 = \pi_2 f_2^{'}.$$

Therefore $\phi : C_1 \to C_2$ is a morphism between vector bundles and the right square of the diagram (2.2) commutes.

Until now we have defined a functor F between the categories $Rep(K_w)^{g.i}$ and $\mathcal{C}(\mathbb{P}^n)$. We need to prove that F is fully faithful and essentially surjective.

(i) F is full:

Given $\phi \in \mathrm{Hom}_{\mathcal{C}(\mathbb{P}^n)}(F(A), F(B))$ we want $f = (f_1, f_2) \in \mathrm{Hom}_{Rep(K_w)^{g.i}}(A, B)$ such that $F(f) = \phi$. Let $\tilde{\phi} = \phi \pi_1 \in \mathrm{Hom}(F^b, C_2)$. Let us apply the left exact covariant functor $\mathrm{Hom}(F^b, -)$ to the lower sequence on (2.3).

$$\begin{tikzcd} 0 \arrow[r] & E^a \arrow[r, "\alpha_1"] \arrow[d, dashed, "f_1^{'}"'] & F^b \arrow[r, "\pi_1"] \arrow[d, dashed, "f_2^{'}"'] & C_1 \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & E^c \arrow[r, "\alpha_2"] & F^d \arrow[r, "\pi_2"] & C_2 \arrow[r] & 0 \end{tikzcd} \tag{2.3}$$

We have

$$0 \longrightarrow \mathrm{Hom}(F^b, E^c) \xrightarrow{\rho_1} \mathrm{Hom}(F^b, F^d) \xrightarrow{\rho_2} \mathrm{Hom}(F^b, C_2) \longrightarrow \mathrm{Ext}^1(F^b, E^c)$$

since $\mathrm{Hom}(F^b, E^c) = \mathrm{Ext}^1(F^b, E^c) = 0$ then

$$\rho_2 : \mathrm{Hom}(F^b, F^d) \to \mathrm{Hom}(F^b, C_2) \tag{2.4}$$

is an isomorphism, so there is a morphism $f_2^{'} \in \mathrm{Hom}(F^b, F^d)$ such that

$$\rho_2(f_2^{'}) = \pi_2 f_2^{'} = \phi\pi_1$$

with $f_2^{'} = f_2 \otimes 1_F$ and $f_2 \in \mathrm{Hom}(\mathbb{C}^b, \mathbb{C}^d)$.

Consider $\tilde{\tilde{\phi}} = f_2^{'}\alpha_1 \in \mathrm{Hom}(E^a, F^d)$. Applying the left exact covariant functor $\mathrm{Hom}(E^a, -)$ to the lower sequence on (2.3) we get

$$0 \longrightarrow \mathrm{Hom}(E^a, E^c) \xrightarrow{\gamma_1} \mathrm{Hom}(E^a, F^d) \xrightarrow{\gamma_2} \mathrm{Hom}(E^a, C_2) \longrightarrow \cdots$$

Once we have an exact sequence,

$$\gamma_2(f_2^{'}\alpha_1) = \pi_2 f_2^{'}\alpha_1 = \phi\pi_1\alpha_1 = 0$$

then $f_2^{'}\alpha_1 \in \ker\gamma_2 = \mathrm{im}\,\gamma_1$, and there is a map $f_1^{'} \in \mathrm{Hom}(E^a, E^c)$ such that $\gamma_1(f_1^{'}) = \alpha_2 f_1^{'} = f_2^{'}\alpha_1$ and $f_1^{'} = f_1 \otimes 1_E$ with $f_1 \in \mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^c)$.

Since $\alpha_1 = \sum_{i=1}^{w} A_i \otimes \sigma_i$, $\alpha_2 = \sum_{i=1}^{w} B_i \otimes \sigma_i$, $\alpha_2 f_1^{'} = f_2^{'}\alpha_1$, and σ is a basis then $f_2 A_i = B_i f_1, i = 1, \cdots, w$, thus $f = (f_1, f_2) \in \mathrm{Hom}_{Rep(K_w)^{g.i}}(A, B)$.

Now we need to prove that $F(f) = \phi$. Suppose $F(f) = \overline{\phi}$ such that $\overline{\phi}\pi_1 = \pi_2 f_2^{'} = \phi\pi_1$. Then $(\overline{\phi} - \phi)\pi_1 = 0$ and $C_1 = \mathrm{im}\,\pi_1 \subset \ker(\overline{\phi} - \phi)$ therefore $\overline{\phi} = \phi$.

(ii) F is faithful:

We must show that $F : \mathrm{Hom}_{Rep(K_w)^{g.i}}(A, B) \to \mathrm{Hom}_{\mathcal{C}(\mathbb{P}^n_{\mathbb{C}})}(F(A), F(B))$ is injective. Let $f = (f_1, f_2)$, $g = (g_1, g_2) \in \mathrm{Hom}(A, B)$ be morphisms such that $F(f) = \phi_1 = \phi_2 = F(g)$, that is, $\phi_1 - \phi_2 = 0$.

$$\begin{tikzcd} 0 \arrow[r] & E^a \arrow[r, "\alpha_1"] \arrow[d, "f_1^{'} - g_1^{'}"'] & F^b \arrow[r, "\pi_1"] \arrow[d, "f_2^{'} - g_2^{'}"'] & C_1 \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & E^c \arrow[r, "\alpha_2"] & F^d \arrow[r, "\pi_2"] & C_2 \arrow[r] & 0 \end{tikzcd} \tag{2.5}$$

Given $\phi_1 - \phi_2 = 0 \in \mathrm{Hom}(C_1, C_2)$, doing the same construction as before,

$$0\pi_1 = 0 \in \mathrm{Hom}(F^b, C_2) \simeq \mathrm{Hom}(F^b, F^d)$$

with isomorphism given by ρ_2 in (2.4). Since

$$\rho_2(f_2^{'} - g_2^{'}) = \pi_2 \circ (f_2^{'} - g_2^{'}) = 0$$

then $f_2^{'} - g_2^{'} = 0$ and so $f_2^{'} = g_2^{'}$. Similarly, $0\alpha_1 = 0 \in \mathrm{Hom}(E^a, F^d)$ and

$$\gamma_1(f_1^{'} - g_1^{'}) = \alpha_2(f_1^{'} - g_1^{'}) = 0\alpha_1 = 0.$$

Since γ_1 injective, $f_1^{'} - g_1^{'} = 0$, then $f_1^{'} = g_1^{'}$. Therefore F is faithful.

(iii) F is essentially surjective:

Given C an object of $\mathcal{C}(\mathbb{P}^n)$ we want a globally injective representation A such that $F(A)$ is isomorphic to C.

Since C is a cokernel bundle it is given by resolution

$$0 \longrightarrow E^a \xrightarrow{\alpha} F^b \longrightarrow C \longrightarrow 0\,.$$

We have σ basis of $\mathrm{Hom}(E,F)$ and $\alpha \in \mathrm{Hom}(E^a, F^b) \simeq \mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^b) \otimes \mathrm{Hom}(E,F)$ so

$$\alpha = \sum_{i=1}^{w} A_i \otimes \sigma_i, \text{ with } A_i \in \mathrm{Hom}(\mathbb{C}^a, \mathbb{C}^b), i = 1, \cdots, w.$$

Since C is locally free, $\alpha(P) = \sum_{i=1}^{w} A_i \otimes \sigma_i(P)$ is injective for each $P \in \mathbb{P}^n$, then $A = (\{\mathbb{C}^a, \mathbb{C}^b\}, \{A_i\}_{i=1}^{w})$ is a globally injective representation of $Rep(K_w)$ and $F(A) = C$.

Thus $F : Rep(K_w)^{g.i} \to \mathcal{C}(\mathbb{P}^n)$ is an equivalence of categories.

□

Observation 2.1.2. *Note that the functor F depends on the choice of basis σ. Let σ' be another basis for $Hom(E,F)$. Let F' be the equivalence functor equivalence between the category of (E,F,σ')−globally injective representations of K_w and the cokernel bundles on $\mathbb{P}^n$. Then if G' is the inverse functor of F' we have that the functor $G' \circ F$ gives an equivalence between the categories (E,F,σ) and (E,F,σ')−globally injective representations of K_w.*

2.2 Properties of $Rep(K_w)^{g.i}$

As above, we still have E, F and σ fixed. Let us see some nice properties of the category of globally injective representations of K_w.

Lemma 2.2.1. *The category $Rep(K_w)^{g.i}$ is closed under subobjects.*

Proof. We want to prove that every subrepresentation of a globally injective representation of K_w is also globally injective. Let $R = (\{V_1, V_2\}, \{A_i\}) \in Rep(K_w)^{g.i}$ and $R' = (\{U_1, U_2\}, \{B_i\}) \subset R$ subrepresentation of R. Let h_1, h_2 be the inclusion of U_1 and U_2 on V_1 and V_2 respectively, $P \in \mathbb{P}^n$ and $u \in U_1$ such that

$$(\sum_{i=1}^{w} B_i \otimes \sigma_i(P))(u) = 0.$$

we have

$$0 = h_2(\sum_{i=1}^{w} B_i \otimes \sigma_i(P))(u) = (\sum_{i=1}^{w} h_2 B_i \otimes \sigma_i(P))(u) =$$

$$(\sum_{i=1}^{w} A_i h_1 \otimes \sigma_i(P))(u) = (\sum_{i=1}^{w} A_i \otimes \sigma_i(P))(h_1(u))$$

since R is globally injective and h_1 is an inclusion then $u = 0$ and R' is globally injective as we wanted.

□

Lemma 2.2.2. *Let $f : A \to B$ be a morphism of globally injective representations of K_w. Then* $\ker f$ *and* $\mathrm{im} f$ are both globally injective.

Proof. Once $\ker f$ and $\mathrm{im} f$ are subrepresentations, it follows from the previous Lemma.

□

Definition 2.2.3. *An abelian category $\mathcal{A}$ is said closed under direct summands if given $A \in \mathrm{Obj}(\mathcal{A})$ such that $A = A_1 \oplus A_2$ then $A_1, A_2 \in \mathrm{Obj}(\mathcal{A})$. And $\mathcal{A}$ is closed under extensions if given $A_1, A_2 \in \mathrm{Obj}(\mathcal{A})$ such that*

$$0 \longrightarrow A_1 \longrightarrow A \longrightarrow A_2 \longrightarrow o$$

is an exact sequence, then $A \in \mathrm{Obj}(\mathcal{A})$.

Definition 2.2.4. *A category $\mathcal{A}$ is exact if it is a full subcategory of an abelian category such that $\mathcal{A}$ is closed under direct summands and closed under extensions.*

Lemma 2.2.5. *The category $Rep(K_w)^{g.i}$ is exact.*

Proof. We have that $Rep(K_w)^{g.i}$ is a full subcategory of the abelian category $Rep(K_w)$. We need to prove that $Rep(K_w)^{g.i}$ is closed under direct summands and extensions.

Let $R \in Rep(K_w)^{g.i}$ such that $R = R_1 \oplus R_2$. We want to show that $R_1, R_2 \in Rep(K_w)^{g.i}$. It follows from Lemma 2.2.1 because R_1 and R_2 are subrepresentations of R.

Now if $R_1, R_2 \in Rep(K_w)^{g.i}$ and $R \in Rep(K_w)$ are such that

$$0 \longrightarrow R_1 \xrightarrow{f} R \xrightarrow{g} R_2 \longrightarrow 0$$

we want to prove that $R \in Rep(K_w)^{g.i}$.

Let $R_1 = (\{U_1, U_2\}, \{A_i\})$, $R_2 = (\{V_1, V_2\}, \{B_i\})$, $R = (\{W_1, W_2\}, \{C_i\})$ be the representations and let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be the morphisms. We have the diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ U_1 \arrow[r, "A_i"] \arrow[d, "f_1"'] & U_2 \arrow[d, "f_2"] \\ W_1 \arrow[r, "C_i"] \arrow[d, "g_1"'] & W_2 \arrow[d, "g_2"] \\ V_1 \arrow[r, "B_i"] \arrow[d] & V_2 \arrow[d] \\ 0 & 0 \end{tikzcd}$$

where $f_2 A_i = C_i f_1$ and $g_2 C_i = B_i g_1$, $i = 0, \cdots, n$. Let $P \in \mathbb{P}^n$ and $u \in W_1$ such that $(\sum_{i=0}^{w} C_i \otimes \sigma_i(P))(u) = 0$. So

$$g_2(\sum_{i=0}^{w} C_i \otimes \sigma_i(P))(u) = \sum_{i=0}^{w} g_2 C_i \otimes \sigma_i(P)(u) = \sum_{i=0}^{w} B_i g_1 \otimes \sigma_i(P)(u) = \sum_{i=0}^{w} B_i \otimes \sigma_i(P)(g_1(u)).$$

Since R_2 is globally injective $g_1(u) = 0$ and $u \in \ker g_1 = \operatorname{im} f_1$. Then $u = f_1(u'), u' \in U_1$.

$$0 = \sum_{i=0}^{w} C_i \otimes \sigma_i(P)(u) = \sum_{i=0}^{w} C_i \otimes \sigma_i(P)(f_1(u')) =$$

$$\sum_{i=0}^{w} C_i f_1 \otimes \sigma_i(P)(u') = \sum_{i=0}^{w} f_2 A_i \otimes \sigma_i(P)(u') = f_2(\sum_{i=0}^{w} A_i \otimes \sigma_i(P)(u')).$$

Thus since f_2 is injective $\sum_{i=0}^{w} A_i \otimes \sigma_i(P)(u') = 0$ and since R_1 is globally injective, $u' = 0$, therefore $u = 0$ and follows the result.

□

2.3 Properties of the functor

In this section we prove some properties of the functor equivalence of categories. Again (E, F, σ) are fixed as in the previous sections.

Lemma 2.3.1. *Let R be a globally injective representation of K_w. If R is decomposable, then $F(R)$ is decomposable.*

Proof. Let us suppose $R = R_1 \oplus R_2$. We want to prove that $F(R) \simeq F(R_1) \oplus F(R_2)$. By Lemma 2.2.1, the representations $R_1 = (\{V_1, V_2\}, \{A_i\})$ and $R_2 = (\{W_1, W_2\}, \{B_i\})$ are globally injective. Let $S_1 : \{P \to \operatorname{coker} \alpha_1(P)\}$ and $S_2 : \{P \to \operatorname{coker} \alpha_2(P)\}$ be the associated cokernel bundles, where $\alpha_1(P) = \sum_{i=1}^{w} A_i \otimes \sigma_i(P)$, $\alpha_2(P) = \sum_{i=1}^{w} B_i \otimes \sigma_i(P)$, $P \in \mathbb{P}^n$.

It is enough to prove that $\operatorname{coker}(\alpha_1 \oplus \alpha_2) \simeq \operatorname{coker} \alpha_1 \oplus \operatorname{coker} \alpha_2$.

We have

$$\operatorname{coker}(\alpha_1 \oplus \alpha_2) = \frac{V_2 \oplus W_2}{\operatorname{im}(\alpha_1 \oplus \alpha_2)}$$

It is easy to check that $\operatorname{im}(\alpha_1 \oplus \alpha_2) = \operatorname{im} \alpha_1 \oplus \operatorname{im} \alpha_2$. Let us prove that

$$\frac{V_2 \oplus W_2}{\operatorname{im} \alpha_1 \oplus \operatorname{im} \alpha_2} \simeq \frac{V_2}{\operatorname{im} \alpha_1} \oplus \frac{W_2}{\operatorname{im} \alpha_2}.$$

Consider

$$\begin{array}{rcl} \varphi : \dfrac{V_2 \oplus W_2}{\operatorname{im} \alpha_1 \oplus \operatorname{im} \alpha_2} & \longrightarrow & \dfrac{V_2}{\operatorname{im} \alpha_1} \oplus \dfrac{W_2}{\operatorname{im} \alpha_2} \\ (v, w) + (\operatorname{im} \alpha_1 \oplus \operatorname{im} \alpha_2) & \longmapsto & (v + \operatorname{im} \alpha_1, w + \operatorname{im} \alpha_2) \end{array}$$

Clearly φ is well defined, injective and surjective therefore F preserves direct sums.

$\square$

Lemma 2.3.2. *The functor F is additive.*

Proof. We must prove that $F : \mathrm{Hom}_{\mathcal{R}}(R_1, R_1) \to \mathrm{Hom}_{\mathcal{C}(\mathbb{P}^n_{\mathbb{C}})}(F(R_1), F(R_2))$ is a group homomorphism. In fact, let $R_1 = (\{\mathbb{C}^a, \mathbb{C}^b\}, \{A_i\})$, $R_2 = (\{\mathbb{C}^c, \mathbb{C}^d\}, \{B_i\})$ be globally injective representations and let $f, g \in \mathrm{Hom}(R_1, R_2)$ be a morphism. Let $\varphi = F(f)$ and $\psi = F(g)$. We want to prove that $F(f+g) = \varphi + \psi$. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & E^{\oplus a} \arrow[r, "\alpha_1"] \arrow[d, "1_E \otimes (f_1+g_1)"] & F^{\oplus b} \arrow[r, "\pi_1"] \arrow[d, "1_F \otimes (f_2+g_2)"] & C_1 \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & E^{\oplus c} \arrow[r, "\alpha_2"'] & F^{\oplus d} \arrow[r, "\pi_2"'] & C_2 \arrow[r] & 0
\end{tikzcd}$$

As we did in Theorem 2.1.1, applying the covariant left exact functor $\mathrm{Hom}(-, C_2)$ to the upper sequence, there is a morphism $\theta \in \mathrm{Hom}(C_1, C_2)$ such that $\theta\pi_1 = \pi_2(1_F \otimes (f_2 + g_2))$. On the other way

$$\pi_2(1_F \otimes (f_2 + g_2)) = \pi_2(1_F \otimes f_2 + 1_F \otimes g_2) = \pi_2(1_F \otimes f_2) + \pi_2(1_F \otimes g_2)$$
$$= \varphi\pi_1 + \psi\pi_1 = (\varphi + \psi)\pi_1.$$

Then $\theta\pi_1 = (\varphi + \psi)\pi_1$. Since F is equivalence of categories $\theta = \varphi + \psi$ thus F is additive.

$\square$

Lemma 2.3.3. *The functor F is exact.*

Proof. Let us prove that F preserves exact sequences. Let $R_1 = (\{\mathbb{C}^{a_1}, \mathbb{C}^{b_1}\}, \{A_i\})$, $R_2 = (\{\mathbb{C}^{a_2}, \mathbb{C}^{b_2}\}, \{B_i\})$ and $R_3 = (\{\mathbb{C}^{a_3}, \mathbb{C}^{b_3}\}, \{C_i\})$ be globally injective representations of K_w and let $f : R_1 \to R_2$ and $g : R_2 \to R_3$ be morphisms such that

$$0 \longrightarrow R_1 \xrightarrow{f} R_2 \xrightarrow{g} R_3 \longrightarrow 0$$

is exact. We want to prove that

$$0 \longrightarrow C_1 \xrightarrow{\varphi} C_2 \xrightarrow{\psi} C_3 \longrightarrow 0$$

is exact, where $C_i = F(R_i), i = 1, 2, 3$ and $\varphi = F(f), \psi = F(g)$. From the exact sequence of representations we get

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d, dashed] & \\
0 \arrow[r] & E^{\oplus a_1} \arrow[r, "\alpha_1"] \arrow[d, "1_E \otimes f_1"] & F^{\oplus b_1} \arrow[r, "\pi_1"] \arrow[d, "1_F \otimes f_2"] & C_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & E^{\oplus a_2} \arrow[r, "\alpha_2"] \arrow[d, "1_E \otimes g_1"] & F^{\oplus b_2} \arrow[r, "\pi_2"] \arrow[d, "1_F \otimes g_2"] & C_2 \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & E^{\oplus a_3} \arrow[r, "\alpha_3"] \arrow[d] & F^{\oplus b_3} \arrow[r, "\pi_3"] \arrow[d] & C_3 \arrow[r] \arrow[d, dashed] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

we need to show that φ is injective and ψ is surjective.

- ψ is surjective:

  It follows from the fact that $\pi_3(1_F \otimes g_2)$ is injective.

- φ is injective.

  Let us suppose $\varphi(s) = 0, s \in C_1$. Then $s = \pi_1(v), v \in F^{b_1}$ and

  $$0 = \varphi\pi_1(v) = \pi_2(1_F \otimes f_2)(v).$$

  Since $\ker \pi_2 = \mathrm{im}\alpha_2$, there is $u \in E^{a_2}$ such that

$$(1_F \otimes f_2)(v) = \alpha_2(u) \tag{2.6}$$

Note that

$$\alpha_3(1_E \otimes g_1)(u) = (1_F \otimes g_2)(\alpha_2)(u) = (1_F \otimes g_2)(1_F \otimes f_2)(v) = 0$$

and since α_3 is injective, $(1_E \otimes g_1)(u) = 0$ so $u = (1_E \otimes f_1)(u')$ with $u' \in E^{a_1}$. We have

$$\alpha_2(u) = \alpha_2(1_E \otimes f_1)(u') = (1_F \otimes f_2)\alpha_1(u').$$

From (2.6) we have $(1_F \otimes f_2)(v) = (1_F \otimes f_2)(\alpha_1(u'))$. Since $(1_F \otimes f_2)$ is injective, it follows that $v = \alpha_1(u')$ therefore

$$s = \pi_1(v) = \pi_1\alpha_1(u') = 0.$$

$\square$

In this section we apply the results of quivers to give a criteria of simplicity for cokernel bundles. This result was originally proved by Brambilla in [9], below we provide a new proof based on representations of quivers.

Theorem 2.3.4. *Let C be a generic cokernel bundle given by resolution* (2.1)*. Are equivalent.*

(i) C is simple;

(ii) $a^2 + b^2 - wab \leq 1$.

Proof. Let C be a generic cokernel bundle given by resolution (2.1) and suppose C is simple. By Theorem 2.1.1 there is a globally injective representation R of K_w such that $C = F(R)$. Since F is full, $\mathbb{C} = \mathrm{Hom}(C, C) \simeq \mathrm{Hom}(R, R)$, thus R is simple and therefore R is indecomposable. By Proposition 1.3.6, $q(a, b) = a^2 + b^2 - wab \leq 1$. Reciprocally, since $C = F(R)$ is generic, if $a^2 + b^2 - wab \leq 1$ by Proposition 1.3.6, R is Schur, then $\mathrm{Hom}(C, C) \simeq \mathrm{Hom}(R, R) = \mathbb{C}$. Therefore C is simple.

□

From the previous Theorem, one concludes that if $q(a, b) > 1$ then C is not simple. However, more is true. Using quivers, it is not difficult to establish the following stronger statement.

Theorem 2.3.5. *Let C be a cokernel bundle given by resolution* (2.1)*. If $a^2 + b^2 - wab > 1$ then C is decomposable.*

Proof. Let $C = F(R)$ be a cokernel bundle and R the associated globally injective representation with dimension vector (a, b). If $q(a, b) > 1$ then by Proposition 1.3.6, R is decomposable. Thus by Lemma 2.3.1, C is decomposable. □

We also can tell when the cokernel bundle is exceptional.

Definition 2.3.6. *A vector bundle E on $\mathbb{P}^n$ is exceptional if it is simple and $\mathrm{Ext}^p(E, E) = 0$ for $p \geq 1$.*

Observation 2.3.7. *Since the path algebra associated to a Kronecker quiver Q is hereditary, then $\mathrm{Ext}^p(R, R) = 0$ for every representation R of Q and $p \geq 2$, see [17] for references. Since the functor equivalence of categories is exact, we have $\mathrm{Ext}^1(R, R) \simeq \mathrm{Ext}^1(F(R), F(R))$. Now we know from [39] that*

$$q(a, b) = \dim \mathrm{Hom}(R, R) - \dim \mathrm{Ext}^1(R, R)$$

hence if C is exceptional then $\mathrm{Ext}^1(R, R) = 0$ and $q(a, b) = 1$.

Proposition 2.3.8. *Let C be a generic cokernel bundle given by resolution* (2.1). *If C is exceptional then $q(a,b) = 1$.*

Proof. Let $C = F(R)$ be a generic cokernel bundle. If C is exceptional,

$$\mathrm{Ext}^1(F(R), F(R)) \simeq \mathrm{Ext}^1(R,R) = 0$$

and $q(a,b) = 1$ because C is simple. □

The converse of the Proposition is not true. For instance, consider the generic cokernel bundle given by exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(4)^{35} \longrightarrow C \longrightarrow 0\,.$$

We have $q(1,35) = 1 + 35^2 - 35.1.35 = 1$, but from the long exact sequence of cohomologies, $\mathrm{Ext}^2(C,C) \simeq \mathbb{C}^{35}$ hence C is not exceptional.

2.4 Steiner bundles

As an example of Cokernel bundles we have the Steiner bundles of type (F_0, F_1) introduced by Soares and Miró-Roig in [35], that generalizes the definition of Steiner bundles in the sense of Dolgachev and Kapranov. The definition is the following.

Let X be a smooth irreducible algebraic variety X over an algebraically closed field k of characteristic zero. Let $\mathcal{D} = D^b(\mathcal{O}_X - mod)$ be the bounded derived category of the abelian category of coherent sheaves of $\mathcal{O}_X$−modules on X.

Definition 2.4.1. *An object $E \in \mathcal{D}$ is called* exceptional *if*

$$\mathrm{Ext}^p(E,E) = \begin{cases} k & if\ p = 0 \\ 0 & if\ p \neq 0 \end{cases}$$

An ordered collection of objects of $\mathcal{D}$, $(E_1, \cdots, E_m)$ is an exceptional collection *if all objects E_i are exceptional and*

$$\mathrm{Ext}^p(E_i, E_j) = 0\ for\ all\ i > j\ and\ all\ p \in \mathbb{Z}.$$

If in addition,

$$Ext^P(E_j, E_i) = 0\ for\ all\ j \leq i\ and\ p \neq 0,$$

the collection $(E_1, \cdots, E_m)$ is called a strongly exceptional collection.

A vector bundle E on X is called a *Steiner bundle of type* (F_0, F_1) if it is defined by an exact sequence of the form

$$0 \longrightarrow F_0^s \xrightarrow{\varphi} F_1^t \longrightarrow E \longrightarrow 0 \tag{2.7}$$

where $s, t \geq 1$ and (F_0, F_1) is an ordered pair of vector bundles on X satisfying the following two conditions:

(i) (F_0, F_1) is strongly exceptional;

(ii) $F_0^* \otimes F_1$ is globally generated.

Note that the Steiner bundles of type (F_0, F_1) where $\dim \mathrm{Hom}(F_0, F_1) \geq 3$, are a particular case of cokernel bundles, since F_0, F_1 are simple, $\mathrm{Hom}(F_1, F_0) = \mathrm{Ext}^1(F_1, F_0) = 0$ and $F_0^* \otimes F_1$ is globally generated.

Miró-Roig and Soares gave the following cohomological characterization for these bundles in [35, Theorem 2.4].

Theorem 2.4.2. *Let X be a smooth irreducible projective variety of dimension n with an $n-$block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n), \mathcal{E}_i = (E_1^i, \cdots, E_{\alpha_i}^i)$, of locally free shaves on X which generate $\mathcal{D}$. Let $E_{i_0}^a \in \mathcal{E}_a$, $E_{j_0}^b \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$ and let E be a locally free sheaf on X.*

Then E is a Steiner bundle of type $(E_{i_0}^a, E_{j_0}^b)$ defined by an exact sequence of the form

$$0 \longrightarrow (E_{i_0}^a)^s \longrightarrow (E_{j_0}^b)^t \longrightarrow E \longrightarrow 0\ ,$$

if and only if $(E_{i_0}^a)^* \otimes E_{j_0}^b$ is globally generated and all $\mathrm{Ext}^k(R^{(m)}E_i^{n-m}, E)$ vanish, with the only exceptions of

$$\mathrm{Ext}^{n-a-1}(R^{(n-a)}E_{i_0}^a, E) = s; \tag{2.8}$$

$$\mathrm{Ext}^{n-b}(R^{(n-b)}E_{j_0}^b, E) = t. \tag{2.9}$$

Consider the category of Steiner bundles of type $(E^a_{i_0}, E^b_{j_0})$ on X, with $X, E^a_{i_0}$ and $E^b_{j_0}$ like in the previous Theorem. Let us denote it by $\mathcal{S}(X)_{(E^a_{i_0}, E^b_{j_0})}$. This is a full subcategory of the category of coherent sheaves on X. Soares proved in [35, Corollary 2.5] that the category is closed under extensions.

Now let $X = \mathbb{P}^n$, with an $n-$block collection $\mathcal{B} = (\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_n), \mathcal{E}_i = (E^i_1, \cdots, E^i_{\alpha_i})$, of locally free shaves on $\mathbb{P}^n$ which generate $\mathcal{D}$. Let $E^a_{i_0} \in \mathcal{E}_a$, $E^b_{j_0} \in \mathcal{E}_b$, where $0 \leq a < b \leq n$ and $1 \leq i_0 \leq \alpha_a$, $1 \leq j_0 \leq \alpha_b$ and consider the category $\mathcal{S}(\mathbb{P}^n)_{(E^a_{i_0}, E^b_{j_0})}$. Let $w = \dim \mathrm{Hom}(E^a_{i_0}, E^b_{j_0})$. As a particular case of Theorem 2.1.1 we have.

Theorem 2.4.3. *There is an equivalence between the category of Steiner bundles of type $(E^a_{i_0}, E^b_{j_0})$ and the category of globally injective representations of K_w, $Rep(K_w)^{g.i}$.*

As corollaries of the Theorem 2.3.4 we have:

Corollary 2.4.4. *Let S be a generic Steiner bundle of type $(E^a_{i_0}, E^b_{j_0})$ on $\mathbb{P}^n$, $n \geq 2$, given by resolution*

$$0 \longrightarrow (E^a_{i_0})^s \longrightarrow (E^b_{j_0})^t \longrightarrow S \longrightarrow 0$$

Then are equivalent

- (i) *S is simple;*
- (ii) $q(s,t) = s^2 + t^2 - (w)st \leq 1$.

Corollary 2.4.5. *Let S be a Steiner bundle with resolution as before. If $q(a,b) > 1$ then S is decomposable.*

Corollary 2.4.6. *If The generic Steiner bundle S with dimension vector (s,t) is exceptional, then (s,t) is a real root, that is, $q(s,t) = 1$.*

Remark 2.4.7. *Soares proved in [38] that a generic Steiner bundle of type (F_0, F_1) given by short exact sequence* (2.7) *is exceptional if and only if $q(s,t) = 1$.*

Now consider the Steiner bundles defined by Dolgachev and Kapranov [14].
A vector bundle S on $\mathbb{P}^n$ is called *Steiner* if it is given by the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^s \longrightarrow \mathcal{O}^t_{\mathbb{P}^n} \longrightarrow S \longrightarrow 0\,.$$

Jardim and Martins in [26, Theorem 4.2] gave a characterization for these bundles in terms of the the cohomology.

Theorem 2.4.8. *A locally free sheaf on $\mathbb{P}^n$, $n \geq 3$ is Steiner if and only if*

(i) $H^0(S(-1)) = H^{n-1}(S(1-n)) = H^n(S(-n)) = 0;$

(ii) $H^p(S(l)) = 0, 1 \leq p \leq n-2, \forall l.$

Let us denote by $\mathcal{S}(\mathbb{P}^n)$ this category of Steiner bundles. We can prove the following.

Lemma 2.4.9. *The category $\mathcal{S}(\mathbb{P}^n)$ is exact.*

Proof. It is not difficult to see, using Theorem (2.4.8) that the category is closed under extensions and closed under direct summands.

□

Using the fact that $\mathcal{S}(\mathbb{P}^n)$ is closed under direct summands we have.

Proposition 2.4.10. *Let $S \in \mathcal{S}(\mathbb{P}^n)$ be a Steiner bundle with $S = F(R)$, for $R \in Rep(K_w)^{g.i}$. Then S is indecomposable if and only if R is indecomposable.*

Proof. From Lemma 2.3.1 we have that if R is decomposable, then S is decomposable. Suppose $S = S_1 \oplus S_2$. Since the category $\mathcal{S}(\mathbb{P}^n)$ is closed under direct summands, $S_1 = F(R_1)$ and $S_2 = F(R_2)$ where $R_1, R_2 \in Rep(K_w)^{g.i}$ and F is the functor equivalence of categories. Then

$$S = F(R) = F(R_1) \oplus F(R_2) \simeq F(R_1 \oplus R_2)$$

by Lemma 2.3.1 . Thus $R \simeq R_1 \oplus R_2$.

□

Chapter 3

Syzygy bundles

Now we are going to study a different class of bundles, and see how we can relate them to quivers. As a reference for Syzygy bundles we have [32].

Let α be a surjective map of sheaves on $\mathbb{P}^n$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) : \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{a_2} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_m)^{a_n} \to \mathcal{O}_{\mathbb{P}^n}^c$$

where d_i are positive distinct integers, let us assume $d_1 > d_2 > \ldots > d_m$.

Let $S = S(d_1, d_2, \ldots, d_m)$ be the kernel of α. We have the short exact sequence.

$$0 \longrightarrow S \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(-d_m)^{a_m} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^c \longrightarrow 0 \tag{3.1}$$

The sheaf S given by resolution (3.1) is called *syzygy sheaf.* When S is locally free we call it a *syzygy bundle.* It has rank $\operatorname{rk} S = a_1 + \ldots + a_m - c \geq n$ by [8].

Observation 3.0.11. *Note that the dual of syzygy bundles are not cokernel bundles. Taking the dual on the sequence* (3.1) *we have* S^* *given by a cokernel, but the bundle* $F = \mathcal{O}_{\mathbb{P}^n}(d_1)^{a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(d_m)^{a_m}$ *is not simple when* $m \geq 2$*.*

3.1 Relation between the categories $\mathcal{S}yz(d_1, d_2)$ and $Rep(K_{k_1,k_2})^{g.s}$

Let $m = 2$ and consider the exact sequence

$$0 \longrightarrow S(d_1,d_2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \xrightarrow{\alpha_1,\alpha_2} \mathcal{O}_{\mathbb{P}^n}^c \longrightarrow 0 \tag{3.2}$$

with $d_1 > d_2$. We denote by $\mathcal{S}yz(d_1,d_2)$ the category of such Syzygy bundles.

Fix $\mathcal{B}_1 = \{f_1^1,\dots,f_{k_1}^1\}$ a basis of $H^0(\mathcal{O}_{\mathbb{P}^n}(d_1))$ and $\mathcal{B}_2 = \{f_1^2,\dots,f_{k_2}^2\}$ basis of $H^0(\mathcal{O}_{\mathbb{P}^n}(d_2))$ where $k_1 = \binom{n+d_1}{d_1}$ and $k_2 = \binom{n+d_2}{d_2}$. Consider the quiver $Q = A_{k_1,k_2}$ bellow

$$\bullet \; \begin{matrix} \xrightarrow{a_1} \\ \vdots \\ \xrightarrow[a_{k_1}]{} \end{matrix} \; \bullet \; \begin{matrix} \xleftarrow{b_1} \\ \vdots \\ \xleftarrow[b_{k_2}]{} \end{matrix} \; \bullet \tag{3.3}$$

Let $R = (\{\mathbb{C}^a,\mathbb{C}^b,\mathbb{C}^c\},\{A_i\}_1^{k_1},\{B_j\}_1^{k_2})$ be a representation of Q, where A_i is a matrix $c\times a$ and B_j a matrix $c \times b$ on $\mathbb{C}$. We define

$$\alpha_1 = \sum_{i=1}^{k_1} A_i \otimes f_i^1 \text{ and } \alpha_2 = \sum_{j=1}^{k_2} B_j \otimes f_j^2$$

and we have a map

$$(\alpha_1,\alpha_2) : \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \to \mathcal{O}_{\mathbb{P}^n}^c$$

Definition 3.1.1. *We say that R is $(\mathcal{B}_1,\mathcal{B}_2)$-*locally surjective *if the map between the fibers $(\alpha_1(P),\alpha_2(P))$ is surjective for some $P \in \mathbb{P}^n$. We say R is $(\mathcal{B}_1,\mathcal{B}_2)$-*globally surjective *if $(\alpha_1(P),\alpha_2(P))$ is surjective for every point $P \in \mathbb{P}^n$.*

Now let us build a functor G between the category of globally surjective representations of A_{k_1,k_2} and the category of syzygy bundles $\mathcal{S}yz(d_1,d_2)$ given by exact sequence (3.2).

Let $R = (\{\mathbb{C}^a,\mathbb{C}^b,\mathbb{C}^c\},\{A_i\}_1^{k_1},\{B_j\}_1^{k_2})$ be a globally surjective representation of Q. We define the sheaf $G(R) = Syz(d_1,d_2)$ as the sheaf given by the kernel of $(\alpha_1,\alpha_2) : \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \to \mathcal{O}_{\mathbb{P}^n}^c$ where α_1 and α_2 are as in definition above. Note that $Syz(d_1,d_2)$ is a vector bundle, since R is globally surjective and $Syz(d_1,d_2)$ is given by exact sequence (3.2).

Now let $\{g_1,g_2,h\}$ be a morphism between the globally surjective representations $R = (\{\mathbb{C}^a,\mathbb{C}^b,\mathbb{C}^c\},\{A_i\}_1^{k_1},\{B_j\}_1^{k_2})$ and $R' = (\{\mathbb{C}^{a'},\mathbb{C}^{b'},\mathbb{C}^{c'}\},\{A'_i\}_1^{k_1},\{B'_j\}_1^{k_2})$. The following diagram commutes for $i = 1,\dots,k_1$ and $j = 1,\dots,k_2$.

$$
\begin{tikzcd}
\mathbb{C}^a \arrow[rr, bend left, "g_1"] \arrow[dr, "A_i"] & & \mathbb{C}^{a'} \arrow[dr, "A'_i"] & \\
& \mathbb{C}^c \arrow[rr, "h"] & & \mathbb{C}^{c'} \\
\mathbb{C}^b \arrow[ur, "B_i"] \arrow[rr, bend right, "g_2"'] & & \mathbb{C}^{b'} \arrow[ur, "B'_j"] &
\end{tikzcd}
\tag{3.4}
$$

it induces the diagram

$$
\begin{tikzcd}
0 \arrow[r] & Syz(d_1,d_2) \arrow[r, "i_1"] \arrow[d, dashed, "\phi"] & \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \arrow[r, "\alpha_1{,}\alpha_2"] \arrow[d, "M"] & \mathcal{O}_{\mathbb{P}^n}^c \arrow[r] \arrow[d, "h\otimes 1_{\mathcal{O}_{\mathbb{P}^n}}"] & 0 \\
0 \arrow[r] & Syz'(d_1,d_2) \arrow[r, "i_2"] & \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a'} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b'} \arrow[r, "\alpha'_1{,}\alpha'_2"] & \mathcal{O}_{\mathbb{P}^n}^{c'} \arrow[r] & 0
\end{tikzcd}
\tag{3.5}
$$

where

$$
M = \begin{pmatrix} g_1 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}(-d_1)} & 0 \\ 0 & g_2 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}(-d_2)} \end{pmatrix}
$$

Since the diagram (3.4) commutes, then the right square of (3.5) commutes. We define ϕ as the following:

given $s \in Syz(d_1, d_2)$ we have $M(i_1(s)) \in \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a'} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b'}$. Since

$$
(\alpha'{}_1, \alpha'_2)(M(i_1(s))) = h \otimes 1_{\mathcal{O}_{\mathbb{P}^n}}(\alpha_1, \alpha_2)(i_1(s)) = 0,
$$

then $M(i_1(s)) \in \ker(\alpha'_1, \alpha'_2) = Syz'(d_1, d_2)$. Then we define $\phi(s) := M(i_1(s))$, so $G(g_1, g_2, h) = \phi$ and G is a functor.

Lemma 3.1.2. *The functor G is faithful.*

Proof. We prove that $\mathrm{Hom}(R, R') \to \mathrm{Hom}(G(R), G(R'))$ is injective. Let $\{g_1, g_2, h\}$ be a morphism between R and R' such that $G(\{g_1, g_2, h\}) = 0$, that is, $\phi = 0$. Since the diagram (3.5) commutes if $\phi = 0$ then $g_1 = g_2 = h = 0$ hence G is faithful. $\square$

Lemma 3.1.3. *The functor G is essentially surjective.*

Proof. Let S be a syzygy bundle with resolution

$$0 \longrightarrow S \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \xrightarrow{\alpha_1,\alpha_2} \mathcal{O}_{\mathbb{P}^n}^c \longrightarrow 0$$

and fix basis $\mathcal{B}_1 = \{f_1^1, \ldots, f_{k_1}^1\}$ and $\mathcal{B}_2 = \{f_1^2, \ldots, f_{k_2}^2\}$ of $H^0(\mathcal{O}_{\mathbb{P}^n}(d_1))$ and $H^0(\mathcal{O}_{\mathbb{P}^n}(d_2))$ respectively. Then the maps α_1 and α_2 are given by

$$\alpha_1 = \sum_{i=1}^{k_1} A_i \otimes f_i^1 \text{ and } \alpha_2 = \sum_{j=1}^{k_2} B_j \otimes f_j^2$$

with $A_i \in Hom(\mathbb{C}^a, \mathbb{C}^c)$ and $B_j \in Hom(\mathbb{C}^b, \mathbb{C}^c)$. Therefore $R = (\{\mathbb{C}^a, \mathbb{C}^b, \mathbb{C}^c\}, \{A_i\}_1^{k_1}, \{B_j\}_1^{k_2})$ is a globally surjective representation of (3.3) such that $G(R) = S$. □

Observation 3.1.4. *The functor G may not be full. It happens because the map $M \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b, \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a'} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b'})$ may not be given by a matrix of the form*

$$M = \begin{pmatrix} f_1 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}(-d_1)} & 0 \\ 0 & f_2 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}(-d_2)} \end{pmatrix}$$

in general the entry m_{21} of the matrix is nonzero.

If we consider the category of syzygy *sheaves*, all the previous and next proofs are the same just changing "globally" surjective representations by " locally " surjective.

Lemma 3.1.5. *The category of globally surjective representations of Q is closed under quotients.*

Proof. Let $R = (\{\mathbb{C}^a, \mathbb{C}^b, \mathbb{C}^c\}, \{A_i\}_1^{k_1}, \{B_j\}_1^{k_2})$ be a ($\mathcal{B}_1$, $\mathcal{B}_2$)-globally surjective representation of Q and $R' = (\{\mathbb{C}^{a'}, \mathbb{C}^{b'}, \mathbb{C}^{c'}\}, \{A'_i\}, \{B'_j\})$ be a subrepresentation of R. We want to prove that the quotient $R/R' = (\{\mathbb{C}^a/\mathbb{C}^{a'}, \mathbb{C}^b/\mathbb{C}^{b'}, \mathbb{C}^c/\mathbb{C}^{c'}\}, \{C_i\}, \{D_j\})$, where C_i and D_j are the maps induced by A_i and B_j respectively, is globally surjective, that is,

$$\gamma_1, \gamma_2 : \mathcal{O}_{\mathbb{P}^n}(-d_1)^{(a-a')} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{(b-b')} \to \mathcal{O}_{\mathbb{P}^n}^{(c-c')}$$

is surjective for every point $P \in \mathbb{P}^n$ where $\gamma_1 = \sum_{i=1}^{k_1} C_i \otimes f_i^1$ and $\gamma_2 = \sum_{j=1}^{k_2} D_j \otimes f_j^2$. We have the diagram

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\mathbb{C}^{a'} \arrow[r, bend left, "A'_i"] \arrow[r, bend right] \arrow[r, phantom, "\vdots"] \arrow[d, "l_1"'] & \mathbb{C}^{c'} \arrow[d, "l_3"'] & \mathbb{C}^{b'} \arrow[l, bend right, "B'_j"'] \arrow[l, bend left] \arrow[l, phantom, "\vdots"] \arrow[d, "l_2"'] \\
\mathbb{C}^{a} \arrow[r, bend left, "A_i"] \arrow[r, bend right] \arrow[r, phantom, "\vdots"] \arrow[d, "p_1"'] & \mathbb{C}^{c} \arrow[d, "p_3"'] & \mathbb{C}^{b} \arrow[l, bend right, "B_j"'] \arrow[l, bend left] \arrow[l, phantom, "\vdots"] \arrow[d, "p_2"'] \\
\mathbb{C}^{a}/\mathbb{C}^{a'} \arrow[r, bend left, "C_i"] \arrow[r, bend right] \arrow[r, phantom, "\vdots"] \arrow[d] & \mathbb{C}^{c}/\mathbb{C}^{c'} \arrow[d] & \mathbb{C}^{b}/\mathbb{C}^{b'} \arrow[l, bend right, "D_j"'] \arrow[l, bend left] \arrow[l, phantom, "\vdots"] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
$$

where l_i are the inclusions and p_i the projections $i = 1, 2, 3$. Consider the commutative diagram

$$
\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^n}(-d_1)^a \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^b \arrow[r, "\alpha_1,\alpha_2"] \arrow[d, "M"] & \mathcal{O}_{\mathbb{P}^n}^c \arrow[d, "p_3 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}}"] \\
\mathcal{O}_{\mathbb{P}^n}(-d_1)^{(a-a')} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{(b-b')} \arrow[r, "\gamma_1,\gamma_2"] & \mathcal{O}_{\mathbb{P}^n}^{(c-c')}
\end{tikzcd}
$$

where

$$
M = \begin{pmatrix} p_1 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}(-d_1)} & 0 \\ 0 & p_2 \otimes 1_{\mathcal{O}_{\mathbb{P}^n}(-d_2)} \end{pmatrix}
$$

Since p_i is surjective and (α_i, α_2) is surjective for every $P \in \mathbb{P}^n$, we have the map γ_1, γ_2 is surjective for every point $P \in \mathbb{P}^n$, hence the representation quotient R/R' is globally surjective.

$\square$

Lemma 3.1.6. *Let R be a ($\mathcal{B}_1$,$\mathcal{B}_2$)-globally surjective representation of Q. If $R = R_1 \oplus R_2$ then R_1 and R_2 are globally surjective.*

Proof. Let us suppose $R = R_1 \oplus R_2$, with R globally surjective. We have the exact sequence

$$
0 \longrightarrow R_1 \longrightarrow R_1 \oplus R_2 \longrightarrow R_2 \longrightarrow 0
$$

since $(R_1 \oplus R_2)/R_1 \simeq R_2$. By the previous Lemma, R_2 is globally surjective. Analogously we prove that R_1 is globally surjective.

$\square$

Lemma 3.1.7. *Let R be a decomposable globally surjective representation of Q. Then $G(R)$ is decomposable.*

Proof. Let $R = R_1 \oplus R_2$ be a decomposable globally surjective representation. From Lemma 3.1.5 we have that $R_1 = (\{\mathbb{C}^{a_1}, \mathbb{C}^{b_1}, \mathbb{C}^{c_1}\}, \{A_i^1\}, \{B_j^1\})$ and $R_2 = (\{\mathbb{C}^{a_2}, \mathbb{C}^{b_2}, \mathbb{C}^{c_2}\}, \{A_i^2\}, \{B_j^2\})$ are globally surjective. Let $S_i = G(R_i)$, $i = 1, 2$ be given by resolution

$$0 \longrightarrow S_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_1} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b_1} \xrightarrow{\alpha_1,\alpha_2} \mathcal{O}_{\mathbb{P}^n}^{c_1} \longrightarrow 0$$

$$0 \longrightarrow S_2 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_1)^{a_2} \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2)^{b_2} \xrightarrow{\beta_1,\beta_2} \mathcal{O}_{\mathbb{P}^n}^{c_2} \longrightarrow 0$$

Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. Since

$$S = G(R) = \ker(\alpha \oplus \beta) = \ker \alpha \oplus \ker \beta = G(R_1) \oplus G(R_2),$$

it follows that S is decomposable. $\square$

Proposition 3.1.8. *Let S be a syzygy bundle given by resolution* (3.2)*. If $q(a, b, c) > 1$ where $q(x, y, z)$ is the Tits form associated to the quiver Q, then S is decomposable.*

Proof. Since G is essentially surjective, $S = G(R)$, where R is a ($\mathcal{B}_1$,$\mathcal{B}_2$)-globally surjective representation of Q with dimension vector (a, b, c). If $q(a, b, c) > 1$, then by Proposition 1.3.6, R is decomposable and by the previous Lemma, S is decomposable. $\square$

Observation 3.1.9. *All the results can be generalized for syzygy bundles with $m \geq 2$. To build the associated quiver we do the same. For each term $\mathcal{O}_{\mathbb{P}^n}(-d_i)^{\oplus a_i}$, we add a vertex to the quiver with $\dim H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))$ arrows from this vertex to the vertex associated to $\mathcal{O}_{\mathbb{P}^n}^{\oplus c}$.*

Note that functor G depends on a choice of basis. When we choose another basis we don't know yet what happens to the functor.

Chapter 4

Applications

In this chapter we want to construct simple vector bundles on $\mathbb{P}^n$ with a high homological dimension and a low rank. We recall some definitions and we give some results of Jardim and Martins, and Bohnhorst and Spindler, that will be useful for our main results.

In section 4.1 we prove one of our main results, that shows how to construct a simple vector bundle on $\mathbb{P}^n$ with any homological dimension.

In section 4.2 we study minimal free resolutions from the Koszul complex. We prove simplicity of some vector bundles using previous results of section 2.3.

In section 4.3 we give a new proof for simplicity of a particular case of the weighted Tango bundles. These bundles were introduced by Cascini in [11].

In section 4.4 we apply the same ideas of previous sections to free resolutions of a different algebra. We prove that some bundles are simple, using the minimal free resolution of compressed Gorenstein Atinian graded algebras of embedding dimension $n+1$ and even socle degree. This algebras have been studied by Migliore, Miró-Roig and Nagel in [33].

The definition and results below can be found in [8, 26].

Let E be a vector bundle over $\mathbb{P}^n$. A *resolution* of E is an exact sequence

$$0 \longrightarrow F_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where every F_i splits as a direct sum of line bundles.

The minimal number d of such resolution is called *homological dimension* of E and denoted by $\mathrm{hd}(E)$.

From Bohnhorst and Spindler's paper [8], we can see that every vector bundle on $\mathbb{P}^n$ admits a resolution.

Jardim and Martins on [26, Proposition 5.3] gave a resolution for every locally free sheaf on a ACM variety.

Proposition 4.0.10 (JM). *Let X be an ACM projective variety of dimension $n \geq 2$. Every locally free sheaf on X admits a resolution of the form*

$$0 \longrightarrow K_d \longrightarrow F_{d-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where $d \leq n-1$, each F_i splits as a sum of line bundles, $F_i = \oplus_{i=1}^{\mathrm{rkF_i}} \mathcal{O}_{\mathbb{P}^n}(l_{i,k})$ and K_d is an ACM locally free sheaf. Moreover, for $d \leq n-2$ the above sequence exists if and only if $H^p_*(E) = 0$ for $1 \leq p \leq n-d-1$.

Since ACM bundles on $\mathbb{P}^n$ are split, $K_d = F_d$ and the last part of this Proposition tell us that for $d \leq n-2$,

$$\mathrm{hd}(E) \leq d \iff H^p_*(E) = 0,\ 1 \leq p \leq n-d-1.$$

This was also proved by Bohnhorst and Spindler, cf. [8, Proposition 1.4]. We can also see that the biggest $\mathrm{hd}(E)$ of a bundle E on $\mathbb{P}^n$ is $n-1$, and E splits if and only if $\mathrm{hd}(E) = 0$.

Bohnhorst and Spindler proved a relation between $\mathrm{rk}(E)$ and $\mathrm{hd}(E)$.

Proposition 4.0.11 (BS). *Let E be a non splitting vector bundle over $\mathbb{P}^n$. Then*

$$\mathrm{rk}(E) \geq n + 1 - \mathrm{hd}(E).$$

Proof. See [8], Corollary 1.7.

□

This inequality tells us if we can construct a non splitting vector bundle with high homological dimension then it is possible to have a low rank vector bundle. Let us see some examples of bundles, their ranks and homological dimension.

Example 4.0.12. *The* cokernel bundle

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^a \longrightarrow \mathcal{O}^b_{\mathbb{P}^n} \longrightarrow S' \longrightarrow 0$$

also have $\mathrm{hd}(S') = 1$ *and* $\mathrm{rk}(S') = b - a \geq n$.

Example 4.0.13. *The* Tango bundle *E over $\mathbb{P}^n$ with $n \geq 3$ is given by resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathcal{O}^{2n-1}_{\mathbb{P}^n} \longrightarrow E \longrightarrow 0$$

$\mathrm{hd}(E) = 2$ *and* $\mathrm{rk}(E) = n - 1$.

Example 4.0.14. *The* Instanton bundle E *on* $\mathbb{P}^n$ *of rank* r *and charge* c *is the cohomology of the monads*

$$\mathcal{O}_{\mathbb{P}^n}(-1)^c \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^c$$

that is, $E = \frac{\ker \alpha}{\mathrm{Im} \beta}$*. It is not difficult to check that* $c = H^1(E(-1))$*, see for instance [25].*

Lemma 4.0.15. *Let* E *be an instanton bundle on* $\mathbb{P}^n$*,* $n \geq 3$*. Then* $\mathrm{hd(E) = n - 1}$*.*

Proof. Since E is an instanton, its charge $c = H^1(E(-1)) \neq 0$, then by Proposition 4.0.10, $\mathrm{hd}(E) > n - 2$, hence $\mathrm{hd}(E) = n - 1$.

□

Example 4.0.16. *The* Horrocks parent bundle *of rank* 3 *on* $\mathbb{P}^5$ *also has* $H^1_*(E) \neq 0$ *therefore* $\mathrm{hd}(E) = 4$*, see for instance [22].*

Example 4.0.17. *The* Horrocks-Mumford bundle E *on* $\mathbb{P}^4$ *has rank* 2*, then* $\mathrm{hd}(E) = 3$*, see [23] for reference.*

In fact, by Proposition 4.0.11, $\mathrm{hd}(E) \geq n + 1 - \mathrm{rkE}$*, then* $\mathrm{hd}(E) \geq 3$*, therefore* $\mathrm{hd}(E) = 3$.

Example 4.0.18. *The* generalized null correlation bundle E *has rank* 2 *on* $\mathbb{P}^3$*. It is given by the cohomology of the monad*

$$\mathcal{O}(-c) \longrightarrow \mathcal{O}(b) \oplus \mathcal{O}(a) \oplus \mathcal{O}(-a) \oplus \mathcal{O}(-b) \longrightarrow \mathcal{O}(c)$$

where $c > b \geq a \geq 0$*. See [15]. Then by Proposition 4.0.11,* $\mathrm{hd}(E) = 2$*.*

This bundle generalizes the null correlation bundle of rank 2 on $\mathbb{P}^3$, [7]. They are given by the cohomology of the monad above when $a = b = 0$ and $c = 1$.

The *Horrocks' splitting criterion* tell us when a bundle splits.

Theorem 4.0.19. *Let* E *be a vector bundle of rank* r *on* $\mathbb{P}^n$*. Then* E *splits if and only if*

$$H^i(\mathbb{P}^n, E(k)) = 0, 1 \leq i \leq n - 1 \;\forall k \in \mathbb{Z}.$$

Proof. See [36, Theorem 2.3.1].

□

It is extremely difficult to construct vector bundles on $\mathbb{P}^n_{\mathbb{C}}$ with rank between 2 an $n-2$. The only known examples are the Horrocks-Mumford of rank 2 on $\mathbb{P}^4$, see [23], and the Horrocks parent bundle of rank 3 on $\mathbb{P}^5$, cf. [22], from the ′70s. We will try to construct an indecomposable vector bundle with a high homological dimension, then using Proposition 4.0.11 we might have a low rank vector bundle. To do this we need some results.

Lemma 4.0.20. *Let C_1 and C_2 be generic cokernel bundles given by exact sequences*

$$0 \longrightarrow E_1 \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}^{b_1} \longrightarrow C_1 \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{b_2} \xrightarrow{\alpha_2} E_2 \longrightarrow C_2 \longrightarrow 0\,.$$

Then we have

(a) *C_1 is simple if and only if $h^0(E_1^*) \geq b_1$;*

(b) *C_2 is simple if and only if $h^0(E_2) \geq b_2$.*

Proof. From Theorem 2.3.4 we have that C_1 is simple if and only if $q(1, b_1) \leq 1$. Since $q(1, b_1) = 1 + b_1^2 - h^0(E_1^*)b_1$, we are done. The proof of item (b) is similar. □

Lemma 4.0.21. *Let E be a vector bundle over $\mathbb{P}^n$. Then the following conditions are equivalent*

(i) *E^* is simple;*

(ii) *E is simple;*

(iii) *$E(k)$ is simple for all $k \in \mathbb{Z}$;*

(iv) *$E(k)$ is simple for some $k \in \mathbb{Z}$.*

Proof. $(i) \Leftrightarrow (ii)$

$$\mathrm{Hom}(E^*, E^*) \simeq \mathrm{End}(E^*) \simeq \mathrm{End}(E) \simeq \mathrm{Hom}(E, E)$$

$(ii) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv)$

$$\mathrm{Hom}(E, E) \simeq \mathrm{Hom}(E(k), E(k))$$

□

4.1 Rank n bundles with any hd

In this section we prove one of our main results. We show how to construct rank n vector bundles E on $\mathbb{P}^n$ with any homological dimension $1 \leq \mathrm{hd}(E) \leq n-1$.

Theorem 4.1.1. *Given $n \in \mathbb{Z}$, with $n \geq 4$, for each $1 \leq l \leq n-1$, there exists a simple vector bundle E over $\mathbb{P}^n$ with $\mathrm{rk(E)} = \mathrm{n}$ and $\mathrm{hd}(E) = l$ given by resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}(d_2)^{2n} \xrightarrow{\alpha_2} \cdots$$

$$\cdots \xrightarrow{\alpha_{l-1}} \mathcal{O}_{\mathbb{P}^n}(d_l)^{2n} \xrightarrow{\alpha_l} E \longrightarrow 0$$

for some integers $d_0, d_1, \cdots, d_l$ with $d_1 = 0 < d_2 < \cdots < d_l$, $d_0 > 0$.

Proof. For each $1 \leq l \leq n-1$, let us prove by induction on l that there exists $d_0, d_1, \cdots, d_l$ with $d_1 = 0 < d_2 < \cdots < d_l$, $d_0 > 0$, such that E is given by exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}(d_2)^{2n} \xrightarrow{\alpha_2} \cdots.$$

$$\cdots \xrightarrow{\alpha_{l-1}} \mathcal{O}_{\mathbb{P}^n}(d_l)^{2n} \xrightarrow{\alpha_l} E \longrightarrow 0$$

(*a*) For $l = 1$, consider the vector bundle E_1 given by exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow E_1 \longrightarrow 0$$

where α_0 is a generic map given by $n+1$ forms of degree d_0. Clearly $\mathrm{hd(E_1)} = 1$, E_1 has rank n and by Theorem 2.7 of [8] E_1 is stable, then it is simple.

(*b*) Let us suppose it is true for $2 \leq l \leq t-1$. Let E_{t-1} be a simple vector bundle with rank n and homological dimension $t-1$ with exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\alpha_1} \cdots \tag{4.1}$$

$$\cdots \xrightarrow{\alpha_{t-2}} \mathcal{O}_{\mathbb{P}^n}(d_{t-1})^{2n} \xrightarrow{\alpha_{t-1}} E_{t-1} \longrightarrow 0$$

Let $d_t > \max\{d_{t-1}, D_t\}$ where E^*_{t-1} is D_t-regular and such that $h^0(E^*(d_t)) \geq 2n$. For a generic map $\alpha : E_{t-1} \to \mathcal{O}_{\mathbb{P}^n}(d_t)^{2n}$ the degeneracy locus of α is the set

$$\Delta_\alpha = \{P \in \mathbb{P}^n \mid \alpha_P : E_{t-1P} \to \mathcal{O}_{\mathbb{P}^n}(d_t)^{2n}_P \text{ is not injective }\}.$$

This set has codimension $2n-n+1$, see [5, 13], hence Δ_α is empty and therefore E_t =cokerα is a cokernel bundle. In fact, we have

(1) E_{t-1} and $\mathcal{O}_{\mathbb{P}^n}(d_t)$ are simple;

(2) $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(d_t), E_{t-1}) \simeq H^0(\mathbb{P}^n, E_{t-1}(-d_t)) = 0$.
It is true because taking short exact sequences in (4.1) and twisting by $\mathcal{O}_{\mathbb{P}^n}(-d_t)$ we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0 - d_t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(-d_t) \longrightarrow E_1(-d_t) \longrightarrow 0$$

$$0 \longrightarrow E_1(-d_t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_2 - d_t)^{2n} \longrightarrow E_2(-d_t) \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow E_{t-2}(-d_t) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d_{t-1} - d_t)^{2n} \longrightarrow E_{t-1}(-d_t) \longrightarrow 0$$

then by the long exact sequence of cohomologies

$$0 \longrightarrow H^0(E_{t-2}(-d_t)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(d_{t-1} - d_t)^{2n}) \longrightarrow$$

$$\longrightarrow H^0(E_{t-1}(-d_t)) \longrightarrow H^1(E_{t-2}(-d_t)) \longrightarrow 0$$

since $H^0(\mathcal{O}_{\mathbb{P}^n}(d_{t-1} - d_t) = 0$, hence $H^0(E_{t-1}(-d_t)) \simeq H^1(E_{t-2}(-d_t))$. By Proposition 4.0.10, since $\mathrm{hd}(E_{t-2}) = t-2 \leq n-2$ we have $H^1(E_{t-2}(-d_t)) = 0$.

(3) $\mathrm{Ext^1(\mathcal{O}_{\mathbb{P}^n}(d_t), E_{t-1}) \simeq H^1(\mathbb{P}^n, E_{t-1}(-d_t)) = 0}$ again it follows by Proposition 4.0.10, since $\mathrm{hd}(E_{t-1}) = t - 1 \leq n - 2$.

(4) $E^*_{t-1} \otimes \mathcal{O}_{\mathbb{P}^n}(d_t)$ is globally generated by the choice of d_t.

(5) $\dim \mathrm{Hom}(E_{t-1}, \mathcal{O}_{\mathbb{P}^n}(d_t)) = h^0(E^*_{t-1}(d_t)) \geq 2n$ by the choice of d_t.

Therefore by definition, E_t is a cokernel bundle, and since

$$q(1, 2n) = 1 + 4n^2 - h^0(E^*(d_t))2n \leq 1,$$

by Theorem 2.3.4, E_t is simple.

Then E_t has resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d_0) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^n}^{2n}(d_2) \xrightarrow{\alpha_2} \cdots \tag{4.2}$$

$$\xrightarrow{\alpha_{t-2}} \mathcal{O}_{\mathbb{P}^n}(d_{t-1})^{2n} \xrightarrow{\alpha_{t-1}} \mathcal{O}_{\mathbb{P}^n}(d_t)^{2n} \longrightarrow E_t \longrightarrow 0$$

Now we need to check that $\mathrm{hd}(E_t) = t$. By resolution (4.2), $\mathrm{hd}(E_t) \leq t$. Suppose $\mathrm{hd}(E_t) \leq t - 1$. Using Proposition 4.0.10 we have

$$\mathrm{H^q(E_t(l)) = 0,\ 1 \leq q \leq n - (t-1) - 1,\ \forall l \in \mathbb{Z}.}$$

Consider the short exact sequences of (4.2). We have the sequence

$$0 \longrightarrow E_{t-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d_t)^{2n} \longrightarrow E_t \longrightarrow 0$$

since $2 \leq t \leq n-1$, applying the long exact sequence of cohomologies we have

$$0 \longrightarrow H^{n-t}(E_t(l)) \longrightarrow H^{n-t+1}(E_{t-1}(l)) \longrightarrow 0 \ \ \forall l \in \mathbb{Z}$$

then $H^{n-t}(E_t(l)) \simeq H^{n-t+1}(E_{t-1}(l))$. We know that $\mathrm{hd}(E_{t-1}) = t-1$, then by Proposition 4.0.10,

$$\exists\ l_0 \in \mathbb{Z};\ H^{n-t+1}(E_{t-1}(l_0)) \neq 0,$$

hence $H^{n-t}(E_t(l_0)) \neq 0$, which contradicts $\mathrm{hd}(E_t) \leq t-1$, so $\mathrm{hd}(E_t) = t$.

$\square$

4.2 Minimal free resolutions from the Koszul complex

From now on let $R = k[x_0, \ldots, x_n]$ be the ring of polynomials in $n+1$ variables, $\{f_1, \ldots, f_{n+1}\}$ a generic regular sequence of forms of degree d, let $I = (f_1, \ldots, f_{n+1})$ be the ideal generated by these forms and $\mathbb{P}^n = \mathrm{Proj}(R)$. The Koszul complex $K(f_1, \cdots, f_{n+1})$ is given by

$$0 \longrightarrow \bigwedge^{n+1} T \longrightarrow \bigwedge^{n} T \longrightarrow \cdots$$

$$\longrightarrow \bigwedge^{1} T \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where $T = R(-d)^{n+1}$. From Lemma 1.2.3, we know this complex is a free resolution of R/I. We have

$$0 \longrightarrow R(-(n+1)d) \longrightarrow R(-nd)^{n+1} \longrightarrow \cdots$$

$$\longrightarrow R(-2d)^{\binom{n+1}{2}} \longrightarrow R(-d)^{n+1} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

Sheafifying we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \xrightarrow{\alpha^t} \mathcal{O}_{\mathbb{P}^n}(-nd)^{n+1} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0 \tag{4.3}$$

where $\alpha : \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \to \mathcal{O}_{\mathbb{P}^n}$ is the map given by the forms.

Observation 4.2.1. *From [34, Proposition 1.1.27], we can see that the complex above is self dual up to twist.*

Lemma 4.2.2. *Consider the short exact sequences in the complex* (4.3).

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-nd)^{\binom{n+1}{n}} \longrightarrow F_1 \longrightarrow 0$$

$$0 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n-1)d)^{\binom{n+1}{n-1}} \longrightarrow F_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-2} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d)^{\binom{n+1}{2}} \longrightarrow F_{n-1} \longrightarrow 0$$

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Then $\operatorname{hd}(F_i) = i$, *for* $1 \leq i \leq n-1$.

Proof. Let us prove it by induction on i.

(i) For the case $i = 1$, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-nd)^{\binom{n+1}{n}} \longrightarrow F_1 \longrightarrow 0$$

therefore $\operatorname{hd}(F_1) \leq 1$.

Since

$$H^{n-1}(F_1(n(d-1))) \simeq H^n(\mathcal{O}_{\mathbb{P}^n}(-n-d)) \neq 0,$$

by Horrocks criterion, Theorem 4.0.19, F_1 does not split, hence $\operatorname{hd}(F_1) = 1$.

(ii) Suppose it is true for $2 \leq j \leq t-1$, $t \leq n-2$. From the complex we have

$$0 \longrightarrow F_{t-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n-t+1)d)^{\binom{n+1}{n-t+1}} \longrightarrow F_t \longrightarrow 0$$

Therefore, from the long exact sequence of cohomologies

$$\cdots \longrightarrow H^i(F_t) \longrightarrow H^{i+1}(F_{t-1}) \longrightarrow H^{i+1}(\mathcal{O}_{\mathbb{P}^n}(-(n-t+1)d))^{\binom{n+1}{n-t+1}} \longrightarrow \cdots$$

hence

$$H^i(F_t) \simeq H^{i+1}(F_{t-1}), \text{ for } 1 \leq i \leq n-2.$$

Since $\operatorname{hd}(F_{t-1}) = t-1$, by Proposition 4.0.10,

$$H^p_*(F_{t-1}) = 0, 1 \leq p \leq n-t,$$

therefore

$$H^p_*(F_t) = 0, 1 \leq p \leq n-t-1.$$

Again by Proposition 4.0.10, $\operatorname{hd}(F_t) \leq t$. Now suppose $\operatorname{hd}(F_t) \leq t-1$, then

$$H^p_*(F_t) = 0, 1 \leq p \leq n-t.$$

Since $H^{n-t}(F_t) \simeq H^{n-t+1}(F_{t-1})$ and $\operatorname{hd}(F_{t-1}) = t-1$, there is $l_0 \in \mathbb{Z}$ such that $H^{n-t}(F_t(l_0)) \simeq H^{n-t+1}(F_{t-1}(l_0)) \neq 0$. Hence $\operatorname{hd}(\mathrm{F_t}) = t$.

$\square$

Theorem 4.2.3. *Consider the short exact sequences in the complex* (4.3)*.*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-nd)^{\binom{n+1}{n}} \longrightarrow F_1 \longrightarrow 0$$

$$0 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n-1)d)^{\binom{n+1}{n-1}} \longrightarrow F_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-2} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d)^{\binom{n+1}{2}} \longrightarrow F_{n-1} \longrightarrow 0$$

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Then F_i is simple of rank $\mathrm{rk}(F_i) = \binom{n}{i}$ $1 \leq i \leq n-1$.

Proof. Since the Koszul complex is self-dual up to twist, is sufficient to prove that F_i is simple for $1 \leq i \leq \frac{n-1}{2}$. Let us prove by induction on i.

(*a*) $i = 1$

Consider the first exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-nd)^{\binom{n+1}{n}} \longrightarrow F_1 \longrightarrow 0$$

By [8, Theorem 2.7], we have that F_1 is stable, then it is simple.

Now suppose that the statement is true for $2 \leq k \leq i-1$. Let us prove it is true for $k = i$. Consider now the Koszul complex twisted with $\mathcal{O}_{\mathbb{P}^n}((n-i+1)d)$, let $\overline{F}_{i-1} = F_{i-1}((n-i+1)d)$ and let $\alpha_i : \overline{F}_{i-1} \to \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{i}}$ be a generic map. Take $\overline{F}_i = \mathrm{coker}\alpha_{i-1}$. We have

(1) $\overline{F}_{i-1}$ and $\mathcal{O}_{\mathbb{P}^n}$ are simple;

(2) $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, \overline{F}_{i-1}) = 0$

(3) $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, \overline{F}_{i-1}) = 0$

(4) $\overline{F}^*_{i-1}$ is globally generated;

(5) $h^0(\overline{F}^*_{i-1}) = \binom{n+1}{i}$

In fact, (1) is obvious. To prove (2) note that

$$H^0(\mathbb{P}^n, \overline{F}_{i-1}) \simeq H^j(\mathbb{P}^n, \overline{F}_{i-(j+1)}),\ 0 \leq j \leq i-2$$

and $H^{i-2}(\mathbb{P}^n, \overline{F}_1) = 0$. To prove (3), from Lemma 4.2.2, $\mathrm{hd}(\overline{F}_{i-1}) \leq i-1$, then by Proposition 4.0.10,

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, \overline{F}_{i-1}) = H^1(\mathbb{P}^n, \overline{F}_{i-1}) = 0.$$

To see (4), take the sequence

$$0 \longrightarrow \overline{F}^*_i \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{i}} \xrightarrow{\alpha^*_{i-1}} \overline{F}^*_{i-1} \longrightarrow 0$$

it follows that $\overline{F}^*_{i-1}$ is globally generated, and by Koszul complex, $h^0(\overline{F}^*_{i-1}) = \binom{n+1}{i}$, which proves (4). Then by Lemma 4.0.20 it follows that $\overline{F}_i$ is simple. Now to see that F_i is simple, note that from Koszul complex we have the sequences

$$0 \longrightarrow \overline{F}^*_i \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{i}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d)^{\binom{n+1}{i+1}} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}((i-1)d)^{n+1} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(id) \longrightarrow 0$$

and

$$0 \longrightarrow F_{n-i}(id) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{i}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d)^{\binom{n+1}{i+1}} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}((i-1)d)^{n+1} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}(id) \longrightarrow 0\ .$$

This show us that $\overline{F}^*_i$ and $F_{n-1}(id)$ are $i-$syzygies of I then

$$\overline{F}^*_i \simeq F_{n-i}(id) \simeq F_i((n-i+1)d)$$

hence F_i is simple.

Now we use induction on i to prove $\mathrm{rk}(F_i) = \binom{n}{i}$, $1 \leq i \leq n-1$. From the short exact sequences of Koszul complex we can see that

$$\mathrm{rk}(F_{n-k}) = \binom{n+1}{k} - \mathrm{rk}(F_{n-(k-1)}),\ \ 2 \leq k \leq n.$$

For the case $i = 1$ we have the resolution

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

and $\mathrm{rk} F_{n-1} = n = \binom{n}{1}$. Let us suppose that it is true for $2 \leq j \leq i-1$. We have

$$\mathrm{rk} F_{n-i} = \binom{n+1}{i} - \mathrm{rk} F_{n-(i-1)} = \binom{n+1}{i} - \binom{n}{i-1} = \binom{n}{i}.$$

Since F_i is isomorphic to F_{n-i}^* up to twist, follows that $\mathrm{rk} F_i = \binom{n}{i}$ for $1 \leq i \leq n-1$. $\square$

Observation 4.2.4. *Using the Koszul complex we can show on a different way how to construct a simple vector bundle E given by exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-ld) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(l-1)d)^{n+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{\binom{n+1}{l-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n}{l-1}+n} \longrightarrow E \longrightarrow 0$$

such that $\mathrm{hd}(E) = l$ *and* $\mathrm{rk} E = n$ *for* $l, d, n \in \mathbb{Z}$, $1 \leq l \leq n-1$.

Consider the ring R and the ideal I like above for $n \geq 4$, and the Koszul complex of R/I. Twist the complex (4.3) with $\mathcal{O}_{\mathbb{P}^n}((n-l+1)d)$ and take the short exact sequences. From Theorem 4.2.3 we know that for $1 \leq l \leq n-1$, F_l is simple, then $G_l = F_l((n-l+1)d)$ is simple, it is given by exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-ld) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(l-1)d)^{n+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{\binom{n+1}{l-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{l}} \longrightarrow G_l \longrightarrow 0$$

and has rank $\binom{n}{l}$. Since G_l is globally generated, by Lemma 4.3.1 of [36], there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n}{l}-n} \longrightarrow G_l \longrightarrow E_l \longrightarrow 0$$

Choose $\beta_l : \mathcal{O}_{\mathbb{P}^n}^{\binom{n}{l}-n} \to G_l$ a generic map. Then we have

(1) $\mathcal{O}_{\mathbb{P}^n}, G_l$ are simple

(2) $\mathrm{Hom}(G_l, \mathcal{O}_{\mathbb{P}^n}) = H^0(G_l^*) = 0$

In fact, from the short exact sequences we have

$$H^0(G_l^*) \simeq H^{n-l-1}(G_{n-1}^*) = 0$$

(3) $\mathrm{Ext}^1(G_l, \mathcal{O}_{\mathbb{P}^n}) = H^1(G_l^*) = 0$

In fact by the complex (4.3) we can see $G_l^* \simeq F_{n-l}(ld)$ and since from Lemma 4.2.2 $\mathrm{hd}(F_{n-l}) \leq n-l$, by Proposition 4.0.10 we have $H^1(G_l^*) = 0$.

(4) G_l is globally generated;

(5) Then since $h^0(G_l) = \binom{n+1}{l} \geq \binom{n}{l} - n$, from Lemma 4.0.20 we have that E_l is simple, and using mapping cone procedure E_l is a rank n vector bundle given by exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-ld) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(l-1)d)^{n+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{\binom{n+1}{l-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n}{l-1}+n} \longrightarrow E_l \longrightarrow 0$$

and by Lemma 4.2.2, $\mathrm{hd}E_l = l$.

4.3 Studying generalizations of Tango bundle

Cascini in 2001, see [11], introduced a new class of rank $n-1$ bundles on $\mathbb{P}^n$ that generalizes the Tango bundle, cf. [41]. Below we have the definition.

Let α, γ be integers such that $\gamma > n\alpha \geq 0$. The bundles $F_{\alpha,\gamma}(\gamma)$ given by exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2\gamma) \longrightarrow \oplus_{k=0}^{n}\mathcal{O}_{\mathbb{P}^n}((n-2k)\alpha - \gamma) \longrightarrow$$
$$\longrightarrow \oplus_{k=0}^{2(n-1)}\mathcal{O}_{\mathbb{P}^n}((2n-1-2k)\alpha) \longrightarrow F_{\alpha,\gamma}(\gamma) \longrightarrow 0$$

are called *weighted Tango bundles of weights α and γ.*

The techniques developed in the previous chapters give us another way to construct the weighted Tango bundles of weights 0 and d.

Theorem 4.3.1. *Given $n \in \mathbb{Z}_+$ and $d \geq 1$, there is a simple vector bundle E over $\mathbb{P}^n$ with $\mathrm{rk}E = n-1$ and resolution*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{2n-1} \longrightarrow E \longrightarrow 0$$

Proof. Consider $R = k[x_0, \cdots, x_n]$ and the ideal $I = (f_1, \cdots, f_{n+1})$ generated by a regular sequence of forms of degree d. Take the sheafification of the Koszul complex of R/I. We have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(n+1)d) \xrightarrow{\alpha^t} \mathcal{O}_{\mathbb{P}^n}(-nd)^{n+1} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0 \tag{4.4}$$

where $\alpha : \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \to \mathcal{O}_{\mathbb{P}^n}$ is the map given by the forms.

Twist the complex (4.4) with $\mathcal{O}_{\mathbb{P}^n}((n-1)d)$ and take the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{2}} \longrightarrow T_1 \longrightarrow 0\ .$$

We know from Theorem 4.2.3 that T_1 is simple and has rank $\binom{n}{2}$. Since it is globally generated, by [36] there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n}{2}-n} \longrightarrow T_1 \longrightarrow T_2 \longrightarrow 0 \tag{4.5}$$

Then we have $\mathrm{rk(T_2) = n}$ and since T_1 is globally generated, T_2 is also globally generated.

Lemma 4.3.2. *The top Chern class of T_2 is zero.*

Proof. **Proof of Lemma:**

Consider the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{2}} \longrightarrow T_1 \longrightarrow 0$$

then the Chern polynomial of T_1 is given by

$$c_t(T_1) = \frac{1-2dt}{(1-dt)^{n+1}}$$

We want to prove that the n-th coefficient of this polynomial is zero. Making a change of variables $dt = s$, we want to prove that the $n-$th coefficient of the polynomial

$p(s) = \frac{1-2s}{(1-s)^{n+1}}$ is zero. Consider the ring $R = k[x_0, \dots, x_n]$ as a graded vector space over k. Then the Hilbert-Poincaré series of R is given by

$$\frac{1}{(1-s)^{n+1}} = \sum_{i \geq 0} \binom{n+i}{i} s^i$$

therefore

$$\frac{1-2s}{(1-s)^{n+1}} = \sum_{i \geq 0} \binom{n+i}{i} s^i - 2 \sum_{i \geq 0} \binom{n+i}{i} s^{i+1}$$

and the $n-$th coefficient of this series is given by

$$\binom{2n}{n} - 2\binom{2n-1}{n-1} = 0$$

which completes the proof.

□

Now since $c_n(T_2) = 0$ and T_2 is globally generated, by Lemma 4.3.2 of [36] there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0 \tag{4.6}$$

Applying mapping cone procedure we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{2n-1} \longrightarrow T_3 \longrightarrow 0 \tag{4.7}$$

Then we must prove that T_3 is simple. Consider the sequences (4.5) and (4.6). We want to prove that

$$\mathrm{h}^0(\mathbb{P}^n, T_3^* \otimes T_3) = 1.$$

Tensoring the dual of (4.6) with T_3 we get

$$0 \longrightarrow T_3^* \otimes T_3 \longrightarrow T_2^* \otimes T_3 \longrightarrow T_3 \longrightarrow 0$$

then

$$\mathrm{h}^0(\mathbb{P}^n, T_3^* \otimes T_3) \leq \mathrm{h}^0(\mathbb{P}^n, T_2^* \otimes T_3).$$

Tensoring the dual of (4.5) with T_3 we get

$$0 \longrightarrow T_2^* \otimes T_3 \longrightarrow T_1^* \otimes T_3 \longrightarrow T_3^{\binom{n}{2}-n} \longrightarrow 0$$

and

$$\mathrm{h}^0(\mathbb{P}^n, T_2^* \otimes T_3) \leq \mathrm{h}^0(\mathbb{P}^n, T_1^* \otimes T_3)$$

Then we need to show that $\mathrm{h}^0(\mathbb{P}^n, T_1^* \otimes T_3) = 1$. Tensoring (4.5) and (4.6) with T_1^* we have

$$0 \longrightarrow T_1^* \longrightarrow T_1^* \otimes T_2 \longrightarrow T_1^* \otimes T_3 \longrightarrow 0$$

and

$$0 \longrightarrow T_1^{*\binom{n}{2}-n} \longrightarrow T_1^* \otimes T_1 \longrightarrow T_1^* \otimes T_2 \longrightarrow 0$$

Looking at the long cohomology exact sequence, we see that it is enough to prove that T_1 is simple and

$$\mathrm{h}^0(\mathbb{P}^n, T_1^*) = \mathrm{h}^1(\mathbb{P}^n, T_1^*) = 0.$$

But T_1 is simple by Theorem 4.2.3 and taking the short exact sequences in the Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+1} \longrightarrow T_0 \longrightarrow 0$$

and

$$0 \longrightarrow T_0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\binom{n+1}{2}} \longrightarrow T_1 \longrightarrow 0$$

we have

$$\mathrm{H}^0(\mathbb{P}^n, T_0^*) \simeq \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})^{\binom{n+1}{2}}$$

then $\mathrm{H}^0(\mathbb{P}^n, T_1^*) = \mathrm{H}^1(\mathbb{P}^n, T_1^*) = 0$ and follows the result.

□

Motivated by Theorem 4.3.1, we may consider generalizing the result by allowing more summands on the first term of resolution, that is, we want to find out for which integers a, d, n the following resolution exists

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+a} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(ad-d)^{2n-1} \longrightarrow E \longrightarrow 0\,. \quad (4.8)$$

We can construct a vector bundle E' of rank n given by resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{n+a} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(ad-d)^{2n} \longrightarrow E' \longrightarrow 0\,. \tag{4.9}$$

In fact, consider a generic cokernel bundle C of rank n on $\mathbb{P}^n$ given by resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2d)^a \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d)^{a+n} \longrightarrow C \longrightarrow 0\,.$$

Since C is stable, by [8, Theorem 2.7], C is simple. For a generic map $\beta : C \to \mathcal{O}_{\mathbb{P}^n}(ad-d)^{2n}$, the degeneracy locus of β is empty, and taking the cokernel of β, the vector bundle $E = \mathrm{coker}\beta$ is given by resolution (4.9).

In order to apply Lemma 4.3.2 of [36] and the mapping cone procedure thus constructing a new bundle E of rank $n-1$, we must have $c_n(E) = 0$. Therefore, we need to find for which values of a, d, the top Chern class of E' is zero. This is equivalent to find a such that the coefficient of t^n in the series

$$c(E') = \frac{(1-2dt)^a(1+(ad-d)t)^{2n}}{(1-dt)^{n+a}} \tag{4.10}$$

is zero. Until now we have the following.

For $a = 1$, we have the case of Tango bundle. Using the software Mathematica, we can see that for $2 \leq n \leq 10$, $1 \leq a \leq 2^n$ and for any d, the only solution of $c_n(E) = 0$ is $a = 1$. This leads us to believe that there are no resolutions of the form (4.8) for $a \neq 1$. One of our future goal is to try proving this claim.

4.4 Minimal free resolution of CGAG algebras

In this section we apply the same ideas of section 4.2 to find other examples of simple vector bundles on $\mathbb{P}^n$. Now we use a different minimal free resolution. For the definitions below see [24, 33].

Let R be a local ring with maximal ideal M over a field k. Let $I \subset R$ be an ideal such that $A = R/I$ is Artinian with maximal ideal $m = M + I$. The *socle* of A, denoted Soc$A$, is the ideal $(0 : m) \subset A$.

The Artinian algebra A is *Gorenstein* (or I is Gorenstein) if and only if $(0 : m)$ has lenght one. In this case, the largest integer j such that $m^j \neq 0$ is the *socle-degree* of A.

Now let k be a field and $R = k[x_0, \cdots, x_n]$ be the ring of polynomials. An Artinian $k-$algebra $A = R/I$ has *socle degrees* $(s_1, \cdots, s_t)$, if the minimal generators of its socle (as $R-$module) have degrees $s_1 \leq \cdots \leq s_t$.

The Hilbert-function of A is denoted by $h_A(t) := \dim_k A_t$. If the Artinian algebra A has socle degrees $(s_1, \cdots, s_t)$, then $s = \max\{s_1, \cdots, s_t\}$ is called the *socle degree* of A. For fixed socle degrees, a graded Artinian algebra is *compressed*, if it has maximal Hilbert function among all graded Artinian algebras with that socle degrees.

Let $\mathbb{P}^n = \mathrm{Proj}(R)$. Let $I = (f_1, \ldots, f_{\alpha_1})$ be an ideal generated by α_1 forms of degree $t+1$, such that the algebra $A = R/I$ is a compressed Gorenstein Artinian graded algebra of embedding dimension $n+1$ and socle degree $2t$. Thus, by Proposition 3.2 of [33], the minimal free resolution of A is

$$0 \longrightarrow R(-2t-n-1) \longrightarrow R(-t-n)^{\alpha_n} \longrightarrow R(-t-n+1)^{\alpha_{n-1}} \longrightarrow \cdots$$

$$\cdots \longrightarrow R(-t-p)^{\alpha_p} \longrightarrow \cdots \longrightarrow R(-t-2)^{\alpha_2} \longrightarrow R(-t-1)^{\alpha_1} \longrightarrow R \longrightarrow A \longrightarrow 0$$

where

$$\alpha_i = \binom{t+i-1}{i-1}\binom{t+n+1}{n+1-i} - \binom{t+n-i}{n+1-i}\binom{t+n}{i-1}, \; for\, i = 1, \cdots, n.$$

Sheafifying the complex above we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n)^{\alpha_n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n+1)^{\alpha_{n-1}} \longrightarrow \cdots \tag{4.11}$$

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-p)^{\alpha_p} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-2)^{\alpha_2} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

where β is the map given by the α_1 forms of degree $t+1$.

Consider the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n)^{\alpha_n} \longrightarrow F_1 \longrightarrow 0 \tag{4.12}$$

$$0 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n+1)^{\alpha_{n-1}} \longrightarrow F_2 \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-p} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-p)^{\alpha_p} \longrightarrow F_{n-(p-1)} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

Lemma 4.4.1. *For each* $1 \leq i \leq n-1$*, we have* $\mathrm{hd}(F_i) = i$.

Proof. Let us prove it by induction on i.

(i) For $i = 1$, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n)^{\alpha_n} \longrightarrow F_1 \longrightarrow 0\,.$$

Hence $\mathrm{hd}(F_1) \leq 1$. Since $H^{n-1}(F_1(-t)) \simeq H^n(\mathcal{O}_{\mathbb{P}^n}(-n-1-t)) \neq 0$, by Horrocks' splitting criterion, Theorem 4.0.19 F_1 does not split. Therefore $\mathrm{hd}(F_1) = 1$.

(ii) Suposse for $2 \leq j \leq l-1$, $hd(E_j) = j$. Consider the short exact sequence

$$0 \longrightarrow F_{l-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n+l-1)^{\alpha_{n-l+1}} \longrightarrow F_l \longrightarrow 0$$

then by the long exact sequence of cohomologies

$$\cdots \longrightarrow H^k(\mathcal{O}_{\mathbb{P}^n}(-t-n+l-1))^{\alpha_{n-l+1}} \longrightarrow H^k(F_l) \longrightarrow H^{k+1}(F_{l-1}) \longrightarrow \cdots$$

we have

$$H^k(F_l) \simeq H^{k+1}(F_{l-1}), 1 \leq k \leq n-2.$$

Since $\mathrm{hd}(F_{l-1}) = l-1$ by Proposition 4.0.10, $H^p_*(F_{l-1}) = 0, 1 \leq p \leq n-l$ and there is $t_0 \in \mathbb{Z}$ such that $H^{n-l+1}(F_{l-1}(t_0)) \neq 0$. Therefore we have

$$H^p_*(F_l) = 0, 1 \leq p \leq n-l-1$$

hence $\mathrm{hd}(F_l) \leq l$ by Proposition 4.0.10. Since $H^{n-l}(F_l(t_0)) \simeq H^{n-l+1}(F_{l-1}(t_0)) \neq 0$, we can not have $\mathrm{hd}(F_l) \leq l-1$, otherwise

$$H^p_*(F_l) = 0, 1 \leq p \leq n-l.$$

Therefore $\mathrm{hd}(F_l) = l$ and we are done.

$\square$

Lemma 4.4.2. *For each* $1 \leq i \leq n-1$, $h^0(F_i^*(-t-n+i)) = \alpha_{n-i}$.

Proof. Since the complex (4.11) is self-dual up to twist, see [34] Theorem 1.1.27, we only need to check it for $1 \le i \le \frac{n-1}{2}$. Proving the Lemma is equivalent to show that

$$\mathrm{h}^0(F^*_{n-j}(-t-j)) = \alpha_j, \text{ for } 1 \le j \le \frac{n-1}{2}.$$

Taking the duals in (4.12) we have

$$0 \longrightarrow F_1^* \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+n)^{\alpha_n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(2t+n+1) \longrightarrow 0 \tag{4.13}$$

$$0 \longrightarrow F_2^* \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+n-1)^{\alpha_{n-1}} \longrightarrow F_1^* \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F^*_{n-p+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+p)^{\alpha_p} \longrightarrow F^*_{n-p} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F^*_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+2)^{\alpha_2} \longrightarrow F^*_{n-2} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+1)^{\alpha_1} \longrightarrow F^*_{n-1} \longrightarrow 0$$

By induction on j we have

- for $j = 1$, twisting (4.13) with $\mathcal{O}_{\mathbb{P}^n}(-t-1)$ gives us

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_1} \longrightarrow F^*_{n-1}(-t-1) \longrightarrow 0$$

thus

$$\dim \mathrm{H}^0(F^*_{n-1}(-t-1)) = \alpha_1.$$

- Now suppose the Lemma is true for $2 \leq l \leq n-2$. Let us prove that $\mathrm{h}^0(F^*_{n-(l+1)}(-t-(l+1))) = \alpha_{l+1}$. Twist (4.13) with $\mathcal{O}_{\mathbb{P}^n}(-t-l-1)$, then we have

$$0 \longrightarrow F^*_{n-l}(-t-l-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_{l+1}} \longrightarrow F^*_{n-l-1}(-t-l-1) \longrightarrow 0$$

From the long exact sequence of cohomologies

$$0 \longrightarrow H^0(F^*_{n-l}(-t-l-1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n})^{\alpha_{l+1}} \longrightarrow$$

$$\longrightarrow H^0(F^*_{n-l-1}(-t-l-1)) \longrightarrow H^1(F^*_{n-l}(-t-l-1)) \longrightarrow 0$$

Since $\mathrm{hd}(F^*_{n-l}) = l \leq n-2$, by Proposition 4.0.10 we have $H^1(F^*_{n-l}(-t-l-1)) = 0$. Now we want to check that $H^0(F^*_{n-l}(-t-l-1)) = 0$. Twisting (4.13) with $\mathcal{O}_{\mathbb{P}^n}(-t-l-1)$ we have

$$0 \longrightarrow F^*_{n-l+1}(-t-l-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\alpha_l} \longrightarrow F^*_{n-l}(-t-l-1) \longrightarrow 0$$

then $H^0(F^*_{n-l}(-t-l-1)) \simeq H^1(F^*_{n-l+1}(-t-l-1))$. Again by Proposition 4.0.10, $\mathrm{hd}(F^*_{n-l+1}) = l-1 \leq n-2$, then $H^1(F^*_{n-l+1}(-t-l-1)) = 0$ which implies that $H^0(F^*_{n-l}(-t-l-1)) = 0$ and therefore $\dim H^0(F^*_{n-l-1}(-t-l-1))) = \alpha_{l+1}$.

□

Theorem 4.4.3. *All F_i are simple for $i = 1, \cdots, n$.*

Proof. We know that the resolution is self dual up to twist, thus we only need to check for F_i, $1 \leq i \leq \frac{n-1}{2}$. Let us prove by induction on i. We can not apply Theorem 2.3.4 because the map $\beta : \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \to \mathcal{O}_{\mathbb{P}^n}$ is not generic. Thus we need an ad hoc proof.

Since the complex (4.11) is self dual, we have

$$F_{n-1} \simeq F_1^*(-2t-n-1).$$

Thus is sufficient to prove that F_{n-1} is simple. Applying $\mathrm{Hom}(-, F_{n-1})$ to the sequence

$$0 \longrightarrow F_{n-1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0$$

we have

$$0 \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, F_{n-1}) \longrightarrow \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1}, F_{n-1}) \longrightarrow \tag{4.14}$$

$$\longrightarrow \mathrm{Hom}(F_{n-1}, F_{n-1}) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, F_{n-1}) \longrightarrow \cdots$$

In order to prove that F_{n-1} is simple, is enough to check

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1}, F_{n-1}) = H^0(F_{n-1}(t+1))^{\alpha_1} = 0,$$

since the exact sequence (4.14) reduces to

$$0 \longrightarrow \mathrm{Hom}(F_{n-1}, F_{n-1}) \longrightarrow k$$

and $1 \in \mathrm{Hom}(F_{n-1}, F_{n-1})$ then $\mathrm{Hom}(F_{n-1}, F_{n-1}) \simeq k$ and F_{n-1} is simple. Indeed, twist all sequences of (4.12) by $\mathcal{O}_{\mathbb{P}^n}(t+1)$. We have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n+1)^{\alpha_n} \longrightarrow F_1(t+1) \longrightarrow 0 \tag{4.15}$$

$$0 \longrightarrow F_1(t+1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n+2)^{\alpha_{n-1}} \longrightarrow F_2(t+1) \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-p}(t+1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-p+1)^{\alpha_p} \longrightarrow F_{n-p+1}(t+1) \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow F_{n-1}(t+1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\alpha_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+1) \longrightarrow 0$$

and applying the long exact sequence of cohomology in the sequences above, one concludes that

$$H^0(F_{n-1}(t+1)) \simeq H^1(F_{n-2}(t+1)) \simeq H^j(F_{n-j-1}(t+1)) \text{ for } 1 \leq j \leq n-2.$$

However,

$$H^{n-2}(F_1(t+1)) \simeq H^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-t-n)) = 0.$$

Thus $H^0(F_{n-1}(t+1))^{a_1} = 0$ therefore F_{n-1} is simple. It follows that

- $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}, F_{n_1}) \simeq H^0(\mathbb{P}^n, F_{n-1}) = 0$
- $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}, F_{n-1}) \simeq H^1(\mathbb{P}^n, F_{n-1}) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$.

Now we know that F_1 is simple, let us prove by induction on i that F_i is simple for $2 \leq i \leq \frac{n-1}{2}$. Suppose F_{k-1} is simple, $2 \leq k \leq \frac{n-1}{2}$. Let γ_k be a generic map

$$\gamma_k : F_{k-1} \to \mathcal{O}_{\mathbb{P}^n}(-t-(n-k+1))^{\alpha_{n-k+1}}.$$

We have

(1) F_{k-1} and $\mathcal{O}_{\mathbb{P}^n}(-t-n+k-1)$ are simple;

(2) $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-t-n+k-1), F_{k-1}) \simeq H^0(F_{k-1}(t+n-k+1)) = 0$

In fact tensoring (4.12) by $\mathcal{O}_{\mathbb{P}^n}(t+n-k+1)$ we have the sequence

$$0 \longrightarrow F_{k-2}(t+n-k+1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\alpha_{n-k+2}} \longrightarrow F_{k-1}(t+n-k+1) \longrightarrow 0 \,.$$

Therefore

$$H^0(F_{k-1}(t+n-k+1)) \simeq H^1(F_{k-2}(t+n-k+1)).$$

by Lemma 4.4.1, $\mathrm{hd}(F_{k-2}) = k-2$, thus by Proposition 4.0.10, $H^1(F_{k-2}(t+n-k+1)) = 0$.

(3) $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^n}(-t-n+k-1), F_{k-1}) \simeq H^1(F_{k-1}(t+n-k+1)) = 0$

Indeed, since we have $\mathrm{hd}(F_{k-1}) = k-1$, then $H^1(F_{k-1}(t+n-k-1)) = 0$ by Proposition 4.0.10.

(4) $F_{k-1}^* \otimes \mathcal{O}_{\mathbb{P}^n}(-t-n+k-1) \simeq F_{k-1}^*(-t-n+k-1)$ is globally generated.

It is true because from the complex (4.12) we have

$$F_j^* \simeq F_{n-j}(2t+n+1).$$

Thus $F_{k-1}^*(-t-n+k-1) \simeq F_{n-k+1}(t+k)$. Twisting (4.12) by $\mathcal{O}_{\mathbb{P}^n}(t+k)$ we see that $F_{n-k+1}(t+k)$ is globally generated.

(5) $\dim \mathrm{Hom}(F_{k-1}, \mathcal{O}_{\mathbb{P}^n}(-t-n+k-1)) \simeq h^0(F_{k-1}^*(-t-n+k-1)) = \alpha_{n-k+1}$.

It follows from Lemma 4.4.2. Now we have by definition that $\overline{F}_k = \text{coker}\alpha_k$ is a cokernel bundle. Since

$$q(1, \alpha_{n-k+1}) = 1 + \alpha_{n-k+1}^2 - \alpha_{n-k+1}(h^0(F_{k-1}^*(-t-n+k-1))) = 1,$$

it follows from Theorem 2.3.4 that $\overline{F}_k$ is simple. Hence we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2t-n+1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n)^{\alpha_n} \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-n+k-1)^{\alpha_{n-k+1}} \longrightarrow \overline{F}_k \longrightarrow 0$$

Taking dual,

$$0 \longrightarrow \overline{F}_k^* \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+n-k+1)^{\alpha_{n-k+1}} \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(t+n)^{\alpha_n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(2t+n+1) \longrightarrow 0\ .$$

Twisting with $\mathcal{O}_{\mathbb{P}^n}(-2t-n-1)$

$$0 \longrightarrow \overline{F}_k^*(-2t-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-k)^{\alpha_{n-k+1}} \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_n} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0\ .$$

However, from the complex (4.11) we have the sequence

$$0 \longrightarrow F_{n-k} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-k)^{\alpha_k} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-t-1)^{\alpha_1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0\ .$$

Since $\overline{F}_k^*(-2t-n-1)$ and F_{n-k} are the $k-$syzygies of β, they are isomorphic. Therefore $F_{n-k} \simeq \overline{F}_k^*(-2t-n-1)$ is simple and we are done.

□

Example 4.4.4. *Let* $R = k[x_0, x_1, x_2, x_3]$, $\mathbb{P}^3 = \text{Proj}(R)$. *Consider five points* $P_1, \cdots, P_5$ *on* $\mathbb{P}^3$. *They are given by five quadrics* $f_1, \cdots, f_5$. *Let* $I = (f_1, \cdots, f_5)$ *be the ideal generated by the quadrics. Take* J *the Artinian reduction of* I *(see [34, Remark 1.1.23]). Then* $A = R/J$ *is Artinian with minimal free resolution*

$$0 \longrightarrow R(-5) \longrightarrow R(-3)^5 \longrightarrow R(-2)^5 \longrightarrow R \longrightarrow R/J \longrightarrow 0\ .$$

Then A *is an Artinian graded algebra with Hilbert function* $h_A(0) = 1, h_A(1) = 3, h_A(2) = 1, h_A(i) = 0, i \geq 3$. *Then* A *has socle degree* 2, *and from the resolution it is Gorenstein (see [34, Definition 1.1.26]). Therefore* A *is a compressed Gorenstein Artinian graded algebra with embedding dimension* 4 *and socle degree* 2.

We now intend to study the simple bundles arising from the resolutions (4.3) and (4.11). For instance, we will try to prove whether or not they are stable.

We also want to apply the techniques of Theorem 4.1.1 to dual resolutions of syzygy bundles, hoping to find new examples of simple or stable bundles, and maybe even of low rank bundles.

Bibliography

[1] H. Abo, W. Decker, N. Sasakura. *An elliptic conic bundle in* $\mathbb{P}^4$ *arising from a stable rank-*3 *vector bundle*. Mathematische Zeitschrift 229 (1998), no.4, 725-741

[2] L. Álvarez-Cónsul, A. King. *A functorial construction of moduli of sheaves*. Invent. math. 168 (2007), 613-666.

[3] V. Ancona, G. Ottaviani. *Stability of special instanton bundles on* $\mathbb{P}^{2n+1}$. Trans. Amer. Math. Soc. 341 (1994), no.2, 677-693.

[4] V. Ancona, G. Ottaviani. *Unstable hyperplanes for Steiner bundles and multidimensional matrices*. Adv. Geom. 1 (2001), no.2, 165-192.

[5] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris. *Geometry of algebraic curves*. Springer - Verlag, New York, 1985.

[6] I. Assem, D. Simson, A. Skowrónski. *Elements of the representation theory of associative algebras*. Volume 1 Techniques of Representation Theory, London Mathematical Society Student Texts 65 , Cambridge University Press, 2006.

[7] W. Barth. *Some properties of stable rank* 2 *bundles on* $\mathbb{P}^n$. Math. Ann., 226 (1977), 125-150.

[8] G. Bohnhorst, H. Spindler. *The Stability of certain vector bundles on* $\mathbb{P}^n$. Complex algebraic varieties (Bayreuth, 1990), Lecture Notes in Mathematics 1507, Springer, Berlin (1992), 39-50.

[9] M.C. Brambilla. *Cokernel bundles and Fibonacci bundles*. Math. Nach. 281 (2008), no.4, 499 - 516.

[10] M. C. Brambilla. *Simplicity of generic Steiner bundles*. Boll. Unione Mat. Ital. Sez. B, Artic. Ric. Mat. (8) 8 (2005), no.3, 723 - 735.

[11] P. Cascini. *Weighted Tango bundles on $\mathbb{P}^n$ and their moduli spaces.* Forum Math. 13 (2001), no.2, 251-260.

[12] P. Cascini. *On the moduli space of the Schwarzenberger bundles.* Pacific J. Math. 205 (2002), no.2, 311-323.

[13] M. C. Chang. *A filtered Bertini-type Theorem.* J. reine angew. Math. 397 (1989), 214 - 219.

[14] I. Dolgachev, M. Kapranov. *Arrangements of hyperplanes and vector bundles on* $\mathbb{P}^n$. Duke Math. J. 71 (1993), no.3, 633-664.

[15] L. Ein. *Generalized null correlation bundles.* Nagoya Math. J. 111 (1988), 13-24.

[16] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry.* Graduate Texts in Mathematics 150. Springer - Verlag, 1995.

[17] P. Gabriel. *Auslander-Reiten sequences and representation finite algebras* Representation theory I (eds V. Dlab and P. Gabriel), Lecture Notes in Mathematics 831, Springer, Berlin (1980), 1-71.

[18] P. Griffiths, J. Harris. *Principles of algebraic geometry.* Wiley, New York, 1978.

[19] A. Grothendieck. *Sur la classification des fibrés holomorphes sur la sphère de Riemman.* Amer. J. Math. 79 (1957), 121-138.

[20] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics 52. Springer, 1977.

[21] R. Hartshorne. *Varieties of small codimension in projective space.* Bulletin of the American Mathematical Society. Vol. 80 (1974), no. 6, 1017-1032.

[22] G. Horrocks. *Examples of rank three vector bundles on five-dimensional projective space.* The Journal of the London Mathematical Society. Second Series 18 (1) (1978), 15-27.

[23] G. Horrocks, D. Mumford. *A rank* 2 *vector bundle on Image* 4 *with* 15000 *symmetries.* Topology, Vol 12, Issue 1, (1973), 63-81.

[24] A. A. Iarrobino. *Associated graded algebra of a Gorenstein Artin algebra.* Memoirs of the American Mathematical Society, 514, Vol 107, 1994.

[25] M. Jardim. *Instanton sheaves on complex pojective spaces*. Collect. Math. 57 (2006), no.1, 69-91.

[26] M. Jardim, R.V. Martins. *Linear and Steiner bundles on projective varieties*. Communications of Algebra, 38 (2010), 2249-2270.

[27] V. G. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, 1990.

[28] V. G. Kac. *Infinite root systems, representations of graphs and invariant theory*. Inventiones Math. 56 (1980), no.1, 57-92.

[29] V. G. Kac. *Root systems, representations of quivers and invariant theory*.Lecture Notes in Mathematics, vol. 996. Invariant Theory (Montecatini, 1982), Springer, Berlin (1983), 74-108.

[30] N. M. Kumar, C. Peterson, A. P. Rao. *Construction of low rank vector bundles on $\mathbb{P}^4$ and $\mathbb{P}^5$*. J. Algebraic Geometry, 11 (2002), no. 2, 203-217.

[31] S. Lang. *Algebra*. Graduate Texts in Mathematics 211. Springer, 2003.

[32] P. M. Marques. *Stability and moduli space of syzygy bundles*. Ph.d Thesis, University of Barcelona. arXiv:1008.2733v1 [math.AG], 2009.

[33] J. C. Migliore, R. M. Miró-Roig, U. Nagel. *Minimal resolution of relatively compressed level algebras*. J. Algebra 284 (2005), no.1, 333-370.

[34] R. M. Miró-Roig. *Determinantal ideals*. Progress in Mathematics 264, Birkhauser, 2008.

[35] R. M. Miró-Roig, H. Soares. *Cohomological characterisation of Steiner bundles*. Forum Math. 21 (2009), no.5, 871-891.

[36] C. Okonek, M. Schneider, H. Spindler. *Vector bundles on complex projective spaces*. Prog. in Math. 3, Birkhauser Boston, 1980.

[37] H. Soares. *Steiner bundles on the hyperquadric $Q_n \subset \mathbb{P}^{n+1}, n \geq 3$*. Comm. Algebra 35 (2007), no.12, 4116-4136.

[38] H. Soares. *Steiner vector bundles on algebraic varieties*. Ph.d Thesis, University of Barcelona, 2008.

[39] A. Schofield. *General representations of quivers*. Proc. London Math. Soc. (3) 65 (1992), no.1, 46-64.

[40] I. Shafarevich. *Basic algebraic geometry 2. Schemes and complex manifolds.* Springer-Verlag, 2 ed., 1997.

[41] H. Tango. *An example of indecomposable vector bundle of rank $n-1$ on $\mathbb{P}^n$.* J. Math. Kyoto Univ., vol 16, no.1 (1976), 137-141.

[42] C.A. Weibel. *An introduction to homological algebra.* Cambridge University Press, 1994.