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AUTOMATA GENERATING FREE PRODUCTS OF GROUPS OF ORDER 2

AUTÔMATOS QUE GERAM PRODUTOS LIVRES DE GRUPOS DE ORDEM 2

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Resumo

Um *autômato* (ou um *autômato de Mealy*) $\mathcal{A}$ consiste numa quádrupla (Q, X, π, λ), em que Q é um conjunto de estados, X é um alfabeto finito, $\pi : Q \times X \to Q$ é uma função de transição e $\lambda : Q \times X \to X$ é uma função de saída. Dentre todos os tipos de autômatos, destacamos os autômatos finitos inversíveis para, assim, podermos definir um grupo de autômatos. Uma família especial de autômatos que possui importância fundamental para o nosso trabalho é a família de autômatos de Bellaterra, estudados inicialmente durante a escola de verão em grupos de autômatos na Universidade de Barcelona, em Bellaterra, no ano de 2004 (este é o motivo pelo qual tal família de autômatos recebe este nome); tais autômatos são definidos por recursão entrelaçada. Nesta dissertação, construímos uma família de autômatos (os autômatos de Bellaterra) com $n \geqslant 4$ estados que agem numa árvore binária enraizada e mostramos que os grupos gerados por estes autômatos (sob determinadas condições) são isomorfos à produtos livres de grupos cíclicos de ordem 2. Este estudo é baseado no artigo [18].

Palavras-chave: Autômatos, Grupos de autômatos, Autômatos de Bellaterra, Produtos livres de grupos, Recursão entrelaçada.

Abstract

An *automaton* (or a *Mealy automaton*) $\mathcal{A}$ consists of a tuple (Q, X, π, λ), in which Q is a set of states, X is a finite alphabet, $\pi : Q \times X \to Q$ is a transition function and $\lambda : Q \times X \to X$ is an output function. Among all the types of automata, we highlight finite invertible automaton in order to define an automata group. A special family of automata which has fundamental importance to our work is the Bellaterra automata family, first studied during the summer school in automata groups at the University of Barcelona in Bellaterra, in 2004 (this is why these automata receive this name); such automata are defined by wreath recursion. In this dissertation, we construct a family of automata (the Bellaterra automata) with $n \geqslant 4$ states acting on a rooted binary tree and we show that the groups generated by these automata are isomorphic to free products of cyclic groups of order 2 (under certain conditions). This study is based on the article [18].

Key words: Automata, Automata groups, Bellaterra automata, Free products of groups, Wreath recursion.

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Introduction

Given a finite invertible automaton $\mathcal{A}$, an automaton group is the group generated by the functions defined by all states of $\mathcal{A}$. Such functions act on the set X^* of finite words over the alphabet X; the set X^* can be regarded as a regular rooted tree. Automata groups started to be mentioned in articles during the 1960s by M. Glushkov (see [10]) and J. Hořejš (see [13]). However, only in the 1980s these objects gained more attention, after some mathematicians have shown that automata groups contain counterexamples to the general Burnside problem (see [1], [11] and [12]). Automata groups have also been used to produce other remarkable groups, including a group without uniform exponential growth, and exotic amenable groups (see [23]). Moreover, all the automata groups discussed in this dissertation are synchronous and so they are residually finite (a large class of groups). Several well-known groups can be generated by synchronous automata, including free groups (see [21]), $\mathrm{GL}_n(\mathbb{Z})$ and its subgroups (see [7]), the solvable Baumslag-Solitar groups $\mathrm{BS}(1, m)$ (see [3]), and the lamplighter groups $R \wr \mathbb{Z}$ with R a finite ring (see [19]). In this dissertation we study how construct free products of the cyclic group C_2 as automata groups.

Around 2004, the Bellaterra automata emerged as good examples of bireversible automata (invertible automata whose dual and the dual of its inverse are invertible) and they formed a good source for free groups and free products generated by automata. The first Bellaterra automaton was discovered while classifying all bireversible 3-state automata over a 2-letter alphabet (see [6]), so we consider such automata acting on the set X^* of finite words over $X = \{0, 1\}$. All preliminaries about automata and actions on trees are introduced in Chapter 1.

In Chapter 2, we study in details the Bellaterra automaton $\mathcal{B}_4$, the 4-state automaton of the family, expliciting the wreath recursions and the Moore diagrams of such automaton and its inverse. Those elements are used in the proofs of some auxiliary propositions and lemmas which lead to the main theorem of this chapter that summarizes the first important result proved in this dissertation: letting $\mathcal{G}$ be the group generated by all 4 states of the automaton $\mathcal{B}_4$ given by the wreath recursion

$$\begin{aligned} a &= (c, b), \\ b &= (b, c), \\ c &= (d, d)\sigma, \\ d &= (a, a)\sigma, \end{aligned}$$

and considering the notation C_2 for the cyclic group of order 2, then

Theorem 0.0.1 (D. Savchuk, Y. Vorobets, [18])**.** The group $\mathcal{G}$ is isomorphic to the free product $C_2 * C_2 * C_2 * C_2$.

We then prove that a 4-state Bellaterra automaton satisfying some conditions about the permutations defined on its wreath recursion generates the free product of 4 groups of order 2. In Chapter 3 we show how to prove its generalization for $n > 4$ (with a few restrictions); in other words, considering the automaton $\mathcal{B}^{(n)}$, the Bellaterra automaton with n states, with transition and output functions defined by the wreath recursion

$$\begin{aligned}
a_n &= (c_n, b_n), \\
b_n &= (b_n, c_n), \\
c_n &= (q_{n1}, q_{n1})\sigma, \\
q_{n,i} &= (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, \quad i = 1, \cdots, n-5, \\
q_{n,n-4} &= (d_n, d_n)\sigma_{n,n-4}, \\
d_n &= (a_n, a_n)\sigma,
\end{aligned}$$

then we establish that

Theorem 0.0.2 (D. Savchuk, Y. Vorobets, [18])**.** The group $\mathcal{G}^{(n)}$, generated by the automaton $\mathcal{B}^{(n)}$, is isomorphic to the free product of n copies of the cyclic group of order 2.

The proof is somehow similar to the case of $n = 4$; it starts by adapting the definition of $\mathcal{B}_n$ ($n > 4$) from the one of $\mathcal{B}_4$. We then use an approach similar to the one used in the proof of the first case due to a result (Lemma 3.2.4) that establishes a relation between the actions of the groups Γ and $\Gamma^{(n)}$ (dual groups to $\mathcal{G}$ and $\mathcal{G}^{(n)}$, respectively).

It is worth reminding that we will indeed follow the construction of the original article [18].

1 Preliminaries

"Mathematicians do not study objects, but relations between objects."

Henri Poincaré, 1854–1912

In this first chapter we will show the main introductory definitions and results which are somehow necessary to get a better understanding of our work.

It was a preference of the author to start from the very beginning in order to try to make this dissertation self-contained. Thus, before diving into the group theory applied to world of automata, we will record some fundamentals about groups, graphs, words and automata.

1.1 Basic tools

In the first part of this preliminary chapter, we explore three elementary objects which are essential to our research: graphs, stabilizers and words.

Since we can say, in a rough way, that automata can be seen as graphs with certain properties and those objects constitute the core of this work, we need to understand well what they are. We begin this section defining graphs and some of their important characteristics that turn out to be useful for this work.

Definition 1.1.1. A *graph* Π consists of two sets: a set $V(\Pi)$ of *vertices* and a set $E(\Pi)$ of *edges*, which are unordered pairs of vertices. We say that an element $e \in E(\Pi)$ gives a pair of *adjacent vertices*.

It is possible to have multiple distinct edges with the same associated pair of vertices as well as having loops, which are edges which start in a vertex and end at the same vertex. If a graph has neither loops nor multiple edges, it is called a *simple* graph.

Graphs are mainly visualized by drawing diagrams consisting of points, which represent the vertices, and arcs connecting two vertices; such arcs represent the edges. An example of such diagram is shown in Example 1.1.3.

Definition 1.1.2. The *degree* (or *valence*) of a vertex v is the number of edges incident with v; in other words, it is the number of the edges containing v. If v is a vertex for a loop, then this loop adds 2 to the degree of v.

Example 1.1.3. Consider the graph Π shown below. According to the Definition 1.1.1, we have that $V(\Pi) = \{1, 2, 3, 4\}$ and $E(\Pi)$ has 6 elements. Furthermore, Π is simple and every vertex of Π has degree 3.

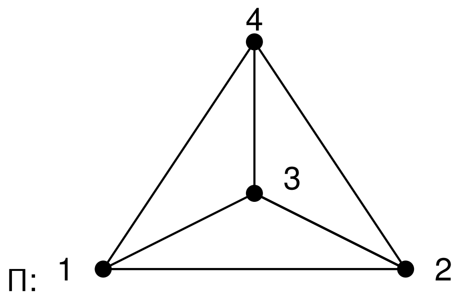


Figure 1 – A simple graph Π.

Definition 1.1.4. A *path* in a graph is an alternating sequence of vertices and edges $\{v_0, e_1, v_1, \cdots, v_{n-1}, e_n, v_n\}$ such that $v_i v_{i+1} = e_{i+1} \in E(\Pi)$ for all $i = 1, 2, \cdots, n-1$. A *cycle* or a *circuit* is a path $\{v_0, e_1, v_1, \cdots, v_{n-1}, e_n, v_n\}(n \geqslant 3)$ along with the edge $v_n v_0$ and $v_i \neq v_j$ for all $i, j = 1, \cdots, n$; that is, a non-trivial path in which first and last vertices are the same but no other vertex is repeated. A graph is *connected* if for any two vertices v and w, there is a path with v as the first vertex and w as the last one.

Definition 1.1.5. A *tree* is a connected graph with no cycles. A *rooted tree* is a tree with a distinguished vertex singled out, called the *root*.

The vertices in a rooted tree form a hierarchy, with the root at the highest level, and the level of every other vertex determined by its distance from the root. Some familiar terms are often used to describe relationships between vertices in a rooted tree: if v and w are adjacent vertices and v lies closer to the root than w, then v is the parent of w, and w is a child of v. Furthermore, a vertex v of degree 1 in a tree is named a *leaf*.

Definition 1.1.6. An *n-ary tree* is a rooted tree such that each vertex has n children, except the leaves.

Definition 1.1.7. If an edge set $E(\Pi)$ of a graph Π is constituted by ordered pairs of vertices, then Π is said to be a *directed* graph. In the diagram representation of a graph, a directed edge is drawn with a directed arrow. A *directed path* is a path where each e_i is the ordered pair (v_i, v_{i+1}).

The Definitions 1.1.5 and 1.1.7 are particularly simple; however, they play an important role in this work since trees are somehow related to monoids, as we will explain in a more detailed way in Subsection 1.2.2. An automaton, an important object described in Section 1.3, is actually a directed graph and the following Example 1.1.8 works as a preview.

Example 1.1.8. For the time being, the figure below appears to us only as a directed graph. In Section 1.3, we will notice that it is much more than this.

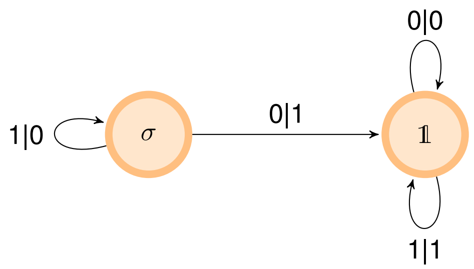


Figure 2 – A directed graph.

Another primary concept is the one of a stabilizer which will appear in Chapter 2. In order to define a stabilizer, we first recall what a group action is.

Definition 1.1.9. Let X be a set. An *action* of a group G on X is a group homomorphism $\rho : G \to \mathrm{Sym}(X)$, in which $\mathrm{Sym}(X)$ denotes the symmetric group over the set X. Equivalently, it is a map $\varphi : G \times X \to X$ such that

1. $\varphi(e, x) = x$ for all $x \in X$; and
2. $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ for all $g, h \in G$ and $x \in X$.

We write $g \cdot x$ in lieu of $\varphi(g, x)$ to simplify the notation. Moreover, we use the notation $G \curvearrowright X$ to express that G acts on X.

Definition 1.1.10. Let X be a set and let G act on X. For all $x \in X$, the *stabilizer* of x is defined by the set

$$\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}.$$

Proposition 1.1.11. $\mathrm{Stab}(x)$ is a subgroup of G, for any $x \in X$ and for whatever group G such that $G \curvearrowright X$.

Definition 1.1.12. Let G act on X. Then, $G \curvearrowright X$ is a *free action* if, for every $x \in X$, we have $\mathrm{Stab}(x) = \{e\}$. An element x with the property that $\mathrm{Stab}(x) = \{e\}$ is *moved freely* by the action of G.

Later on in this work we will study stabilizers in the context of group actions on trees.

To finish this section, we give some attention to words. The notion of word that we adopt here is simple but powerful concerning to the main subject of this work: automata that generate free products of groups. Also, words are important to the construction of the topics covered in this preliminary chapter since they are related to free groups and they are the first entry into formal language theory and its relation with groups.

Definition 1.1.13. Let X be a non-empty set. In the context of this research, X is called an *alphabet* and its elements are called *letters*. Then, a *word* consists on a finite sequence of elements of X, possibly with repetition. It also includes the *empty word*, that is the word consisting of a sequence of no elements of X (represented by ε).

Definition 1.1.14. The collection of all words over the alphabet X, including the empty word, is denoted by X^*.

Definition 1.1.15. Given an alphabet $X = \{x_1, \cdots, x_n\}$, a *language* $\mathcal{L}$ is meant to be any subset $\mathcal{L} \subset X^*$.

A non-empty word w on the alphabet X is represented, in general, with the form $w = x_1 x_2 \cdots x_n$ with $x_1, x_2, \cdots, x_n \in X$. Proceeding along this way of thinking, we say that the *length* of w is n while the length of the empty word is 0.

Note that the set X^* can be endowed with a structure of a n-ary tree. More explicitly, we declare that:

- Any $w \in X^*$ is adjacent to wx for any $x \in X$;
- The word ε (empty word) corresponds to the root of the tree;
- X^n corresponds to the n-th level of the tree.

All formal inverses to the elements of X make up a set, denoted by X^{-1}; for example, if $X = \{x_1, x_2\}$ then $X^{-1} = \{x_1^{-1}, x_2^{-1}\}$. Thus, we are able to construct the set $\{X \cup X^{-1}\}^*$ whose elements are finite strings of elements from X and their formal inverses. From those assumptions, it should be natural that $(x^{-1})^{-1} = x$ and if a word w can be written in the form $w = x_1 x_2 \cdots x_{k-1} x_k \in \{X \cup X^{-1}\}^*$ then we have $w^{-1} = x_k^{-1} x_{k-1}^{-1} \cdots x_2^{-1} x_1^{-1} \in \{X \cup X^{-1}\}^*$.

We can, also, define a product in the alphabet X, called concatenation.

Definition 1.1.16. Given two non-empty words $w_1 = x_1 \cdots x_n$ and $w_2 = y_1 \cdots y_n$, with $w_1, w_2 \in X^*$, we define the *concatenation* $w_1 w_2$ by

$$w_1 w_2 = (x_1 \cdots x_n)(y_1 \cdots y_n) = x_1 \cdots x_n y_1 \cdots y_n.$$

The empty word works as the identity element with respect to this product. With the operation defined in Definition 1.1.16, X^* becomes a monoid (called the *free monoid* generated by X) that satisfies the universal property:

Proposition 1.1.17. Let $\varphi : X \to M$ be a function from X to a monoid M. Then, there is a unique monoid homomorphism $\Phi : X^* \to M$ that extends φ. This means that the diagram below commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, hook] & M \\ X^* \arrow[ur, dashed, "\Phi"'] & \end{tikzcd}$$

Proof. Define $\Phi : X^* \to M$ by $\Phi(w) = \Phi(x_1 \cdots x_n) = \varphi(x_1) \cdots \varphi(x_n)$; moreover, consider $w_1 = x_1 \cdots x_n$ and $w_2 = y_1 \cdots y_n$ words on X^*. Then,

$$\begin{aligned} \Phi(w_1 w_2) &= \Phi(x_1 \cdots x_n y_1 \cdots y_n) \\ &= \varphi(x_1) \cdots \varphi(x_n)\varphi(y_1) \cdots \varphi(y_n) \\ &= \Phi(x_1 \cdots x_n)\Phi(y_1 \cdots y_n) \\ &= \Phi(w_1)\Phi(w_2) \end{aligned}$$

and

$$\begin{aligned} \Phi(\varepsilon) &= \Phi(\varepsilon \cdot \varepsilon) \\ &= \varphi(\varepsilon)\varphi(\varepsilon) \\ &= \Phi(\varepsilon)\Phi(\varepsilon), \end{aligned}$$

which means that $\Phi(\varepsilon) = \varepsilon$; therefore, Φ is a monoid homomorphism. Uniqueness follows easily as well. □

1.2 Free groups

In any group, one always has simplifications obtained by cancelling elements with their inverses; in other words, considering a group G and an element $a \in G$, the equations $aa^{-1} = a^{-1}a = \varepsilon$ hold. Informally, we would think of a free group as a group in which there are no other simplifications.

Still informally, consider $X = \{x_1, \cdots, x_n\}$ as a set of elements of a group F. We say that a word $w \in \{X \cup X^{-1}\}^*$ is *freely reduced* if it does not contain a subword (word contained on w and composed by a sequence of consecutive letters of $X \cup X^{-1}$) consisting of an element adjacent to its formal inverse. Using such intuition, we can define a free group as follows:

Definition 1.2.1. A group F is a *free group with basis* X if $X = \{x_1, \cdots, x_n\}$ is a set of generators for F and the identity ε can not be represented by a freely reduced word in $x_1, \cdots, x_n$ and their inverses.

It can be proved that, equivalently, every nontrivial element $w \in X$ can be uniquely represented as a product $w = x_1 \cdots x_k$, in which $x_i \in X \cup X^{-1}$ and $x_i x_{i+1} \neq \varepsilon$ for all $i = 1, \cdots, k$.

Based on the Definition 1.2.1, one can define the rank of a free group.

Definition 1.2.2. The *rank* of a free group F with basis $X = \{x_1, \cdots, x_n\}$ is the number of elements in X. That is to say, the rank of F is the cardinality of its basis X.

More formally, we give the following definition for a free group:

Definition 1.2.3. A group F is *freely generated* by a subset $X \subseteq F$ if, for any group G and any map $\phi : X \to G$, there is a unique homomorphism $\widehat{\phi} : F \to G$ extending ϕ; that is, we have $\widehat{\phi}(w) = \phi(w) \ \forall \ w \in X$.

Note that, by the definition above, the subset X, which makes part of the characterization of a free group, does not need to be finite. Also, we can say that F satisfies the *universal property* so it makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, hook] & G \\ F \arrow[ur, dashed, "\widehat{\phi}"'] & \end{tikzcd} .$$

Proposition 1.2.4. Definitions 1.2.1 and 1.2.3 of a free group are equivalent.

Proof. See [5], page 54. □

Given a subset X of a group F, there are two features of the free group generated by X: it is unique and it always exists.

Theorem 1.2.5 (Uniqueness of free groups)**.** Let X be a set. Then, up to canonical isomorphism, there is at most one group freely generated by X.

Theorem 1.2.6 (Existence of free groups)**.** Let X be a set. Then there exists a group freely generated by X. (By Theorem 1.2.5, this group is unique up to isomorphism.)

A proof of Theorems 1.2.5 and 1.2.6 can be found in [14], pages 20-24.

Definition 1.2.7. Considering $n \in \mathbb{N}$ and $X = \{x_1, \cdots, x_n\}$ a finite set with n distinct elements, the group F_n, freely generated by X, is "the" *free group of rank n*.

1.2.1 Free products of groups

In this subsection we define the free product of two groups A and B. The construction used in this product can be easily extended in order to provide a good definition for the free product of n groups. By taking isomorphic copies of such groups, we may assume that $A \cap B = \{e\}$.

Definition 1.2.8. A *normal form* is an expression of the form $x_1 \cdots x_n$ in which we have $x_i \in (A \cup B) \backslash \{e\}$, with $i = 1, \cdots, n$, and every two adjacent factors x_i and x_{i+1} belong to distinct groups.

The *length* of a normal form is exactly the number n; moreover, the identity e is identified with the normal form of length zero.

Consider the set of all normal forms and define a multiplication $\cdot$ on this set: let $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$ be normal forms such that $n, m \geqslant 1$; next, define

$$e \cdot x = x \cdot e = x, \tag{1.1}$$

for all normal form x, and

$$x \cdot y = \begin{cases} x_1 \cdots x_n y_1 \cdots y_m & \text{if } x_n \in A, y_1 \in B \quad \text{or} \quad x_n \in B, y_1 \in A, \\ x_1 \cdots x_{n-1} z y_2 \cdots y_m & \text{if } x_n, y_1 \in A \quad \text{or} \quad x_n, y_1 \in B \text{ and } z = x_n y_1 \neq e, \\ x_1 \cdots x_{n-1} y_2 \cdots y_m & \text{if } x_n, y_1 \in A \quad \text{or} \quad x_n, y_1 \in B \text{ and } x_n y_1 = e. \end{cases} \tag{1.2}$$

The set of normal forms endowed with the multiplication defined above is a group. Due to this construction, we can define the free product of two groups A and B.

Definition 1.2.9. The *free product* of two groups A and B is the group defined by all normal forms in A and B along with the multiplication defined by relations (1.1) and (1.2). It is denoted by $A * B$.

Notice that the groups A and B are embedded into the group $A * B$. The next proposition formalizes how this happens; its proof is straightforward.

Proposition 1.2.10. Consider A and B subgroups of a group G. Given a nontrivial element $g \in G$ such that g can be written uniquely as a product $g = g_1 \cdots g_n$, with $g_i \in (A \cup B) \backslash \{e\}$ $(i = 1, \cdots, n)$, and every two adjacent factors g_i and g_{i+1} belong to distinct groups, then $G \simeq A * B$.

We recall that, for a set X and a subset R of the free group $F(X)$ on X, the notation $\langle X \mid R \rangle$ is called a *group presentation* and denotes the quotient group $F(X)/R^{F(X)}$ in which $R^{F(X)}$ denotes the smallest normal group containing all products of elements the type frf^{-1} with $r \in R$ and $f \in F(X)$. We will not go into details as it is not used later in this dissertation, but we notice that elements of $R^{F(X)}$ are called *relations* or *relators*. The following theorem gives another important characterization of free products of groups.

Theorem 1.2.11. Let A and B be groups such that $A = \langle X \mid R \rangle$, $B = \langle Y \mid S \rangle$ and $X \cap Y = \varnothing$. Then, $A * B = \langle X \cup Y \mid R \cup S \rangle$.

Proof. See [5], page 72. □

1.2.2 Groups and actions on trees

Groups and graphs appear together often in Geometric Group Theory. By Cayley's Theorem, we learn that every group G generated by a finite set X can be represented as a symmetry group of a connected, directed and finite graph, namely the *Cayley graph of G with respect to X*. R. Frucht also proved that all finitely generated groups can be realized as label and orientation preserving symmetries of finite, directed graphs (see [9]). Then, analyzing groups by their actions on graphs seems to be a very interesting approach to the study of Geometric Group Theory. This is the approach used in our work as well.

Recalling what was said in Section 1.1, the free monoid X^* can be seen as a vertex set of a rooted tree. Considering n the cardinality of X, X^* can be naturally endowed with a structure of a rooted n-ary tree by declaring that the empty word ε is the root of such tree and that two arbitrary words w_1 and w_2 ($w_1, w_2 \in X$) are connected by an edge if and only if the equation $w_2 = w_1 x$ holds, for some $x \in X$. Furthermore, a vertex labelled by w ($w \in X^*$) has $|X|$ children whose labels are wx_i for each $x_i \in X$.

Example 1.2.12. The main alphabet used in this work is the set $X = \{0, 1\}$ so we show here, as an example of what was stated above, the set X^* described as a rooted tree with all words on this alphabet constituting the vertex set of that corresponding tree.

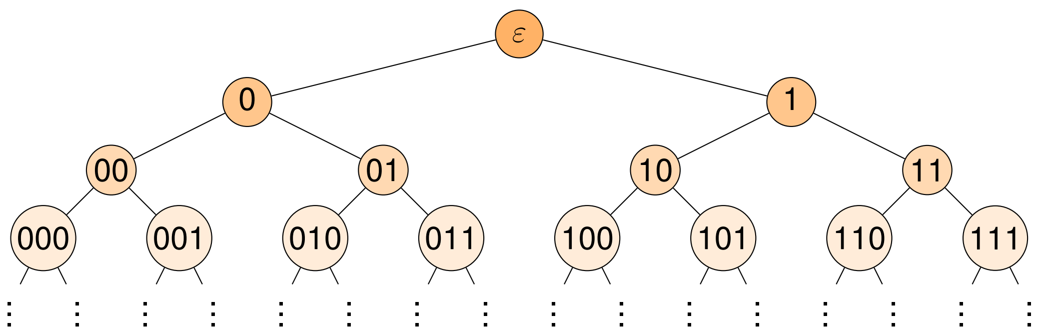


Figure 3 – The set $X^* = \{0, 1\}^*$ viewed as a rooted binary tree.

It is convenient not to distinguish between a vertex and its label so we can refer to "the vertex w" instead of saying "the vertex labeled by w".

Definition 1.2.13. A map $f : X^* \to X^*$ is an *endomorphism of the tree* X^* if it preserves the root and the adjacency of the vertices. Formally, given a word $w \in X$, for any two adjacent vertices $w, wx \in X^*$, the vertices $f(w)$ and $f(wx)$ are adjacent as well; this means that there exist $w' \in X^*$ and $x' \in X$ such that $f(w) = w'$ and $f(wx) = w'x'$.

Denoting the n-th level of the tree X^* by $X^n \subset X^*$, we establish the following proposition:

Proposition 1.2.14. If $f : X^* \to X^*$ is an endomorphism then $f(X^n) \subseteq X^n$.

Proof. Use induction on n. □

Definition 1.2.15. An *automorphism* is a bijective endomorphism.

Here are some important theorems related to actions of groups on trees:

Theorem 1.2.16. A group G is free if and only if it acts freely on a tree.

Proof. See [15], page 70. □

Corollary 1.2.17 (Nielsen-Schreier Theorem)**.** Every subgroup of a free group is itself free.

Proof. See [15], page 73. □

Our interest in this work is to study the groups of automorphisms and the semigroups of endomorphisms of a rooted tree X^*. Such endormorphisms can be defined by an initial automaton, which will be introduced on Section 1.3. We shall denote by Aut X^* the group of all automorphisms of the rooted tree X^*.

Using the automorphisms defined before, one can adapt the Definition 1.1.9 for actions on trees.

Definition 1.2.18. Consider the tree (monoid) X^* and let Aut X^* be the group of all tree automorphisms of X^*. We say that a group G *acts by automorphisms* on X^* if there exists a group homomorphism $\rho : G \to \text{Aut } X^*$. The homomorphism ρ gives an action in the same sense of Definition 1.1.9.

We recall that a group action $G \times X \to X$ is *transitive* if it has only a single group orbit; that is, given a pair of elements $x_1, x_2 \in X$, there exists an element $g \in G$ such that $gx_1 = x_2$.

Definition 1.2.19. A *level transitive* action of a group G by automorphisms of the tree X^* is a transitive action on every level X^n of the tree X^*.

Definition 1.2.20. An element $g \in \mathrm{Aut}\, X^*$ is spherically transitive if, for each n, $\langle g \rangle$ acts transitively on all vertices at distance n from the root; in other words, g acts transitively spherically if $\langle g \rangle$ acts transitively on the set of words of length n.

Definition 1.2.20 is somehow related to Definition 1.2.19 since it can be seen as a "restriction" for level transitivity.

A group acting on a rooted tree has some important subgroups which we introduce next.

Definition 1.2.21. Let $G \leqslant \mathrm{Aut}\, X^*$ be an automorphism group of the rooted tree X^*. Then, we define that

1. The subgroup $G_w = \{g \in G \mid g(w) = w\}$, in which $w \in X^*$ is a vertex, is called a *vertex stabilizer*.
2. The subgroup $\mathrm{Stab}_G(n) = \bigcap_{w \in X^n} G_w$ is called the *n-th level stabilizer*.

1.3 Automata

Automata form the mathematical model that constitutes the core of the article studied in this research. As stated in [16], one needs a nice way to define automorphisms of rooted trees in order to be able to perform computations with them; one of the approaches for this involves automata. In this section, some main definitions and properties of automata are described.

Based on the book [15], we give the following definition.

Definition 1.3.1. An *automaton* consists of a directed graph $\mathcal{A}$, an associated alphabet X, a subset of vertices called the *start states* and a subset of states called the *accept states*. Also, the directed edges are labelled by elements of the alphabet X.

Definition 1.3.1 shows a somehow easy way to think of an automaton; however, this work requires a more formal approach of such an object. The following definitions related to automata given in this subsection are based mainly on [2] and [18]. Now, we give an equivalent but more formal definition for an automaton.

Definition 1.3.2. An *automaton* (or a *Mealy automaton*) $\mathcal{A}$ consists of a tuple (Q, X, π, λ), in which Q is a set of states, X is a finite alphabet, $\pi : Q \times X \to Q$ is a transition function and $\lambda : Q \times X \to X$ is an output function.

Definition 1.3.3. If the set Q is finite, then the automaton $\mathcal{A}$ is said to be *finite* (finite-state automaton - FSA). If a given state $q \in Q$ is selected to be an initial state, then $\mathcal{A}_q$ is an *initial automaton* (see the figure of Example 1.3.6). Furthermore, $\mathcal{A}$ is *invertible* if, for every state $q \in Q$, the output function $\lambda(q, x)$ induces a permutation on X.

If $\mathcal{A}$ is an invertible automaton, then its inverse is the automaton $\mathcal{A}^{-1}$ whose states are in a bijective correspondence $\mathcal{A}^{-1} \to \mathcal{A}$ with the set of states of A, given by $q^{-1} \mapsto q$. If $\pi(q_1, w_1) = q_2$ and $\lambda(q_1, w_1) = w_2$ in $\mathcal{A}$, then $\pi^{-1}(q_1^{-1}, w_2) = q_1$ and $\lambda^{-1}(q_1^{-1}, w_2) = w_1$ in $\mathcal{A}^{-1}$.

Definition 1.3.4. The *language $L(\mathcal{A})$ accepted by an automaton* is the set of all words $w \in X^*$ corresponding to directed paths p_w that begin at a start state and end at an accept state of the automaton $\mathcal{A}$.

Automata, also, can be represented by *Moore diagrams*.

Definition 1.3.5. Given an automaton $\mathcal{A} = (Q, X, \pi, \lambda)$, the *Moore diagram* of $\mathcal{A}$ is a directed graph such that its vertices are the states from Q and its edges have form $q \xrightarrow{x|\lambda(q,x)} \pi(q, x)$ for $q \in Q$ and $x \in X$.

Example 1.3.6. Consider the following Moore diagram of an automaton:

$\mathcal{D}$:

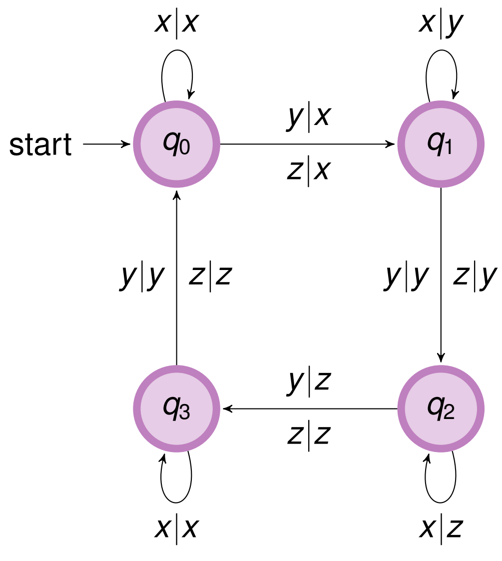


Figure 4 – Moore diagram of an initial automaton with initial state q_0.

Then, the state set of the initial automaton $\mathcal{D}$ is given by $Q = \{q_0, q_1, q_2, q_3\}$, the alphabet by $X = \{x, y, z\}$ and functions π and λ are explicitly given by

$$
\begin{aligned}
\pi(q_i, x) &= q_i, i = 1, 2, 3, 4,\\
\pi(q_0, y) &= \pi(q_0, z) = q_1,\\
\pi(q_1, y) &= \pi(q_1, z) = q_2,\\
\pi(q_2, y) &= \pi(q_2, z) = q_3,\\
\pi(q_3, y) &= \pi(q_3, z) = q_0
\end{aligned}
\tag{1.3}
$$

and

$$
\begin{aligned}
\lambda(q_0, \bullet) &= x,\\
\lambda(q_1, \bullet) &= y,\\
\lambda(q_2, \bullet) &= z,\\
\lambda(q_3, \bullet) &= \bullet.
\end{aligned}
\tag{1.4}
$$

According to [17], automata can be interpreted as devices transforming words on the following manner: if an automaton $\mathcal{A}$ on an alphabet X is in a state $q \in A$ and it gets a finite word as an input $w = x_1 \cdots x_n \in X^*$ and then $\mathcal{A}$ reads the first letter x_1 of w returning the letter $q(x) = \lambda(q, x)$ as an output and going to the state $q|_{x_1} = \pi(q, x_1)$. Then it is finally ready to process the rest of the word w in a similar fashion, reading letter by letter. At the end it will stop at some state of $\mathcal{A}$ and the output will be a word on X^*. In other words, given a word $w = x_1 \cdots x_n$, an initial automaton takes the letter x_1 as an input and returns a letter $\lambda(x_1)$ as an output; then the rest of the word is manipulated in a similar way by the automaton $\mathcal{A}_{\pi(x_1)}$.

We can rewrite the sentences above in a more formal way: the functions π and λ can be extended naturally to the functions $\pi : Q \times X^* \to Q$ and $\lambda : Q \times X^* \to X^*$, as follows:

$$\pi(q, w) = \pi(q, x_1 \cdots x_n) = \pi(\pi(q, x_1), x_2 \cdots x_n) \tag{1.5}$$

$$\lambda(q, w) = \lambda(q, x_1 \cdots x_n) = \lambda(q, x_1)\lambda(\pi(q, x_1), x_2 \cdots x_n). \tag{1.6}$$

Then, by construction, using the relations (1.5) e (1.6), we realize that any initial automaton acts on X^* as a rooted tree endomorphism. If the automaton is invertible, then it acts as a rooted tree automorphism.

Moreover, notice that since the automaton outputs exactly one letter for each letter that it reads, then, expressing by $|w|$ the length of the word w, we see that $|\lambda(q, w)| = |w|$.

Before going trough some theory related to automata groups, we show some examples of calculations involving automata with the interest of clarifying our ideas.

Example 1.3.7. Consider the initial automaton $\mathcal{D}$ given in the Example 1.3.6:

$\mathcal{D}$:

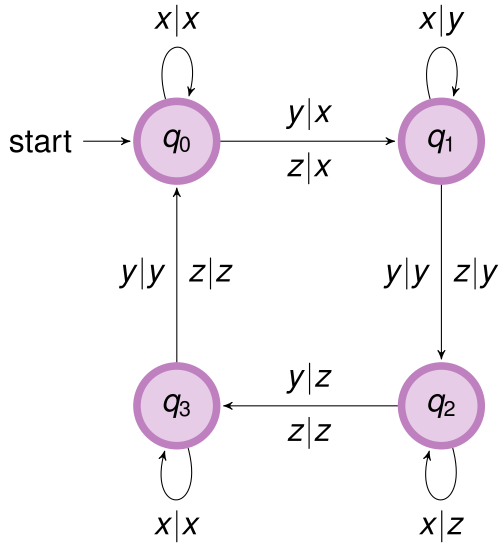


Figure 5 – Moore diagram of the initial automaton of the Example 1.3.6.

Remember that, in this case, $Q = \{q_0, q_1, q_2, q_3\}$ and $X = \{x, y, z\}$. Then, let yyy be a word in X^*; since q_0 is the initial state of the automaton $\mathcal{D}$, we obtain, using relations (1.3) and (1.4):

$$\begin{aligned}
\pi(q_0, yyy) &= \pi(\pi(q_0, y), yy) \\
&= \pi(q_1, yy) \\
&= \pi(\pi(q_1, y), y) \\
&= \pi(q_2, y) \\
&= q_3
\end{aligned}$$

and

$$\begin{aligned}
\lambda(q_0, yyy) &= \lambda(q_0, y)\lambda(\pi(q_0, y), yy) \\
&= x\lambda(q_1, yy) \\
&= x\lambda(q_1, y)\lambda(\pi(q_1, y), y) \\
&= xy\lambda(q_2, y) \\
&= xyz.
\end{aligned}$$

Thus, the automaton $\mathcal{D}$ reads the word yyy starting at the initial state q_0 and outputs the word xyz. The sequence of states visited during such computation (the associated path of yyy) is q_0, q_1, q_2, q_3. Note that $|yyy| = |xyz| = 3$.

Example 1.3.8 (given in [8]). Consider the following automaton $\mathcal{H} = (Q, X, \pi, \lambda)$:

$\mathcal{H}$:

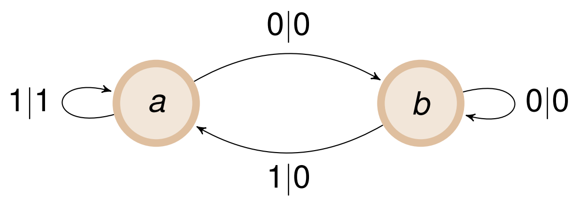


Figure 6 – Moore diagram of the automaton $\mathcal{H}$.

From the figure above, we see that the automaton's state set is $Q = \{a, b\}$; also, its alphabet X is the set $\{0, 1\}$. The transformations π and λ of the Definition 1.3.5 are determined by

$$\begin{aligned} \pi(\bullet, 0) &= b \\ \pi(\bullet, 1) &= a \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \lambda(b, \bullet) &= \lambda(a, 0) = 0 \\ \lambda(a, 1) &= 1. \end{aligned} \tag{1.8}$$

Let $w = 0011 \in X^*$ be a word on the alphabet X. Using relations (1.7) and (1.8) we shall compute $\pi(a, w)$ and $\lambda(a, w)$. Then,

$$\begin{aligned} \pi(a, 0011) &= \pi(b, 011) \\ &= \pi(b, 11) \\ &= \pi(a, 1) \\ &= a \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} \lambda(a, 0011) &= 0\lambda(b, 011) \\ &= 00\lambda(b, 11) \\ &= 000\lambda(a, 1) \\ &= 0001. \end{aligned} \tag{1.10}$$

As a further example of calculations, denoting $1^i = 1 \cdots 1$ (length i) for any $i \in \mathbb{N}$, still by relations (1.7) and (1.8) we get $\lambda(a, 1^i) = 1^i$ and $\lambda(b, 1^i) = 01^{i-1}$ for any $i \in \mathbb{N}$.

Doing similar computations like the ones done in (1.9) and (1.10), we obtain $\pi(b, 0011) = a$ and $\lambda(b, 0011) = 0001$ as well. However, keeping the same graph, but changing the initial state of an automaton can return a different action on an element of X^*. The following example show this.

Example 1.3.9 (given in [2]). Let $\mathcal{A} = (Q, X, \pi, \lambda)$ be the automaton with state set $Q = \{\sigma, \mathbb{1}\}$ and alphabet $X = \{0, 1\}$ given by

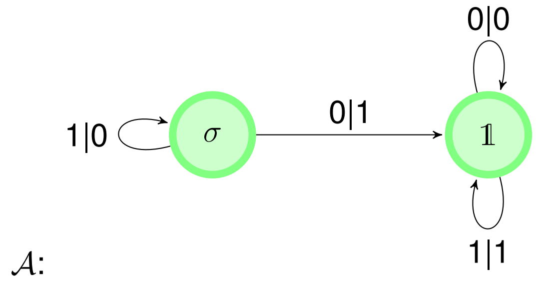


Figure 7 – Moore diagram of the automaton $\mathcal{A}$.

The transformation associated with the state $\mathbb{1}$ is the identity transformation, since any path starting from $\mathbb{1}$ is a loop with same input and output. Also, in this case, the transformations π and λ are given by

$$\begin{aligned} \pi(\mathbb{1}, \bullet) &= \pi(\sigma, 0) = \mathbb{1}, \\ \pi(\sigma, 1) &= \sigma \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} \lambda(\mathbb{1}, \bullet) &= \bullet, \\ \lambda(\sigma, 0) &= \lambda(\sigma, 1) = 1. \end{aligned} \tag{1.12}$$

Notice that the transformation associated with the state σ always changes the input by λ. For example, considering the word $111001 \in X^*$ and σ as an initial state, using relations (1.11) and (1.12), one gets

$$\begin{aligned} \pi(\sigma, 111001) &= \pi(\sigma, 11001) \\ &= \pi(\sigma, 1001) \\ &= \pi(\sigma, 001) \\ &= \pi(\mathbb{1}, 01) \\ &= \pi(\mathbb{1}, 1) \\ &= \mathbb{1} \end{aligned}$$

and

$$\begin{aligned}
\lambda(\sigma, 111001) &= 0\lambda(\sigma, 11001)\\
&= 00\lambda(\sigma, 1001)\\
&= 000\lambda(\sigma, 001)\\
&= 0001\lambda(\mathbb{1}, 01)\\
&= 00010\lambda(\mathbb{1}, 1)\\
&= 000101.
\end{aligned}$$

This means that the associated path consists of three loops at σ, the edge to $\mathbb{1}$ and two loops at $\mathbb{1}$. Further, the output to the word $111001 \in X^*$ by the automaton $\mathcal{A}$ is the word 000101.

Observe that, while $\pi(\mathbb{1}, 111001) = \pi(\sigma, 111001) = \mathbb{1}$, one obtains that $\lambda(\mathbb{1}, 111001) = 111001$ so $\lambda(\mathbb{1}, 111001) \neq \lambda(\sigma, 111001)$. This indicates that choosing the initial state of an automaton to do computations using functions π and λ has crucial importance with respect to the input and to the path associated to the action of Q on a word $w \in X^*$.

1.3.1 Automata groups and semigroups

Automaton groups are groups of automorphisms of rooted trees generated by actions of automata. They were introduced, firstly, by R. Grigorčhuk and his infinite periodic group in [11]; it provided some inspiration to other authors like N. Gupta and S. Sidki that, in the article [12], presented the Gupta-Sidki group. Such groups were made known at first as some interesting examples with "special" properties; despite this fact, a considerable theory has developed since then and nowadays it is clear that such groups have much more than specific "fancy" properties. This theory, also, has being generalized to semigroups, with relatively recent results.

The two following definitions are given in [18]:

Definition 1.3.10. An *automaton group* is the group generated by all states of an automaton $\mathcal{A}$. It is denoted by $\mathcal{G}(\mathcal{A})$.

Definition 1.3.11. An *automaton semigroup* is the semigroup generated by all states of an automaton $\mathcal{A}$. It is denoted by $\mathcal{S}(\mathcal{A})$.

Given $w \in X^*$ and $q \in Q$, we use the convention $w \cdot q = \lambda(q, w)$ in order to agree with the notations of the articles [8] and [18].

Since Definition 1.3.10 and Definition 1.3.11 do matter to this study, inspired by [8] we make some considerations that converge to similar definitions of automata groups and automata semigroups.

Having in mind that X^* can correspond to a rooted tree, we analyze the action of λ on X^*, observing the action of a state $q \in Q$: such functions can be wiewed, thus, as transformations of such tree; the function λ sends the vertex w to the vertex $\lambda(q, w)$ and q can be visualized as an endomorphism $X^* \to X^*$.

We affirm that if $w_1x_1 \cdot q = w_2x_2$ (with $w_1, w_2 \in X^*$ and $x_1, x_2 \in X$), then $w_1 \cdot q = w_2$. In fact, we have

$$\begin{aligned} w_2x_2 &= w_1x_1 \cdot q \\ &= \lambda(q, w_1x_1) \\ &= \lambda(q, w_1)\lambda(\pi(q, w_1), x_1). \end{aligned} \tag{1.13}$$

Analyzing the last equality of (1.13), since $x_1 \in X$ then $\lambda(\pi(q, w_1), x_1)$ must be a letter of X; thus, by the order of the letters of w_2x_2 we have that $\lambda(\pi(q, w_1), x_1) = x_2$ necessarily. Therefore, $\lambda(q, w_1) = w_2$, which means that $w_1 \cdot q = w_2$.

The previous assertion indicates that, concerning the transformation λ on the tree, if a vertex w_1 is the parent of a vertex w_1x_1, then their images under the action of λ (w_2 and w_2x_2 regarding the assertion) are also parent and child vertices. In other words, the action of λ on the tree corresponding to X^* preserves adjacency; therefore, it is an endomorphism of such tree. Besides, the function λ preserves lengths of words on X^* and, more generally, it preserves levels of the tree.

Remark 1.3.12. Let Q^+ be the free monoid such that all its words are of the form $\alpha = q_1 \cdots q_n$, with $q_i \in Q$ for all $i = 1, \cdots, n$. Observe that each state q of Q determines a function $q : X^* \to X^*$ arising from the function λ; thus, a word $\alpha \in Q^+$ induces an endomorphism $X^* \to X^*$ which is a composition of functions determined by each element of α.

Here we introduce an important notation for the endomorphism discussed in the above paragraph that will be used throughout this subsection: we denote by $\alpha(w)$ or $\alpha \cdot w$ the operation corresponding to the endomorphism which is a composition of functions determined by α, with $\alpha \in Q^+$ and $w \in X^*$. Using this notation, $\alpha \in Q^+$ acts on $w \in X^*$ by $\alpha(w) = q_n(q_{n-1}(\cdots(q_2(q_1(w)))))$. Note, also, that if $\alpha = q_1 \cdots q_n$, with $q_i \in Q$, then $\alpha(w) = \alpha \cdot w = \lambda(q_n, \lambda(q_{n-1}, \lambda(\cdots, \lambda(q_2, \lambda(q_1, w)))))$.

If End X^* denotes the endomorphism semigroup of the tree X^*, then, since Q^+ is also a semigroup, there exists, by the explanation given on Remark 1.3.12, a natural semigroup endomorphism $\varphi : Q^+ \to \text{End } X^*$ given by $\varphi(\alpha) = \alpha(w)$.

Seeing that φ is a semigroup endomorphism, its image is necessarily a semigroup; such image is denoted $\Sigma(\mathcal{A})$. This helps us to define that

Definition 1.3.13. A semigroup $\mathcal{S}(\mathcal{A})$ is called an *automaton semigroup* if there exists an automaton $\mathcal{A}$ such that $\mathcal{S}(\mathcal{A}) \simeq \Sigma(\mathcal{A})$.

Definition 1.3.13 is strictly related to Definition 1.3.11 in the sense that the semigroup $\Sigma(\mathcal{A})$ matches with the one described on the definition given before, since the elements of End X^*, by the previous discussion, can be seen as states of an automaton.

Recalling the Definition 1.3.3, an automaton $\mathcal{A}$ is invertible if its associated function $\lambda : Q \times X^* \to X^*$ induces a permutation on X^*; this means, also, that each state q induces an automorphism $X^* \to X^*$. In order to construct a definition for automata groups, we think of a similar approach to the states/applications q being bijections by looking for automata that produce invertible transformations of the set X^*. In some articles, like [8], the definition of *invertible automaton* says that an automaton is invertible if its states induce invertible transformations of X^*.

Consider an invertible automaton $\mathcal{A} = (Q, X, \pi, \lambda)$. Since the map q (the endomorphism $X^* \to X^*$ associated to q) is invertible, given $q \in Q$ and $\gamma \in X^*$, there exists a unique $w \in X^*$ such that $w \cdot q = \gamma$. For each state $q \in Q$, define the state q^{-1} by its action on X^*: $\gamma \cdot q^{-1} = w$ if and only if $w \cdot q = \gamma$. Note that, for any $q \in Q$ and $w \in X^*$, one has the following relation:

$$w \cdot qq^{-1} = w \cdot q^{-1}q = w. \tag{1.14}$$

Denote by Q^{-1} the set $Q^{-1} = \{q^{-1} \mid q \in Q\}$ and denote by $(Q \cup Q^{-1})^+$ the free monoid which its elements are words $\alpha = q_1 \cdots q_n$, with $q_i \in Q \cup Q^{-1}$, $\forall\ i = 1, \cdots, n$. By the construction above, one obtains that there exists a natural homomorphism $\varphi : (Q \cup Q^{-1})^+ \to$ End X^*. Note that, by relation (1.14), we verify that the Im φ is a subgroup of Aut X^*, the automorphism group of X^*; this image is denoted $\Delta(\mathcal{A})$. Notice that $\Delta(\mathcal{A})$ is well defined only when the automaton $\mathcal{A}$ is invertible. By this construction, we are able to define that

Definition 1.3.14. A group $\mathcal{G}(\mathcal{A})$ is said to be an *automaton group* if there exists an automaton $\mathcal{A}$ such that $\mathcal{G}(\mathcal{A}) \simeq \Delta(\mathcal{A})$.

Definitions 1.3.14 and 1.3.10 are related in the same way as it was explained before that definitions 1.3.13 and 1.3.11 are connected to each other.

Example 1.3.15. Consider the automaton $\mathcal{H}$ given in the Example 1.3.8. In this example our aim is to find what the semigroup generated by $\mathcal{H}$ is.

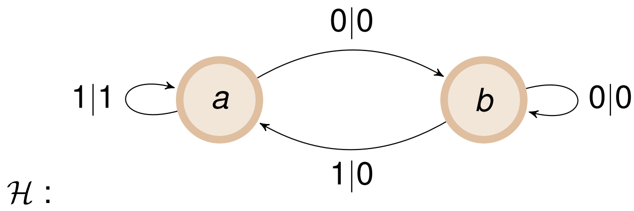


Figure 8 – Moore diagram of the automaton of the Example 1.3.8.

Observe that $\mathcal{H}$ is not invertible so it does not generate a group. Further, in this case, $Q = \{a, b\}$ and $X = \{0, 1\}$.

We have that the automaton $\mathcal{H}$ acts on the set X^*; it means that the semigroup $\mathcal{S}(\mathcal{H}) = \Sigma(\mathcal{H})$ generated by $\mathcal{H}$ is a subsemigroup of End X^* generated by the states a and b.

Now, one needs information about how the automaton works in order to get information regarding $\mathcal{S}(\mathcal{H})$. Such data can be obtained by studying the actions of a and b on words. Before going through this study, we introduce some useful notation to this example: X^ω represents the set of infinite sequences over X.

Also, we denote by γ^ω the infinite word expressed in countably many repetitions of the finite word $\gamma \in X^*$.

Consider $\gamma^\omega \in X^\omega$, an infinite word on X. The first and important clue related to the action of the states of the automaton $\mathcal{H}$ is that $\gamma \cdot b$ must begin with a 0, regardless of whether its first letter is 0 or 1. This happens because, by the definition of $\mathcal{H}$, 0 is the output of all elements of X by the state b. Thus, we can write $\gamma \cdot b = 0\beta$, with $\beta \in X^\omega$. Using the definition of $\mathcal{H}$, mind that

$$(0\beta) \cdot a = 0(\beta \cdot b) \tag{1.15}$$

and

$$(0\beta) \cdot b = 0(\beta \cdot b). \tag{1.16}$$

So, by relations (1.15) and (1.16), we realize that

$$\begin{aligned}(0\beta) \cdot a = (0\beta) \cdot b &\Longleftrightarrow \gamma \cdot b \cdot a = \gamma \cdot b \cdot b \\ &\Longleftrightarrow \gamma \cdot ba = \gamma \cdot b^2.\end{aligned}$$

This holds for all $\gamma \in X^\omega$; hence, $ba = b^2$ in the semigroup $\mathcal{S}(\mathcal{H})$. Due to this relation, all the occurences of ba in words of $\mathcal{S}(\mathcal{H})$ can be replaced by b^2 so every element of this semigroup can be written as a product of the form $a^i b^j$, with $i, j \in \mathbb{N}$.

At this moment, our intention is to show that every element of $\mathcal{S}(\mathcal{H})$ can be uniquely expressed in this way. In the first place, observe that, for $k, n \in \mathbb{N}$, $i, j \in \mathbb{N} \cup \{0\}$ and $n > j$, the following relations hold:

$$\begin{aligned} 0^k 1^\omega \cdot a &= 0(0^{k-1} 1^\omega \cdot b) \\ &= 00^{k-1}(1^\omega \cdot b) \\ &= 0^k 0(1^\omega \cdot a) \\ &= 0^{k+1} 1^\omega; \end{aligned} \tag{1.17}$$

$$\begin{aligned} 0^k 1^\omega \cdot b &= 0^k(1^\omega \cdot b) \\ &= 0^k 0(1^\omega \cdot a) \\ &= 0^{k+1} 1^\omega; \end{aligned} \tag{1.18}$$

$$\begin{aligned} 1^n 0^\omega \cdot a &= 1^n(0^\omega \cdot a) \\ &= 1^n 0(0^\omega \cdot b) \\ &= 1^n 00^\omega \\ &= 1^n 0^\omega; \end{aligned} \tag{1.19}$$

$$\begin{aligned} 1^n 0^\omega \cdot b &= 0(1^{n-1} 0^\omega \cdot a) \\ &= 01^{n-1}(0^\omega \cdot a) \\ &= 01^{n-1} 0^\omega. \end{aligned} \tag{1.20}$$

Looking at the automaton $\mathcal{H}$, we note that an input 0 never becomes an output 1; then, the language accepted by $\mathcal{H}$ consists on words which have no more than one occurrence of 01. This means that, to find the result we want, it is enough to analyze the action of $a^i b^j$ on words of the form 01^ω and $1^n 0^\omega$.

Using relations (1.17) and (1.18), we find

$$\begin{aligned} 01^\omega \cdot a^i b^j &= (01^\omega \cdot a) \cdot a^{i-1} b^j \\ &= 0^2 1^\omega \cdot a^{i-1} b^j \\ &= 0^{i+1} 1^\omega \cdot b^j \\ &= 0^{i+j+1} 1^\omega. \end{aligned} \tag{1.21}$$

The last two equalities of the relation above come from the fact that, by induction, $01^\omega \cdot a^i = 0^{i+1} 1^\omega$ and $0^k 1^\omega \cdot b^j = 0^{j+k} 1^\omega$ for $k \in \mathbb{N}$ and $i, j \in \mathbb{N} \cup \{0\}$.

On the other hand, using relations (1.19) and (1.20), we get

$$\begin{aligned} 1^n0^\omega \cdot a^ib^j &= (1^n0^\omega \cdot a) \cdot a^{i-1}b^j \\ &= 1^n0^\omega \cdot a^{i-1}b^j \\ &= 1^n0^\omega \cdot b^j \\ &= (1^n0^\omega \cdot b) \cdot b^{j-1} \\ &= 01^{n-1}0^\omega \cdot b^{j-1} \\ &= 0^j1^{n-j}0^\omega. \end{aligned} \tag{1.22}$$

As in the previous case, we used two equalities which are also proved by induction: $1^n0^\omega \cdot a^i = 1^n0^\omega$ and $1^n0^\omega \cdot b^j = 0^j1^{n-j}0^\omega$.

Then, suppose that $a^ib^j = a^kb^l$, with $i, j, k, l \in \mathbb{N}$. By (1.21) and (1.22), one obtains $0^{i+j+1}1^\omega = 0^{k+l+1}1^\omega$ and $0^j1^{n-j}0^\omega = 0^l1^{n-l}0^\omega$, which implies that $i+j+1 = k+l+1$ and $j = l$; consequently, $i = k$. This guarantees the uniqueness of the product expressing every element of $\mathcal{S}(\mathcal{H})$. Since we have been able to express all elements in a unique normal form, we have that the semigroup $\mathcal{S}(\mathcal{H})$ generated by the automaton $\mathcal{H}$ is presented by $Sg\,\langle a, b \mid ba = b^2\rangle$ (this notation means the semigroup generated by the letters a, b and with the single relation $ba = b^2$).

Automaton groups and automaton semigroups are also known as *self-similar groups* and *self-similar semigroups*, respectively (see [16]).

1.3.2 Wreath product and wreath recursion

Here we introduce wreath recursions and show how they are important for some calculations and for defining automata by such relations. In order to understand what wreath recursions are, we remind the definition of wreath product since this concept is crucial to us. The notion of wreath recursion gives a convenient language and notation for automorphisms of the tree X^*.

Then, before defining the wreath product, one needs to define the concept of a semidirect product of groups.

Definition 1.3.16. A group G is called a *split extension* of a group H by a group F if $H \lhd G$ and G contains a subgroup F_1 such that $F_1 \simeq F$, $H \cap F_1 = \{e\}$ and $HF_1 = G$. Alternatively, by this construction, one says that G is a *semidirect product* of H by F. The notation is $G = H \rtimes F$.

Definition 1.3.17. Let H be a group and let X be a finite set such that $H \rightharpoondown X$ (right action) by permutations; further, let G be an arbitrary group. The *(permutational) wreath product* between G and H, denoted by $G \wr H$, is the semi-direct product $G^X \rtimes H$, in which H acts on the direct power $G^X = G \times \cdots \times G$ ($|X|$ times) by permuting the direct factors.

Each element of the wreath product $G \wr H$ can be written in the form gh, with $g \in G^X$ and $h \in H$. Fixing some indexing $\{x_1, \cdots, x_d\}$ of the set X provides a more suitable notation for g: we write $g = (g_1, \cdots, g_d)$ for $g_i \in G$ for all $i = 1, \cdots, d$; we say that g_i is the coordinate of g, corresponding to x_i.

Two elements $(g_1, \cdots, g_d)\alpha, (f_1, \cdots, f_d)\beta \in G \wr H$ are multiplied according to the following rule:

$$(g_1, \cdots, g_d)\alpha(f_1, \cdots, f_d)\beta = (g_1 f_{\alpha(1)}, \cdots, g_d f_{\alpha(d)})\alpha\beta, \tag{1.23}$$

with $g_i, f_i \in G$ and $\alpha, \beta \in H$; furthermore, $\alpha(i)$ is the image of i under the action of α (the index satisfying $\alpha(x_i) = x_{\alpha(i)}$).

In the previous pages we learned that an automaton can generate a group or a semigroup by its states which, in turn, generate endomorphisms of the tree X^*. Now we explain that, conversely, given a rooted endomorphism of the tree X^*, it is possible to create an initial automaton whose action on X^* matches exactly that of a given endomorphism. First, we need to define the section of a rooted tree endomorphism at a vertex of X^*; this is done based on the upcoming construction.

Consider $g : X^* \to X^*$, an endomorphism of the tree X^*, let $x \in X$ be a letter and let $w \in X^*$ be a word. Then, the equality

$$g(xw) = g(x)w' \tag{1.24}$$

holds, for some $w' \in X^*$. The application $g|_x : X^* \to X^*$, defined by

$$g|_x(w) = w', \tag{1.25}$$

with w' as in (1.24), is also a rooted tree endomorphism of X^*.

Definition 1.3.18. A *section* of a tree endomorphism $g : X^* \to X^*$ at vertex x is the endomorphism $g|_x : X^* \to X^*$ defined by (1.25).

Definition 1.3.18 can be extended to words $x_1 \cdots x_n \in X^*$ in the following manner: we define

$$g|_{x_1 \cdots x_n} = g|_{x_1} \cdots g|_{x_n}.$$

Now, given $g : X^* \to X^*$ rooted tree endomorphism of X^*, we construct an initial automaton $\mathcal{A}(g)$ whose action coincides with the one defined by g. The set Q of states of the automaton $\mathcal{A}(g)$ is defined by $Q = \{g|_w \mid w \in X^*\}$, in which $g|_w$

represent different sections of g at the vertices w of the tree X^*. The transition function $\pi : Q \times X \to Q$ and the transition function $\lambda : Q \times X \to X$ are defined by

$$\pi(g|_w, x) = g|_{wx}$$
$$\lambda(g|_w, x) = g|_w(x).$$

Notice that the initial automaton $\mathcal{A}(g)$ has infinitely many states, since there is a state for each word $w \in X^*$.

From now on, we will use the following convention: if g and h are elements of some (semi)group acting on the set X and $x \in X$, then

$$gh(x) = h(g(x)). \tag{1.26}$$

Taking into consideration the convention (1.26), we can compute sections of any element g of an automaton semigroup: in case that $g = g_1 \cdots g_n$ and $w \in X^*$, then we have

$$g|_w = g_1|_w \cdot g_2|_{g_1(w)} \cdots g_n|_{g_1 \cdots g_{n-1}(w)}. \tag{1.27}$$

Definition 1.3.19. Let g be an automorphism of the tree X^*. Then, its *portrait* is the tree X^* in which every vertex $w \in X^*$ is labeled by $\sigma_w \in \mathrm{Sym}(X)$ that represents the action of $g|_w$ on X. The *depth* of a portrait is the number of the drawn levels of it.

Note that if $|X| = 2$ then we just need to distinguish the vertices whose action $g|_w$ is non-trivial. Also, we observe that the portrait determines uniquely the automorphism g because, given a word $w = x_1 \cdots x_n$, we have

$$g(w) = g(x_1 \cdots x_n) = g(x_1)g|_{x_1}(x_2)g|_{x_1x_2}(x_3) \cdots g|_{x_1 \cdots x_{n-1}}(x_n).$$

In terms of sections, V. Nekrashevych gives in [16] an equivalent and convenient definition of automaton groups:

Definition 1.3.20. For X finite set, an automorphism group G of the rooted tree X^* is an *automaton group* (*self-similar group*) if for every $g \in G$ and $w \in X^*$ we have $g|_w \in G$.

Proposition 1.3.21. Let G be an automaton group. Then, for any G there exists a natural embedding $\Psi : G \hookrightarrow G \wr \mathrm{Sym}(X)$ defined by

$$\Psi(g) = (g_1, \cdots, g_d)\lambda_g, \tag{1.28}$$

in which $g \in G$, $(g_1, \cdots, g_d)\lambda_g \in G \wr \mathrm{Sym}(X) = G \times \cdots \times G \rtimes \mathrm{Sym}(X)$, $g_1, \cdots, g_d$ are the sections of g at the vertices of the first level (the cardinality of X is d) and, finally, λ_g is a permutation of X induced by the action of g on the first level of the tree X^* (note that $\lambda_g = (\lambda_g(1), \cdots, \lambda_g(d))$).

In general, the embedding defined by (1.28) exists also for automaton semigroups; however, since we will work only with groups throughout this study, it is not necessary to cover the general case. In Section 1.3.5 we will show why the embedding (1.28) is useful to define automaton groups. We will use the same notation as [18]: if $\Psi(g) = (g_1, \cdots, g_d)\lambda_g$, then we write $g = (g_1, \cdots, g_d)\lambda_g$.

Definition 1.3.22. The description of the action of each g in an automaton group G, denoted by $g = (g_1, \cdots, g_d)\lambda_g$, is called a *wreath recursion* defining G.

One relevant comment concerning a wreath recursion consists of saying that it is a reasonable mechanism used to compute sections of products of automorphisms. Let $g = (g_1, \cdots, g_d)\lambda_g$ and $h = (h_1, \cdots, h_d)\lambda_h$ be two rooted automorphisms defined by wreath recursion. By the relation given in (1.23), we obtain

$$\begin{aligned} gh &= (g_1, \cdots, g_d)\lambda_g(h_1, \cdots, h_d)\lambda_h \\ &= (g_1 h_{\lambda_g(1)}, \cdots, g_d h_{\lambda_g(d)})(\lambda_g\lambda_h). \end{aligned} \tag{1.29}$$

Then, using (1.29) and the notation of [18], we will be able to prove the Proposition 1.3.21.

Proof of Proposition 1.3.21. Let g and h be automorphisms of the automaton group G. We want to prove that Ψ, as defined before, is an injective homomorphism.

In fact, Ψ is an homomorphism, since

$$\begin{aligned} \Psi(g)\Psi(h) &= (g_1, \cdots, g_d)\lambda_g(h_1, \cdots, h_d)\lambda_h \\ &= (g_1 h_{\lambda_g(1)}, \cdots, g_d h_{\lambda_g(d)})(\lambda_g\lambda_h) \\ &= ((gh)_1, \dots, (gh)_d)\lambda_{gh} \\ &= \Psi(gh). \end{aligned}$$

Also, remembering convention (1.26), since the permutation λ_{gh} is clearly given by applying first λ_g and then λ_h, and similarly, we have that the automorphism $(gh)_i$ is computed by first applying g_i and then applying $h_{\lambda_g(i)}$. Also, Ψ is injective, since if we have $\Psi(g) = (e_1, \cdots, e_d)e_{\mathrm{Sym}(X)}$ (we denoted by $e_\bullet$ the identities appearing in this relation), then g itself is the identity automorphism, since its sections and the permutation are trivial. □

As V. Nekrashevych defines automaton groups by sections of automorphisms (see Definition 1.3.20), in [16] he gives a definition for automaton groups based on the elements of Proposition 1.3.21.

Definition 1.3.23. An automorphism group $G \neq \operatorname{Aut} X^*$ is *self-similar* if it satisfies $G \leqslant G \wr \operatorname{Sym}(X)$. (Recall that $G \leqslant G \wr \operatorname{Sym}(X)$ means that $\Psi(G) \leqslant G \wr \operatorname{Sym}(X)$, in which Ψ is an application $\Psi : \operatorname{Aut} X^* \to \operatorname{Aut} X^* \wr \operatorname{Sym}(X)$.)

Example 1.3.24. Consider the automaton $\mathcal{H}$ explored in Example 1.3.8 and Example 1.3.15 whose Moore diagram is given by

$\mathcal{H}$:

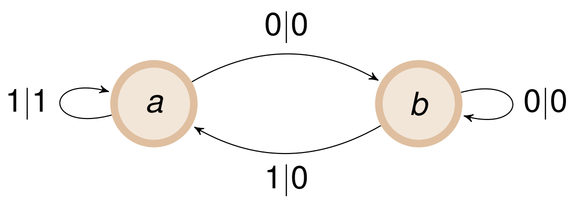


Figure 9 – Moore diagram of the automaton of the Example 1.3.8 and Example 1.3.15.

By virtue of the latter considerations, we get that the wreath recursions corresponding to the states of the automaton $\mathcal{H}$ are

$$\begin{aligned} a &= (b, a)\mathbb{1} = (b, a) \\ b &= (b, a)\lambda, \end{aligned}$$

in which $\mathbb{1}$ is the identity map $\mathbb{1} : X \to X$ ($X = \{0, 1\}$) and $\lambda : X \to X$ is defined by $\lambda(0) = \lambda(1) = 0$. Notice that λ is not a bijection, but we are now dealing with an automaton semigroup in the current example.

Using the product given in (1.29), one gets the following relations:

$$\begin{aligned} a^2 &= (b, a)\mathbb{1}(b, a)\mathbb{1} \\ &= (bb_{\mathbb{1}}, aa_{\mathbb{1}})\mathbb{1}\mathbb{1} \\ &= (b^2, a^2); \end{aligned}$$

$$\begin{aligned} ab &= (b, a)\mathbb{1}(b, a)\lambda \\ &= (bb_{\mathbb{1}}, aa_{\mathbb{1}})\mathbb{1}\lambda \\ &= (b^2, a^2)\lambda; \end{aligned}$$

$$\begin{aligned} ba &= (b, a)\lambda(b, a)\mathbb{1} \\ &= (bb_{\lambda}, aa_{\lambda})\lambda\mathbb{1} \\ &= (b^2, ab)\lambda; \end{aligned} \tag{1.30}$$

$$\begin{aligned} b^2 &= (b, a)\lambda(b, a)\lambda \\ &= (bb_{\lambda}, aa_{\lambda})\lambda\lambda \\ &= (b^2, ab)\lambda, \end{aligned} \tag{1.31}$$

in which a_λ and b_λ denote the endomorphisms induced by the action of λ. Note that (1.30) and (1.31) corroborate the deduction of $ba = b^2$, done in the Example 1.3.15.

1.3.3 Dual automata

The definition of an automaton is symmetric because, given any finite automaton, one can construct a new automaton by interchanging the state set with the alphabet as well as interchanging the output function with the transition function. Such construction determines a *dual automaton*.

Definition 1.3.25. Let $\mathcal{A} = (Q, X, \pi, \lambda)$ be a finite automaton. The *dual automaton* of the automaton $\mathcal{A}$ is a finite automaton $\widehat{\mathcal{A}} = (X, Q, \widehat{\lambda}, \widehat{\pi})$ such that, for any $x \in X$ and $q \in Q$, we have

$$\hat{\lambda}(x, q) = \lambda(q, x)$$
$$\hat{\pi}(x, q) = \pi(q, x).$$

Notice that, by the definition above, the dual of the dual of an automaton $\mathcal{A}$ is exactly $\mathcal{A}$. If the alphabet X is larger than the set of states Q, then it may be more convenient to draw the *dual Moore diagram*, which is the Moore diagram of the dual automaton $\widehat{\mathcal{A}}$, instead of the usual Moore diagram of $\mathcal{A}$.

Since the alphabet X and the state set Q are switched in the automaton $\widehat{\mathcal{A}}$, the arrows of the dual Moore diagram of $\widehat{\mathcal{A}}$ show the actions of elements of the alphabet X while the labels show the transitions between letters of the the states of Q. More accurately, for every $(x, q) \in X \times Q$ there exists an arrow starting from x, ending on $\widehat{\pi}(x, q)$ and labeled by $q|\widehat{\lambda}(x, q)$.

Despite the fact that it may sometimes be more useful to work with the dual Moore diagram (rather than working with the Moore diagram itself), we need to be a little careful because some properties are not carried over from the automaton to its dual. For example, it is possible to find an automaton that is invertible while its dual is not and vice versa. The next example shows that there exists the possibility of having a dual automaton that is invertible while the original automaton is not.

Example 1.3.26. Consider the automaton $\mathcal{D}$ analyzed in Examples 1.3.6 and 1.3.7.

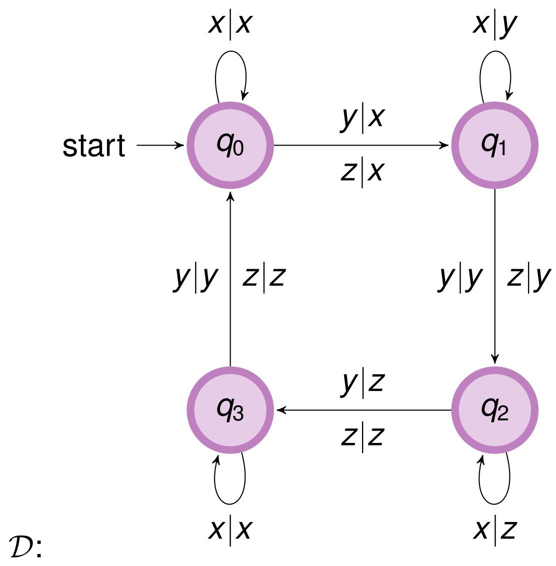


Figure 10 – Moore diagram of the initial automaton of Examples 1.3.6 and 1.3.7.

Then, applying Definition 1.3.25, one obtains the dual automaton $\hat{\mathcal{D}}$ whose Moore diagram is given by

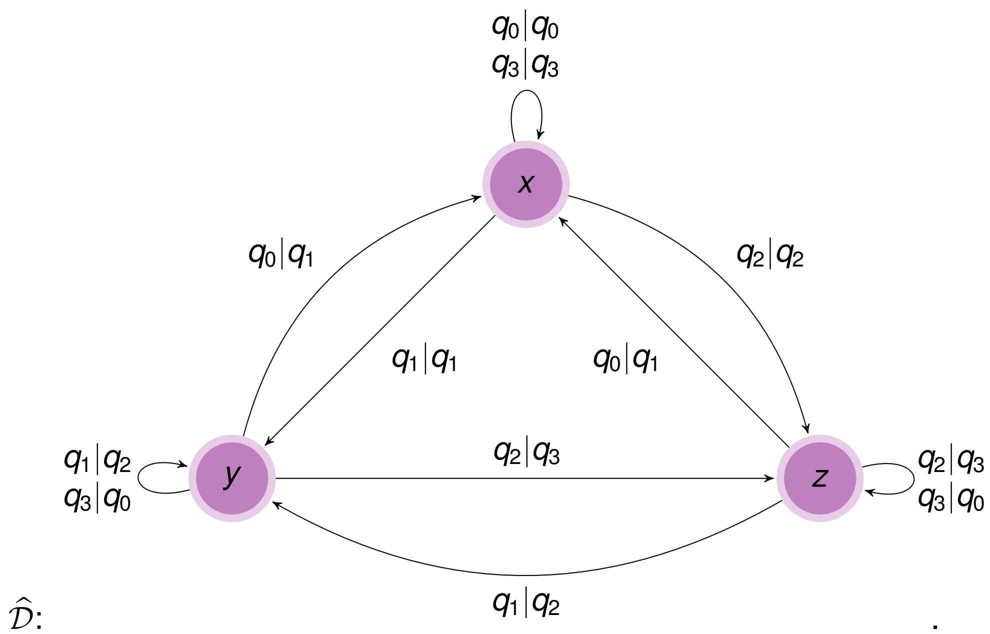


Figure 11 – Dual Moore diagram of the automaton $\mathcal{D}$.

Note that the states of the automaton $\mathcal{D}$ do not induce permutations while the states of the automaton $\widehat{\mathcal{D}}$ do and so the automaton $\widehat{\mathcal{D}}$ is invertible even though the automaton $\mathcal{D}$ is not invertible.

Example 1.3.27. Consider again the automaton $\mathcal{H}$ of Examples 1.3.8, 1.3.15 and 1.3.24 whose Moore diagram is given by

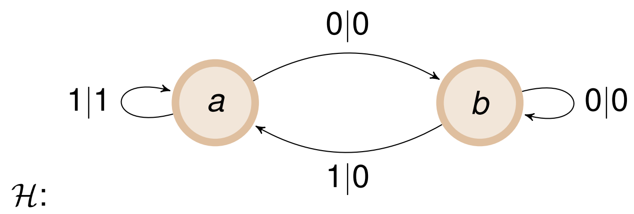


Figure 12 – Moore diagram of the automaton of Examples 1.3.8, 1.3.15 and 1.3.24.

Hence, the dual automaton of $\mathcal{H}$ is the automaton $\widehat{\mathcal{H}}$ whose Moore diagram is

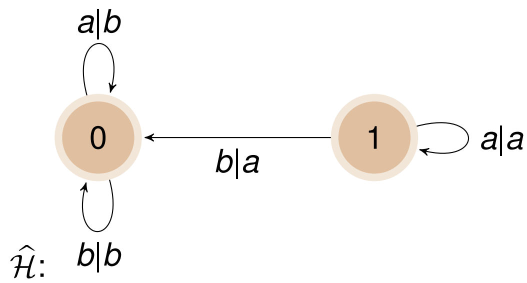


Figure 13 – Dual Moore diagram of the automaton $\mathcal{H}$.

Since the states of an automaton $\mathcal{A}$ generate a semigroup, the states of the dual automaton $\widehat{\mathcal{A}}$ of $\mathcal{A}$ generate a new semigroup as well; these two semigroups are related to each other, as the next definition shows.

Definition 1.3.28. Let G be an automaton semigroup generated by the states of an automaton $\mathcal{A}$. The *dual semigroup* of G, denoted by $\widehat{G}$, is a semigroup generated by the states of the dual automaton $\widehat{\mathcal{A}}$.

A relevant class of automata which involves dual automata is the class of bireversible automata.

Definition 1.3.29. $\mathcal{A}$ is a *bireversible* automaton if $\mathcal{A}$ itself, its dual $\widehat{\mathcal{A}}$ and the dual of $\mathcal{A}^{-1}$ are all invertible.

Concerning the next proposition, we denote by the same symbol both the element of a free monoid and its image under canonical epimorphism onto its correspondent semigroup.

Proposition 1.3.30. Consider G, an automaton semigroup generated by a finite set S such that $G \curvearrowright X^*$ and consider $\widehat{G}$, a dual semigroup to G acting on S^*. Then,

1. For any $g \in G$ and $w \in X^*$, $g|_w = w(g)$ in G.
2. For any $g \in S^*$ and $w \in \widehat{G}$, $w|_g = g(w)$ in $\widehat{G}$.

Proof. See article [21]. □

1.3.4 Automata generating C_2 and $C_2 * C_2$

In the introduction of this dissertation we asserted the theorems that will be proved in the next chapters, all of them related to automata generating free products of groups of order 2. One should note, however, that such theorems do give the answers to the question related to automata generating free products of n copies of C_2 for $n \geqslant 4$ only. What about the cases C_2, $C_2 * C_2$ and $C_2 * C_2 * C_2$?

The first automaton which opened ways to the study of automata generating a free product of groups of order 2 was the Bellaterra automaton $\mathcal{B}_3$ that generates the free product $C_2 * C_2 * C_2$. Such automaton, along with the others from the Bellaterra family, will be introduced in Section 1.3.5.

The groups C_2 and $C_2 * C_2$ are also automata groups. It can be easily seen that the automaton with one state only in which such state acts as σ on the alphabet $\{0, 1\}$ generates the group C_2 and so the Moore diagram of such automaton is given by

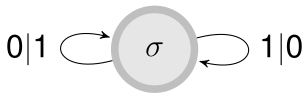


Figure 14 – Moore diagram of the automaton that generates C_2.

We have that $C_2 * C_2$ is isomorphic to the dihedral group D_∞. In the article [23], it is cited that the automaton which generates $C_2 * C_2 \simeq D_\infty$ has the following Moore diagram:

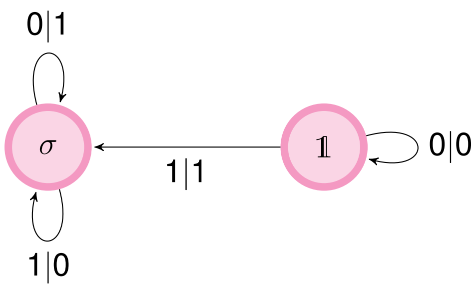


Figure 15 – Moore diagram of the automaton generating $C_2 * C_2 \simeq D_\infty$.

Another interesting group generated by the states of a 2-state automaton is the infinite cyclic group $\mathbb{Z}$; this group is isomorphic to the group generated by the automaton whose Moore diagram is exactly

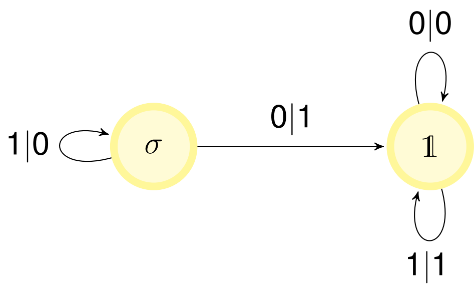


Figure 16 – Moore diagram of the automaton generating the infinite cyclic group $\mathbb{Z}$.

The automata groups described above are some of the few ones that are generated by automata on two states over an alphabet of two letters, as the next theorem shows:

Theorem 1.3.31. The only groups generated by automata on two states over an alphabet of two letters are

- the trivial group;
- the group C_2;
- the Klein group $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$;
- the infinite cyclic group $\mathbb{Z}$;
- the infinite dihedral group D_∞ ($\simeq C_2 * C_2$);
- the lamplighter group $\bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z}$.

Proof. See [23], page 7. □

1.3.5 Bellaterra automata

One family of automata which deserves our attention, since it composes the answer to the main problem of this study, is the Bellaterra automata family. The first Bellaterra automaton ($\mathcal{B}_3$) was discovered while classifying all bireversible 3-state automata over a 2-letter alphabet so this automaton is a component of the class of bireversible automata, which appears to be a natural source for automata generating free products and free groups. Also, the automaton $\mathcal{B}_3$ is a nice and seemingly simple self-similar example; that is, all states of such automaton generate a group.

The first important result about the Bellaterra automata family was proved by Y. Muntyan and D. Savchuk; it was about the group generated by $\mathcal{B}_3$.

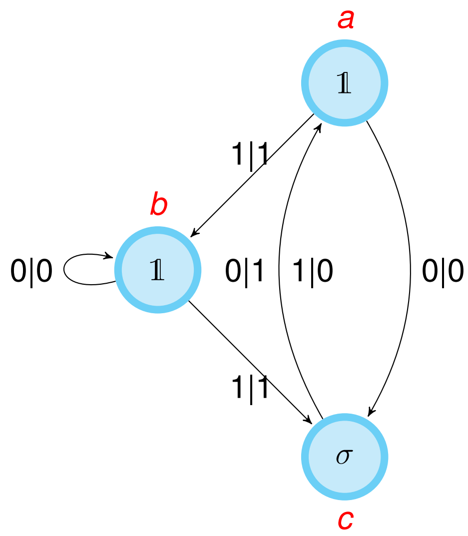


Figure 17 – Moore diagram of the Bellaterra automaton $\mathcal{B}_3$.

Theorem 1.3.32 (Muntyan, Y. and Savchuk, D.)**.** The automaton $\mathcal{B}_3$ generates a group isomorphic to $C_2 * C_2 * C_2$.

Proof. See [6] or [16] (Nekrashevych's book gives a more detailed proof, on page 24). $\square$

Considering the alphabet $X = \{0, 1\}$, we have that the functions a, b and c defined by the states of the Bellaterra automaton $\mathcal{B}_3$ act on the set X^* of finite words over X. Those transformations are uniquely determined by wreath recursion, namely:

$$\begin{aligned} a &= (c, b), \\ b &= (b, c), \\ c &= (a, a)\sigma. \end{aligned} \tag{1.32}$$

Remark 1.3.33. $t = (t_0, t_1)$ means that, according to the relation (1.32), for any $w \in X^*$, we have $t(0w) = 0t_0(w)$ and $t(1w) = 1t_1(w)$; furthermore, for any $w \in X^*$, $t = (t_0, t_1)\sigma$ means that $t(0w) = 1t_0(w)$ and $t(1w) = 0t_1(w)$.

Notice that the convention (1.26) works fine with the notation of the wreath recursion defined previously. In fact, writing $t = (t_0, t_1)\sigma$, for example, corresponds to first acting on X^* by (t_0, t_1) and then permuting the first letter of a word $w \in X^*$ by the permutation σ, regarded as a permutation of X^*.

Starting from the Bellaterra automaton $\mathcal{B}_3$, it is possible to produce a family of bireversible automata such that all of their states define involutions (applications which are their own inverses); this setup is relatively simple: looking at the automaton $\mathcal{B}_3$, we insert new states on the path from the state c to the state a.

Now we introduce the Bellaterra automaton $\mathcal{B}_4$, which has 4 states. Such automaton has great importance to this study since one of the main results is related to it (see Theorem 2.1.1). The Moore diagram of $\mathcal{B}_4$ is given by

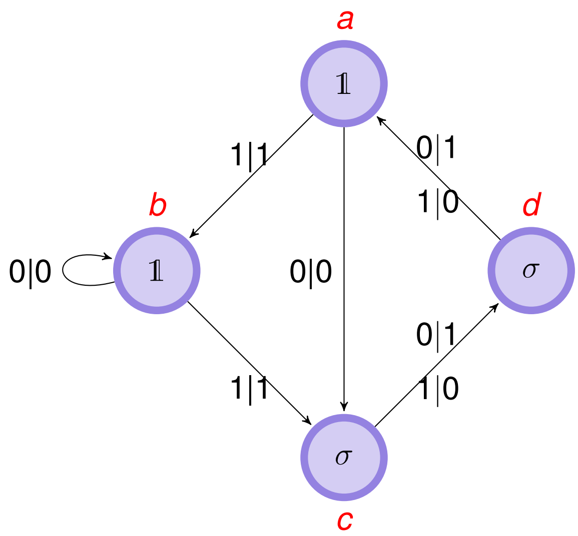


Figure 18 – Moore diagram of the Bellaterra automaton $\mathcal{B}_4$.

The states of $\mathcal{B}_4$, also, are defined by the following wreath recursion:

$$\begin{aligned} a &= (c, b), \\ b &= (b, c), \\ c &= (d, d)\sigma, \\ d &= (a, a)\sigma. \end{aligned} \tag{1.33}$$

It is worth reminding that Remark 1.3.33 also works for the relation (1.33).

Still having in mind the idea of inserting new states on the path from the state c to the state a of the automaton $\mathcal{B}_3$, we can give rise to the family of the Bellaterra automata, establishing the wreath recursion that defines each Bellaterra automaton denoted by $\mathcal{B}^{(n)}$, $n > 4$.

Then, $\mathcal{B}^{(n)}$ ($n > 4$) is an n-state automaton whose Moore diagram representing this Bellaterra automaton is the following:

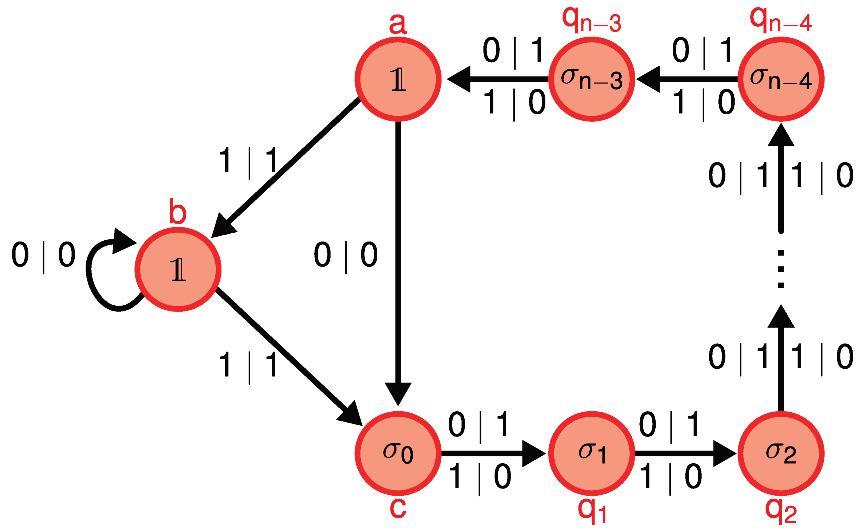


Figure 19 – Moore diagram of the Bellaterra automaton $\mathcal{B}^{(n)}$ (courtesy of Altair Santos).

Also, the automaton $\mathcal{B}^{(n)}$ is defined by the wreath recursion

$$\begin{aligned} a &= (c, b), \\ b &= (b, c), \\ c &= (q_1, q_1)\sigma_0, \\ q_i &= (q_{i+1}, q_{i+1})\sigma_i, \quad i = 1, \cdots, n-4, \\ q_{n-3} &= (a, a)\sigma_{n-3}. \end{aligned} \tag{1.34}$$

All the $\sigma_i \in \mathrm{Sym}(\{0, 1\})$ are chosen arbitrarily. As well as in the cases of $\mathcal{B}_3$ and $\mathcal{B}_4$, the assertions of Remark 1.3.33 are valid for the equalities of the wreath recursion (1.34) defining $\mathcal{B}^{(n)}$.

Remark 1.3.34. Note that, in the previous figure representing the Moore diagram of the automaton $\mathcal{B}^{(n)}$, we chose conveniently the permutations $\sigma_i \in \mathrm{Sym}(\{0, 1\})$, with $i = 1, \cdots, n-4$, in order to show a better representation of such automaton. We emphasize, again, that in the Bellaterra automaton $\mathcal{B}^{(n)}$ all the permutations σ_i are chosen arbitrarily.

Researchers believe that the Bellaterra automata family has a significant property related to free product of groups: for them, each automaton $\mathcal{B}^{(n)}$, $n \geqslant 3$, having at least one of the σ_i nontrivial, generates the free product of groups of order 2. M. Vorobets and Y. Vorobets, in [22], found the first result endorsing such conjecture: they showed that if the number of the states of the automaton is odd and $\sigma_i = (0\ 1)$ for all $i = 0, \cdots, n-3$, then the conjecture holds. If the automata has an even number of states and the number of nontrivial σ_i is odd, then the supposition is also true; it was proved in a subsequent paper submitted by the same authors and B. Steinberg (see [20]).

We intend, in this dissertation, to prove that any n-state Bellaterra automaton ($n \geqslant 4$) satisfying $\sigma_0 = (0\ 1)$ and $\sigma_{n-3} = (0\ 1)$ generates the free product of groups of order 2. The next two chapters are devoted to the proof of the case $n = 4$ (see Chapter 2) and to the proof of the general case $n > 4$ (see Chapter 3).

2 Automaton generating $C_2 * C_2 * C_2 * C_2$

> "The art of doing mathematics consists in finding that special case which contains all the germs of generality."
>
> — David Hilbert, 1862–1943

This chapter is dedicated to the particular case of the Bellaterra automaton $\mathcal{B}_4$ and to the proof that it generates a free product of four groups of order 2. Such case plays an important role in our work since the steps used in the construction of the main proof serve as a way to the next chapter's main proof which gives a "generalization" for the case of the automata $\mathcal{B}^{(n)}$. First, we give some information about the automaton $\mathcal{B}_4$ itself and then we establish some results converging to the proof that the group generated by the states of $\mathcal{B}_4$ is isomorphic to $C_2 * C_2 * C_2 * C_2$. We follow the exposition from [18].

2.1 Considerations about the automaton $\mathcal{B}_4$

First of all, we shall recall the 4-state Bellaterra automaton $\mathcal{B}_4$. In Section 1.3.5 it was shown that its Moore diagram is given by

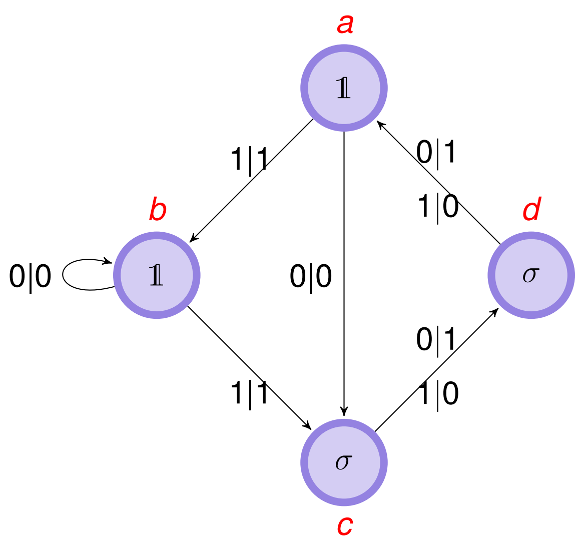


Figure 20 – Moore diagram of the Bellaterra automaton $\mathcal{B}_4$.

We recall, also, that its transition and output functions are given by the wreath recursion defined on (1.33).

Let $\mathcal{G}$ be the group generated by the states of the Bellaterra automaton $\mathcal{B}_4$. The following theorem is the main result of this chapter.

Theorem 2.1.1 (D. Savchuk, Y. Vorobets, [18])**.** The group $\mathcal{G}$ is isomorphic to the free product $C_2 * C_2 * C_2 * C_2$.

However, before proving Theorem 2.1.1, we need to prove several preliminary facts in order to obtain a structure for the main proof. All of them are proved in the next section.

2.2 Construction of the proof of the Theorem 2.1.1

In the first place, observe that the inverse of the Bellaterra automaton $\mathcal{B}_4$ is the automaton itself; that is, we have that $\mathcal{B}_4^{-1} = \mathcal{B}_4$. In fact, to create the inverse automaton of an invertible automaton $\mathcal{A}$ by using the Moore diagram of $\mathcal{A}$, we simply switch the input and the output of each edge; however, in the case of $\mathcal{B}_4$, switching input and output of each edge does not change any information contained on it, since all states of such automaton induce involutions (applications which are thir own inverses).

The fact that $\mathcal{B}_4^{-1} = \mathcal{B}_4$ helps us to show that the automaton $\mathcal{B}_4$ is bireversible: it suffices to show that the dual automaton $\widehat{\mathcal{B}_4}$ is invertible. Indeed, the Moore representation of $\widehat{\mathcal{B}_4}$ is given by

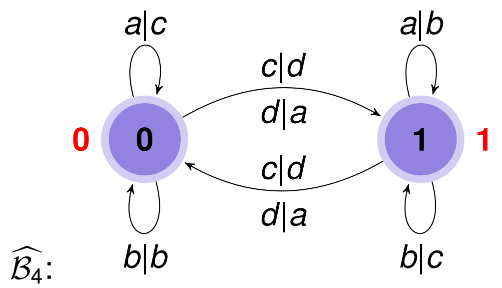


Figure 21 – Moore diagram of the dual automaton of $\mathcal{B}_4$.

Since states **0** and **1** induce permutations on the set $\{a, b, c, d\}$, the dual automaton $\mathcal{B}_4$ is invertible. By switching inputs and outputs of all edges of the Moore diagram of $\widehat{\mathcal{B}_4}$, we find the Moore diagram of its inverse automaton $\widehat{\mathcal{B}_4}^{-1}$ which is given by

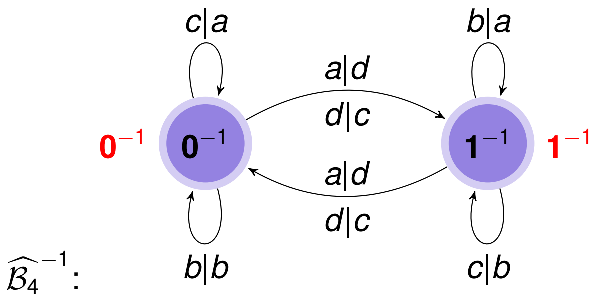


Figure 22 – Moore diagram of the inverse automaton of $\widehat{\mathcal{B}_4}$.

Therefore, the Bellaterra automaton $\mathcal{B}_4$ is bireversible.

The dual automaton $\widehat{\mathcal{B}_4}$ can be represented by a wreath recursion; in such case, it is given by

$$\begin{aligned} \mathbf{0} &= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(acd), \\ \mathbf{1} &= (\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(abcd). \end{aligned} \tag{2.1}$$

Let Γ be the dual group to the group $\mathcal{G}$ (defined in Section 2.1); in other words, Γ is the group generated by the recursions of (2.1). This group acts on a rooted 4-ary tree, denoted by T, whose vertices are labeled by words over the alphabet $\{a, b, c, d\}$. Using the product described on (1.29), we get

$$\begin{aligned} a^2 &= (c, b)\mathbb{1}(c, b)\mathbb{1} \\ &= (c, b)(c, b)\mathbb{1}\mathbb{1} \\ &= (c^2, b^2); \end{aligned} \tag{2.2}$$

$$\begin{aligned} b^2 &= (b, c)\mathbb{1}(b, c)\mathbb{1} \\ &= (b, c)(b, c)\mathbb{1}\mathbb{1} \\ &= (b^2, c^2); \end{aligned} \tag{2.3}$$

$$\begin{aligned} c^2 &= (d, d)\sigma(d, d)\sigma \\ &= (d, d)(d, d)\sigma\sigma \\ &= (d^2, d^2); \end{aligned} \tag{2.4}$$

$$\begin{aligned} d^2 &= (a, a)\sigma(a, a)\sigma \\ &= (a, a)(a, a)\sigma\sigma \\ &= (a^2, a^2). \end{aligned} \tag{2.5}$$

Due to relations (2.2), (2.3), (2.4) and (2.5), we have

$$a^2 = b^2 = c^2 = d^2 = e \tag{2.6}$$

in $\mathcal{G}$, which means that, in fact, a, b, c and d are involutions. One explanation for this fact is that the permutation of a^2, b^2, c^2 and d^2 on the first level corresponds to the identity and by the wreath recursion defining each of these squares, on the subsequent levels they act as the identity as well, as can be seen by checking the entries on the right hand sides of equations (2.2), (2.3), (2.4) and (2.5). Furthermore, recalling that Γ acts on the words over $\{a, b, c, d\}$, we get that the image under any element of Γ of any word containing a^2, b^2, c^2 or d^2 as subwords will also contain a^2, b^2, c^2 or d^2 as subwords. Then, by excluding all words over $\{a, b, c, d\}$ which have a^2, b^2, c^2, d^2 as subwords, we create a subtree $\widehat{T}$ of T that is invariant under the action of Γ and which contains the words on $\{a, b, c, d\}$ not having subwords of the form a^2, b^2, c^2 or d^2. Note that ε, root of $\widehat{T}$, has four children while all other vertices in $\widehat{T}$ have three; one visualizes this on the next figure, in which tree T and subtree $\widehat{T}$ (represented in red) are drawn.

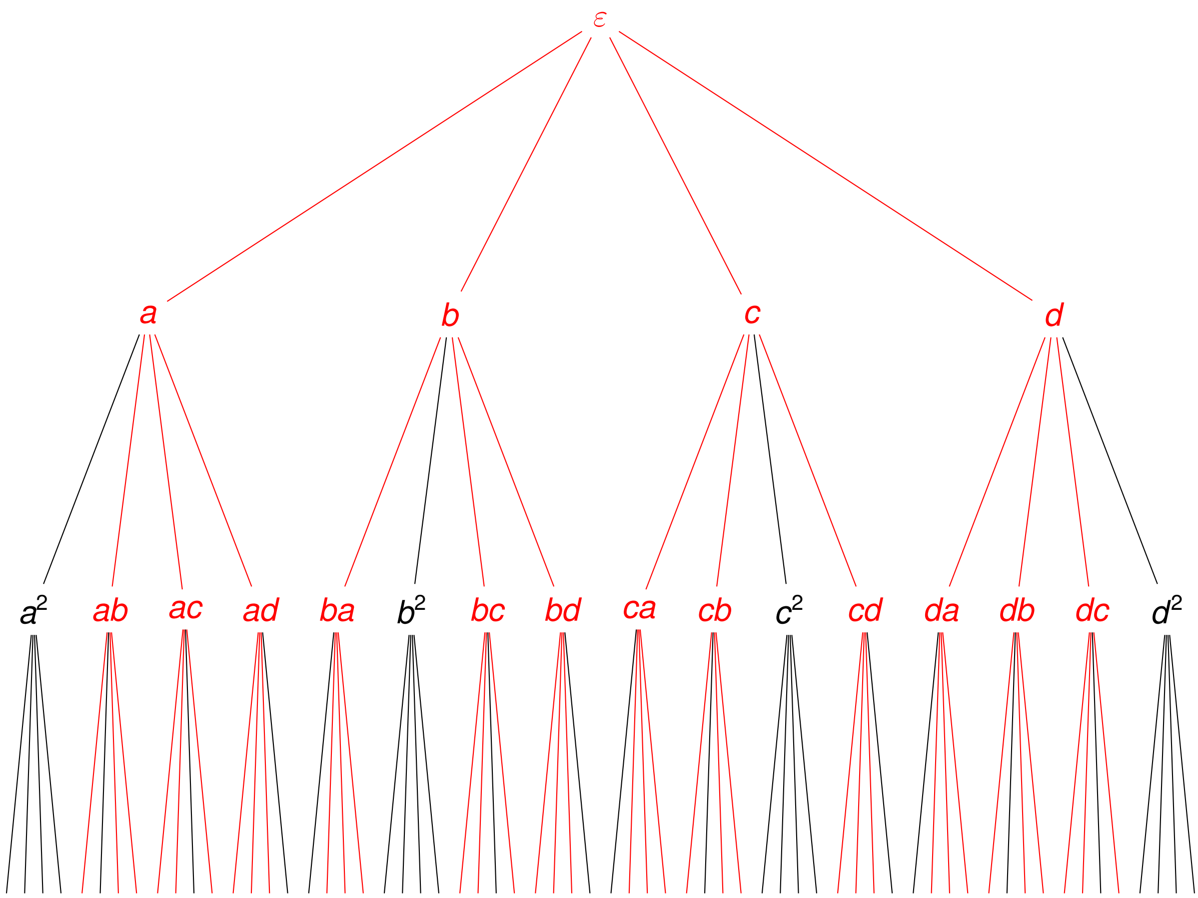


Figure 23 – Tree T and its subtree $\widehat{T}$.

Proposition 2.2.1 (Šunić, Z., [18])**.** Let G be any semigroup generated by a finite automaton and $\widehat{G}$ be its dual semigroup. Then, G is finite if and only if $\widehat{G}$ is finite.

Proof. Remember that the dual of the dual of an automaton $\mathcal{A}$ is the automaton $\mathcal{A}$ itself and so the dual of the dual of group G coincides with G. Then, assuming one direction true (for example, "G finite implies in $\widehat{G}$ finite"), then the other one follows (in our example, replacing G by $\widehat{G}$ assures that "$\widehat{G}$ finite implies in $\widehat{\widehat{G}} = G$ finite") so it is enough to show the implication in only one direction.

Indeed, suppose G is a finite semigroup. Let $v \in \widehat{G}$ be an element of the dual semigroup and let g be a vertex of the tree the semigroup $\widehat{G}$ acts on. By Proposition 1.3.30 we have $v|_g = g(v)$ in $\widehat{G}$; this implies that the number of different sections of v depends on the number of elements of G, and this number is bounded since G is finite. As there are only finitely many different automata with a fixed number of states (elements of G), we conclude that the possibilities for $v \in \widehat{G}$ are limited and so the dual semigroup $\widehat{G}$ is finite. $\square$

Lemma 2.2.2. Let H be the subgroup of the group $\mathcal{G}$ (generated by the states of the Bellaterra automaton $\mathcal{B}_4$) defined by $H = \langle ab, bc, cd, da \rangle$. Then, the quotient group $H \,/\, \mathrm{Stab}_H(2)$ is cyclic of order 4 and the portrait of depth 2 of every element (see Definition 1.3.19 for the definition of portrait) of H must coincide with one of the following portraits:

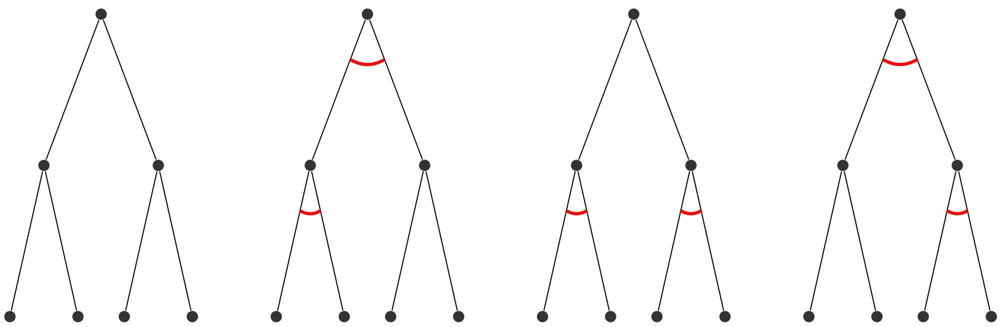


Figure 24 – Possible portraits of elements of H of order 2.

Proof. In order to prove this lemma, we establish how the elements of $H \,/\, \mathrm{Stab}_H(2)$ act on the two first levels of the tree $\{0, 1\}^*$ by showing how the elements of the generator set of H ab, bc, cd, da act; further, we show the portraits of depth 2 of each element of such set in order to guarantee that they match with the ones of the statement of the lemma.

First of all, we represent the elements ab, bc, cd and da by wreath recursions, using the ones of (1.33). Then, we obtain

$$\begin{aligned} ab &= (c, b)\mathbb{1}(b, c)\mathbb{1} \\ &= (c, b)(b, c)\mathbb{1}\mathbb{1} \\ &= (cb, bc); \end{aligned} \tag{2.7}$$

$$\begin{aligned} bc &= (b, c)\mathbb{1}(d, d)\sigma \\ &= (b, c)(d, d)\mathbb{1}\sigma \\ &= (bd, cd)\sigma; \end{aligned} \tag{2.8}$$

$$\begin{aligned} cd &= (d, d)\sigma(a, a)\sigma \\ &= (d, d)(a, a)\sigma\sigma \\ &= (da, da); \end{aligned} \tag{2.9}$$

$$\begin{aligned} da &= (a, a)\sigma(c, b)\mathbb{1} \\ &= (a, a)(b, c)\sigma\mathbb{1} \\ &= (ab, ac)\sigma. \end{aligned} \tag{2.10}$$

Summarizing the results found in (2.7), (2.8), (2.9) and (2.10), we have

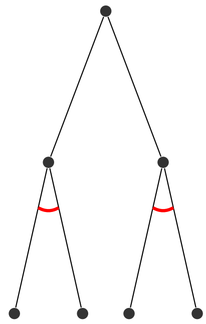
$$\begin{aligned} ab &= (cb, bc), \\ bc &= (bd, cd)\sigma, \\ cd &= (da, da), \\ da &= (ab, ac)\sigma. \end{aligned} \tag{2.11}$$

Let us construct the portrait of depth 2 of the element ab: by relation (2.11), ab acts as $\mathbb{1}$ on the first level of the tree $\{0, 1\}^*$. On the second level, its sections are cb and bc. Since $cb = cd\ da\ ab$ (product of generators of H), we obtain that the wreath recursion defining cb is given by

$$\begin{aligned} cb &= (da, da)(ab, ac)\sigma(cb, bc) \\ &= (da, da)(ab, ac)(bc, cb)\sigma \\ &= (da^2b^2c, da^2c^2b)\sigma \\ &= (dc, db)\sigma \end{aligned}$$

so it acts as σ on the first level (here it corresponds to the second level of ab); also, still by (2.11), bc acts as σ on the first level. Having in mind that the first level of cb and bc

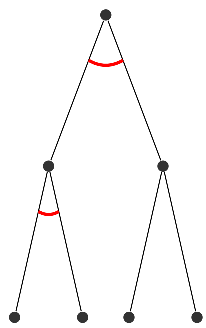
correspond to the second level of *ab*, we conclude that the portrait of depth 2 of *ab* is given by



The construction of the portraits of *bc*, *cd* and *da* is similar to the one of the element *ab*. In the case of *bc*, using (2.11) we get that *bc* acts as σ on the first level of $\{0, 1\}^*$. Because the sections of *bc* are *bd* and *cd* and using the fact that the wreath recursion defining *bd* is

$$\begin{aligned} bd &= bc\ cd \\ &= (bd, cd)\sigma(da, da) \\ &= (bd, cd)(da, da)\sigma \\ &= (ba, ca)\sigma \end{aligned}$$

and $cd = (da, da)$, we conclude that *bd* acts as σ and *cd* acts as $\mathbb{1}$ on the first level which means that *bc* acts as σ on the first half of the second level while it acts as $\mathbb{1}$ on its second half. Thus, the portrait of depth 2 corresponding to the action of *bc* on the first two levels of the tree is

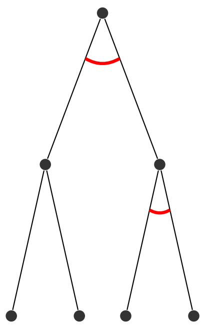


The element $cd = (da, da)$, by (2.11), acts on the first level as $\mathbb{1}$ and acts on the entire second level as σ since $da = (ab, ac)\sigma$ so the portrait of depth 2 of *cd* is essentially the same as the one of *ab*.

Finally, the element $da = (ab, ac)\sigma$ acts as σ on the first level. Due to the relations on (2.11), we get $ab = (cb, bc)$ and

$$\begin{aligned} ac &= ab\ bc \\ &= (cb, bc)(bd, cd)\sigma \\ &= (cd, bd)\sigma; \end{aligned}$$

then, ab acts as $\mathbb{1}$ on the left half while ac acts as σ on the right half of the second level of the tree. This implies that the portrait of depth 2 of da is given by



It can be shown that the product of any two of the portraits of the statement yields another such portrait, showing that $H \,/\, \mathrm{Stab}_H(2)$ is a group of order 4, so we get that these portraits are all of the possible ones. Instead, remembering that we have, by definition, $H \,/\, \mathrm{Stab}_H(2) = \langle ab, bc, cd, da\rangle \,/\, \mathrm{Stab}_H(2)$, we will show that the quotient group $H \,/\, \mathrm{Stab}_H(2) = \langle bc\rangle \,/\, \mathrm{Stab}_H(2)$ is cyclic and has order 4. Then, we analyze the actions of the powers of the element bc.

Note that $bc = (bd, cd)\sigma$; then, we obtain

$$\begin{aligned} (bc)^2 &= (bd, cd)\sigma(bd, cd)\sigma \\ &= (bd, cd)(cd, bd)\sigma\sigma \\ &= (bdcd, cdbd), \end{aligned} \tag{2.12}$$

$$\begin{aligned} (bc)^3 &= (bdcd, cdbd)(bd, cd)\sigma \\ &= (bdcdbd, cdbdcd)\sigma \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} (bc)^4 &= (bdcdbd, cdbdcd)\sigma(bd, cd)\sigma \\ &= (bdcdbd, cdbdcd)(cd, bd)\sigma\sigma \\ &= ((bdcd)^2, (cdbd)^2). \end{aligned} \tag{2.14}$$

Analyzing the actions of these powers of bc on the first two levels of $\{0, 1\}^*$ by constructing the portraits of depth 2 of those elements, we get the following results:

$(bc)^2$ acts as $\mathbb{1}$ on the first level (see (2.12)) and since its sections are *bdcd* and *cdbd*, with wreath recursions respectively given by

$$\begin{aligned} bdcd &= (ba, ca)\sigma(da, da) \\ &= (ba, ca)(da, da)\sigma \\ &= (bada, cada)\sigma \end{aligned}$$

and

$$\begin{aligned} cdbd &= (da, da)(ba, ca)\sigma \\ &= (daba, daca)\sigma, \end{aligned}$$

then $(bc)^2$ acts as σ on the second level of the tree so the portrait of depth 2 corresponding to the action of such element on the first and second levels of $\{0, 1\}^*$ coincides with the portrait of *ab*. Also, $(bc)^3$ acts as σ on the first level (see (2.13)) and, since *bdcdbd* and *cdbdcd* are the sections of $(bc)^3$ defined by

$$\begin{aligned} bdcdbd &= (bada, cada)\sigma(ba, ca)\sigma \\ &= (bada, cada)(ca, ba)\sigma\sigma \\ &= (badaca, cadaba) \end{aligned}$$

and

$$\begin{aligned} cdbdcd &= (daba, daca)\sigma(da, da) \\ &= (daba, daca)(da, da)\sigma \\ &= (dabada, dacada)\sigma, \end{aligned}$$

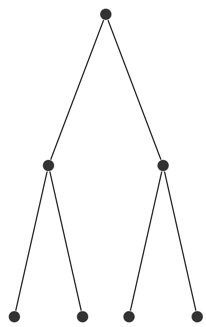
respectively, we infer that the action of $(bc)^3$ on the two first levels of the tree $\{0, 1\}^*$ is the same as the action of the element *da* on the considered levels; therefore, its portraits of depth 2 are the same. Lastly, $(bc)^4$ acts as $\mathbb{1}$ on the first level (by (2.14)) and on the second level as well, because

$$\begin{aligned} (bdcd)^2 &= (bada, cada)\sigma(bada, cada)\sigma \\ &= (bada, cada)(cada, bada)\sigma\sigma \\ &= (badacada, cadabada) \end{aligned}$$

and

$$\begin{aligned} (cdbd)^2 &= (daba, daca)\sigma(daba, daca)\sigma \\ &= (daba, daca)(daca, daba)\sigma\sigma \\ &= (dabadaca, dacadaba); \end{aligned}$$

thus, its portrait of depth 2 corresponds to the identity portrait that is given by



As a result, the powers of bc restricted to the two first levels of $\{0, 1\}^*$ coincide with the restrictions of the elements of the generator set of H; then, bc, $(bc)^2$, $(bc)^3$ and $(bc)^4$ generate $H \,/\, \mathrm{Stab}_H(2)$. Furthermore, since $(bc)^4$ acts on the two first levels of the tree as the identity, we conclude that $H \,/\, \mathrm{Stab}_H(2)$ is cyclic and has order 4. □

We recall the following result without proof.

Proposition 2.2.3. An automorphism g of a rooted binary tree acts transitively on levels if and only if on each level the number of sections of g at the vertices of this level acting nontrivially on the first level below is odd.

Proof. See [4], page 118. □

Lemma 2.2.4. The group $\mathcal{G}$ is infinite.

Proof. The goal of this proof is to show that $\mathcal{G}$ acts transitively on each level of the tree $\{0, 1\}^*$. By proving this and by definition of a transitive group action we will have that for every two elements w_1 and w_2 in a level of the tree $\{0, 1\}^*$, which is infinite, we find an element $g \in \mathcal{G}$ such that $g \cdot w_1 = w_2$ so we get infinite elements g in $\mathcal{G}$.

Consider the subgroup $H = \langle ab, bc, cd, da \rangle$ of $\mathcal{G}$, introduced in Lemma 2.2.2; such subgroup has other important features. First, we note that H consists of all elements of $\mathcal{G}$ represented as words of even length in a, b, c and d: this happens because the generators of $\mathcal{G}$ are involutions by (2.6) so every product of an even number of generators of $\mathcal{G}$ has possible cancellations of pairs of equal letters and can thus be written as a word of even length where every letter has exponent 1 and so can be written as a product of generators of H and, conversely, every product of generators of H has even length. Moreover, H is a subgroup of index 2 in $\mathcal{G}$ by the previous consideration: since all words of even length belong to H, we get exactly two cosets of H in $\mathcal{G}$ (one containing all words on $\{a, b, c, d\}$ of even length and other one containing all words of odd length).

We claim that H is a self-similar group. By (2.11), ab, bc, cd and da are described in a wreath recursion; observing each one of those elements and using the fact that $cb = cd\ da\ ab$, $bd = bc\ cd$ and $ac = ab\ bc$, which implies that $cb, bd, ac \in H$, we have that all sections of generators of H belong to such group, for all $w \in X^*$. Moreover, by performing calculations similar to those seen in Lemma 2.2.2, we see that the sections of any element belong to H. Thus, by Definition 1.3.20, H is self-similar.

By Proposition 2.2.3 an automorphism $g \in H$ of the rooted binary tree acts level transitively if and only if on each level the number of sections of g at the vertices of this level acting nontrivially below the first level of the section is odd. The proof that the number of sections of g at the vertices of this level acting nontrivially below the first level of the section is odd happens by induction of levels: first, by induction we have that each element $g \in H$ acting nontrivially on the first level acts spherically transitively, so the result is trivially true. Now, suppose that the number of sections of the level k acting nontrivially on the first level (below the k-th level) is odd. It follows that, using the fact that H is self-similar, we get that all sections of g are also elements of H; this means that, from level k onwards, the tree has same behavior as the one of first levels so each of the sections will produce exactly one switch on the $(k+1)-th$ level while the sections acting trivially on the first level produce either none or two switches on the level $k+1$.

Then, the total of switches on the level $k+1$ is an odd number and, thus, $g \in H$ acts transitively on $\{0,1\}^*$. Therefore, since H is a self-similar subgroup of $\mathcal{G}$, the group $\mathcal{G}$ itself acts level transitively on the tree. The result follows. □

Corollary 2.2.5. The group Γ, dual of $\mathcal{G}$, is infinite.

Proof. This Corollary follows immediately from Lemma 2.2.4 and Proposition 2.2.1. By Lemma 2.2.4, the group $\mathcal{G}$ is infinite; then, by a contrapositive argument applied to Proposition 2.2.1, we obtain that the dual group Γ is infinite. □

Corollary 2.2.6. The stabilizers of levels of T in Γ are pairwise different.

Proof. We have that all stabilizers have infinite order; this happens because Γ is infinite (see Corollary 2.2.5) and all stabilizers of levels are finite index subgroups of Γ. Consider $g \in \mathrm{Stab}_\Gamma(n)$, a nontrivial stabilizer and let $m \geqslant n+1$ be the smallest level on which g acts nontrivially (its existance is guaranteed due to the nontriviality of g). Then, there exists a vertex $v = x_1 x_2 \cdots x_{m-1}$ of the tree $\{0,1\}^*$ such that $g|_v$ acts nontrivially on the first level below the $(m-1)$-th level. This implies that g fixes all elements until the level $m-1$ and then $g|_{x_1}$ fixes all elements until the level $(m-1)-1 = m-2$, $g|_{x_1 x_2}$ fixes all elements until the level $(m-1)-2 = m-3$ and so on, until we reach that the element $g|_{x_1 x_2 \cdots x_{m-n-1}}$ fixes all elements until the level $(m-1)-(m-n-1) = n$ and it acts nontrivially on the first level below the level n. This means that $g|_{x_1x_2\cdots x_{m-n-1}} \in \mathrm{Stab}_\Gamma(n)$; however, this element does not belong to $\mathrm{Stab}_\Gamma(n+1)$. Consequently, we get that $\mathrm{Stab}_\Gamma(n+1) \subset \mathrm{Stab}_\Gamma(n)$ so such stabilizers are different for all $n \in \mathbb{N}$. $\square$

Lemma 2.2.7. Let $\widehat{T}_n$ be the subtree of the tree $\widehat{T}$ consisting of its first n levels. Then, $\mathrm{Stab}_\Gamma(n) = \mathrm{Stab}_\Gamma(\widehat{T}_n)$.

Proof. Note that $\mathrm{Stab}_r(n) \subseteq \mathrm{Stab}_\Gamma(\widehat{T}_n)$ since the n-th level of the tree T contains all elements of the n-th level of $\widehat{T}_n$. We are left to prove, then, that $\mathrm{Stab}_r(n) \supseteq \mathrm{Stab}_\Gamma(\widehat{T}_n)$.

Suppose that there exists $v \in \mathrm{Stab}_r(\widehat{T}_n) \setminus \mathrm{Stab}_\Gamma(n)$. Then, we can consider a vertex g of the n-th level of T which does not belong to $\widehat{T}$ and is not fixed under v. Using the fact that v fixes every element of $\widehat{T}_n$ and writing $g = ftth$ (since $g \in T \setminus \widehat{T}$), with $f, h \in \mathcal{G}$ and $t \in \{a, b, c, d\}$, we obtain $v(g) = ftth'$, with $h' \in \mathcal{G}$. Then,

$$\begin{aligned} v(fh) &= v(f)v|_f(h) \\ &= f \cdot (v|_f)|_{tt}(h) \\ &= fv|_{ftt}(h) \\ &= fh'. \end{aligned} \tag{2.15}$$

The second equality in (2.15) holds because $t^2 = e$ for any $t \in \{a, b, c, d\}$ and so, for any $w \in \Gamma$, by Proposition 1.3.30 we get $w|_{tt} = (tt)w = w$ in Γ and $w|_{tt}(h) = (tt)w(h) = w(h)$, for any word $h \in T$.

We can repeat this procedure until we get an element of $\widehat{T}_n$ not fixed under the action of v, obtaining a contradiction. Therefore, $\mathrm{Stab}_r(n) \supseteq \mathrm{Stab}_\Gamma(\widehat{T}_n)$ and, then, we get that $\mathrm{Stab}_r(n) = \mathrm{Stab}_\Gamma(\widehat{T}_n)$. $\square$

Corollary 2.2.8. For any $n \geqslant 1$ there exists an element of Γ that fixes $\widehat{T}_n$ but moves some vertex in $\widehat{T}_{n+1}$.

Proof. In the proof of Corollary 2.2.6 we constructed an element which makes part of $\mathrm{Stab}_\Gamma(n) = \mathrm{Stab}_\Gamma(\widehat{T}_n)$, but does not belong to $\mathrm{Stab}_\Gamma(n+1) = \mathrm{Stab}_\Gamma(\widehat{T}_{n+1})$. $\square$

Lemma 2.2.9. The sections of any element of $\mathrm{Stab}_\Gamma(n)$ at the vertices of the n-th level act on the first level below the n-th level by even permutations.

Proof. The first part of this proof is dedicated to show that if $v \in \Gamma$ fixes a given vertex $d = x_1 \cdots x_n$, then the parities of the actions of v and $v|_d$ on the first level coincide. In order to do this, we construct a new generating set for Γ as follows: consider an element $x \in \mathrm{Sym}\{(a, b, c, d)\}$ and let $\overline{x}$ be the automorphism of T defined by

$$\overline{x} = (\overline{x}, \overline{x}, \overline{x}, \overline{x})x. \tag{2.16}$$

By its definition, the portrait of $\overline{x}$ has x at each vertex of its tree.

We claim that

$$(\mathbf{0}\mathbf{1}^{-1})^2 = (\mathbf{0}^{-1}\mathbf{1})^2 = e. \tag{2.17}$$

First of all, based on the automaton $\widehat{\mathcal{B}_4}^{-1}$ (described at the beginning of this chapter), we are able to describe $\mathbf{0}^{-1}$ and $\mathbf{1}^{-1}$ by wreath recursions, finding that

$$\begin{aligned} \mathbf{0}^{-1} &= (\mathbf{1}^{-1},\mathbf{0}^{-1},\mathbf{0}^{-1},\mathbf{1}^{-1})(adc) \\ \mathbf{1}^{-1} &= (\mathbf{0}^{-1},\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1})(adcb). \end{aligned} \tag{2.18}$$

Before proving the equality (2.17), we verify that $\mathbf{0}^{-1}$ and $\mathbf{1}^{-1}$ are the inverses of $\mathbf{0}$ and $\mathbf{1}$, respectively. Using the wreath recursions defining each one of these elements (given in (2.18) and (2.1)), we get

$$\begin{aligned} \mathbf{0}\mathbf{0}^{-1} &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})(acd)(\mathbf{1}^{-1},\mathbf{0}^{-1},\mathbf{0}^{-1},\mathbf{1}^{-1})(adc) \\ &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})(\mathbf{0}^{-1},\mathbf{0}^{-1},\mathbf{1}^{-1},\mathbf{1}^{-1})(acd)(adc) \\ &= (\mathbf{0}\mathbf{0}^{-1},\mathbf{0}\mathbf{0}^{-1},\mathbf{1}\mathbf{1}^{-1},\mathbf{1}\mathbf{1}^{-1}); \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathbf{0}^{-1}\mathbf{0} &= (\mathbf{1}^{-1},\mathbf{0}^{-1},\mathbf{0}^{-1},\mathbf{1}^{-1})(adc)(\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})(acd) \\ &= (\mathbf{1}^{-1},\mathbf{0}^{-1},\mathbf{0}^{-1},\mathbf{1}^{-1})(\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{1})(adc)(acd) \\ &= (\mathbf{1}^{-1}\mathbf{1},\mathbf{0}^{-1}\mathbf{0},\mathbf{0}^{-1}\mathbf{0},\mathbf{1}^{-1}\mathbf{1}); \end{aligned} \tag{2.20}$$

$$\begin{aligned} \mathbf{1}\mathbf{1}^{-1} &= (\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0})(abcd)(\mathbf{0}^{-1},\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1})(adcb) \\ &= (\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0})(\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1},\mathbf{0}^{-1})(abcd)(adcb) \\ &= (\mathbf{1}\mathbf{1}^{-1},\mathbf{1}\mathbf{1}^{-1},\mathbf{0}\mathbf{0}^{-1},\mathbf{0}\mathbf{0}^{-1}); \end{aligned} \tag{2.21}$$

$$\begin{aligned} \mathbf{1}^{-1}\mathbf{1} &= (\mathbf{0}^{-1},\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1})(adcb)(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0})(abcd) \\ &= (\mathbf{0}^{-1},\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1})(\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{0})(adcb)(abcd) \\ &= (\mathbf{0}^{-1}\mathbf{0},\mathbf{1}^{-1}\mathbf{1},\mathbf{1}^{-1}\mathbf{1},\mathbf{0}^{-1}\mathbf{0}). \end{aligned} \tag{2.22}$$

Notice that, by relations (2.19), (2.20), (2.21) and (2.22), the elements $\mathbf{0}\mathbf{0}^{-1}$, $\mathbf{0}^{-1}\mathbf{0}$, $\mathbf{1}\mathbf{1}^{-1}$ and $\mathbf{1}^{-1}\mathbf{1}$ act as $\mathbb{1}$ on their first levels; further, all of their sections are constituted of elelents belonging to the same set $\{\mathbf{0}\mathbf{0}^{-1},\mathbf{0}^{-1}\mathbf{0},\mathbf{1}\mathbf{1}^{-1},\mathbf{1}^{-1}\mathbf{1}\}$ so on the subsequent levels they also act as the identity. Thus, $\mathbf{0}\mathbf{0}^{-1} = \mathbf{0}^{-1}\mathbf{0} = \mathbf{1}\mathbf{1}^{-1} = \mathbf{1}^{-1}\mathbf{1} = e$.

Now, we verify the equality (2.17). Using (2.1) and (2.18) again, we obtain

$$\begin{aligned} \mathbf{0}\mathbf{1}^{-1} &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})(acd)(\mathbf{0}^{-1},\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1})(adcb) \\ &= (\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})(\mathbf{1}^{-1},\mathbf{1}^{-1},\mathbf{0}^{-1},\mathbf{0}^{-1})(acd)(adcb) \\ &= (\mathbf{0}\mathbf{1}^{-1},\mathbf{0}\mathbf{1}^{-1},\mathbf{1}\mathbf{0}^{-1},\mathbf{1}\mathbf{0}^{-1})(ab), \end{aligned} \tag{2.23}$$

which implies that

$$\begin{aligned}(\mathbf{0}\mathbf{1}^{-1})^2 &= (\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{1}\mathbf{0}^{-1}, \mathbf{1}\mathbf{0}^{-1})(ab)(\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{1}\mathbf{0}^{-1}, \mathbf{1}\mathbf{0}^{-1})(ab)\\ &= (\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{1}\mathbf{0}^{-1}, \mathbf{1}\mathbf{0}^{-1})(\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{1}\mathbf{0}^{-1}, \mathbf{1}\mathbf{0}^{-1})(ab)(ab)\\ &= ((\mathbf{0}\mathbf{1}^{-1})^2, (\mathbf{0}\mathbf{1}^{-1})^2, (\mathbf{1}\mathbf{0}^{-1})^2, (\mathbf{1}\mathbf{0}^{-1})^2);\end{aligned} \tag{2.24}$$

also,

$$\begin{aligned}\mathbf{0}^{-1}\mathbf{1} &= (\mathbf{1}^{-1}, \mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1})(adc)(\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(abcd)\\ &= (\mathbf{1}^{-1}, \mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1})(\mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0})(adc)(abcd)\\ &= (\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0})(bc);\end{aligned} \tag{2.25}$$

which implies that

$$\begin{aligned}(\mathbf{0}^{-1}\mathbf{1})^2 &= (\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0})(bc)(\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0})(bc)\\ &= (\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0})(\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0})(bc)(bc)\\ &= ((\mathbf{1}^{-1}\mathbf{0})^2, (\mathbf{0}^{-1}\mathbf{1})^2, (\mathbf{0}^{-1}\mathbf{1})^2, (\mathbf{1}^{-1}\mathbf{0})^2).\end{aligned} \tag{2.26}$$

Looking at the sections of the wreath recursions defining $(\mathbf{0}\mathbf{1}^{-1})^2$ (see (2.24)) and $(\mathbf{0}^{-1}\mathbf{1})^2$ (see (2.26)), we calculate $(\mathbf{1}\mathbf{0}^{-1})^2$ and $(\mathbf{1}^{-1}\mathbf{0})^2$, finding that

$$\begin{aligned}(\mathbf{1}\mathbf{0}^{-1})^2 &= \mathbf{1}\mathbf{0}^{-1}\mathbf{1}\mathbf{0}^{-1}\\ &= (\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(abcd)(\mathbf{1}^{-1}, \mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1})(adc)\\ &\qquad (\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(abcd)(\mathbf{1}^{-1}, \mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1})(adc)\\ &= (\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(\mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1})(ab)(\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(abcd)(\mathbf{1}^{-1}, \mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1})(adc)\\ &= (\mathbf{1}\mathbf{0}^{-1}, \mathbf{1}\mathbf{0}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1})(\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(acd)(\mathbf{1}^{-1}, \mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1})(adc)\\ &= (\mathbf{1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}\mathbf{1}^{-1}\mathbf{0})(\mathbf{0}^{-1}, \mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1})(acd)(adc)\\ &= ((\mathbf{1}\mathbf{0}^{-1})^2, (\mathbf{1}\mathbf{0}^{-1})^2, (\mathbf{0}\mathbf{1}^{-1})^2, (\mathbf{0}\mathbf{1}^{-1})^2);\end{aligned} \tag{2.27}$$

$$\begin{aligned}(\mathbf{1}^{-1}\mathbf{0})^2 &= \mathbf{1}^{-1}\mathbf{0}\mathbf{1}^{-1}\mathbf{0}\\ &= (\mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1}, \mathbf{0}^{-1})(adcb)(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(acd)\\ &\qquad (\mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1}, \mathbf{0}^{-1})(adcb)(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(acd)\\ &= (\mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1}, \mathbf{0}^{-1})(\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1})(bc)(\mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1}, \mathbf{0}^{-1})(adcb)(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(acd)\\ &= (\mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0}, \mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1})(\mathbf{0}^{-1}, \mathbf{1}^{-1}, \mathbf{1}^{-1}, \mathbf{0}^{-1})(adc)(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(acd)\\ &= (\mathbf{0}^{-1}\mathbf{1}\mathbf{0}^{-1}, \mathbf{1}^{-1}\mathbf{0}\mathbf{1}^{-1}, \mathbf{1}^{-1}\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}^{-1}\mathbf{1}\mathbf{0}^{-1})(\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1})(adc)(acd)\\ &= ((\mathbf{0}^{-1}\mathbf{1})^2, (\mathbf{1}^{-1}\mathbf{0})^2, (\mathbf{1}^{-1}\mathbf{0})^2, (\mathbf{0}^{-1}\mathbf{1})^2).\end{aligned} \tag{2.28}$$

By (2.27) and (2.28) we have that the sections of $(\mathbf{0}\mathbf{1}^{-1})^2$ and $(\mathbf{0}^{-1}\mathbf{1})^2$ act as $\mathbb{1}$ and their subsequent sections have the same behavior. Then, by a similar argument than the one given in the affirmation $\mathbf{0}\mathbf{0}^{-1} = \mathbf{0}^{-1}\mathbf{0} = \mathbf{1}\mathbf{1}^{-1} = \mathbf{1}^{-1}\mathbf{1} = e$, we conclude that $(\mathbf{0}\mathbf{1}^{-1})^2 = (\mathbf{0}^{-1}\mathbf{1})^2 = e$.

We will use the now proved equality (2.17) to verify two important equalities: $\mathbf{1}\mathbf{0}^{-1} = \mathbf{0}\mathbf{1}^{-1}$ and $\mathbf{1}^{-1}\mathbf{0} = \mathbf{0}^{-1}\mathbf{1}$. In fact,

$$\begin{aligned} \mathbf{1}\mathbf{0}^{-1} &= \mathbf{0}\mathbf{0}^{-1}\mathbf{1}\mathbf{0}^{-1}\mathbf{1}\mathbf{1}^{-1} \\ &= \mathbf{0}(\mathbf{0}^{-1}\mathbf{1})^2\mathbf{1}^{-1} \\ &= \mathbf{0}\mathbf{1}^{-1} \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} \mathbf{1}^{-1}\mathbf{0} &= \mathbf{0}^{-1}\mathbf{0}\mathbf{1}^{-1}\mathbf{0}\mathbf{1}^{-1}\mathbf{1} \\ &= \mathbf{0}^{-1}(\mathbf{0}\mathbf{1}^{-1})^2\mathbf{1} \\ &= \mathbf{0}^{-1}\mathbf{1}. \end{aligned} \tag{2.30}$$

Then, by (2.23), (2.29) and (2.16),

$$\begin{aligned} \mathbf{0}\mathbf{1}^{-1} &= (\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{1}\mathbf{0}^{-1}, \mathbf{1}\mathbf{0}^{-1})(ab) \\ &= (\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1})(ab) \\ &= \overline{(ab)}. \end{aligned} \tag{2.31}$$

Also, by (2.25), (2.30) and (2.16),

$$\begin{aligned} \mathbf{0}^{-1}\mathbf{1} &= (\mathbf{1}^{-1}\mathbf{0}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{1}^{-1}\mathbf{0})(bc) \\ &= (\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1})(bc) \\ &= \overline{(bc)}. \end{aligned} \tag{2.32}$$

Observe that $\overline{(ac)} \in \Gamma$. Indeed, by using the results of (2.31) and (2.32),

$$\begin{aligned} \overline{(ab)} \cdot \overline{(bc)} = \mathbf{0}\mathbf{1}^{-1}\mathbf{0}^{-1}\mathbf{1} &= (\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1})(ab)(\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1})(bc) \\ &= (\mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1}, \mathbf{0}\mathbf{1}^{-1})(\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1}, \mathbf{0}^{-1}\mathbf{1})(ab)(bc) \\ &= (\mathbf{0}\mathbf{1}^{-1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}\mathbf{1}^{-1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}\mathbf{1}^{-1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{0}\mathbf{1}^{-1}\mathbf{0}^{-1}\mathbf{1})(ac) \\ &= \overline{(ac)}. \end{aligned} \tag{2.33}$$

Because $\overline{(ab)} = \mathbf{0}\mathbf{1}^{-1}$ and $\overline{(bc)} = \mathbf{0}^{-1}\mathbf{1}$ (by (2.31) and (2.32)), we have $\overline{(ab)}, \overline{(bc)} \in \Gamma$. Thus, $\overline{(ab)} \cdot \overline{(bc)} = \overline{(ac)} \in \Gamma$.

Define $\alpha = \mathbf{1}\overline{(ac)}$ and $\beta = \mathbf{0}\overline{(ac)}$. Then, using (2.33), one gets

$$\begin{aligned}
\alpha &= \mathbf{1}\overline{(ac)} \\
&= (\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(abcd)(\overline{(ac)}, \overline{(ac)}, \overline{(ac)}, \overline{(ac)})(ac) \\
&= (\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0})(\overline{(ac)}, \overline{(ac)}, \overline{(ac)}, \overline{(ac)})(abcd)(ac) \\
&= (\mathbf{1}\overline{(ac)}, \mathbf{1}\overline{(ac)}, \mathbf{0}\overline{(ac)}, \mathbf{0}\overline{(ac)})(ab)(cd) \\
&= (\alpha, \alpha, \beta, \beta)(ab)(cd)
\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}
\beta &= \mathbf{0}\overline{(ac)} \\
&= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(acd)(\overline{(ac)}, \overline{(ac)}, \overline{(ac)}, \overline{(ac)})(ac) \\
&= (\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1})(\overline{(ac)}, \overline{(ac)}, \overline{(ac)}, \overline{(ac)})(acd)(ac) \\
&= (\mathbf{0}\overline{(ac)}, \mathbf{0}\overline{(ac)}, \mathbf{1}\overline{(ac)}, \mathbf{1}\overline{(ac)})(cd) \\
&= (\beta, \beta, \alpha, \alpha)(cd).
\end{aligned} \tag{2.35}$$

Note that $\alpha^2 = \beta^2 = e$. In fact, due to (2.34) and (2.35),

$$\begin{aligned}
\alpha^2 &= (\alpha, \alpha, \beta, \beta)(ab)(cd)(\alpha, \alpha, \beta, \beta)(ab)(cd) \\
&= (\alpha, \alpha, \beta, \beta)(\alpha, \alpha, \beta, \beta)(ab)(cd)(ab)(cd) \\
&= (\alpha^2, \alpha^2, \beta^2, \beta^2)
\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}
\beta^2 &= (\beta, \beta, \alpha, \alpha)(cd)(\beta, \beta, \alpha, \alpha)(cd) \\
&= (\beta, \beta, \alpha, \alpha)(\beta, \beta, \alpha, \alpha)(cd)(cd) \\
&= (\beta^2, \beta^2, \alpha^2, \alpha^2).
\end{aligned} \tag{2.37}$$

Relations (2.36) and (2.37) lead us to say that all the sections of the wreath recursions defining α^2 and β^2 correspond to the action of $\mathbb{1}$ not only at the first level but in all the levels, since the sections of α^2 and β^2 have only such elements. Therefore, $\alpha^2 = \beta^2 = e$.

We also claim that $\beta\alpha^{-1} = \overline{(ab)}$. In order to do this, we determine the element α^{-1} by using the definition of α. Due to (2.34) and (2.35) we have

$$\begin{aligned}
\alpha &= (\alpha, \alpha, \beta, \beta)(ab)(cd) \\
&= (\mathbf{101}^{-1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{101}^{-1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{001}^{-1}\mathbf{0}^{-1}\mathbf{1}, \mathbf{001}^{-1}\mathbf{0}^{-1}\mathbf{1})(ab)(cd)
\end{aligned} \tag{2.38}$$

so α^{-1} is given by

$$\alpha^{-1} = (\mathbf{1}^{-1}\mathbf{010}^{-1}\mathbf{1}^{-1}, \mathbf{1}^{-1}\mathbf{010}^{-1}\mathbf{1}^{-1}, \mathbf{1}^{-1}\mathbf{010}^{-1}\mathbf{0}^{-1}, \mathbf{1}^{-1}\mathbf{010}^{-1}\mathbf{0}^{-1})(ab)(cd). \tag{2.39}$$

In fact,

$$\begin{aligned}
\alpha\alpha^{-1} &= (\mathbf{101^{-1}0^{-1}1}, \mathbf{101^{-1}0^{-1}1}, \mathbf{001^{-1}0^{-1}1}, \mathbf{001^{-1}0^{-1}1})(ab)(cd) \\
&\qquad (\mathbf{1^{-1}010^{-1}1^{-1}}, \mathbf{1^{-1}010^{-1}1^{-1}}, \mathbf{1^{-1}010^{-1}0^{-1}}, \mathbf{1^{-1}010^{-1}0^{-1}})(ab)(cd) \\
&= (\mathbf{101^{-1}0^{-1}11^{-1}010^{-1}1^{-1}}, \mathbf{101^{-1}0^{-1}11^{-1}010^{-1}1^{-1}}, \mathbf{001^{-1}0^{-1}11^{-1}010^{-1}0^{-1}}, \\
&\qquad \mathbf{001^{-1}0^{-1}11^{-1}010^{-1}0^{-1}})(ab)(cd)(ab)(cd) \\
&= (\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})
\end{aligned}$$

and

$$\begin{aligned}
\alpha^{-1}\alpha &= (\mathbf{1^{-1}010^{-1}1^{-1}}, \mathbf{1^{-1}010^{-1}1^{-1}}, \mathbf{1^{-1}010^{-1}0^{-1}}, \mathbf{1^{-1}010^{-1}0^{-1}})(ab)(cd) \\
&\qquad (\mathbf{101^{-1}0^{-1}1}, \mathbf{101^{-1}0^{-1}1}, \mathbf{001^{-1}0^{-1}1}, \mathbf{001^{-1}0^{-1}1})(ab)(cd) \\
&= (\mathbf{1^{-1}010^{-1}1^{-1}101^{-1}0^{-1}1}, \mathbf{1^{-1}010^{-1}1^{-1}101^{-1}0^{-1}1}, \mathbf{1^{-1}010^{-1}0^{-1}001^{-1}0^{-1}1}, \\
&\qquad \mathbf{1^{-1}010^{-1}0^{-1}001^{-1}0^{-1}1})(ab)(cd)(ab)(cd) \\
&= (\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}),
\end{aligned}$$

which give the equality $\alpha\alpha^{-1} = \alpha^{-1}\alpha = e$.

Then, by (2.35), (2.38), (2.39) and (2.29) we find that

$$\begin{aligned}
\beta\alpha^{-1} &= (\beta, \beta, \alpha, \alpha)(cd) \\
&= (\mathbf{001^{-1}0^{-1}1}, \mathbf{001^{-1}0^{-1}1}, \mathbf{101^{-1}0^{-1}1}, \mathbf{101^{-1}0^{-1}1})(cd) \\
&\qquad (\mathbf{1^{-1}010^{-1}1^{-1}}, \mathbf{1^{-1}010^{-1}1^{-1}}, \mathbf{1^{-1}010^{-1}0^{-1}}, \mathbf{1^{-1}010^{-1}0^{-1}})(ab)(cd) \\
&= (\mathbf{001^{-1}0^{-1}11^{-1}010^{-1}1^{-1}}, \mathbf{001^{-1}0^{-1}11^{-1}010^{-1}1^{-1}}, \mathbf{101^{-1}0^{-1}11^{-1}010^{-1}0^{-1}}, \\
&\qquad \mathbf{101^{-1}0^{-1}11^{-1}010^{-1}0^{-1}})(cd)(ab)(cd) \\
&= (\mathbf{01^{-1}}, \mathbf{01^{-1}}, \mathbf{10^{-1}}, \mathbf{10^{-1}})(ab) \qquad (2.40) \\
&= (\mathbf{01^{-1}}, \mathbf{01^{-1}}, \mathbf{01^{-1}}, \mathbf{01^{-1}})(ab) \\
&= \overline{(ab)}.
\end{aligned}$$

Consider the set $\langle \alpha, \beta, \overline{(bc)} \rangle$. By the definition of each one of the generators α, β and $\overline{(bc)}$ (see (2.34), (2.35) and (2.32)), we have $\alpha, \beta, \overline{(bc)} \in \Gamma$ so $\langle \alpha, \beta, \overline{(bc)} \rangle \subseteq \Gamma$. Since $\beta\alpha^{-1} = \overline{(ab)}$ (by relation (2.40)) then $\overline{(ab)}, \overline{(bc)} \in \langle \alpha, \beta, \overline{(bc)} \rangle$; consequently, we have that $\overline{(ab)} \cdot \overline{(bc)} = \overline{(ac)} \in \langle \alpha, \beta, \overline{(bc)} \rangle$. Further, because $\alpha = \mathbf{1}\overline{(ac)}$, one has that $\mathbf{1} = \alpha\overline{(ac)}^{-1} \in \langle \alpha, \beta, \overline{(bc)} \rangle$ as well as the fact that $\beta = \mathbf{0}\overline{(ac)}$ implies that one has $\mathbf{0} = \beta\overline{(ac)}^{-1} \in \langle \alpha, \beta, \overline{(bc)} \rangle$ so we learn that $\Gamma = \langle \mathbf{0}, \mathbf{1} \rangle \subseteq \langle \alpha, \beta, \overline{(bc)} \rangle$. Therefore, $\Gamma = \langle \alpha, \beta, \overline{(bc)} \rangle$.

Let $v \in \Gamma$ be an arbitrary element fixing vertex d. Using the fact proved above that $\Gamma = \langle \alpha, \beta, \overline{(bc)} \rangle$, we represent v as a word over $\{\alpha, \beta, \overline{(bc)}\}$, obtaining $v = v_1 \cdots v_k$. By (1.27), we have

$$v|_d = v_1|_d \cdot v_2|_{v_1(d)} \cdots v_k|_{v_1 \cdots v_{k-1}(d)}.$$

We claim that the parity of the action of v_i on the first level is different from the one of $v_i|_{v_1\cdots v_{i-1}(d)}$ only if one of the four following cases happens:

1. $v_i = \alpha$ and $v_1 v_2 \cdots v_{i-1}(d) = c$;

2. $v_i = \alpha$ and $v_1 v_2 \cdots v_{i-1}(d) = d$;

3. $v_i = \beta$ and $v_1 v_2 \cdots v_{i-1}(d) = c$;

4. $v_i = \beta$ and $v_1 v_2 \cdots v_{i-1}(d) = d$.

Recall that

$$\begin{aligned} \alpha &= (\alpha, \alpha, \beta, \beta)(ab)(cd); \\ \beta &= (\beta, \beta, \alpha, \alpha)(cd); \\ \overline{(bc)} &= (\overline{(bc)}, \overline{(bc)}, \overline{(bc)}, \overline{(bc)})(bc). \end{aligned} \tag{2.41}$$

Observing the sections of the definition of $\overline{(bc)}$ as part of the wreath recursion (2.41) defining $\Gamma = \langle \alpha, \beta, \overline{(bc)} \rangle$, which are all $\overline{(bc)}$'s, we note that there is no chance of changing parities by the action of such element since all permutations of the sections of $\overline{(bc)}$ are equal to (bc) and then there is no way of getting new parities by this action. Thus, the parities of v_i and $v_i|_{v_1\cdots v_{i-1}(d)}$ can be distinct only if v_i is not $\overline{(bc)}$. When this happens, if $v_1 v_2 \cdots v_{i-1}(d) = c$ then $v_1 v_2 \cdots v_i(d) = d$ as well as if $v_1 v_2 \cdots v_{i-1}(d) = d$ then $v_1 v_2 \cdots v_i(d) = c$ (by definition of α and β in (2.41)). The converse is also true in the following sense: if one has $v_1 v_2 \cdots v_{i-1}(d) \neq d$ and $v_1 v_2 \cdots v_i(d) = d$ then $v_1 v_2 \cdots v_{i-1}(d) = c$ and v_i is either α or β as well as if $v_1 v_2 \cdots v_{i-1}(d) = d$ and $v_1 v_2 \cdots v_i(d) \neq d$ then $v_1 v_2 \cdots v_{i-1}(d) = c$ and v_1 is either α or β. This means that the parity of the action of v_i on the first level is different from the action of the action of $v_i|_{v_1 v_2\cdots v_{i-1}(d)}$ when there is a change from d to anything else or from something to d in the sequence $\{d, v_1(d), \cdots, v_1 v_2 \cdots v_k(d)\}$. Further, note that the parity of the action of α is even while the parity of the action of β is odd so the parity of v_i changes in relation to the one of $v_i|_{v_1 v_2\cdots v_{i-1}(d)}$ when it changes from the action of α to the action of β and vice-versa. Observing the sections of α and β, we verify that such change happens when $v_i = \alpha$ and $v_1 v_2 \cdots v_{i-1}(d) = c$ or $v_i = \alpha$ and $v_1 v_2 \cdots v_{i-1}(d) = d$ or $v_i = \beta$ and $v_1 v_2 \cdots v_{i-1}(d) = c$ or $v_i = \beta$ and $v_1 v_2 \cdots v_{i-1}(d) = d$.

We are considering $v = v_1 v_2 \cdots v_k \in \Gamma$ such that v fixes d so we have that $v_1 v_2 \cdots v_k(d) = v(d) = d$; then, since we start with d and end with d, there must be an even number of changes in the sequence $\{d, v_1(d), \cdots, v_1 v_2 \cdots v_k(d)\}$. By the considerations about the parities of v_i and $v_i|_{v_1 v_2\cdots v_{i-1}(d)}$ given above, we conclude that the parity is different in an even number of places of the sequences and thus the parities of the actions of v and $v|_d$ on the first level (below the first convenient level) coincide.

Let $g = (g|_a, g|_b, g|_c, g|_d) \in \mathrm{Stab}_\Gamma(1)$ be an element stabilizing all elements of the first level. By the considerations above, in the case of g we have that, since $g \in \mathrm{Stab}_\Gamma(1)$ then the permutation on the definition of g is the identity, which is even. Thus, the action of $g|_d$ on the first level is also even.

Define the conjugation

$$g^{\bullet} = \bullet^{-1} g \bullet,$$

with $\bullet \in \Gamma$. Because conjugation of a permutation preserves its parity, the parities of g and $g^{\bullet}$ coincide.

Computing g^{β} and using the fact that $\beta^{-1} = \beta$ and $\alpha^{-1} = \alpha$ (we proved previously that $\alpha^2 = \beta^2 = e$), by (2.35) one gets

$$\begin{aligned} g^{\beta} &= \beta^{-1} g \beta \\ &= (\beta, \beta, \alpha, \alpha)(cd)(g|_a, g|_b, g|_c, g|_d)(\beta, \beta, \alpha, \alpha)(cd) \\ &= (\beta, \beta, \alpha, \alpha)(g|_a, g|_b, g|_d, g|_c)(cd)(\beta, \beta, \alpha, \alpha)(cd) \\ &= (\beta, \beta, \alpha, \alpha)(g|_a, g|_b, g|_d, g|_c)(\beta, \beta, \alpha, \alpha)\mathbb{1} \\ &= (\beta g|_a \beta, \beta g|_b \beta, \alpha g|_d \alpha, \alpha g|_c \alpha) \\ &= (\beta^{-1} g|_a \beta, \beta^{-1} g|_b \beta, \alpha^{-1} g|_d \alpha, \alpha^{-1} g|_c \alpha) \\ &= ((g|_a)^{\beta}, (g|_b)^{\beta}, (g|_d)^{\alpha}, (g|_c)^{\alpha}), \end{aligned} \tag{2.42}$$

which belongs to $\mathrm{Stab}_\Gamma(1)$. Observing equation (2.42), note that $g^{\beta}|_d = (g|_c)^{\alpha}$ so, by the past considerations about parities of actions on sections, we have that $(g|_c)^{\alpha}$ and, consequently, $g|_c$ act on the first level by an even permutation (since conjugation preserves parity).

In order to establish some concerns about $g|_a$ and $g|_b$, we compute $g^{\overline{(ac)}}$ and $g^{\overline{(bc)}}$, obtaining

$$\begin{aligned} g^{\overline{(ac)}} &= \overline{(ac)}^{-1} g \overline{(ac)} \\ &= (\overline{(ac)}^{-1}, \overline{(ac)}^{-1}, \overline{(ac)}^{-1}, \overline{(ac)}^{-1})(ac)(g|_a, g|_b, g|_c, g|_d)(\overline{(ac)}, \overline{(ac)}, \overline{(ac)}, \overline{(ac)})(ac) \\ &= (\overline{(ac)}^{-1}, \overline{(ac)}^{-1}, \overline{(ac)}^{-1}, \overline{(ac)}^{-1})(g|_c, g|_b, g|_a, g|_d)(\overline{(ac)}, \overline{(ac)}, \overline{(ac)}, \overline{(ac)})\mathbb{1} \\ &= ((g|_c)^{\overline{(ac)}}, (g|_b)^{\overline{(ac)}}, (g|_a)^{\overline{(ac)}}, (g|_d)^{\overline{(ac)}}) \end{aligned} \tag{2.43}$$

and

$$\begin{aligned} g^{\overline{(bc)}} &= \overline{(bc)}^{-1} g \overline{(bc)} \\ &= (\overline{(bc)}^{-1}, \overline{(bc)}^{-1}, \overline{(bc)}^{-1}, \overline{(bc)}^{-1})(bc)(g|_a, g|_b, g|_c, g|_d)(\overline{(bc)}, \overline{(bc)}, \overline{(bc)}, \overline{(bc)})(bc) \\ &= (\overline{(bc)}^{-1}, \overline{(bc)}^{-1}, \overline{(bc)}^{-1}, \overline{(bc)}^{-1})(g|_a, g|_c, g|_b, g|_d)(\overline{(bc)}, \overline{(bc)}, \overline{(bc)}, \overline{(bc)})\mathbb{1} \\ &= ((g|_a)^{\overline{(bc)}}, (g|_c)^{\overline{(bc)}}, (g|_b)^{\overline{(bc)}}, (g|_d)^{\overline{(bc)}}). \end{aligned} \tag{2.44}$$

Since $g^{\overline{(ac)}}|_c = (g|_a)^{\overline{(ac)}}$ (see (2.43)) and $g^{\overline{(bc)}}|_c = (g|_b)^{\overline{(bc)}}$ (see (2.44)), due to the fact that $g|_c$ acts on the first level by an even permutation, we infer that $(g|_a)^{\overline{(ac)}}$ and, consequently, $g|_a$ also act by an even permutation on the first level; the same happens with $(g|_b)^{\overline{(bc)}}$ and, therefore, with $g|_b$. This implies that all sections of g at the vertices of the first level act on the first level by even permutations.

By self-similarity it is enough to prove the case $n = 1$, which we just did, and so the result follows. □

Lemma 2.2.10. The group Γ acts transitively on the levels of $\widehat{T}$.

Proof. We use induction on levels to prove this lemma. Since $\mathcal{G}$ is generated by a, b, c and d, the transitivity on the first level is guaranteed. Now, assume that Γ acts transitively on the n-th level of $\widehat{T}$. By Corollary 2.2.8, there exists an element $v \in \Gamma$ such that v fixes $\widehat{T}_n$ but it acts nontrivially on $\widehat{T}_{n+1}$; in other words, there exists a vertex $g \in \widehat{T}_n$ such that $v(g) = g$ and $v|_g$ acts nontrivially on the first level below the n-th level. Since $v \in \mathrm{Stab}_\Gamma(\widehat{T}_n)$, by Lemma 2.2.9 the section $v|_g$ acts on the first level by an even permutation; since $v|_g$ on the first level is even, it must be a cycle of length 3 (only option for a nontrivial even permutation of four elements a, b, c and d). Since the vertex g has three children in the tree $\widehat{T}$, then $v|_g$ is transitive on the first level.

Without loss of generality, assume that $g \in \widehat{T}_n$ ends with c. By the hypothesis of induction, Γ acts transitively on the n-th level of $\widehat{T}$ which indicates that, letting the word $h_1 h_2 \cdots h_n h_{n+1} \in \widehat{T}_{n+1}$ be a vertex, there is an element $w \in \Gamma$ moving g to $h_1 h_1 \cdots h_n$. Pick $v^k w \in \Gamma$ with $k \in \{0, 1, 2\}$; then, due to the fact that the permutation induced by $v|_g$ is a 3-cycle, we obtain that there exists k such that $v^k w(gn) = h_1 h_2 \cdots h_n h_{n+1}$, with $n \in \{a, b, d\}$. Therefore, Γ acts transitively on $\widehat{T}_{n+1}$. □

All lemmas, propositions and corollaries of this chapter build to the proof of Theorem 2.1.1, which turns out to be relatively elementary after using all that was proved on this subsection.

Proof of Theorem 2.1.1. First, we claim that there exists $h \in \mathcal{G}$, $h \neq e$, with the property that h lies in the n-th level of $\widehat{T}$, with $n \geqslant 1$. Indeed, just consider an element h given by

$$h = \begin{cases} (ab)^{\frac{n-1}{2}} c & \text{if } n \text{ is odd,} \\ (ab)^{\frac{n}{2}-1} ac & \text{if } n \text{ is even.} \end{cases}$$

Recalling Lemma 2.2.10, we have that the group Γ acts transitively on each level of $\widehat{T}$. Let g be an arbitrary word belonging to the n-th level of $\widehat{T}$. Then, g has length n and does not contain a^2, b^2, c^2 or d^2.

Now, suppose $g \equiv_{\mathcal{G}} e$; that is, suppose that g is equivalent to the identity word in $\mathcal{G}$. Since g is a word from the level n of $\widehat{T}$, by Lemma 2.2.10 and Proposition 1.3.30 there is $v \in \Gamma$ satisfying

$$e = g|_v = v(g) = h \neq e,$$

that gives a contradiction. Hence, there are no relations in the group $\mathcal{G}$ except the relation $a^2 = b^2 = c^2 = d^2 = e$ (via (2.6)) which implies that $\mathcal{G}$ is isomorphic to a free product of four groups of order 2. $\square$

3 A family of automata generating free products of C_2

"Mathematicians are like managers - they want improvement without change."

Edsger Dijkstra, 1930–2002

The last chapter of this dissertation is structured so as to provide a "generalization" of what was shown in Chapter 2. We make a connection with the previous chapter in order to prove that the group generated by the Bellaterra automaton $\mathcal{B}^{(n)}$, for any $n \in \mathbb{Z}, n > 4$, with some declared permutations of some states, generates a free product of n groups of order 2. As well as in the last chapter, after showing a few observations about $\mathcal{B}^{(n)}$, we verify several results that lead to the main proof. We follow, as in Chapter 2, the exposition from [18].

3.1 Considerations about the automaton $\mathcal{B}^{(n)}$

We invoke, one more time, the Subsection 1.3.5 before providing more specific commentaries related to the Bellaterra automata family. We recall that, concerning its Moore diagram, a Bellaterra automaton $\mathcal{B}^{(n)}$ can be created by the insertion of new states on the path from state c to state a of the automaton $\mathcal{B}_3$. Given such Moore diagram, we are able to give explicitly the wreath recursion defining the automaton $\mathcal{B}^{(n)}$.

Consider the following construction of $\mathcal{B}^{(n)}$ for $n \in \mathbb{Z}$, $n \geqslant 4$: for any permutation $\sigma_{n,i} \in \mathrm{Sym}(\{0,1\})$, $i = 1, \cdots, n-4$, let $\mathcal{B}^{(n)}$ be the automaton with n states (namely a_n, b_n, c_n, q_{n1}, q_{n2}, $\cdots$, $q_{n,n-4}$, d_n) whose transition and output functions are given by the wreath recursion

$$\begin{aligned}
a_n &= (c_n, b_n),\\
b_n &= (b_n, c_n),\\
c_n &= (q_{n1}, q_{n1})\sigma,\\
q_{n,i} &= (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}, \quad i = 1, \cdots, n-5,\\
q_{n,n-4} &= (d_n, d_n)\sigma_{n,n-4},\\
d_n &= (a_n, a_n)\sigma.
\end{aligned} \tag{3.1}$$

The Moore diagram of $\mathcal{B}^{(n)}$, in this case, is

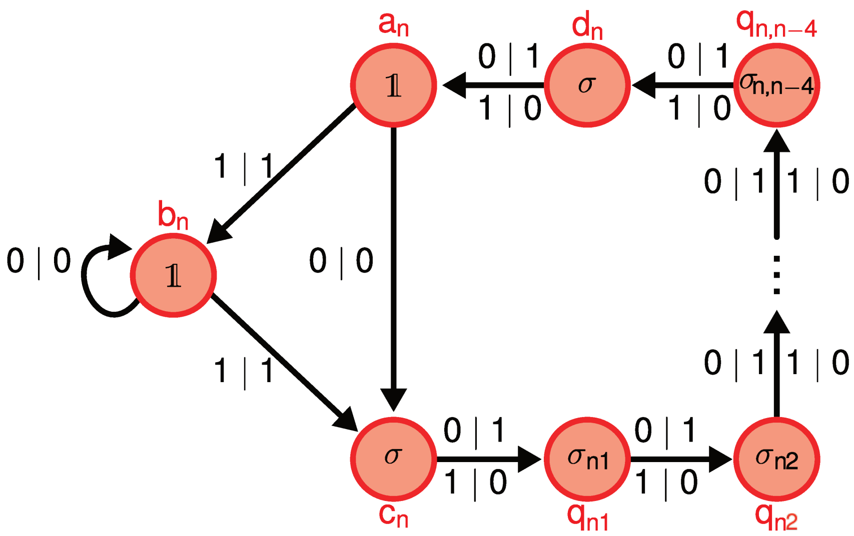


Figure 25 – Moore diagram of the Bellaterra automaton $\mathcal{B}^{(n)}$ (courtesy of Altair Santos).

Note that, although the graph structure of such automaton is the same as the one of the automaton $\mathcal{B}^{(n)}$ shown in the Subsection 1.3.5, the wreath recursions defining the two automata (see (1.34) and (3.1)) are slightly different in the sense that now we have chosen the permutation of the state c_n and the state d_n (which corresponds to the state q_{n-3} in the wreath recursion (1.34)); $\sigma = (0\ 1)$ is the permutation picked to figure in the definition of both states, matching with what was said in the end of the Subsection 1.3.5. Being aware of the choice of permutations $\sigma_{n,i}$ and of the slight abuse of notation implied by this decision, we follow the article [18] and still consider the automaton defined on this Section as the Bellaterra automaton $\mathcal{B}^{(n)}$. The Remark 1.3.34 is also valid for the Moore diagram of $\mathcal{B}^{(n)}$ introduced above.

Let $\mathcal{G}^{(n)}$ be the group generated by all states of the Bellaterra automaton $\mathcal{B}^{(n)}$ with wreath recursion given in (3.1). This chapter is dedicated to the proof of the following theorem:

Theorem 3.1.1 (D. Savchuk, Y. Vorobets, [18])**.** The group $\mathcal{G}^{(n)}$, generated by the automaton $\mathcal{B}^{(n)}$, is isomorphic to the free product of n copies of the cyclic group of order 2.

As stated at the end of the Section 2.1, in order to prove this theorem it is necessary to prove some results which open ways to the main proof. The approach is very similar to the one used on Chapter 2 and then it turns out that the proof of the main theorem of this chapter relies on the conclusions given in Section 2.2. The lemmas needed to prove Theorem 3.1.1 along with the theorem itself are proved in the next section.

3.2 Construction of the proof of the Theorem 3.1.1

The arrangement made in Section 2.1 is the basis for the outline to the proof of Theorem 3.1.1. We prove that the dual automaton of $\mathcal{B}^{(n)}$ acts transitively on the invariant subtree consisting of words with no double letters. All considerations contribute to the construction of the group $\mathcal{G}^{(n)}$ and its characterization.

In the same manner as we treated the case of the automaton $\mathcal{B}_4$, we make some observations about the Bellaterra automaton $\mathcal{B}^{(n)}$. In the first place, observe that the inverse of $\mathcal{B}^{(n)}$ is the automaton itself; the reason why this happens is the same as the one given in the case of $\mathcal{B}_4$.

We also want to show that $\mathcal{B}^{(n)}$ is bireversible: since $(\mathcal{B}^{(n)})^{-1} = \mathcal{B}^{(n)}$, we are left to verify that the dual automaton $\widehat{\mathcal{B}^{(n)}}$ is invertible; then, our first step towards the proof of this assertion consists on finding such dual automaton, based on the wreath recursion (3.1) and its consequent Moore diagram. Using those data, one gets that $\widehat{\mathcal{B}^{(n)}}$ has the following Moore diagram:

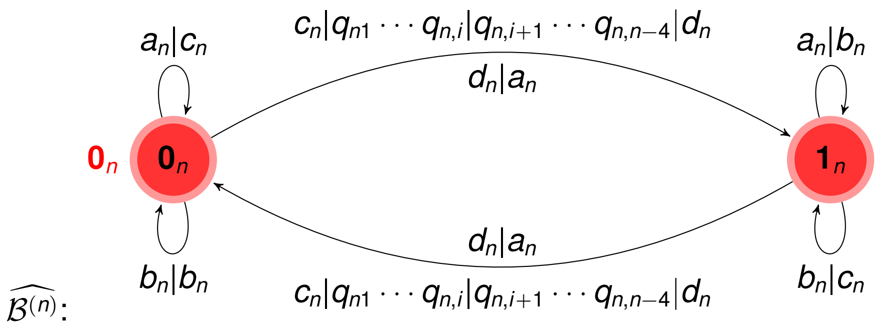


Figure 26 – Moore diagram of the dual automaton of $\mathcal{B}^{(n)}$.

Because the states $\mathbf{0}_n$ and $\mathbf{1}_n$ induce permutations on our new alphabet $Y^{(n)} = \{a_n, b_n, c_n, q_{n1}, \cdots, q_{n,n-4}, d_n\}$, the Bellaterra automaton $\widehat{\mathcal{B}^{(n)}}$ is invertible; thus, $\mathcal{B}^{(n)}$ is bireversible and the dual group $\Gamma^{(n)}$ is well defined.

By the Moore diagram given above, the wreath recursion defining $\widehat{\mathcal{B}^{(n)}}$ is given by

$$\begin{aligned} \mathbf{0}_n &= (\mathbf{0}_n, \mathbf{0}_n, \mathbf{1}_n, \mathbf{K}_{n1}, \cdots, \mathbf{K}_{n,n-4}, \mathbf{1}_n)(a_n c_n q_{n1} q_{n2} \cdots q_{n,n-4} d_n), \\ \mathbf{1}_n &= (\mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_n, \mathbf{L}_{n1}, \cdots, \mathbf{L}_{n,n-4}, \mathbf{0}_n)(a_n b_n c_n q_{n1} q_{n2} \cdots q_{n,n-4} d_n), \end{aligned} \tag{3.2}$$

with $\mathbf{K}_{n,i} = \mathbf{0}_n$ and $\mathbf{L}_{n,i} = \mathbf{1}_n$ if $\sigma_{n,i} = \mathbb{1}$ and $\mathbf{K}_{n,i} = \mathbf{1}_n$ and $\mathbf{L}_{n,i} = \mathbf{0}_n$ otherwise.

Again, from the Moore diagram of $\widehat{\mathcal{B}^{(n)}}$, we obtain the one describing the inverse automaton $(\widehat{\mathcal{B}^{(n)}})^{-1}$.

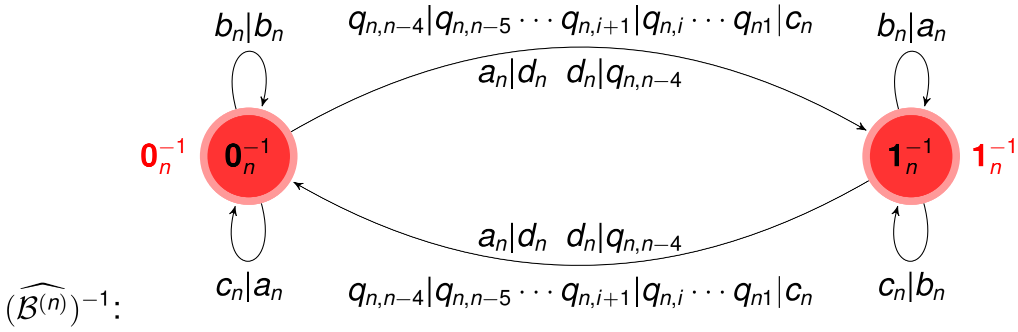


Figure 27 – Moore diagram of the inverse automaton of $\widehat{\mathcal{B}^{(n)}}$.

Then, the wreath recursion defining $\widehat{\mathcal{B}^{(n)}}$ is

$$\begin{aligned}\mathbf{0}_n^{-1} &= (\mathbf{1}_n^{-1},\mathbf{0}_n^{-1},\mathbf{0}_n^{-1},\mathbf{1}_n^{-1},\mathbf{M}_{n1},\ \cdots\ ,\mathbf{M}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_n),\\ \mathbf{1}_n^{-1} &= (\mathbf{0}_n^{-1},\mathbf{1}_n^{-1},\mathbf{1}_n^{-1},\mathbf{0}_n^{-1},\mathbf{N}_{n1},\ \cdots\ ,\mathbf{N}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_nb_n),\end{aligned} \tag{3.3}$$

with $\mathbf{M}_{n,i} = \mathbf{0}_n^{-1}$ and $\mathbf{N}_{n,i} = \mathbf{1}_n^{-1}$ if $\sigma_{n,i} = \mathbb{1}$ and $\mathbf{M}_{n,i} = \mathbf{1}_n^{-1}$ and $\mathbf{N}_{n,i} = \mathbf{0}_n^{-1}$ otherwise.

Let $\widehat{T}^{(n)}$ be the subtree of the tree $T^{(n)}$ consisting of all words over the alphabet $Y^{(n)} = \{a_n, b_n, c_n, q_{n1}, q_{n2},\ \cdots\ , q_{n,n-4}, d_n\}$ with no double letters. Such subtree is represented in red in the figure below:

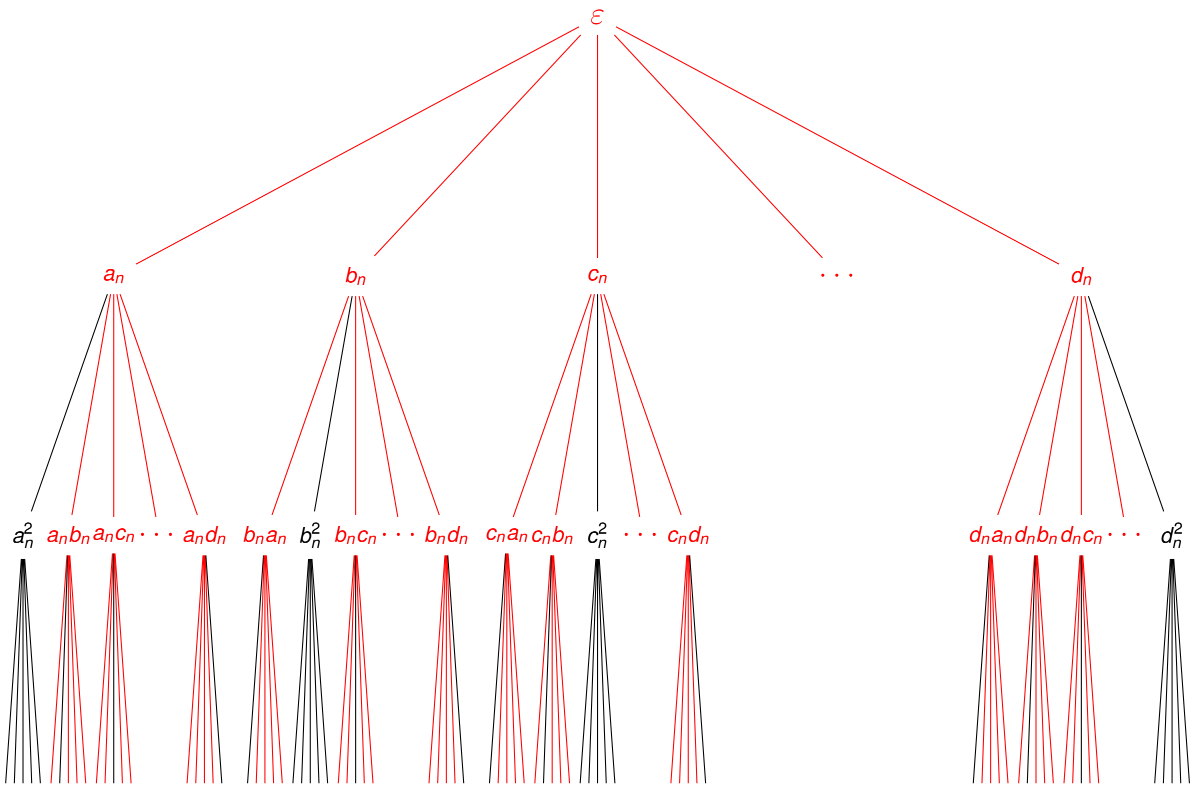


Figure 28 – Tree $T^{(n)}$ and its subtree $\widehat{T}^{(n)}$.

Notice that the empty word ε, root of $\widehat{T}^{(n)}$, has n descendants while all other vertices have $n-1$ descendants; in addition, $\widehat{T}^{(n)}$ is invariant under the action of $\Gamma^{(n)}$.

Many results from Chapter 2 will be generalized to the case of $\mathcal{B}^{(n)}$. First, note that all generators of $\mathcal{G}^{(n)}$ are involutions; that is,

$$a_n^2 = b_n^2 = c_n^2 = q_{n1}^2 = \cdots = q_{n,n-4}^2 = d_n^2 = e. \tag{3.4}$$

In fact, by the wreath recursion (3.1) of $\mathcal{B}^{(n)}$, one obtains that

$$\begin{aligned} a_n^2 &= (c_n, b_n)(c_n, b_n) \\ &= (c_n^2, b_n^2), \end{aligned}$$

$$\begin{aligned} b_n^2 &= (b_n, c_n)(b_n, c_n) \\ &= (b_n^2, c_n^2), \end{aligned}$$

$$\begin{aligned} c_n^2 &= (q_{n1}, q_{n1})\sigma(q_{n1}, q_{n1})\sigma \\ &= (q_{n1}, q_{n1})(q_{n1}, q_{n1})\sigma\sigma \\ &= (q_{n1}^2, q_{n1}^2), \end{aligned}$$

$$\begin{aligned} q_{n,i}^2 &= (q_{n,i+1}, q_{n,i+1})\sigma_{n,i}(q_{n,i+1}, q_{n,i+1})\sigma_{n,i} \\ &= (q_{n,i+1}, q_{n,i+1})(q_{n,i+1}, q_{n,i+1})\sigma_{n,i}\sigma_{n,i} \\ &= (q_{n,i+1}^2, q_{n,i+1}^2), \end{aligned}$$

$$\begin{aligned} q_{n,n-4}^2 &= (d_n, d_n)\sigma_{n,n-4}(d_n, d_n)\sigma_{n,n-4} \\ &= (d_n, d_n)(d_n, d_n)\sigma_{n,n-4}\sigma_{n,n-4} \\ &= (d_n^2, d_n^2), \end{aligned}$$

$$\begin{aligned} d_n^2 &= (a_n, a_n)\sigma(a_n, a_n)\sigma \\ &= (a_n, a_n)(a_n, a_n)\sigma\sigma \\ &= (a_n^2, a_n^2). \end{aligned}$$

Note that, since $\sigma_{n,i} \in \mathrm{Sym}(\{0,1\})$ then $\sigma_{n,i}\sigma_{n,i} = \mathbb{1}$ for all $\sigma_{n,i}$. Also, all sections of all squares of elements of $Y^{(n)}$ act as the identity on the first level and, because such sections are still squares of elements of $Y^{(n)}$, in all sections the action is the identity one. Therefore, (3.4) holds.

Now, we settle some results equivalent to the ones found in the Section 2.2. First, we have that

$$\mathbf{0}_n\mathbf{1}_n^{-1} = \overline{(a_n b_n)} \tag{3.5}$$

and

$$\mathbf{0}_n^{-1}\mathbf{1}_n = \overline{(b_nc_n)}. \tag{3.6}$$

We follow the same steps as we did in Section 2.2 and verify that

$$\mathbf{0}_n\mathbf{0}_n^{-1} = \mathbf{0}_n^{-1}\mathbf{0}_n = \mathbf{1}_n\mathbf{1}_n^{-1} = \mathbf{1}_n^{-1}\mathbf{1}_n = e, \tag{3.7}$$

$$(\mathbf{0}_n\mathbf{1}_n^{-1})^2 = (\mathbf{0}_n^{-1}\mathbf{1}_n)^2 = e \tag{3.8}$$

and

$$\mathbf{1}_n\mathbf{0}_n^{-1} = \mathbf{0}_n\mathbf{1}_n^{-1} \text{ and } \mathbf{1}_n^{-1}\mathbf{0}_n = \mathbf{0}_n^{-1}\mathbf{1}_n \tag{3.9}$$

in order to prove (3.5) and (3.6).

By using relations (3.2) and (3.3) we obtain

$$\begin{aligned}\mathbf{0}_n\mathbf{0}_n^{-1} &= (\mathbf{0}_n, \mathbf{0}_n, \mathbf{1}_n, \mathbf{K}_{n1}, \cdots, \mathbf{K}_{n,n-4}, \mathbf{1}_n)(a_nc_nq_{n1}q_{n2}\cdots q_{n,n-4}d_n)\\ &\qquad (\mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{M}_{n1}, \cdots, \mathbf{M}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_n)\\ &= (\mathbf{0}_n, \mathbf{0}_n, \mathbf{1}_n, \mathbf{K}_{n1}, \cdots, \mathbf{K}_{n,n-4}, \mathbf{1}_n)(\mathbf{0}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{M}_{n1}, \cdots, \mathbf{M}_{n,n-4}, \mathbf{1}_n^{-1})\mathbb{1} \qquad (3.10)\\ &= (\mathbf{0}_n\mathbf{0}_n^{-1}, \mathbf{0}_n\mathbf{0}_n^{-1}, \mathbf{1}_n\mathbf{1}_n^{-1}, \mathbf{K}_{n1}\mathbf{M}_{n1}, \cdots, \mathbf{K}_{n,n-4}\mathbf{M}_{n,n-4}, \mathbf{1}_n\mathbf{1}_n^{-1}),\end{aligned}$$

$$\begin{aligned}\mathbf{0}_n^{-1}\mathbf{0}_n &= (\mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{M}_{n1}, \cdots, \mathbf{M}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_n)\\ &\qquad (\mathbf{0}_n, \mathbf{0}_n, \mathbf{1}_n, \mathbf{K}_{n1}, \cdots, \mathbf{K}_{n,n-4}, \mathbf{1}_n)(a_nc_nq_{n1}q_{n2}\cdots q_{n,n-4}d_n)\\ &= (\mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{M}_{n1}, \cdots, \mathbf{M}_{n,n-4})(\mathbf{1}_n, \mathbf{0}_n, \mathbf{0}_n, \mathbf{1}_n, \mathbf{K}_{n1}, \cdots, \mathbf{K}_{n,n-4})\mathbb{1} \qquad (3.11)\\ &= (\mathbf{1}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{0}_n, \mathbf{0}_n^{-1}\mathbf{0}_n, \mathbf{1}_n^{-1}\mathbf{1}_n, \mathbf{M}_{n1}\mathbf{K}_{n1}, \cdots, \mathbf{M}_{n,n-4}\mathbf{K}_{n,n-4}),\end{aligned}$$

$$\begin{aligned}\mathbf{1}_n\mathbf{1}_n^{-1} &= (\mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_n, \mathbf{L}_{n1}, \cdots, \mathbf{L}_{n,n-4}, \mathbf{0}_n)(a_nb_nc_nq_{n1}q_{n2}\cdots q_{n,n-4}d_n)\\ &\qquad (\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{N}_{n1}, \cdots, \mathbf{N}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_nb_n)\\ &= (\mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_n, \mathbf{L}_{n1}, \cdots, \mathbf{L}_{n,n-4}, \mathbf{0}_n)(\mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{N}_{n1}, \cdots, \mathbf{N}_{n,n-4}, \mathbf{0}_n^{-1})\mathbb{1} \qquad (3.12)\\ &= (\mathbf{1}_n\mathbf{1}_n^{-1}, \mathbf{1}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{0}_n^{-1}, \mathbf{L}_{n1}\mathbf{N}_{n1}, \cdots, \mathbf{L}_{n,n-4}\mathbf{N}_{n,n-4}, \mathbf{0}_n\mathbf{0}_n^{-1}),\end{aligned}$$

and

$$\begin{aligned}\mathbf{1}_n^{-1}\mathbf{1}_n &= (\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{N}_{n1}, \cdots, \mathbf{N}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_nb_n)\\ &\qquad (\mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_n, \mathbf{L}_{n1}, \cdots, \mathbf{L}_{n,n-4}, \mathbf{0}_n)(a_nb_nc_nq_{n1}q_{n2}\cdots q_{n,n-4}d_n)\\ &= (\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}, \mathbf{0}_n^{-1}, \mathbf{N}_{n1}, \cdots, \mathbf{N}_{n,n-4})(\mathbf{0}_n, \mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_n, \mathbf{L}_{n1}, \cdots, \mathbf{L}_{n,n-4})\mathbb{1} \qquad (3.13)\\ &= (\mathbf{0}_n^{-1}\mathbf{0}_n, \mathbf{1}_n^{-1}\mathbf{1}_n, \mathbf{1}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{0}_n, \mathbf{N}_{n1}\mathbf{L}_{n1}, \cdots, \mathbf{N}_{n,n-4}\mathbf{L}_{n,n-4}).\end{aligned}$$

Inspecting (3.10), (3.11), (3.12) and (3.13), we note that all sections of $\mathbf{0}_n\mathbf{0}_n^{-1}$, $\mathbf{0}_n^{-1}\mathbf{0}_n$, $\mathbf{1}_n\mathbf{1}_n^{-1}$ and $\mathbf{1}_n^{-1}\mathbf{1}_n$ contain only these same elements; this is true even with the elements of the form $\mathbf{K}_{n,i}\mathbf{M}_{n,i}$, $\mathbf{M}_{n,i}\mathbf{K}_{n,i}$, $\mathbf{L}_{n,i}\mathbf{N}_{n,i}$ or $\mathbf{N}_{n,i}\mathbf{L}_{n,i}$ due to the way that $\mathbf{K}_{n,i}$, $\mathbf{L}_{n,i}$, $\mathbf{M}_{n,i}$ and $\mathbf{N}_{n,i}$ were defined. Since all such elements act as the identity at the first level, the action of the identity happens in all levels of $\mathbf{0}_n\mathbf{0}_n^{-1}$, $\mathbf{0}_n^{-1}\mathbf{0}_n$, $\mathbf{1}_n\mathbf{1}_n^{-1}$ and $\mathbf{1}_n^{-1}\mathbf{1}_n$. Hence, (3.7) holds.

In order to get the equality (3.8), we calculate $\mathbf{0}_n\mathbf{1}_n^{-1}$ and $\mathbf{0}_n^{-1}\mathbf{1}_n$ and then we determine their squares. Still by (3.2) and (3.3), one obtains

$$\begin{aligned}\mathbf{0}_n\mathbf{1}_n^{-1} &= (\mathbf{0}_n,\mathbf{0}_n,\mathbf{1}_n,\mathbf{K}_{n1},\ \cdots,\mathbf{K}_{n,n-4},\mathbf{1}_n)(a_nc_nq_{n1}q_{n2}\cdots q_{n,n-4}d_n)\\ &\qquad(\mathbf{0}_n^{-1},\mathbf{1}_n^{-1},\mathbf{1}_n^{-1},\mathbf{0}_n^{-1},\mathbf{N}_{n1},\ \cdots,\mathbf{N}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_nb_n)\\ &= (\mathbf{0}_n,\mathbf{0}_n,\mathbf{1}_n,\mathbf{K}_{n1},\ \cdots,\mathbf{K}_{n,n-4},\mathbf{1}_n)(\mathbf{1}_n^{-1},\mathbf{1}_n^{-1},\mathbf{0}_n^{-1},\mathbf{N}_{n1},\ \cdots,\mathbf{N}_{n,n-4},\mathbf{0}_n^{-1})(a_nb_n)\\ &= (\mathbf{0}_n\mathbf{1}_n^{-1},\mathbf{0}_n\mathbf{1}_n^{-1},\mathbf{1}_n\mathbf{0}_n^{-1},\mathbf{K}_{n1}\mathbf{N}_{n1},\ \cdots,\mathbf{K}_{n,n-4}\mathbf{N}_{n,n-4},\mathbf{1}_n\mathbf{0}_n^{-1})(a_nb_n) \qquad (3.14)\end{aligned}$$

and, thus,

$$\begin{aligned}(\mathbf{0}_n\mathbf{1}_n^{-1})^2 &= (\mathbf{0}_n\mathbf{1}_n^{-1},\mathbf{0}_n\mathbf{1}_n^{-1},\mathbf{1}_n\mathbf{0}_n^{-1},\mathbf{K}_{n1}\mathbf{N}_{n1},\ \cdots,\mathbf{K}_{n,n-4}\mathbf{N}_{n,n-4},\mathbf{1}_n\mathbf{0}_n^{-1})(a_nb_n)\\ &\qquad(\mathbf{0}_n\mathbf{1}_n^{-1},\mathbf{0}_n\mathbf{1}_n^{-1},\mathbf{1}_n\mathbf{0}_n^{-1},\mathbf{K}_{n1}\mathbf{N}_{n1},\ \cdots,\mathbf{K}_{n,n-4}\mathbf{N}_{n,n-4},\mathbf{1}_n\mathbf{0}_n^{-1})(a_nb_n)\\ &= ((\mathbf{0}_n\mathbf{1}_n^{-1})^2,(\mathbf{0}_n\mathbf{1}_n^{-1})^2,(\mathbf{1}_n\mathbf{0}_n^{-1})^2,(\mathbf{K}_{n1}\mathbf{N}_{n1})^2,\ \cdots,(\mathbf{K}_{n,n-4}\mathbf{N}_{n,n-4})^2,(\mathbf{1}_n\mathbf{0}_n^{-1})^2)\\ &\qquad (3.15)\end{aligned}$$

Further, one gets that

$$\begin{aligned}\mathbf{0}_n^{-1}\mathbf{1}_n &= (\mathbf{1}_n^{-1},\mathbf{0}_n^{-1},\mathbf{0}_n^{-1},\mathbf{1}_n^{-1},\mathbf{M}_{n1},\ \cdots,\mathbf{M}_{n,n-4})(a_nd_nq_{n,n-4}q_{n,n-5}\cdots q_{n2}q_{n1}c_n)\\ &\qquad(\mathbf{1}_n,\mathbf{1}_n,\mathbf{0}_n,\mathbf{L}_{n1},\ \cdots,\mathbf{L}_{n,n-4},\mathbf{0}_n)(a_nb_nc_nq_{n1}q_{n2}\cdots q_{n,n-4}d_n)\\ &= (\mathbf{1}_n^{-1},\mathbf{0}_n^{-1},\mathbf{0}_n^{-1},\mathbf{1}_n^{-1},\mathbf{M}_{n1},\ \cdots,\mathbf{M}_{n,n-4})(\mathbf{0}_n,\mathbf{1}_n,\mathbf{1}_n,\mathbf{0}_n,\mathbf{L}_{n1},\ \cdots,\mathbf{L}_{n,n-4}) \qquad (3.16)\\ &= (\mathbf{1}_n^{-1}\mathbf{0}_n,\mathbf{0}_n^{-1}\mathbf{1}_n,\mathbf{0}_n^{-1}\mathbf{1}_n,\mathbf{1}_n^{-1}\mathbf{0}_n,\mathbf{M}_{n1}\mathbf{L}_{n1},\ \cdots,\mathbf{M}_{n,n-4}\mathbf{L}_{n,n-4})(b_nc_n)\end{aligned}$$

and, thus,

$$\begin{aligned}(\mathbf{0}_n^{-1}\mathbf{1}_n)^2 &= (\mathbf{1}_n^{-1}\mathbf{0}_n,\mathbf{0}_n^{-1}\mathbf{1}_n,\mathbf{0}_n^{-1}\mathbf{1}_n,\mathbf{1}_n^{-1}\mathbf{0}_n,\mathbf{M}_{n1}\mathbf{L}_{n1},\ \cdots,\mathbf{M}_{n,n-4}\mathbf{L}_{n,n-4})(b_nc_n)\\ &\qquad(\mathbf{1}_n^{-1}\mathbf{0}_n,\mathbf{0}_n^{-1}\mathbf{1}_n,\mathbf{0}_n^{-1}\mathbf{1}_n,\mathbf{1}_n^{-1}\mathbf{0}_n,\mathbf{M}_{n1}\mathbf{L}_{n1},\ \cdots,\mathbf{M}_{n,n-4}\mathbf{L}_{n,n-4})(b_nc_n)\\ &= ((\mathbf{1}_n^{-1}\mathbf{0}_n)^2,(\mathbf{0}_n^{-1}\mathbf{1}_n)^2,(\mathbf{0}_n^{-1}\mathbf{1}_n)^2,(\mathbf{1}_n^{-1}\mathbf{0}_n)^2,(\mathbf{M}_{n1}\mathbf{L}_{n1})^2,\ \cdots,(\mathbf{M}_{n,n-4}\mathbf{L}_{n,n-4})^2).\\ &\qquad (3.17)\end{aligned}$$

Remark 3.2.1. The elements $\mathbf{K}_{n,i}\mathbf{N}_{n,i}$ in (3.14) and (3.15) are either $\mathbf{0}_n\mathbf{1}_n^{-1}$ or $\mathbf{1}_n\mathbf{0}_n^{-1}$ while the elements $\mathbf{M}_{n,i}\mathbf{L}_{n,i}$ in (3.16) and (3.17) are either $\mathbf{0}_n^{-1}\mathbf{1}_n$ or $\mathbf{1}_n^{-1}\mathbf{0}_n$ by definition of $\mathbf{K}_{n,i}$, $\mathbf{L}_{n,i}$, $\mathbf{M}_{n,i}$ and $\mathbf{N}_{n,i}$.

With respect to proving (3.15) and (3.17) we see that, by an argument similar to the one given previously (in the proof of the equality (3.7)) and by Remark 3.2.1, all actions on all sections of $(\mathbf{0}_n\mathbf{1}_n^{-1})^2$ and $(\mathbf{0}_n^{-1}\mathbf{1}_n)^2$ in all levels correspond to the identity. Therefore, (3.8) holds.

Now, for the sake of proving (3.9), we use (3.8) to obtain

$$\begin{aligned}\mathbf{1}_n\mathbf{0}_n^{-1} &= \mathbf{0}_n\mathbf{0}_n^{-1}\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n\mathbf{1}_n^{-1}\\ &= \mathbf{0}_n(\mathbf{0}_n^{-1}\mathbf{1}_n)^2\mathbf{1}_n^{-1}\\ &= \mathbf{0}_n\mathbf{1}_n^{-1}\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}\mathbf{1}_n^{-1}\mathbf{0}_n &= \mathbf{0}_n^{-1}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{1}_n\\ &= \mathbf{0}_n^{-1}(\mathbf{0}_n\mathbf{1}_n^{-1})^2\mathbf{1}_n\\ &= \mathbf{0}_n^{-1}\mathbf{1}_n.\end{aligned}\tag{3.19}$$

Then, by (3.14), (3.18) and by the definition in (2.16),

$$\begin{aligned}\mathbf{0}_n\mathbf{1}_n^{-1} &= (\mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{1}_n\mathbf{0}_n^{-1}, \mathbf{K}_{n1}\mathbf{N}_{n1}, \cdots, \mathbf{K}_{n,n-4}\mathbf{N}_{n,n-4}, \mathbf{1}_n\mathbf{0}_n^{-1})(a_nb_n)\\ &= (\mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \cdots, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1})(a_nb_n)\\ &= \overline{(a_nb_n)}.\end{aligned}\tag{3.20}$$

Remark 3.2.2. Having in mind Remark 3.2.1 and using (3.18), we are able to switch all occurences of $\mathbf{K}_{n,i}\mathbf{N}_{n,i} = \mathbf{1}_n\mathbf{0}_n^{-1}$ by $\mathbf{0}_n\mathbf{1}_n^{-1}$, leading to the second equality in (3.20).

Also, by (3.16), (3.19) and by the definition in (2.16), one gets

$$\begin{aligned}\mathbf{0}_n^{-1}\mathbf{1}_n &= (\mathbf{1}_n^{-1}\mathbf{0}_n, \mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{1}_n^{-1}\mathbf{0}_n, \mathbf{M}_{n1}\mathbf{L}_{n1}, \cdots, \mathbf{M}_{n,n-4}\mathbf{L}_{n,n-4})(b_nc_n)\\ &= (\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n^{-1}\mathbf{1}_n, \cdots, \mathbf{0}_n^{-1}\mathbf{1}_n)(b_nc_n)\\ &= \overline{(b_nc_n)}.\end{aligned}\tag{3.21}$$

Similarly to the case explored in Remark 3.2.2, using Remark 3.2.1 and (3.19), all occurrences of $\mathbf{M}_{n,i}\mathbf{L}_{n,i} = \mathbf{1}_n^{-1}\mathbf{0}_n$ are switched by $\mathbf{0}_n^{-1}\mathbf{1}_n$ and then the second equality in (3.21) holds.

By (3.20) and (3.21), the relations (3.5) and (3.6) are verified. Furthermore, by calculations analogous to the ones in (2.33), we get that $\overline{(a_nc_n)} \in \Gamma^{(n)}$. Indeed,

$$\begin{aligned}\overline{(a_nb_n)} \cdot \overline{(b_nc_n)} = \mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n &= (\mathbf{0}_n\mathbf{1}_n^{-1}, \cdots, \mathbf{0}_n\mathbf{1}_n^{-1})(a_nb_n)(\mathbf{0}_n^{-1}\mathbf{1}_n, \cdots, \mathbf{0}_n^{-1}\mathbf{1}_n)(b_nc_n)\\ &= (\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \cdots, \mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n)(a_nc_n)\\ &= \overline{(a_nc_n)},\end{aligned}\tag{3.22}$$

which implies that $\overline{(a_nc_n)} \in \Gamma^{(n)}$, since $\overline{(a_nb_n)}, \overline{(b_nc_n)} \in \Gamma^{(n)}$ (by (3.20) and (3.21)).

Also similarly to (2.34) and (2.35), we define the elements $\alpha_n = \mathbf{1}_n \cdot \overline{(a_n c_n)}$ and $\beta_n = \mathbf{0}_n \cdot \overline{(a_n c_n)}$, finding that

$$\begin{aligned}
\alpha_n &= \mathbf{1}_n \cdot \overline{(a_n c_n)} \\
&= (\mathbf{1}_n, \mathbf{1}_n, \mathbf{0}_n, \mathbf{L}_{n1}, \cdots, \mathbf{L}_{n,n-4}, \mathbf{0}_n)(a_n b_n c_n q_{n1} \cdots q_{n,n-4} d_n) \\
&\qquad\qquad (\overline{(a_n c_n)}, \overline{(a_n c_n)}, \overline{(a_n c_n)}, \overline{(a_n c_n)})(a_n c_n) \\
&= (\mathbf{1}_n\overline{(a_n c_n)}, \mathbf{1}_n\overline{(a_n c_n)}, \mathbf{0}_n\overline{(a_n c_n)}, \mathbf{L}_{n1}\overline{(a_n c_n)}, \cdots, \mathbf{L}_{n,n-4}\overline{(a_n c_n)}, \mathbf{0}_n\overline{(a_n c_n)})(a_n b_n) \\
&\qquad\qquad (c_n q_{n1} \cdots q_{n,n-4} d_n) \\
&= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \cdots, \gamma_{n,n-4}, \beta_n)(a_n b_n)(c_n q_{n1} \cdots q_{n,n-4} d_n) \qquad (3.23)
\end{aligned}$$

and

$$\begin{aligned}
\beta_n &= \mathbf{0}_n \cdot \overline{(a_n c_n)} \\
&= (\mathbf{0}_n, \mathbf{0}_n, \mathbf{1}_n, \mathbf{K}_{n1}, \cdots, \mathbf{K}_{n,n-4}, \mathbf{1}_n)(a_n c_n q_{n1} \cdots q_{n,n-4} d_n) \\
&\qquad\qquad (\overline{(a_n c_n)}, \overline{(a_n c_n)}, \overline{(a_n c_n)}, \overline{(a_n c_n)})(a_n c_n) \\
&= (\mathbf{0}_n\overline{(a_n c_n)}, \mathbf{0}_n\overline{(a_n c_n)}, \mathbf{1}_n\overline{(a_n c_n)}, \mathbf{K}_{n1}\overline{(a_n c_n)}, \cdots, \mathbf{K}_{n,n-4}\overline{(a_n c_n)}, \mathbf{1}_n\overline{(a_n c_n)}) \\
&\qquad\qquad (c_n q_{n1} \cdots q_{n,n-4} d_n) \\
&= (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \cdots, \delta_{n,n-4}, \alpha_n)(c_n q_{n1} \cdots q_{n,n-4} d_n), \qquad (3.24)
\end{aligned}$$

in which $\gamma_{n,i} = \alpha_n$ and $\delta_{n,i} = \beta_n$ if $\mathbf{L}_{n,i} = \mathbf{1}_n$ or $\gamma_{n,i} = \beta_n$ and $\delta_{n,i} = \alpha_n$ otherwise.

In Section 2.2, by showing that $\beta\alpha^{-1} = \overline{(ab)}$ we described a new generating set for the dual group Γ, that is $\{\alpha, \beta, \overline{(bc)}\}$. Related to this, our aim in this section is to show that $\Gamma^{(n)} = \langle \alpha_n, \beta_n, \overline{(b_n c_n)} \rangle$ by showing that

$$\beta_n \alpha_n^{-1} = \overline{(a_n b_n)}. \qquad (3.25)$$

In fact, using (3.23) and (3.22), we obtain explicitly

$$\begin{aligned}
\alpha_n &= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \cdots, \gamma_{n,n-4}, \beta_n)(a_n b_n)(c_n q_{n1} \cdots q_{n,n-4} d_n) \\
&= (\mathbf{1}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{1}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \\
&\qquad \mathbf{P}_{n1}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \cdots, \mathbf{P}_{n,n-4}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n)(a_n b_n)(c_n q_{n1} \cdots q_{n,n-4} d_n),
\end{aligned} \qquad (3.26)$$

with $\mathbf{P}_{n,i} = \mathbf{1}_n$ if $\gamma_{n,i} = \alpha_n$ and $\mathbf{P}_{n,i} = \mathbf{0}_n$ if $\gamma_{n,i} = \beta_n$.

Whence, α_n^{-1} is given by

$$\begin{aligned}
\alpha_n^{-1} = (&\mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{0}_n^{-1}, \\
&\mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{Q}_{n1}, \cdots, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{Q}_{n,n-4})(a_n b_n)(c_n d_n q_{n,n-4} \cdots q_{n1}),
\end{aligned} \qquad (3.27)$$

with $\mathbf{Q}_{n,i} = \mathbf{1}_n^{-1}$ if $\mathbf{P}_{n,i} = \mathbf{1}_n$ and $\mathbf{Q}_{n,i} = \mathbf{0}_n^{-1}$ if $\mathbf{P}_{n,i} = \mathbf{0}_n$.

It is easily verified that $\alpha_n\alpha_n^{-1} = \alpha_n^{-1}\alpha_n = e$ by equations (3.26) and (3.27). The computation is straightforward; however, this verification becomes very cumbersome and is therefore omitted.

Using the definition of β_n given in (3.24) together with (3.22) and defining $\mathbf{R}_{n,i} = \mathbf{1}_n^{-1}$ if $\mathbf{K}_{n,i} = \mathbf{1}_n$ and $\mathbf{R}_{n,i} = \mathbf{0}_n^{-1}$ if $\mathbf{K}_{n,i} = \mathbf{0}_n$, we see that

$$\begin{aligned}
\beta_n\alpha_n^{-1} &= (\mathbf{0}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{1}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{R}_{n1}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \cdots, \\
&\qquad \mathbf{R}_{n,n-4}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{1}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n)(c_nq_{n1}\cdots q_{n,n-4}d_n)(\mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1}, \\
&\qquad \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{Q}_{n1}, \cdots, \\
&\qquad \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{Q}_{n,n-4})(a_nb_n)(c_nd_nq_{n,n-4}\cdots q_{n1}) \\
&= (\mathbf{0}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{0}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{1}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{R}_{n1}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \cdots, \\
&\qquad \mathbf{R}_{n,n-4}\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n, \mathbf{1}_n\mathbf{0}_n\mathbf{1}_n^{-1}\mathbf{0}_n^{-1}\mathbf{1}_n)(\mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1}, \\
&\qquad \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{0}_n^{-1}, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{Q}_{n1}, \cdots, \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{Q}_{n,n-4}, \\
&\qquad \mathbf{1}_n^{-1}\mathbf{0}_n\mathbf{1}_n\mathbf{0}_n^{-1}\mathbf{1}_n^{-1})(a_nb_n) \\
&= (\mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{1}_n\mathbf{0}_n^{-1}, \mathbf{R}_{n1}\mathbf{Q}_{n1}, \cdots, \mathbf{R}_{n,n-4}\mathbf{Q}_{n,n-4}, \mathbf{1}_n\mathbf{0}_n^{-1}) \\
&= (\mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1}, \cdots, \mathbf{0}_n\mathbf{1}_n^{-1}, \mathbf{0}_n\mathbf{1}_n^{-1})(a_nb_n) \qquad (3.28) \\
&= \overline{(a_nb_n)}.
\end{aligned}$$

Remark 3.2.3. Due to the way that $\mathbf{R}_{n,i}$ and $\mathbf{Q}_{n,i}$ were defined, we have that either $\mathbf{R}_{n,i} = \mathbf{0}_n\mathbf{1}_n^{-1}$ or $\mathbf{R}_{n,i} = \mathbf{1}_n\mathbf{0}_n^{-1}$. If this last equality occurs, equation (3.18) makes the fourth equality in (3.28) become true.

In conclusion, equality (3.25) holds. This implies that, by similar arguments to those given in Section 2.2, we get a new generating set for $\Gamma^{(n)}$ which can be described, then, as

$$\Gamma^{(n)} = \langle \alpha_n, \beta_n, \overline{(b_nc_n)} \rangle.$$

The importance of the following lemma to this work comes from the fact that such result establishes a relation between the actions of the groups Γ and $\Gamma^{(n)}$ so it promotes a link between the results of this chapter and all results shown in the previous chapter. Considering that the tree $\widehat{T}$ naturally embeds in the tree $\widehat{T}^{(n)}$ via an homomorphism of monoids induced by $a \mapsto a_n$, $b \mapsto b_n$, $c \mapsto c_n$, $d \mapsto d_n$ and knowing that the action on the letters not in the image of $\widehat{T}$ is defined to be trivial, then we get that Γ also acts on $\Gamma^{(n)}$.

Lemma 3.2.4. For any $v \in \Gamma$, there exists $v' \in \Gamma^{(n)}$ with the following property: for any word g over $\{a_n, b_n, c_n\}$ such that $v(g)$ is also a word over $\{a_n, b_n, c_n\}$, we have $v(g) = v'(g)$.

Proof. Based on the homomorphism of monoids cited above we can build a correspondence $a \to a_n$, $b \to b_n$, $c \to c_n$, $d \to d_n$, so we can consider $x_1x_2\cdots x_n \in \langle \alpha, \beta, \overline{(b_nc_n)}\rangle$, a word over $\{\alpha, \beta, \overline{(b_nc_n)}\}$, representing the vertex v. We also determine a word $y_1y_2\cdots y_n \in \langle \alpha_n, \beta_n, \overline{(b_nc_n)}\rangle$ in which each y_i is related to x_i and it is defined by the following rule:

1. If $x_i = \overline{(b_nc_n)}$, then define $y_i = x_i$;
2. If $x_i = \alpha$ (respectively $x_i = \beta$), then compute the total number of α and β among $x_1x_2\cdots x_{i-1}$. If such number is even, then define $y_i = \alpha_n$ (resp. $y_i = \beta_n$); if this number is odd, put $y_i = \alpha_n^{-1}$ (resp. $y_i = \beta_n^{-1}$).

Now, let $g \in \langle a_n, b_n, c_n\rangle$ be the arbitrary word matching with the word given in the statement of this lemma. Our intention is to show by induction on i that $y_1y_2\cdots y_i(g)$ can be obtained from $x_1x_2\cdots x_i(g)$ by replacing all occurrences of d_n by q_{n1} when the total number of α and β among $x_1x_2\cdots x_{i-1}$ is odd and it coincides with $x_1x_2\cdots x_i(g)$ if this number is even.

If $i = 0$, the assertion is true so we are left to prove the induction step. Suppose, then, that the considerations of the last paragraph hold for $y_1y_2\cdots y_i(g)$ and $x_1x_2\cdots x_i(g)$. Note that, if $x_{i+1} = y_{i+1} = \overline{(b_nc_n)}$ then, since $\overline{(b_nc_n)}$ fixes letters d_n and q_{n1}, the relation between $y_1y_2\cdots y_{i+1}(g)$ and $x_1x_2\cdots x_{i+1}(g)$ is the same as between $y_1y_2\cdots y_i(g)$ and $x_1x_2\cdots x_i(g)$ so in this case we already have the result. Thus, we can assume $x_{i+1} = \alpha$ or $x_{i+1} = \beta$.

Agreeing with we want to show, we divide the total number of occurrences of α and β among $x_1x_2\cdots x_i$ in two cases: first, suppose that this number is odd. By induction assumption, $y_1y_2\cdots y_i(g)$ is obtained from $x_1x_2\cdots x_i(g)$ by replacing all occurrences of d_n by q_{n1} so $y_1y_2\cdots y_i(g) \in \langle a_n, b_n, c_n, q_{n1}\rangle$. If $x_{i+1} = \alpha$ ($x_{i+1} = \beta$), then $y_{i+1} = \alpha_n^{-1}$ ($y_{i+1} = \beta_n^{-1}$) by construction. We claim that the following relations hold for α_n^{-1} and β_n^{-1}:

$$\begin{aligned}
\alpha_n^{-1} &= (\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \beta_n^{-1}, \gamma_{n1}^{-1}, \cdots, \gamma_{n,n-4}^{-1})(a_nb_n)(c_nd_nq_{n,n-4}\cdots q_{n1}), \\
\beta_n^{-1} &= (\beta_n^{-1}, \beta_n^{-1}, \alpha_n^{-1}, \alpha_n^{-1}, \delta_{n1}^{-1}, \cdots, \delta_{n,n-4}^{-1})(c_nd_nq_{n,n-4}\cdots q_{n1}),
\end{aligned} \tag{3.29}$$

with $\gamma_{n,i}$ and $\delta_{n,i}$ defined in (3.23) and (3.24), respectively.

Indeed,

$$\begin{aligned}
\alpha_n\alpha_n^{-1} &= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \cdots, \gamma_{n,n-4}, \beta_n)(a_nb_n)(c_nq_{n1}\cdots q_{n,n-4}d_n) \\
&\qquad (\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \beta_n^{-1}, \gamma_{n1}^{-1}, \cdots, \gamma_{n,n-4}^{-1})(a_nb_n)(c_nd_nq_{n,n-4}\cdots q_{n1}) \\
&= (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \cdots, \gamma_{n,n-4}, \beta_n)(\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \gamma_{n1}^{-1}, \cdots, \gamma_{n,n-4}^{-1}, \beta_n^{-1})\mathbb{1} \\
&= (\alpha_n\alpha_n^{-1}, \alpha_n\alpha_n^{-1}, \beta_n\beta_n^{-1}, \gamma_{n1}\gamma_{n1}^{-1}, \cdots, \gamma_{n,n-4}\gamma_{n,n-4}^{-1}, \beta_n\beta_n^{-1}),
\end{aligned}$$

$$\begin{aligned}
\alpha_n^{-1}\alpha_n &= (\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \beta_n^{-1}, \gamma_{n1}^{-1}, \cdots, \gamma_{n,n-4}^{-1})(a_nb_n)(c_nd_nq_{n,n-4}\cdots q_{n1}) \\
&\qquad (\alpha_n, \alpha_n, \beta_n, \gamma_{n1}, \cdots, \gamma_{n,n-4}, \beta_n)(a_nb_n)(c_nq_{n1}\cdots q_{n,n-4}d_n) \\
&= (\alpha_n^{-1}, \alpha_n^{-1}, \beta_n^{-1}, \beta_n^{-1}, \gamma_{n1}^{-1}, \cdots, \gamma_{n,n-4}^{-1})(\alpha_n, \alpha_n, \beta_n, \beta_n, \gamma_{n1}, \cdots, \gamma_{n,n-4})\mathbb{1} \\
&= (\alpha_n^{-1}\alpha_n, \alpha_n^{-1}\alpha_n, \beta_n^{-1}\beta_n, \beta_n^{-1}\beta_n, \gamma_{n1}^{-1}\gamma_{n1}, \cdots, \gamma_{n,n-4}^{-1}\gamma_{n,n-4}),
\end{aligned}$$

$$\begin{aligned}
\beta_n\beta_n^{-1} &= (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \cdots, \delta_{n,n-4}, \alpha_n)(c_nq_{n1}\cdots q_{n,n-4}d_n) \\
&\qquad (\beta_n^{-1}, \beta_n^{-1}, \alpha_n^{-1}, \alpha_n^{-1}, \delta_{n1}^{-1}, \cdots, \delta_{n,n-4}^{-1})(c_nd_nq_{n,n-4}\cdots q_{n1}) \\
&= (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \cdots, \delta_{n,n-4}, \alpha_n)(\beta_n^{-1}, \beta_n^{-1}, \alpha_n^{-1}, \delta_{n1}^{-1}, \cdots, \delta_{n,n-4}^{-1}, \alpha_n^{-1})\mathbb{1} \\
&= (\beta_n\beta_n^{-1}, \beta_n\beta_n^{-1}, \alpha_n\alpha_n^{-1}, \delta_{n1}\delta_{n1}^{-1}, \cdots, \delta_{n,n-4}\delta_{n,n-4}^{-1}, \alpha_n\alpha_n^{-1}),
\end{aligned}$$

$$\begin{aligned}
\beta_n^{-1}\beta_n &= (\beta_n^{-1}, \beta_n^{-1}, \alpha_n^{-1}, \alpha_n^{-1}, \delta_{n1}^{-1}, \cdots, \delta_{n,n-4}^{-1})(c_nd_nq_{n,n-4}\cdots q_{n1}) \\
&\qquad (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \cdots, \delta_{n,n-4}, \alpha_n)(c_nq_{n1}\cdots q_{n,n-4}d_n) \\
&= (\beta_n^{-1}, \beta_n^{-1}, \alpha_n^{-1}, \alpha_n^{-1}, \delta_{n1}^{-1}, \cdots, \delta_{n,n-4}^{-1})(\beta_n, \beta_n, \alpha_n, \alpha_n, \delta_{n1}, \cdots, \delta_{n,n-4})\mathbb{1} \\
&= (\beta_n^{-1}\beta_n, \beta_n^{-1}\beta_n, \alpha_n^{-1}\alpha_n, \alpha_n^{-1}\alpha_n, \delta_{n1}^{-1}\delta_{n1}, \cdots, \delta_{n,n-4}^{-1}\delta_{n,n-4})
\end{aligned}$$

and, since all sections of $\alpha_n\alpha_n^{-1}$, $\alpha_n^{-1}\alpha_n$, $\beta_n\beta_n^{-1}$ and $\beta_n^{-1}\beta_n$ act by the identity and so on, we conclude that $\alpha_n\alpha_n^{-1} = \alpha_n^{-1}\alpha_n = \beta_n\beta_n^{-1} = \beta_n^{-1}\beta_n = e$; therefore, the equalities for α_n^{-1} and β_n^{-1} in (3.29) hold.

Remark 3.2.5. Note that α^{-1} was already defined in (3.27); however, the simpler definition of such element given in (3.29) fits better in this proof.

Back to the current case of the induction proof, by (3.29) the images of $y_1y_2\cdots y_i(g)$ under the actions of α_n^{-1} and β_n^{-1} coincide with the images of $x_1x_2\cdots x_i(g)$ under the actions of α and β, respectively. This happens because, by induction hypothesis, all d_n's are switched by q_{n1} and both actions of α_n^{-1} and β_n^{-1} take q_{n1} to c_n, mirroring the actions of α and β (see (2.41)), since c_n is taken to d_n by the actions of α_n^{-1} and β_n^{-1}. Thus, we have that $y_1y_2\cdots y_{i+1}(g) = x_1x_2\cdots x_{i+1}(g)$ which agrees with what we wanted to prove, considering that the total number of α and β among $x_1, x_2, \cdots, x_{i+1}$ is even (odd total of α and β among $x_1, x_2, \cdots, x_i$ and $x_{i+1} = \alpha$ or $x_{i+1} = \beta$).

Suppose now, that the number of the occurrences of α and β among $x_1, x_2, \cdots, x_i$ is even. By inductive hypothesis, $y_1y_2\cdots y_i(g) = x_1x_2\cdots x_i(g)$ and by such equality we have that $y_1y_2\cdots y_i(g)$ is a word over $\{a_n, b_n, c_n, d_n\}$. Furthermore, by construction, $y_{i+1} = \alpha_n$ or $y_{i+1} = \beta_n$. Observing the definitions of α_n and β_n in (3.23) and (3.24), one gets that the action of y_{i+1} on the letters of $y_1y_2\cdots y_i(g) \in \langle a_n, b_n, c_n, d_n\rangle$ coincides with the action of x_{i+1} except that both α_n and β_n move c_n to q_{n1} instead of moving it to d_n; the argument for this is similar to the given in the previous case: the actions of α and β match with the actions of α_n and b_n except in the case of c_n as explained above. Therefore, all instances of a_n, b_n and c_n are kept and, then, the resulting word $y_1y_2\cdots y_{i+1}(g)$ can be obtained from $x_1x_2\cdots x_{i+1}(g)$ by switching all d_n's in $y_1y_2\cdots y_i(g)$ by q_{n1}'s. Note that this agrees with what we need since the total number of α and β among $x_1, x_2, \cdots, x_i$ turns to be odd.

To finish the main proof by using the induction proved above, define a word $v' = y_1y_2\cdots y_k$ and notice that, if $v(g)$ is a word over $\{a_n, b_n, c_n\}$, then we have that $v'(g) = y_1y_2\cdots y_k(g)$ must coincide with $v(g)$ regardless the total number of α and β in the word representing v because of the relation between y_i and x_i and the arguments given in the induction assertion proved previously in this lemma. □

Lemma 3.2.6. The group $\Gamma^{(n)}$ acts transitively on the levels of $\widehat{T}^{(n)}$.

Proof. This lemma is also proved by induction on levels. We have that $\Gamma^{(n)}$ acts transitively on the first level of $\widehat{T}^{(n)}$ since this level is composed by all letters of $Y^{(n)}$ (see the beginning of this subsection for the definition of $Y^{(n)}$) and the generators of $\Gamma^{(n)}$ induce permutations on such set. Hence, suppose that $\Gamma^{(n)}$ acts transitively on the level m. In order to prove that such transitivity happens by using the previous lemmas, we will show that an arbitrary vertex of the $(m+1)$-th level can be moved to the vertex $a_nb_na_nb_n\cdots b_na_n$ or $a_nb_na_nb_n\cdots a_nb_n$ depending on the parity of m.

Let g be an arbitrary vertex of the level $m+1$ of $\widehat{T}^{(n)}$. Then, we can write $g = ht$, in which h is a vertex of the m-th level and $t \in Y^{(n)}$. Without loss of generality, assume that m is even. By the induction hypothesis, $\Gamma^{(n)}$ acts transitively on the level m of $\widehat{T}^{(n)}$ which means that there exists $v \in \Gamma^{(n)}$ such that $v(h) = a_nb_na_nb_n\cdots a_nb_n$. Thus,

$$v(g) = v(ht) = v(h)t' = a_nb_na_nb_n\cdots a_nb_nt',$$

with $t' \in Y^{(n)}$.

Recall that $\beta_n = (\beta_n, \beta_n, \alpha_n, \delta_{n1}, \cdots, \delta_{n,n-4}, \alpha_n)(c_nq_{n1}\cdots q_{n,n-4}d_n)$. Whence, by definition of the permutation in the definition of β_n, we have that β_n fixes the vertex $a_nb_na_nb_n\cdots a_nb_n$; further, one obtains that $\beta_n|_{a_nb_na_nb_n\cdots a_nb_n} = \beta_n$ by the sections of β_n. It implies that there exists a power k of β_n, $k = 1, \cdots, n-1$, satisfying $\beta_n^k(g) = a_nb_na_nb_n\cdots a_nb_nt''$, with $t'' \in \{a_n, b_n, c_n\}$, so we can assume that $t' \in \{a_n, b_n, c_n\}$. Considering the correspondence $a \to a_n$, $b \to b_n$, $c \to c_n$, $d \to d_n$ described right before Lemma 3.2.4 and by Lemma 2.2.10 we get that Γ acts transitively on the tree

$\widehat{T}^{(n)}$ so there exists $w \in \Gamma$ such that $w(a_nb_na_nb_n\cdots a_nb_nt') = a_nb_na_nb_n\cdots a_nb_na_n$. Since $a_nb_na_nb_n\cdots a_nb_n \in \langle a_n, b_n, c_n\rangle$, by Lemma 3.2.4 we obtain that there is $w' \in \Gamma^{(n)}$ satisfying

$$w'(a_nb_na_nb_n\cdots a_nb_nt') = w(a_nb_na_nb_n\cdots a_nb_nt') = a_nb_na_nb_n\cdots a_nb_na_n.$$

In the case that m is odd, by similar arguments as above we are able to find $w'' \in \Gamma^{(n)}$ satisfying $w''(a_nb_na_nb_n\cdots a_nb_na_ns') = a_nb_na_nb_n\cdots a_nb_n$ in the level $m+1$, with $s' \in Y^{(n)}$. Then, this proves our assertion so every word on the $(m+1)$-th level of $\widehat{T}^{(n)}$ can be taken to the word $a_nb_na_nb_n\cdots a_nb_na_n$ if m is even, and it can be taken to the word $a_nb_na_nb_n\cdots a_nb_n$ if m is odd.

Finally, suppose that g_1 and g_2 are arbitrary vertices of the tree $\widehat{T}^{(n)}$ belonging to the level $m+1$. By what we proved above, there exist $w_1, w_2 \in \Gamma^{(n)}$ such that the equality $w_1(g_1) = w_2(g_2) = a_nb_na_nb_n\cdots a_nb_na_n$ or $a_nb_na_nb_n\cdots a_nb_n$ holds (depending on the parity of m), which implies that $w_1w_2^{-1}(g_1) = g_2$. Since $w_1w_2^{-1} \in \Gamma^{(n)}$, the transitivity of $\Gamma^{(n)}$ on the levels of $\widehat{T}^{(n)}$ is guaranteed. $\square$

This last lemma implies that the Theorem 3.1.1 is proved by using Lemma 3.2.4 exactly in the same manner that Theorem 2.1.1 was proved by using Lemma 2.2.10, since the essence of Lemma 3.2.4 and Lemma 2.2.10 is the same: it is sufficient to replace Γ and $\widehat{T}$ by $\Gamma^{(n)}$ and $\widehat{T}^{(n)}$, respectively, and conveniently replace the elements of such groups.

Final remarks

The first chapter of this dissertation, devoted to the preliminaries of our work, is longer in comparison to the subsequent chapters; however, we thought it was better to present all the theory needed to completely understand the connection between groups and automata. Many examples were given in order to figure out how an automaton works and how its states are related to transformations that, in turn, are related to some specific groups. The intention behind this long introduction was to show nothing but a few important considerations about the Bellaterra automata and the main proofs in the subsequent chapters.

Chapters 2 and 3, which showed that the states from automata belonging to the Bellaterra automata family generate groups isomorphic to the free products of cyclic groups of order 2, were based on [18]. We followed the thoughts used in the proofs of the lemmas, corollaries, propositions and theorems of such article with the purpose of expanding and explaining all computations and techniques used in order to clarify completely the ideas behind them. Even the simplest relations were illustrated and explained.

The hope of the author is that this dissertation serves as a simple but good reference to beginners in Group Theory applied to automata like the author was, not a long time ago. Besides all the construction done concerning automata groups, the Bellaterra automata family example of groups generated by automata, introduced in the Savchuk and Vorobets' article and explored in details in this work, provided a meaningful contribution to the comprehension of the relation between automata and Geometric Group Theory.

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