UNIVERSIDADE ESTADUAL DE CAMPINAS

Instituto de Matemática, Estatística e Computação Científica

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Control sets of linear control systems on solvable, non-nilpotent Lie groups of dimension 3

Conjuntos de controle de sistemas lineares sobre grupos de Lie solúveis, não nilpotentes de dimensão 3

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2023

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutor em Matemática.

Thesis presented to the Institute of Mathematics, Statistics and Scientific Computing of the University of Campinas in partial fulfillment of the requirements for the degree of Doctor in Mathematics.

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Este trabalho corresponde à versão final da Tese defendida pelo aluno Alejandro Otero Robles e orientada pelo Prof. Dr. Lino Anderson da Silva Grama.

Campinas

2023

Ficha catalográfica
Universidade Estadual de Campinas
Biblioteca do Instituto de Matemática, Estatística e Computação Científica
Ana Regina Machado - CRB 8/5467

Ot2c

Otero Robles, Alejandro, 1992-
Control sets of linear control systems on solvable, non-nilpotent Lie groups of dimension 3 / Alejandro Otero Robles. – Campinas, SP : [s.n.], 2023.

Orientador: Lino Anderson da Silva Grama.
Coorientador: Adriano João da Silva.
Tese (doutorado) – Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computação Científica.

1. Sistemas de controle linear. 2. Teoria do controle. 3. Grupos de Lie. I. Grama, Lino Anderson da Silva, 1981-. II. Silva, Adriano João da, 1985-. III. Universidade Estadual de Campinas. Instituto de Matemática, Estatística e Computação Científica. IV. Título.

Informações Complementares

Título em outro idioma: Conjuntos de controle de sistemas lineares sobre grupos de Lie solúveis, não nilpotentes de dimensão 3
Palavras-chave em inglês:
Linear control systems
Control theory
Lie groups
Área de concentração: Matemática
Titulação: Doutor em Matemática
Banca examinadora:
Lino Anderson da Silva Grama [Orientador]
Pedro Jose Catuogno
Victor Alberto José Ayala Bravo
Alexandre José Santana
Eyüp Kizil
Data de defesa: 28-09-2023
Programa de Pós-Graduação: Matemática

Identificação e informações acadêmicas do(a) aluno(a)
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- Currículo Lattes do autor: http://lattes.cnpq.br/0072207655066376

Tese de Doutorado defendida em 28 de setembro de 2023 e aprovada pela banca examinadora composta pelos Profs. Drs.

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A Ata da Defesa, assinada pelos membros da Comissão Examinadora, consta no SIGA/Sistema de Fluxo de Dissertação/Tese e na Secretaria de Pós-Graduação do Instituto de Matemática, Estatística e Computação Científica.

This work is dedicated to all those people who helped me get to this point.

Acknowledgements

I want to especially thank to my Ph.D advisor Professor Adriano da Silva, for his assistance during my doctorate. His willingness to discuss research issues despite the challenges, his patience in explaining complex matters, and his understanding during the toughest moments throughout these years.

I am immensely grateful to my wife, Isabel, for her invaluable support during the time we've been together, especially during these years. Without her, I wouldn't have been able to reach this point.

I am grateful to Professor Lucas Catão for his tremendous help during my initial stay in Brazil. Those were challenging times, and he was there to assist both, me and my wife.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001

Resumo

O objetivo desta tese é estudar os conjuntos de controle dos sistemas de controle linear em grupos de Lie solúveis não nilpotentes de dimensão três. Para fazer isto, dividimos os sistemas com respeito a seu nil-rank, o posto da derivação associada ao drift do sistema restrita ao nilradical da algebra de Lie do espaço base. No caso que o nil-rank é dois, mostramos que um sistema deste tipo é equivalente por difeomorfismo ao produto de um sistema homogêneo sobre $\mathbb{R}$ por um sistema afim sobre $\mathbb{R}^2$, assim, estudando os conjuntos de controle deste sistema afim sobre $\mathbb{R}^2$, mostramos que existe um único conjunto de controle sobre o sistema inicial que tem o elemento neutro do grupo no fecho. Para o caso em que o nil-rank é um, mostramos que o LARC é equivalente à condição de ad-rank, o que implica a existência de conjuntos de controle com interior não vazio. Finalmente, no caso em que o nil-rank é igual a zero, mostramos que a maioria dos sistemas tem uma quantidade infinita de conjuntos de controle de interior não vazio.

Palavras-chave: Conjunto de controle. Sistema de controle linear. Grupos de Lie solúveis não nilpotentes. LARC.

Abstract

The purpose of this thesis is to study control sets of linear control systems on solvable, non-nilpotent Lie groups of dimension three. To achieve this, we divide the systems according to their nil-rank, the rank of the derivation associated with the system's drift restricted to the nilradical of the Lie algebra of the base space. In the case where the nil-rank is two, we prove that such system is diffeomorphically equivalent to the product of a homogeneous system on $\mathbb{R}$ by an affine system on $\mathbb{R}^2$. By examining the control sets of this affine system on $\mathbb{R}^2$, we establish that there exists a unique control set for the initial system that includes the identity element of the group in its closure. For the case where the nil-rank is one, we establish that LARC (Lie Algebra Rank Condition) is equivalent to the ad-rank condition, implying the existence of control sets with non-empty interiors. Finally, in the case where the nil-rank is zero, we show that the most systems have an infinite number of control sets with non-empty interiors.

Keywords: Control set. Linear control system. Solvable nonnilpotent Lie groups. LARC.

Contents

Introduction

Control systems have been studied for a long time, in particular, linear control systems on $\mathbb{R}^n$ have many physical applications ((LEITMANN, 1962; PONTRYAGIN et al., 1962; SHELL, 1968)). Roughly speaking, a control system is a system characterized by a base space (differentiable manifold) and a dynamics characterized by a family of differential equations parameterized by functions known as controls.

The first generalization of a linear control system was carried out by L. Markus in (MARKUS, 1980) for a group of matrices. Subsequently, V. Ayala and J. Tirao in (AYALA; TIRAO, 1999) introduced the concept of linear control systems on an arbitrary Lie group. One of the main reasons to study control systems on Lie groups was provided by P. Juan in (JOUAN, 2010), where it is shown that every affine control system with complete vector fields that generate a finite-dimensional Lie algebra is diffeomorphically equivalent to a linear control system on a Lie group or a homogeneous space.

On the other hand, one of the main concepts to be studed in control systems is controllability: given any two points in the base space, can we connect these points through solutions of the family of differential equations in positive time? For this question, there are some answers that depend on the algebra and geometry of the control system. For instance, in (DA-SILVA, 2016), it is shown that a control system on a nilpotent Lie group is controllable if and only if the reachable set from the identity is open, and the eigenvalues of the derivation associated to the drift of the system have zero real part. Additionally, V. Ayala and A. Da Silva demonstrated in (AYALA; DA-SILVA, 2015) that $Spec(\mathcal{D}) \cap \mathbb{R} = 0$, where $\mathcal{D}$ is the derivation associated to the drift of the system, implies controllability if the base space of the system has finite semisimple center, i.e any Lie group admitting a maximal semisimple Lie subgroup with finite center.

Unfortunately, the controllability of a system is a rare property; therefore, it makes more sense to approach the problem in a more realistic way by considering control sets – regions in the base space where approximate controllability holds. Thus, in this direction, one might inquire whether given a control system, there exist control sets. If so, how many exist, what are its properties and how to make them depend on the geometry of the base space? For example, in (AYALA; DA-SILVA; ZSIGMOND, 2017), it is shown that a linear control system on a solvable Lie group has a unique control set with a non-empty interior.

Motivated by the above contextualization and by the controllability results on linear control systems on solvables non-nilpotent Lie groups of dimension 3 (see (AYALA; DA-SILVA, 2019)) in this thesis we study, control sets of linear control systems over non-nilpotent solvable Lie groups of dimension 3 are studied and characterized. These groups are characterized through their Lie algebras, described in (ONISHCHIK; VINBERG, 1994).

This work is organized as follows:

In the chapter 1, we formally define the objects of study for the thesis and describe the main tools for studying these objects. First we define an affine control system and control sets, then we describe properties of the control sets. After this, we introduce LARC and establish conditions under which two affine control systems are conjugate and equivalent. In this chapter, we also show that given a linear control system on a Lie group, there exists a unique linear control system on the universal covering group that is conjugate to it. In other words, we can lift the given control system to a control system on a simply connected group. Additionally, we show that it is possible to project the initial linear control system onto a linear control system on a homogeneous space. We also discuss various results related to lienar control systems oon two-dimensional Lie groups (abelian and non-abelian cases). Finally, we describe solvable non-nilpotent Lie groups of dimension 3, following the approach of (ONISHCHIK; VINBERG, 1994), and prove that under certain conditions on the associated linear vector field, an LCS on these groups is equivalent to the product of a homogeneous system on $\mathbb{R}$ by an affine control system on $\mathbb{R}^2$.

In the chapter 2, we study LCS with a nil-rank equal to two. This means that the rank of the derivation associated with the system's drift, restricted to the nilradical of the Lie algebra, is 2. This condition, as proved in the previous chapter, allows us to establish an equivalence between these systems and the product of a homogeneous system on $\mathbb{R}$ by an affine system on $\mathbb{R}^2$ ($\Sigma_{\mathbb{R}\times\mathbb{R}^2}$). Because of this, we begin the chapter by studying the properties of control sets for these systems $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$. To examine these properties, we analyze affine control systems on $\mathbb{R}^2$ that are conjugated to $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ through the canonical projection. Finally, we prove that LCS on solvable non-nilpotent Lie groups of dimension 3 have a unique non-empty interior control set containing the group's identity element at the closure of the set.

In the chapter 3, we study the cases where the nilrank of a LCS is equal to zero or one. We show that for nil-rank one control systems, the LARC is enough to assure existence of control sets with nonempty interior. Moreover, in case of systems with nil-rank zero, we show that they admit an infinite numbers of control sets with empty interior.

In the chapter 4, we study the LCSs on 3D solvable non-nilpotnet Lie groups G that are not simply connected. By (ONISHCHIK; VINBERG, 1994, Chapter 7), the only connected, solvable, nonnilpotent 3D Lie groups are the group of rigid motions, its n-fold covers and the group $\mathrm{Aff}(2)\times S^1$.

Finally, in the chapter 5, we give some examples of control sets in order to illustrate the results of the work.

1 Preliminaries

In this section we introduce the basic ideas about affine control systems, control sets, linear control systems and solvable nonnilpotent Lie groups of dimension 3.

1.1 Affine control systems and control sets

In this subsection we will define the main objects that we will work on this text, namely, affine control systems and its control sets. The affine control systems are generally characterized by a base space or state space and a dynamics on this space, determined by a family of differential equations parameterized by a set of functions that we will call controls or control functions. On the other hand, control sets are intuitively maximal regions of the state space in which we have approximate controllability, i.e., from the system dynamics, we can take an element of this region and bring it as close as we want to any other point in the same region.

Definition 1. *A affine control system on a connected finite-dimensional differential manifold denoted by M, is a family of ordinary differential equations*

$$\dot{x}(s) = f_0(x(s)) + \sum_{i=1}^{m} u_i(s) f_i(x(s)),\ \mathbf{u} = (u_1, \ldots, u_m) \in \mathcal{U}. \tag{Σ_M}$$

where $f_0, \ldots, f_m$ are smooth vector fields on M and $\mathcal{U}$ is the set of all piecewise constant functions such that $\mathbf{u}(s) \in \Omega$ where $\Omega \subset \mathbb{R}^m$ is a compact, convex subset with $0 \in \operatorname{int}\Omega$. The functions $\mathbf{u}$ are called control functions.

Let us assume that, for each $u \in \mathbb{R}^m$, the map

$$F_u : x \to F(x,u) \quad \text{with} \quad F(x,u) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$$

from M to TM be a C^∞-vector field on M. Thus, for each $x \in M$ and for each $u \in \Omega$, the differential equation $\dot{x} = F(x,u)$ has a locally unique solution $s \mapsto \phi(s,x,u)$ with $\phi(0,x,u) = 0$. In addition to this, we assume that the fields F_u, for all $u \in \Omega$, are complete, i.e., the solutions of the associated differential equation exist globally for all $t \in \mathbb{R}$. Therefore, we can define the following map,

$$\phi : \mathbb{R} \times M \times \mathcal{U} \to M, \quad (s, x, \mathbf{u}) \to \phi(s, x, \mathbf{u})$$

that satisfies the cocycle property:

$$\phi(s+t,x,\mathbf{u}) = \phi(s,\phi(t,x,\mathbf{u}),\Theta_t\mathbf{u})$$

for all $s,t \in \mathbb{R}$, $x \in M$, $\mathbf{u} \in \mathcal{U}$. The map $\Theta_s\mathbf{u}$ is known as the *shift flow* on $\mathcal{U}$ defined as follows:

$$(\Theta_t\mathbf{u})(s) := \mathbf{u}(s+t).$$

From the cocycle property, it follows that:

- There exists an inverse map of the diffeomorphism $\phi(s,\cdot,u)$, and it is given by $\phi(-s,\cdot,\Theta_s u)$.
- The fact that for any $s>0$, the value $\phi(s,x,\mathbf{u})$ depends only on $\mathbf{u}|_{[0,s]}$, implies that

  $$\phi(s_2,\phi(s_1,x,\mathbf{u}_1),\mathbf{u}_2) = \phi(s_1+s_2,x,\mathbf{u})$$

  with

  $$\mathbf{u}(s) = \begin{cases} \mathbf{u}_1(s) & \text{for } \quad \tau \in [0,s_1] \\ \mathbf{u}_2(s-s_1) & \text{for } \quad \tau \in [s_1,s_1+s_2]. \end{cases}$$

For any $x \in M$, we define the set of points reachable from x and the set of points controllable to x at time $S>0$ as

$$\mathcal{O}^+_S(x) := \{y \in M | \text{ there are, } \mathbf{u} \in \mathcal{U} \text{ with } y = \phi(S,x,\mathbf{u})\}$$

$$\mathcal{O}^-_S(x) := \{y \in M | \text{ there are, } \mathbf{u} \in \mathcal{U} \text{ with } x = \phi(S,y,\mathbf{u})\}$$

respectively. The set of points reachable from x or positive orbit of x and the set of points controllable to x or negative orbit of x are

$$\mathcal{O}^+(x) := \bigcup_{S>0} \mathcal{O}^+_S(x) \quad \text{and} \quad \mathcal{O}^-(x) := \bigcup_{S>0} \mathcal{O}^-_S(x)$$

respectively.

On the other hand, we say that the system is controllable if for all $x \in M$ it holds that $M = \mathcal{O}^+(x)$.

The next result relates the positive and negative orbits of the system.

Lemma 1. *Let us consider the system Σ_M and its corresponding sets $\mathcal{O}^+_{\Sigma_M}(x)$ and $\mathcal{O}^-_{\Sigma_M}(x)$. Take also the reversed time system*

$$\dot{x}(s) = -f_0(x(s)) - \sum_{i=1}^{m} v_i(s) f_i(x(s)),\ \mathbf{v} = (v_1,\ldots,v_m) \in \mathcal{U}. \qquad (-\Sigma_M)$$

with the corresponding sets $\mathcal{O}^+_{-\Sigma_M}(x)$ and $\mathcal{O}^-_{-\Sigma_M}(x)$. Therefore, $\mathcal{O}^+_{\Sigma_M}(x) = \mathcal{O}^-_{-\Sigma_M}(x)$ and $\mathcal{O}^-_{\Sigma_M}(x) = \mathcal{O}^+_{-\Sigma_M}(x)$.

Proof. If $y \in \mathcal{O}^{+}_{\Sigma_M}(x)$, then there exists $\mathbf{u} \in \mathcal{U}$ and $s_1 > 0$ such that $y = \phi(s_1, x, \mathbf{u})$. We define the curve,

$$\alpha(s) := \phi(-s, y, \Theta_{s_1}\mathbf{u}), \qquad s \in [0, s_1].$$

It satisfies $\alpha(0) = y$ and by cocycle property $\alpha(s_1) = \phi(-s_1, \phi(s_1, x, \mathbf{u}), \Theta_{s_1}\mathbf{u}) = x$. Moreover,

$$\frac{d}{ds}\alpha(s) = \frac{d}{ds}\phi(-s, y, \Theta_{s_1}\mathbf{u})$$

$$-f_0(\phi(-s, y, \Theta_{s_1}\mathbf{u}(-s))) - \sum_{i=1}^{m}(\Theta_{s_1}u_i)(-s)f_i(\phi(-s, y, \Theta_{s_1}\mathbf{u})) = -f_0(\alpha(s)) - \sum_{i=1}^{m} v_i(s)f_i(\alpha(s)),$$

where $\mathbf{v} := \Theta_{s_1}\mathbf{u} \in \mathcal{U}$. Therefore, $\mathcal{O}^{+}_{\Sigma_M}(x) \subset \mathcal{O}^{-}_{-\Sigma_M}(x)$. The other inclusion follows analogously. □

Now we define the accessibility properties and the Lie algebra rank condition.

Definition 2. *We say that the control system Σ_M is*

- *locally accessible from x if for all $S > 0$ the sets*

$$\mathcal{O}^{+}_{\leqslant S}(x) := \bigcup_{0<s\leqslant S} \mathcal{O}^{+}_{s}(x) \quad \text{and} \quad \mathcal{O}^{-}_{\leqslant S}(x) := \bigcup_{0<s\leqslant S} \mathcal{O}^{-}_{s}(x)$$

  have nonempty interior.

- *locally accessible if it is locally accessible from every $x \in M$*

Definition 3. *We say that the control system Σ_M satisfies the Lie algebra rank condition (LARC) if $\mathcal{L}(x) = T_xM$ for any $x \in M$, where*

$\mathcal{L}$ is the smallest Lie subalgebra containing any F_u, with $u \in \Omega$.

By the general theory of control systems, the LARC implies local accessibility and these two concepts are actually equivalent if the system is analytic, i.e. system composed for analytical fields.

In what follows, we formally define the concept of control sets. Such subsets contain most of the information related with the controllability property of the system and are the object of study of this thesis.

Definition 4. *A subset D of M is called a control set of the system Σ_M if it satisfies the following properties*

(i) (Weak invariance) For every $x \in M$, there exists a $\mathbf{u} \in \mathcal{U}$ such that $\phi(\mathbb{R}^+, x, \mathbf{u}) \subset D$;

(ii) (Approximate controllability) $D \subset \overline{\mathcal{O}^+(x)}$ *for every* $x \in D$;

(iii) (Maximality) D *is maximal with respect to properties (i) and (ii).*

Definition 5. *Let* L *be a subset of* M. *We say that* L *is positively invariant if* $\mathcal{O}^+(x) \subset L$ *for every* $x \in L$. *Similarly,* L *is negatively invariant if* $\mathcal{O}^-(x) \subset L$ *for every* $x \in L$.

The following theorem summarizes some of the most important properties of control sets with nonempty interior. Their proofs can be found in Chapter 3 of (COLONIUS; KLIEMANN, 2000)

Theorem 1. *Let* Σ_M *be a locally accessible system on* M. *If* D *is a control set with nonempty interior, then*

- D *is connected and* $\overline{\operatorname{int} D} = \overline{D}$;
- $\operatorname{int} D \subset \mathcal{O}^+(x)$ *for any* $x \in D$. *For any* $y \in \operatorname{int} D$

$$D = \overline{\mathcal{O}^+(y)} \cap \mathcal{O}^-(y).$$

  In particular, controllability holds on $\operatorname{int} D$;

- *Assume that* $\varphi(s, x, \mathbf{u})$ *is a periodic trajectory, that is,* $\varphi(s + \tau, x, \mathbf{u}) = \varphi(s, x, \mathbf{u})$ *for some* $\tau > 0$ *and all* $s \in \mathbb{R}$. *If* $x \in \operatorname{int} D$ *then* $\varphi(s, x, \mathbf{u}) \in \operatorname{int} D$ *for all* $s \in \mathbb{R}$;
- D *is closed iff* D *is invariant in positive time iff* $D = \overline{\mathcal{O}^+(x)}$ *for any* $x \in D$;
- D *is open iff* D *is invariant in negative time iff* $D = \mathcal{O}^-(x)$ *for any* $x \in D$.

The following result shows that if the positive, or negative, orbit of a point $x \in M$ is open, then x is contained in a control set. Although this result is a direct consequence of the previous theorem, for the sake of completeness, we prove it here.

Proposition 1. *Let us assume that* Σ_M *satisfies the LARC and consider* $x \in M$. *If* $\mathcal{O}^+(x)$ *or* $\mathcal{O}^-(x)$ *are open sets, then*

$$D = \overline{\mathcal{O}^+(x)} \cap \mathcal{O}^-(x)$$

is a control set with nonempty interior.

Proof. Let us show that D satisfies the properties in the definition of control set.

(i) *Weak invariance:* For any $y \in \mathcal{O}^+(x) \cap \mathcal{O}^-(x)$, there exists $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ and $s_1, s_2 > 0$ such that

$$\phi(s_1, y, \mathbf{u}_1) = x \quad \text{and} \quad \phi(s_2, x, \mathbf{u}_2) = y.$$

By considering the $(s_1 + s_2)$-periodic control function given by

$$\mathbf{u}_*(s) = \begin{cases} \mathbf{u}_1(s) & 0 \leqslant s \leqslant s_1 \\ \mathbf{u}_2(s) & s_1 < s \leqslant s_2 \end{cases},$$

allows us to conclude that $s \in \mathbb{R} \mapsto \phi(s, y, \mathbf{u}_*) \in M$ is a $(s_1 + s_2)$-periodic curve that is contained in D by construction.

Now, for any $z \in D$, the fact that $z \in \mathcal{O}^-(x)$ implies the existence of $\mathbf{u} \in \mathcal{U}$ and $s > 0$ such that $\varphi(s, z, \mathbf{u}) = x$. By concatenating this curve with the previous one, we obtain a trajectory starting in z that is contained in D.

(ii) *Approximate controllability:* For any $y \in D$ it holds that

$$y \in \mathcal{O}^-(x) \quad \Longleftrightarrow \quad x \in \mathcal{O}^+(y) \quad \Longrightarrow \quad D \subset \overline{\mathcal{O}^+(x)} \subset \overline{\mathcal{O}^+(y)}.$$

(iii) *Maximality:* Let us assume that there exists a control set $\hat{D}$ such that $D \subset \hat{D}$. Since $x \in \hat{D}$ it holds that $\hat{D} \subset \overline{\mathcal{O}^+(x)}$. On the other hand, for any $y \in \hat{D}$, we get that

$$x \in \hat{D} \subset \overline{\mathcal{O}^+(y)} \quad \Longrightarrow \quad \mathcal{O}^-(x) \cap \mathcal{O}^+(y) \neq \emptyset \quad \Longrightarrow \quad y \in \mathcal{O}^-(x).$$

As the choosen $y \in \hat{D}$ was arbitrary, we conclude that $\hat{D} \subset \mathcal{O}^-(x)$ and hence $\hat{D} \subset D$, concluding the proof.

□

Suppose N is another smooth manifold, and

$$\dot{x}(s) = g_0(x(s)) + \sum_{i=1}^{m} u_i(s) g_i(x(s)), \quad \mathbf{u} = (u_1, \ldots, u_m) \in \mathcal{U} \tag{Σ_N}$$

is an affine control system on N. If $\psi : M \to N$ is a surjective smooth map, we say that Σ_M and Σ_N are ψ-conjugate if its respective vector fields are ψ-conjugate, i.e

$$\psi_* f_i = g_i \circ \psi, \quad j = 0, \ldots m.$$

If such ψ exists, we say that Σ_M and Σ_N are ψ-conjugate. If ψ is a diffeomorphism, we say that Σ_M and Σ_N are equivalent.

Proposition 2. *Let Σ_M and Σ_N be ψ-conjugated systems. It holds:*

1. *If D is a control set of Σ_M, there exists a control set E of Σ_N such that $\psi(D) \subset E$;*
2. *If for some $y_0 \in \operatorname{int} E$ it holds that $\psi^{-1}(y_0) \subset \operatorname{int} D$, then $D = \psi^{-1}(E)$.*

Proof. 1. For any $x \in D$ there exists $\mathbf{u} \in \mathcal{U}$ such that $\phi_M(\mathbb{R}^+, x, \mathbf{u}) \subset D$. Since ψ conjugate the systems Σ_M and Σ_N, their respective solutions are conjugate and hence

$$\phi_N(\mathbb{R}^+, \psi(x), \mathbf{u}) = \psi(\phi_M(\mathbb{R}^+, x, \mathbf{u})) \subset \psi(D).$$

On the other hand, the continuity of ψ and the conjugations property gives us that

$$D \subset \overline{\mathcal{O}_M^+(x)} \implies \psi(D) \subset \psi\left(\overline{\mathcal{O}_M^+(x)}\right) \subset \overline{\mathcal{O}_N^+(\psi(x))},$$

showing that $\psi(D)$ satisfies properties (i) and (ii) in the Definition 4. Consequently, there exists a control set E of Σ_N such that $\psi(D) \subset E$, showing the item.

2. Let us show that $\psi^{-1}(E)$ satisfies conditions (i) and (ii) in Definition 4.

(i) *Weak invariance:* Let $x \in \psi^{-1}(E)$. Then, $\psi(x) \in E$ and there exists $\mathbf{u} \in \mathcal{U}$ such that $\phi_N(\mathbb{R}^+, \psi(x), \mathbf{u}) \subset E$ and by the conjugation property

$$\psi(\phi_M(\mathbb{R}^+, x, \mathbf{u})) = \phi_N(\mathbb{R}^+, \psi(x), \mathbf{u}) \subset E \qquad \iff \qquad \phi_M(\mathbb{R}^+, x, \mathbf{u}) \subset \psi^{-1}(E).$$

(ii) *Approximate controllability:* Let us start by showing that controllability holds in $\psi^{-1}(\operatorname{int} E)$. For this, let $x_1, x_2 \in \psi^{-1}(\operatorname{int} E)$. Since $\psi(x_1), \psi(x_2), y_0 \in \operatorname{int} E$ there exist, by controllability, $\mathbf{u}_1, \mathbf{u_2} \in \mathcal{U}$ and $s_1, s_2 > 0$ such that

$$\phi_N(s_1, \psi(x_1), \mathbf{u}_1) = y_0 = \phi_N(-s_2, \psi(x_2), \mathbf{u}_2).$$

By the conjugation property, the previous imply that

$$\phi_M(s_1, x_1, \mathbf{u}_1) \in \psi^{-1}(y_0) \qquad \text{and} \qquad \phi_M(-s_2, x_2, \mathbf{u}_2) \in \psi^{-1}(y_0).$$

Since, by hypothesis, $\psi^{-1}(y_0) \subset \operatorname{int} D$ and controllability holds in $\operatorname{int} D$, there exists $\mathbf{u_3} \in \mathcal{U}$ and $s_3 > 0$ such that

$$\phi_M(s_3, \phi(s_1, x_1, \mathbf{u}_1), \mathbf{u}_3) = \phi(-s_2, x_2, \mathbf{u}_2)$$

$$\implies \phi(s_2, \phi_M(s_3, \phi(s_1, x_1, \mathbf{u}_1), \mathbf{u}_3), \Theta_{-s_2}\mathbf{u}_2) = x_2,$$

showing that controllability holds in $\psi^{-1}(\operatorname{int} E)$. Moreover, by continuity it holds that

$$\overline{E} = \overline{\operatorname{int} E} \qquad \implies \qquad \overline{\psi^{-1}(E)} = \overline{\psi^{-1}(\operatorname{int} E)},$$

and by the previous

$$\psi^{-1}(E) \subset \overline{\mathcal{O}_M^+(x)}, \quad \forall x \in \psi^{-1}(\operatorname{int} E).$$

On the other hand, if $x \in \psi^{-1}(E)$ then $\psi(x) \in E$ and hence

$$\operatorname{int} E \subset \overline{\mathcal{O}_N^+(\psi(x))} \quad \Longrightarrow \quad \exists \mathbf{u} \in \mathcal{U}, s > 0, \quad \psi(\phi_M(s, x, \mathbf{u})) = \phi(s, \psi(x), \mathbf{u}) \in \operatorname{int} E,$$

and hence

$$\phi_M(s, x, \mathbf{u}) \in \psi^{-1}(\operatorname{int} E) \quad \Longrightarrow \quad \psi^{-1}(E) \subset \overline{\mathcal{O}_M^+(\phi_M(s, x, \mathbf{u}))} \subset \overline{\mathcal{O}_M^+(x)},$$

showing that $\psi^{-1}(E)$ satisfies (ii) in Definition 4.

By the previous, $\psi^{-1}(E)$ has to be contained in a control set of Σ_M. Since $\psi^{-1}(E) \cap D \neq \emptyset$, we conclude that $\psi^{-1}(E) \subset D$ and by item 1. the equality actually holds.

□

Before finishing the section, let us make a simple proposition that will help us ahead.

Proposition 3. *Let $f : M \to \mathbb{R}$ be a continuous function and Σ_G a affine control system on M. If $x, y \in M$ are such that,*

(i) $f(x) < f(y)$,

(ii) For all $\mathbf{u} \in \mathcal{U}$, the function $g_{\mathbf{u}}(s) := f(\phi(s, y, \mathbf{u}))$ satisfies $g_{\mathbf{u}}(0) < g_{\mathbf{u}}(s)$ for all $s > 0$.

then $x \notin \overline{\mathcal{O}^+(y)}$. In particular, it cannot exists a control set of Σ_M that contains both x and y, since the condition (ii) of Definition 4 cannot be satisfied.

Proof. Let us suppose that $x \in \overline{\mathcal{O}^+(y)}$. In this case, there exists a sequence $y_n \in \mathcal{O}^+(y)$ such that $y_n \to x$ as $n \to +\infty$. Writting,

$$y_n = \phi(s_n, y, \mathbf{u}_n), \quad \text{for} \quad s_n > 0, \mathbf{u}_n \in \mathcal{U},$$

we get, by the continuity of f, that

$$f(\phi(s_n, y, \mathbf{u}_n)) = f(y_n) \to f(x).$$

In particular, by considering $\varepsilon = f(y) - f(x) > 0$ there exists $N \in \mathbb{N}$ such that

$$f(\phi(s_N, y, \mathbf{u}_N)) \in (f(x) - \varepsilon, f(x) + \varepsilon)$$

and hence,

$$g_{\mathbf{u}_N}(s_N) = f(\phi(s_N, y, \mathbf{u}_N)) < f(y) = f(\phi(0, y, \mathbf{u}_n)) = g_{\mathbf{u}_N}(0),$$

showing the result.

□

1.2 Linear vector fields and linear control systems on Lie groups

In this chapter, we will define a linear vector field in the context of control systems on Lie groups, discuss some of its properties, and define a linear control systems. Furthermore, we will explore the properties satisfied by linear control systems on solvable Lie groups. Specifically, we will focus on linear control systems on solvable, non-nilpotent Lie groups of dimension three. We will characterize the linear vector fields in these systems and determine the general form of the system of ordinary differential equations (ODEs) that characterizes these systems.

Lastly, we will prove that every linear control system on a solvable, non-nilpotent Lie group of dimension three, under certain regularity condition on the associated linear vector field, is equivalent to the product of a homogeneous system on $\mathbb{R}$ and a specific affine control system on $\mathbb{R}^2$. The latter one will be studied in the following chapter.

We begin with the definition of a linear vector field on a Lie group.

Definition 6. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, identity element e, and η be the normalizer of $\mathfrak{g}$ in the algebra of all smooth vector fields over G, i.e.,*

$$\eta := norm_{X(G)}(\mathfrak{g}) := \{F \in X(G) \mid \forall Y \in \mathfrak{g}, \;\; [F, Y] \in \mathfrak{g}\}.$$

A vector field $\mathcal{X}$ on G is linear if it belongs to η and $\mathcal{X}(e) = 0$.

The following theorem (JOUAN, 2014) establishes equivalent conditions for a vector field on G to be linear.

Theorem 2. *Let $\mathcal{X}$ be a vector field on a connected Lie group G. The following conditions are equivalent:*

(1) $\mathcal{X}$ is linear;

(2) The flow $\{\varphi_s\}_{s\in\mathbb{R}}$ of $\mathcal{X}$ is a one parameter group of automorphisms of G, that is,

$$\forall s \in \mathbb{R}, \qquad \varphi_s(gh) = \varphi_s(g)\varphi_s(h);$$

(3) $\mathcal{X}(gh) = (dL_g)_h\mathcal{X}(h) + (dR_h)_g\mathcal{X}(g)$ for all $g, h \in G$.

The vector field $\mathcal{X}$ is complete and is uniquely associated to a derivation $\mathcal{D}$ of $\mathfrak{g}$ defined by

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \quad \forall Y \in \mathfrak{g}.$$

As $\mathcal{X}$ is linear, the diffeomorphism φ_s is an automorphism of G for any $s \in \mathbb{R}$. Therefore, $d(\varphi_s)_e$ is an automorphism of $\mathfrak{g}$ and hence

$$(d\varphi_s)_e = e^{s\mathcal{D}}, \quad \forall s \in \mathbb{R}. \tag{1.1}$$

This nice relationship between the flow of $\mathcal{X}$ and the derivation $\mathcal{D}$ have several important applications, mainly due to the subgroups/subalgebras associated with them as we define next.

Let us consider an eigenvalue $\alpha \in \mathbb{C}$ of the derivation $\mathcal{D}$. The real generalized eigenspaces of $\mathcal{D}$ are the subspaces of $\mathfrak{g}$ defined as

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : (\mathcal{D} - \alpha I)^n X = 0 \text{ for some } n \geqslant 1\}, \quad \text{if} \quad \alpha \in \mathbb{R} \quad \text{and}$$

$$\mathfrak{g}_\alpha = \operatorname{span}\{\operatorname{Re}(v), \operatorname{Im}(v); \quad v \in \bar{\mathfrak{g}}_\alpha\}, \quad \text{if} \quad \alpha \in \mathbb{C},$$

where $\bar{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g}$ stands for the complexification of $\mathfrak{g}$, and $\bar{\mathfrak{g}}_\alpha$ the generalized eigenspace of the extension $\bar{\mathcal{D}} = \mathcal{D} + i\mathcal{D}$ of $\mathcal{D}$ to $\bar{\mathfrak{g}}$.

By Proposition 3.1 of (SAN-MARTIN, 2010), it holds that

$$[\bar{\mathfrak{g}}_\alpha, \bar{\mathfrak{g}}_\beta] \subset \bar{\mathfrak{g}}_{\alpha+\beta},$$

where $\bar{\mathfrak{g}}_{\alpha+\beta} = \{0\}$ if $\alpha + \beta$ is not an eigenvalue of $\mathcal{D}$. Therefore, if we put

$$\mathfrak{g}_\lambda := \bigoplus_{\alpha; \operatorname{Re}(\alpha) = \lambda} \mathfrak{g}_\alpha,$$

the previous imply that $[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_2}] \subset \mathfrak{g}_{\lambda_1+\lambda_2}$ when $\lambda_1 + \lambda_2 = \operatorname{Re}(\alpha)$ for some eigenvalue α of $\mathcal{D}$ and zero otherwise, where $\mathfrak{g}_\lambda = \{0\}$ if $\lambda \in \mathbb{R}$ is not the real part of any eigenvalue of $\mathcal{D}$.

The *unstable, central,* and *stable* subalgebras of $\mathfrak{g}$ are given, respectively, by

$$\mathfrak{g}^+ = \bigoplus_{\alpha:\, \operatorname{Re}(\alpha)>0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}^0 = \bigoplus_{\alpha:\, \operatorname{Re}(\alpha)=0} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{g}^- = \bigoplus_{\alpha:\, \operatorname{Re}(\alpha)<0} \mathfrak{g}_\alpha.$$

Since the previous subalgebras are given as sum of all the (generalized) eigenspaces of $\mathcal{D}$, it holds that $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-$. Moreover, these subalgebras are invariant by the derivation $\mathcal{D}$ with $\mathfrak{g}^+$ and $\mathfrak{g}^-$ nilpotent subalgebras.

The relationship between the previous subalgebras with the dynamics of the $\mathcal{X}$ are obtained through their subgroups. Let us denote by G^+, G^-, G^0, $G^{+,0}$, and $G^{-,0}$, the connected Lie subgroups whose associated Lie algebras are given, respectively, by $\mathfrak{g}^+$, $\mathfrak{g}^-$, $\mathfrak{g}^0$, $\mathfrak{g}^{+,0} := \mathfrak{g}^+ \oplus \mathfrak{g}^0$ and $\mathfrak{g}^{-,0} := \mathfrak{g}^- \oplus \mathfrak{g}^0$. By Proposition 2.9 of (DA-SILVA, 2016), all of the previous subgroups are invariant by the flow of $\mathcal{X}$, closed, and their intersection are trivial, that is,

$$G^+ \cap G^- = G^+ \cap G^{-,0} = \ldots = \{e\}.$$

Moreover, G^+ and G^- are connected, simply connected, nilpotent Lie groups. Analogously as the algebra level, the subgroups G^+, G^0 and G^- are called, respectively, the *unstable, central,* and *stable subgroups* of $\mathcal{X}$.

Next we define linear control systems. A *linear control system (LCS for short)* on G is determined by the family of ODEs

$$\dot{g}(s) = \mathcal{X}(g(s)) + \sum_{i=1}^{m} u_i(s) Y^i(g(s)), \tag{Σ_G}$$

where $\mathcal{X}$ is a linear vector field, Y^i are left-invariant vector fields, and $\mathbf{u} = (u_1, \ldots, u_m) \in \mathcal{U}$ are control functions as defined previously.

By choosing $Y^1, \ldots, Y^m$ left-invariant, the solutions of Σ_G satisfy

$$\phi(t, gh, \mathbf{u}) = \varphi_t(g)\phi(t, h, \mathbf{u}), \qquad \forall g, h \in G, t \in \mathbb{R}, \mathbf{u} \in \mathcal{U}.$$

A consequence of the previous formula is the following proposition(see (JOUAN, 2014, Proposition 2))

Proposition 4. *For a LCS, it holds that*

1. $\mathcal{O}^+_{S_1+S_2}(e) = \mathcal{O}^+_{S_2}(e)\varphi_{S_1}(\mathcal{O}^+_{S_1}(e))$, *for all* $S_1, S_2 > 0$*;*
2. $\mathcal{O}^+_{S_1}(e) \subset \mathcal{O}^+_{S_2}(e)$, *for all* $0 < S_1 < S_2$*;*
3. $\mathcal{O}^+_{\leqslant S}(e) = \mathcal{O}^+_{S}(e)$, *for all* $S > 0$.

Also, due to the symmetry present on Lie groups, the LARC for linear control systems is determined at the origin of the group, in fact, the linear control system Σ_G satisfies the LARC if $\mathfrak{g}$ is the smallest $\mathcal{D}$-invariant subalgebra containing the vectors $\{Y^1, \ldots, Y^m\}$.

Moreover, we say that Σ_G satisfies the *ad-rank condition* if $\mathfrak{g}$ is the smallest $\mathcal{D}$-invariant subspace containing the vectors $\{Y^1, \ldots, Y^m\}$. The ad-rank condition is much stronger than the LARC, once that the previous contain all the brackets among the elements in $\{Y^1, \ldots, Y^m\}$ and the former not necessarily. Also, if the system satisfies the ad-rank condition, it is locally controllable, that is,

$$e \in \operatorname{int} \mathcal{O}^+_S(e) \cap \operatorname{int} \mathcal{O}^-_S(e), \qquad \forall S > 0.$$

Following (AYALA; DA-SILVA; ZSIGMOND, 2017; DA-SILVA, 2016), there is a nice relationship between the subgroups G^+, G^0 and G^- associated with the drift $\mathcal{X}$ of a LCS Σ_G and its dynamics. In fact, the following result holds:

Theorem 3. *Let* Σ_G *be a LCS on a connected Lie group* G *and assume that* $\mathcal{O}^+(e)$ *is a neighborhood of the identity element* $e \in G$*. Then,*

$$G^{-,0} \subset \mathcal{O}^+(e) \quad \text{and} \quad G^{+,0} \subset \mathcal{O}^-(e).$$

Moreover,

$$\mathcal{C}_G = \overline{\mathcal{O}^+(e)} \cap \mathcal{O}^-(e),$$

is a control set of Σ_G with nonempty interior satisfying $G^0 \subset \operatorname{int} \mathcal{C}_G$. If G is solvable, $\mathcal{C}_G$ is the unique control set of Σ_G with nonempty interior and, Σ_G is controllable if $G = G^0$ or, equivalently, if the derivation $\mathcal{D}$ has only eigenvalues with zero real parts.

We finish this section with a definition that will be useful to divide our analyzes in cases.

Definition 7. *Let Σ_G be a LCS on a connected Lie group G with Lie algebra $\mathfrak{g}$. We define the nil-rank of Σ_G as the rank of the restriction $\mathcal{D}|_{\mathfrak{n}}$, where $\mathcal{D}$ is the derivation associated with the drift of Σ_G and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$.*

1.3 Lifting a LCS to simply connected groups

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and denote by $\widetilde{G}$ its simply connected covering group, choosen in such a way that the canonical projection $\pi : \widetilde{G} \to G$ satisfies $(d\pi)_{\tilde{e}} = \mathrm{id}_{\mathfrak{g}}$, and hence $\pi \circ \widetilde{\exp} = \exp$, where $\widetilde{\exp}$ and exp stands, respectively, for the exponential maps of $\widetilde{G}$ and G.

Now, if $\mathcal{X}$ be a linear vector field on G and denote by $\mathcal{D}$ its associated derivation. By the previous choice and Theorem 2.2 of (AYALA; TIRAO, 1999), the derivation $\mathcal{D}$ is associate with a unique linear vector field $\widetilde{\mathcal{X}}$ on $\widetilde{G}$. If we denote by $\{\widetilde{\varphi}_s\}_{s\in\mathbb{R}}$ and $\{\varphi_s\}_{s\in\mathbb{R}}$, respectively, the flows of $\widetilde{\mathcal{X}}$ and $\mathcal{X}$, then for all $X \in \mathfrak{g}, s \in \mathbb{R}$,

$$\pi\left(\widetilde{\varphi}_s(\widetilde{\exp}X)\right) = \pi\left(\widetilde{\exp}\left(\mathrm{e}^{s\mathcal{D}}X\right)\right) = \exp\left(\mathrm{e}^{s\mathcal{D}}X\right) = \varphi_s(\exp X) = \varphi_s(\pi(\widetilde{\exp}X)).$$

By the connectedness of $\widetilde{G}$ we conclude that

$$\forall s \in \mathbb{R}, \qquad \pi \circ \widetilde{\varphi}_s = \varphi_s \circ \pi,$$

and hence $\widetilde{\mathcal{X}}$ and $\mathcal{X}$ are π-related vector fields.

In the same way, the property $\pi \circ \widetilde{\exp} = \exp$ implies that

$$\forall s \in \mathbb{R}, Y \in \mathfrak{g}, g \in \widetilde{G}, \qquad \pi(g\, \widetilde{\exp}\, sY) = \pi(g) \exp sY,$$

which implies that the left-invariant vector fields $\widetilde{Y}$ on $\widetilde{G}$ and Y on G determined by Y are π-conjugated. As a consequence, any LCS Σ_G on G can be lifted (uniquely) to a LCS $\widetilde{\Sigma}_{\widetilde{G}}$ on $\widetilde{G}$ in such a way that $\widetilde{\Sigma}_{\widetilde{G}}$ and Σ_G are π-conjugated.

1.4 Projecting a LCS to homogeneous spaces

Let $H \subset G$ be a closed subgroup. Following (JOUAN, 2010), a vector field f on the homogeneous space $H\backslash G$ is said to be linear if it is π-conjugated to a linear vector field $\mathcal{X}$ on G, where $\pi : G \to H\backslash G$ is the canonical projection, that is,

$$f \circ \pi = \pi_* \circ \mathcal{X}.$$

Moreover, by Proposition 4 of (JOUAN, 2010) the previous is equivalent to the invariance of the subgroup H by the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of $\mathcal{X}$. In fact, if

$$\forall s \in \mathbb{R} \qquad \varphi_s(H) \subset H,$$

the relation,

$$\Phi_s(\pi(g)) = \pi(\varphi_s(g)) \qquad \forall g \in G, s \in \mathbb{R},$$

defines a flow $\Phi : \mathbb{R} \times H\backslash G \to H\backslash G$ on the homogeneous space $H\backslash G$, whose associated vector field

$$f(x) := \frac{d}{ds}_{|s=0} \Phi_s(x),$$

is π-conjugated to $\mathcal{X}$.

As a consequence, the LCS Σ_G is π-conjugated to a affine control system $\Sigma_{H\backslash G}$ (also called linear control system), if the subgroup H is invariant by the flow of the drift $\mathcal{X}$ of Σ_G.

1.5 Low dimensional Lie groups

In this section we introduce the main properties of linear vector fields and linear control systems on dimensions two and three. As we will see ahead, the dynamics on the class of the 3D Lie groups we are interested can be recover from their counterpart on 2D Lie groups and homogeneous spaces.

1.5.1 2D Lie groups

Up to isomorphisms, there are only two possible connected Lie groups of dimension two, an abelian and a solvable nonabelian. In the following, we briefly comment the results concerning LCSs on such groups. Some examples of control sets for these systems can be found at the last section.

The abelian case

Up to isomorphisms, the only connected, simply connected Lie group of dimension two $(\mathbb{R}^2, +)$, where the product is the usual sum of vectors in $\mathbb{R}^2$. In this case, left-invariant vector fields and linear vector fields are given, respectively by

$$Y(v) = \eta \quad \text{and} \quad \mathcal{X}(v) = Av, \quad v \in \mathbb{R}^2,$$

where $\eta \in \mathbb{R}^2$ is a nonzero vector and $A \in \mathfrak{gl}(2, \mathbb{R})$ a nontrivial 2×2 matrix.

A (one-input) linear control system in $\mathbb{R}^2$ is given by

$$\dot{v} = Av + u\eta, \ u \in \Omega, \tag{$\Sigma_{(\mathbb{R}^2,+)}$}$$

with $\Omega = [u^-, u^+]$ and $u^- < 0 < u^+$. The LARC in this case, is equivalent to $\langle A\eta, R\eta \rangle \neq 0$. Controllability and control sets of $\Sigma_{(\mathbb{R}^2,+)}$ are very well know (see (COLONIUS; KLIEMANN, 2000, Example 3.2.16)) and are summarized in the next result.

Theorem 4. *If $\Sigma_{(\mathbb{R}^2,+)}$ satisfies the LARC, it admits a unique control set $\mathcal{C}_{(\mathbb{R}^2,+)}$ containing the identity element in its interior and satisfying:*

1. *$\mathcal{C}_{(\mathbb{R}^2,+)} = \mathbb{R}^2$ if A has only eigenvalues with zero real parts;*
2. *$\mathcal{C}_{(\mathbb{R}^2,+)}$ is open if A has only eigenvalues with positive real parts;*
3. *$\mathcal{C}_{(\mathbb{R}^2,+)}$ is closed if A has only eigenvalues with negative real parts;*

The solvable nonabelian case

For the solvable case, we consider the following interpretation: Let us denote by $\mathrm{Aff}(\mathbb{R})$ the Lie subgroup $(\mathbb{R}^2, *)$ with product given by

$$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, y_1 + \mathrm{e}^{x_1} y_2).$$

The Lie algebra $\mathfrak{aff}(\mathbb{R})$ of $\mathrm{Aff}(\mathbb{R})$ is $(\mathbb{R}^2, [\cdot, \cdot])$ with bracket given by

$$[(\alpha_1, \beta_1), (\alpha_2, \beta_2)] = (0, \alpha_1\beta_2 - \alpha_2\beta_1).$$

A left-invariant vector field and a linear vector field on $\mathrm{Aff}(\mathbb{R})$ are given, respectively, by

$$Y(x, y) = (\alpha, \mathrm{e}^x \beta) \quad \text{and} \quad \mathcal{X}(x, y) = (0, by + (\mathrm{e}^x - 1)a),$$

where $(\alpha, \beta), (a, b) \in \mathbb{R}^2$.

As a consequence, a (one-input) LCS on $\mathrm{Aff}(\mathbb{R})$ is defined by the family of ODE's

$$\begin{cases} \dot{x} = u\alpha \\ \dot{y} = by + (\mathrm{e}^x - 1)a + u\mathrm{e}^x\beta \end{cases}, \quad \text{where} \quad u \in \Omega, \tag{$\Sigma_{\mathrm{Aff}(\mathbb{R})}$}$$

with Ω as previously. Moreover, $\Sigma_{\mathrm{Aff}(\mathbb{R})}$ satisfies the LARC if and only if $\alpha(a\alpha + b\beta) \neq 0$ (see (AYALA; DA-SILVA, 2020, Section 2.2)). The next result, whose proof can be found in (AYALA; DA-SILVA, 2020, Section 3.4), summarizes the controllability and control set of $\Sigma_{\mathrm{Aff}(\mathbb{R})}$.

Theorem 5. *If $\Sigma_{\mathrm{Aff}(\mathbb{R})}$ satisfies the LARC, it admits exactly one control set $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})}$ containing the identity element in its interior and such that:*

1. $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})} = \mathrm{Aff}(\mathbb{R})$ *if* $b = 0$*;*
2. $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})}$ *is closed if* $b < 0$ *and open if* $b > 0$.

1.5.2 3D solvable, nonnilpotent Lie groups

We will consider here only the connected and simply connected Lie groups that are solvable and non-nilpotent of dimension 3, since from them, we can determine the connected Lie groups as $G = \widetilde{G}/Z$ where Z is a central discrete subgroup of $\widetilde{G}$

By (ONISHCHIK; VINBERG, 1994, Theorem 1.4, Chapter 7), the real Lie algebras of dimension 3 are given as semidirect product of $\mathbb{R}$ with $\mathbb{R}^2$, i.e. $\mathfrak{g}(\theta) := \mathbb{R} \times_\theta \mathbb{R}^2$ where $\theta(t) := t\theta$, with θ a 2×2 matrix, which can take one of the following forms:

- $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $\theta = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$ with $|\gamma| \leqslant 1$
- $\theta = \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix}$ with $\gamma \in \mathbb{R}$

The bracket of these Lie algebras is defined by

$$[(t_1, v_1), (t_2, v_2)] = (0, \theta \cdot (t_1 v_2 - t_2 v_1))$$

which is completely determined by the equation

$$[(t, 0), (0, v)] = (0, t\theta v)$$

Up to isomorphism, for the Lie algebra $\mathfrak{g}(\theta)$ the unique connected, simply connected Lie group associated, is given by the semi direct product $\mathbb{R} \times_\rho \mathbb{R}^2$, with $\rho(t) = \mathrm{e}^{t\theta}$ and the product of the group as

$$(t_1, v_1)(t_2, v_2) = (t_1 + t_2, v_1 + \rho(t_1) v_2)$$

with $(t_1, v_1), (t_2, v_2) \in \mathbb{R} \times_\rho \mathbb{R}^2$. Therefore, we go to denote the Lie group associate to the Lie algebra $\mathbb{R} \times_\theta \mathbb{R}^2$ with $\rho(1) = \mathrm{e}^\theta$ as $G(\theta)$.

In particular, for all $(t, v) \in G(\theta)$ and $w \in \mathbb{R}^2$, we have that

$$(t, v)(0, w)(t, v)^{-1} = (t, v + \rho_t w)(-t, -\rho_{-t} v) = (0, v + \rho_t w + \rho_t(-\rho_{-t} v) = (0, \rho_t w). \quad (1.2)$$

Let us use the previous computations to calculate some homogeneous spaces of $G(\theta)$. Precisely, let $w \in \mathbb{R}^2$ be a nonzero vector and consider the subgroup $H = \mathbb{R}(0, w)$. From the above calculations, we get that

$$H \cdot (t_1, v_1) = H \cdot (t_2, v_2) \quad \Longleftrightarrow \quad t_1 = t_2 \quad \text{and} \quad \langle v_1, Rw \rangle = \langle v_2, Rw \rangle,$$

where R represents the clockwise rotation of $\frac{\pi}{2}$ radians; implying that the canonical projection is given by

$$\pi : G(\theta) \to H \backslash G(\theta), \qquad \pi(t, v) = (t, \langle v, Rw \rangle),$$

that is, π coincides with the projection onto the first two components of the basis $\{(1, 0), (0, Rw), (0, w))\}$, when $G(\theta)$ is seen as the vector space $\mathbb{R}^3$. Moreover, $H \backslash G(\theta)$ is also a Lie group if and only if H is a normal subgroup of $G(\theta)$ and for (1.2) H is a normal subgroup of $G(\theta)$ if and only if w is an eigenvector of θ.

On the other hand, the subgroup $S = \mathbb{R}(1, 0)$ is such that

$$(t, 0)(t_1, v_1) = (t_2, v_2) \quad \Longleftrightarrow \quad t = t_2 - t_1 \quad \text{and} \quad \rho_t v_1 = v_2,$$

and hence

$$S \cdot (t_1, v_1) = S \cdot (t_2, v_2) \quad \Longleftrightarrow \quad \rho_{-t_1} v_1 = \rho_{-t_2} v_2.$$

As a consequence, the homogeneous space $S \backslash G(\theta)$ is naturally identified with $\mathbb{R}^2$ and its canonical projection is given by

$$\pi : G(\theta) \to S \backslash G(\theta), \quad \pi(t, v) = \rho_{-t} v.$$

In what follows, we define an operator which is closely related with the product in the group. Such operator appears in the definition of the group exponential and also in the definition of linear vector fields on the group $G(\theta)$.

Let A be a 2×2 matrix and define

$$\Lambda^A : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (t, v) \mapsto \int_0^t \mathrm{e}^{sA} v ds. \quad (1.3)$$

The previous operator satisfies the following:

(1) $\Lambda_0^A = 0$ (2) $\dfrac{d}{dt}\Lambda_t^A = \mathrm{e}^{tA}$ (3) $\Lambda_{t+s}^A = \Lambda_t^A + \mathrm{e}^{tA}\Lambda_s^A$

(4) $\mathrm{e}^{tA} - A\Lambda_t^A = \mathrm{Id}_{\mathbb{R}^2}$ (5) $\mathrm{e}^{sA}\Lambda_t^A = \Lambda_t^A\mathrm{e}^{sA}$ (6) $\Lambda_t^A = (\mathrm{e}^{tA} - \mathrm{Id}_{\mathbb{R}^2})A^{-1}$ if $\det A \neq 0$

The first property is clear from the definition of Λ^A. The second one, follows from the Fundamental Theorem of Calculus. For the third one, derivation of Λ_{t+s}^A and $\Lambda_t^A + \mathrm{e}^{tA}\Lambda_s^A$ with respect to s, shows that both satisfy the same differential equation. Since they coincide for $s = 0$, the equality holds.

Analogously, one obtain the equality in the fourth and fifth items. Now, for the last one can be obtained from the fourth under the assumption on the determinant of the matrix A.

Following (AYALA; DA-SILVA, 2019), the left-invariant and linear vector fields on the groups $G(\theta)$ are given, respectively, as

$$Y^L(t,v) = (\alpha, \rho_t\eta) \quad \text{and} \quad \mathcal{X}(t,v) = (0, Av + \Lambda_t^\theta\xi), \tag{1.4}$$

where $(\alpha,\eta) \in \mathbb{R}\times\mathbb{R}^2$, $\xi \in \mathbb{R}^2$ and $A \in \mathfrak{gl}(2,\mathbb{R})$ satisfies $A\theta = \theta A$.

For the sake of completeness, we show it here. The follow curve, $\gamma(s) = (t_2, v_2) + s(t,v) \in G(\theta)$ satisfies that $\gamma(0) = (t_2, v_2)$ and $\gamma'(0) = (t,v)$, therefore, for the left translation, we have that

$$d(L_{(t_1,v_1)})_{(t_2,v_2)}(t,v) = \frac{d}{ds}|_{s=0}L_{(t_1,v_1)}(\gamma(s)) = \frac{d}{ds}|_{s=0}(t_1,v_1)(t_2+st, v_2+sv),$$

and since $(t_1,v_1)(t_2+st, v_2+sv) = (t_1+t_1+st, v_1+\rho_{t_1}(v_2+sv))$, we conclude that,

$$d(L_{(t_1,v_1)})_{(t_2,v_2)}(t,v) = \frac{d}{ds}|_{s=0}L_{(t_1,v_1)}(\gamma(s)) = \frac{d}{ds}|_{s=0}(t_1,v_1)(t_2+st, v_2+sv) = (t, \rho_{t_1}v). \tag{1.5}$$

Now, let Y^L be a left-invariant field, then,

$$Y^L(t,v) = d(L_{(t,v)})_e(\alpha,\eta), \quad \text{with} \quad (\alpha,\eta) \in T_eG(\theta),$$

and by the previous calculations, we have that

$$Y^L(t,v) = (\alpha, \rho_t\eta).$$

In the same way, for the right translations, we have

$$d(R_{(t_1,v_1)})_{(t_2,v_2)}(t,v) = \frac{d}{ds}|_{s=0}R_{(t_1,v_1)}(\gamma(s)) = \frac{d}{ds}|_{s=0}(t_2+st, v_2+sv)(t_1,v_1),$$

and since $(t_2+st, v_2+sv)(t_1,v_1) = (t_2+st+t_1, v_2+sv+\rho_{t_2+st}v_1)$, we conclude that

$$d(R_{(t_1,v_1)})_{(t_2,v_2)}(t,v) = \frac{d}{ds}|_{s=0}R_{(t_1,v_1)}(\gamma(s)) = \frac{d}{ds}|_{s=0}(t_2+st, v_2+sv)(t_1,v_1) = (t, v+t\theta\rho_{t_2}v_1). \tag{1.6}$$

Before continuing with linear fields, follow (AYALA; DA-SILVA, 2019), we note that for any $(\alpha,\eta) \in \mathfrak{g}(\theta)$, the exponential map is given that

$$\exp(\alpha,\eta) = \begin{cases} (0,\eta), & \text{if } \alpha = 0, \\ \left(\alpha, \frac{1}{\alpha}\Lambda^{\theta}_{\alpha}\eta\right), & \text{if } \alpha \neq 0; \end{cases} \tag{1.7}$$

in fact, we consider $(\alpha,\eta) \in \mathfrak{g}(\theta)$ and define the curve

$$\Gamma(s) = \begin{cases} (0,s\eta), & \text{if } \alpha = 0, \\ \left(s\alpha, \frac{1}{\alpha}\Lambda^{\theta}_{s\alpha}\eta\right), & \text{if } \alpha \neq 0. \end{cases}$$

since in both cases $\Gamma(0) = 0$ and

$$\Gamma'(s) = \begin{cases} (0,\eta) = d(L_{\Gamma(s)})_{(0,0)}(0,\eta), & \text{if } \alpha = 0, \\ (\alpha,\rho_{s\alpha}\eta) = d(L_{\Gamma(s)})_{(0,0)}(\alpha,\eta), & \text{if } \alpha \neq 0. \end{cases}$$

On the other hand, $\frac{d}{ds}\exp(s(\alpha,\eta)) = d(L_{\exp(s(\alpha,\eta))})_{(0,0)}(\alpha,\eta)$, therefore, for uniqueness, the two curves are equals.

Let $\mathcal{X}$ be a linear field on $G(\theta)$ and $\mathcal{D}$ its associated derivation. Because, $\mathcal{D}$ is a derivation and $\mathbb{R}^2$ is the nilradical of $\mathfrak{g}(\theta) = \mathbb{R} \times_\theta \mathbb{R}^2$, we know that $\mathcal{D}(\mathfrak{g}(\theta)) \subset \mathbb{R}^2$, therefore, exist a linear map $\mathcal{D}^* : \mathbb{R}^2 \mapsto \mathbb{R}^2$ satisfying $\mathcal{D}(0,v) = (0,\mathcal{D}^*v)$ for any $v \in \mathbb{R}^2$. Let v be element of $\mathbb{R}^2$, it hold that

$$(0,\mathcal{D}^*\theta v) = \mathcal{D}(0,\theta v) = \mathcal{D}[(1,0),(0,v)] = [\mathcal{D}(1,0),(0,v)] + [(1,0),\mathcal{D}(0,v)] = (0,\theta\mathcal{D}^*v),$$

because, $(0,\theta v) = [(1,0),(0,v)]$, $\mathcal{D}$ is a derivation and $[\mathcal{D}(1,0),(0,v)] = 0$. Now, we can conclude that $\mathcal{D}^*\theta = \theta\mathcal{D}^*$ and $\mathcal{D}^*\rho_t = \rho_t\mathcal{D}^*$ for any $t \in \mathbb{R}$.

Finally, since $(t,v) = (t,0)(0,\rho_{-t}v)$ it follow that

$$\mathcal{X}(t,v) = \mathcal{X}((t,0)(0,\rho_{-t}v))$$

and for the Theorem 2, we have that

$$\mathcal{X}((t,0)(0,\rho_{-t}v)) = d(L_{(t,0)})_(0,\rho_{-t}v)\mathcal{X}(0,\rho_{-t}v) + d(R_{(0,\rho_{-t}v)})_{(t,0)}\mathcal{X}(t,0); \tag{1.8}$$

from the general form of exponential map, previously calculated it follow that $(t,0) = \exp(t,0)$ and $(0,\rho_{-t}v) = \exp(0,\rho_{-t}v)$, therefore,

$$\varphi_s(0,\rho_{-t}v) = \varphi_s(\exp(0,\rho_{-t}v))$$

and since, $(d\varphi_s)_e = \mathrm{e}^{s\mathcal{D}}, \quad \forall s \in \mathbb{R}.$,

$$\varphi_s(\exp(0,\rho_{-t}v)) = \exp(\mathrm{e}^{s\mathcal{D}}(0,\rho_{-t}v)) = \exp((0,\mathrm{e}^{s\mathcal{D}}\rho_{-t}v)) = (0,\mathrm{e}^{s\mathcal{D}}\rho_{-t}v)$$

in the same way

$$\varphi_s(t,0) = \varphi_s(\exp(t,0)) = \exp(\mathrm{e}^{s\mathcal{D}}(t,0)),$$

considering $\mathcal{D}(1,0) = (0,\xi)$, we observed that $\mathrm{e}^{s\mathcal{D}}(t,0) = t\left(1, \sum_{j\geqslant 0} \frac{s^{j+1}}{(j+1)!}(\mathcal{D}^*)^j\xi\right)$ by definition of $\mathcal{D}^*$; and by 1.7, we have that

$$\varphi_s(t,0) = \exp t\left(1, \sum_{j\geqslant 0} \frac{s^{j+1}}{(j+1)!}(\mathcal{D}^*)^j\right) = \left(t, \sum_{j\geqslant 0} \frac{s^{j+1}}{(j+1)!}\Lambda_t^\theta((\mathcal{D}^*)^j\xi)\right)$$

. Now, we can calculate $\mathcal{X}(0,\rho_{-t}v)$ and $\mathcal{X}(t,0)$.

$$\mathcal{X}(0,\rho_{-t}v) = \frac{d}{ds}|_{s=0}\varphi_s(0,\rho_{-t}v) = (0,\mathcal{D}^*\rho_{-t}v)$$

and

$$\mathcal{X}(t,0) = \frac{d}{ds}|_{s=0}\varphi_s(t,0) = (0,\Lambda_t\xi).$$

With the previous and using the formulas 1.5 and 1.6 in 1.8, we get that

$$\mathcal{X}(t,v) = (0, Av + \Lambda_t^\theta\xi).$$

Since the linear vector field $\mathcal{X}$ is fully characterized by the matrix A and the vector ξ, we will usually write $\mathcal{X} = (A,\xi)$ to represent the linear vector field. The same holds for a left-invariant vector field, which we usually denote by $Y = (\alpha,\eta)$.

The flow associated with the linear vector field $\mathcal{X} = (A,\xi)$ can also be explicitly calculated and is given by

$$\varphi_s(t,v) = (t, \mathrm{e}^{sA}v + \Lambda_t^\theta \Lambda_s^A \xi). \tag{1.9}$$

In fact, derivation of $\varphi_s(t,v)$ on s gives us that

$$\varphi_s'(t,v) = (0, A\mathrm{e}^{sA}v + \mathrm{e}^{sA}\Lambda_t^\theta \xi).$$

However,

$$Ae^{sA}v + \mathrm{e}^{sA}\Lambda_t^\theta\xi = Ae^{sA}v + \mathrm{e}^{sA}\Lambda_t^\theta\xi - \Lambda_t^\theta\xi + \Lambda_t^\theta\xi = Ae^{sA}v + (\mathrm{e}^{sA} - \mathrm{Id}_{\mathbb{R}^2})\Lambda_t^\theta\xi + \Lambda_t^\theta\xi$$

and using the property (4), we obtain

$$Ae^{sA}v + \mathrm{e}^{sA}\Lambda_t^\theta\xi = Ae^{sA}v + A\Lambda_s^A\Lambda_t^\theta\xi + \Lambda_t^\theta\xi = A(\mathrm{e}^{sA}v + \Lambda_t^\theta\Lambda_s^A\xi) + \Lambda_t^\theta\xi = \mathcal{X}(\varphi_s(t,v)).$$

Furthermore, the associated derivation of $\mathcal{X} = (A,\xi)$ is given by $\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix}$; in fact, we calculate

$$d(\varphi_s)_e = \left.\begin{pmatrix} 1 & 0 \\ \rho_t\Lambda_s^A\xi & \mathrm{e}^{sA} \end{pmatrix}\right|_e = \begin{pmatrix} 1 & 0 \\ \Lambda_s^A\xi & \mathrm{e}^{sA} \end{pmatrix}$$

and for the relation (1.1), we know that

$$d(\varphi_s)_e = \mathrm{e}^{s\mathcal{D}} \quad \forall\, s \in \mathbb{R};$$

therefore,

$$\frac{d}{ds}(\mathrm{e}^{s\mathcal{D}})|_0 = \mathcal{D},$$

so, we can conclude that

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix}$$

The next proposition, whose proof can be found at (AYALA; DA-SILVA; HERNÁNDEZ, 2023, Proposition 2.2), describes the set of automorphisms of $G(\theta)$. It will be very useful when conjugating LCSs.

Proposition 5. *For the Lie group $G(\theta)$, it holds that:*

$$\mathrm{Aut}(G(\theta)) = \left\{\psi(t,v) = (\varepsilon t, Pv + \varepsilon\Lambda_{\varepsilon t}\eta), \;\; \eta \in \mathbb{R}^2, P \in \mathrm{Gl}(2,\mathbb{R}), \;\; \textit{and } P\theta = \varepsilon\theta P\right\},$$

where $\varepsilon = 1$ if $\operatorname{tr}\theta \neq 0$ and $\varepsilon = \pm 1$ if $\operatorname{tr}\theta = 0$.

1.6 Linear control systems on $G(\theta)$

In this section we define the linear control system on the groups $G(\theta)$, we will study, and discuss some of their properties.

Definition 8. *A (one-input) linear control system on $G(\theta)$ is given, in coordinates, by the family of ODE's given by*

$$\begin{cases} \dot{t} = u\alpha \\ \dot{v} = Av + \Lambda_t^{\theta}\xi + u\rho_t\eta \end{cases} \qquad (\Sigma_{G(\theta)})$$

with $\Omega = [u^-, u^+]$ and $u^- < 0 < u^+$.

In what follows, we calculated the LARC and the ad-rank condition for the system $\Sigma_{G(\theta)}$ is terms A and ξ, where $\mathcal{X} = (A, \xi)$ is the drift of $\Sigma_{G(\theta)}$.

Proposition 6. *For the LCS $\Sigma_{G(\theta)}$ it holds:*

1. *$\Sigma_{G(\theta)}$ satisfies the ad-rank condition if and only if*

$$\alpha\Big\langle A(\alpha\xi + A\eta), R(\alpha\xi + A\eta)\Big\rangle \neq 0.$$

2. *$\Sigma_{G(\theta)}$ satisfies the LARC if and only if*

$$\alpha\left(\Big\langle A(\alpha\xi + A\eta), R(\alpha\xi + A\eta)\Big\rangle^2 + \Big\langle \theta(\alpha\xi + A\eta), R(\alpha\xi + A\eta)\Big\rangle^2\right) \neq 0;$$

where R is the clockwise rotation of $\dfrac{\pi}{2}$ radians.

Proof. Let us assume that $\mathcal{X} = (A, \xi)$ and $Y = (\alpha, \eta)$ are, respectively, the linear and the left-invariant vector fields of $\Sigma_{G(\theta)}$. The derivation associated with $\mathcal{X}$ is then $\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix}$. Therefore,

$$\mathcal{D}Y = (0, \alpha\xi + A\eta), \quad \mathcal{D}^2Y = (0, A(\alpha\xi + A\eta)) \quad \text{and} \quad [Y, \mathcal{D}Y] = (0, \alpha\theta(\alpha\xi + A\eta)).$$

1. By definition, $\Sigma_{G(\theta)}$ satisfies ad-rank condition, if and only if,

$$\mathfrak{g}(\theta) = \operatorname{span}\{Y, \mathcal{D}Y, \mathcal{D}^2Y\}.$$

Therefore, if $a_1, a_2, a_3 \in \mathbb{R}$ are such that

$$a_1(\alpha, \eta) + a_2(0, \alpha\xi + A\eta) + a_3(0, A(\alpha\xi + A\eta)) = 0,$$

then, in coordinates,

$$\begin{pmatrix} \alpha & 0 & 0 \\ \langle \eta, \mathbf{e}_1 \rangle & \langle \alpha\xi + A\eta, \mathbf{e}_1 \rangle & \langle A(\alpha\xi + A\eta), \mathbf{e}_1 \rangle \\ \langle \eta, \mathbf{e}_2 \rangle & \langle \alpha\xi + A\eta, \mathbf{e}_2 \rangle & \langle A(\alpha\xi + A\eta), \mathbf{e}_2 \rangle \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, $\Sigma_{G(\theta)}$ satisfies the ad-rank condition, if and only if,

$$\det \begin{pmatrix} \alpha & 0 & 0 \\ \langle \eta, \mathbf{e}_1 \rangle & \langle \alpha\xi + A\eta, \mathbf{e}_1 \rangle & \langle A(\alpha\xi + A\eta), \mathbf{e}_1 \rangle \\ \langle \eta, \mathbf{e}_2 \rangle & \langle \alpha\xi + A\eta, \mathbf{e}_2 \rangle & \langle A(\alpha\xi + A\eta), \mathbf{e}_2 \rangle \end{pmatrix} \neq 0,$$

if and only if

$$0 \neq \alpha \left\{ \langle \alpha\xi + A\eta, \mathbf{e}_1 \rangle \langle A(\alpha\xi + A\eta), \mathbf{e}_2 \rangle - \langle \alpha\xi + A\eta, \mathbf{e}_2 \rangle \langle A(\alpha\xi + A\eta), \mathbf{e}_1 \rangle \right\}$$

$$= \alpha \langle A(\alpha\xi + A\eta), R(\alpha\xi + A\eta) \rangle,$$

showing the assertion.

2. Calculations as the one previously done, gives us that $\{Y, \mathcal{D}Y, [Y, \mathcal{D}Y]\}$ is linearly independent, if and only if,

$$\det \begin{pmatrix} \alpha & 0 & 0 \\ \langle \eta, \mathbf{e}_1 \rangle & \langle \alpha\xi + A\eta, \mathbf{e}_1 \rangle & \alpha\langle \theta(\alpha\xi + A\eta), \mathbf{e}_1 \rangle \\ \langle \eta, \mathbf{e}_2 \rangle & \langle \alpha\xi + A\eta, \mathbf{e}_2 \rangle & \alpha\langle \theta(\alpha\xi + A\eta), \mathbf{e}_2 \rangle \end{pmatrix} \neq 0,$$

or equivalent, if and only if $\alpha\langle \theta(\alpha\xi + A\eta), R(\alpha\xi + A\eta) \rangle$. Now, $\Sigma_{G(\theta)}$ satisfies the LARC if and only if it satisfies the ad-rank condition or if

$$\mathfrak{g}(\theta) = \operatorname{span}\{Y, \mathcal{D}Y, [Y, \mathcal{D}Y]\}.$$

Therefore, the LARC is equivalent to

$$\alpha \left(\left\langle A(\alpha\xi + A\eta), R(\alpha\xi + A\eta) \right\rangle^2 + \left\langle \theta(\alpha\xi + A\eta), R(\alpha\xi + A\eta) \right\rangle^2 \right) \neq 0,$$

as stated. □

The next proposition shows that one can use automorphisms to conjugate a given LCS to a second one, in order to simplify the expressions of linear or left-invariant vector fields associated. Such results will be useful ahead.

Proposition 7. *Let $\Sigma_{G(\theta)}$ be a LCS on $G(\theta)$ with associated linear vector field $\mathcal{X} = (A, \xi)$ and left-invariant vector field $Y = (\alpha, \eta)$. Assume that $\Sigma_{G(\theta)}$ satisfies the LARC. It holds:*

1. *There exists $\tilde{\psi} \in \operatorname{Aut}(G(\theta))$ conjugating $\Sigma_{G(\theta)}$ to a LCS $\tilde{\Sigma}_{G(\theta)}$ with left-invariant vector field $\tilde{Y} = (\alpha, 0)$;*

2. *If* $\det A \neq 0$*, there exists* $\hat{\psi} \in \mathrm{Aut}(G(\theta))$ *conjugating* $\Sigma_{G(\theta)}$ *to a LCS* $\hat{\Sigma}_{G(\theta)}$ *with linear vector field* $\hat{\mathcal{X}} = (A, 0)$*.*

Proof. 1. By the LARC, $\alpha \neq 0$ and we can define the map

$$\tilde{\psi} : G(\theta) \to G(\theta), \qquad \tilde{\psi}(t, v) = (t, v - \alpha^{-1}\Lambda_t^\theta \eta).$$

By Proposition 5, $\tilde{\psi} \in \mathrm{Aut}(G(\theta))$ and a simple calculation show that its differential is given by

$$(d\tilde{\psi})_{(t,v)}(a, w) = (a, w - a\alpha^{-1}\rho_t\eta).$$

Therefore,

$$(d\tilde{\psi})_{(t,v)}Y^L(t, v) = (d\tilde{\psi})_{(t,v)}(\alpha, \rho_t\eta) = (\alpha, \rho_t\eta - \alpha\alpha^{-1}\rho_t\eta) = (\alpha, 0) = \tilde{Y}(\tilde{\psi}(t, v)),$$

and, if $\tilde{\mathcal{X}} = (A, \xi + \alpha^{-1}A\eta)$, then

$$\begin{aligned}(d\tilde{\psi})_{(t,v)}\mathcal{X}(t, v) &= (d\tilde{\psi})_{(t,v)}(0, Av + \Lambda_t^\theta\xi) = (0, Av + \Lambda_t^\theta\xi) \\ &= \big(0, A(v - \alpha^{-1}\Lambda_t^\theta\eta) + \Lambda_t^\theta(\xi + \alpha^{-1}A\eta)\big) = \tilde{\mathcal{X}}(\tilde{\psi}(t, v)),\end{aligned}$$

showing that $\Sigma_{G(\theta)}$ is $\tilde{\psi}$-conjugated to the system $\tilde{\Sigma}_{G(\theta)}$ determined by $\tilde{\mathcal{X}}$ and $\tilde{Y}$.

2. Since $\det A \neq 0$ the map

$$\hat{\psi} : G(\theta) \to G(\theta), \qquad \hat{\psi}(t, v) = (t, v + \Lambda_t A^{-1}\xi),$$

is a well defined automorphism of $G(\theta)$ satisfying

$$\forall (a, w) \in T_{(t,v)}G(\theta), \quad (d\hat{\psi})_{(t,v)}(a, w) = (a, a\rho_t A^{-1}\xi + w).$$

On the other hand,

$$A\theta = \theta A \quad \text{implies that} \quad \forall t \in \mathbb{R}, w \in \mathbb{R}^2, \quad A\Lambda_t^\theta w = \Lambda_t^\theta A w,$$

and hence,

$$(d\hat{\psi})_{(t,v)}\mathcal{X}(t, v) = \hat{\mathcal{X}}(\hat{\psi}(t, v)) \quad \text{and} \quad (d\hat{\psi})_{(t,v)}Y(t, v) = \hat{Y}(\hat{\psi}(t, v)),$$

where $\hat{\mathcal{X}}(t, v) = (0, Av)$ and $\hat{Y} = (\alpha, \alpha A^{-1}\xi + \eta)$, showing that $\Sigma_{G(\theta)}$ is equivalent to the LCS $\hat{\Sigma}_{G(\theta)}$ determined by $\hat{\mathcal{X}}$ and $\hat{Y}$. □

For a LCS $\Sigma_{G(\theta)}$ with associated linear vector field $\mathcal{X} = (A, \xi)$, the nil-rank of $\Sigma_{G(\theta)}$ is exactly the rank of the matrix A. In fact, since the derivation $\mathcal{D}$ associated to $\mathcal{X} = (A, \xi)$ is $\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \xi & A \end{pmatrix}$ and the nilradical of $\mathfrak{g}(\theta)$ is $\mathfrak{n} = \{0\} \times \mathbb{R}^2$, we have that $\mathcal{D}|_{\mathfrak{n}} = A$. The next result shows that the dynamics of a LCS on $G(\theta)$, with nil-rank two, is the same as the dynamics of the product of a homogeneous system on $\mathbb{R}$ with a particular class of control-affine system on $\mathbb{R}^2$.

Proposition 8. *Let $\Sigma_{G(\theta)}$ be a LCS with nil-rank two on $G(\theta)$. Then, $\Sigma_{G(\theta)}$ is equivalent to a control-affine system on $\mathbb{R} \times \mathbb{R}^2$ of the form*

$$\begin{cases} \dot{t} = u\alpha \\ \dot{v} = (A - u\alpha\theta)v + u\eta \end{cases} \tag{$\Sigma_{\mathbb{R}\times\mathbb{R}^2}$}$$

Proof. Since $\Sigma_{G(\theta)}$ has nil-rank two, we have that $\det A \neq 0$ and by the previous proposition we can assume w.l.o.g. that $\Sigma_{G(\theta)}$ is determined by the vectors $\mathcal{X} = (A, 0)$ and $Y = (\alpha, \eta)$.

Define the map

$$\psi : G(\theta) \to \mathbb{R} \times \mathbb{R}^2, \qquad \psi(t, v) = (t, \rho_{-t}v).$$

It holds that ψ is a diffeomorphism and it satisfies

$$\forall (a, w) \in T_{(t,v)}G(\theta), \qquad (d\psi)_{(t,v)}(a, w) = (a, -a\theta\rho_{-t}v + \rho_{-t}w).$$

Consequently,

$$(d\psi)_{(t,v)}\mathcal{X}(t, v) = \mathcal{X}(\psi(t, v)) \quad \text{and} \quad (d\psi)_{(t,v)}Y^L(t, v) = Z(\psi(t, v)),$$

where $Z(t, v) = (\alpha, -\alpha\theta v + \eta)$.

Therefore, $\Sigma_{G(\theta)}$ is equivalent to the control-affine system on $\mathbb{R} \times \mathbb{R}^2$ given by

$$\begin{cases} \dot{t} = u\alpha \\ \dot{v} = (A - u\alpha\theta)v + u\eta \end{cases}, \quad u \in \Omega \tag{$\Sigma_{\mathbb{R}\times\mathbb{R}^2}$}$$

concluding the proof. □

Remark 1. *Although the subjacent manifold of $G(\theta)$ is $\mathbb{R} \times \mathbb{R}^2$, the change in notations made in the previous result is to emphasize the fact that the system $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ is not a linear control system. When working with such system we will always use the previous change in notation.*

As a direct consequence of the above discussion, we have the following:

Corollary 1. *Let $\Sigma_{G(\theta)}$ be a LCS with nil-rank two on $G(\theta)$, then the induced system on the homogeneous space $S\backslash G(\theta)$ is given by*

$$\dot{v} = (A - u\alpha\theta)v + u\eta, \; u \in \Omega \tag{$\Sigma_{\mathbb{R}^2}$}$$

where $S = \mathbb{R} \cdot (1, 0)$.

Proof. Let $\Sigma_{G(\theta)}$ be a LCS with nil-rank two and assume w.l.o.g. that $\mathcal{X} = (A, 0)$ and $Y = (\alpha, \eta)$ are the vector fields of $\Sigma_{G(\theta)}$. By the previous proposition, the diffeomorphism $\psi(t, v) = (t, \rho_{-t}v)$ conjugates $\Sigma_{G(\theta)}$ to the control-affine system

$$\begin{cases} \dot{t} = u\alpha \\ \dot{v} = (A - u\alpha\theta)v + u\eta \end{cases}, \quad u \in \Omega \tag{$\Sigma_{\mathbb{R}\times\mathbb{R}^2}$}$$

By considering

$$\pi_2 : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2, \qquad \pi_2(t, v) = v,$$

the composition $\pi_2 \circ \psi$ is a conjugation between $\Sigma_{G(\theta)}$ and the control-affine system on $\mathbb{R}^2$ given by

$$\dot{v} = (A - u\alpha\theta)v + u\eta, \ u \in \Omega. \qquad (\Sigma_{\mathbb{R}^2})$$

However, by the discussion in the beginning of the Section 1.5.2, the map $\pi(t, v) = \rho_{-t}v$ is the canonical projection $\pi : G(\theta) \to S\backslash G(\theta)$, and since $\pi = \pi_2 \circ \psi$, the result follows. $\square$

2 LCSs with nilrank two on $G(\theta)$

In this part we study the LCSs on the groups $G(\theta)$ having nilrank-two. In order to do that, we start with a full investigation of a particular class of control-affine systems on $\mathbb{R}^2$, whose dynamics is intrinsically associated with the LCSs on $G(\theta)$.

2.1 A particular class of control-affine systems on $\mathbb{R}^2$

In this section, we study the class of control-affine systems on $\mathbb{R}^2$ given by the family of differential equations.

$$\dot{v} = (A - u\theta)v + u\eta, \qquad u \in \Omega \tag{$\Sigma_{\mathbb{R}^2}$}$$

where $\eta \in \mathbb{R}^2$ is a fixed nonzero vector, $A, \theta \in \mathrm{gl}(2, \mathbb{R})$, with the restrictions, $\det A \neq 0$ and $[A, \theta] = 0$.

A more general study of control-affine systems on higher dimensional Euclidean spaces was done recently in (COLONIUS; SANTANA; SETTI, 2021). Despite this fact, a full independent analysis of the system $\Sigma_{\mathbb{R}^2}$ is done here by considering the dynamics of 2×2 matrices.

Let us define $A(u) := A - u\theta$. If $\det A(u) \neq 0$, the solutions of $\Sigma_{\mathbb{R}^2}$ are builded through concatenations of the solutions for constant controls $u \in \Omega$

$$\phi(s, v, u) = \mathrm{e}^{sA(u)}(v - v(u)) + v(u), \qquad \text{where} \qquad v(u) := -uA(u)^{-1}\eta.$$

The points $v(u) := -uA(u)^{-1}\eta$ in $\mathbb{R}^2$ are called equilibrium points because they satisfy the equation

$$A(u)v + u\eta = 0, \tag{2.1}$$

they have this form because $\det A(u) \neq 0$. However, in general, we can define the set of equilibrium points $\mathcal{E}$ as follows:

$$\mathcal{E} = \{v \in \mathbb{R}^2 \mid A(u)v + u\eta = 0 \quad \text{for some} \quad u \in \Omega\}$$

The importance of such a set comes from the fact that any point in $\mathcal{E}$ satisfies conditions 1. and 2. of Definition 4 and hence, is contained in a control set of $\Sigma_{\mathbb{R}^2}$.

Since we are assuming that $\det A \neq 0$, the subset of Ω given by

$$\Omega_0 := \{u \in \operatorname{int}\Omega;\ \det A(u) \neq 0\},$$

is an open neighborhood of $0 \in \mathbb{R}$. The map $v : u \in \Omega_0 \mapsto v(u) = -uA(u)^{-1}\eta \in \mathbb{R}^2$ is a smooth regular curve in $\mathbb{R}^2$.

In fact, a simple derivation of the equation 2.1, gives us that

$$-\theta v(u) + A(u)v'(u) = -\eta \qquad \Longrightarrow \qquad v'(u) = -A(u)^{-1}(\eta - \theta v(u)).$$

Therefore,

$$v'(u) = 0 \qquad \overset{\det A(u)\neq 0}{\Longleftrightarrow} \qquad \eta = \theta v(u) = -uA(u)^{-1}\theta\eta$$

and this last equation is equivalent to

$$A\eta - u\theta\eta = A(u)\eta = -u\theta\eta,$$

from which we conclude that

$$v'(u) = 0 \qquad \Longleftrightarrow \qquad A\eta = 0.$$

Since we are assuming that $\det A \neq 0$ we conclude that $v'(u) \neq 0$, showing the regularity of the curve.

In what follows, we describe explicitly the solutions of $\Sigma_{\mathbb{R}^2}$

Proposition 9. *Let $0 = s_0 < s_1 < \cdots < s_n$ and $u_1, \ldots, u_j \in \Omega_0$ and define*

$$s = \sum_{j=0}^{n} s_j \qquad \textit{and} \qquad \mathbf{u} \in \mathcal{U} \qquad \textit{with} \qquad \mathbf{u}(t) = u_j \in \Omega, t \in [s_j, s_{j+1}).$$

Then,

$$\phi(s, v, \mathbf{u}) = \mathrm{e}^{\sum_{i=1}^{n} s_i A(u_i)} v + \sum_{i=1}^{n} \mathrm{e}^{\sum_{k=i+1}^{n+1} s_k A(u_k)} (1 - e^{s_i A(u_i)}) v(u_i)$$

with $s_{n+1} = 0$. Consequently,

$$\phi(s, v, \mathbf{u}) = \mathrm{e}^{\sum_{i=1}^{n} s_i A(u_i)} v + \phi(s, 0, \mathbf{u})$$

Proof. Let us proceed by induction on the natural $n \in \mathbb{N}$. For $n = 1$, we have that

$$\phi(s, v, u) = \mathrm{e}^{sA(u)}(v - v(u)) + v(u) = \mathrm{e}^{sA(u)} v + (1 - \mathrm{e}^{sA(u)}) v(u),$$

and the result follows. Let us then assume that the result is true for $n = N$ and consider $n = N + 1$. Defining

$$\hat{s}_i = s_i, \quad i = 1, \ldots, N, \quad \hat{s} = \sum_{j=0}^{N} \hat{s}_k \quad \text{and} \quad \hat{\mathbf{u}} = \mathbf{u}_{|[0, s_N)},$$

gives us, by the induction hypothesis, that

$$\phi(\hat{s}, v, \hat{\mathbf{u}}) = \mathrm{e}^{\sum_{i=1}^{N} \hat{s}_i A(u_i)} v + \sum_{i=1}^{N} \mathrm{e}^{\sum_{k=i+1}^{N+1} \hat{s}_k A(u_k)} (1 - \mathrm{e}^{\hat{s}_i A(u_i)}) v(u_i), \qquad \text{with} \qquad \hat{s}_{N+1} = 0.$$

Hence,

$$\phi(s, v, \mathbf{u}) = \phi(s_{N+1}, \phi(\hat{s}, v, \hat{\mathbf{u}}), u_{N+1})$$

$$= \mathrm{e}^{s_{N+1} A(u_{k+1})} \left(\mathrm{e}^{\sum_{i=1}^{N} \hat{s}_i A(u_i)} v + \sum_{i=1}^{N} \mathrm{e}^{\sum_{k=i+1}^{N+1} \hat{s}_k A(u_k)} (1 - \mathrm{e}^{\hat{s}_i A(u_i)}) v(u_i) \right) + (1 - \mathrm{e}^{s_{N+1} A(u_{N+1})}) v(u_{N+1})$$

$$= \mathrm{e}^{\sum_{i=1}^{N+1} s_i A(u_i)} v + \sum_{i=1}^{N} \mathrm{e}^{\sum_{k=i+1}^{N+1} s_k A(u_k)} (1 - \mathrm{e}^{s_i A(u_i)}) v(u_i) + (1 - \mathrm{e}^{s_{N+1} A(u_{N+1})}) v(u_{N+1})$$

$$= \mathrm{e}^{\sum_{i=1}^{N+1} s_i A(u_i)} v + \sum_{i=1}^{N+1} \mathrm{e}^{\sum_{k=i+1}^{N+2} s_k A(u_k)} (1 - \mathrm{e}^{s_i A(u_i)}) v(u_i), \qquad \text{with } s_{N+2} := 0,$$

concluding the proof. □

The next result analyses the LARC for the system $\Sigma_{\mathbb{R}^2}$.

Lemma 2. *The system $\Sigma_{\mathbb{R}^2}$ satisfies the LARC if and only if η is not a common eigenvector of A and θ.*

Proof. First we calculate the Lie bracket of F_{u_1} and F_{u_2} with $F_u(x) := A(u)x + u\eta$. By definition, the Lie bracket between two fields is

$$[X, Y](x) = \frac{d}{ds} \left(d(\varphi_{-s})_{\varphi_s(x)} (Y(\varphi_s(x))) \right)_{s=0},$$

where φ_s is the flow of the vector field X. For $u_1, u_2 \in \Omega$, it holds that

$$d(\varphi_{-s}^{u_1})_{\varphi_s^{u_1}(x))} = \mathrm{e}^{-sA(u)} \qquad \text{and} \qquad F_{u_2}(\varphi_s^{u_1}(x)) = A(u_2)\varphi_s^{u_1}(x) + u_2\eta.$$

Hence,

$$[F_{u_2}, F_{u_1}](x) = \frac{d}{ds} \left(\mathrm{e}^{-sA(u_2)} [A(u_1)\varphi_s^{u_2}(x) + u_1\eta] \right)_{s=0} =$$

$$= \left(-A(u_2) \left(\mathrm{e}^{-sA(u_2)} (A(u_1)\varphi_s^{u_2}(x) + u_1\eta) \right) + \mathrm{e}^{-sA(u_2)} (A(u_1)(A(u_2)\varphi_s^{u_2}(x) + u_2\eta)) \right)_{s=0}$$

$$= -A(u_2)(A(u_1)x + u_1\eta) + A(u_1)(A(u_2)x + u_2\eta) = (u_2 - u_1)A\eta,$$

where for the last equality we used that A and θ commutes.

In the same way,

$$[F_{u_3}, [F_{u_2}, F_{u_1}]](x) = \frac{d}{ds}\left(\mathrm{e}^{-sA(u_3)}[(u_2 - u_1)A\eta]\right)_{s=0} = -(u_2 - u_1)A(u_3)A\eta,$$

and inductively

$$[F_{u_n}, [F_{u_{n-1}}, [F_{u_{n-2}}, \cdots [F_{u_2}, F_{u_1}] \cdots]]](x)$$
$$= (-1)^n (u_2 - u_1)A(u_n)A(u_{n-1})A(u_{n-2}) \cdots A(u_3)A\eta.$$

Since the product $A(u_n)A(u_{n-1})A(u_{n-2}) \cdots A(u_3)A$ only depends on A and θ and $[A, \theta] = 0$, the result follows. □

2.1.1 The control sets of $\Sigma_{\mathbb{R}^2}$

In this subsection we show the existence of control sets with nonempty interior for the control-affine system $\Sigma_{\mathbb{R}^2}$ introduced previously. We start with a result concerning to the positive and negative orbits at equilibrium points of the system.

Proposition 10. *It holds that*

$$\mathcal{O}^+(v(u)) \quad \textit{and} \quad \mathcal{O}^-(v(u)),$$

are open sets for any $u \in \Omega_0$ with the exception of, at the most, one $u^* \in \Omega_0$ where $\langle A(u^*)\eta, R\eta \rangle = 0$.

Proof. The proofs for the positive and negative orbits are analogous, hence, we show only the positive case. In order to prove the result for this case, we first show that $\mathcal{O}^+(v(u))$ is open if, and only if $v(u) \in \mathrm{int}\mathcal{O}^+(v(u))$ and afterwards, we prove that $v(u) \in \mathrm{int}\,\mathcal{O}^+(v(u))$.

Claim 1: $\mathcal{O}^+(v(u))$ is open if, and only if $v(u) \in \mathrm{int}\mathcal{O}^+(v(u))$.

We note that $\mathrm{int}\mathcal{O}^+(v)$ is positively invariant for all $v \in \mathbb{R}^2$, in fact, we know that,

- $\phi(t, \mathrm{int}\mathcal{O}^+(v), u) \subset \mathcal{O}^+(v)$
- $\phi(t, \cdot, u)$ is a open map for all $t \geqslant 0$ and $u \in \Omega$,

hence, we conclude that $\phi(t, \mathrm{int}\mathcal{O}^+(v), u) \subset \mathrm{int}\mathcal{O}^+(v)$ for all $t \geqslant 0$ and $u \in \Omega$, i.e., $\mathcal{O}^+(v) \subset \mathrm{int}\mathcal{O}^+(v)$ for all $v \in \mathrm{int}\mathcal{O}^+(v)$.

Let us now consider $s > 0$ and $u \in \Omega_0$, and define the map

$$f : \Omega_0^2 \to \mathbb{R}^2, \quad f(u_1, u_2) := \mathrm{e}^{sA(u_2)} \left(\mathrm{e}^{sA(u_1)} \left(v(u) - v(u_1) \right) + v(u_1) - v(u_2) \right) + v(u_2).$$

We observe that,

$$f(u,u) = v(u) \qquad \text{and} \qquad f(u_1,u_2) = \phi(s, \phi(s, v(u), u_1), u_2)),$$

implying that

$$f(\Omega_0^2) \subset \mathcal{O}^+(v(u)).$$

As a consequence, $v(u) \in \operatorname{int} \mathcal{O}^+(v(u))$ if the differencial of f is surjective on $u \in \Omega$. Calculating the partial derivatives of f, we obtain that

$$\frac{\partial f}{\partial u_1}(u_1,u_2) = \mathrm{e}^{sA(u_2)} \left(-s\theta \mathrm{e}^{sA(u_1)} (v(u) - v(u_1)) + (1 - \mathrm{e}^{sA(u_1)}) v'(u_1) \right) \qquad \text{and,}$$

$$\frac{\partial f}{\partial u_2}(u_1,u_2) = -s\theta \mathrm{e}^{sA(u_2)} \left(\mathrm{e}^{sA(u_1)} (v(u) - v(u_1)) + v(u_1) - v(u_2) \right) + (1 - \mathrm{e}^{sA(u_2)}) v'(u_2).$$

Evaluating at u gives us,

$$\frac{\partial f}{\partial u_1}(u,u) = \mathrm{e}^{sA(u)} (1 - \mathrm{e}^{sA(u)}) v'(u) \qquad \text{and} \qquad \frac{\partial f}{\partial u_2}(u,u) = (1 - \mathrm{e}^{sA(u)}) v'(u).$$

Let R be the counterclockwise rotation of $\pi/2$. Then,

$$\left\langle \frac{\partial f}{\partial u_1}(u,u), R \frac{\partial f}{\partial u_2}(u,u) \right\rangle = \left\langle \mathrm{e}^{sA(u)} (1 - \mathrm{e}^{sA(u)}) v'(u), R(1 - \mathrm{e}^{sA(u)}) v'(u) \right\rangle,$$

and

$$\left\langle \mathrm{e}^{sA(u)} (1 - \mathrm{e}^{sA(u)}) v'(u), R(1 - \mathrm{e}^{sA(u)}) v'(u) \right\rangle = \left\langle (1 - \mathrm{e}^{sA(u)}) \mathrm{e}^{sA(u)} v'(u), R(1 - \mathrm{e}^{sA(u)}) v'(u) \right\rangle$$

since $\mathrm{e}^{sA(u)}$ and $(1 - \mathrm{e}^{sA(u)})$ commute.

Moreover, since $B(x,y) = \langle x, Ry \rangle$ is an alternating bi-linear form,

$$\left\langle (1 - \mathrm{e}^{sA(u)}) \mathrm{e}^{sA(u)} v'(u), R(1 - \mathrm{e}^{sA(u)}) v'(u) \right\rangle = \det(1 - \mathrm{e}^{sA(u)}) \left\langle \mathrm{e}^{sA(u)} v'(u), R v'(u) \right\rangle$$

and hence,

$$\left\langle \frac{\partial f}{\partial u_1}(u,u), R \frac{\partial f}{\partial u_2}(u,u) \right\rangle \qquad \text{is linearly independent if and only if}$$

$$\det(1 - \mathrm{e}^{sA(u)}) \left\langle \mathrm{e}^{sA(u)} v'(u), R v'(u) \right\rangle \neq 0. \tag{2.2}$$

Claim 2: For any $u \in \Omega_0$, there exists $\delta > 0$ such that $\det(1 - \mathrm{e}^{sA(u)}) \neq 0$ for all $s \in (0, \delta)$.

In fact, if $s_0 > 0$ and $u \in \Omega_0$ are such that $\det(1 - \mathrm{e}^{s_0 A(u)}) = 0$ then we have two possibilities:

1. $e^{s_0 A(u)} \neq I$: In this case, we conclude that one root of the characteristic polynomial of $e^{s_0 A(u)}$ is 1 and the eigenspace associated with the eigenvalue 1 has dimension 1. Therefore, there exists a nonzero vector v such that $e^{s_0 A(u)} v = v$.

   Since $A(u)$ and $e^{s_0 A(u)}$ commute, v is also an eigenvector of $A(u)$. However, $A(u)v = av$ implies that
$$v = e^{s_0 A(u)} v = e^{s_0 a} v \qquad \Longrightarrow \qquad a = 0,$$
   contradicting the assumption that $\det A(u) \neq 0$.

2. $e^{s_0 A(u)} = I$: Since $\det e^{s_0 A(u)} = e^{s_0 \operatorname{tr} A(u)}$, we obtain in this case that $\operatorname{tr} A(u) = 0$. Furthermore, the condition $\det A(u) \neq 0$, allows us to conclude that $A(u)$ has a pair of purely imaginary eigenvalues.

By the previous, $\det(1 - e^{s_0 A(u)}) = 0$ implies the existence of $a \neq 0$ such that
$$A(u) = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \qquad \Longrightarrow \qquad e^{sA(u)} = \begin{pmatrix} \cos sa & -\sin sa \\ \sin sa & \cos sa \end{pmatrix}.$$
Therefore,
$$\det(1 - e^{sA(u)}) = 2(1 - \cos sa) = 0 \qquad \Longleftrightarrow \qquad as = 2k\pi, k \in \mathbb{Z}.$$
and so,
$$\det(1 - e^{sA(u)}) \neq 0, \quad \forall s \in (0, \delta), \quad \text{for any } 0 < \delta < \frac{2\pi}{a},$$
showing the claim.

Claim 3: With the exception of, at most, one $u \in \Omega_0$, there exists $\epsilon = \epsilon(u)$ such that
$$\left\langle e^{sA(u)} v'(u), Rv'(u) \right\rangle \neq 0, \qquad \forall s \in (0, \epsilon).$$

Let us assume that $u \in \Omega_0$ does not satisfies the previous condition. That is,
$$\exists (s_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+, \quad \text{with} \quad s_n \to 0 \quad \text{and} \quad \left\langle e^{s_n A(u)} v'(u), Rv'(u) \right\rangle = 0. \tag{2.3}$$

Hence,
$$0 = \lim_{n \to \infty} \frac{1}{s_n} \left\langle e^{s_n A(u)} v'(u) - v'(u), Rv'(u) \right\rangle$$
$$= \left.\frac{d}{ds}\right|_{s=0} \left\langle e^{sA(u)} v'(u), Rv'(u) \right\rangle = \langle A(u) v'(u), Rv'(u) \rangle.$$

In particular,
$$A(u) v'(u) = \lambda v'(u), \qquad \text{for some} \qquad \lambda \neq 0.$$

Since $v'(u) = -A(u)^{-1}(\eta - \theta v(u))$, the previous gives us that

$$\eta - \theta v(u) = \lambda A(u)^{-1}(\eta - \theta v(u)) \quad \iff \quad A(u)(\eta - \theta v(u)) = \lambda(\eta - \theta v(u)).$$

Remembering that $A(u) = A - u\theta$ and $A(u)v(u) = -u\eta$, allows us to obtain

$$A\eta = A\eta - u\theta\eta + u\theta\eta = \lambda\eta - \lambda\theta v(u),$$

and, by applying $A(u)$ to the previous equality, we get that

$$A(u)A\eta = A(u)(\lambda\eta - \lambda\theta v(u)) \quad \overset{[A,A(u)]=0}{\iff} \quad AA(u)\eta = \lambda A\eta - \lambda u\theta\eta + \lambda u\theta\eta = \lambda A\eta,$$

or, equivalently,

$$A(A(u)\eta - \lambda\eta) = 0.$$

Since we are assuming that $\det A \neq 0$, we get

$$A(u)\eta = \lambda\eta \quad \text{or equivalently} \quad \langle A(u)\eta, R\eta\rangle = 0. \tag{2.4}$$

Now, if

$$u_1 \neq u_2 \quad \text{and} \quad \langle A(u_1)\eta, R\eta\rangle = \langle A(u_2)\eta, R\eta\rangle = 0,$$

then, by the definition of $A(u)$, we get that

$$\langle A\eta, R\eta\rangle = u_1\langle\theta\eta, R\eta\rangle \quad \text{and} \quad \langle A\eta, R\eta\rangle = u_2\langle\theta\eta, R\eta\rangle,$$

and hence

$$0 = \langle A\eta, R\eta\rangle - \langle A\eta, R\eta\rangle = \underbrace{(u_1 - u_2)}_{\neq 0}\langle\theta\eta, R\eta\rangle \quad ,$$

implying that

$$\langle A\eta, R\eta\rangle = \langle\theta\eta, R\eta\rangle = 0.$$

As a consequence, if (2.3) holds for two differents elements in Ω, then necessarily η is a common eigenvector of A and θ, which contradicts the assumption that $\Sigma_{\mathbb{R}^2}$ satisfies the LARC, which implies the claim.

By the claims 2. and 3. for all $u \in \Omega_0$, with exception of at most one, there exists $\epsilon^* = \min(\delta, \epsilon)$, such that

$$\det(1 - \mathrm{e}^{sA(u)})\left\langle \mathrm{e}^{sA(u)}v'(u), Rv'(u)\right\rangle \neq 0 \quad \forall s \in (0, \epsilon^*),$$

implying that the differential of f at the point $u \in \Omega_0$ is surjective. In particular, f is an open map around such value $u \in \Omega_0$, that is, there exists an open neighborhood $U \subset \Omega_0$ with $u \in U$ and such that $f(U^2)$ is open. Since $v(u) = f(u, u) \in f(U^2) \subset \mathcal{O}^+(v(u))$ we get $v(u) \in \operatorname{int}\mathcal{O}^+(v(u))$ which implies that $\mathcal{O}^+(v(u))$ is open and concludes the proof.

□

Remark 2. *Note that*

$$\langle A(u^*)\eta, R\eta\rangle = 0 \quad \Longleftrightarrow \quad \eta \text{ is an eigenvector of } A(u^*).$$

Therefore, if the system satisfies the LARC, the only possibility for the previous is that $A(u^*)$ is an scalar matrix, that is,

$$\langle A(u^*)\eta, R\eta\rangle = 0 \quad \Longleftrightarrow \quad A(u^*) = \lambda(u^*)I_2.$$

Next we show the existence of a control set with nonempty interior for the system $\Sigma_{\mathbb{R}^2}$ that contains $0 \in \mathbb{R}^2$ in its closure.

Proposition 11. *The system $\Sigma_{\mathbb{R}^2}$ admits a control set with nonempty interior $\mathcal{C}_{\mathbb{R}^2}$ such that*

$$v(\Omega_0) \subset \operatorname{int} \mathcal{C}_{\mathbb{R}^2},$$

with the exception of, at most, one $u^* \in \Omega_0$ satisfying $\langle A(u^*)\eta, R\eta\rangle = 0$.

Proof. By Propositions 10 and 1, with the exception of at most one $u^* \in \Omega_0$, there exists a control set $\mathcal{C}_u$ satisfying

$$\mathcal{C}_u = \overline{\mathcal{O}^+(v(u))} \cap \mathcal{O}^-(v(u)) \quad \text{and} \quad v(u) \in \operatorname{int} \mathcal{C}_u.$$

Let us define the sets

$$\Omega_0^- := \{u \in \Omega_0;\ u < u^*\} \quad \text{and} \quad \Omega_0^+ := \{u \in \Omega_0;\ u > u^*\}.$$

Since $\Omega_0^\pm$ are open intervals their images $v(\Omega_0^\pm)$ are connected subset of $\mathbb{R}^2$. Moreover, the fact that $v(u) \in \operatorname{int} \mathcal{C}_u$ for all $u \in \Omega_0^\pm$ implies necessarily that

$$\mathcal{C}_{u_1} = \mathcal{C}_{u_2} \quad \text{for all} \quad u_1, u_2 \in \Omega_0^\pm \tag{2.5}$$

In fact, if $u_1, u_2 \in \Omega_0^+$ with $u_1 < u_2$ satisfies

$$v(u_2) \in \mathbb{R}^2 \setminus \operatorname{int} \mathcal{C}_{u_1} \quad \text{then} \quad v([u_1, u_2]) \cap \partial \mathcal{C}_{u_1} \neq \emptyset.$$

Therefore, there exists $\bar{u} \in \Omega_0^+$ such that $v(\bar{u}) \in \partial \mathcal{C}_{u_1}$. On the other hand,

$$\bar{u} \in \Omega_0^+ \quad \Longrightarrow \quad v(\bar{u}) \in \operatorname{int} \mathcal{C}_{\bar{u}} \quad \Longrightarrow \quad \mathcal{C}_{\bar{u}} \cap \mathcal{C}_{u_1} \neq \emptyset$$

$$\Longrightarrow \quad \mathcal{C}_{\bar{u}} = \mathcal{C}_{u_1} \quad \Longrightarrow \quad v(\bar{u}) \in \operatorname{int} \mathcal{C}_{u_1},$$

which is a contradiction, showing that relation (2.5) holds.

In order to conclude the proof, we have to show that

$$\exists u_1 \in \Omega_0^+, u_2 \in \Omega_0^-; \quad \mathcal{C}_{u_1} = \mathcal{C}_{u_2}.$$

By Remark 2 it holds that $A(u^*) = \lambda(u^*)I_2$. In particular, the fact that $\lambda(u^*)^2 = \det A(u^*) \neq 0$, implies the existence of $u_1 < u^* < u_2$ such that $A(u_1)$ and $A(u_2)$ has a pair of eigenvalues with real parts of the same sign as $\lambda(u^*)$. Therefore,

$$\phi(s, v(u_1), u_2) \to v(u_2) \quad \text{and} \quad \phi(s, v(u_2), u_1) \to v(u_1), \quad \lambda(u^*)s \to -\infty.$$

Since $v(u_i) \in \operatorname{int} \mathcal{C}_{u_i}$, the previous allows us to construct a periodic chain passing through $\mathcal{C}_{u_1}$ and $\mathcal{C}_{u_2}$ which implies $\mathcal{C}_{u_1} = \mathcal{C}_{u_2}$ and concludes the proof.

□

Remark 3. *The equilibrium $v(u^*) \in \mathcal{E}$ associated with $u^* \in \Omega_0$ can be in the boundary of $\mathcal{C}_{\mathbb{R}^2}$ (see Example 4).*

Proposition 12. *Let us assume that* $\det A(u) > 0$ *for all $u \in \Omega_0$. Then, for the control set $\mathcal{C}_{\mathbb{R}^2}$, it holds that:*

1. $\operatorname{tr} A(u) > 0$ *for all $u \in \Omega_0$ and $\mathcal{C}_{\mathbb{R}^2}$ is open;*
2. $\operatorname{tr} A(u) < 0$ *for all $u \in \Omega_0$ and $\mathcal{C}_{\mathbb{R}^2}$ is closed;*
3. $\operatorname{tr} A(u) = 0$ *for some $u \in \Omega_0$ and $\mathcal{C}_{\mathbb{R}^2} = \mathbb{R}^2$;*

Proof. Assume that $A(u)$ has a pair of eigenvalues with negative real parts. Then, for any $v \in \mathbb{R}^2$ we have that

$$\phi(s, v, u) = \mathrm{e}^{sA(u)}(v - v(u)) + v(u) \to v(u), \quad s \to +\infty.$$

If $\mathcal{O}^-(v(u))$ is open, there exists $s_0 > 0$ such that

$$\phi(s_0, v, u) \in \mathcal{O}^-(v(u)) \quad \Longrightarrow \quad v \in \phi_{-s_0,u}(\mathcal{O}^-(v(u))) \subset \mathcal{O}^-(v(u)),$$

implying that $\mathcal{O}^-(v(u)) = \mathbb{R}^2$. In same way, if $A(u)$ has a pair of eigenvalues with positive real parts and $\mathcal{O}^+(v(u))$ is open, then $\mathcal{O}^+(v(u)) = \mathbb{R}^2$.

Since $\det A(u) > 0$, if $\operatorname{tr} A(u) \neq 0$ for all $u \in \Omega_0$, then necessarily $A(u)$ has a pair of eigenvalues with positive real parts if $\operatorname{tr} A(u) > 0$ and negative real parts if $\operatorname{tr} A(u) < 0$, which by the previous implies items 1. and 2.

On the other hand, since $\operatorname{tr} A(u) = \operatorname{tr} A + u \operatorname{tr} \theta$, if $\operatorname{tr} A(u) = 0$ for some $u \in \Omega_0$, we have the following possibilities:

- $\operatorname{tr} A(u_0) = 0$ and $\operatorname{tr} A(u_1) \operatorname{tr} A(u_2) < 0$ for any $u_1, u_2 \in \Omega_0$ satisfying $u_1 < u_0 < u_2$.

In this case, there exist $u_1, u_2 \in \Omega_0$ such that $A(u_1)$ has a pair of eigenvalues with positive real parts and $A(u_2)$ has a pair of eigenvalues with negative real parts. As a consequence,

$$\mathbb{R}^2 = \mathcal{O}^+(v(u_1)) \quad \text{and} \quad \mathcal{O}^-(v(u_2)) = \mathbb{R}^2,$$

implying that

$$\mathcal{O}^-(v(u_1)) = \overline{\mathcal{O}^+(v(u_1))} \cap \mathcal{O}^-(v(u_1)) = \mathcal{C}_{\mathbb{R}^2} = \overline{\mathcal{O}^+(v(u_2))} \cap \mathcal{O}^-(v(u_2)) = \overline{\mathcal{O}^+(v(u_2))}.$$

Therefore, $\mathcal{C}_{\mathbb{R}^2}$ is open and closed in $\mathbb{R}^2$ which implies $\mathcal{C}_{\mathbb{R}^2} = \mathbb{R}^2$.

- $\operatorname{tr} A(u) = 0$ for all $u \in \Omega_0$;

For this case, for any $u \in \Omega_0 \backslash \{\mu\}$ we have that the solutions of $\Sigma_{\mathbb{R}^2}$ are given by concatenations of the curves

$$\phi(s, v, u) = R_{s(\mu-u)}(v - v(u)) + v(u), \quad \text{where} \quad v(u) = -uA(u)^{-1}\eta = \frac{u}{\mu - u}R\eta,$$

and $R_{s(\mu-u)}$ stands for the rotation of $s(\mu - u)$-degrees. In particular,

$$|\phi(s, v, u) - v(u)| = |R_{s(\mu-u)}(v - v(u))| = |v - v(u)|,$$

shows that the solution curve $s \mapsto \varphi(s, v, u)$ lies onto the circumference $C_{u,v}$ with radius $|v - v(u)|$ and center $v(u)$.

Let us show the controllability of $\Sigma_{\mathbb{R}^2}$ by explicitly constructing a periodic trajectory from an arbitrary point $v_0 \in \mathbb{R}^2$ to the origin as follows:

(a) Let $\rho > 0$ and define the set $\Omega_\rho := [-\rho, \rho]$ satisfying

$$\mu \notin \Omega_\rho \quad \text{and} \quad \Omega_\rho \subset \Omega_0,$$

which is possible since $\mu \neq 0$.

(b) Since the circumference C_{ρ,v_0} intersects the line $\mathbb{R} \cdot R\eta$ in two points, let us denote by v_1 the point of this intersection that is closer to $v(-\rho)$. Consider $s_1 > 0$ such that $v_1 = \phi(s_1, v, \rho)$;

(c) If $v_1 \notin v(\Omega_\rho)$, we do the same of item (a) with the circumference $C_{-\nu,v_1}$, in order to obtain a point

$$v_2 = \phi(s_2, v_1, -\rho) \in C_{-\rho,v_1} \cap \mathbb{R} \cdot R\eta.$$

(d) Inductively, if a point $v_n \notin v(\Omega_\rho)$ was obtained, we consider the point v_{n+1} in the intersection of the circumference $C_{(-1)^n\rho,v_n}$ with the line $\mathbb{R} \cdot R\eta$. The radius ρ_n of the circumference $C_{(-1)^n\rho,v_n}$ satisfies

$$\rho_n = |v_n - v((-1)^n\rho)| = |v_0 - v_\rho| - n \cdot \operatorname{diam} \cdot v(\Omega_\rho).$$

Therefore, for some $N \in \mathbb{N}$ large enough, we obtain that $v_N \in v(\Omega_\rho) \cap \mathbb{R} \cdot R\eta$.

(e) Since $v_N \in v(\Omega_\rho)$, the continuity of the curve $u \mapsto v(u)$ assures the existence of $u_N \in \Omega_\rho$ such that $|v(u_N)| = |v_N - v(u_N)|$. As a consequence, the circumference C_{u_N,v_N} contains v_N and the origin 0, and hence, there exists $s_N > 0$ such that $\varphi(s_N, v_N, u_N) = 0$. By concatenation, we obtain a trajectory from v_0 to 0 (Figure 1, blue path).

(f) By choosing the "complementay half" of the circumferences $C_{(-1)^n\rho,v_n}$, $n = 0, \dots N-1$ and C_{u_N,v_N}, obtained in the previous items, we obtain a trajectory from 0 to v_0 (1, red path).

The previous steps shows how to construct a periodic trajectory passing through v_0 and 0. By the arbitrariness of $v_0 \in \mathbb{R}^2$, the system $\Sigma_{\mathbb{R}^2}$ has to be controllable, concluding the proof.

□

Remark 4. *Example 3 shows that the control set $\mathcal{C}_{\mathbb{R}^2}$ is not necessarily the only one with nonempty interior.*

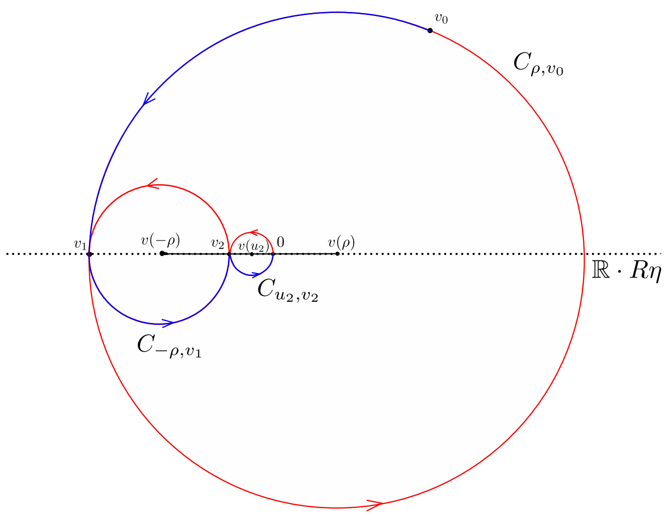

Figure 1 – Periodic trajectory passing through 0 and v_0.

2.2 The control sets of LCSs with nil-rank two on $G(\theta)$

In this section we prove our main results characterizing the control sets of LCSs on $G(\theta)$ with nil-rank two. The idea is basically use Proposition 8 and the results in the previous sections. We start with the following result characterizing the control sets of the system $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$.

Proposition 13. *Let us consider the control system*

$$\begin{cases} \dot{t} = u\alpha \\ \dot{v} = (A - u\alpha\theta)v + u\eta \end{cases}, \quad u \in \Omega. \tag{$\Sigma_{\mathbb{R}\times\mathbb{R}^2}$}$$

If $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ satisfies the LARC, then it admits a control set with nonempty interior $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}$ satisfying

$$\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2} = \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2},$$

where $\mathcal{C}_{\mathbb{R}^2}$ is the control set containing $v(\Omega_0)$ of the associated control-affine system

$$\dot{v} = A(u)v + \alpha u\eta, \; u \in \Omega. \tag{$\Sigma_{\mathbb{R}^2}$}$$

Proof. Let us first notice that the solutions of system $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ satisfy

$$\phi(s, (t_0, v_0), \mathbf{u}) = (\phi_1(s, t_0, \mathbf{u}), \phi_2(s, v_0, \mathbf{u})), \qquad \forall s \in \mathbb{R}, (t_0, v_0) \in \mathbb{R} \times \mathbb{R}^2, \mathbf{u} \in \mathcal{U},$$

and hence, they are given by concatenations of the solutions

$$\phi(s, (t_0, v_0), u) = (t_0 + u\alpha s, \mathrm{e}^{sA(u)}(v_0 - v(u)) + v(u)), .$$

Let us assume w.l.o.g. that $\alpha = 1$, otherwise we change the control range Ω by $\alpha\Omega$. Let us show that $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}$ satisfies the three conditions in Definition 4.

(i) *Weak invariance:* Let (t_0, v_0) be a point in $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}$. Then, $v_0 \in \mathcal{C}_{\mathbb{R}^2}$ and there exists $\mathbf{u} \in \mathcal{U}$ such that

$$\phi_2(\mathbb{R}^+, v_0, \mathbf{u}) \qquad \Longrightarrow \qquad \phi(\mathbb{R}^+, v_0, \mathbf{u}) \subset \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2} = \mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}.$$

(ii) *Approximate controllability:* Let us start by showing that exact controllability holds in $\mathbb{R} \times \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$. Let then $(t_1, v_1), (t_2, v_2) \in \mathbb{R} \times \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$ and consider $v(u_1), v(u_2) \in \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$ with $u_1 < 0 < u_2$.

Since controllability holds in $\operatorname{int} \mathcal{C}_{\mathbb{R}^2}$, there exists $t_1', t_2' \in \mathbb{R}$, $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}$ and $s_1, s_2 > 0$ such that

$$\phi(s_1, (t_1, v_1), \mathbf{u}_1) = (t_1', v(u_1)) \qquad \text{and} \qquad \phi(s_2, (t_2', v(u_2)), \mathbf{u}_2) = (t_2, v_2).$$

Analogously, there exists $s_3 > 0$ and $\mathbf{u}_3 \in \mathcal{U}$ such that

$$\phi(s_3, (t_1', v(u_1)), \mathbf{u}_3) = (t_2'', v(u_2)).$$

- If $t_2'' \leqslant t_2'$ we have that

$$s_4 = \frac{t_2' - t_2''}{u_2\alpha} \geqslant 0 \qquad \text{and} \qquad \phi(s_4, (t_2'', v(u_2), u_2) = (t_2'' + u_2\alpha s_4, v(u_2)) = (t_2', v(u_2)).$$

Hence, by concatenation

$$\phi(s_2, \phi(s_4, \phi(s_3, \phi(s_1, (t_1, v_1), \mathbf{u}_1), \mathbf{u}_3), u_2), \mathbf{u}_2) = (t_2, v_2).$$

- If $t_2'' > t_2'$ we have that

$$s_4' = \frac{t_2' - t_2''}{\alpha u_1} > 0$$

and

$$\phi(s_4', (t_1', v(u_1), u_1) = (t_1' + u_1\alpha s_4', v(u_1)) = (t_1' + t_2' - t_2'', v(u_1)),$$

implying that

$$\phi(s_3, \phi(s_4', (t_1', v(u_1), u_1), \mathbf{u}_3) = \phi(s_3, (t_1' + t_2' - t_2'', v(u_1)), \mathbf{u}_3)$$

$$= (t_2' - t_2'', 0) + \phi(s_3, (t_1', v(u_1)), \mathbf{u}_3) = (t_2' - t_2'', 0) + (t_2'', v(u_2)) = (t_2', v(u_2)).$$

Therefore, by concatenation,

$$\phi(s_2, \phi(s_3, \phi(s_4', \phi(s_1, (t_1, v_1), \mathbf{u}_1), u_1), \mathbf{u}_3), \mathbf{u}_2) = (t_2, v_2),$$

showing that controllability holds inside $\mathbb{R} \times \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$ (see Figure 2).

As a consequence, we get that

$$\forall (t, v) \in \mathbb{R} \times \operatorname{int} \mathcal{C}_{\mathbb{R}^2}, \quad \mathbb{R} \times \operatorname{int} \mathcal{C}_{\mathbb{R}^2} \subset \mathcal{O}^+(t, v) \quad \Longrightarrow \quad \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2} \subset \overline{\mathcal{O}^+(t, v)}.$$

On the other hand, if $v \in \mathcal{C}_{\mathbb{R}^2}$ and $v_0 \in \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$ we have that

$$v \in \mathcal{O}_2^-(v_0) \quad \Longleftrightarrow \quad v_0 \in \mathcal{O}_2^+(v).$$

Therefore, for any $(t, v) \in \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2}$ there exists $(t_0, v_0) \in \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2}$ such that $(t_0, v_0) \in \mathcal{O}^+(t, v)$ implying by the previous that

$$\mathbb{R} \times \mathcal{C}_{\mathbb{R}^2} \subset \overline{\mathcal{O}^+(t_0, v_0)} \subset \overline{\mathcal{O}^+(t, v)},$$

showing that $\mathbb{R} \times \mathcal{C}_{\mathbb{R}^2}$ satisfies condition 2. in the Definition 4.

- *Maximality:* If $D \subset \mathbb{R} \times \mathbb{R}^2$ is a control set such that $\mathbb{R} \times \mathcal{C}_{\mathbb{R}^2} \subset D$, then $\pi_2(D)$ is contained in a control set of $\Sigma_{\mathbb{R}^2}$. Since $\mathcal{C}_{\mathbb{R}^2} \subset \pi_2(D)$ we have by maximality that $\mathcal{C}_{\mathbb{R}^2} = \pi_2(D)$ and hence

$$D \subset \pi_2^{-1}(\pi_2(D)) = \pi_2^{-1}(\mathcal{C}_{\mathbb{R}^2}) = \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2},$$

which shows that $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2} = \mathbb{R} \times \mathbb{R}^2$ is a control set of $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$.

□

Next we state and prove our main result.

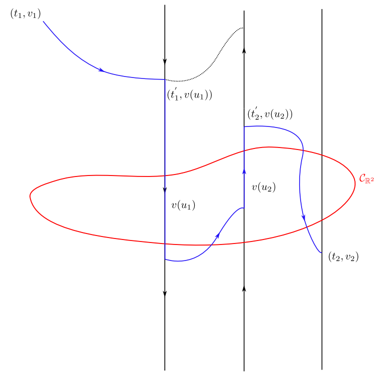

Figure 2 – Trajectory connecting points in the interior of $\mathbb{R} \times \mathcal{C}_{\mathbb{R}^2}$.

Theorem 6. *Any linear control system $\Sigma_{G(\theta)}$ on $G(\theta)$ with nil-rank two satisfying the LARC admits a unique control set $\mathcal{C}_{G(\theta)}$ containing the identity element in its closure. Moreover, if $\mathcal{X} = (A, \xi)$ is the associate linear vector field and*

$$\Omega_0 = \{u \in \Omega; \ \det(A - \alpha u\theta) \neq 0\},$$

we get that:

1. *If $\det A > 0$ and $\operatorname{tr}(A - \alpha u\theta) > 0$ for all $u \in \Omega_0$, then $\mathcal{C}_{G(\theta)}$ is open;*
2. *If $\det A > 0$ and $\operatorname{tr}(A - \alpha u\theta) < 0$ for all $u \in \Omega_0$, then $\mathcal{C}_{G(\theta)}$ is closed;*
3. *If $\det A > 0$ and $\operatorname{tr}(A - \alpha u\theta) = 0$ for some $u \in \Omega_0$, then $\mathcal{C}_{G(\theta)} = G(\theta)$.*

Proof. Since $\Sigma_{G(\theta)}$ has nil-rank two, we get that $\det A \neq 0$, and hence, Proposition 7 and Proposition 8 implies the existence of a diffeomorphism $\psi : G(\theta) \to \mathbb{R} \times \mathbb{R}^2$, fixing the identity element $e = (0, 0)$, and conjugating $\Sigma_{G(\theta)}$ to the control-affine system

$$\left\{ \begin{array}{l} \dot{t} = u\alpha \\ \dot{v} = (A - u\alpha\theta)v + u\eta \end{array} \right. , \ u \in \Omega. \qquad (\Sigma_{\mathbb{R}\times\mathbb{R}^2})$$

By the previous proposition $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ admits a control set $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}$ with nonempty interior and containing $v(\Omega_0)$, implying that

$$\mathcal{C}_{G(\theta)} := \psi^{-1}(\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}),$$

is a control set of $\Sigma_{G(\theta)}$ containing the identity element in its closure. Moreover, by Proposition 12, the properties 1. 2. and 3. hold true. Therefore, it only remains to show the uniqueness of $\mathcal{C}_{G(\theta)}$.

If $e \in \operatorname{int} \mathcal{C}_{G(\theta)}$, Theorem 3 together with the solubility of $G(\theta)$ imply the uniqueness of $\mathcal{C}_{G(\theta)}$. Therefore, let us assume that $\langle A\eta, R\eta \rangle = 0$. If $D \subset \mathbb{R} \times \mathbb{R}^2$ is a control set with nonempty interior of $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ for any $(t, v) \in \operatorname{int} D$ there exists $\tau > 0$ and $\mathbf{u} \in \mathcal{U}$ satisfying

$$\phi(\tau, (t, v), \mathbf{u}) = (t, v) \quad \text{where} \quad \tau = \sum_{i=1}^{n} s_i \quad \text{and} \quad \mathbf{u}_{|[s_{i-1}, s_i)} = u_i \in \Omega, i = 1, \ldots, n.$$

By Proposition 9 we get that

$$\sum_{i=1}^{n} s_i u_i = 0 \quad \text{and} \quad \phi_2(\tau, v, \mathbf{u}) = \mathrm{e}^{\sum_{i=1}^{n} s_i A(u_i)} v + \phi_2(s, 0, \mathbf{u}) = v$$

with $s_{n+1} = 0$. Moreover,

$$\sum_{i=1}^{n} s_i A(u_i) = \sum_{i=1}^{n} s_i (A - u_i \theta) = \tau A - \sum_{i=1}^{n} s_i u_i \theta = \tau A,$$

implies that

$$\phi_2(\tau, v, \mathbf{u}) = \mathrm{e}^{\tau A} v + \phi_2(\tau, 0, \mathbf{u}) = v.$$

On the other hand, the assumption $\langle A\eta, R\eta \rangle = 0$ implies that $A = \lambda I_2$ (see Remark 2) and hence,

$$\phi_2(\tau, v, \mathbf{u}) = \mathrm{e}^{\tau\lambda} v + \phi_2(\tau, 0, \mathbf{u}).$$

Let us assume w.l.o.g. that $\lambda < 0$. In this case, the periodicity of (t, v) implies that

$$v = \phi_2(n\tau, v, \mathbf{u}) = \mathrm{e}^{n\tau\lambda} v + \phi_2(n\tau, 0, \mathbf{u}),$$

and since

$$\mathrm{e}^{n\tau\lambda} \to 0 \quad \Longrightarrow \quad \phi_2(n\tau, 0, \mathbf{u}) \to v.$$

Since $0 \in \overline{\mathcal{C}_{\mathbb{R}^2}}$, there exists by continuity $v_1 \in \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$ such that $\phi_2(n\tau, v_1, \mathbf{u}) \in \pi(\operatorname{int} D)$. On the other hand, for any $u \in \Omega_0$ with $u \neq 0$ it holds that

$$\phi_2(s, v, u) \to v(u), \quad s \to +\infty \quad \Longrightarrow \quad \phi_2(s_0, v, u) \in \operatorname{int} \mathcal{C}_{\mathbb{R}^2},$$

for some $s_0 > 0$. Since $\pi_2(\operatorname{int} D)$ has to be contained the interior of a control set $D' \subset \mathbb{R}^2$ of $\Sigma_{\mathbb{R}^2}$ (see Proposition 2), the previous imply (by exact controllability in $\operatorname{int} D'$) the existence of an orbit starting and finishing in $\operatorname{int} \mathcal{C}_{\mathbb{R}^2}$ and passing by $\operatorname{int} D'$, forcing the equality $D' = \mathcal{C}_{\mathbb{R}^2}$. Therefore,

$$D \subset \pi_2^{-1} \pi_2(D) \subset \pi^{-1}(\mathcal{C}_{\mathbb{R}^2}) = \mathbb{R} \times \mathcal{C}_{\mathbb{R}^2} = \mathcal{C}_{\mathbb{R}\times\mathbb{R}^2},$$

showing the uniqueness of $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}$ and hence, $\mathcal{C}_{G(\theta)}$ is the unique control set with nonempty interior of $\Sigma_{G(\theta)}$, concluding the proof. □

Remark 5. *The previous result shows that, even though, the projected system $\Sigma_{\mathbb{R}^2}$ can have more than one control set with nonempty interior (see Example 3), the control system$\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ admits a unique one. Moreover, by Example 4, the identity element of $G(\theta)$ is not always in the interior of $\mathcal{C}_{G(\theta)}$.*

3 LCSs with nil-rank smaller than two on $G(\theta)$

In this section the cases where the nilrank of a LCS is equal to zero or one. As we will see, for control systems with nil-rank one, the LARC is enough to assure existence of control sets with nonempty interior. On the other hand, the dynamics of LCSs with rank zero strongly depends of the group structure.

3.1 Nil-rank one LCSs

Lemma 3. *For a LCS with nil-rank one, the LARC is equivalent to the ad-rank condition.*

Proof. Up to isomorphism, we can consider $\Sigma_{G(\theta)}$ to be a LCS on $G(\theta)$ satisfying the LARC and such that $\mathcal{X} = (A, \xi)$ and $Y = (\alpha, 0)$, with $\alpha \neq 0$. By Proposition 6,

$$\text{LARC is equivalent to } \langle A\xi, R\xi\rangle^2 + \langle \theta\xi, R\xi\rangle^2 \neq 0,$$

and

$$\text{ad-rank is equivalent to } \langle A\xi, R\xi\rangle \neq 0.$$

However, the assumption that $\Sigma_{G(\theta)}$ has nil-rank one, implies necessarily that $\dim\ker A = 1$. As a consequence, we have the following possibilities:

1. $\xi \in \ker A$ which by the commutativity of A and θ imply that $\langle \theta\xi, R\xi\rangle = 0$ and hence LARC holds only if the ad-rank condition holds;
2. $\xi \notin \ker A$ and hence $\mathbb{R}\xi$ is a one dimensional eigenspace of A with $\mathbb{R}^2 = \mathbb{R}\xi \oplus \ker A$. Consequently,
$$\langle A\xi, R\xi\rangle = 0 \quad \Longrightarrow \quad \langle \theta\xi, R\xi\rangle = 0,$$
showing again that the LARC imply the ad-rank condition.

In both cases the LARC implies the ad-rank condition, proving the result. □

As a consequence of the previous result and Theorem 3, any LCSs with nil-rank one admits a unique control set $\mathcal{C}_{G(\theta)}$ with nonempty interior, such that $G^0 \subset \operatorname{int}\mathcal{C}_{G(\theta)}$.

On the other hand, the assumption that the nil-rank of $\Sigma_{G(\theta)}$ is equal to one, implies necessarily that $\dim\ker A = 1$ and, by the commutativity of A and θ, the subgroup $H := \{0\} \times \ker A \subset G(\theta)$ is a closed normal subgroup. Consequently, we have a well induced linear control system on the 2D Lie group $H\backslash G$ and the following holds:

Theorem 7. *Any LCS with nil-rank one satisfying the LARC admits a unique control set with nonempty interior $\mathcal{C}_{G(\theta)}$ satisfying*

$$\mathcal{C}_{G(\theta)} = \pi^{-1}(\mathcal{C}_{H\backslash G(\theta)}) = \mathcal{C}_{H\backslash G(\theta)} \times H,$$

where $\pi : G(\theta) \to H\backslash G(\theta)$ is the canonical projection. Moreover, if $\mathcal{X} = (A, \xi)$ is the drift of the LCS, then $\mathcal{C}_{G(\theta)}$ is open if $\operatorname{tr} A > 0$ and closed if $\operatorname{tr} A < 0$, or $\operatorname{tr} A = 0$ and $\mathcal{C}_{G(\theta)} = G(\theta)$.

Proof. By the previous discussion, $H = \{0\} \times \ker A$ is a one-dimensional normal subgroup of $G(\theta)$ and, $H\backslash G(\theta)$ is a 2-dimensional Lie group.

Since the induced LCS $\Sigma_{H\backslash G(\theta)}$ on $H\backslash G(\theta)$ satisfies the LARC, there exists a unique control set $\mathcal{C}_{H\backslash G(\theta)}$ that contains the identity element of $H\backslash G(\theta)$ in its interior. Moreover,

$$\pi^{-1}(eH) = H = \{0\} \times \ker A \subset \operatorname{int} \mathcal{C}_{G(\theta)},$$

since H is contained in the set of fixed points of the flow of $\mathcal{X} = (A, \xi)$. By Proposition 2 we obtain that $\mathcal{C}_{G(\theta)} = \pi^{-1}(\mathcal{C}_{H\backslash G(\theta)})$, proving the first equality.

On the other hand, if $w \in \ker A$ is an unitary vector, and we consider the basis $\{(1,0), (0, Rw), (0, w)\}$, the canonical projection is given by

$$\pi : G(\theta) \to (t, \langle v, Rw \rangle),$$

which coincides with projection onto the first two components of the chosen basis. As a consequence, the control set $\mathcal{C}_{H\backslash G(\theta)}$ is naturally identified inside $G(\theta)$ as the subset $\mathcal{C}_{H\backslash G(\theta)} \times \{0\}$ of the plane generated by $\{(1,0), (0, Rw)\}$. Since

$$(0, s_1 w)(t, s_2 Rw) = (t, s_1 w + s_2 Rw), \qquad \forall t, s_1, s_2 \in \mathbb{R},$$

we get that

$$\pi^{-1}(\mathcal{C}_{H\backslash G(\theta)}) = H\mathcal{C}_{H\backslash G(\theta)} \times \{0\} = \mathcal{C}_{H\backslash G(\theta)} \times H,$$

proving the second equality. Now, since π conjugates $\Sigma_{G(\theta)}$ and $\Sigma_{H\backslash G(\theta)}$, it commutes with the associated derivations, that is,

$$\hat{\mathcal{D}} \circ \pi = \pi \circ \mathcal{D},$$

where $\hat{\mathcal{D}}$ is the derivation associated with the system $\Sigma_{H\backslash G(\theta)}$. Since

$$\{0\} \times \ker A \subset \ker \mathcal{D} \qquad \text{and} \qquad \dim \ker \mathcal{D} = 2,$$

we obtain that $\operatorname{tr} \hat{\mathcal{D}} = \operatorname{tr} \mathcal{D} = \operatorname{tr} A$ is only possible eigenvalue of $\hat{\mathcal{D}}$, which by the results in Section 2.5 implies the result. □

3.2 Nil-rank zero LCSs

In this section we analyze the LCSs with nil-rank zero. As we will see, for most of the classes of the groups $G(\theta)$, the LCSs with nil-rank zero do not admit control sets with nonempty interior. Instead, they admit an infinite numbers of control sets with empty interior.

We start with a lemma which will be used in the proof of the results of this section.

Lemma 4. *Let us assume that* $\theta \neq \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix}$ *with* $\gamma \neq 0$. *If* $\langle \theta\xi, R\xi \rangle \neq 0$, *there exists* $\hat{\xi} \in \mathbb{R}^2$ *such that*

$$g(t) := \langle \Lambda_t^{\theta}\xi, \hat{\xi} \rangle \geqslant 0, \qquad \forall t \in \mathbb{R}.$$

Proof. The lemma is proved case by case.

1. $\theta = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}$, $|\gamma| \in (0, 1]$: In this case, $\langle \theta\xi, R\xi \rangle \neq 0$ implies that $\xi = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2$ with $\xi_1\xi_2 \neq 0$.

Let us assume that $\gamma \in (0, 1)$, since the case $\gamma \in (-1, 0)$ is analogous. Consider $\hat{\xi} = \gamma\xi_1^{-1}\mathbf{e}_1 - \gamma\xi_2^{-1}\mathbf{e}_2$, we obtain

$$g(t) = \langle \Lambda_t^{\theta}\xi, \hat{\xi} \rangle = \left\langle (\mathrm{e}^t - 1)\xi_1\mathbf{e}_1 + \frac{1}{\gamma}(\mathrm{e}^{\gamma t} - 1)\mathbf{e}_2, \gamma\xi_1^{-1}\mathbf{e}_1 - \gamma\xi_2^{-1}\mathbf{e}_2 \right\rangle = \gamma(\mathrm{e}^t - 1) - (\mathrm{e}^{\gamma t} - 1).$$

Derivation, implies that $g'(t) = \gamma(\mathrm{e}^t - \mathrm{e}^{\gamma t})$, and hence

$$g'(t) > 0, t \in (0, +\infty) \quad \text{and} \quad g'(t) < 0, t \in (-\infty, 0),$$

showing that g is strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, +\infty)$. Since $g(0) = 0$ we obtain that $g(t) \geqslant 0$ for all $t \in \mathbb{R}$.

2. $\theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$: In this case, $\langle \theta\xi, R\xi \rangle \neq 0$ implies also that $\xi = \xi_1\mathbf{e}_1 + \xi_2\mathbf{e}_2$ with $\xi_1\xi_2 \neq 0$. Defining $\hat{\xi} = \xi_1^{-1}\mathbf{e}_1 - \xi_2^{-1}\mathbf{e}_2$ gives us

$$g(t) = \langle \Lambda_t^{\theta}\xi, \hat{\xi} \rangle = \left\langle (\mathrm{e}^t - 1)\xi_1\mathbf{e}_1 + t\xi_2\mathbf{e}_2, \xi_1^{-1}\mathbf{e}_1 - \xi_2^{-1}\mathbf{e}_2 \right\rangle = (\mathrm{e}^t - t - 1),$$

As the case before, deriving the function, we obtain $g'(t) = \mathrm{e}^t - 1$, that satisfies

$$g'(t) > 0, t \in (0, +\infty) \quad \text{and} \quad g'(t) < 0, t \in (-\infty, 0),$$

and because $g(0) = 0$, $g(t) \geqslant 0$ for all $t \in \mathbb{R}$.

3. $\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: In this case, $\langle \theta\xi, R\xi \rangle \neq 0$ implies that $\xi = \xi_1 \mathbf{e}_1 + \xi_2 \mathbf{e}_2$ with $\xi_2 \neq 0$. By considering and we should define $\hat{\xi} = \xi_2^{-1}\mathbf{e}_1 - \xi_1\xi_2^{-2}\mathbf{e}_2$. By a simple calculation shows us that

$$g(t) = \langle \Lambda_t^\theta \xi, \hat{\xi} \rangle = -\mathrm{e}^t + 1 + t\mathrm{e}^t.$$

As in the first case, $g'(t) = te^t$ and hence

$$g'(t) > 0, t \in (0, +\infty) \quad \text{and} \quad g'(t) < 0, t \in (-\infty, 0),$$

implying that

$$g(t) \geqslant g(0) = 0, \qquad \forall t \in \mathbb{R}.$$

4. $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: In this case, $\langle \theta\xi, R\xi \rangle \neq 0$ implies $\xi \neq 0$. Defining $\hat{\xi} = -\xi_2 e_1 + \xi_1 e_2$ gives us

$$g(t) = \langle \Lambda_t^\theta \xi, \hat{\xi} \rangle = \Big\langle [\xi_2(\cos t - 1) + \xi_1 \sin t]\, e_1 + [\xi_2 \sin t - \xi_1(\cos t - 1)]\, e_2, -\xi_2 e_1 + \xi_1 e_2 \Big\rangle$$
$$= -\left(\xi_1^2 + \xi_2^2\right)(\cos t - 1) = |\xi|^2(1 - \cos t) \geqslant 0,$$

as required.

□

Using the previous lemma we are able to show the following:

Theorem 8. *Let $\Sigma_{G(\theta)}$ be a LCS with nil-rank zero. It holds:*

1. *If $\theta \neq \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix}$ for all $\gamma \neq 0$, then $\Sigma_{G(\theta)}$ admits an infinite numbers of control sets with empty interior*
2. *If $\theta = \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix}$ for some $\gamma \neq 0$, then $\Sigma_{G(\theta)}$ is controllable.*

Proof. By Proposition 7 we can assume w.l.o.g. that

$$\begin{cases} \dot{t} = u\alpha \\ \dot{v} = \Lambda_t^\theta \xi \end{cases}, u \in \Omega. \qquad (\Sigma_{G(\theta)})$$

1. If $\theta \neq \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix}$ for all $\gamma \neq 0$, there exists by Lemma 4 a vector $\hat{\xi} \in \mathbb{R}^2$ such that $\langle \Lambda_t^\theta \xi, \hat{\xi} \rangle \geqslant 0$ for all $t \in \mathbb{R}$. As a consequence, the smooth function

$$f : G(\theta) \to \mathbb{R}, \qquad f(t, v) := \langle v, \hat{\xi} \rangle,$$

is such that, the functions $g_{\mathbf{u}}(s) = f(\phi_2(s,(t,v),\mathbf{u}))$, satisfies

$$g'_{\mathbf{u}}(s) = \frac{d}{ds}\langle\phi_2(s,(t,v),\mathbf{u}),\hat{\xi}\rangle = \langle\Lambda^\theta_{\phi_1(s,(t,v),\mathbf{u})}\xi,\hat{\xi}\rangle \geqslant 0,$$

and are nondecreasing. As a consequence,

$$(t_1,v_1),(t_2,v_2) \in G(\theta); \qquad \text{with} \qquad f(t_1,v_1) < f(t_2,v_2),$$

cannot be in the same control set of $\Sigma_{G(\theta)}$.

On the other hand, the fact that $(0,v) \in G(\theta)$ are fixed points of the drift $\mathcal{X} = (0,\xi)$, implies that any such point is contained in a control set of $\Sigma_{G(\theta)}$. Therefore, for any $c \in \mathbb{R}$ the plane

$$P_c := \{(t,v), \quad \langle v,\hat{\xi}\rangle = c\},$$

contains (at least) one control of $\Sigma_{G(\theta)}$, concluding the proof.

2. Let us start by noticing that the solutions of $\Sigma_{G(\theta)}$ satisfy

$$\phi(s,(t,v_1+v_2),\mathbf{u}) = \phi(s,(t,v_1),\mathbf{u}) + v_2, \quad \forall \mathbf{u} \in \mathcal{U}. \tag{3.1}$$

Let $H = \mathbb{R}\cdot(0,\theta^{-1}\xi)$ and consider the homogeneous space $H\backslash G(\theta)$. By the calculations in Section 2.5.2, the canonical map is given by

$$\pi : G(\theta) \to H\backslash G(\theta), \qquad \pi(t,v) = (t,\langle v,R\theta^{-1}\xi\rangle).$$

Therefore, the induced system on $\mathbb{R}^2 = H\backslash G(\theta)$ is given by

$$\begin{cases} \dot{t} = u\alpha \\ \dot{x} = \langle\Lambda^\theta_t\xi, R\theta^{-1}\xi\rangle \end{cases}. \tag{$\Sigma_{H\backslash G(\theta)}$}$$

Now, the fact that $\theta = \gamma I_{\mathbb{R}^2} + R$ implies that $\mathrm{e}^{t\theta} = \mathrm{e}^{\gamma t}((\cos t)I_{\mathbb{R}^2} + (\sin t)R)$. As a consequence,

$$\langle\Lambda^\theta_t\xi,R\theta^{-1}\xi\rangle = \langle(\mathrm{e}^{t\theta}-I_{\mathbb{R}^2})\theta^{-1}\xi,R\theta^{-1}\xi\rangle = \langle\mathrm{e}^{t\theta}\theta^{-1}\xi,R\theta^{-1}\xi\rangle$$

$$= \mathrm{e}^{\gamma t}\cos t\langle\theta^{-1}\xi,R\theta^{-1}\xi\rangle + \mathrm{e}^{\gamma t}\sin t\langle R\theta^{-1}\xi,R\theta^{-1}\xi\rangle = |\theta^{-1}\xi|^2\mathrm{e}^{\gamma t}\sin t.$$

The rest of the prove is conclude in XX steps.

Step 1: $\Sigma_{H\backslash G(\theta)}$ is controllable;

It is not hard to see that the solutions $\hat{\phi}$ of $\Sigma_{H\backslash G(\theta)}$ satisfy

$$(i)\ \ \hat{\phi}_1(s,(t_0,x_0),u) = t_0 + u\alpha s, \qquad (ii)\ \ \hat{\phi}_2(s,(t_0,x_0),0) = x_0 + s|\theta^{-1}\xi|^2\mathrm{e}^{\gamma t_0}\sin t_0,$$

$$\text{and} \qquad (iii)\ \ \hat{\phi}_2(s,(t_0,x_0),u) = \hat{\phi}_2(s,(t_0,0),u) + x_0.$$

In particular, property (i) implies that

$$\forall t_1, t_2 \in \mathbb{R},\ \exists u \in \Omega; \qquad \hat{\phi}(s, \{t_1\} \times \mathbb{R}, u) = \{t_2\} \times \mathbb{R}.$$

Therefore, it is always possible to construct a trajectory from any given point (t, x) to the axis $\{0\} \times \mathbb{R}$ in positive and negative time. Hence, $\Sigma_{H\backslash G(\theta)}$ is controllable as soon as any two points in $\{0\} \times \mathbb{R}$ can be connected by a trajectory of the system.

Let us consider $x, y \in \mathbb{R}$ and assume w.l.o.g. that $x < y$. Fix $t_1, t_2 \in \mathbb{R}$ satisfying

$$-\pi/2 < t_1 < 0 < t_2 < \pi/2,$$

and consider $s_{0,1}, s_{2,0}, s_{1,2} \in (0, +\infty)$ and $u_{0,1}, u_{2,0}, u_{1,2} \in \Omega$ such that

$$\hat{\phi}(s_{0,1}, \{0\} \times \mathbb{R}, u_{0,1}) = \{t_1\} \times \mathbb{R}, \quad \hat{\phi}(s_{2,0}, \{t_2\} \times \mathbb{R}, u_{2,0}) = \{0\} \times \mathbb{R}$$

and

$$\hat{\phi}(s_{1,2}, \{t_1\} \times \mathbb{R}, u_{1,2}) = \{t_2\} \times \mathbb{R}.$$

Take $\hat{x}_i, \hat{y}_j$, $i, j = 1, 2$ satisfying

$$\hat{\phi}(s_{0,1}, (0, x), u_{0,1}) = (t_1, \hat{x}_1), \quad \hat{\phi}(s_{0,1}, (0, y), u_{0,1}) = (t_1, \hat{y}_1),$$

$$\hat{\phi}(s_{2,0}, (t_2, \hat{x}_2), u_{2,0}) = (0, x) \quad \text{and} \quad \hat{\phi}(s_{2,0}, (t_2, \hat{y}_2), u_{2,0}) = (0, y).$$

Using the third property, it is straightforward to show that there exist z_1, z_2 satisfying

$$z_1 \leqslant \min\{\hat{x}_1, \hat{y}_1\} \quad z_2 \leqslant \min\{\hat{x}_2, \hat{y}_2\}, \quad \text{and} \quad \hat{\phi}_2(s_{1,2}, (t_1, z_1), u_{1,2}) = z_2.$$

Furthermore, since $\sin t_1 < 0$ and $\sin t_2 > 0$ there exists $s_1, s_2, \hat{s}_1, \hat{s}_2 \in [0, +\infty)$ such that

$$\hat{x}_1 + s_1 |\theta^{-1}\xi|^2 \mathrm{e}^{\gamma t_1} \sin t_1 = z_1, \qquad \hat{y}_1 + \hat{s}_1 |\theta^{-1}\xi|^2 \mathrm{e}^{\gamma t_1} \sin t_1 = z_1,$$

$$z_2 + \hat{s}_2 |\theta^{-1}\xi|^2 \mathrm{e}^{\gamma t_2} \sin t_2 = \hat{x}_2 \quad \text{and} \quad z_2 + s_2 |\theta^{-1}\xi|^2 \mathrm{e}^{\gamma t_2} \sin t_2 = \hat{y}_2.$$

Using the previous, a closed trajectory passing through $(0, x_1)$ to $(0, x_2)$ is obtained as follows:

1. Go from $(0, x)$ to $(t_1, \hat{x}_1)$ in time $s_{0,1}$ with the control $u_{0,1}$;
2. Go from $(t_1, \hat{x}_1)$ to (t_1, z_1) in time s_1 with the control 0;
3. Go from (t_1, z_1) to (t_2, z_2) in time $s_{1,2}$ with the control $u_{1,2}$;

4. Go from (t_2, z_2) to $(t_2, \hat{y}_2)$ in time s_2 with the control 0;
5. Go from $(t_2, \hat{y}_2)$ to $(0, y)$ in time $s_{2,0}$ with the control $u_{2,0}$;
6. Go from $(0, y)$ to $(t_1, \hat{y}_1)$ in time $s_{0,1}$ with the control $u_{1,0}$;
7. Go from $(t_1, \hat{y}_1)$ to (t_1, z_1) in time $\hat{s}_1$ with the control 0;
8. Go from (t_1, z_1) to (t_2, z_2) in time $s_{1,2}$ with the control $u_{1,2}$;
9. Go from (t_2, z_2) to $(t_2, \hat{x}_2)$ in time $\hat{s}_2$ with the control 0;
10. Go from $(t_2, \hat{x}_2)$ to $(0, x)$ in time $s_{2,0}$ with the control $u_{2,0}$;

The previous implies that $\Sigma_{H\backslash G(\theta)}$ is controllable.

Step 2: If $s_0(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^{\pm}(e)$, there exists $s_1 \in \mathbb{R}$ such that

$$s_0 s_1 < 0 \qquad \text{and} \qquad s_1(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^{\pm}(e).$$

Let us show the claim only for the positive orbit, since the proof for the negative orbit is analogous.

By a straightforward calculation we obtain that the second coordinate of the solutions of $\Sigma_{G(\theta)}$, for $u \neq 0$, is given by

$$\phi_2(s, (t_0, v_0), u) = \frac{1}{u\alpha} \mathrm{e}^{t_0\theta}(\mathrm{e}^{u\alpha s\theta} - u\alpha s\theta - I_{\mathbb{R}^2})\theta^{-2}\xi + s\Lambda^{\theta}_{t_0}\xi + v_0.$$

Let us choose $\rho \in \mathbb{R}$ such that $\pm\rho\alpha^{-1} \in \Omega$. Since $\phi_1(s, (t_0, v_0), u) = t_0 + u\alpha s$, we get that

$$\phi(s, \phi(s, s_0(0, \theta^{-1}\xi), \rho\alpha^{-1}), -\rho\alpha^{-1}) = (0, \phi_2(s, \phi(s, s_0(0, \theta^{-1}\xi), \rho\alpha^{-1}), -\rho\alpha^{-1})) \in \{0\} \times \mathbb{R}^2.$$

Therefore,

$$\begin{aligned}\phi_2(s, \phi(s, s_0(0, \theta^{-1}\xi), \rho\alpha^{-1}), -\rho\alpha^{-1}) &= \phi_2\left(s, \frac{1}{\rho}(\mathrm{e}^{\rho s\theta} - \rho s\theta - I_{\mathbb{R}^2})\theta^{-2}\xi + s_0\theta^{-1}\xi, \rho\alpha^{-1}\right) \\ = -\frac{1}{\rho}\mathrm{e}^{\rho s\theta}(\mathrm{e}^{-\rho s\theta} + \rho s\theta - I_{\mathbb{R}^2})\theta^{-2}\xi &+ s\Lambda^{\theta}_{\rho s}\xi + \frac{1}{\rho}(\mathrm{e}^{\rho s\theta} - \rho s\theta - I_{\mathbb{R}^2})\theta^{-2}\xi + s_0\theta^{-1}\xi \\ &= -\frac{2}{\rho}\left(\theta^{-2}\xi + \rho s\theta^{-1}\xi - \mathrm{e}^{\rho s\theta}\theta^{-2}\xi\right) + s_0\theta^{-1}\xi.\end{aligned}$$

Using that $\mathrm{e}^{t\theta} = \mathrm{e}^{\gamma t}((\cos t)I_{\mathbb{R}^2} + (\sin t)R)$ and $(\gamma^2+1)\theta^{-1} = \gamma I_{\mathbb{R}^2} - R$, allows us to write such solution on the orthogonal basis $\{\theta^{-1}\xi, R\{\theta^{-1}\xi\}$ as

$$\phi_2(s, \phi(s, s_0(0, \theta^{-1}\xi), \rho\alpha^{-1}), -\rho\alpha^{-1}) = H_1(s) \cdot \theta^{-1}\xi + H_2(s) \cdot R\theta^{-1}\xi,$$

where

$$H_1(s) = \frac{2}{\rho(\gamma^2+1)}\left[\mathrm{e}^{\gamma\rho s}(\gamma\cos(\rho s) + \sin(\rho s)) - \gamma - s\rho(\gamma^2+1)\right] + s_0,$$

and

$$H_2(s) = \frac{2}{\rho(\gamma^2+1)}\left[\mathrm{e}^{\gamma\rho s}(\gamma\sin(\rho s) - \cos(\rho s)) + 1\right].$$

Let us assume here that $\gamma > 0$ since the other possibility is analogous. Note that

$$H_2(s) = 0 \qquad \Longleftrightarrow \qquad \gamma\mathrm{e}^{\gamma\rho s}\sin(\rho s) + 1 = \mathrm{e}^{\gamma\rho s}\cos(\rho s),$$

implying that

$$H_1(s) = \frac{2}{\rho}\mathrm{e}^{\gamma\rho s}\sin(\rho s) - 2s + s_0, \qquad \text{if} \qquad H_2(s) = 0.$$

On the other hand, for any $k \in \mathbb{Z}$, we get that

$$H_2\left(\frac{2k\pi}{\rho}\right) = \frac{2}{\rho(\gamma^2+1)}(-\mathrm{e}^{\gamma 2k\pi}+1) \qquad \text{and} \qquad H_2\left(\frac{\pi+2k\pi}{\rho}\right) = \frac{2}{\rho(\gamma^2+1)}(\mathrm{e}^{\gamma(\pi+2k\pi)}+1),$$

As a consequence, if $\gamma k > 0$ and $\varepsilon > 0$ is small enough, the function H_2 changes signs on the intervals

$$I_k := \left(\frac{2\pi k+\varepsilon}{\rho}, \frac{\pi+2\pi k-\varepsilon}{\rho}\right) \qquad \text{and} \qquad \hat{I}_k := \left(\frac{\pi+2\pi k+\varepsilon}{\rho}, \frac{2\pi(k+1)-\varepsilon}{\rho}\right).$$

Let us denote by $s_k \in I_k$ and $\hat{s}_k \in \hat{I}_k$ zeros of H_2 when $\gamma k > 0$. For $|k| \to +\infty$, it holds:

1. If $s_0 > 0$ and $\gamma > 0$ we choose $\rho > 0$ and $k \in \mathbb{N}$. In this case,

$$\gamma k > 0, \quad \hat{s}_k \to +\infty, \quad H_2(\hat{s}_k) = 0 \quad \text{and}$$

$$H_1(\hat{s}_k) = \frac{2}{\rho}\mathrm{e}^{\gamma\rho\hat{s}_k}\sin(\rho\hat{s}_k) - 2\hat{s}_k + s_0 \to -\infty, \qquad \text{since} \qquad \sin(\rho\hat{s}_k) \leqslant -\sin(\varepsilon) < 0.$$

2. If $s_0 > 0$ and $\gamma < 0$ we choose $\rho < 0$ and $-k \in \mathbb{N}$. In this case,

$$\gamma k > 0, \quad s_k \to +\infty, \quad H_2(s_k) = 0 \quad \text{and}$$

$$H_1(s_k) = \frac{2}{\rho}\mathrm{e}^{\gamma\rho s_k}\sin(\rho s_k) - 2s_k + s_0 \to -\infty, \qquad \text{since} \qquad \sin(\rho s_k) \geqslant \sin(\varepsilon) > 0.$$

3. If $s_0 < 0$ and $\gamma > 0$ we choose $\rho > 0$ and $k \in \mathbb{N}$. In this case,

$$\gamma k > 0, \quad s_k \to +\infty, \quad H_2(s_k) = 0 \quad \text{and}$$

$$H_1(s_k) = \frac{2}{\rho}\mathrm{e}^{\gamma\rho s_k}\sin(\rho s_k) - 2s_k + s_0 \to +\infty, \qquad \text{since} \qquad \sin(\rho s_k) \geqslant \sin(\varepsilon) > 0.$$

4. If $s_0 < 0$ and $\gamma < 0$ we choose $\rho < 0$ and $-k \in \mathbb{N}$. In this case,

$$\gamma k > 0, \quad \hat{s}_k \to +\infty, \quad H_2(\hat{s}_k) = 0 \quad \text{and}$$

$$H_1(\hat{s}_k) = \frac{2}{\rho}\mathrm{e}^{\gamma\rho\hat{s}_k}\sin(\rho\hat{s}_k) - 2\hat{s}_k + s_0 \to +\infty, \qquad \text{since} \qquad \sin(\rho\hat{s}_k) \leqslant -\sin(\varepsilon) < 0.$$

Therefore, in any case, there exists $S > 0$ such that $H_2(S) = 0$ and $s_0 H_1(S) < 0$. Hence, for $s_1 = H_1(S)$, we get

$$s_1(0, \theta^{-1}\xi) = \phi(S, \phi(S, s_0(0, \theta^{-1}\xi), \rho\alpha^{-1}), -\rho\alpha^{-1}) \in \phi(S, \phi(S, \operatorname{int} \mathcal{O}^+(e), \rho\alpha^{-1}), -\rho\alpha^{-1})$$

and

$$\phi(S, \phi(S, \operatorname{int} \mathcal{O}^+(e), \rho\alpha^{-1}), -\rho\alpha^{-1}) \subset \operatorname{int} \mathcal{O}^+(e),$$

which proves the assertion.

Step 3: If for some $s_0 \in \mathbb{R}$, $s_0(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^{\pm}(e)$ then,

$$\exists R > 0;\ \forall s > R \qquad \Longrightarrow \qquad \begin{cases} s(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^{\pm}(e) & \text{if } s_0 > 0 \\ -s(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^{\pm}(e) & \text{if } s_0 < 0 \end{cases}.$$

Again, let us show the assertion only for the positive orbit. Since $\Sigma_{G(\theta)}$ satisfies the LARC, by (COLONIUS; KLIEMANN, 2000, Lemma 4.5.2) it holds that

$$s_0(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^+(e) \qquad \Longrightarrow \qquad s_0(0, \theta^{-1}\xi) \in \operatorname{int}_{\leqslant S} \mathcal{O}^+(e) \overset{(4}{=} \operatorname{int}_S \mathcal{O}^+(e),$$

for some $S > 0$. Consequently, there exists an interval $(a, b) \subset \mathbb{R}$ satisfying $s_0 \cdot (a, b) \subset (0, +\infty)$ and

$$\forall s \in (a, b), \qquad s(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^+_S(e).$$

On the other hand, the fact that points in $\{0\} \times \mathbb{R}^2$ are fixed points of the drift, allows us to obtain, that

$$(s_1 + s_2)(0, \theta^{-1}\xi) = \varphi_S(s_1(0, \theta^{-1}\xi)) s_2(0, \theta^{-1}\xi) \in \operatorname{int}_{2S} \mathcal{O}^+(e) \subset \operatorname{int} \mathcal{O}^+(e),$$

and inductively we conclude that

$$\forall n \in \mathbb{N}, s \in (na, nb), \qquad s(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^+(e).$$

Since, a, b have the same sign as s_0, there exists $R > 0$ such that

$$\frac{s_0}{|s_0|}(R, +\infty) \subset \bigcup_{n \in \mathbb{N}} (na, nb),$$

the assertion follows.

Step 4: $\Sigma_{G(\theta)}$ is controllable.

Let us consider $(t, v) \in \operatorname{int} \mathcal{O}^+(e)$, which exists by the LARC. By Step 1. there exists $s > 0$ and $\mathbf{u} \in \mathcal{U}$ such that

$$\pi(\phi(s, (t, v), \mathbf{u})) = \hat{\phi}(s, \pi(t, v), \mathbf{u}) = (0, 0) = \pi(e) \qquad \Longrightarrow \qquad \phi(s, (t, v), \mathbf{u}) \in H.$$

Consequently, there exists $s_0 \in \mathbb{R}$ such that

$$s_0(0, \theta^{-1}\xi) = \phi(s, (t, v), \mathbf{u}) \subset \phi(s, \operatorname{int} \mathcal{O}^+(e), \mathbf{u}) \subset \operatorname{int} \mathcal{O}^+(e).$$

By Step 2., the previous implies the existence of $s_1 \in \mathbb{R}$ with $s_0 s_1 < 0$ and such that $s_1(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^+(e)$.

Now, Step 3. applied for s_0 and s_1 assures the existence of $S > 0$ great enough, such that

$$S(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^+(e) \quad \text{and} \quad -S(0, \theta^{-1}\xi) \in \operatorname{int} \mathcal{O}^+(e).$$

Since, by (COLONIUS; KLIEMANN, 2000, Lemma 4.5.2), there exists $S_1, S_2 > 0$ such that

$$S(0, \theta^{-1}\xi) \in \operatorname{int}_{S_1} \mathcal{O}^+(e) \quad \text{and} \quad -S(0, \theta^{-1}\xi) \in \operatorname{int}_{S_2} \mathcal{O}^+(e),$$

we conclude that

$$e = (S(0, \theta^{-1}\xi))(-S(0, \theta^{-1}\xi)) = \varphi_{S_2}(S(0, \theta^{-1}\xi))(-S(0, \theta^{-1}\xi)) \in \varphi_{S_2}(\operatorname{int} \mathcal{O}^+_{S_1}(e)) \operatorname{int} \mathcal{O}^+_{S_2}(e)$$

$$\overset{4}{=} \operatorname{int} \mathcal{O}^+_{S_1+S_2}(e) \subset \operatorname{int} \mathcal{O}^+(e),$$

which by Theorem 3 implies the controllability of $\Sigma_{G(\theta)}$, concluding the proof.

□

4 Connected nonsimply connected groups

In this section we analyze the LCSs on 3D solvable Lie groups G that are not simply connected. The analysis is done by using the lift of LCSs from G to its simply connected cover as commented in Section 2.4.

Due to the characterization present in (ONISHCHIK; VINBERG, 1994, Chapter 7), the only connected, solvable, nonnilpotent 3D Lie groups associated with the Lie algebras $\mathfrak{g}(\theta)$ that are also nonsimply connected, appear when

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the next sections we consider, separately the possible cases for θ.

4.1 The group of rigid motions and its n-fold covers

If $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ then, for each $n \in \mathbb{N}$

$$Z_n := \{(2kn\pi, 0) \in G(\theta), k \in \mathbb{Z}\}$$

is a discrete central subgroup of $G(\theta)$. In particular, the quotients $G(\theta)/Z_n$ are connected Lie groups associated with the Lie algebra $\mathfrak{g}(\theta)$. The group $SE(2) := G(\theta)/Z_1$ is the group of *proper motions* of $\mathbb{R}^2$. It is the connected component of the group of rigid motions of $\mathbb{R}^2$. For $n \geqslant 2$ the group $SE(2)_n := G(\theta)/Z_n$ is a n-fold cover of $SE(2)$.

The canonical projection is given by

$$\pi_n(t, v) = ([t], v), \quad [t] := t + 2n\pi\mathbb{Z}.$$

By the results in Sections 1.4 and 1.5, in order to analyze the behavior of LCSs on $\Sigma_{SE(2)_n}$, it is enough to understand LCSs on $G(\theta)$ whose flow of the drift let the subgroups Z_n invariant. However, using the expression (1.9) we have that

$$\varphi_s(2nk\pi, 0) = (2nk\pi, \mathrm{e}^{sA}0 + \Lambda^\theta_{2nk\pi}\Lambda^A_s\xi) = (2nk\pi, 0), \qquad \text{where we used that} \qquad \Lambda^\theta_{2nk\pi} \equiv 0,$$

implying that a linear vector field on $SE(2)_n$ is given by

$$\mathcal{X}([t], v) = (0, Av + \Lambda^\theta_t\xi).$$

On the other hand, if $\det A \neq 0$, a LCS on $G(\theta)$ is conjugated to the control-affine system $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$ through the diffeomorphism

$$\psi(t, v) = (t, \rho_{-t}(v + \Lambda^\theta_t A^{-1}\xi)).$$

By Proposition 13,

$$\forall u \in \Omega_0, \qquad \mathbb{R} \times \{v(u)\},$$

is contained in the interior of the control set $\mathcal{C}_{\mathbb{R}\times\mathbb{R}^2}$ of $\Sigma_{\mathbb{R}\times\mathbb{R}^2}$. As a consequence,

$$\psi^{-1}(\mathbb{R} \times \{v(u)\}) = \{(t, \rho_t(v(u) - \Lambda_t A^{-1}\xi)), \ \ t \in \mathbb{R}\} \subset \operatorname{int}\mathcal{C}_{G(\theta)}, \qquad \forall u \in \Omega_0.$$

We can now prove our the main result for the control set of LCSs on $SE(2)_n$.

Theorem 9. *Let $\Sigma_{SE(2)_n}$ be a LCS on $SE(2)_n$ that satisfies the LARC. It holds:*

(i) If $A \not\equiv 0$ then $\Sigma_{SE(2)_n}$ admits a unique control set $\mathcal{C}_{SE(2)_n}$ satisfying

$$\pi_n^{-1}\left(\mathcal{C}_{SE(2)_n}\right) = \mathcal{C}_{G(\theta)},$$

where $\pi_n : G(\theta) \to SE(2)_n$ is the canonical projection and $\mathcal{C}_{G(\theta)}$ the unique control set of the LCS on $G(\theta)$ that is π_n-conjugated to $\Sigma_{SE(2)_n}$.

(ii) If $A \equiv 0$ then $\Sigma_{SE(2)_n}$ admits an infinite number of control sets with empty interior.

Proof. (i) By Proposition 2 there exists a control set $\mathcal{C}_{SE(2)_n}$ of the system $\Sigma_{SE(2)_n}$ satisfying $\pi_n\left(\mathcal{C}_{G(\theta)}\right) \subset \mathcal{C}_{SE(2)_n}$ and the equality holds if we show that

$$\pi_n^{-1}(\pi_n(t, v)) \subset \operatorname{int}\mathcal{C}_{G(\theta)}, \qquad \text{for some} \qquad (t, v) \in \operatorname{int}\mathcal{C}_{G(\theta)}.$$

Since, for all $k, n \in \mathbb{N}$ we have that $\rho_{2nk\pi} = \mathrm{id}_{\mathbb{R}^2}$, we get that

$$\forall u \in \Omega_0, \qquad \pi_n^{-1}(\pi_n(0, v(u))) = \{(2nk\pi, v(u)), k \in \mathbb{Z}\}$$
$$= \left\{\left(2nk\pi, \rho_{2nk\pi}(v(u)) - \Lambda^\theta_{2nk\pi}A^{-1}\xi\right)\right), k \in \mathbb{Z}\right\} \subset \psi^{-1}(\mathbb{R} \times \{v(u)\}) \subset \operatorname{int}\mathcal{C}_{G(\theta)},$$

implying that $\pi_n^{-1}(\mathcal{C}_{SE(2)_n}) = \mathcal{C}_{G(\theta)}$ and showing the item.

(ii) The function

$$f : SE(2)_n \to \mathbb{R}, \qquad f([t], v) := \langle v, \hat{\xi}\rangle,$$

where $\hat{\xi}$ is given in Lemma 4, is such that the functions $g_{\mathbf{u}}(s) = \langle \phi(s, ([t], v), \mathbf{u})$ satisfy

$$g'_{\mathbf{u}}(s) = \frac{d}{ds}\langle \phi_2(s, ([t], v), \mathbf{u}), \hat{\xi}\rangle = \langle \Lambda^\theta_{\phi_1(s,([t],v),\mathbf{u})}\xi, \hat{\xi}\rangle \geqslant 0,$$

and are nondecreasing. Therefore, if

$$([t_1], v_1), ([t_2], v_2) \in G(\theta); \qquad \text{satisfy} \qquad f([t_1], v_1) < f([t_2], v_2),$$

they cannot be in the same control set of $\Sigma_{SE(2)_n}$.

On the other hand, the points $([0], v) \in SE(2)_n$ are fixed points of the drift $\mathcal{X} = (0, \xi)$, and hence, any such point is contained in a control set of $\Sigma_{SE(2)_n}$. Therefore, for any $r \in \mathbb{R}$ the cylinder

$$C_r := \{([t], v), \quad \langle v, \hat{\xi} \rangle = r\},$$

contains (at least) one control of $\Sigma_{SE(2)_n}$, concluding the proof.

□

4.2 The group $\mathrm{Aff}(\mathbb{R}) \times S^1$

If $\theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the group $G(\theta)$ can be seen as the Cartesian product $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$. In fact, since $v \in \mathbb{R}^2$ is written uniquely as $v = x\mathbf{e}_1 + y\mathbf{e}_2$, the map

$$(t, v) \in G(\theta) \mapsto ((t, x), y) \in \mathrm{Aff}(\mathbb{R}) \times \mathbb{R}, \tag{4.1}$$

is a diffeomorphims. Moreover,

$$\rho_t v = x\rho_t\mathbf{e}_1 + y\rho_t\mathbf{e}_2 = x\mathrm{e}^t\mathbf{e}_1 + y\mathbf{e}_2,$$

implies that

$$(t_1, v_1)(t_2, v_2) = (t_1 + t_2, v_1 + \rho_{t_1}v_2) = (t_1 + t_2, (x_1 + \mathrm{e}^{t_1}x_2)\mathbf{e}_1 + (y_1 + y_2)\mathbf{e}_2)$$

$$\mapsto ((t_1 + t_2, x_1 + \mathrm{e}^{t_1}x_2), y_1 + y_2) = ((t_1, x_1)(t_2, x_2), y_1 + y_2) = ((t_1, x_1), y_1)((t_2, x_2), y_2),$$

which shows that the map (4.1) is an isomorphisms.

Following (ONISHCHIK; VINBERG, 1994, Chapter 7), up to isomorphisms, the only central discrete central subgroup of $G(\theta) = \mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ is $Z := \{((0, 0), 2k\pi), k \in \mathbb{Z}\}$. Therefore, $G(\theta)/Z = \mathrm{Aff}(\mathbb{R}) \times S^1$ is the unique connected, nonsimply connected, Lie group with the Lie algebra $\mathfrak{g}(\theta) = \mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. The canonical projection is given by

$$\pi : \mathrm{Aff}(\mathbb{R}) \times \mathbb{R} \to \mathrm{Aff}(\mathbb{R}) \times S^1, \quad ((t, x), y) \mapsto ((t, x), [y]).$$

By Section 1.3, the linear vector fields of $\mathrm{Aff}(\mathbb{R}) \times S^1$ are completely determined by the linear vector fields on $G(\theta) = \mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ whose flows let the subgroup Z invariant. Let then $\mathcal{X}$ be a linear vector field on $G(\theta)$ with associated flow $\{\varphi_s\}_{s\in\mathbb{R}}$ and assume that

$$\varphi_s(0, 2k\pi e_2) \in Z, \qquad \forall s \in \mathbb{R}, k \in \mathbb{Z}.$$

Using the expression in (1.9) for φ_s, one obtain that,

$$\varphi_s(0, 2k\pi e_2) = (0, 2k\pi\mathrm{e}^{sA}e_2) \in Z \quad \iff \quad \mathrm{e}^{sA}e_2 \in 2\pi\mathbb{Z}e_2 \quad \iff \quad Ae_2 = 0.$$

Theorem 10. *Any linear control system $\Sigma_{\mathrm{Aff}(\mathbb{R})\times S^1}$ on $\mathrm{Aff}(\mathbb{R})\times S^1$ satisfying the LARC, admits a unique control set $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1}$ satisfying:*

$$\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1} = \pi_1^{-1}(\mathcal{C}_{\mathrm{Aff}(\mathbb{R})}) = \mathcal{C}_{\mathrm{Aff}(\mathbb{R})}\times S^1,$$

where $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})}$ is the unique control set of the LCS on $\mathrm{Aff}(\mathbb{R})$ that is π_1-conjugated to $\Sigma_{\mathrm{Aff}(\mathbb{R})\times S^1}$, where $\pi_1 : \mathrm{Aff}(\mathbb{R})\times S^1 \to \mathrm{Aff}(\mathbb{R})$ is the canonical projection. Moreover, $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1}$ is open if $\operatorname{tr} A > 0$, closed if $\operatorname{tr} A < 0$ and equal to $\mathrm{Aff}(\mathbb{R})\times S^1$ if $\operatorname{tr} A = 0$.

Proof. Let us assume w.l.o.g. that the left-invariant vector field of $\Sigma_{\mathrm{Aff}(\mathbb{R})\times S^1}$ is $Y = (\alpha, 0)$, with $\alpha \neq 0$. In this case, our system in $\mathrm{Aff}(\mathbb{R})\times S^1$, is given by

$$\begin{cases} \dot{t} = u\alpha \\ \dot{x} = \lambda x + (\mathrm{e}^t - 1)\xi_1 \\ \dot{[y]} = t\xi_2 \end{cases}, u \in \Omega, \qquad \left(\Sigma_{\mathrm{Aff}(\mathbb{R})\times S^1}\right)$$

and the LARC is equivalent to $\alpha\xi_1\xi_2 \neq 0$.

If $\lambda \neq 0$, the system satisfy the ad-rank condition and hence, there exists a unique control set $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1}$ with nonempty interior. Moreover, since the points in $\{(0,0)\}\times S^1$ are fixed by the flow of $\mathcal{X} = (A, \xi)$, we get that $\{(0,0)\}\times S^1 \subset \operatorname{int}\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1}$ (see Theorem 3). On the other hand, the projection

$$\pi : \mathrm{Aff}(\mathbb{R})\times S^1 \to \mathrm{Aff}(\mathbb{R}), \qquad \pi_1((t,x),y) = (t,x),$$

conjugates $\Sigma_{\mathrm{Aff}(\mathbb{R})\times S^1}$ and the linear control system

$$\begin{cases} \dot{t} = u\alpha \\ \dot{x} = \lambda x + (\mathrm{e}^t - 1)\xi_1 \end{cases}, u \in \Omega. \qquad \left(\Sigma_{\mathrm{Aff}(\mathbb{R})}\right)$$

Since $(0,0) \in \operatorname{int}\mathcal{C}_{\mathrm{Aff}}$ and $\pi_1^{-1}(0,0) = \{(0,0)\}\times S^1 \subset \operatorname{int}\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1}$ we obtain, by Proposition 2, that

$$\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1} = \pi_1^{-1}(\mathcal{C}_{\mathrm{Aff}(2)}) = \mathcal{C}_{\mathrm{Aff}(\mathbb{R})}\times S^1.$$

In particular, $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})\times S^1}$ is open when $\lambda > 0$ and closed when $\lambda < 0$ since the same holds for $\mathcal{C}_{\mathrm{Aff}(\mathbb{R})}$ (see Section 1.5.1)

Let us assume now that $\lambda = 0$. In this case, the canonical projection

$$\pi_1 : \mathrm{Aff}(\mathbb{R})\times S^1 \to \mathrm{Aff}(\mathbb{R}), \qquad ((t,x),y) \mapsto (t,x),$$

conjugates $\Sigma_{\mathrm{Aff}(\mathbb{R})\times S^1}$ to the LCS

$$\begin{cases} \dot{t} = u\alpha \\ \dot{x} = (\mathrm{e}^t - 1)\xi_1 \end{cases}, u \in \Omega, \qquad \left(\Sigma_{\mathrm{Aff}(\mathbb{R})}\right)$$

on $\mathrm{Aff}(\mathbb{R})$. By Theorem 5, we have that $\Sigma_{\mathrm{Aff}(\mathbb{R})}$ is controllable.

Let us consider $P \in \operatorname{int} \mathcal{O}^+(e)$, whose existence is assured by the LARC. The controllability of the projected system $\Sigma_{\mathrm{Aff}(\mathbb{R})}$, implies the existence of $u \in \mathcal{U}$ and $\tau > 0$ such that

$$\pi_1(\phi(\tau, P, u)) = \phi_1(\tau, \pi_1(P), u) = (0, 0),$$

and hence

$$((0,0), [y_0]) = \phi(\tau, P, u) \in \phi_{\tau,u}(\operatorname{int} \mathcal{O}^+(e)) \subset \operatorname{int} \mathcal{O}^+(e).$$

Moreover, the LARC together with (COLONIUS; KLIEMANN, 2000, Lemma 4.5.2) and Proposition 4, imply the existence of $S > 0$ such that $((0,0), [y_0]) \in \operatorname{int} \mathcal{O}^+_S(e)$. Since $\operatorname{int} \mathcal{O}^+_S(e)$ is open, there exists $r \in \mathbb{Q}$ with $((0,0), [r]) \in \operatorname{int} \mathcal{O}^+_S(e)$. On the other hand, the fact that points in $\{(0,0)\} \times S^1$ are fixed points of the drift, allows us to obtain, that

$$((0,0), [2r]) = ((0,0), [r])((0,0), [r]) = ((0,0), [r])\varphi_S((0,0), [r]) \in \operatorname{int}_{2S} \mathcal{O}^+(e),$$

and, inductively, that

$$((0,0), [mr]) \in \operatorname{int}_{mS} \mathcal{O}^+(e), \qquad \forall m \in \mathbb{N}.$$

Since $r \in \mathbb{Q}$, there exists $m_0 \in \mathbb{N}$ such that $m_0 r \in \mathbb{Z}$ implying that

$$e = ((0,0), [0]) = ((0,0), [m_0 r]) \in \operatorname{int} \mathcal{O}^+_{m_0 S}(e) \subset \operatorname{int} \mathcal{O}^+(e).$$

Since the associated derivation $\mathcal{D}$ has only eigenvalues with zero real parts, Theorem 3 imply that $\Sigma_{\mathrm{Aff}(\mathbb{R}) \times S^1}$ is controllable, concluding the proof. □

Remark 6. *Let us note the big difference between π_1-conjugated LCSs on* $\mathrm{Aff}(\mathbb{R}) \times S^1$ *and on its simply connected covering* $\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}$ *when the associated matrix* $A \equiv 0$. *Precisely,* $\Sigma_{\mathrm{Aff}(\mathbb{R}) \times \mathbb{R}}$ *admits an infinite number of control sets with empty interior, while its projection* $\Sigma_{\mathrm{Aff}(\mathbb{R}) \times S^1}$ *is controllable.*

5 Examples

In this section we present some examples of control sets in order to illustrate the results of the work.

5.1 Control sets of LCSs on 2D groups

Example 1. *Let us consider linear control systems on* $(\mathbb{R}^2,+)$

$$\begin{cases} \dot{x} = x+u \\ \dot{y} = -y+u \end{cases} \qquad \Sigma^1_{(\mathbb{R}^2,+)} \qquad \textit{and} \qquad \begin{cases} \dot{x} = x+u \\ \dot{y} = u \end{cases} \qquad \Sigma^2_{(\mathbb{R}^2,+)}$$

where $\Omega = [-1,1]$*. Their respective control sets, are given, respectively, by (see figure 3).*

$$\mathcal{C}^1_{(\mathbb{R}^2,+)} = (-1,1)\times[-1,1] \qquad \textit{and} \qquad \mathcal{C}^2_{(\mathbb{R}^2,+)} = (-1,1)\times\mathbb{R}.$$

Note that the linear vector field $\Sigma^1_{(\mathbb{R}^2,+)}$ *has no singularities, and hence its control set is bounded. In contrast, the existence of a singularity for the linear vector field of* $\Sigma^2_{(\mathbb{R}^2,+)}$ *makes the control set unbounded.*

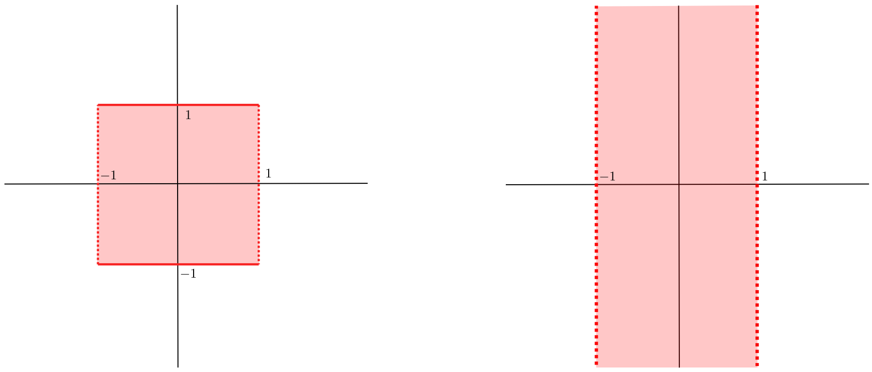

Figure 3 – Control sets of $\Sigma^1_{(\mathbb{R}^2,+)}$ and $\Sigma^2_{(\mathbb{R}^2,+)}$, respectively.

Example 2. *Let us consider*

$$\begin{cases} \dot{x} = u \\ \dot{y} = -y+u\mathrm{e}^x \end{cases}, \qquad \textit{where} \quad u\in\Omega. \tag{$\Sigma_{\mathrm{Aff}(\mathbb{R})}$}$$

For this system we have that, if $\Omega = \left[-\frac{1}{2},\frac{1}{2}\right]$*, the control set of* $\Sigma_{\mathrm{Aff}(\mathbb{R})}$ *is given by*

$$\mathcal{C}_{\mathrm{Aff}(\mathbb{R})} = \left\{(x,y)\in\mathbb{R}^2, \quad -\mathrm{e}^x \leqslant y \leqslant \frac{1}{3}\mathrm{e}^x\right\}.$$

On the other hand, if $\Omega = \left[-1, \frac{1}{2}\right]$*, the control set of* $\Sigma_{\mathrm{Aff}(\mathbb{R})}$ *is given by*

$$\mathcal{C}_{\mathrm{Aff}(\mathbb{R})} = \left\{(x, y) \in \mathbb{R}^2, \quad y \leqslant \frac{1}{3}\mathrm{e}^x\right\}.$$

The control sets of the previous systems are described in the picture below (Figure 4).

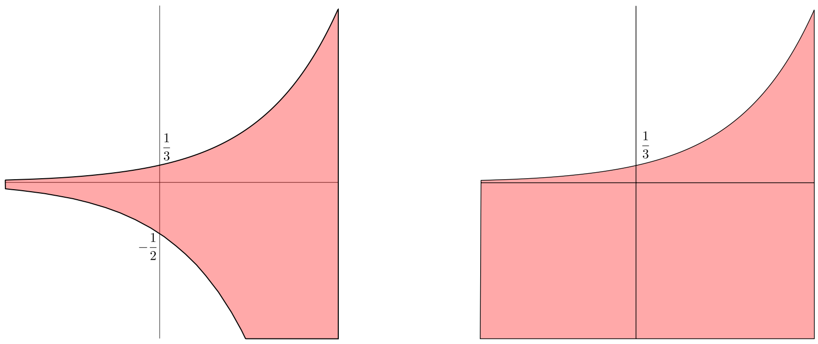

Figure 4 – Control sets of $\Sigma_{\mathrm{Aff}(\mathbb{R})}$ for $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\Omega = \left[-1, \frac{1}{2}\right]$, respectively.

5.2 Control sets of the control-affine system $\Sigma_{\mathbb{R}^2}$

Example 3. *Let us assume that*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \Omega = [-2, 2].$$

It is not hard to see that

$$v(u) = \left(\frac{u}{u-1}, \frac{-u}{u+1}\right) \quad \text{for} \quad u \notin \{-1, 1\}.$$

Moreover, $\Omega_0 = (-1, 1)$ *and* $v(\Omega_0)$ *is an unbounded set given (see Figure 5). By simple calculations, one obtains that*

$$\mathcal{C}_{\mathbb{R}^2} = \left\{(x, y), y \geqslant -\frac{1}{2} \quad \text{or} \quad x \leqslant \frac{1}{2}\right\}.$$

Also, by following the proof of Proposition 10 it is not hard to show that $\mathcal{O}^+(v(u))$ *and* $\mathcal{O}^-(v(u))$ *are actually open sets for any* $u \notin \{-1, 1\}$*, and hence, there exists control sets with nonempty interior* $\mathcal{C}_1, \mathcal{C}_2$ *such that*

$$\{v(u) \in \mathbb{R}^2 | u \in (-2, 1) \subset \Omega\} \subset \mathcal{C}_1 \quad \text{and} \quad \{v(u) \in \mathbb{R}^2 | u \in (1, 2) \subset \Omega\} \subset \mathcal{C}_2,$$

showing that a control-affine system on $\mathbb{R}^2$ *can admit more than one control set with nonempty interior (see Figure 5).*

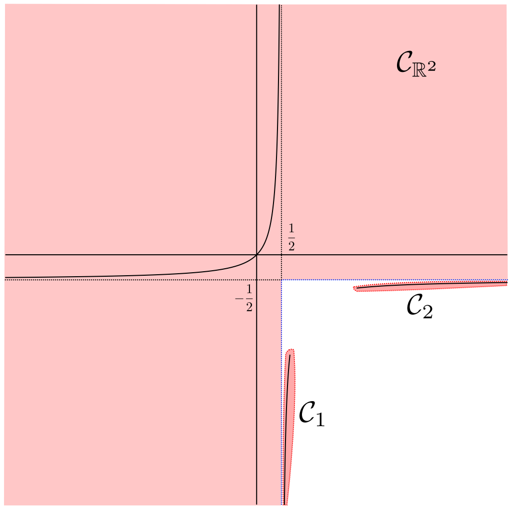

Figure 5 – Control-affine system on $\mathbb{R}^2$ having three control sets with nonempty interior.

Example 4. *The present example shows that the identity element of $G(\theta)$ is not always in the interior of the control set. For the details, the reader can consult (AYALA; DA-SILVA; ROBLES, , Section 3).*

Let us consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Omega = [-1, 1].$$

Since $\det A(u) = 1 + u^2 > 0$*, all the equilibria are given by* $v(u) = \left(\frac{-u}{1+u^2}, \frac{u^2}{1+u^2} \right)$ *and they lie on the circumference with center in* $= (0, -1/2)$ *and radius* $\frac{1}{2}$*. Since the eigenvalues of A are positive, the control set of the previous system is open and is contained in the open ball* $B = \{w \in \mathbb{R}^2; \ |w + \mathbf{e}_2| < 1\}$.

Since $v(0) = (0,0) \in \partial B$*, we have that* $(0,0) \in \partial \mathcal{C}_{\mathbb{R}^2}$ *but* $(0,0) \notin \operatorname{int} \mathcal{C}_{\mathbb{R}^2}$*. In fact, as showed in (AYALA; DA-SILVA; ROBLES, , Proposition 3.8), the singleton* $\{(0,0)\}$ *is a one-point control set of* $\Sigma_{\mathbb{R}^2}$.

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