Space–time defects: Open and closed shells

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A class of space–times with zero curvature tensor except on a hypersurface is derived; in particular, metrics that represent parabolic and cylindrical plates, and parabolic, spherical, spheroidal, and toroidal shells are studied. The motion of particles around and inside the defects is considered. The interpretation of some of these space–times as wormholes is briefly considered. © 1995 American Institute of Physics.

I. INTRODUCTION

There exist a variety of solutions of the Einstein equation characterized by curvature tensors proportional to Dirac distribution with support on a hypersurface in a previous work; by analogy of the defects of a lattice, we argue that these solutions represent pure space–time defects.

These defects appear in different contexts: (a) geometrically, in the zero width limit, strait cosmic strings and plane symmetric domain walls are characterized by space–times with metrics that have null Riemann–Christoffel curvature tensor everywhere except on the lines that represent the strings and on the planes of the walls; (b) limit or particular cases of solutions of the Einstein equation coupled to usual matter, e.g., the Lemos–Letelier family of disks formed by counter-rotating particles. The space–times associated with the topological defect are not necessarily space–time defects, e.g., the $O(3)$ global monopole.

In the present work we derive some new families of solutions of the Einstein equation that represent space–time defects associated to plates and shells of nontrivial shapes. Also we dwell in geometrical as well as in physical aspects of the solutions. In particular, we studied the motion of particles around the defects and found a curious behavior, the particles move as a particle fluid in laminar flow around an obstacle.

In Sec. II we present a very short review of the Lichnerowicz theory of distribution mainly to fix notation and conventions. In the next section we derive the new family of solutions. We find that the density and pressures are proportional to the principal curvature of the shells. In Sec. IV we specialize the solution to the case of parabolic and cylindrical plates, and parabolic, spherical, spheroidal, and toroidal shells. In the penultimate section we study the motion of particles moving in the field of a spherical and cylindrical defect. In the last section we made remarks about the possible interpretation of the solutions as wormholes. Also we speculate about a field theory approach to these solutions.

II. DISTRIBUTION VALUED CURVATURE TENSORS

Following Lichnerowicz we assume that there exists a hypersurface $\phi=0$ in which the metric tensor has a discontinuous derivative, and that $\phi=0$ divides a region $\Omega$ of the space–time into two parts $\Omega^+$ and $\Omega^-$ where $\phi>0$ and $\phi<0$, respectively. The metric tensor $g$ is assumed to be continuous across $\phi=0$, i.e.,

$$[g_{\mu\nu}] \equiv g_{\mu\nu}|_{\phi=0^+} - g_{\mu\nu}|_{\phi=0^-} = 0.$$ (1)

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The discontinuities in the first derivative of the metric tensor are characterized by the tensor $b_{\mu\nu}$, defined by
\[
[g_{\mu\nu},] = \xi_\mu b_{\mu\nu},
\]
where $\xi_\mu = \partial \phi / \partial x^\mu$ is the normal vector to the hypersurface.

With the assumption that $g_{\mu\nu}$ is at least $C^3$ in regions $\Omega^\pm$, we find that the Riemann tensor takes the form
\[
R^\rho_{\sigma\mu\nu} = \delta(\phi) H^\rho_{\sigma\mu\nu} + (R^\rho_{\sigma\mu\nu})^0 + \theta(1 - \theta) J^\rho_{\sigma\mu\nu},
\]
with
\[
2H^\rho_{\sigma\mu\nu} = b_{\nu\xi} \xi_\mu - b_{\mu\xi} \xi_\nu - b_{\sigma\nu} \xi_\mu \xi_\nu + b_{\sigma\mu} \xi_\nu \xi_\nu + b_{\sigma\mu} \xi_\nu \xi_\nu,
\]
\[
J^\rho_{\sigma\mu\nu} = [\Gamma^\rho_{\sigma\mu}] [\Gamma^\rho_{\tau\nu}] - [\Gamma^\rho_{\sigma\tau}] [\Gamma^\rho_{\mu\nu}].
\]

$(R^\rho_{\sigma\mu\nu})^0$ are the usual Riemann–Christoffel tensor defined in $\Omega^\pm$ and $\theta$ denotes the Heaviside function. Note that the last term of Eq. (3) appears multiplied by the distribution $\theta(1 - \theta)$ that vanishes everywhere except on $\phi=0$. Thus, for finite $J^\rho_{\sigma\mu\nu}$ the contribution of these terms to the curvature is identically zero in the sense of distributions. Furthermore, in this work we shall study only space–times that have null Riemannian curvature outside $\phi=0$, i.e.,
\[
R^\pi_{\mu\nu\rho\sigma} = 0.
\]
The energy–momentum tensor is assumed to have the form $T^\mu_{\nu} = \hat{T}^\mu_{\nu} \delta(\phi)$, in other words a distribution with support on the hypersurface $\phi=0$. Hence the Einstein equations, in geometric units, $c = 4 \pi G = 1$, reduce to
\[
H^\mu_{\nu} - \frac{1}{2} g^\mu_{\nu} H = \hat{T}^\mu_{\nu}.
\]

**III. A CLASS OF SPACE–TIME DEFECTS**

To construct metrics that represent pure space–time defects we shall take Eq. (5) as our starting point, in particular, we shall study the metric
\[
ds^2 = dt^2 - Y^2(x,y)(dx^2 + dy^2) - Z^2(x,y)dz^2.
\]
The functions $Y$ and $Z$ are functions of only $x$ and $z$ and are determined by Eq. (5). We shall see that this metric is sufficiently general to allow for a large number of significant particular cases.

We find that Eq. (7) represents a flat space–time when (a) $Z=A$ and $(\ln Y)_{xx} + (\ln Y)_{yy} = 0$ and (b) $Y^2 = A^2(z^2 + Z^2)$ and $Z_{xx} + Z_{yy} = 0$. $A$ is an arbitrary real constant and $(\ )_{xx} = \partial_x$, etc. In order to end up with a metric with discontinuous first derivatives on $\phi=0$ we shall perform different coordinate transformations in each side of the hypersurface.

\[
t \to T_z = M(\tau \pm \phi) + N(\tau \pm \phi), \quad x \to X_z = U(\tau \pm \phi) + V(\tau \pm \phi),
\]
where $M$, $N$, $U$, and $V$ are arbitrary functions of the indicated arguments, $\tau$ and $\phi$ are arbitrary functions of only $t$ and $x$. The coordinates $y$ and $z$ are left unchanged. We find
\[
ds^2 = F_z^2 dt^2 - 2G_z dt \, dx - [(H_z^2 - G_z^2)/F_z^2] dx^2 - Y_z^2 dy^2 - Z_z^2 \, dz^2,
\]
where
\[ Y_\pm = Y(X_\pm, y), \quad Y_\pm = Y(X_\pm, y), \]

\[ F^2_\pm = [(\dot{M}_\pm + \dot{N}_\pm)^2 - Y_\pm^2 (\dot{\bar{U}}_\pm + \dot{\bar{V}}_\pm)^2 \tau_{i, i}] \mp 2[(\dot{M}_\pm - \dot{N}_\pm)^2 - Y_\pm^2 (\ddot{U}_\pm - \ddot{V}_\pm)] \]

\[ \times \tau_{, \phi, \tau} + [(\dot{M}_\pm - \dot{N}_\pm)^2 - Y_\pm^2 (\dot{\bar{U}}_\pm - \dot{\bar{V}}_\pm)^2 \phi_{i, i}^2] \]

\[ H^2_\pm = Y_\pm^2 (\dot{M}_\pm + \dot{N}_\pm)^2 (\tau_{, \phi, \tau} - \tau_{, \phi, i}) \]

and \( G_\pm \) is obtained from \( F^2_\pm \) replacing \( \tau_{i, i}^2 \rightarrow \tau_{i, \phi, \tau}, \tau_{i, i} \rightarrow \tau_{i, \phi, \phi}, \tau_{i, i} \rightarrow \tau_{i, \phi, \phi}, \) and \( \phi_{i, i}^2 \rightarrow \phi_{i, \phi, \phi}. \)

The over dot means derivative with respect to the argument and \( M_\pm \) stands for \( M(\tau_{\pm}, \phi), \) etc.

To have a continuous metric on \( \phi=0 \) we need

\[ \hat{M}^2(\tau(\tau, x), x) - \hat{N}^2(\tau(\tau, x), x) - Y^2(\tau(\tau, x), y)\left[ \hat{U}^2(\tau(\tau, x), y) - \hat{V}^2(\tau(\tau, x), y) \right] = 0. \] (11)

Assuming that \( \partial_\tau Y \neq 0 \) we have that \( \hat{M}^2 = \hat{N}^2 \) and \( \hat{U}^2 = \hat{V}^2. \) In order to keep \( T \) as a time coordinate and \( X \) as a space one, we choose \( M = \hat{N} \) and \( \hat{U} = - \hat{V}. \) Also \( M = 0 \) and \( V = - U + W \) where \( W \) is a constant. With these assumptions we end up with

\[ T_\pm = M(\tau_{\mp}, \phi) + M(\tau_{\mp}, \phi), \quad X_\pm = X(\tau_{\mp}, \phi) + X(\tau_{\mp}, \phi) + W. \] (12)

Note that on the hypersurface \( \phi=0 \) we have \( T=2M(\tau) \) and \( X=W. \)

From Eq. (2) we find

\[ b_{\mu} = 2^3 \dot{U}^3 Y \frac{\partial Y}{\partial X} \phi_{, i}^2, \quad b_{\mu} = 2^3 \dot{U}^3 Y \frac{\partial Y}{\partial X} \phi_{, i}^2, \quad b_{xx} = 2^3 \dot{U}^3 Y \frac{\partial Y}{\partial X} \phi_{, x}^2, \]

\[ b_{yy} = 2^3 \dot{U}^3 Y \frac{\partial Y}{\partial X}, \quad b_{zz} = 2^3 \dot{U}^3 Y \frac{\partial Y}{\partial X}. \]

And from the Einstein equations (6) we get the energy–momentum tensor of the surface \( \phi=0 \)

\[ \hat{T}_{\mu \nu} = \hat{\rho} u_{\mu} u_{\nu} - \hat{\rho} g_{\mu \nu} - \hat{\rho} z_{\mu} z_{\nu}, \]

where the eigenvalues are

\[ \hat{\rho} = - \frac{1}{\dot{U} Y^2} \frac{\partial}{\partial X} \ln(Y Z), \quad \hat{\rho} = - \frac{1}{\dot{U} Y^2} \frac{\partial}{\partial X} \ln(Z), \hat{\rho} = - \frac{1}{\dot{U} Y^2} \frac{\partial}{\partial X} \ln(Y), \]

and the normalized eigenvectors are

\[ u_{\mu} = \frac{1}{\dot{U} H \sqrt{F H \phi_{, \phi} \phi_{, \mu}}} \left[ (F^2 \phi_{, \phi} - G \phi_{, \phi}) e_{\mu}^0 + H \phi_{, \phi} e_{\mu}^1 \right], \]

\[ e_{\mu}^0 = e_{\mu}^2, \quad e_{\mu}^3 = e_{\mu}^3. \]

The quantities \( e_{\mu}^a, \) \( a=0,1,2,3, \) are the orthonormal vierbein associated to Eq. (9)

\[ e_{\mu}^0 = F \delta_{\mu}^0 + (G/F) \delta_{\mu}^1, \quad e_{\mu}^1 = (H/F) \delta_{\mu}^1, \]

\[ e_{\mu}^2 = Y \delta_{\mu}^2, \quad e_{\mu}^3 = Z \delta_{\mu}^3. \] (17)

On the hypersurface \( \phi=0 \) we have the useful expressions
\[
\phi_{,\mu} \phi^{\mu} = -1/(4 \hat{U}^2 Y^2), \quad F^2 = 4(\hat{M}^2 \tau^2 - Y^2 \hat{U}^2 \phi^2),
\]

\[
G = 4(\hat{M}^2 \tau, \tau, x - Y^2 \hat{U}^2 \phi, \phi, x),
\]

\[
F^2 = 16 Y^2 (\hat{M} \hat{U})^2 \langle \tau, \phi, x - \tau, \phi, t \rangle^2.
\]

Before the study of the energy–momentum tensor (14) let us examine the geometrical properties of the hypersurface \( \phi = 0 \). We find from Eq. (9) that its intrinsic metric reduces to

\[
ds_2^2 = dt'^2 + Y^2 (w, y) dy^2 - Z^2 (w, y) dZ^2,
\]

where \( dt' = S(t) dt \) and \( S(t) = 2 \hat{M} (\tau, \phi, \phi, t) \big|_{\phi = 0} \). Therefore, the intrinsic metric of the hypersurface is always static. Hence there is no loss of generality in doing \( \phi = \phi(x) \) and \( \tau = \pi(t) \).

It is also interesting to evaluate the principal curvatures associated to the spatial part of the hypersurface. To do this we shall use Cartan method that make use of the three dimensional metric written in local Cartesian coordinates

\[
ds^2 = (dx)^2 + (dy)^2 + (dz)^2,
\]

where \( d\theta^i = Y \, dx, \quad d\theta^j = Y \, dy, \) and \( d\theta^k = Z \, dz \). The Cartan first structure equation, \( d\omega + \omega \wedge \omega = 0 \), can be solved for the connection one-forms \( \omega \); we get

\[
\omega_{x} = \frac{\partial Y}{Y} \, \theta^x, \quad \omega_{y} = \frac{\partial Z}{Z} \, (\theta^y - \theta^x).
\]

From the above equations we read that the principal curvatures are

\[
k_y = \frac{\partial Y}{Y}, \quad k_z = \frac{\partial Z}{Z}.
\]

\( k_y, k_z \) is the curvature of the line formed by the intersection of the surfaces \( z = \text{const}, y = \text{const} \) with the surface of constant \( X \) (that represents the surface \( \phi = 0 \)).

Now we can write the density and the principal surface tensions in the more appealing form

\[
\hat{\rho} = - (\hat{U} Y)^{-1} (k_y + k_z), \quad \hat{\rho}_y = - (\hat{U} Y)^{-1} k_z, \quad \hat{\rho}_z = - (\hat{U} Y)^{-1} k_y.
\]

We shall require always positive density; from Eq. (15) we have that positive \( \hat{\rho}_y (\hat{\rho}_z) \) means tension along the directions \( \partial_y (\partial_z) \). Negative \( \hat{\rho}_{x,y} \) represent pressure. The density is proportional to the mean curvature of the shell, a similar property is known for null shells (shells made of a flux of null matter), but in this late case we have zero pressure or tension. A striking property of these shells is that

\[
\hat{\rho}_N = \hat{\rho} - \hat{\rho}_{x} - \hat{\rho}_{y} = 0.
\]

In other words the related Newtonian density is null, i.e., these shells do not have a Newtonian analog. They can be thought of as formed by two orthogonal families of Nambu strings; we recall that the state equation of these strings is \( \text{pressure} = \text{tension} \).

### IV. PARTICULAR PLATES AND SHELLS

In this section we shall study specializations of the metric (9) that represents simple plates and shells. From our previous discussion we have that without loss of generality we can take \( M = \pi 2 \).
and $t=t$. The function $U$ will be taken as $U=-\epsilon \phi^2$ and $\phi=x-x_0$, with $x_0$ constant and $\epsilon$ a sign function $\epsilon=\pm 1$; for a more general choice of functions see Ref. 1. With these assumptions we have that the coordinate transformation (8) reduces to

$$T_\pm = t, \quad X = -\epsilon |x-x_0| + \omega.$$  \hspace{1cm} (25)

### A. Cylindrical plates

The Minkowski space in cylindrical coordinates

$$x = e^{r'} \cos \varphi, \quad y = e^{r'} \sin \varphi$$

is

$$ds^2 = dr^2 - e^{2r'}(dr'^2 + d\varphi^2) - dz^2.$$  \hspace{1cm} (26)

This system of coordinates as well as the other used in this section is studied in great detail in Ref. 7. The equation $\phi = r' - r_0' = 0$ for $r_0'$ constant represents a cylinder of radius $r_0 = e^{r_0'}$. Hence

$$ds^2 = dr^2 - e^{2\epsilon (r'-r_0'+\omega)}(dr'^2 + d\varphi^2) - dz^2$$  \hspace{1cm} (27)

is the metric associated to a cylindrical plate of equation of state $p_z = 2e^{-2\omega} \delta (r' - r_0')$. Or in the usual cylindrical coordinates, $r = e^{r'}$, we get

$$ds^2 = dr^2 - dr^2 - [||r-r_0|-\epsilon w]^2 d\varphi^2 - dz^2,$$  \hspace{1cm} (28)

with $p_z = 2e^{-\epsilon w} \delta (r - r_0)$, that is the cylindrical plate studied in Ref. 1. In order to have positive density we need $\epsilon w > 0$. Even though this cylinder can be considered as made of parallel fibers with the same equation of state as the Nambu strings, we have that the metrics (27) or (28) are not asymptotically conic as the one representing the strings. As a matter of fact when $r \to +\infty$ we have the usual Minkowski space as a limit. We shall come back to this point in the next sections.

### B. Parabolic plates

The Minkowski metric in parabolic coordinates

$$x = a(\lambda^2 - \mu^2), \quad y = a \lambda \mu$$

($a$ is a constant), reads

$$ds^2 = dt^2 - a^2(\lambda^2 + \mu^2)(d\lambda^2 + d\mu^2) - dz^2.$$  \hspace{1cm} (29)

The equation $\mu = \mu_0$ describes the parabolic surface $x = y^2/(2a\mu_0^2) - a\mu_0^2/2$. Therefore the metric

$$ds^2 = dt^2 - a^2[\lambda^2 + (|\mu - \mu_0| - \epsilon w)^2](d\lambda^2 + d\mu^2) - dz^2$$  \hspace{1cm} (30)

represent a parabolic plate with equation of state

$$p_z = p_e = \frac{2\epsilon w a^2}{(\lambda^2 + \mu^2)^2} \delta (\mu - \mu_0).$$  \hspace{1cm} (31)
As before we require $\epsilon w > 0$. This last metric describes a plate made of fibers parallel to the $z$ axis that pierce the $x,y$ plane along a parabola $x = y^2/(2aw^2) - a/w^2/2$. Each fiber has the same equation of state as Nambu strings. Note that the fibers have greater density and tension wherein the plate has greater curvature.

C. Parabolic shells (bowls)

The Minkowski metric in cylindrical parabolic coordinates

$$x = a\lambda \mu \cos \varphi, \quad y = a\lambda \mu \sin \varphi, \quad z = a(\lambda^2 - \mu^2)/2$$

reads

$$ds^2 = dr^2 - a^2(\lambda^2 + \mu^2)(d\lambda^2 + d\mu^2) - (a\lambda \mu)^2 dz^2. \quad (32)$$

The equation $\mu = \mu_0$ describes the parabolic surface of revolution $x = (x^2 + y^2)/(2a\mu_0^2) - a\mu_0^2/2$. Therefore, the metric

$$ds^2 = dr^2 - a^2[\lambda^2 + (|\mu - \mu_0| - \epsilon w)^2](d\lambda^2 + d\mu^2) - a^2\lambda^2(\mu - \mu_0 - \epsilon w)^2 dz^2 \quad (33)$$

represents a cylindrical parabolic shell (bowl) with equation of state

$$p = p_\lambda + p_\varphi, \quad p_\varphi = \frac{2\epsilon w a^{-2}}{(\lambda^2 + \mu^2)^2} \delta(\mu - \mu_0), \quad p_\lambda = \frac{2\epsilon w a^{-2}}{\lambda^2 + w^2} \delta(\mu - \mu_0). \quad (34)$$

We can think that the “bowl” is made of fibers of parabolic shape that are held together by fibers of circular shape. Again each fiber has the same equation of state “pressure=tension.” In general, such a distribution of tensions can be made quite stable. An every day example is a dome. Note that the fibers have greater density and tension wherein the shell has greater curvature.

D. Spherical shells

Using the same methodology as in the cylindrical case we find that the metric

$$ds^2 = dt^2 - dr^2 - (r - r_0)^2(\sin^2 \vartheta d\vartheta^2 + 
\sin^2 \varphi \, d\vartheta^2) \quad (35)$$

represent a spherical shell with equation of state

$$\rho/2 = p_\vartheta - p_\varphi - 2(\epsilon/w) \delta(r - r_0). \quad (36)$$

This metric was also studied in Ref. 1. The remarks about stability of the previous shell also apply to this case and to the next.

E. Oblate and prolate spheroidal shells

The Minkowski metric in oblate spheroidal coordinates

$$x = a \cosh \mu \sin \theta \cos \varphi, \quad y = a \cosh \mu \sin \theta \cos \varphi, \quad z = a \sinh \mu \cos \theta$$

reads

$$ds^2 = dt^2 - a^2(\sinh^2 \mu + \cos^2 \vartheta)(d\mu^2 + d\vartheta^2) - (a \cosh \mu \sin \theta)^2 \, d\varphi^2. \quad (37)$$

The equation $\mu = \mu_0$ describes the surface of revolution.
Therefore the metric
\[ ds^2 = dt^2 - a^2 P^2 \left( d\mu^2 + d\varphi^2 \right) - a^2 Q^2 \, d\varphi^2, \]
with
\[ P^2 = \sinh^2(\mu - \mu_0) - \epsilon \omega + \cos^2 \vartheta, \]
\[ Q = \cosh(\mu - \mu_0) - \epsilon \omega \sin \vartheta \]
represents a shell with the shape of an oblate spheroid with equation of state
\[ \rho = p_\vartheta + p_\varphi, \]
\[ p_\vartheta = \frac{\epsilon \sinh(2\omega) a^{-2}}{(\sinh^2 \omega + \cos^2 \vartheta)^{3/2}} \delta(\mu - \mu_0), \]
\[ p_\varphi = \frac{\epsilon \sinh(2\omega) a^{-2}}{\sinh^2 \omega + \cos^2 \vartheta} \delta(\mu - \mu_0). \]

The prolate spheroid is obtained by changing \( \cosh \mu \rightarrow \sinh \mu \) and \( \sinh \mu \rightarrow \cosh \mu \) in the transformation of coordinates above. The metric for the shell of a prolate spheroid is similar to Eq. (38) but now
\[ P^2 = \sinh^2(\mu - \mu_0) - \epsilon \omega + \sin^2 \vartheta, \]
\[ Q = \sinh(\mu - \mu_0) - \epsilon \omega \sin \vartheta. \]

The corresponding density and tensions are
\[ \rho = p_\vartheta + p_\varphi, \]
\[ p_\vartheta = \frac{\epsilon \sinh(2\omega) a^{-2}}{(\sinh^2 \omega + \sin^2 \vartheta)^{3/2}} \delta(\mu - \mu_0), \]
\[ p_\varphi = \frac{\epsilon \sinh(2\omega) a^{-2}}{\sinh^2 \omega + \sin^2 \vartheta} \delta(\mu - \mu_0). \]

The limit situation of an oblate spheroid is a disk and a prolate spheroid is a spindle. So, in principle, these metrics can be used to describe disklike and spindlelike defects.

**F. Toroidal shells**

Simple toroidal coordinates are
\[ x = (b + r \cos \vartheta) \cos \varphi, \quad y = (b + r \cos \vartheta) \sin \varphi, \quad z = r \cos \vartheta. \]

Now we can proceed as in the cylindrical shell case: first we change \( r \rightarrow e' \) in order to have a metric of the form (7), then we apply the formalism to compute the density and the tensions, then we come back to the usual coordinates. We find that the metric that describes a toroidal shell is...
\begin{align*}
  ds^2 &= dt^2 - dr^2 - (|r - r_0| - \epsilon w)^2 d\vartheta^2 + \left[ b + (\epsilon w - |r - r_0|) \cos \vartheta \right]^2 d\varphi^2. \quad (43)
\end{align*}

The density and tensions are
\begin{align*}
  \rho &= \rho_{\varphi} + \rho_{\vartheta}, \quad \rho_{\vartheta} = \frac{2\epsilon}{b + \omega \cos \vartheta} \delta(r - r_0), \quad \rho_{\varphi} = -\frac{2\epsilon}{\omega} \delta(r - r_0). \quad (44)
\end{align*}

The only allowed singularities in this formalism are delta functions appearing in the density and tensions. This restricts us the value of the constants \( w \) and \( b \), to \( b > w \). We have that the tension in the directions along the polar angle \( \vartheta \) is like the one for the cylinder, but along \( \varphi \) we have, for \( \epsilon = 1 \), negative values in the region \( \pi/2 < \vartheta < 3\pi/2 \), and for \( \epsilon = -1 \) we also have negative values, but now for \( -\pi/2 < \vartheta < \pi/2 \). A shell with tensions and pressures does not look very stable.

**V. THE INTERACTION WITH TEST PARTICLES**

In this section we study the motion of test particles in the presence of spherical and cylindrical defects studied above. Equation (35) with \( \epsilon = 1 \) and \( w > 0 \) tells us that the metric at each side of the surface of discontinuity \( r = r_0 \) is different
\begin{align*}
  ds^2 &= dt^2 - dr^2 - (r - b)^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad r > r_0, \quad (45) \\
  ds^2 &= dt^2 - dr^2 - (r - a)^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad r < r_0. \quad (46)
\end{align*}

Due to the spherical symmetry we shall study the geodesic equations in the plane \( \vartheta = \pi/2 \); for the first of these metrics, we find
\begin{align*}
  \dot{t} &= 0, \quad \dot{r} - (r - b)^2 \dot{\varphi}^2 = 0, \quad \left[ (r - b)^2 \dot{\varphi} \right] \dot{r} = 0, \quad (47)
\end{align*}

where the over dot denote derivative with respect to the parameter \( s \). These equations can be easily solved, we find \( t = \tilde{E}s \), \( (r - b)^2 \dot{\varphi} = \tilde{l} \), and
\begin{align*}
  r &= b + \eta \frac{l}{\sqrt{2E}} \left[ 1 + \left( \frac{2Es}{l} \right)^{2} \right]^{1/2}, \quad \varphi = \varphi_0 + \tan^{-1}(2Es/l), \quad (48)
\end{align*}

where \( 2E = \tilde{E}^2 - 1 > 0 \) and \( \eta = \pm 1 \). Thus these are not closed trajectories; the open trajectories are easily studied in Cartesian coordinates, we find
\begin{align*}
  \pm x &= \eta \frac{l}{\sqrt{2E}} \frac{b}{\sqrt{1 + (2Es/l)^2}}, \quad \pm y = \eta \sqrt{2Es + \frac{2bEs/l}{\sqrt{1 + (2Es/l)^2}}}. \quad (49)
\end{align*}

The \( \pm \) in front of \( x \) and \( y \) are due to the reflection symmetry of the situation, so without loosing generality we shall take the solution with \( + \) sign. For \( s = 0 \) we get, \( x = \eta(l/\sqrt{2E}) + b \) and \( y = 0 \), thus to have a solution outside \( b \) we take \( \eta = 1 \) and for solutions inside the sphere \( r = a \) we change \( b \) by \( a \) and take \( \eta = -1 \) in Eq. (49). For large \( s \to \pm \infty \) we get \( x = \eta(l/\sqrt{2E}) \) and \( y = \pm \infty \).

In Fig. 1 we show several particles being scattered by the spherical surface \( r = b \), with \( b = 1 \), \( E = 1/8 \), and \( l = 0.004, 0.3, 0.505, 0.8 \), and 1.05. We see that the behavior resembles one of streamlines of a usual fluid in the presence of a spherical obstacle. We note that the interaction really takes place around a few radii of the sphere. Also for photons we have the same situation, in this case \( 2E = \tilde{E}^2 \) and \( s \) is an affine parameter; we do not have the formation of double images due to the presence of the defect. In Fig. 2 we show particles moving in the interior of the sphere...
FIG. 1. The trajectories of particles with equal energy moving in the gravitational filed of a spherical defect of effective radius $b = 1$ are shown in the $xy$ plane for increasing values of initial angular momentum.

with $b = 0.9$, $\sqrt{2E} = -1/2$, and $I = -0.025$, $-0.1$, and $-0.45$. We see that the surface $r = b(r = a)$ acts as an effective gravitational radius outside (inside) of the defect, the singularity is in the middle. We will come back to this point in the next section.

The case of a cylindrical plate is similar; the trajectory in the $x,y$ plane is given by Eq. (49) and for $z$ we have $z = u_0 s + z_0$, in this case $2E = E^2 - u_0^2 - 1 > 0$. We do not have the formation of double images in this case. We note that even though we have the equation of state associated to straight cosmic strings, in the present case the asymptotic conic structure is absent.

Because of the peculiarities of the metrics under consideration we have that particles moving
around convex shells will behave in the same way as in the case of the sphere; we will have significant interaction only at a distance of a few “characteristic sizes” of the shell and after that the motion will remain unchanged.

VI. CONCLUDING REMARKS

Let us come back to the metrics (45) and (46) that represent the space–time associated to a spherical symmetric defect. We can interpret this metric as the result of cutting a spherical slab of radius $a$ and thickness $b-a$ of the three dimensional space, and then gluing together the exterior $r>b$ with the interior $r<a$ we have introduced a mild curvature singularity in the middle of the slab, $(a+b)/2 = r_0$. This represents the shell of matter of density (36). When $r<a$ we do not need to stop at $r=0$, in principle we can go beyond and take negative values of $r$, in this case we can interpret the solution as a wormhole. We have two locally isometric Minkowski spaces united by a throat made of a material with equation of state (36). Note that both surfaces $r=b$ and $r=a$ have zero area. So we can also interpret this solution as arising by the blowing of the coordinate singularity of the usual Minkowski space in spherical coordinates up to a sphere of radius $b$ and then putting inside of this sphere another infinite Minkowski space. The same interpretation can be made of all the compact shells presented in Sec. III. We note that the equation of state of the “throats” (36), (40), and (42) satisfies all the energy conditions.\(^8\)
Due to the local character of the metric, it is well known that to a single metric one can relate different global topologies. An example is the Vilenkin domain wall that is isometric to a sphere.\textsuperscript{1,2} We note that in metric (27) both variables $r$ and $\phi$ can be taken to be unbounded. In this case the metric represents a plane plate parallel to the $z$ axis located in the new Cartesian coordinate $r = r_0$.

In this work we have studied space-time defects without questioning its origin. A next step is to relate the energy–momentum tensor (14) with a field theory, or better to specific solutions. These are necessarily very special solutions of the field equations because not only its energy–momentum tensor has support in a hypersurface but also it does not curve the space–time outside this surface. An example is provided by the kink solution of the $\lambda \Phi^4$ model for plane domain walls. Another type of solution of the Einstein equation coupled to matter field equations are gravitational ghosts. These arc solutions that have zero energy–momentum tensor. A known example of this class of ghost fields is some solutions of the massless Dirac equation, the so-called ghost neutrinos studied some time ago.\textsuperscript{3} In these solutions the spinor representing the neutrinos are covariantly constant, then its associated energy–momentum tensor is null, but its probability current is different from zero. We hope to come back to this point on another occasion.

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\textsuperscript{2}P. S. Shellard and A. Vilenkin, Cosmic Strings and Other Topological Defects (Cambridge University, Cambridge, England, 1994).
\textsuperscript{5}See, for instance, T. J. Willmore, An Introduction to Differential Geometry (Clarendon, Oxford, 1959).